Analytical studies of standing waves in three NLS models

By

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Abstract

In this work, we present analytical studies of standing waves in three NLS models. We first consider the spectral stability of ground states of fourth order semi-linear Schrödinger and Klein-Gordon equations and semi-linear Schrödinger and Klein-Gordon equations with fractional dispersion. We use Hamiltonian index counting theory, together with the information from a variational construction to develop sharp conditions for spectral stability for these waves. The second case is about the existence and the stability of the vortices for the NLS in higher dimensions. We extend the existence and stability results of Mizumachi from two-space dimensions to n space dimensions. Finally, the third equation we consider is a nonlocal NLS which comes from modeling nonlinear waves in Parity-time symmetric systems. Here again, we investigate the spectral stability of standing waves of its $\mathcal{PT}$ symmetric solutions.

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Contents

Chapter 1

Introduction

1.1 Basic Sobolev Spaces

For functions $u : \mathbb{R} \mapsto \mathbb{C}$, we define the L^p -norm for any $p \geq 1$,

$$\|u\|_p := \left(\int_{\mathbb{R}} |u(x)|^p \, dx \right)^{1/p}.$$

The L^∞ -norm is realized as the $p \to \infty$ limit of the L^p -norm, and is given for smooth functions by

$$\|u\|_\infty := \sup_{x \in \mathbb{R}} |u(x)|.$$

For any $p \geq 1$ the Bananch space $L^p(\mathbb{R})$ is given by

$$L^p(\mathbb{R}) := \{u : \|u\|_p < \infty\}.$$

For differentiable functions we define the $W^{k,p}$-norm

$$\|u\|_{W^{k,p}} := \left(\sum_{j=0}^{k} \|\partial_x^j u\|_p^p \right)^{\frac{1}{p}},$$

and the associated space

$$W^{k,p}(\mathbb{R}) := \{u : \|u\|_{W^{k,p}} < \infty\}.$$

The Hilbert spaces $H^k := W^{k,2}$ are used frequently throughout the text. We introduce the inner product

$$\langle f,g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx,$$

where the overbar denote complext conjugation. The spaces $H^k(\mathbb{R})$ are Hilbert spaces, since their norm is induced by the inner product

$$\|u\|_{H^k}^2 = \sum_{j=0}^{k} \left\langle \partial_x^j u, \partial_x^j u \right\rangle.$$

Introducing the Fourier transform of u,

$$\hat{u}(\eta) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\eta x} u(x)\,dx,$$

and its inverse,

$$u(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\eta x} \hat{u}(\eta)\,d\eta,$$

we have Plancherel's equality

$$\|u\|_2 = \|\hat{u}\|_2,$$

and one particularly useful property of the Fourier transform:

$$\widehat{\partial_x^l u} = (ik)^l \hat{u}.$$

Moreover, for each $k > 0$ the following norm is equivalent to the usual norm on $H^k(\mathbb{R})$,

$$\|u\|_{H^k}^2 = \int_{\mathbb{R}} \left(1 + |\eta|^{2k}\right) |\hat{u}(\eta)|^2\,d\eta.$$

1.2 The Point Spectrum: Sturm-Liouville Theory

1.2.1 Sturm-Liouville Operators on a Bounded Domain

A Sturm-Liouville operator L takes the form

$$Lp := \partial_x^2 p + a_1(x)\partial_x p + a_0(x)p,$$

and will also be called a Sturmian operator.

Consider L to be defined on the bounded interval $[-1,1]$, subject to boundary conditions

$$b_1^- p(-1) + b_2^- \partial_x p(-1) = 0, \;\; b_1^+ p(+1) + b_2^+ \partial_x p(+1) = 0. \tag{1.2.1}$$

Assume that $(b_1^\pm)^2 + (b_2^\pm)^2 > 0$, and the coefficients $a_1(x)$ and $a_0(x)$ in L are $\mathbb{C}^1$ and real-valued. The spectral problem is naturally posed on $H^2_{bc}[-1,+1]$, where

$$H^2_{bc}[-1,+1] := \{u \in H^2[-1,+1] : b_1^\pm u(\pm 1) + b_2^\pm \partial_x u(\pm 1) = 0\}.$$

The operator L is self-adoint in the weighted inner product $\langle u,v\rangle_p := \int_{-1}^1 u(x)\overline{v(x)}\rho(x)\; dx$, with associated norm $\|\cdot\|_\rho$, where the weighted function is

$$\rho(x) := e^{\int_0^x a_1(s)\; ds} > 0.$$

The associated eigenvalue problem is

$$Lp = \lambda p, \tag{1.2.2}$$

Theorem 1.2.1. *Consider the Sturmian eigenvalue problem 1.2.2 with separated boundary conditions 1.2.1 on the space $H^2_{bc}([-1,+1])$. All of the eigenvalues are real-valued and simple, and can* *be enumerated in a strictly descending order*

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots, \quad \lim_{n\to\infty} \lambda_n = -\infty.$$

The eigenfunctions $p_j(x)$ *associated with the eigenvalue* λ_j *for* $j = 0, 1, 2, \cdots$, *can be normalized so that*

(a) p_j *has* j *simple zeros in the open interval* $(-1, +1)$.

(b) The eigenfunctions are orthonormal in the ρ*-weighted inner product,*

$$\langle p_j, p_k \rangle_\rho = \delta_{jk},$$

where δ *is the Kronecker delta.*

(c) The eigenfunctions form a complete orthnormal basis of $L^2[-1, 1]$ *in the* ρ*-weighted inner product. That is , any* $u \in L^2[-1.1]$ *can be expressed as*

$$u = \sum_{j=0}^{\infty} u_j p_j,$$

where the sum on the right-hand side converges in $\|\cdot\|_\rho$ *and* $u_j := \langle u, p_j \rangle_\rho$ *and*

$$\|u\|_\rho^2 = \sum_{j=0}^{\infty} |u_j|^2.$$

(d) The ground-state eigenvalue can be characterized as the supremum of the bilinear form associated to L,

$$\lambda_0 = \sup_{\|u\|_\rho = 1} \langle Lu, u \rangle_\rho,$$

moreover the supremum is achieved at $u = p_0$*, which has no zeros on* $(-1, 1)$.

1.2.2 Sturm-Liouville Operators on the Real Line

Consider the Sturmian operator L acting on $H^2(\mathbb{R})$ with smooth coefficients $a_0(x)$ and $a_1(x)$, which decay exponentially to constants at $x = \pm\infty$, i.e.,

$$\lim_{x\to\pm\infty} e^{\nu|x|}|a_1(x) - a_1^{\pm}| = 0, \quad \lim_{x\to\pm\infty} e^{\nu|x|}|a_0(x) - a_0^{\pm}| = 0, \tag{1.2.3}$$

for some $\nu > 0$ and constants $a_1^{\pm}\ a_0^{\pm} \in \mathbb{R}$. The operator L is self-adjoint in the ρ-weighted inner product, and the weight has the finite asymptotic values

$$\rho_{\pm} := \lim_{x\to\pm\infty} e^{-a_1^{\pm}x}\rho(x).$$

Moreover, the following theorem holds.

Theorem 1.2.2. *Consider the eigenvalue problem 1.2.2 on the space $H^2(\mathbb{R})$, where the coefficients satisfy 1.2.3. The point spectrum, $\sigma_{pt}(L)$, consists of a finite number, possibly zero, of simple eigenvalues, which can be enumerated in a strictly descending order*

$$\lambda_0 > \lambda_1 > \cdots > \lambda_N > b := max\{a_0^-, a_0^+\}.$$

For $j = 0, \cdots, N$ the eigenfunction $p_j(x)$ associated with the eigenvalue λ_j can be normalized so that:

(a) p_j has j simple zeros.

(b) The eigenfunctions are orthonormal in the ρ-weighted inner product.

(c) The ground-state eigenvalue, if it exists, can be characiterized as the supremum of the bilinear form associated to L, $\lambda_0 = \sup_{\|u\|_\rho=1} \langle Lu, u\rangle_\rho$, and the supremum is acheived at $u = p_0$, which has no zeros.

1.3 Total positivity theory

In this section, we present some basic results from John Albert's positivity theory, [23]

Let $\mathcal{T}$ be the operator defined on a dense subspace of $L^2(\mathbb{R})$ by

$$\mathcal{T}g(x) = Mg(x) + \omega g(x) - \varphi^p(x)g(x),$$

where $p \geq 1$ is an integer, ω is a real parameter, φ is real-valued solution of

$$(M+\omega)\varphi = \frac{1}{p+1}\varphi^{p+1}.$$

having a suitable decay at infinity, and M is defined as a Fourier multiplier operator by

$$\widehat{Mg}(\xi) = m(\xi)\hat{g}(\xi).$$

Here circumflexes denotes the Fourier transform, $m(\xi)$ is a measurable, locally bounded, even function on $\mathbb{R}$ satisfying

(1) $A_1|\xi|^\mu \leq m(\xi) \leq A_2|\xi|^\mu$ for $|\xi| \geq \xi_0$;

(2) $m(\xi) \geq b$;

where A_1 and A_2 are positive real constants, $\mu \geq 1$, and ξ_0 and b are real numbers. Throughout, it is assumed that $\omega > -b$. Under the above assumptions, we have the following.

Lemma 1.3.1. *The operator $\mathcal{T}$ is a closed, unbounded, self-adjoint operator on $L^2(\mathbb{R})$ whose spectrum consists of the interval $[\omega,\infty)$ together with a finite number of discrete eigenvalues in the interval $(-\infty,\omega]$, in which all of them have finite algebraic multiplicity. In addition, zero is an eigenvalue of $\mathcal{T}$ with eigenfunction φ'.*

Proof.

Proposition 1.3.2. *The essential spectrum is conserved under a relatively compact perturbation. More precisely, let $\mathcal{T} \in \mathcal{L}(x)$ and Let A be $T-compact$. Then $\mathcal{T}$ and $\mathcal{T}+A$ have the same essential spectrum.*

Consider $\mathcal{T}g(x) = Mg(x) + \omega g(x) - \varphi^p(x)g(x)$, first, note that the essential spectrum of the operator $M+\omega$ is the interval $[\omega, \infty)$, while the operator $\mathcal{T}$ is a perturbation of $M+\omega$ by a relatively compact operator. Therefore, by the Proposition above, the essential spectrum of $\mathcal{T}$ is also $[\omega, \infty)$. We also know that the dimensions of the null space and deficiency of $\mathcal{T} - \lambda I$ are independent of λ if $\lambda \notin [\omega, \infty)$, with the possible exception if a set of isolated points $\{\lambda_n\}$.

Remark 1.3.3. *Let $\mathcal{T} \in \mathcal{L}(X,Y)$, the graph $G(\mathcal{T})$ of $\mathcal{T}$ is the closed linear manifolds of $X \times Y$ consisting of all elements $\{u, \mathcal{T}u\}$ where $u \in D(\mathcal{T})$. Note that: $N(\mathcal{T}) = G(\mathcal{T}) \cap X$, $R(\mathcal{T}) + X = G(\mathcal{T}) + X$.*

$$null\ \mathcal{T} = dim(G(\mathcal{T}) \cap X) = null\ (G(\mathcal{T}), X),$$

$$def\ \mathcal{T} = codim(G(\mathcal{T}) + X) = def\ (G(\mathcal{T}), X).$$

Proposition 1.3.4. *A closed symmetric operator $\mathcal{T}$ has deficiency index $(0,0)$ if and only if $\mathcal{T}$ is self-adjoint.*

Since $\mathcal{T}$ is self-adjoint, then we have $null\ (\mathcal{T} - \lambda I) = def\ (\mathcal{T} - \lambda I) = 0$ for $\lambda \notin \{\lambda_n\} \cup [\omega, \infty)$, which means that λ_n are isolated eigenvalues of $\mathcal{T}$. Furthermore, to show that the set of all λ_n is finite, it suffices to show that the spectrum of $\mathcal{T}$ is bounded below. Then it will be shown that if $K = (|\varphi|_\infty^p) + \omega$, then $spec(\mathcal{T})$ does not intersect the interval $(-\infty, -K)$. Let $\lambda < -K$, and consider $\mathcal{T} - \lambda I$. $\mathcal{T} - \lambda I = (M - \lambda I) + (\omega - \varphi^p)$, since $M - \lambda I$ has symbol $(\alpha(k) - \lambda)$ and $\lambda < 0$, then $M - \lambda I$ is invertible as an operator on L^2. We can obtain

$$\|(M - \lambda I)^{-1}\|_{2,2} = \sup_{k \in \mathbb{R}} |\frac{1}{\alpha(k) - \lambda}| = \frac{1}{\lambda},$$

Further,

$$\|w - \varphi^p\|_{2,2} \leq K < |\lambda| = \frac{1}{\|(M - \lambda I)^{-1}\|_{2,2}}.$$

Hence, the Neumann series for the inverse of $(M-\lambda I)+(\omega-\varphi^p)$ converges, so that $(\mathcal{T}-\lambda I)^{-1}$ exists and is bounded. Hence $\lambda \notin spec(\mathcal{T})$. Thus the spectrum of $\mathcal{T}$ is bounded below and this completes the proof of the lemma. □

In order to obtain additional spectral properties of $\mathcal{T}$, let us introduce the family of operators $S_\theta,\ \theta \geq 0$ on $L^2(\mathbb{R})$.

$$S_\theta g(x) = \frac{1}{\omega_\theta(x)} \int_{\mathbb{R}} K(x-y)g(y)\, dy,$$

where $K(x) = \widehat{\varphi^p}(x)$ and $\omega_\theta(x) = m(x)+\theta+\omega$. These operators act on the Hilbert space

$$X = \{g \in L^2(\mathbb{R}); \|g\|_{X,\theta} = \left(\int_{\mathbb{R}} |g(x)|^2 \omega_\theta(x)\, dx\right)^{1/2} < \infty\}.$$

Proposition 1.3.5. *(a) If $g \in L^2$ is an eigenfunction of S_θ for a non-zero eigenvalue, then $g \in X$.*

(b) The restriction of S_θ to X is a compact, self-adjoint operator on X with respect to the norm $\|\cdot\|_{X,\theta}$.

The following two corollaries are immediate consequence of the Proposition above and the spectral theorem for self-adjoint compact operators on a Hilbert space.

Corollary 1.3.6. *Suppose $\theta \geq 0$. Then $-\theta$ is an eigenvalue of $\mathcal{T}$ (as an operator on $L^2(\mathbb{R})$) with eigenfunction g if and only if, 1 is an eigenvalue of S_θ (as an operator on X) with eigenfunction $\hat{g}$. In particular, both eigenvalues have the same multiplicity.*

Corollary 1.3.7. *For every $\theta \geq 0$, S_θ has a family of eigenvectors $\{\psi_i(\theta)\}_{i=0}^{\infty}$ forming an orthonormal basis of X with respect to the norm $\|\cdot\|_{X,\theta}$. Moreover, the corresponding eigenvalues $\{\lambda_i(\theta)\}_{i=0}^{\infty}$ are real and can be numbered in order of decreasing absolute value:*

$$|\lambda_0(\theta)| \geq |\lambda_1(\theta)| \geq \cdots \geq 0.$$

We also have the third result which is a Krein-Rutman-type theorem.

Lemma 1.3.8. *The eigenvalue $\lambda_0(0)$ of S_0 is positive, simple, and has a strictly positive eigenfunction $\psi_{0,0}(x)$. Moreover, $\lambda_0(0) > |\lambda_1(0)|$.*

Recall that a function $h : \mathbb{R} \to \mathbb{R}$ is said to be in the class $PF(2)$ if :

(1) $h(x) > 0$ for all $x \in \mathbb{R}$;

(2) for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$, there holds $h(x_1 - y_1)h(x_2 - y_2) - h(x_1 - y_2)h(x_2 - y_1) \geq 0$;

(3) strict inequality holds in (2) whenever the intervals (x_1, x_2) and (y_1, y_2) intersect.

Theorem 1.3.9. *Suppose $\hat{\varphi} > 0$ on $\mathbb{R}$ and $\widehat{\varphi^p} =: K \in PF(2)$. Then $\mathcal{T}$ satisfies the following.*

(1) $\mathcal{T}$ has a simple, negative eigenvalue κ;

(2) $\mathcal{T}$ has no negative eigenvalue other than κ;

(3) the eigenvalue 0 of $\mathcal{T}$ is simple.

Proof. The stated assumptions on φ and K imply that for all $\theta \geq 0$, the eigenvalues $\lambda_0(\theta)$ and $\lambda_1(\theta)$ of S_θ are distinct, positive and simple. Moreover, by classical perturbation theory, $\lambda_0(\theta)$ and $\lambda_1(\theta)$ depend differentiably on θ in $[0, \infty)$; and corresponding eigenfunctions $\psi_0 = \psi_0(\theta) \in X$ and $\psi_1 = \psi_1(\theta) \in X$ may be chosen which also depend differentiably on θ in $[0, \infty)$ and which satisfy $\|\psi_0(\theta)\|_{X,\theta} = \|\psi_1(\theta)\|_{X,\theta} = 1$ for all $\theta \geq 0$.
Then we claim that

$$\frac{d}{d\theta}(\lambda_i(\theta)) < 0 \; for \; i = 0, 1 \; and \; \theta \geq 0.$$

Consider the following:

$$\begin{aligned}
\frac{d\lambda_i}{d\theta} &= \frac{d}{d\theta}(\lambda_i \|\psi_i\|_{X,\theta}) = \frac{d}{d\theta}\{\lambda_i \int_{-\infty}^{\infty} (\psi_i(x))^2 w_\theta(x)\, dx\} = \frac{d}{d\theta}\{\int_{-\infty}^{\infty} (S_\theta \psi_i(x))\psi_i(x) w_\theta(x)\, dx\} \\
&= \frac{d}{d\theta}\{\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} K(x-y)\psi_i(y)\, dy)\psi_i(x)\, dx\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-y)\{\frac{d\psi_i}{d\theta}(y)\psi_i(x) + \psi_i(y)\frac{d\psi_i}{d\theta}(x)\}\, dxdy = 2\int_{-\infty}^{\infty} \frac{d\psi_i}{d\theta}(x)(S_\theta \psi_i(x)) w_\theta(x)\, dx
\end{aligned}$$

$$= 2\lambda_i \int_{-\infty}^{\infty} \frac{d\psi_i}{d\theta}(x)\psi_i(x)w_\theta(x)dx = 2\lambda_i\{\frac{d}{d\theta}(\frac{1}{2}\int_{-\infty}^{\infty}(\psi_i(x))^2 w_\theta(x)\, dx) - \frac{1}{2}\int_{-\infty}^{\infty}(\psi_i(x))^2\, dx\}$$
$$= 2\lambda_i\{0 - \frac{1}{2}\int_{-\infty}^{\infty}(\psi_i(x))^2\, dx\} = -\lambda_i \int_{-\infty}^{\infty}(\psi_i(x))^2\, dx < 0$$

$\lim_{\theta\to\infty}\lambda_0(\theta) = 0$, furthermore, we have $\lambda_1(0) = 1$. Hence $\lambda_0(0) > 1$. Then it follows that there exists a unique $\theta_0 \in (0,\infty)$ such that $\lambda_0(\theta_0) = 1$. Set $\kappa = -\theta_0$ and then $-\theta_0$ is an eigenvalue of $\mathcal{T}$. Also, for $i \geq 1$ and $\theta > 0$, one has $\lambda_i(\theta) \leq \lambda_1(\theta) < \lambda_1(0) = 1$, showing that 1 can not be an eigenvalue of S_θ, besides $\theta = \theta_0$. Furthermore, 0 is an eigenvalue of $\mathcal{T}$, which means that 1 is an eigenvalue of S_0. Thus 0 is simple. □

1.4 Properties of Solitary Wave solutions

The equation

$$u_t + u^p u_x - (Mu)_x = 0, \tag{1.4.1}$$

where $p > 0$ is an integer, and M is defined as a Fourier multiplier operator by

$$\widehat{Mg}(\xi) = m(\xi)\hat{g}(\xi).$$

for all $k\ in\mathbb{R}$.

Here circumflexes denotes the Fourier transform, $m(\xi)$ is a measurable, locally bounded, even function on $\mathbb{R}$ satisfying

(1) $A_1|\xi|^\mu \leq m(\xi) \leq A_2|\xi|^\mu$ for $|\xi| \geq \xi_0$;

(2) $m(\xi) \geq b$;

where A_1 and A_2 are positive real constants, $\mu \geq 1$, and ξ_0 and b are finite real numbers. Let $u(x,t) = \varphi(x - Ct)$ be a travelling-wave solution of (1.4.1). Substituting the form of $u(x,t)$ into

(1.4.1) and integrating once (with zero boundary conditions imposed at infinity), one obtains

$$(M+C)\varphi = (\frac{1}{p+1})\varphi^{p+1}. \tag{1.4.2}$$

Any solution φ of (1.4.2) is an even function and lies in the space $H^{\mu/2}$. Also assume $C > -b$, hence $M+C$ represents a positive operator. In studying the stability of the solitary wave φ, it has been found useful to consider the linear operator $L : L^2 \to L^2$ defined by $Lu = (M+C)u - \varphi^p u$.

Similarly, let's consider the solitary-wave solutions of equations of the form

$$u_t + u^p u_x - (M_{n,p}(u))_x = 0,$$

where $M_{n,p}$ is differential operator of order $2n$. The solitary waves in question are of the form $\varphi(x) = (sech(x))^r$, where $r = \frac{2n}{p}$. The operator $M_{n,p}$ will be defined by means of the following Proposition.

Proposition 1.4.1. *Let n be a given positive integer and p a given positive real number. Then there exists a unique vector* $A = (a_0, a_1, \cdots, a_n)$ *in* $\mathbb{R}^{n+1}$ *such that*

$$\sum_{i=0}^{n} a_i(\partial^{2i}\varphi) = \frac{\varphi^{p+1}}{p+1}.$$

Proof. For each natural number i, one has $\partial^{2i}\varphi = \sum_{j=0}^{i} b_{ij} sech^{r+2j}(x)$, where the b_{ij} are non-zero real numbers depending only on r. Define B to be the $(n+1)\times(n+1)$ matrix $\{b_{ij}\}_{i,j=0,n}$, where b_{ij} is set equal to zero for $i < j$. Since $\varphi^{p+1}(x) = sech^{r+2n}(x)$, it holds if and only if $AB = D$; where $D = (0,0,\cdots,0,\frac{1}{p+1}) \in \mathbb{R}^{n+1}$. But B is non-singular, as it is a lower-diagonal matrix with non-zero elements on the diagonal. Hence there is a unique A in $\mathbb{R}^{n+1}$ for which the above holds. □

Now, for given n and p, define the differential operator $M_{n,p}$ by

$$M_{n,p} = \sum_{i=1}^{n} a_i \partial^{2i};$$

where the a_i are the constants . Also define $C_{n,p} = a_0$. Then one has

$$(M_{n,p}+C_{n,p})(\varphi) = \frac{1}{p+1}(\varphi)^{p+1}.$$

Thus, for $\varphi(x) = (sech(x))^r$, and $L = (M_{n,p}+C_{n,p}-\varphi^p)$, where $r = \frac{2n}{p}$, first we need to compute the sign of $I_1 := (L^{-1}\varphi,\varphi)$. We will compute by means of a spectral analysis of the operators T_0 and S_0 introduced above in Section 1. The notation

$$\lambda_m = \frac{\Gamma(r+m)}{\Gamma(r+1)} \cdot \frac{\Gamma(r+2n+1)}{\Gamma(r+2n+m)} \quad (m \geq 0)$$

will be used later.

Lemma 1.4.2. *For any integer $m \geq 0$, there exist constants c_{mj} $(0 \leq j \leq m-1)$, depending only on n and p, such that*

$$\partial^m(\frac{\varphi^{p+1}}{p+1}) = \varphi^p\{(\frac{1}{\lambda_m})(\partial^m\varphi) + \sum_{j=0}^{m-1} c_{m,j}(\partial^j\varphi)\}.$$

Proof. The proof is by induction. It clearly holds for $m = 0$. Assume that it holds for m. Then for any integer $j \geq 0$, there exist constant β_{jl} $(0 \leq l \leq j)$ such that

$$(\partial\varphi)(\partial^j\varphi) = \varphi\{(\frac{r}{r+j})(\partial^{j+1}\varphi) + \sum_{l=0}^{j}\beta_{jl}(\partial^j\varphi)\}.$$

Then by the inductive hypothesis,

$$\begin{aligned}
\partial^{m+1}(\frac{\varphi^{p+1}}{p+1}) = \partial\partial^m(\frac{\varphi^{p+1}}{p+1}) &= (\frac{1}{\lambda_m})\partial(\varphi^p(\partial^m\varphi)) + \sum_{j=0}^{m-1} c_{mj}\partial(\varphi^p(\partial^j\varphi)) \\
&= (\frac{1}{\lambda_m})[p\varphi^{p-1}(\partial\varphi)(\partial^m\varphi) + \varphi^p(\partial^{m+1}\varphi)] + \sum_{j=0}^{m-1} c_{m,j}[p\varphi^{p-1}(\partial\varphi)(\partial^j\varphi) + \varphi^p(\partial^{j+1}\varphi)] \\
&= (\frac{1}{\lambda_m})[p\varphi^{p-1}(\partial\varphi)(\partial^m\varphi) + \varphi^p(\partial^{m+1}\varphi)] + \sum_{j=0}^{m-1} c_{m,j}[p\varphi^{p-1}(\partial\varphi)(\partial^j\varphi) + \varphi^p(\partial^{j+1}\varphi)]
\end{aligned}$$

$$= (\frac{1}{\lambda_m})(\frac{pr}{r+m}+1)\varphi^p(\partial^{m+1}\varphi) + \varphi^p\sum_{j=0}^{m-1}\{\left(\frac{pc_{mj}r}{r+j}+1\right)(\partial^{j+1}\varphi) + \sum_{l=0}^{j} pc_{mj}\beta_{jl}(\partial^l\varphi)\}.$$

Since $\frac{1}{\lambda_m}(\frac{pr}{r+m}+1) = (\frac{1}{\lambda_m+1})$, it proves the statement of the Lemma for $(m+1)$. □

Lemma 1.4.3. *For any integer $m \geq 0$, there exist constants γ_{mi} $(0 \leq i \leq m)$, depending only on n and p, such that the function $q_m = \sum_{i=0}^m \gamma_{mi}(\partial^i\varphi)$ satisfies $(M_{n,p}+C_{n,p})(q_m) = (\frac{1}{\lambda_m})\varphi^p q_m$.*

Proof. Define the matrix $G = \{g_{ij}\}_{0\leq i,j\leq m}$ by

$$g_{ij} = \begin{cases} c_{ij} & \text{if } 0 \leq j \leq i-1 \\ \frac{1}{\lambda_i} & \text{if } j = i \\ 0 & \text{if } i+1 \leq j \leq m \end{cases} \tag{1.4.3}$$

If $0 \leq i \leq m$ then

$$\partial^i(\frac{\varphi^{p+1}}{p+1}) = \varphi^p\sum_{j=0}^m g_{ij}(\partial^j\varphi).$$

Since G is a lower- diagonal matrix with diagonal entries $g_{ij} = \frac{1}{\lambda_i}$, then $\frac{1}{\lambda_i}$ is an eigenvalue of G for each $0 \leq i \leq m$. Define $(\gamma_{m0}, \cdots, \gamma_{mm})$ to be a left eigenvector of G for the eigenvalue $\frac{1}{\lambda_m}$, so that $\sum_{i=0}^m \gamma_{mi}g_{ij} = (\frac{1}{\lambda_m})\gamma_{mj}$ for each $0 \leq j \leq m$. Since

$$(M_{n,p}+C_{n,p})(\varphi) = \frac{1}{p+1}(\varphi)^{p+1}.$$

It follows that

$$(M_{n,p}+C_{n,p})(q_m) = \sum_{i=0}^m \gamma_{mi}\partial^i(M_{n,p}+C_{n,p})(\varphi) = \sum_{i=0}^m \gamma_{mi}\partial^i(\frac{\varphi^{p+1}}{p+1}) = (\varphi^p)\sum_{i=0}^m \gamma_{mi}\sum_{j=0}^m g_{ij}(\partial^j\varphi)$$
$$= (\varphi^p)\sum_{j=0}^m(\sum_{i=0}^m \gamma_{mi}g_{ij})(\partial^j\varphi) = (\varphi^p)\sum_{j=0}^m(\frac{1}{\lambda_m}\gamma_{mj})(\partial^j\varphi) = \frac{1}{\lambda_m}\varphi^p q_m.$$

□

Theorem 1.4.4. *If $\varphi(x) = (sech(x))^r$, and $L = (M_{n,p}+C_{n,p}-\varphi^p)$, where $r = \frac{2n}{p}$, then the quantity $I_1 = (L^{-1}\varphi,\varphi)$ is given by the formula*

$$a\sum_{j=0}^{\infty}\left(\frac{\lambda_{2j}}{1-\lambda_{2j}}\right)\left\{\frac{\Gamma(2j+1)\cdot(2j+n+r-\frac{1}{2})}{\Gamma(2j+2n+2r-1)}\right\}\left\{\frac{\Gamma(j+n)\Gamma(j+n+r-\frac{1}{2})}{\Gamma(j+1)\Gamma(j+r+\frac{1}{2})}\right\}^2,$$

where $a=\left(\frac{2^{n+r-1}\Gamma(r)}{\pi\Gamma(n)}\right)^2$.

Proof. Denote $\{T_\theta\}_{\theta\geq 0}$ by

$$T_\theta g=(M+C+\theta)^{-1}(\varphi^p\cdot g).$$

Let's consider the following statements.

Proposition 1.4.5. *Suppose* $ker(T_\theta)=0$. *Let* $\{\zeta_i\}_{i=0}^{\infty}$ *be a complete orthonormal set of eigenfunctions of* T_θ *in Y, with* $T_\theta\zeta_i=\lambda_i\zeta_i$ *for* $i\geq 0$. *Then* $\{\sqrt{\lambda_i}\hat{\zeta}_i\}_{i=0}^{\infty}$ *is a complete orthonormal set of eigenfunctions for* S_θ *in X, with* $S_\theta\hat{\zeta}_i=\lambda_i\hat{\zeta}_i$.

Lemma 1.4.6. *Let* $p=r+n-\frac{1}{2}$. *For each integer* $m\geq 0$, *the functions* $\zeta_m(x)=\varphi(x)C_m^p(tanhx)$ *is an eigenfunction of* T_0 *for the eigenvalue* λ_m. *Furthermore,* $\{\zeta_m\}_{m\geq 0}$ *forms a complete set of eigenfunctions for* T_0 *in Y.*

Then we begin our proof. First we define $e_i=\sqrt{\lambda_i}(\frac{\hat{\zeta}_i}{\|\zeta_i\|_Y})$ for $i\geq 0$. Then by lemma above $\{e_i\}_{i\geq 0}$ is a complete orthonormal set of eigenfunctions for S_0 in X. Define a function η by

$$\eta=\sum_{i=0}^{\infty}(\frac{1}{1-\lambda_i})\langle\frac{\hat{\varphi}}{w_0},e_i\rangle_{X,0}e_i.$$

Since $\sum_{i=0}^{\infty}(\frac{1}{1-\lambda_i})^2\|\langle\frac{\hat{\varphi}}{w_0},e_i\rangle_{X,0}\|^2\leq A\sum_{i=0}^{\infty}\|\langle\frac{\hat{\varphi}}{w_0},e_i\rangle_{X,0}\|^2=A\|\frac{\hat{\varphi}}{w_0}\|_{X,0}^2=A\int_{-\infty}^{\infty}|\hat{\varphi}|^2\,dx=A\|\varphi\|_0^2$ the series for η converges in X, and so $\eta\in X\subset L^2$. We choose $\psi\in L^2$ so that $\hat{\psi}=\eta$. Then we have

$$\begin{aligned}(L\psi)\hat{}&=[(M_{n,p}+C_{n,p})\psi-\varphi^p\psi]\hat{}=w_0(\eta-T_0\eta)\\&=w_0\sum_{i=0}^{\infty}\langle\frac{\hat{\varphi}}{w_0},e_i\rangle_{X,0}e_i=\hat{\varphi}\end{aligned}$$

Hence $L\psi = \varphi$ and so $I = \langle \psi, \varphi \rangle_0$. Applying the inverse Fourier transform to η gives

$$\psi = \sum_{i=0}^{\infty} (\frac{\lambda_i}{1-\lambda_i})(\int_{-\infty}^{\infty} \varphi(t)\zeta_i(t)\, dt)\frac{\zeta_i}{\|\zeta_i\|_Y^2}.$$

Then we note that the Gegenbauer polynomials $\{C_m^p\}_{m=0}^{\infty}$ are defined by

$$C_m^p(\xi) = \sum_{s=0}^{[m/2]} (-1)^s \frac{\Gamma(m+p-s)}{s!(m-2s)!\Gamma(p)}(2\xi)^{m-2s}.$$

where $p > -1/2$.And the expression for the coefficients of C_m^p is not defined if $p = 0$. Let $L_{2,p} = L_{2,p}([-1,1])$ be the space of all measurable functions h on $[-1,1]$ such that $\|h\|_{2,p} = \left(\int_{-1}^{1} |h(\xi)|^2(1-\xi^2)^{p-1/2}\, d\xi\right)^{1/2} < \infty$. Then $L_{2,p}$ is a Hilbert space with the inner product $\langle g,h\rangle_{2,p} = \int_{-1}^{1} g(\xi)h(\bar{\xi})(1-\xi^2)^{p-1/2}\, d\xi$; and $\{C_m^p\}_{m=0}^{\infty}$ forms a complete orthogonal set in $L_{2,p}$; with normalizing constants given by

$$\|C_m^p\|_{2,p} = \{\frac{\pi 2^{1-2p}\Gamma(m+2p)}{\Gamma(p)^2(m+p)m!}\}^{1/2}.$$

(Here Γ denotes Euler's Gamma function.) If $\{P_m\}_{m=0}^{\infty}$ is any other set of orthogonal polynomials in $L_{2,p}$ such that $deg(P_m) = m$ for all $m \geq 0$, then each P_m must be a constant multiple of C_m^p. Thus for all $p, \sigma > -1/2$ one has the identity as follows:

For m odd,

$$\int_{-1}^{1} C_m^p(\xi)(1-\xi^2)^{\sigma-1/2}\, d\xi = 0,$$

For m even, $\int_{-1}^{1} C_m^p(\xi)(1-\xi^2)^{\sigma-1/2}\, d\xi = [\frac{\Gamma(\sigma+1/2)}{\sqrt{\pi}\Gamma(p)\Gamma(p-\sigma)}][\frac{\Gamma(m/2+p-\sigma)\Gamma(m/2+p)}{\Gamma(m/2+1)\Gamma(m/2+\sigma+1)}]$

Therefore

$$\begin{aligned} I = \langle \psi, \varphi \rangle_0 &= \sum_{i=0}^{\infty} (\frac{\lambda_i}{1-\lambda_i})\frac{(\int_{-\infty}^{\infty} \varphi(t)\zeta_i(t)\, dt)^2}{\|\zeta_i\|_Y^2} = \sum_{i=0}^{\infty} (\frac{\lambda_i}{1-\lambda_i})\frac{(\int_{-\infty}^{\infty} C_i^p(\tanh x) sech^{2r}(x)\, dx)^2}{\int_{-\infty}^{\infty} (C_i^p(\tanh x))^2 \operatorname{sech}^{2n+2r}(x)\, dx} \\ &= \sum_{i=0}^{\infty} (\frac{\lambda_i}{1-\lambda_i})\frac{(\int_{-1}^{1} C_i^p(z)(1-z^2)^{r-1}\, dz)^2}{\int_{-1}^{1} (C_i^p(z))^2(1-z^2)^{n+r-1}\, dz} \end{aligned}$$

□

1.5 Instability Index Count Theory

We consider the Hamiltonian system

$$\partial_t u = J \frac{\delta \mathcal{H}}{\delta u}(u) \tag{1.5.1}$$

and impose the following conditions on J and $\mathcal{H}$.

Hypothesis: Hamiltonian Framework

(a) $\mathcal{H} : Y \mapsto \mathbb{R}$ has two continuous derivatives, $\frac{\delta \mathcal{H}}{\delta u} : Y \mapsto Y^*$, and $\frac{\delta^2 \mathcal{H}}{\delta u^2} : D \subset X \mapsto X$ generates a continuous bilinear form on Y. In particular for each critical point ϕ of $\mathcal{H}$, there exists $C > 0$ such that $| \left\langle \frac{\delta^2 \mathcal{H}}{\delta u^2}(\phi) v, v \right\rangle | \leq c \|v\|_Y^2$ and moreover

$$\|\mathcal{H}(\phi + v) - \mathcal{H}(\phi) - \left\langle \frac{\delta^2 \mathcal{H}}{\delta u^2}(\phi) v, v \right\rangle \| \leq C \|v\|_Y^3,$$

for all $\|v\|_Y$.

(b) $J : Y \mapsto X$ is skew-symmetric and there exists $M > 0$ such that

$$\ker(J) = span\{\psi_1, \cdots, \psi_M\}$$

with the $\psi_j' s$ orthonormal in X.

(c) Both J and $\mathcal{H}$ possess an $N-$ dimensional symmetry group $T(\gamma) : Y \mapsto Y$ satisfying

$$\begin{aligned} &(a) T_j(0) = I, \\ &(b) T_j(s+t) = T_j(s) T_j(t) = T_j(t) T_j(s), \\ &(c) T_i(\gamma_i) T_j(\gamma_j) = T_j(\gamma_i) T_i(\gamma_j) \\ &(d) T_j(\gamma_j) \mathcal{F}(u) = \mathcal{F}(T_j(\gamma_j) u), \mathcal{F} = J \frac{\delta \mathcal{H}}{\delta u}, \\ &(e) T_j' := \lim_{\gamma_j \to 0} \frac{T_j(\gamma_j) - T_j(0)}{\gamma_j}. \end{aligned} \tag{1.5.2}$$

(c) In particular

$$\mathcal{H}(T(\gamma)u) = \mathcal{H}(u), \quad JT(\gamma) = T(\gamma)J,$$

for all $u \in Y$ and $\gamma \in \mathbb{R}^N$.

(d) The symmetry T is an isometry on X; that is,

$$\langle T(\gamma)u, T(\gamma)v \rangle_X = \langle u, v \rangle_X$$

for all $u, v \in X$ and all $\gamma \in \mathbb{R}^N$.

(e) For each $j = 1, \cdots, N$, the symmetry generator $T_j^{'} : Y \subset X \to \mathrm{ker(J)}^{\perp} \subset \mathrm{X}$. Moreover for each j the operator $J^{-1}T_j : X \mapsto Y^*$ is bounded.

(f) There is an open, connected set $\Omega \subset \mathbb{R}^N$ such that for all $c \in \Omega$ there is a ϕ_c that solves

$$\frac{\delta E}{\delta u}(\phi_c) = 0;$$

where $E(u;c) := \mathcal{H}(u) + \sum_{i=1}^{N} c_i Q_i(u)$, $Q_i(u) := \frac{1}{2}\left\langle J^{-1}T_i^{'}u, u \right\rangle, \quad i = 1, \cdots, N.$

(g) $\mathrm{dimker(L) = N}$ and $\mathrm{Ker(L) = span\{T_1^{'}\phi_c, \cdots, T_n^{'}\phi_c\}}$.

(h) The dimension of the negative space of L is finite.

(i) There exists $\delta > 0$ such that $b[v, v] \geq \delta \|v\|_Y^2$, $v \in P(L)$, where $P(L)$ is the largest subspace $K \subset Y \subset X$ over which the bilinear form b associated with L is positive, $b[v, v] := \langle Lv, v \rangle$.

(j) the essential spectrum of JL is a strict subset of the imaginary axis, that is, ther exists $\omega_0 > 0$ such that

$$\sigma_{\text{ess}}(JL) \subset (-i\infty, -i\omega_0] \cup [i\omega_0, i\infty).$$

We investigate the spectrum of the full linearization, JL of (1.5.1) about a critical point ϕ_c of E. We have $\sigma_{pt}(JL)$ satisfies as below.

Proposition 1.5.1. *Consider the linear operator JL associated with the linearization of the real Hamiltonian system (1.5.1) about a critical point ϕ_c. The point spectrum $\sigma_{pt}(JL)$ is symmetric with respect to the real and imginary axes. That is, if $\lambda \in \sigma_{pt}(JL)$, then the quartet $\{\pm\lambda, \pm\bar{\lambda}\} \subset \sigma_{pt}(JL)$.*

Definition 1.5.2. *(Negative Krein index) Let Hypothesis $(a)-(j)$ hold with $L = \delta^2\mathcal{H}/\delta u^2$. Let $\lambda \in i\mathbb{R}\backslash\{0\}$ be a purely imaginary nonzero eigenvalue with associated generalized eigenspace E^λ and basis given by $\{v_1^\lambda, \cdots, v_k^\lambda\}$. For $\lambda \in (i\mathbb{R}\backslash\{0\}) \cap \sigma_{pt}(JL)$, we introduce the negative Krein index $k_i^-(\lambda) := n(E^\lambda)$, and define the total negative Krein index $k_i^- = k_i^-(JL) := \sum_{\mathrm{Re}\lambda=0} k_i^-(\lambda)$. If $k_i^-(\lambda) \geq 1$, then the eigenvalue λ is said to have a negative Krein signature; otherwise, it has a positive Krein signature.*

Then we introduce the instability indices counting formulas, which in many cases can be used to determine accurately both stability and instability regimes for the waves under consideration. For a Hamiltonian eigenvalue problem in the form

$$JLu = \lambda u,$$

where J is skew symmetric and L is a self-adjoint linear differential operator with domain $D(L) = H^s(\mathbb{R})$ for some $s \geq 0$. Assume the spectrum of L is such that

(1) there are $n(L) = N < +\infty$ negative eigenvalues (counting multiplicity), so that each of the corresponding eigenvectors $\{f_j\}_{j=1}^N$ belongs to $H^{s+1}(\mathbb{R})$.

(2) there is a $\kappa > 0$, such that $\sigma_{ess}(L) \subset [\kappa^2, +\infty)$.

(3) $dim[kerL] < \infty$.

Let us define various quantities that will appear in the index count.

- Let k_r represent the number of positive real eigenvalues (counting multiplicities).
- k_c is the number of complex eigenvalues with positive real part.

- k_i^- is the number of pairs of purely imaginary eigenvalues with negative Krein signature.
- We will henceforth denote by $n(M)$ the number of negative eigenvalues (counting multiplicities) of a self-adjoint operator M. Again by symmetries, k_i and k_c are even.
- The total Hamilton-Krein index is then defined

$$K_{Ham} := k_r + k_c + k_i.$$

Theorem 1.5.3. *For the eigenvalue problem*

$$JLu = \lambda u \quad u \in L^2(\mathbb{R}),$$

where J is assumed to be bounded, invertible and skew-symmetric ($J^* = -J$), while $(L, D(L))$ is self-adjoint($L^* = L$) and not necessarily bounded, with finite dimensional kernel $Ker[L]$. In addition, we assume that $J^{-1} : Ker[L] \to Ker[L]^\perp$. Here, the orthogonality is understood with respect to the dot product of the underlying Hilbert space $H : D(L) \subset H$. Further L satisfies $D(L) = H^s(\mathbb{R})$ for some $s > 0$, assume that conditions $(1), (2)$ and (3) above hold. Introduce the matrix D as follows. Let $Ker[L] = \{\phi_1, \ldots, \phi_n\}$, then

$$D_{ij} := \langle L^{-1}[J^{-1}\phi_i], J^{-1}\phi_j \rangle. \tag{1.5.3}$$

Note that the last formula makes sense, since $\mathcal{J}^{-1}\phi_i \in Ker[L]^\perp$. Thus $L^{-1}[\mathcal{J}^{-1}\phi_i]$ is well-defined. The index counting theorem, see Theorem 1, [25] states that if $det(D) \neq 0$, then

$$k_r + 2k_c + 2k_i^- = n(L) - n(D). \tag{1.5.4}$$

Chapter 2

Ground States of NLS and KG with fractional dispersion

In this chapter, we consider standing wave solutions of various dispersive models with non-standard form of the dispersion terms. Using index count calculations, together with the information from a variational construction, we develop sharp conditions for spectral stability of these waves.

2.1 Introduction and statement of the main results

For $s \in (0,1]$ and $d \geq 1$, we consider the focusing fractional Schrödinger equation

$$iu_t - (-\Delta)^s u + |u|^\alpha u = 0, (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d \tag{2.1.1}$$

In addition, we shall be interested in the fractional Klein-Gordon equation

$$u_{tt} + (-\Delta)^s u + u - |u|^\alpha u = 0, (t,x) \in \mathbf{R} \times \mathbf{R}^d \tag{2.1.2}$$

These nonlocal equations arise in a variety of models in mathematical physics, see many examples in [1] and the references therein. Also, a similar model

$$iu_t + (-\Delta)^s u + |u|^\alpha u = 0, \tag{2.1.3}$$

has been introduced by Laskin in quantum physics [20], and it is a fundamental equation of fractional quantum mechanics, a generalization of the standard quantum mechanics extending the Feynman path integral to Levy processes[20]. Further, in [13], Hong and Sire have discussed the local well-posedness and ill-posedness in Sobolev spaces, and in [12], Guo and Huo focused on the global well-posedness for the Cauchy problem of the 1-D fractional nonlinear Schrödinger equation with data in $L^2(\mathbf{R})$. Regarding well-posedness in the natural energy space, one has local and hence global well-posedness for Cauchy data in $H^s(\mathbf{R}^d)$, provided $\alpha < \frac{4s}{d}$, due to the conservation law. Generally, some solutions will blow up for $\alpha > \frac{4s}{d}$, [8].

Additionally, we will be interested in two higher order dispersion models, which are outside of the scope of (3.1.1) and (2.1.2). Namely, we consider the fourth order cubic Schrödinger equation, in one spatial dimension

$$iu_t + u_{xx} - u_{xxxx} + |u|^2 u = 0 \tag{2.1.4}$$

and the fourth order cubic Klein-Gordon equation

$$u_{tt} + u_{xxxx} - u_{xx} + u - |u|^2 u = 0, \tag{2.1.5}$$

The fourth order Schrödinger equation was introduced in [17] and [18], and it has an important role in modeling the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Moreover, the equation was also used in nonlinear fiber optics and the theory of optical solitons in gyrotropic media. In this chapter, we are interested in the existence and linear stability of standing wave solutions for these equations.

2.1.1 Standing wave solutions for fractional models

The existence of special solutions is an important feature of the fractional models. More precisely, we seek solutions of the fractional NLS equation (i.e. (3.1.1)) in the form $u_\omega(t,x) = e^{i\omega t} Q_\omega(x), \omega > 0$, with $Q_\omega > 0$. We obtain the following profile equation

$$\omega Q_\omega + (-\Delta)^s Q_\omega - Q_\omega^{\alpha+1} = 0, x \in \mathbf{R}^d \tag{2.1.6}$$

For the fractional Klein-Gordon equation, we have the profile equation

$$(1-\omega^2)R_\omega + (-\Delta)^s R_\omega - R_\omega^{\alpha+1} = 0, x \in \mathbf{R}^d. \tag{2.1.7}$$

where we require that $|\omega| < 1, R_\omega > 0$. Clearly (2.1.6) and (2.1.7) are closely related to each other. Indeed, setting for each $\omega \in (-1,1)$, $\gamma := 1-\omega^2 > 0$, whence $R_\omega = Q_\gamma$. Thus, we proceed to describe the properties of Q_ω, keeping in mind this relationship.

Note that the equation (2.1.6) enjoys a nice scaling property, which allows one to explicitly describe the solutions Q_ω of (2.1.6) in terms of a single function. To this end, consider (2.1.6) with $\omega = 1$,

$$(-\Delta)^s Q + Q - Q^{\alpha+1} = 0, x \in \mathbf{R}^d, \tag{2.1.8}$$

where we henceforth adopt for brevity the notation $Q = Q_1$. If one establishes that (2.1.8) has a unique (modulo symmetries) solution Q, then all solutions of (2.1.6) (modulo symmetries) are given by the formula $Q_\omega = w^{\frac{1}{\alpha}} Q(w^{\frac{1}{2s}} x)$.

The equation (2.1.8) has been well-studied, at least in the classical case $s = 1$, in the last thirty years. First, it is well-known that for $s = 1, d = 1, \alpha > 0$, such solutions are explicitly given in terms of powers of the sech functions. Clearly, one cannot hope for such solutions to be explicit outside of the cases mentioned above. In the case $s = 1$, $d \geq 1$, $\alpha > 0$, it has been shown in the classical paper [28] that such $Q : Q > 0$ is unique, modulo the translational symmetries. In the fractional case, i.e. $s \in (0,1)$, this difficult problem was resolved recently. It has been shown (in [10] for the case $d = 1$ and subsequently in [11] for the case $d \geq 2$) that (2.1.8) possesses a unique positive radial solution[1], provided

$$0 < \alpha < \alpha_*(s,d) = \begin{cases} \frac{4s}{d-2s} & s < \frac{d}{2} \\ \infty & s > \frac{d}{2}. \end{cases}$$

On the other hand, Pokhozaev type arguments for the elliptic equation (2.1.8) show that smooth

[1] which we refer to, with a slight abuse of notation, by $Q(|x|)$

and localized solutions Q do not exist, when $\alpha > \alpha_*(s,d)$.

In addition to the uniqueness, a number of additional properties of Q were established, which will be important for us as well and we discuss them below. The main tool in establishing all these important results has been the heavy use of the fact that a variant of (2.1.8) is in fact the Euler-Lagrange equation of a particular constrained minimization problem and Q is its minimizer.

2.1.2 Standing waves for fourth order models

It is clear that the fourth order case (which roughly corresponds to the case $s=2, d=1$ of our fractional family of equations) does not fit in the Frank-Lenzmann theory. Indeed, important ingredients of their proofs break down, such as maximum principle and positivity of the heat kernels of the corresponding semigroups, to mention a few. Nevertheless, it is an interesting question whether there exist any reasonable solutions of the profile equations and if so, what are their stability properties. More precisely, we again consider solutions in the form $u = e^{i\alpha t}\phi$ of (2.1.4), which yields the profile equation

$$\phi'''' - \phi'' + \alpha\phi - \phi^3 = 0. \tag{2.1.9}$$

The ansatz $\phi(x) = asech^2(bx)$ produces, for $\alpha = \frac{4}{25}$, the solution

$$\phi(x) = \sqrt{\frac{3}{10}} sech^2\left(\frac{x}{\sqrt{20}}\right). \tag{2.1.10}$$

Here, the solution displayed in (3.3.1) serves as a standing wave to the fourth order Schrödinger equation (2.1.4). A simple modification provides a solution to the fourth order Klein-Gordon equation as well. Indeed, a direct verification shows that

$$u(x,t) = e^{i\frac{\sqrt{21}}{5}t}\phi(x) \tag{2.1.11}$$

is a solution to (2.1.5). One of the main difficulties associated with the stability analysis of (3.3.1) ((2.1.11) respectively) is the fact that no explicit solution is available for values of $\alpha \neq \frac{4}{25}$. In other words, since we lack a one parameter family of solutions, the spectral computations become quite delicate. In particular, the standard approach to computing certain quantities related to stability depends on taking a derivative (in the explicit solution) in terms of α. This is the usual presentation of the Vakhitov-Kolokolov criteria, which in this case necessarily fails, due to the fact that such an explicit formula in terms of α is simply unavailable. We overcome these issues by resorting to the positivity theory as developed in [23], [5], [6].

In the next sections, we consider the linearized problems associated with the stability of these solitary waves.

2.1.3 The eigenvalue problem for the fractional NLS model

We first linearize around the standing wave $Q = Q_1$ in (3.1.1). Using the ansatz

$$u = e^{it}\{Q + (\varphi + i\psi)\}, \tag{2.1.12}$$

and taking real and imaginary parts leads us to

$$\begin{aligned} \varphi_t &= (-\Delta)^s \psi + \psi - Q^\alpha \psi \\ -\psi_t &= (-\Delta)^s \varphi + \varphi - (\alpha+1) Q^\alpha \varphi. \end{aligned}$$

Introduce the skew symmetric portion is $\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the self-adjoint portion of the linearized operator $\mathcal{L} := \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$ with

$$\begin{aligned} L_1 &= (-\Delta)^s + 1 - (\alpha+1) Q^\alpha, \\ L_2 &= (-\Delta)^s + 1 - Q^\alpha. \end{aligned}$$

both acting on the domain $H^{2s}(\mathbf{R}^d)$. It is now clear that the eigenvalue problem is in the form

$$\partial_t \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \mathcal{J}\mathcal{L} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{2.1.13}$$

Standard scaling argument shows that stability for Q_1 is equivalent to the stability for $Q_\omega, \omega > 0$, whence we henceforth concentrate on this particular case.

2.1.4 The eigenvalue problem for the fractional Klein-Gordon model

For the fractional KG model, (2.1.2), we linearize at the solution $e^{iwt}(1-w^2)^{\frac{1}{\alpha}} Q((1-w^2)^{\frac{1}{2s}} x)$. More precisely, we take the ansatz

$$u = (1-w^2)^{\frac{1}{\alpha}} e^{iwt}\{Q((1-w^2)^{\frac{1}{2s}} x) + v((1-w^2)^{\frac{1}{2s}} x, t)\},$$

Ignoring all second and higher order terms leads us to the eigenvalue problem

$$\begin{aligned} iwv_t &+ v_{tt} - w^2(Q+v) + (1-w^2)(-\Delta)^s Q + \\ &+ ((1-w^2)(-\Delta)^s v + Q + v - (1-w^2)(Q^{\alpha+1} + Q^\alpha v + \alpha Q^\alpha \Re(v)) = 0, \end{aligned}$$

Letting $v = \begin{pmatrix} \Re v \\ \Im v \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ allows us to rewrite the eigenvalue problem in the following matrix form

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_{tt} + \begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}_t + (1-w^2)\mathcal{L} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0 \tag{2.1.14}$$

where $\mathcal{L}$ is already defined in (2.1.13). Equivalently, writing $\varphi \to e^{\lambda t}\varphi, \psi \to e^{\lambda t}\psi$, one can write the last second order model as a first order system in the form

$$
\partial_t \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix} = \mathcal{J}\mathcal{L} \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix}, \tag{2.1.15}
$$

where

$$
\mathcal{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & w \\ 0 & -1 & -w & 0 \end{pmatrix}, \mathcal{L} = \begin{pmatrix} (1-w^2)L_1 & 0 & 0 & 0 \\ 0 & (1-w^2)L_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.1.16}
$$

are a skew-symmetric and a self-adjoint operators respectively.

2.1.5 The eigenvalue problem of the fourth order models

We now derive the relevant eigenvalue problem for the fourth order Schrödinger model (2.1.4).

In order to consider the stability of the wave $e^{i\alpha t}\phi$, with $\alpha = \frac{4}{25}$ and ϕ given by (3.3.1). We take

$$
u = e^{i\alpha t}[\phi + v + iw], \tag{2.1.17}
$$

for real-valued functions v, w and plug it into (2.1.4). Ignoring the contributions of terms in the form $O(v^2), O(w^2)$ and some algebra leads us to the the eigenvalue problem

$$
\partial_t \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x^4 - \partial_x^2 + \alpha - 3\phi^2 & 0 \\ 0 & \partial_x^4 - \partial_x^2 + \alpha - \phi^2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \tag{2.1.18}
$$

As usual, we denote $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$ with

$$\begin{cases} L_1 = & \partial_x^4 - \partial_x^2 + \alpha - 3\phi^2, \\ L_2 = & \partial_x^4 - \partial_x^2 + \alpha - \phi^2. \end{cases} \tag{2.1.19}$$

Finally, we discuss the linearization (and subsequently the eigenvalue problem) associated with the solution (2.1.11) to the fourth order cubic equation (2.1.5). To introduce proper notations, let $\beta = \frac{\sqrt{21}}{5}$, so that the wave is exactly $e^{i\beta t}\phi(x) = e^{i\beta t}\sqrt{\frac{3}{10}}sech^2(\sqrt{\frac{1}{20}}x)$. Setting

$$u = e^{i\beta t}(\phi + \varphi + i\psi),$$

plugging this ansatz into (2.1.5), ignoring the contributions of the type $O(\varphi^2), O(\psi^2)$ and taking real and imaginary parts, we obtain

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_{tt} + \begin{pmatrix} 0 & -2\beta \\ 2\beta & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}_t + \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0, \tag{2.1.20}$$

where L_1, L_2 are exactly the operators introduced in (2.1.19). We can also write a further equivalent formula

$$\partial_t \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -L_1 & 0 & 0 & 2\beta \\ 0 & -L_2 & -2\beta & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix} =: \mathcal{H} \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix} \tag{2.1.21}$$

We note that in addition

$$\mathcal{H} = \mathcal{J}\mathcal{L} = \begin{pmatrix} 0 & I_2 \\ -I_2 & B \end{pmatrix} \begin{pmatrix} \widetilde{\mathcal{L}} & 0 \\ 0 & I_2 \end{pmatrix} \tag{2.1.22}$$

$$\widetilde{\mathcal{L}} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, B = \begin{pmatrix} 0 & -2\beta \\ 2\beta & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.1.23}$$

2.1.6 Main results

We are now ready to state our results, first for the fractional NLS.

Theorem 2.1.1. *The standing waves $e^{i\omega t}Q_\omega$ of the fractional NLS* (3.1.1) *are linearly and orbitally stable for $\alpha < \frac{4s}{d}$. Moreover, they are linearly unstable for $\alpha > \frac{4s}{d}$.*

For the fractional Klein-Gordon model, the soliton $e^{i\omega t}(1-\omega^2)^{\frac{1}{\alpha}}Q((1-\omega^2)^{\frac{1}{2s}}x)$ is spectrally stable if and only if

$$\alpha < \frac{4s}{d}, \ \sqrt{\frac{4s\alpha}{4s\alpha+4s-\alpha d}} < |\omega| < 1$$

Our next result concerns the stability of the waves for the fourth order Schrödinger and Klein-Gordon equations.

Theorem 2.1.2. *The wave $e^{i\alpha t}\phi$ (with $\alpha = \frac{4}{25}$ and ϕ given by* (3.3.1)*) is spectrally stable solution of* (2.1.4)*. The wave $e^{i\beta t}\phi$, with $\beta = \frac{\sqrt{21}}{5}$ is spectrally unstable as a solution to the fourth order Klein-Gordon model* (2.1.5).

2.1.7 Spectral information regarding the operators L_1, L_2

By the representations of the Hamiltonian in both (2.1.13) and (2.1.16), it is clear that the spectral properties of the operators L_1, L_2 will play substantial role in our analysis.

Proposition 2.1.3. *The operator L_1 defined in* (2.1.13) *has a unique negative eigenvalue, which is simple. The eigenvalue zero is of multiplicity d, with $Ker[L_1] = span\{\partial_1 Q, \dots, \partial_d Q\}$. The operator L_2 satisfies $L_2 \geq 0$, with an eigenvalue at zero, which corresponds to the eigenfunction Q. As such the eigenvalue at zero is simple. Moreover, the essential spectrum for both operators is $[1,\infty)$.*

Proof. For L_1, we refer to the paper [11], where it was shown that $n(L_1) = 1$, while $Ker[L_1] = \{\partial_1 Q, \dots, \partial_d Q\}$.

Next, we clearly have $L_2[Q] = 0$, by construction of Q. We now show that L_2 has no negative eigenvalues. Assume for a contradiction that L_2 has a negative eigenvalue, say we pick the smallest such eigenvalue $-\sigma^2$. Then, there is an $F \neq 0$, so that $L_2[F] = -\sigma^2 F, \|F\| = 1$. According to the Rayleigh characterization of e-values, $-\sigma^2 = \inf_{\|G\|=1} \langle L_2 G, G\rangle$ and so F is a solution of this problem. Rewrite this constrained minimization problem in the form

$$\begin{cases} \langle L_2 G, G\rangle = \|(-\Delta)^{s/2} G\|_{L^2}^2 + \|G\|_{L^2}^2 - \int_{\mathbf{R}^d} Q^\alpha(x) G^2(x) dx \to \min \\ \int_{\mathbf{R}^d} G^2(x) dx = 1 \end{cases} \tag{2.1.24}$$

We now need to refer to recent results on the multi-dimensional Polya-Szegö inequality, which imply that the functional $\langle L_2 G, G\rangle$ is minimized by its decreasing rearrangement. More precisely, for $s \in (0,1)$, there is the generalized Polya-Szegö inequality

$$1\|(-\Delta)^{s/2} G\|_{L^2} \geq \|(-\Delta)^{s/2} G^*\|_{L^2},$$

where G^* is the decreasing rearrangement of the function G. Moreover, since Q^α is radially decreasing, there is the simple inequality

$$\int_{\mathbf{R}^d} Q^\alpha(x) G^2(x) dx \leq \int_{\mathbf{R}^d} Q^\alpha(x) [G^*]^2(x) dx \tag{2.1.25}$$

where the equality in (2.1.25) is achieved only if $G = G^*$. This is a simple consequence of $\int fg \leq \int f^* g^*$, see Theorem 3.4, [21]. In addition, an elementary property of the decreasing rearrangement says that $\|G\|_{L^p} = \|G^*\|_{L^p}$ for $p \in (0, \infty)$ and in particular for $p = 2$. It follows that $\langle L_2 G, G\rangle \geq \langle L_2 G^*, G^*\rangle$, with equality possible only if $G = G^*$, while clearly $\|G\|_{L^2} = \|G^*\|_{L^2}$. Thus, the eigenfunction F, corresponding to the lowest eigenvalue $-\sigma^2$ must necessarily be such that $F = F^*$ (since it isa solution to the constrained minimization problem (2.1.24)). In particular $F \geq 0$. But then $\langle F, Q\rangle = 0$, since any two e-functions corresponding to two different eigenvalues of L_2 must be orthogonal. This however leads to a contradiction, since $F \geq 0$, $Q > 0$. Thus, 0 is the lowest eigenvalue for L_2, whence $L_2 \geq 0$.

□

2.2 On the stability of the standing waves for the fractional NLS and Klein-Gordon equations: Proof of Theorem 2.1.1

We consider the cases of NLS and Klein-Gordon separately, although there is quite a few calculations that will appear in both.

2.2.1 Stability of fNLS waves

For the stability of the eigenvalue problem (2.1.13), we take the standard transformation $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \to e^{\lambda t} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$. In addition, due to the results of Proposition 2.1.3, the self-adjoint operator $\mathcal{L}$ satisfies $n(\mathcal{L}) = 1$ and

$$Ker[\mathcal{L}] = \left\{ \begin{pmatrix} 0 \\ Q \end{pmatrix}, \begin{pmatrix} \partial_1 Q \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \partial_d Q \\ 0 \end{pmatrix} \right\} =: \{Q_0, Q_1, \dots, Q_d\}. \tag{2.2.1}$$

In addition, it is clear that $\mathcal{J}^{-1} = -\mathcal{J} : Ker[\mathcal{L}] \to Ker[\mathcal{L}]^{\perp}$, whence the matrix $D \in \mathcal{M}_{(d+1)\times(d+1)}$ may be defined as in (1.5.3). Obviously, for $j \geq 1$, $D_{0j} = 0$. Next, note that for $i \geq 1, j \geq 1, i \neq j$, we have

$$D_{ij} = \langle L_2^{-1} \partial_i Q, \partial_j Q \rangle = 0, \tag{2.2.2}$$

since $\partial_i Q$ is odd in the i^{th} variable (and then so[2] is $L_2^{-1}[\partial_i Q]$), while $\partial_j Q$ is odd in the j^{th} variable. On the other hand, for $i = 1, \dots d$,

$$D_{ii} = \langle L_2^{-1} \partial_i Q, \partial_i Q \rangle > 0,$$

due to the positivity of L_2^{-1} on $Ker[L_2]^{\perp}$ and the fact that $\partial_i Q \perp Ker[L_2] = span[Q]$. Clearly now, $n(D) = n(\langle \mathcal{L}^{-1} \mathcal{J}^{-1} Q_0, \mathcal{J}^{-1} Q_0 \rangle = n(\langle L_1^{-1}[Q], Q \rangle)$.

In order to compute this quantity, we use the standard scaling properties of the profile equation

[2]Note that the space of functions which are odd in the $j^{th}, j = 1, \dots, d$ variable, is an invariant subspace for L_2^{-1}

(2.1.8). Namely, $Q_\mu := \mu^{\frac{1}{\alpha}} Q(\mu^{\frac{1}{2s}} x)$ solves

$$\mu Q_\mu + (-\Delta)^s Q_\mu - Q_\mu^{\alpha+1} = 0.$$

Taking derivative in μ yields the relation $L_1[\frac{\partial Q_\mu}{\partial \mu}] = -Q_\mu$, whence since $Q_\mu \perp Ker[L_1]$, we derive $L_1^{-1}[Q_\mu] = -\frac{\partial Q_\mu}{\partial \mu}$, whence

$$\langle L_1^{-1}[Q], Q \rangle = -\frac{1}{2} \partial_\mu \|Q_\mu\|^2|_{\mu=1} = -\frac{1}{2} \left(\frac{2}{\alpha} - \frac{d}{2s} \right) \|Q\|^2. \tag{2.2.3}$$

The fact that $n(\mathcal{L}) = 1$, the spectral stability of fNLS waves is equivalent to $\langle L_1^{-1}[Q], Q \rangle < 0$, or $\frac{2}{\alpha} - \frac{d}{2s} > 0$. This is easily seen to be equivalent to $\alpha < \frac{4s}{d}$ as stated. Due to the structure of $Ker[\mathcal{L}]$, namely (2.2.1), all the elements of the $Ker[\mathcal{L}]$ are accounted for by invariances of the model, so by the results of [37] (Theorem 5.2.11) and the well-posedness of the Cauchy problem in the energy space $H^s(\mathbb{R}^d)$ established in [9], the waves are orbitally stable as well.

2.2.2 Stability of the fKG waves

The relevant eigenvalue problem for the stability of the fractional Klein-Gordon waves is $\mathcal{J}\mathcal{L}\vec{X} = \lambda \vec{X}$, where $\mathcal{J}, \mathcal{L}$ are given by (2.1.16). By the form of $\mathcal{L}$, we have that $n(\mathcal{L}) = n(L_1) = 1$, owing to Proposition 2.1.3. The description of $Ker[\mathcal{L}]$ is again explicit, thanks again to Proposition 2.1.3. More precisely, we have

$$Ker[\mathcal{L}] = \{Q_0, Q_1, \ldots, Q_d\}, Q_0 = \begin{pmatrix} 0 \\ Q \\ 0 \\ 0 \end{pmatrix}, Q_j = \begin{pmatrix} \partial_j Q \\ 0 \\ 0 \\ 0 \end{pmatrix}, j = 1, \ldots, d.$$

Since $\mathcal{J}^{-1} = \begin{pmatrix} 0 & \omega & -1 & 0 \\ -\omega & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, we have by (1.5.3), that for $i \geq 1, j \geq 1, i \neq j$,

$$D_{ij} = \frac{\omega^2}{1-\omega^2} \langle L_2^{-1}[\partial_i Q], \partial_j Q \rangle = 0,$$

by (2.2.2). Similarly, $D_{i0} = D_{0i} = 0$ by our previous arguments for the fNLS case. Thus, the matrix D has only diagonal potentially non-zero entries. In fact, the entries $D_{ii}, i = 1, \ldots, n$ are positive due to the positivity of L_2^{-1} on $Ker[L_2]^{\perp}$. Indeed,

$$\begin{aligned} D_{ii} &= \langle \mathcal{L}^{-1}\mathcal{J}^{-1}Q_i, \mathcal{J}^{-1}Q_i \rangle = \\ &= \langle \begin{pmatrix} \frac{L_1^{-1}}{1-\omega^2} & 0 & 0 & 0 \\ 0 & \frac{L_2^{-1}}{1-\omega^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\omega\partial_i Q \\ \partial_i Q \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\omega\partial_i Q \\ \partial_i Q \\ 0 \end{pmatrix} \rangle = \\ &= \frac{\omega^2}{1-\omega^2} \langle L_2^{-1}[\partial_i Q], \partial_i Q \rangle + \|\partial_i Q\|^2 > 0. \end{aligned}$$

Thus, as before, matters have been reduced to D_{00}, more precisely, $n(D) = n(D_{00})$. The stability condition, according to (1.5.4) is exactly $D_{00} < 0$. We have, according to (2.2.3)

$$\begin{aligned} D_{00} &= \langle \mathcal{L}^{-1}\mathcal{J}^{-1}Q_0, \mathcal{J}^{-1}Q_0 \rangle = \frac{\omega^2}{1-\omega^2} \langle L_1^{-1}[Q], Q \rangle + \|Q\|^2 = \\ &= \left[\frac{\omega^2}{1-\omega^2} \left(\frac{d}{4s} - \frac{1}{\alpha} \right) + 1 \right] \|Q\|^2. \end{aligned}$$

It is clear that the stability condition is satisfied only if $\alpha < \frac{4s}{d}$ and then,

$$\frac{\omega^2}{1-\omega^2} > \frac{4s\alpha}{4s - \alpha d}.$$

Resolving this last inequality yields the condition

$$\omega^2 > \frac{4s\alpha}{4s\alpha + 4s - \alpha d}.$$

Since we have initially required $|\omega| < 1$ for the existence of the waves, we can finally formulate the necessary and sufficient condition for stability as follows

$$\frac{4s\alpha}{4s\alpha + 4s - \alpha d} < \omega^2 < 1.$$

Note that this last inequality implicitly requires $\alpha < \frac{4s}{d}$, since otherwise the double inequality will not have any solutions in ω.

2.3 On the stability of the standing waves for the fourth order models

We start this section with a discussion about the spectral properties of the self-adjoint operators L_1, L_2, defined in (2.1.19). We have the following result.

Proposition 2.3.1. *The operator L_1 with domain $H^4(\mathbb{R}) \times H^4(\mathbb{R})$ has a unique negative eigenvalue, which is simple. The eigenvalue zero is of dimension exactly $d = 1$, with associated eigenfunctions $\partial_j \phi, j = 1, \dots d$. L_2 has no negative eigenvalues, it has eigenvalue at zero, which is simple. Moreover the essential spectrum is the interval $[\alpha, \infty)$.*

2.3.1 Computing the Vakhitov-Kolokolov type quantities for $sech^r$ solutions using Albert's approach

For $\varphi(x) = (sech(x))^r, r = \frac{2n}{p}$, it was established that (see Lemma 4.7, [23]) there exist unique $(n+1)$ tuple $a_0, \dots, a_n$, so that

$$\sum_{i=0}^{n} a_i(\partial^{2i}\varphi) = \frac{\varphi^{p+1}}{p+1}.$$

Thus, upon introducing the differential operator $M_{n,p} := \sum_{i=1}^{n} \partial^{2i}$, and denoting $C_{n,p} := a_0$, we see that φ satisfies the profile equation

$$(M_{n,p} + a_0)\varphi = \frac{\varphi^{p+1}}{p+1}. \tag{2.3.1}$$

With this notations, Albert has shown (see Theorem 4.10 in [23]) the following formula

$$\langle (M_{n,p} + C_{n,p} - \varphi^p)^{-1} \varphi, \varphi \rangle = a \sum_{j=0}^{\infty} b_j \tag{2.3.2}$$

where $a = \left(\frac{2^{n+r-1}\Gamma(r)}{\pi \Gamma(n)} \right)^2 > 0$, $\lambda_m = \frac{\Gamma(r+m)}{\Gamma(r+1)} \frac{\Gamma(r+2n+1)}{\Gamma(r+2n+m)}$ and

$$b_j = \left(\frac{\lambda_{2j}}{1-\lambda_{2j}} \right) \left\{ \frac{\Gamma(2j+1) \cdot (2j+n+r-\frac{1}{2})}{\Gamma(2j+2n+2r-1)} \right\} \left\{ \frac{\Gamma(j+n)\Gamma(j+n+r-\frac{1}{2})}{\Gamma(j+1)\Gamma(j+r+\frac{1}{2})} \right\}^2.$$

2.3.2 Stability of the wave of the fourth order Schrödinger equation (2.1.4)

Matters are reduced to the number of negative eigenvalues of D. As we have previously observed on the related fractional NLS model,

$$D = \begin{pmatrix} \langle L_2^{-1} \phi', \phi' \rangle & 0 \\ 0 & \langle L_1^{-1} \phi, \phi \rangle \end{pmatrix},$$

which in view of the positivity of L_2^{-1} on $Ker[L_2]^{\perp}$ reduces to the consideration of the quantity $\langle L_1^{-1}\phi, \phi \rangle$. The stability is then characterized by the condition $\langle L_1^{-1}\phi, \phi \rangle < 0$. Recalling that $L_1 = \partial_x^4 - \partial_x^2 + \alpha - 3\phi^2$, with ϕ given by (3.3.1), we apply the Albert's theory for the quantity $\langle L_1^{-1}\phi, \phi \rangle$, see Section 2.3.1 and (2.3.2) below. More specifically, in the notations there, we take $n = 2, r = 2, p = 2$, which yields the formula

$$\lambda_{2j} = \frac{\Gamma(2j+2)}{\Gamma(3)} \cdot \frac{\Gamma(7)}{\Gamma(2j+6)} = \frac{6!}{2!} \cdot \frac{(2j+1)!}{(2j+5)!}. \tag{2.3.3}$$

and hence

$$b_j = \frac{360(2j+7/2)(j+1)^2(j+5/2)^2(2j)!}{((2j+2)(2j+3)(2j+4)(2j+5)-360)\,(2j+6)!}. \tag{2.3.4}$$

Then we have[3] $\sum_{j=1}^{\infty} b_j \approx 0.0118141$, $b_0 = -0.045573$, whence

$$\langle L_1^{-1}\phi, \phi\rangle = a\sum_{j=0}^{\infty} b_j < 0,$$

whence we conclude the stability of the wave (3.3.1).

2.3.3 On the instability of the wave (2.1.11) of the fourth order Klein-Gordon model

We need to consider the eigenvalue problem (2.1.21), with $\mathcal{L}, \mathcal{J}$ given in (2.1.23). Based on the index counting theory and the fact that $n(\mathcal{L}) = 1$ by proposition 2.3.1, we are interested in the number of negative eigenvalues of the matrix

$$D = \begin{pmatrix} \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_1, \mathcal{J}^{-1}\phi_1\rangle & \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_1, \mathcal{J}^{-1}\phi_2\rangle \\ \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_2, \mathcal{J}^{-1}\phi_1\rangle & \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_2, \mathcal{J}^{-1}\phi_2\rangle \end{pmatrix}$$

where the two elements of the kernel are given by

$$\phi_1 = (\phi', 0, 0, 0)^T, \quad \phi_2 = (0, \phi, 0, 0)^T$$

Since

$$\mathcal{J}^{-1} = \begin{pmatrix} 0 & 2\beta & -1 & 0 \\ -2\beta & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.3.5}$$

[3] Here we have used Mathematica for an approximation of the value of the series

We have

$$
\begin{aligned}
D_{11} &= \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_1, \mathcal{J}^{-1}\phi_1\rangle = \\
&= \left\langle \begin{pmatrix} L_1^{-1} & 0 & 0 & 0 \\ 0 & L_2^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2\beta\phi' \\ \phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2\beta\phi' \\ \phi' \\ 0 \end{pmatrix} \right\rangle = \\
&= 4\beta^2\langle L_2^{-1}\phi', \phi'\rangle + \|\phi'\|^2 > 0,
\end{aligned}
$$

since L_2^{-1} is positive on $Ker[L_2]^\perp = span[\phi]^\perp$. A quick inspection shows $D_{12} = D_{21} = 0$, while

$$D_{22} = 4\beta^2\langle L_1^{-1}\phi, \phi\rangle + \|\phi\|^2.$$

Thus, we have reduced matters to the sign of D_{22}, as usual. It turns out that $D_{22} > 0$, which implies a real instability, since then $n(D) = 0$, while $n(\mathcal{L}) = 1$. Thus, it remains to show that $D_{22} > 0$. We apply again Albert's theory.

We have in fact just evaluated the quantity $\langle L_1^{-1}\phi, \phi\rangle$ in our Schrödinger calculations. With the same λ_{2j} and b_j as in (2.3.3), (2.3.4) respectively, we find

$$\langle L_1^{-1}\phi, \phi\rangle = \frac{1}{3}\left(\sqrt{\frac{9}{10}}\right)^2 \frac{1}{\sqrt{1/20}} \left(\frac{2^3\Gamma(2)}{\pi\Gamma(2)}\right)^2 \left(\sum_{j=0}^{\infty} b_j\right) \approx -0.0979003,$$

However, for the function ϕ defined in (3.3.1), $\|\phi\|^2 \sim 1.7888543...$ whence

$$D_{22} = 4\beta^2\langle L_1^{-1}\phi, \phi\rangle + \|\phi\|^2 \sim 0.802019... > 0.$$

Chapter 3

Spectral stability of vortices for the NLS in n dimensions

3.1 Introduction and statement of the main results

We consider the focusing nonlinear Schrödinger equation

$$iu_t + \triangle u + |u|^{p-1}u = 0, (t,x) \in \mathbf{R}_+ \times \mathbf{R}^n \tag{3.1.1}$$

We will be particularly interested in the stability properties of its vortices in arbitrary spatial dimensions, $n \geq 2$. Clearly, a solution in the form $e^{i\omega t}\Psi(x)$ satisfies the standard profile equation

$$-\Delta\Psi + \omega\Psi - |\Psi|^{p-1}\Psi = 0, x \in \mathbf{R}^n \tag{3.1.2}$$

3.1.1 Vortices of NLS

We are interested in vortex type solutions for (3.1.2). In order to introduce the relevant form of these special solutions, let us focus for the moment on the case of two spatial dimensions, $n = 2$. In this case, we are looking for standing wave solutions in the form $\Psi = e^{im\theta}\phi(r)$, where $\omega > 0$, $m \in \mathbb{N}$ and (r,θ) are the polar coordinates in $\mathbb{R}^2$. In terms of the radial variable, ϕ satisfies the ODE

$$-[\partial_r^2 + \frac{1}{r}\partial_r]\phi_\omega + \left(\omega + \frac{m^2}{r^2}\right)\phi_\omega - |\phi_\omega|^{p-1}\phi_\omega = 0, \quad r > 0. \tag{3.1.3}$$

The general case of vortices in arbitrary spatial dimension is handled as follows. In even space dimensions, $n = 2l$, standing wave solutions are sought in the form

$$\Psi = e^{i\Sigma_{k=1}^{l} m_k\theta_k}\phi_\omega(r_1, r_2, \cdots, r_l),$$

where (r_k, θ_k) are polar coordinates for (x_{2k-1}, x_{2k}), $m_k \in \mathbb{N}\cup\{0\}$, $k = 1, 2, \cdots, l$. Clearly, the profile equation for such a ϕ is as follows

$$-\sum_{k=1}^{l}\Delta_{r_k}\phi_\omega + (\omega + \sum_{k=1}^{l}\frac{m_k^2}{r_k^2})\phi_\omega - |\phi_\omega|^{p-1}\phi_\omega = 0, \tag{3.1.4}$$

where we have used the notation $\Delta_r = \partial_r^2 + \frac{1}{r}\partial_r$ for the radial Laplacian in two spatial dimensions. In odd spatial dimensions, say $n = 2l+1, l \geq 1$, the waves are

$$\Psi = e^{i\Sigma_{k=1}^{l} m_k\theta_k}\phi_w(r_1, r_2, \cdots, r_l, z),$$

where (r_k, θ_k) are in $\mathbb{R}^2$, $m_k \in \mathbb{N}\cup\{0\}$, $k = 1, 2, \cdots, l-1$, and $(r_l, \theta_l, z), z = x_n$ are cylindrical coordinates in $\mathbb{R}^3$. The corresponding profile equation is

$$-[\partial_z^2 + \sum_{k=1}^{l}\Delta_{r_k}]\phi_\omega + (\omega + \sum_{k=1}^{l}\frac{m_k^2}{r_k^2})\phi_\omega - |\phi_\omega|^{p-1}\phi_\omega = 0 \tag{3.1.5}$$

Before we discuss the known results for the existence and uniqueness of such vortex solutions, let us take the time to quickly review the classical ground states. These are solutions in the form $e^{i\omega t}\Psi(\rho)$ for the profile equation (3.1.2), which are well-understood in the literature. In fact, its existence and uniqueness ([26], [27],[28]) in all dimensions and appropriate values, namely $p \in (1, p_n^*)$, $p_n^* := \begin{cases} +\infty & n = 1,2 \\ 1 + \frac{4}{n-2} & n \geq 3 \end{cases}$ was shown in [28]. Further, the stability behavior of these (unique) solutions is also well-known, [29], this is in fact one of the main class of examples that was worked out within the Grillakis-Shatah-Strauss formalism, [29], [30]. Concisely, these waves are spectrally/orbitally/linearly stable for $p \in (1, 1 + \frac{4}{n})$ and unstable for $p \in (1 + \frac{4}{n}, p_n^*)$ [31]. It is worth mentioning however that the ground states conform to our setup only in the case of $n = 2$, as ρ there is the global radial variable, whereas we are proposing here an ansatz with $[\frac{n}{2}]$ pairs of radial variables.

We now turn to the problem at hand, namely the existence and stability of solutions to (3.1.4) and (3.1.5). In the two dimensional case, $n = 2$ and $m = 0$, these are the ground states, for which we have a complete picture, including uniqueness and stability analysis. For the case $n = 2, m \neq 0$, the existence of the solutions of (3.1.3) are also well-studied. In the work [32], the authors have provided a detailed study of the elliptic problem (3.1.3) - in particular, they proved the existence of smooth solutions to (3.1.3) with any prescribed number of zeros. If $\phi_{\omega,m}$ is nonnegative, then $e^{i(m\theta+wt)}\phi_\omega(r)$ is a ground state in the class $X_m = \{e^{im\theta}v(r)|v \in H^1_{rad}(\mathbb{R}^2), v \in L^2_{rad}(\mathbb{R}^2)\}$, and Mizumachi, [33, 34] proved that the standing wave solution $e^{i(m\theta+\omega t)}\phi_\omega(r)$ is orbitally stable in the class $\mathcal{X}_m$ if $1 < p < 3$.

Muzumachi showed the uniqueness of positive solutions to (3.1.3) with $m \neq 0$,using the classification theorem developed by Yanagida and Yotsutani [35]. He has also considered their stability, when the perturbation is in the same form as the soliton, namely $v = e^{i(m\theta+\omega t)}h(t,r)$. His results can be summarized as follows - these solutions are unique for all $p \in (1,\infty)$ and they are orbitally stable for $p \in (1,3)$ and unstable otherwise. Here, it is worth discussing the situation in more details. Recall that this is indeed in line with the expectations and the case $m = 0$, which predicts stability for two spatial dimensions exactly for $p \in (1,3)$. On the other hand, the perturbations for the vortices are only in the special form described above, so it remained an open question whether or not such stability holds for arbitrary perturbations. In addition, the ODE techniques in [33, 34] seem to apply to the two dimensional case only.

3.1.2 Function spaces and harmonics

We will work with the Lebesgue spaces $L^p(\mathbf{R}^n), 1 \leq p \leq \infty$, defined through the norms $\|f\|_p = \left(\int_{\mathbf{R}^n} |f(x)|^p dx\right)^{1/p}$. More generally, $L^p(w(x)dx)$ for a positive weight w, is defined by the norm $\|f\|_{L^p(w(x)dx)} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)dx\right)^{1/p}$. In addition, there are the corresponding Sobolev spaces,

$W^{k,p}(\mathbf{R}^n)$, defined through $f^{(\alpha)} \in L^p(\mathbf{R}^n) : |\alpha| \leq k$, with their respective natural norms.

Next, we introduce the following decomposition of $L^2(\mathbf{R}^2)$ - we can identify an arbitrary function $f \in L^2(\mathbf{R}^2)$ with a sequence $\{f_n\}_{n=-\infty}^{\infty}$ by

$$f = \sum_{m=-\infty}^{\infty} f_m(\rho) e^{im\theta}$$

so that $\|f\|^2_{L^2(\mathbf{R}^2)} = \sum_{m=-\infty}^{\infty} \|f_m\|^2_{L^2((0,\infty),\rho d\rho)}$. This is nothing but an instance of a decomposition in spherical harmonics, valid in all dimensions, which takes this particularly simple form in two spatial dimensions. We will denote the L^2 subspaces $\mathcal{X}_m := span\{g(\rho)e^{im\theta} | g \in H^1_{rad}(\mathbf{R}^2), g \in L^2_{rad}(\mathbf{R}^2)\}$. Clearly, in the standard dot product of $L^2(\mathbf{R}^2)$, $\mathcal{X}_l \perp \mathcal{X}_m$, as long as $l \neq m$, so $L^2(\mathbf{R}^2) = \oplus_{m=-\infty}^{\infty} \mathcal{X}_m$.

Clearly, the Laplacian Δ on $\mathcal{X}_m$ takes the form

$$\Delta[f(\rho)e^{im\theta}] = [\partial^2_\rho f + \frac{1}{\rho}\partial_\rho f - \frac{m^2}{\rho^2} f]e^{im\theta} = [\Delta_r f - \frac{m^2}{\rho^2} f(\rho)]e^{im\theta}. \tag{3.1.6}$$

Note that the evolution of the NLS (3.1.1) leaves the spaces $\mathcal{X}_m$ invariant, in the sense that whenever $u_0 \in \mathcal{X}_m$, the corresponding solution $u(t,\cdot) \in \mathcal{X}_m$ for any later time $t > 0$. In view of this, it is worth considering the Cauchy problem for (3.1.1) in the spaces $\mathcal{X}_m, m = 0, 1, \ldots$. In particular, Mizumachi's results state that the two dimensional solutions of (3.1.3), $\phi_{\omega,m}$ are orbitally stable on $\mathcal{X}_m$.

In the higher dimensional situations, $n \geq 3$, we can similarly consider spaces

$$\mathcal{X}_{\vec{m}}, \vec{m} = (m_1, \ldots, m_l), l = [\frac{n}{2}].$$

$$\mathcal{X}_{\vec{m}} := span\{f_{\vec{m}}(\rho_1, \ldots, \rho_l)e^{i(m_1\theta_1 + \ldots + m_l\theta_l)} | f_{\vec{m}} \in H^1_{rad}(\mathbf{R}^l), f_{\vec{m}} \in L^2_{rad}(\mathbf{R}^l)\},$$

with the appropriate norm. In the case $n = 2l$, we can write

$$f = \sum_{m_1,\ldots,m_l=-\infty}^{\infty} f_{\vec{m}}(\rho_1, \ldots, \rho_l) e^{i(m_1\theta_1 + \ldots + m_l\theta_l)}$$

with a norm $\|f\|^2_{L^2(\mathbf{R}^2)} = \sum_{m_1,\ldots,m_l=-\infty}^{\infty} \|f_{\vec{m}}\|^2_{L^2((0,\infty)^l,\rho_1\ldots\rho_l d\rho_1\ldots d\rho_l)}$.

In the odd dimensional case, $n = 2l+1$, we simply take [21]

$$f = \sum_{m_1,\ldots,m_l=-\infty}^{\infty} f_{\vec{m}}(\rho_1,\ldots,\rho_l,z_n)e^{i(m_1\theta_1+\ldots+m_l\theta_l)}$$

with a norm $\|f\|^2_{L^2(\mathbf{R}^2)} = \sum_{m_1,\ldots,m_l=-\infty}^{\infty} \|f_{\vec{m}}\|^2_{L^2((0,\infty)^l,\rho_1\ldots\rho_l d\rho_1\ldots d\rho_l)L^2_z}$. For future reference, introduce the subspace of $L^2(\mathbf{R}^n)$

$$L^2_r := \{f = f(\rho_1,\ldots,\rho_l) : \|f\|^2_{L^2_r} = \int_0^\infty |f(\rho_1,\ldots,\rho_l)|^2 \rho_1\ldots\rho_l d\rho_1\ldots d\rho_l\}$$

in the case $n = 2l$, while in the odd dimensional case, $n = 2l+1$,

$$L^2_r := \{f = f(\rho_1,\ldots,\rho_l,x_n) : \|f\|^2_{L^2_r} = \int_0^\infty |f(\rho_1,\ldots,\rho_l,x_n)|^2 \rho_1\ldots\rho_l d\rho_1\ldots d\rho_l dx_n\},$$

and the corresponding Sobolev spaces $H^2_r = \{f \in L^2_r : \partial^2_{\rho_k} f \in L^2_r, k = 1,\ldots,l\}$, $H^2_r = \{f \in L^2_r : \partial^2_{\rho_k} f, \partial^2_{x_n} f \in L^2_r, k = 1,\ldots,l\}$

We are now ready to give a precise formulation of the main results. As usual, the spectral stability of a wave is determined by its linearized operator. It simply means that the linearized operator around the wave lacks spectrum in the open right-half of the complex plane. Otherwise, we refer to the wave as (spectrally) unstable.

3.1.3 Main results

We start with the two dimensional case. This case was considered by Mizumachi in [33, 34], but we include our approach and results here in order to illustrate the variational method we use in the higher dimensional cases. Our result states the following.

Theorem 3.1.1. *Let $n = 2$, $1 < p < 3$, $\omega > 0$, $m \in \mathcal{Z}$. Then, the equation* (3.1.3) *has classical and positive solutions $\phi_{\omega,m}$, which are constructed as (multiples of) the minimizers of the following* *constrained minimization problem*

$$\begin{cases} \mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx \to \min \\ \\ \mathcal{K}(u) = \int_{\mathbb{R}^2} |u|^2\, dx = \lambda, u \in \mathcal{X}_m \end{cases} \tag{3.1.7}$$

Such solutions are spectrally stable, for $p \in (1,3)$, *with respect to perturbations in* $\mathcal{X}_m$.

Remark: Mizumachi, [33] has shown the uniqueness, by ODE methods, for positive solutions of (3.1.3). Thus the solutions produced by Theorem 3.1.1 are exactly the same as the ones in [33]. Thus, the results of Theorem 3.1.1 are not really new, but the proof follows along a completely different line of argument. Basically, we do not need to study the spectral properties of the linearized operators, arising out of the solutions of the ODE (3.1.3). Instead, we rely on the variational construction, which yields the same properties in a more direct way.

On the other hand, it is worth noting that Mizumachi's result is stronger, namely the orbital stability in the case $p \in (1,3)$, while our results concern only the spectral stability. For the purposes of the proof (see Theorem 7.1.5, [37]), orbital stability is the same as spectral stability plus it requires in addition that the linearized operator L_+ (see Proposition 3.2.4 below for a definition) to satisfy $Ker[L_+] = span[\nabla\phi_{\omega,m}]$. We do not verify here this property, sometimes referred to as non-degeneracy of $\phi_{\omega,m}$.

In the higher dimensional cases, the statement remains essentially unchanged, except with the appropriate dependence of the index p on the dimension. The results here are new, but in fact they follow along the ideas of the proof of Theorem 3.1.1. Again, orbital stability will follow, once one can establish the non-degeneracy of the waves.

Theorem 3.1.2. *Let* $n \geq 3$, $\vec{m} \in \mathcal{Z}^{[\frac{n}{2}]}$, $p \in (1, 1+\frac{4}{n})$ *and* $\omega > 0$. *Then, the equation* (3.1.3) *has classical and positive solutions* $\phi_{\omega,\vec{m}}$, *constructed as multiples of the minimizers of the following* *constrained variational problem*

$$\begin{cases} \mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}\,dx \to \min \\ \\ \mathcal{K}(u) = \int_{\mathbb{R}^n} |u|^2\,dx = \lambda, u \in \mathcal{X}_{\vec{m}} \end{cases} \tag{3.1.8}$$

Such solutions are spectrally stable, when $p \in (1, 1+\frac{4}{n})$*, with respect to perturbations in the space* $\mathcal{X}_{\vec{m}}$.

3.2 The vortices in $\mathbf{R}^2$ and their stability properties

We start with the variational construction. in addition to establishing the existence of these waves[1], this approach will give us helpful information regrading the spectral properties of the associated linearized operators, which in turn will be helpful in our stability considerations.

3.2.1 Variational construction of the vortices in $\mathbf{R}^2$

We consider the minimization problem (3.1.7) and show that the minimizer exists and is a weak solution of the associated Euler-Lagrange equation (3.1.3). We compute the action of the functional

$$\mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\,dx$$

on functions $u \in \mathcal{X}_m$, where $u(x) = u(\rho,\theta) = \phi(\rho)e^{im\theta}$ to get the following convenient form

$$\mathcal{H}(u) = \mathcal{H}(\phi) = \frac{1}{2}\int_0^\infty |\phi'|^2\,\rho d\rho + \frac{1}{2}m^2\int_0^\infty \frac{|\phi|^2}{\rho}\,d\rho - \frac{1}{p+1}\int_0^\infty |\phi|^{p+1}\rho d\rho,$$

where $\rho = |x|$ is the radial variable in dimension two and $u \in H^1_r(\mathbb{R}^2)$ is a radial function. Fix m. Let $I_\lambda = \inf_{u\in\mathcal{X}_m, \mathcal{K}(u)=\lambda} \mathcal{H}(u)$ with $\lambda > 0$. We will show that a minimizer $u = \phi(|x|)$ exists and is a weak solution of (3.1.3) for some ω. To do this, we will use a concentration compactness argument

[1] which has already been established with other methods, e.g. [32], [33], [34]

and a few preliminary lemmas.

Proposition 3.2.1. *If $1 < p < 3$, and $\lambda > 0$, then $-\infty < I_\lambda < 0$. In addition, there exists a constrained minimizer in $\mathcal{X}_m$.*

Proof. Fix $\lambda > 0$, and $u \in \mathcal{X}_m$ with $\|u\|_{L^2(\mathbb{R}^2)} = \lambda$. Consider dilations $u_\mu(x) = \mu u(\mu x)$, $\mu > 0$. Since $\|u_\mu\|_{L^2} = \|u\|_{L^2}$, $\|\nabla u_\mu\|_{L^2} = \mu\|\nabla u\|_{L^2}$, $\|u_\mu\|_{L^{p+1}} = \mu^{\frac{p-1}{p+1}}\|u\|_{L^{p+1}}$, we have that

$$\mathcal{H}(u_\mu) = \frac{\mu^2}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \frac{\mu^{p-1}}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx.$$

For $p < 3$, $\mathcal{H}(u_\mu) < 0$ for $\mu > 0$ sufficiently small. Thus $I_\lambda < 0$. We will also show $I_\lambda > -\infty$. By Gagliardo-Nirenberg-Sobolev Inequality,

$$\|u\|_{L^q} \le C_{n,q}\|\nabla u\|_{L^2(\mathbb{R}^n)}^{n(1/2-1/q)}\|u\|_{L^2(\mathbb{R}^n)}^{1-n(1/2-1/q)} \tag{3.2.1}$$

if $n > 2$, $2 \le q \le \frac{2n}{n-2}$, and if $n = 2$, $2 \le q < \infty$.

$$\begin{aligned}\mathcal{H}(u) &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx\\ &\ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \frac{C_{2,p}^{p+1}}{p+1}\|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{2(p-1)}{2}}\lambda^b\\ &\ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \frac{C_p^{p+1}}{p+1}\|\nabla u\|_{L^2(\mathbb{R}^2)}^{p-1}\lambda^b\end{aligned}$$

where $b = \frac{1}{2}(1-2(1/2-1/p+1))(p+1)$. Then $\mathcal{H}(u) \ge g(\|\nabla u\|_{L^2})$, where $g(R) = \frac{1}{2}R^2 - \frac{C_{2,p}^{p+1}}{p+1}\lambda^b R^{p-1}$ for $p < 3$. Hence $I_\lambda \ge g_{min} > -\infty$. It follows that $-\infty < I_\lambda < 0$.

Next, we will use a standard concentration compactness argument in order to establish the existence of u. We just indicate the main steps, as the argument mirrors a well-known construction.[2]

Since $-\infty < I_\lambda < 0$, we can find a minimizing sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{X}_m$, such that $\|u_k\|_{L^2}^2 = \lambda$ and $\mathcal{H}(u_k) \to I_\lambda$ as $k \to \infty$. Since $I_\lambda < 0$, thus $\mathcal{H}(u_k) < \infty$ for k large enough. Further, by Sobolev-

[2] except at the final phase, when the tightness is established. There, one needs to show the non-trivial fact that the translates guaranteed by tightness are actually bounded in $\mathbf{R}^2$, whence it easily follows that there is a minimizer in $\mathcal{X}_m$

Garliardo-Nirenberg, for large k, $g(\|\nabla u_k\|_{L^2}) \leq \mathcal{H}(u_k) < 0$. Then it follows that $\|\nabla u_k\|_{L^2(\mathbb{R}^2)} \leq R_0$ for k sufficiently large.

Without loss of generality, we can assume that $\|\nabla u_k\|_{L^2(\mathbb{R}^2)} \leq R_0$ for any $k > 0$. Thus $\{u_k\}_{k=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^2)$. By concentration compactness, we have either "convergence of translates", "vanishing" or "splitting".

First let's rule out vanishing. Since $\mathcal{H}(u_k) \leq \frac{I_\lambda}{2} < 0$ for k sufficiently large,

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_k|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u_k|^{p+1} \, dx \leq \frac{I_\lambda}{2} < 0.$$

for k sufficiently large. In particular, we obtain

$$\frac{1}{p+1}\int_{\mathbb{R}^2} |u_k|^{p+1} \, dx \geq -\frac{I_\lambda}{2} > 0.$$

for k sufficiently large. If vanishing occurred, by Gagliardo-Nirenberg-Sobolev, there will exist a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $u_{kj} \to 0$ in $L^q(\mathbb{R}^2)$, for any $2 < q < \infty$, a contradiction.

Then we rule out splitting. If splitting occurred, then there exists $\gamma \in (0, \lambda)$ and a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$, and bounded sequences $\{v_j\}_{j=1}^{\infty}$ and $\{w_j\}_{j=1}^{\infty}$ in $\mathcal{X}_m$ with

$$\begin{aligned}
&\lim_j \|v_j\|^2_{L^2(\mathbb{R}^2)} = \gamma, \;\; \lim_j \|w_j\|^2_{L^2(\mathbb{R}^2)} = \lambda - \gamma, \\
&dist(spt(v_j), spt(w_j)) \to \infty \\
&\lim_j \int_{\mathbb{R}^2} \left(|u_{k_j}|^{p+1} - |v_j|^{p+1} - |w_j|^{p+1}\right) \, dx = 0 \\
&\liminf_{j\to\infty} \int_{\mathbb{R}^n} \left(|\nabla u_{k_j}|^2 - |\nabla v_j|^2 - |\nabla w_j|^2\right) \, dx \geq 0,
\end{aligned}$$

where $spt(v_j) = \{x \in \mathbf{R}^2 | v_j(x) \neq 0\}$ and $spt(w_j) = \{x \in \mathbf{R}^2 | w_j(x) \neq 0\}$. Fix $\varepsilon > 0$, we have that for all sufficiently large j,

$$I_\lambda + 5\varepsilon \geq \mathcal{H}(u_{k_j}) + 4\varepsilon \geq \mathcal{H}(v_j) + \mathcal{H}(w_j) + \varepsilon.$$

Now there exist sequences $\{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty} : \lim_j a_j = \lim_j b_j = 1$, so that $\|a_j v_j\|_{L^2}^2 = \gamma$, $\|b_j w_j\|_{L^2}^2 = \lambda - \gamma$. Thus $\mathcal{H}(a_j v_j) \geq I_\gamma$ and $\mathcal{H}(b_j w_j) \geq I_{\lambda-\gamma}$, while for j large enough, we have

$$\mathcal{H}(v_j) \geq \mathcal{H}(a_j v_j) - \varepsilon/2, \;\; \mathcal{H}(w_j) \geq \mathcal{H}(b_j w_j) - \varepsilon/2,$$

since $\lim_j[\mathcal{H}(a_j v_j) - \mathcal{H}(v_j)] = 0$. Thus for large j, $I_\lambda + 5\varepsilon \geq \mathcal{H}(a_j v_j) + \mathcal{H}(b_j w_j) \geq I_\gamma + I_{\lambda-\gamma}$, whence

$$I_\lambda \geq I_\gamma + I_{\lambda-\gamma}.$$

However, similar to the classical case, the map $\lambda \to I_\lambda$ is strictly subadditive. In fact, we have the following lemma to that effect. Once Lemma 3.2.2 is established, a contradiction is reached and we will have shown that splitting cannot occur.

Lemma 3.2.2. *The map $\lambda \to I_\lambda$ is strictly subadditive, i.e $I_\lambda < I_\gamma + I_{\lambda-\gamma}$, $\forall \gamma \in (0, \lambda)$.*

Proof. Note for every $\theta > 1$,

$$\begin{aligned}\mathcal{H}(\theta u) &= \frac{\theta^2}{2}\int_{\mathbf{R}^2} |\nabla u|^2 \, dx - \frac{\theta^{p+1}}{p+1}\int_{\mathbb{R}^2} |u|^{p+1} \, dx \\ &= \theta^2\Big(\frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2 \, dx - \frac{\theta^{p-1}}{p+1}\int_{\mathbb{R}^2} |u|^{p+1} \, dx\Big) < \theta^2 \mathcal{H}(u)\end{aligned}$$

Thus $I_{\theta^2\lambda} < \theta^2 I_\lambda$, for all $\theta > 1$. Hence, for all $\gamma \in (\frac{\lambda}{2}, \lambda)$,

$$I_\lambda = I_{\frac{\lambda}{\gamma}\gamma} < \frac{\lambda}{\gamma} I_\gamma = I_\gamma + \Big(\frac{\lambda-\gamma}{\gamma}\Big) I_\gamma = I_\gamma + \frac{\lambda-\gamma}{\gamma} I_{\frac{\gamma}{\lambda-\gamma}(\lambda-\gamma)} < I_\gamma + I_{\lambda-\gamma}$$

If $\gamma \in (0, \frac{\lambda}{2}]$, repeat the steps above with replacing γ with $\lambda - \gamma$. □

By concentration compactness, there is a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and a sequence $\{y_j\}_{j=1}^{\infty} \subset \mathbb{R}^2$, such that $u_{k_j}(\cdot - y_j) \to u_0$ in L^2, for some $u_0 \in H^1(\mathbb{R}^2)$. We will show first that $\{y_j\}_{j=1}^{\infty}$ is a bounded sequence. In fact, we have the following

Lemma 3.2.3. *The sequence $\{y_j\} \subset \mathbf{R}^2$ is bounded.*

Proof. We argue by contradiction. Let $\{y_j\}$ be unbounded (and after picking a subsequence, denoted again by $\{y_j\}$), so that $\lim_j |y_j| = \infty$ and $\frac{y_j}{|y_j|} \to (\cos(\theta_0), \sin(\theta_0)) \in \mathcal{S}^1$. Without loss of generality, $u_{k_j}(x) = \phi_{k_j}(|x|)e^{im\theta}$, for real-valued functions ϕ_{k_j}. Let $\varepsilon > 0$. Then, there is N and j_0, so that for all $j \geq j_0$,

$$\|u_0\|_{L^2(|x|>N)} < \varepsilon, \|u_{k_j}(\cdot - y_j)\|_{L^2(|x|>N)} < \varepsilon.$$

We introduce the function $\theta_j(x) : x - y_j = (|x-y_j|\cos\theta_j(x), |x-y_j|\sin\theta_j(x))$ such that

$$\lim_{j\to\infty} \|\phi_{k_j}(|\cdot - y_j|)e^{im\theta_j(\cdot)} - u_0(\cdot)\|_{L^2(|x|<N)} = 0.$$

Since $|x| < N$, we have

$$e^{i\theta_j(x)} = \left(\frac{x_1 - y_j^1}{|x-y_j|} + i\frac{x_2 - y_j^2}{|x-y_j|}\right) \to -\left(\frac{y_j^1}{|y_j|} + i\frac{y_j^2}{|y_j|}\right) = -e^{i\theta_0}.$$

Thus, $\lim_j e^{-im\theta_j(x)} = (-1)^m e^{-im\theta_0} =: e^{i\alpha_m}$. It follows that

$$\lim_{j\to\infty} \|\phi_{k_j}(|\cdot - y_j|) - e^{i\alpha_m}u_0(\cdot)\|_{L^2(|x|<N)} = 0$$

Then there is j_0, so that whenever $j \geq j_0$,

$$\|\phi_{k_j}(|\cdot - y_j|) - e^{i\alpha_m}u_0(\cdot)\| \leq 3\varepsilon.$$

We conclude that $e^{i\alpha_m}u_0$ is real-valued. Without loss of generality (namely, if we have picked $e^{i\alpha_m}u_{k_j}(\cdot - y_j) \to e^{i\alpha_m}u_0$), we have reduced to the case where $u_{k_j}(\cdot - y_j) \to u_0$ and u_0 is real-valued. It follows that for $u_{k_j}(x) = \phi_{k_j}(|x|)e^{im\theta}$ (here ϕ_{k_j} is not necessarily real valued!), we have

$$\lim_j \|\phi_{k_j}(|\cdot|)e^{im\theta} - u_0(\cdot - y_j)\|_{L^2} = 0. \tag{3.2.2}$$

Letting $\phi_{k_j}(|x|) = p_j(|x|) + iq_j(|x|)$ and taking the imaginary part of the function in (3.2.2)

$$\lim_j \|p_j(|x|)\sin(m\theta) + q_j(|x|)\cos(m\theta)\|_{L^2(\mathbf{R}^2)} = 0.$$

But by the constraint,

$$\|p_j(|x|)\sin(m\theta) + q_j(|x|)\cos(m\theta)\|^2_{L^2(\mathbf{R}^2)} = \int_0^\infty \int_0^{2\pi} [p_j^2(\rho)\sin^2(m\theta) + q_j^2(\rho)\cos^2(m\theta)]d\theta\rho d\rho$$
$$= \pi \int_0^\infty [p_j(\rho)^2 + q_j^2(\rho)]\rho d\rho = \frac{1}{2}\|\phi_{k_j}\|^2_{L^2(\mathbf{R}^2)} = \frac{\lambda}{2}.$$

Thus, we have reached a contradiction with $\lim |y_j| = \infty$. □

Since y_j is a bounded, after taking a subsequence (denoted the same), we have $y_j \to y_0 \in \mathbf{R}^2$. It now easily follows that $\lim_j \|u_{k_j} - u_0(\cdot - y_0)\|_{L^2} = 0$. Clearly, since $\mathcal{X}_m$ is a closed subspace, we have that $u_0(x - y_0) =: v_0 = \phi_0(|x|)e^{im\theta}$ and $\|v_0\|^2 = \lim_j \|u_{k_j}\|^2 = \lambda$. By (3.2.1), it follows that $\lim_j \|u_{k_j} - v_0\|_{L^q} = 0, 2 < q < \infty$, in particular for $q = p+1$.

Clearly, $u_{k_j} \to v_0$ weakly in $H^1(\mathbf{R}^2)$, whence using lower-weak semicontinuity of $u \to \int_{\mathbb{R}^2} |\nabla u|^2\, dx$, we conclude

$$I_\lambda = \liminf_{j\to\infty} \mathcal{H}(u_{k_j}) \geq \mathcal{H}[v_0],$$

while $\|v_0\|^2 = \lambda$. Thus, $v_0 \in \mathcal{X}_m$ is a minimizer of (3.1.7).

□

Proposition 3.2.4. *A constrained minimizer of* (3.1.7)*, ϕ satisfies the Euler-Lagrange equation*

$$-\Delta\phi + \omega_\lambda \phi - |\phi|^{p-1}\phi = 0, \quad \omega_\lambda = \frac{\|\phi\|^{p+1}_{L^{p+1}(\mathbf{R}^2)} - \|\nabla\phi\|^2_{L^2(\mathbf{R}^2)}}{\lambda}. \tag{3.2.3}$$

Moreover, the following scaling identities hold

$$\phi^{\lambda} = \lambda^{\frac{1}{3-p}}\phi^{1}(\lambda^{\frac{p-1}{2(3-p)}}x),\ I_{\lambda} = \lambda^{\frac{2}{3-p}}I_{1},\ J_{\lambda} = \|\nabla\phi^{\lambda}\|^{2} = \lambda^{\frac{2}{3-p}}\frac{2(p-1)}{p-3}I_{1} \qquad (3.2.4)$$

$$K_{\lambda} = \int_{\mathbb{R}^{2}}|\phi^{\lambda}|^{p+1}\,dx = \lambda^{\frac{2}{3-p}}\frac{2(p+1)}{p-3}I_{1},\ \ \omega_{\lambda} = \lambda^{\frac{p-1}{3-p}}\frac{4}{p-3}I_{1} \qquad (3.2.5)$$

Finally, the linearized operator

$$L_{+} := -\Delta + \omega_{\lambda} - p|\phi|^{p-1}$$

is non-negative on the co-dimension one subspace $\{\phi e^{im\theta}\}^{\perp}$ *of the space* $\mathcal{X}_m$*. That is,*

$$\langle L_{+}h, h\rangle \geq 0,\ \ h \in \{\phi e^{im\theta}\}^{\perp}, h \in \mathcal{X}_m \cap domain(L_{+}).$$

Equivalently, the operator $L_{+}^{rad} = -\Delta_r + \frac{m^2}{|x|^2} + \omega_{\lambda} - p|\phi|^{p-1}$ *acting on the subspace* $H^2_{rad.} \cap \{\phi\}^{\perp}$ *is non-negative.*

Proof. $\phi^{\lambda} = \lambda^{b}\phi(\lambda^{a}x)$ where $\phi = \phi^{1}$ with $\int_{\mathbb{R}^2}|\phi(x)|^2\,dx = 1$. Set $J_{\lambda}(\phi^{\lambda}) = J_1(\phi^{\lambda}) + J_2(\phi^{\lambda})$, where

$$J_1(\phi^{\lambda}) := \int_{\mathbb{R}^2}|\nabla\phi^{\lambda}|^2\,dx = \int_{\mathbb{R}^2}|\lambda^{a+b}\nabla\phi(\lambda^{a}x)^2|\,dx = \lambda^{2b}\int_{\mathbb{R}^2}|\nabla\phi(x)|^2\,dx = \lambda^{2b}J_1(\phi).$$

Further, we also have

$$J_2(\phi^{\lambda}) = m^2\int_{\mathbb{R}^2}\frac{|\phi^{\lambda}(x)|}{|x|^2}\,dx = m^2\lambda^{2b}\int_{\mathbb{R}^2}\frac{|\phi(\lambda^{a}x)|}{|x|^2}\,dx = m^2\lambda^{2b}\int_{\mathbb{R}^2}|\phi(x)|^2\,dx = \lambda^{2b}J_2(\phi).$$

$$\begin{aligned} K_{\lambda}(\phi^{\lambda}) &= \int_{\mathbb{R}^2}|\phi^{\lambda}(x)|^{p+1}\,dx = \lambda^{b(p+1)}\int_{\mathbb{R}^2}|\phi(\lambda^{a}x)|^{p+1}\,dx \\ &= \lambda^{b(p+1)-2a}\int_{\mathbb{R}^2}|\phi(x)|^{p+1}\,dx = \lambda^{b(p+1)-2a}K(\phi). \end{aligned}$$

So we have

$$\lambda^{2b} = \lambda^{b(p+1)-2a} \Rightarrow \; b + 2a = bp.$$

Further

$$\lambda = \int_{\mathbb{R}^2} \lambda^{2b} |\phi(\lambda^a x)|^2 \, dx = \lambda^{2b-2a} \int_{\mathbb{R}^2} |\phi(x)|^2 \, dx = \lambda^{2b-2a} \Rightarrow \; 2b - 2a = 1.$$

Clearly we obtain $a = \frac{p-1}{2(3-p)}$ and $b = \frac{1}{3-p}$. Hence $\phi^\lambda = \lambda^{\frac{1}{3-p}} \phi_1(\lambda^{\frac{p-1}{2(3-p)}} x)$, $\mathcal{H}(\phi) = \frac{1}{2} J(\phi) - \frac{1}{p+1} K(\phi)$ and $\mathcal{H}_\lambda = \lambda^{2b} \mathcal{H}_1 = \lambda^{\frac{2}{3-p}} \mathcal{H}_1$. Let $J := J_1$ and $K := K_1$we obtain,

Thus it suffices to prove the results for the case $\lambda = 1$. So fix $\lambda = 1$. Let $\phi = \phi^1$ be a minimizer. For any $\delta > 0$, consider $u_\delta = \phi + \delta h$. We have that

$$\mathcal{H}(\frac{u_\delta}{\|u_\delta\|}) \geq \mathcal{H}_1.$$

Note that

$$\|u_\delta\| = \sqrt{\|\phi\|^2 + 2\delta \langle \phi, h \rangle + O(\delta^2)} = 1 + \delta \langle \phi, h \rangle + O(\delta^2).$$

$$\begin{aligned}
&\frac{1}{2} \left\| \frac{\nabla u_\delta}{\|u_\delta\|} \right\|^2 = \frac{1}{2} \frac{\|\nabla \phi + \delta \nabla h\|^2}{\|u_\delta\|^2} \\
&= \frac{1}{2} \frac{\|\nabla \phi\|^2 + 2\delta \langle \nabla \phi, \nabla h \rangle + O(\delta^2)}{1 + 2\delta \langle \phi, h \rangle + O(\delta^2)} = \frac{1}{2} \frac{\|\nabla \phi\|^2 - 2\delta \langle \triangle \phi, h \rangle + O(\delta^2)}{1 + 2\delta \langle \phi, h \rangle + O(\delta^2)} \\
&= \frac{1}{2} \frac{\|\nabla \phi\|^2 (1 + 2\delta \langle \phi, h \rangle) - 2\delta \langle \phi, h \rangle \|\nabla \phi\|^2 (1 + 2\delta \langle \phi, h \rangle + O(\delta^2)) - 2\delta \langle \triangle \phi, h \rangle + O(\delta^2)}{1 + 2\delta \langle \phi, h \rangle + O(\delta^2)} \\
&= \frac{1}{2} J_1 - \delta \left(\langle \triangle \phi, h \rangle + J_1 \langle \phi, h \rangle \right) + O(\delta^2)
\end{aligned}$$

Then we come to compute the $p+1$ order term as below,

$$-\frac{1}{p+1} \int_{\mathbb{R}^2} \frac{|u_\delta|^{p+1}}{\|u_\delta\|^{p+1}} \, dx = -\frac{1}{p+1} \int_{\mathbb{R}^2} \frac{|\phi + \delta h|^{p+1}}{(1 + \delta \langle \phi, h \rangle + O(\delta^2))^{p+1}} \, dx,$$

Then we simplify,

$$
\begin{aligned}
&-\frac{1}{p+1}\int_{\mathbb{R}^2}\frac{|u_\delta|^{p+1}}{\|u_\delta\|^{p+1}}\,dx = -\frac{1}{p+1}\int_{\mathbb{R}^2}\frac{\left(|\phi+\delta h|^2\right)^{\frac{p+1}{2}}}{(1+\delta\langle\phi,h\rangle+O(\delta^2))^{p+1}}\,dx\\
&=-\frac{1}{p+1}\int_{\mathbb{R}^2}\frac{|\phi|^{p+1}+\delta(p+1)|\phi|^{p-1}\phi h+O(\delta^2)}{1+(p+1)\delta\langle\phi,h\rangle+O(\delta^2)}\,dx\\
&=-\frac{1}{p+1}\int_{\mathbb{R}^2}\frac{|\phi|^{p+1}(1+(p+1)\delta\langle\phi,h\rangle)-\delta(p+1)|\phi|^{p+1}\langle\phi,h\rangle+\delta(p+1)|\phi|^{p-1}\phi h+O(\delta^2)}{1+(p+1)\delta\langle\phi,h\rangle+O(\delta^2)}\,dx\\
&=-\frac{1}{p+1}\int_{\mathbb{R}^2}|\phi|^{p+1}\,dx+\delta\int_{\mathbb{R}^2}|\phi|^{p+1}\langle\phi,h\rangle-\delta\left\langle|\phi|^{p-1}\phi,h\right\rangle+O(\delta^2)\\
&=-\frac{1}{p+1}K+\delta\left(K\langle\phi,h\rangle-\left\langle|\phi|^{p-1}\phi,h\right\rangle\right)+O(\delta^2)
\end{aligned}
$$

For the term containing m^2, we also obtain,

$$
\begin{aligned}
\frac{1}{2}m^2\int_{\mathbb{R}^2}\frac{|u_\delta|^2}{|x|^2\|u_\delta\|^2}\,dx \quad &= \frac{1}{2}m^2\int_{\mathbb{R}^2}\frac{|\phi+\delta h|^2}{|x|^2(1+2\delta\langle\phi,h\rangle+O(\delta^2))}dx\\
&=\frac{1}{2}m^2\int_{\mathbb{R}^2}\frac{|\phi|^2(1+2\delta\langle\phi,h\rangle)-|\phi|^2(2\delta\langle\phi,h\rangle+O(\delta^2))+2\delta\phi h+O(\delta^2)}{|x|^2(1+2\delta\langle\phi,h\rangle+O(\delta^2))}dx\\
&=\frac{1}{2}J_2-J_2\delta\langle\phi,h\rangle+\delta\frac{m^2}{|x|^2}\langle\phi,h\rangle+O(\delta^2)=\frac{1}{2}J_2+\delta\left(\frac{m^2}{|x|^2}\langle\phi,h\rangle-J_2\langle\phi,h\rangle\right)+O(\delta^2)
\end{aligned}
$$

Since $\frac{1}{2}J-\frac{1}{p+1}K=\mathcal{H}_1$ and $\mathcal{H}(\frac{u_\delta}{\|u_\delta\|})\geq\mathcal{H}_1$, we obtain
$\frac{1}{2}J-\delta\left(\langle\triangle\phi,h\rangle-\frac{m^2}{|x|^2}\langle\phi,h\rangle+J\langle\phi,h\rangle\right)-\frac{1}{p+1}K+\delta\left(K\langle\phi,h\rangle-\left\langle|\phi|^{p-1}\phi,h\right\rangle\right)+O(\delta^2)\geq\frac{1}{2}J-\frac{1}{p+1}K$
We conclude

$$\delta\left\langle-\triangle\phi+\frac{m^2}{|x|^2}\phi-\phi^p+(K-J)\phi,h\right\rangle+O(\delta^2)\geq 0$$

Since it is true for all $\delta\in\mathbb{R}$ and for all test functions h, we conclude that ϕ satisfies

$$-\triangle\phi+\frac{m^2}{|x|^2}\phi-|\phi|^{p-1}\phi+(K-J)\phi=0$$

which is the Euler-Lagrange equation (3.1.3), with a scalar $\omega=K-J$. Finally, there is the Pokhozaev's identity, which we derive in the following way.

Set

$$z_\mu(x) = \mu\phi(\mu x).$$

Since $\int_{\mathbb{R}^2} z_\mu^2(x)\, dx = \int_{\mathbb{R}^2} \phi^2\, dx = 1$, z_μ satisfies

$$\mathcal{H}(z_\mu) = \frac{\mu^2}{2}J - \frac{\mu^{p-1}}{p+1}K.$$

Since the scalar valued function $\mu \to \mathcal{H}_\mu(z)$ achieves its minimum at $\mu = 1$, we must have $\frac{d\mathcal{H}(z_\mu)}{d\mu}|_{\mu=1} = 0$. This is the Pokhozaev's identity

$$J - \frac{p-1}{p+1}K = 0.$$

Hence we obtain the formulas

$$J = \frac{2(p-1)}{p-3}\mathcal{H}_1 \tag{3.2.6}$$

$$K = \frac{2(p+1)}{p-3}\mathcal{H}_1 \tag{3.2.7}$$

$$\omega = \frac{4}{p-3}\mathcal{H}_1. \tag{3.2.8}$$

We then establish the non-coercivity of L_+ on the codimension subspace $\{\phi\}^\perp$. Note that for every test function h, the function

$$g(\delta) = \mathcal{H}(\frac{\phi + \delta h}{\|\phi + \delta h\|})$$

has a minimum at $\delta = 0$, which means that we must have $g'(0) = 0$, and $g''(0) \geq 0$.

We take $h : \langle h, \phi \rangle = 0$, and $\|h\| = 1$. Note that under this restriction

$$\|\phi + \delta h\|^2 = \|\phi\|^2 + 2\delta\,\langle \phi, h\rangle + \delta^2\|h\|^2 = 1 + \delta^2,$$

$$\|\phi + \delta h\| = (1+\delta^2)^{1/2} = 1 + \frac{\delta^2}{2} + O(\delta^3).$$

Consider $g(\delta)$:

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\nabla(\phi+\delta h)}{\|\phi+\delta h\|}\right\|^2 = \frac{1}{2}\frac{\|\nabla\phi\|^2+2\delta\langle\nabla\phi,\nabla h\rangle+\delta^2\|\nabla h\|^2}{1+\delta^2} \\
&= \frac{1}{2}\frac{\|\nabla\phi\|^2(1+\delta^2)-\delta^2\|\nabla\phi\|^2(1+\delta^2)-2\delta\langle\triangle\phi,h\rangle(1+\delta^2)+\delta^2\langle-\triangle h,h\rangle(1+\delta^2)+O(\delta^3)}{1+\delta^2} \\
&= \frac{1}{2}J_1-\frac{1}{2}\delta^2 J_1-\delta\langle\triangle\phi,h\rangle+\frac{\delta^2}{2}\langle-\triangle h,h\rangle+O(\delta^3)
\end{aligned}
$$

Further, for the $p+1$ order term, we can compute,

$$
\begin{aligned}
&-\frac{1}{p+1}\int_{\mathbb{R}^2}\frac{|\phi+\delta h|^{p+1}}{\|\phi+\delta h\|^{p+1}}\,dx = \frac{-1}{p+1}\int_{\mathbb{R}^2}\frac{|\phi|^{p+1}+\delta(p+1)|\phi|^{p-1}\phi h+\frac{p(p+1)}{2}\delta^2|\phi|^{p-1}h^2+O(\delta^3)}{(1+\delta^2)^{\frac{p+1}{2}}}\,dx \\
&= \frac{-1}{p+1}\int_{\mathbb{R}^2}\frac{|\phi|^{p+1}(1+\frac{p+1}{2}\delta^2)-\frac{p+1}{2}\delta^2|\phi|^{p+1}+\delta(p+1)|\phi|^{p-1}\phi h+\frac{p(p+1)}{2}\delta^2|\phi|^{p-1}h^2+O(\delta^3)}{1+\frac{p+1}{2}\delta^2+O(\delta^3)}\,dx \\
&= -\frac{1}{p+1}K+\frac{\delta^2}{2}K-\delta\left\langle|\phi|^{p-1}\phi,h\right\rangle-\frac{p}{2}\delta^2\left\langle|\phi|^{p-1}h,h\right\rangle+O(\delta^3)
\end{aligned}
$$

In addition,

$$
\frac{1}{2}m^2\int_{\mathbb{R}^2}\frac{|\phi+\delta h|^2}{|x|^2\|\phi+\delta h\|^2}\,dx = \frac{1}{2}m^2\int_{\mathbb{R}^2}\frac{|\phi|^2+2\delta\phi h+\delta^2 h^2}{|x|^2(1+\delta^2)}\,dx = \frac{1}{2}J_2-\frac{1}{2}\delta^2 J_2+\frac{\delta^2 m^2}{2|x|^2}\langle h,h\rangle+O(\delta^3)
$$

Thus

$$
g(\delta) = g(0)-\frac{\delta^2}{2}\left\langle (J-K)h+\triangle h-\frac{m^2}{|x|^2}h+p|\phi|^{p-1}h,h\right\rangle+O(\delta^3)
$$

Recall that $\omega = K-J$. Since $g(\delta)\geq g(0)$ for all small enough δ, it follows that the operator L_+ defined by $L_+ = -\triangle_r+\omega+\frac{m^2}{|x|^2}-p|\phi|^{p-1}$ satisfies $\langle L_+h,h\rangle \geq 0$, which completes the proof. □

Theorem 3.2.5. *The vortex solution $u = e^{i(m\theta+\omega t)}\phi(r)$ constructed through the variational procedure above is positive. In addition, it is spectrally stable with respect to perturbations in the same class $\mathcal{X}_m$, when $1<p<3$.*

Proof. Linearize around the solution ϕ, consider $u = e^{i(m\theta+\omega t)}(\phi+\varphi+i\psi)$, where φ,ψ are radial functions. This results in the following equations for the perturbation.

$$
\begin{aligned}
-\varphi_t &= \triangle_r \psi - (\omega + \frac{m^2}{|x|^2})\psi + |\phi|^{p-1}\psi \\
\psi_t &= \triangle_r \varphi - (\omega + \frac{m^2}{|x|^2})\varphi + p|\phi|^{p-1}\varphi.
\end{aligned}
$$

This gives the system

$$
\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{3.2.9}
$$

where

$$
\begin{aligned}
L_+ &= -\Delta_r + (\omega + \frac{m^2}{|x|^2}) - p|\phi|^{p-1}, \\
L_- &= -\Delta_r + (\omega + \frac{m^2}{|x|^2}) - |\phi|^{p-1}.
\end{aligned}
$$

Introducing

$$
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix},
$$

we see that the eigenvalue problem (3.2.9) can be presented in the form

$$
JL \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \tag{3.2.10}
$$

We study this problem, using index counting theories. More precisely, by a corollary of the index counting theorem (see Theorem 1, [25] or better yet, Theorem 7.1.5, [37])

$$
n_{unstable}(JL) \leq n(L) - n(D), \tag{3.2.11}
$$

where in our case

$$D = \begin{pmatrix} \langle L_-^{-1}\phi', \phi' \rangle & 0 \\ 0 & \langle L_+^{-1}\phi, \phi \rangle \end{pmatrix}.$$

We proceed to establish that $n(L) = 1 = n(D)$, which would imply the spectral stability, by (3.2.11). Since

$$-\Delta_r \phi + (\omega + \frac{m^2}{|x|^2})\phi - |\phi|^{p-1}\phi = 0,$$

we have that $L_-\phi = 0$ and $L_+\phi = -\triangle_r\phi + (\omega + \frac{m^2}{|x|^2})\phi - p|\phi|^{p-1}\phi = -(p-1)|\phi|^{p-1}\phi$. It follows that

$$\langle L_+\phi, \phi \rangle = -(p-1)\int_0^\infty |\phi(r)|^{p+1} r dr < 0$$

We claim that L_- does not have negative spectrum and zero is a simple eigenvalue. Assume that ψ is such that $\|\psi\| = 1$and $L_-\psi = -\sigma^2\psi$. Then, $\langle \psi, \phi \rangle = 0$, and further

$$-\sigma^2 = \langle L_-\psi, \psi \rangle > \langle L_+\psi, \psi \rangle$$

But this is a contradiction, since we have proved that $L_+|_{\{\phi\}^\perp} \geq 0$. Thus, L_- doesn't have negative spectrum. Similar argument, with $\sigma = 0$, shows that L_- does not have other eigenfunctions at zero except ϕ. Note that by Sturm-Liouville theory for the singular Schrödinger operator $L_- = -\Delta_r + (\omega + \frac{m^2}{r^2}) - |\phi|^{p-1}$, acting on $L^2(rdr)$, we have that $\phi > 0$, as a ground state.

Next, observe that

$$-\triangle\phi - (\omega + \frac{m^2}{|x|^2})\phi + |\phi|^{p-1}\phi = 0$$

has solutions $\phi(r) = \omega^{\frac{1}{p-1}}\phi(\omega^{\frac{1}{2}} r)$. By differentiating the equation above with respect to ω, we have

$$L_+(-\frac{d}{d\omega}\phi) = \phi$$

which implies that

$$\langle L_+^{-1}\phi, \phi \rangle = -\frac{1}{2}\frac{d}{d\omega}\|\phi\|^2.$$

We obtain

$$\langle L_+^{-1}\phi,\phi\rangle = -\frac{1}{2}\frac{d}{d\omega}\int_0^\infty \omega^{\frac{2}{p-1}}|\phi_1(\omega^{\frac{1}{2}}r)|^2 r\,dr = -\frac{1}{2}\frac{3-p}{p-1}\omega^{\frac{4-2p}{p-1}}\int_0^\infty |\phi_1(r)|^2 r\,dr.$$

It follows that $\langle L_+^{-1}\phi,\phi\rangle < 0$, if $1 < p < 3$. where $n(L) = n(L_+) + n(L_-) = 1$ and $n(D) = n(\langle L_1^{-1}\phi,\phi\rangle) = 1$, thus it is spectrally stable, if $1 < p < 3$. □

3.3 The vortices in higher dimensions

The arguments proceed parallel to the two dimensional case, so we just indicate the main points. First, consider the NLS equation in space dimension $n = 2l$

$$iu_t + \triangle u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^{2n}, \;\; t > 0$$

and study the existence and stability of standing wave solutions of the form

$$e^{i\omega t}e^{i\sum_{k=1}^{l} m_k\theta_k}\phi_\omega(r_1, r_2, \cdots, r_l),$$

where (r_k, θ_k) are polar coordinates in $\mathbb{R}^2$, $m_k \in \mathbb{N}\cup\{0\}$, $k = 1, 2, \cdots, l$. Then ϕ_ω satisfies (3.1.4).

The case of odd-dimensional space $\mathbf{R}^{2l+1}$ proceeds the same way by using solutions of the form

$$e^{i\omega t}e^{i\sum_{k=1}^{l} m_k\theta_k}\phi_\omega(r_1, r_2, \cdots, r_l, z),$$

where (r_k, θ_k) are polar coordinates in $\mathbf{R}^2$, $m_k \in \mathbb{N}\cup\{0\}$, $k = 1, 2, \cdots, n-1$, and (r_l, θ_l, z) are the cylindrical coordinates in $\mathbb{R}^3$. The equation for ϕ_ω is then (3.1.5).

3.3.1 Variational construction of the waves

The minimization problem remains the same for both even and odd cases: for $\lambda > 0$ minimize the functional

$$\mathcal{H}(u)=\frac{1}{2}\int_{\mathbb{R}^{2l}}|\nabla u|^2\,dx-\frac{1}{p+1}\int_{\mathbb{R}^{2l}}|u|^{p+1}\,dx$$

subject to the constraint

$$\int_{\mathbb{R}^{2l}}|u|^2\,dx=\lambda, u\in\mathcal{X}_{\vec{m}}.$$

Restricting this problem to the subspaces $\mathcal{X}_{\vec{m}}$, allows us to find a minimizer in $\mathcal{X}_{\vec{m}}$. Let $I_\lambda = \inf_{u\in\mathcal{X}_{\vec{m}},\mathcal{K}(u)=\lambda}\mathcal{H}(u)$.

Proposition 3.3.1. *Let $\vec{m}\in\mathcal{Z}^{[\frac{n}{2}]}$ and $1<p<1+\frac{4}{n}$, and $\lambda>0$, then $-\infty<I_\lambda<0$. In addition, there exists a constrained minimizer in $\mathcal{X}_{\vec{m}}$.*

Proof. Assume that $n\geq 3$, since we have already considered the case $n=2$.

By (3.2.1) and taking into account that $\|u\|_{L^2}^2=\lambda$, we have

$$\mathcal{H}[u]\geq\frac{1}{2}\int_{\mathbf{R}^n}|\nabla u|^2dx-C_{\lambda,p,n}\|\nabla u\|^{\frac{n(p-1)}{2}},$$

we conclude that $\mathcal{H}[u]\geq g(\|\nabla u\|)$, with $g(R)=\frac{R^2}{2}-C_{\lambda,p,n}R^{\frac{n(p-1)}{2}}$, which is bounded from below, if $\frac{n(p-1)}{2}<2$ or equivalently, if $p<1+\frac{4}{n}$. Thus, $I_\lambda>-\infty$. A dilation argument, with $u_\mu=\mu^{\frac{n}{2}}u(\mu\cdot)$: $\|u_\mu\|_{L^2}^2=\|u\|_{L^2}^2=\lambda$, we have

$$H[u_\mu]=\mu^2[\frac{1}{2}\|\nabla u\|^2-\frac{\mu^{\frac{n(p-1)}{2}-2}}{p+1}\|u\|_{L^{p+1}}^{p+1}]$$

which shows that for $\mu<<1$, we have $H[u_\mu]<0$, whence $I_\lambda<0$.

Since $-\infty<I_\lambda<0$, we can find a minimizing sequence $\{u_k\}_{k=1}^\infty\subset H^1(\mathbf{R}^n)$, such that $\|u_k\|_{L^2}^2=\lambda$ and $\mathcal{H}(u_k)\to I_\lambda$ as $k\to\infty$. Since $I_\lambda<0$, thus $\mathcal{H}(u_k)<\infty$ for k large enough. Further, by Gagliardo-Nirenberg-Sobolev, for large k, $g(\|Du_k\|_{L^2})\leq\mathcal{H}(u_k)<0$. Then it follows that $\|Du_k\|_{L^2(\mathbf{R}^n)}\leq R_0$ for k sufficently large. WLOG, we can assume that $\|Du_k\|_{L^2(\mathbf{R}^n)}\leq R_0$ for any $k>0$. Thus $\{u_k\}_{k=1}^\infty$ is bounded in $H^1(\mathbf{R}^n)$. By concentration Compactness. We have either "convergence of translates", "vanishing" or "splitting".

First let's rule out "vanishing".

$\mathcal{H}(u_k) \le \frac{I_\lambda}{2} < 0$ for k sufficently large. Hence

$$\frac{1}{2}\int_{\mathbf{R}^n} |\nabla u|^2 \, dx + \int_{\mathbf{R}^n} \sum_{k=1}^{[\frac{n}{2}]} \frac{m_k^2}{x_{2k-1}^2 + x_{2k}^2} |u|^2 \, dx - \frac{1}{p+1}\int_{\mathbf{R}^n} |u|^{p+1} \, dx \le \frac{I_\lambda}{2} < 0$$

for k sufficently large. Thus we obtain $\frac{1}{p+1}\int_{\mathbf{R}^n} |u_k|^{p+1} \, dx \ge -\frac{I_\lambda}{2} > 0$ for k sufficently large. If "vanishing" occurred, then there exists subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $u_{kj} \to 0$ in $L^q(\mathbf{R}^n)$, for any $2 < q < 2 + \frac{4}{n-2}$, $n \ge 3$. Since $1 < p < 1 + \frac{4}{n-2}$. Thus here $2 < p+1 < 2 + \frac{4}{n-2}$,and satisfying $u_{k_j} \to 0$ in $L^{p+1}(\mathbf{R}^n)$. Contradiction.

Then we rule out splitting.

If splitting occurred, then there exists subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and bounded sequences $\{v_j\}_{j=1}^{\infty}$ and $\{w_j\}_{j=1}^{\infty}$ in $H^1(\mathbf{R}^n)$ with

$$\|v_j\|_{L^2(\mathbf{R}^n)}^2 \to \gamma < \lambda,$$

$$\|w_j\|_{L^2(\mathbf{R}^n)}^2 \to \lambda - \gamma,$$

$$dist(spt(v_j), spt(w_j)) \to \infty$$

and $\int_{\mathbf{R}^n} \left(|u_{k_j}|^q - |v_j|^q - |w_j|^q\right) \, dx \to 0$ as $j \to \infty$ for any $2 < q < \frac{2n}{n-2}$,

and $\liminf_{j\to\infty} \int_{\mathbf{R}^n} \left(|Du_{k_j}|^2 - |Dv_j|^2 - |Dw_j|^2\right) \, dx \ge 0$ Then $\forall \varepsilon > 0$, for j sufficently large,

$I_\lambda + (n+4)\varepsilon \ge \mathcal{H}(v_j) + \mathcal{H}(w_j) + \varepsilon$

Now $\exists$ sequence $\{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty}$ in $\mathbb{R}^+$ such that $\|a_j v_j\|_{L^2}^2 = \gamma$, $\|b_j w_j\|_{L^2}^2 = \lambda - \gamma$ for any j and further $a_j, b_j \to 1$, thus

$$\mathcal{H}(a_j v_j) \ge I_\gamma$$

and

$$\mathcal{H}(b_j w_j) \ge I_{\lambda-\gamma}$$

for any j. Further, for j large enough, we have

$$\mathcal{H}(v_j) \geq \mathcal{H}(a_j v_j) - \varepsilon/2,$$

$$\mathcal{H}(w_j) \geq \mathcal{H}(b_j w_j) - \varepsilon/2,$$

Thus $\forall\ j$ sufficently large $I_\lambda + 5\varepsilon \geq \mathcal{H}(a_j v_j) + \mathcal{H}(b_j w_j) \geq I_\gamma + I_{\lambda-\gamma}$, taking $\varepsilon \to 0^+$, then we obtain

$$I_\lambda \geq I_\gamma + I_{\lambda-\gamma}.$$

for every $\gamma \in (0, \lambda)$, while Lemma 3.2.2 (just replacing $\mathbf{R}^2$ by $\mathbf{R}^n$) implies the opposite inequality. Hence, splitting is ruled out as well. It follows that tightness occurs. In other words, there is a subsequence $\{u_{k_j}\}$ and a sequence of $\{y_j\} \subset \mathbf{R}^n$, so that $u_{k_j}(\cdot - y_j) \to u_0$ in $L^2(\mathbf{R}^n)$. Again, we show that $\{y_j\}$ must be a bounded sequence.

Lemma 3.3.2. *In the even dimensional case, $n = 2l$, the sequence $\{y_j\} \subset \mathbf{R}^n$ is bounded. In the odd dimensional case, $n = 2l+1$, $\tilde{y}_j := (y_j^1, \ldots, y_j^{2l})$ is a bounded in $\mathbf{R}^{2l} = \mathbf{R}^{n-1}$.*

Proof. We start with the case of even dimensions. The proof generally proceeds parallel to Lemma 3.2.3, with a few important technical differences that we outline below.

Assume that unboundedness of $\{y_j\}$. After taking a subsequence (denoted the same), we may and do assume $\lim_j |y_j| = \infty$. Note that $y_j = ((y_j^1, y_j^2), \ldots, (y_j^{2l-1}, y_j^{2l}))$. We consider the variables in pairs $(y_{2k-1}, y_{2k}), k = 1, \ldots l$. Clearly, we may have a situation where some pairs (y_j^{2k-1}, y_j^{2k}), as elements of $\mathbf{R}^2$, are unbounded, while some others are bounded. Up to a permutation of the variables, it is clearly enough to consider the case where $\lim_j |(y_j^1, y_j^2)| = \infty, \ldots, \lim_j |(y_j^{2k_0-1}, y_j^{2k_0})| = \infty$, while the rest of the coordinates $\{|(y_j^{2k-1}, y_j^{2k})|\}_j, k > k_0 \geq 1$ are bounded. After passing to another subsequence, if necessary, we may assume that the bounded coordinates are actually convergent, say to $(y^{2k_0+1}, \ldots, y^{2l})$. Defining $\tilde{y}_j = (y_j^1, \ldots, y_j^{2k_0}, 0, \ldots, 0)$, we see that

$$\lim_j \|u_{k_j}(x - \tilde{y}_j) - u_0(x_1, \ldots, x_{2k_0}, x_{2k_0+1} + y_{2k_0+1}, \ldots, x_{2l} + y_{2l})\|_{L^2(\mathbf{R}^n)} = 0.$$

Recalling that $u_{k_j} \in \mathcal{X}_{\vec{m}}$, it follows that without loss of generality, we may assume that $u_{k_j}(x-y_j)$ has the following representation

$$\phi_{k_j}(|(x_1-y_j^1, x_2-y_j^2)|, \dots, |(x_{2k_0-1}-y_j^{2k_0-1}, x_{2k_0}-y_j^{2k_0})|, |(x_{2k_0+1}, x_{2k_0+2})|, \dots) \times$$
$$\times \quad e^{i(\sum_{k=1}^{k_0} m_k \theta_j^k(x) + \sum_{k=k_0+1}^{n} m_k \theta^k(x))}$$

Thus, after some relabeling, we may without loss of generality assume that again $\lim_j \|u_{k_j}(x-y_j) - u_0\|_{L^2(\mathbf{R}^n)} = 0$, where $y_j = (y_1^j, \dots, y_{2k_0}^j, 0, \dots, 0)$ and $\lim_j |(y_j^1, y_j^2)| = \infty, \dots, \lim_j |(y_j^{2k_0-1}, y_j^{2k_0})| = \infty$. Passing to a further subsequence, we may and do assume that $\frac{(y_j^{2k-1}, y_j^{2k})}{|(y_j^{2k-1}, y_j^{2k})|} \to (\cos(\theta_k), \sin(\theta_k)) \in \mathbf{S}^1$.

Let $\varepsilon > 0$, choose N and j_0, so that for $j \geq j_0$,

$$\|u_0\|_{L^2(|x|>N)} < \varepsilon, \|u_{k_j}(\cdot - y_j)\|_{L^2(|x|>N)} < \varepsilon$$

Now, for the polar angles $\theta_j^k(x)$, corresponding[3] to the pair $(x^{2k-1} - y_j^{2k-1}, x^{2k} - y_j^{2k})$, we have again

$$e^{i\theta_j^k(x)} = \left(\frac{x^{2k-1} - y_j^{2k-1}}{|(x^{2k-1} - y_j^{2k-1}, x^{2k} - y_j^{2k})|} + i \frac{x^{2k} - y_j^{2k}}{|(x^{2k-1} - y_j^{2k-1}, x^{2k} - y_j^{2k})|} \right) \to -e^{i\theta_k},$$

whence $\lim_j e^{-im_k \theta_j^k(x)} = (-1)^m e^{-im_k \theta_k} =: e^{i\alpha_k}$. As in Lemma 3.2.3, it follows that

$$\phi_{k_j}(|(x_1-y_j^1, x_2-y_j^2)|, \dots, |(x_{2k_0-1}-y_j^{2k_0-1}, x_{2k_0}-y_j^{2k_0})|, |(x_{2k_0+1}, x_{2k_0+2})|, \dots)$$

converges in $L^2(|x|<N)$ to

$$e^{i(\sum_{k=1}^{k_0} \alpha_k - \sum_{k=k_0+1}^{n} m_k \theta^k(x))} u_0.$$

The choice of N implies that this convergence is over $L^2(\mathbf{R}^n)$ and hence $e^{i(\sum_{k=1}^{k_0} \alpha_k - \sum_{k=k_0+1}^{n} m_k \theta^k(x))} u_0$ is real-valued, as a limit of real-valued functions.

[3] Here, for $k \geq k_0 + 1$, we simply have that θ^k is the polar angle for (x_{2k-1}, x_{2k})

By a similar argument to Lemma 3.2.3, we may assume without loss of generality that $u_{k_j}(x-y_j) \to u_0$ and u_0 is real-valued. Note that $\lim_j \|u_{k_j}(\cdot)-u_0(x-y_j)\|_{L^2}=0$.

Picking representative, ϕ_{k_j} (which is not necessarily real-valued anymore!), we conclude that

$$\lim_j \|\phi_{k_j}(|(x_1,x_2)|,\ldots,|(x_{2l-1},x_{2l})|)e^{i\sum_{k=1}^l m_k\theta_k(x_{2k-1},x_{2k}))}-u_0(x-y_j)\|_{L^2(\mathbf{R}^n)}=0. \tag{3.3.1}$$

Let $\phi_{k_j}(|(x_1,x_2)|,\ldots,|(x_{2l-1},x_{2l})|)=p_j(r_1,\ldots,r_l)+iq_j(r_1,\ldots,r_l)$ and taking imaginary parts in (3.3.1), we obtain

$$\lim_j \|p_j\sin(\sum_{k=1}^l m_k\theta_k(x_{2k-1},x_{2k}))+q_j\cos(\sum_{k=1}^l m_k\theta_k(x_{2k-1},x_{2k}))\|_{L^2(\mathbf{R}^n)}=0. \tag{3.3.2}$$

But,

$$\begin{aligned} &\|p_j\sin(\sum_{k=1}^l m_k\theta_k(x_{2k-1},x_{2k}))+q_j\cos(\sum_{k=1}^l m_k\theta_k(x_{2k-1},x_{2k}))\|^2_{L^2(\mathbf{R}^n)}= \\ = &\int [p_j^2(r_1,\ldots,r_l)\sin^2(\sum_{k=1}^l m_k\theta_k)+q_j^2(r_1,\ldots,r_l)\cos^2(\sum_{k=1}^l m_k\theta_k)]d\theta\; r_1\ldots r_l dr+ \\ + &\; 2\int p_j(r_1,\ldots,r_l)q_j(r_1,\ldots,r_l)\sin(\sum_{k=1}^l m_k\theta_k)\cos(\sum_{k=1}^l m_k\theta_k)d\theta\; r_1\ldots r_l dr. \end{aligned}$$

But $\int_{[0,2\pi]^l}\sin(\sum_{k=1}^l m_k\theta_k)\cos(\sum_{k=1}^l m_k\theta_k)d\theta_1\ldots d\theta_l=0$, while

$$\int_{[0,2\pi]^l}\sin^2(\sum_{k=1}^l m_k\theta_k)d\theta_1\ldots d\theta_l=\int_{[0,2\pi]^l}\cos^2(\sum_{k=1}^l m_k\theta_k)d\theta_1\ldots d\theta_l=\frac{(2\pi)^l}{2},$$

whence

$$\|p_j\sin(\sum_{k=1}^l m_k\theta_k)+q_j\cos(\sum_{k=1}^l m_k\theta_k)\|^2_{L^2(\mathbf{R}^n)}=\frac{1}{2}\|\phi_{k_j}\|^2_{L^2(\mathbf{R}^n)}=\frac{\lambda}{2}>0.$$

This is a contradiction with (3.3.2), whence the proof of Lemma 3.3.2 in the even dimensional case.

In the odd dimensional case, we proceed similarly. Note the last component may be unbounded. Assuming the unboundedness of $\{\tilde{y}_j\}:\tilde{y}_j=(y_j^1,\ldots,y_j^{n-1},0)$, we take a subsequence (denoted the same) so that $\lim_j |\tilde{y}^j| = \infty$. We have

$$\|u_{k_j}(x - \tilde{y}_j) - u_0(x_1, x_2, \ldots, x_{n-1}, x_n - y_j^n)\|_{L^2} \to 0.$$

Take our initial sequence to be $\tilde{u}_{k_j}(x) := |u_{k_j}(x_1, \ldots, x_{n-1}, x_n + y_j^n)$. Clearly, it still belongs to $\mathcal{X}_{\vec{m}}$ if u_{k_j} does and for which $\|\tilde{u}_{k_j}\|_{L^2}^2 = \|u_{k_j}\|_{L^2}^2 = \lambda$ and $\mathcal{H}[\tilde{u}_{k_j}] = \mathcal{H}[u_{k_j}]$. Thus, we have reduced matters to

$$\|\tilde{u}_{k_j}(\cdot - \tilde{y}_j) - u_0\|_{L^2} \to 0.$$

We rule out the potential unboundedness of the $\tilde{y}_j$ as in the argument for even dimensions, since $\tilde{y}_j$ has even number of non-zero component, for which we apply the polar coordinates etc. Thus, it follows that $\sup_j |\tilde{y}_j| < \infty$ and the proof of Lemma 3.3.2 is complete.

□

We are now ready to finish the proof of Proposition 3.3.1. Since $\{y_j\}$ is bounded (or just $\{\tilde{y}_j\}$ in the odd dimensional case), we may take a convergent subsequence (denoted the same), $y_j \to y_0$ (or $\tilde{y}_j \to \tilde{y}_0$ in the odd dimensions). We have $\lim_j \|u_{k_j} - u_0(\cdot - y_0)\|_{L^2(\mathbf{R}^n)} = 0$ or $\lim_j \|u_{k_j} - u_0(\cdot - (\tilde{y}_0, y_j^n))\|_{L^2(\mathbf{R}^n)} = 0$ in odd dimensions. From this, it follows that $u_0(\cdot - y_0) \in \mathcal{X}_{\vec{m}}$ in the even dimensional case and $u_0(\cdot - (\tilde{y}_0, 0)) \in \mathcal{X}_{\vec{m}}$ in the odd dimensional case. Both of these serve as constrained minimizers of (3.1.8) and Proposition 3.3.1 is established.

□

Next, we need a version of Proposition 3.2.4. We just state it as the proof proceeds in an identical way as in the case $n = 2$.

Proposition 3.3.3. *A constrained minimizer of* (3.1.7)*, ϕ satisfies the Euler-Lagrange equation*

$$-\Delta\phi + \omega_\lambda \phi - |\phi|^{p-1}\phi = 0, \quad \omega_\lambda = \frac{\|\phi\|_{L^{p+1}(\mathbf{R}^n)}^{p+1} - \|\nabla\phi\|_{L^2(\mathbf{R}^n)}^2}{\lambda}. \tag{3.3.3}$$

Alternatively, $\phi_{\vec{m}}$ satisfies either (3.1.4) *in the even case or* (3.1.5) *in the odd case, with $\omega = \omega_\lambda$.*

Moreover, the following scaling identities hold

$$\phi^{\lambda} = \lambda^{\frac{2}{4-n(p-1)}} \phi^{1}(\lambda^{\frac{p-1}{4-n(p-1)}} x),\ I_{\lambda} = \lambda^{\frac{n+2-p(n-2)}{4-n(p-1)}} I_1,\ \omega_{\lambda} = \lambda^{\frac{2(p-1)}{4-n(p-1)}} \frac{2n(p-1)-4(p+1)}{4-n(p-1)} I_1 \quad (3.3.4)$$

The linearized operator

$$L_{+} := -\Delta + \omega_{\lambda} - p|\phi|^{p-1}$$

is non-negative on the co-dimension one subspace $\{\phi e^{i\sum_{k=1}^{[\frac{n}{2}]} m_k \theta_k}\}^{\perp}$ *of the space* $\mathcal{X}_{\vec{m}}$*. That is,*

$$\langle L_{+}h, h\rangle \geq 0, h \in \mathcal{X}_{\vec{m}} \cap \{\phi e^{i\sum_{k=1}^{[\frac{n}{2}]} m_k \theta_k}\}^{\perp} \cap domain(L_{+}),$$

Equivalently, in the even dimensional case, the operator

$$L_{+}^{rad} = -\sum_{k=1}^{\frac{n}{2}} \Delta_{r_k} + \sum_{k=1}^{\frac{n}{2}} \frac{m_k^2}{r_k^2} + \omega_{\lambda} - p|\phi|^{p-1}$$

acting on the subspace $H_r^2 \cap \{\phi\}^{\perp}$ *is non-negative, while in the odd dimensional case*

$$L_{+}^{rad} = -\sum_{k=1}^{\frac{n-1}{2}} \Delta_{r_k} - \partial_{x_n}^2 + \sum_{k=1}^{\frac{n-1}{2}} \frac{m_k^2}{r_k^2} + \omega_{\lambda} - p|\phi|^{p-1}$$

is non-negative on the subspace $H_r^2 \cap \{\phi\}^{\perp}$

Proof. $\phi^{\lambda} = \lambda^{b}\phi(\lambda^{a}x)$ where $\phi = \phi^{1}$ with $\int_{\mathbb{R}^n} |\phi(x)|^2\, dx = 1$. Set $J_{\lambda}(\phi^{\lambda}) = J_1(\phi^{\lambda}) + J_2(\phi^{\lambda})$, where

$$J_1(\phi^{\lambda}) := \int_{\mathbb{R}^n} |\nabla \phi^{\lambda}|^2\, dx = \int_{\mathbb{R}^n} |\lambda^{a+b} \nabla \phi(\lambda^{a}x)^2|\, dx = \lambda^{2b+2a-na} \int_{\mathbb{R}^n} |\nabla \phi(x)|^2\, dx = \lambda^{2b+2a-na} J_1(\phi).$$

$$J_2(\phi^{\lambda}) = \int_{\mathbb{R}^n} \sum_{k=1}^{[\frac{n}{2}]} \frac{m_k^2}{x_{2k-1}^2 + x_{2k}^2} |\phi^{\lambda}|^2\, dx = \lambda^{2b+2a-na} J_2(\phi).$$

$$K_{\lambda}(\phi^{\lambda}) = \int_{\mathbb{R}^n} |\phi^{\lambda}(x)|^{p+1}\, dx = \lambda^{b(p+1)-na} K(\phi).$$

So we have

$$\lambda^{2b+2a-na} = \lambda^{b(p+1)-na} \Rightarrow\ b+2a = bp.$$

Further

$$\lambda = \int_{\mathbb{R}^n} \lambda^{2b} |\phi(\lambda^a x)|^2\, dx = \lambda^{2b-na} \int_{\mathbb{R}^n} |\phi(x)|^2\, dx = \lambda^{2b-na} \Rightarrow\ 2b-na = 1.$$

Clearly we obtain $a = \frac{p-1}{4-n(p-1)}$ and $b = \frac{2}{4-n(p-1)}$. Hence $\phi^\lambda = \lambda^{\frac{2}{4-n(p-1)}} \phi^1(\lambda^{\frac{p-1}{4-n(p-1)}} x)$, $\mathcal{H}(\phi) = \frac{1}{2}J(\phi) - \frac{1}{p+1}K(\phi)$ and $\mathcal{H}_\lambda = \lambda^{\frac{n+2-p(n-2)}{4-n(p-1)}} \mathcal{H}_1$.

Thus it suffices to prove the results for the case $\lambda = 1$. So fix $\lambda = 1$. Let $\phi = \phi^1$ be a minimizer. For any $\delta > 0$, consider $u_\delta = \phi + \delta h$. We have that

$$\mathcal{H}(\frac{u_\delta}{\|u_\delta\|}) \geq \mathcal{H}_1.$$

Note that

$$\|u_\delta\| = \sqrt{\|\phi\|^2 + 2\delta \langle \phi, h\rangle + O(\delta^2)} = 1 + \delta \langle \phi, h\rangle + O(\delta^2).$$

In the odd dimensional case, (with the obvious modifications in the even case)

$$\begin{aligned}
&\frac{1}{2}\left\| \frac{\nabla u_\delta}{\|u_\delta\|} \right\|^2 = \frac{1}{2}\frac{\|\nabla\phi + \delta \nabla h\|^2}{\|u_\delta\|^2} = \frac{1}{2}\frac{\|\nabla\phi\|^2 + 2\delta \langle \nabla\phi, \nabla h\rangle + O(\delta^2)}{1 + 2\delta\langle \phi, h\rangle + O(\delta^2)} = \frac{1}{2}\frac{\|\nabla\phi\|^2 - 2\delta \langle \triangle\phi, h\rangle + O(\delta^2)}{1 + 2\delta\langle \phi, h\rangle + O(\delta^2)}\\
&= \frac{1}{2}\frac{\|\nabla\phi\|^2(1+2\delta\langle\phi,h\rangle) - 2\delta\langle\phi,h\rangle\,\|\nabla\phi\|^2(1+2\delta\langle\phi,h\rangle) - 2\delta\langle\triangle\phi,h\rangle + O(\delta^2)}{1+2\delta\langle\phi,h\rangle + O(\delta^2)}\\
&= \frac{1}{2}J_1 - \delta\,(< \triangle\phi, h > + J_1\langle\phi,h\rangle) + O(\delta^2)\\
&= \frac{1}{2}J_1 - \delta\left(\left\langle (\sum_{k=1}^{\frac{n-1}{2}} \triangle_{r_k} + \triangle_z)\phi, h\right\rangle + J_1\langle\phi,h\rangle\right) + O(\delta^2)
\end{aligned}$$

In addition,

$$\frac{1}{2}\int_{\mathbb{R}^n} \sum_{k=1}^{\frac{n-1}{2}} \frac{m_k^2 |u_\delta|^2}{(x_{2k-1}^2 + x_{2k}^2)\|u_\delta\|^2} = \frac{1}{2}J_2 + \delta\left(\sum_{k=1}^{\frac{n-1}{2}} \frac{m_k^2}{(x_{2k-1}^2 + x_{2k}^2)}\langle\phi,h\rangle - J_2\langle\phi,h\rangle\right) + O(\delta^2)$$

We can also compute the last term with $p+1$ power,

$$
\begin{aligned}
&-\frac{1}{p+1}\int_{\mathbb{R}^n}\frac{|u_\delta|^{p+1}}{\|u_\delta\|^{p+1}}\,dx = \frac{-1}{p+1}\int_{\mathbb{R}^n}\frac{|\phi+\delta h|^{p+1}}{(1+\delta\langle\phi,h\rangle+O(\delta^2))^{p+1}}\,dx \\
&= \frac{-1}{p+1}\int_{\mathbb{R}^n}\frac{|\phi|^{p+1}+\delta(p+1)|\phi|^{p-1}\phi h+O(\delta^2)}{1+(p+1)\delta\langle\phi,h\rangle+O(\delta^2)}\,dx \\
&= \frac{-1}{p+1}\int_{\mathbb{R}^n}\frac{|\phi|^{p+1}(1+(p+1)\delta\langle\phi,h\rangle)-\delta(p+1)|\phi|^{p+1}\langle\phi,h\rangle+\delta(p+1)|\phi|^{p-1}\phi h+O(\delta^2)}{1+(p+1)\delta\langle\phi,h\rangle+O(\delta^2)}\,dx \\
&= \frac{-1}{p+1}\int_{\mathbb{R}^n}|\phi|^{p+1}\,dx+\delta\int_{\mathbb{R}^n}|\phi|^{p+1}\langle\phi,h\rangle-\delta\left\langle|\phi|^{p-1}\phi,h\right\rangle+O(\delta^2) \\
&= -\frac{1}{p+1}K+\delta\left(K\langle\phi,h\rangle-\left\langle|\phi|^{p-1}\phi,h\right\rangle\right)+O(\delta^2)
\end{aligned}
$$

Since $\frac{1}{2}J-\frac{1}{p+1}K=\mathcal{H}_1$ and $\mathcal{H}(\frac{u_\delta}{\|u_\delta\|})\geq\mathcal{H}_1$, We conclude

$$
\delta\left\langle -(\sum_{k=1}^{\frac{n-1}{2}}\triangle_{r_k}+\triangle_z)\phi+\sum_{k=1}^{\frac{n-1}{2}}\frac{m_k^2}{(x_{2k-1}^2+x_{2k}^2)}\phi-|\phi|^{p-1}\phi+(K-J)\phi,h\right\rangle+O(\delta^2)\geq 0
$$

Since it is true for all $\delta\in\mathbb{R}$ and for all test functions h, we conclude that ϕ satisfies

$$
-(\sum_{k=1}^{\frac{n-1}{2}}\triangle_{r_k}+\triangle_z)\phi+\sum_{k=1}^{\frac{n-1}{2}}\frac{m_k^2}{(x_{2k-1}^2+x_{2k}^2)}\phi-|\phi|^{p-1}\phi+(K-J)\phi=0
$$

which is the Euler-Lagrange equation (3.1.5), with a scalar $\omega=K-J$. Finally, there is the Pokhozaev's identity, which we derive in the following way. Set

$$
z_\mu(x)=\mu^{\frac{n}{2}}\phi(\mu x).
$$

Since $\int_{\mathbb{R}^n}z_\mu^2(x)\,dx=\int_{\mathbb{R}^n}\phi^2\,dx=1$, z_μ satisfies

$$
\mathcal{H}(z_\mu)=\frac{\mu^2}{2}J-\frac{\mu^{\frac{n}{2}(p-1)}}{p+1}K.
$$

Since the scalar valued function $\mu\to\mathcal{H}_\mu(z)$ achieves its minimum at $\mu=1$, we must have

$\frac{d\mathcal{H}(z_\mu)}{d\mu}|_{\mu=1} = 0$. This is the Pokhozaev's identity

$$J - \frac{n(p-1)}{2(p+1)}K = 0.$$

Hence we obtain the formulas

$$J = \frac{2p-2+2(n-1)(p-1)}{4-n(p-1)}\mathcal{H}_1 \tag{3.3.5}$$

$$K = \frac{4(p+1)}{4-n(p-1)}\mathcal{H}_1 \tag{3.3.6}$$

$$\omega = \frac{2n(p-1)-4(p+1)}{4-n(p-1)}\mathcal{H}_1. \tag{3.3.7}$$

We then establish the non-coercivity of L_+ on the codimension subspace $\{\phi\}^\perp$. Note that for every test function h, the function

$$g(\delta) = \mathcal{H}(\frac{\phi+\delta h}{\|\phi+\delta h\|})$$

has a minimum at $\delta = 0$, which means that we must have $g'(0) = 0$, and $g''(0) \geq 0$.

We take $h : \langle h, \phi \rangle = 0$, and $\|h\| = 1$. Note that under this restriction

$$\|\phi+\delta h\|^2 = \|\phi\|^2 + 2\delta \langle \phi, h \rangle + \delta^2 \|h\|^2 = 1 + \delta^2,$$

$$\|\phi+\delta h\| = (1+\delta^2)^{1/2} = 1 + \frac{\delta^2}{2} + O(\delta^3).$$

Consider $g(\delta)$:

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\nabla(\phi+\delta h)}{\|\phi+\delta h\|}\right\|^2 = \frac{1}{2}\frac{\|\nabla\phi\|^2 + 2\delta\langle\nabla\phi,\nabla h\rangle + \delta^2\|\nabla h\|^2}{1+\delta^2} \\
&= \frac{1}{2}\frac{\|\nabla\phi\|^2(1+\delta^2) - \delta^2\|\nabla\phi\|^2(1+\delta^2) - 2\delta\langle\triangle\phi,h\rangle(1+\delta^2) + \delta^2\langle-\triangle h,h\rangle(1+\delta^2) + O(\delta^3)}{1+\delta^2} \\
&= \frac{1}{2}J_1 - \frac{1}{2}\delta^2 J_1 - \delta\langle\triangle\phi,h\rangle + \frac{\delta^2}{2}\langle-\triangle h,h\rangle + O(\delta^3)
\end{aligned}$$

Thus it comes to

$$\frac{1}{2}\left\|\frac{\nabla(\phi+\delta h)}{\|\phi+\delta h\|}\right\|^2=\frac{1}{2}J_1-\frac{1}{2}\delta^2 J_1-\delta\left\langle(\sum_{k=1}^{\frac{n-1}{2}}\triangle_{r_k}+\triangle_z)\phi,h\right\rangle+\frac{\delta^2}{2}\left\langle-(\sum_{k=1}^{\frac{n-1}{2}}\triangle_{r_k}+\triangle_z)h,h\right\rangle+O(\delta^3)$$

One also has

$$\frac{1}{2}\int_{\mathbb{R}^n}\sum_{k=1}^{\frac{n-1}{2}}\frac{m_k^2|\phi+\delta h|^2}{(x_{2k-1}^2+x_{2k}^2)\|\phi+\delta h\|^2}\,dx=\frac{1}{2}J_2-\frac{1}{2}\delta^2 J_2+(\sum_{k=1}^{\frac{n-1}{2}}\frac{m_k^2}{x_{2k-1}^2+x_{2k}^2})\langle h,h\rangle+O(\delta^3).$$

Further we obtain

$$\begin{aligned}&-\frac{1}{p+1}\int_{\mathbb{R}^n}\frac{|\phi+\delta h|^{p+1}}{\|\phi+\delta h\|^{p+1}}\,dx\\&=\frac{-1}{p+1}\int_{\mathbb{R}^n}\frac{|\phi|^{p+1}(1+\frac{p+1}{2}\delta^2)-\frac{p+1}{2}\delta^2|\phi|^{p+1}+(p+1)\delta|\phi|^{p-1}\phi+\frac{p(p+1)}{2}\delta^2|\phi|^{p-1}h^2+O(\delta^3)}{1+\frac{p+1}{2}\delta^2+O(\delta^3)}\,dx\\&=\frac{-1}{p+1}K+\frac{\delta^2}{2}K-\delta\left\langle|\phi|^{p-1}\phi,h\right\rangle-\frac{p}{2}\delta^2\left\langle|\phi|^{p-1}h,h\right\rangle+O(\delta^3).\end{aligned}$$

Hence it follows that

$$g(\delta)=g(0)-\frac{\delta^2}{2}\left\langle(J-K)h+(\sum_{k=1}^{\frac{n-1}{2}}\triangle_{r_k}+\triangle_z)h-\sum_{k=1}^{\frac{n-1}{2}}\frac{m_k^2}{x_{2k-1}^2+x_{2k}^2}h+p|\phi|^{p-1}h,h\right\rangle+O(\delta^3).$$

Recall that $\omega=K-J$. Since $g(\delta)\geq g(0)$ for all small enough δ, it follows that the operator L_+ defined by $L_+=-(\sum_{k=1}^{\frac{n-1}{2}}\triangle_{r_k}+\triangle_z)+w+\sum_{k=1}^{\frac{n-1}{2}}\frac{m_k^2}{x_{2k-1}^2+x_{2k}^2}-p|\phi|^{p-1}$ satisfies $\langle L_+h,h\rangle\geq 0$. □

The next theorem gives the spectral stability of the vortices constructed in this section, with respect to perturbations in $\mathcal{X}_{\vec{m}}$.

3.3.2 Stability analysis of the waves

Theorem 3.3.4. *In the even dimensional cases $n=2l$, the vortex soltion $\phi_{\vec{m}}(r_1,\ldots,r_l)e^{i\sum_{k=1}^{l}m_k\theta_k}$ is spectrally stable with respect to perturbations in $\mathcal{X}_{\vec{m}}$, whenever $1<p<1+\frac{4}{n}$.*

In the odd dimensional case, $n = 2l+1$, the vortex soltion $\phi_{\vec{m}}(r_1,\ldots,r_l,x_n)e^{i\sum_{k=1}^{l} m_k\theta_k}$ is spectrally stable with respect to perturbations in $\mathcal{X}_{\vec{m}}$, whenever $1 < p < 1+\frac{4}{n}$.

Proof. The linearized problem that we obtain is exactly in the form (3.2.9). Passing to the radial subspaces L^2_r, the system is in the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

where now in the even case

$$\begin{aligned} L_+ &= -\sum_{k=1}^{\frac{n}{2}} \Delta_{r_k} + \sum_{k=1}^{\frac{n}{2}} \frac{m_k^2}{r_k^2} + \omega_\lambda - p|\phi|^{p-1} \\ L_- &= -\sum_{k=1}^{\frac{n}{2}} \Delta_{r_k} + \sum_{k=1}^{\frac{n}{2}} \frac{m_k^2}{r_k^2} + \omega_\lambda - |\phi|^{p-1}, \end{aligned}$$

with the obvious modifications in the odd case. Recall that according to Proposition 3.3.3, we have that $n(L_+) = 1$, while for L_-, we establish in the similar fashion that $L_- \geq 0$, with an unique eigenvalue at zero, spanned by $\phi_{\vec{m}}$. Thus, spectral stability will be established (see the index counting formula (3.2.11)), once we verify that the quantity $\langle L_+^{-1}\phi_{\vec{m}}, \phi_{\vec{m}}\rangle < 0$.

Similar to the two dimensional case, this computation is done by a scaling argument. Indeed, taking a derivative in λ in the Euler-Lagrange equation (3.3.3), we obtain

$$L_+[\partial_\lambda \phi_\lambda] = -\frac{d\omega_\lambda}{d\lambda}\phi,$$

whence it follows that

$$\langle L_+^{-1}\phi, \phi\rangle = -\frac{1}{\frac{d\omega_\lambda}{d\lambda}}\langle \partial_\lambda \phi_\lambda, \phi_\lambda\rangle = -\frac{1}{\frac{d\omega_\lambda}{d\lambda}}.$$

From the scaling relation (3.3.4), we compute

$$\frac{d\omega_\lambda}{d\lambda} = \frac{2(p-1)}{4-n(p-1)}\lambda^{\frac{p(n+2)-(n+6)}{4-n(p-1)}}\frac{2n(p-1)-4(p+1)}{4-n(p-1)}I_1.$$

One can check that this last expression is positive, since $I_1 < 0$ and $p \in (1, 1+\frac{4}{n})$. Thus, $\langle L_+^{-1}\phi, \phi\rangle < 0$ and the spectral stability is established.

$\square$

Chapter 4

Nonlocal NLS equation $\mathcal{PT}$ symmetric systems

The first integrable nonlinear evolution equation solved by the method of inverse scattering transform was the Korteweg-deVries (KdV) equation [38]. Remarkably, it was shown that solitons corresponded to eigenvalues of the time independent linear Schrödingerequation. Soon thereafter, the concept of Lax pair [39] was introduced and the KdV equation, and others, were expressed as a compatibility condition of two linear equations. A few years later, Zakharov and Shabat [40] used the idea of Lax pair to integrate the nonlinear Schrödinger equation.

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q^*(-x,t), \quad \sigma = \pm 1, \tag{4.0.1}$$

where $*$ is the complex conjugate, and obtain soliton solutions.

In 2013, a new nonlocal reduction of the AKNS scattering problem was found [3], which gave rise to an integrable nonlocal NLS equation (4.0.1). Remarkably, it has a self-induced nonlinear "potential", thus, it is a $\mathcal{PT}$ symmetric equation [7]. In other words, one can view (4.0.1) as a linear Schrödinger equation

$$iq_t(x,t) = q_{xx}(x,t) + V(q,x,t)q(x,t), \tag{4.0.2}$$

with a self-induced potential $V(q,x,t) = -2\sigma q(x,t)q^*(-x,t)$ satisfying the $\mathcal{PT}$ symmetry condition $V(q,x,t) = V^*(q,-x,t)$.

4.1 Nonlocal NLS equation

In this section following the paper by Ablowitz and Musslimani [2], we first consider the following nonlocal NLS equation

$$iq_t(x,t) = q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t), \tag{4.1.1}$$

It is nonlocal in a simple way, since one of the nonlinear terms has to depend on $-x$. The equation can be written as

$$iq_t(x,t) = q_{xx}(x,t) + V(x,t)q(x,t),$$

where $V(x,t) = 2q(x,t)\bar{q}(-x,t)$.

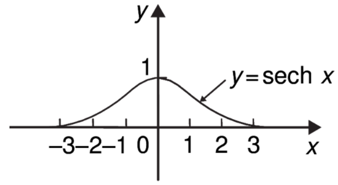
The equation is NLS with a $\mathcal{PT}$ symmetric potential $V(x,t)$, since $V(x,t) = \bar{V}(-x,t)$. Consider the standing waves in the form $q(x,t) = e^{-iwt}\phi(x)$, where $\phi(x) = \phi(-x)$. We obtain

$$i(-iw)e^{-iwt}\phi - e^{-iwt}\phi_{xx} - 2e^{-2iwt}\phi^2 e^{iwt}\phi = 0$$

Thus ϕ satisfies the second order ODE

$$-\phi'' + w\phi - 2\phi^3 = 0. \tag{4.1.2}$$

Thus $\phi'' = w\phi - 2\phi^3$ can be multiplied on both sides by ϕ' to get $\phi''\phi' = w\phi\phi' - 2\phi^3\phi'$. Integrate once to get $\phi' = \pm\phi(w-\phi^2)^{\frac{1}{2}}$. It follows that $\int \frac{d\phi}{\phi(w-\phi^2)^{1/2}} = \pm\int dx$. We have obtained the explicit form of the wave $\phi = \sqrt{w}\,\text{sech}(\sqrt{w}x)$.



To study the linear stability, we linearize around ϕ, set $q = e^{-iwt}(\phi(x) + u(x,t))$ Then it follows that

$we^{-iwt}(\phi+u)+ie^{-iwt}u_t-e^{-iwt}(\phi''+u_{xx})-2e^{-iwt}(\phi^2+2\phi u)(\phi(x)+\bar{u}(-x,t))=0$, and we obtain the equation for a complex solution $u(x,t)$,

$$wu+iu_t-u_{xx}-2\phi^2\bar{u}(-x,t)-4\phi^2u=0.$$

Take $u=u_1+iu_2$, and separate the real and imaginary parts,

$$w(u_1+iu_2)+i(u_1+iu_2)_t-(u_1+iu_2)_{xx}-2\phi^2(u_1(-x,t)-iu_2(-x,t))-4\phi^2(u_1+iu_2)=0.$$

The resulting 2×2 system looks like:

$$\begin{cases} wu_1-(u_2)_t-(u_1)_{xx}-4\phi^2u_1-2\phi^2u_1(-x,t)=0 \\ wu_2+(u_1)_t-(u_2)_{xx}-4\phi^2u_2+2\phi^2u_2(-x,t)=0 \end{cases}$$

We will introduce new variables in order to formally get rid of the nonlocality of this system. Consider variables U_1, V_1, U_2 and V_2 as follows:

$$U_1=\frac{u_1(x,t)+u_1(-x,t)}{2}, \text{ even in } x,$$

$$V_1=\frac{u_1(x,t)-u_1(-x,t)}{2}, \text{ odd in } x$$

$$U_2=\frac{u_2(x,t)+u_2(-x,t)}{2}, \text{ even in } x$$

$$V_2=\frac{u_2(x,t)-u_2(-x,t)}{2}, \text{ odd in } x,$$

Further, $u_1(x,t)=U_1+V_1$, $u_2(x,t)=U_2+V_2$, $u_1(-x,t)=U_1-V_1$ and $u_2(-x,t)=U_2-V_2$. The system becomes:

$$\begin{cases} -(U_2+V_2)_t+w(U_1+V_1)-(U_1+V_1)_{xx}-4\phi^2(U_1+V_1)-2\phi^2(U_1-V_1)=0 \\ (U_1+V_1)_t+w(U_2+V_2)-(U_2+V_2)_{xx}-4\phi^2(U_2+V_2)+2\phi^2(U_2-V_2)=0 \end{cases}$$

Since U_1,U_2 are even and V_1,V_2 are odd, this can be written as a system of four equations:

$$\begin{cases} -(U_2)_t + wU_1 - (U_1)_{xx} - 4\phi^2 U_1 - 2\phi^2 U_1 = 0 \\ -(V_2)_t + wV_1 - (V_1)_{xx} - 4\phi^2 V_1 + 2\phi^2 V_1 = 0 \\ (U_1)_t + wU_2 - (U_2)_{xx} - 4\phi^2 U_2 + 2\phi^2 U_2 = 0 \\ (V_1)_t + wV_2 - (V_2)_{xx} - 4\phi^2 V_2 - 2\phi^2 V_2 = 0 \end{cases}$$

or

$$\begin{cases} (U_2)_t = -(U_1)_{xx} + wU_1 - 6\phi^2 U_1 \\ (V_2)_t = -(V_1)_{xx} + wV_1 - 2\phi^2 V_1 \\ (U_1)_t = (U_2)_{xx} - wU_2 + 2\phi^2 U_2 \\ (V_1)_t = (V_2)_{xx} - wV_2 + 6\phi^2 V_2 \end{cases}$$

for $(U_1, V_1, U_2, V_2) \in L^2_{\text{even}} \times L^2_{\text{odd}} \times L^2_{\text{even}} \times L^2_{\text{odd}}$.

Introduce the operators $L_+ = -\partial_{xx} + w - 6\phi^2$, $L_- = -\partial_{xx} + w - 2\phi^2$ acting on H^2_{even} or H^2_{odd}.

Then the system can be written in the form

$$\begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}_t = JL \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix},$$

since

$$\begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}_t = \begin{pmatrix} -L_- U_2 \\ -L_+ V_2 \\ L_+ U_1 \\ L_- V_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_+ & 0 & 0 & 0 \\ 0 & L_- & 0 & 0 \\ 0 & 0 & L_- & 0 \\ 0 & 0 & 0 & L_+ \end{pmatrix} \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}$$

In here we use $J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $L = \begin{pmatrix} L_+ & 0 & 0 & 0 \\ 0 & L_- & 0 & 0 \\ 0 & 0 & L_- & 0 \\ 0 & 0 & 0 & L_+ \end{pmatrix}$.

Since ϕ is a solution to (4.1.2), $L_-\phi = 0$. Also $\phi = \sqrt{w}\,\text{sech}(\sqrt{w}x)$ does not have zeros. Using Sturm-Liouville theory, we deduce that $L_- \geq 0$.

Again using (4.1.2) and differenting with respect to w on both sides, it follows that $L_+(-\frac{d\phi}{dw}) = \phi$. Thus $\langle L_+^{-1}\phi, \phi\rangle = -\frac{1}{2}\frac{d}{dw}\|\phi_w\|^2 = -\frac{1}{2}\frac{d}{dw}\int_{-\infty}^{\infty} w\,\text{sech}^2(\sqrt{w}x)\,dx = -\frac{1}{2\sqrt{w}} < 0$.

Further $\phi'' = w\phi - 2\phi^3$, by taking derivative on both sides, we have $L_+\phi' = 0$.

$\phi' = -w\tanh(\sqrt{w}x)\,\text{sech}(\sqrt{w}x)$, $\phi' = 0$ as $x = 0$, and ϕ' changes sign once. $L_+\phi = -4\phi^3$, $\langle L_+\phi, \phi\rangle = -4\int_{-\infty}^{\infty}\phi^4\,dx < 0$.

Using Sturm-Liouville theory again, L_+ has a simple negative eigenvalue.

$$Ker(L) = Ker\begin{pmatrix} L_+ & 0 & 0 & 0 \\ 0 & L_- & 0 & 0 \\ 0 & 0 & L_- & 0 \\ 0 & 0 & 0 & L_+ \end{pmatrix} = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \phi' \end{pmatrix}, \begin{pmatrix} \phi' \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \phi \\ 0 \end{pmatrix}\right\}$$

Then we have

(1) L_+ is defined on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$, has a unique, simple negative eigenvalue whose eigenfunction is even; zero is simple with associated eigenfunction ϕ', and the essential spectrum is $[w, \infty)$.

(2) L_- is defined on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$, has no negative eigenvalue ; zero is simple with associated eigenfunction ϕ, and the essential spectrum is $[w, \infty)$.

(3) J is bounded, invertible and skew-symmetric ($J^* = -J$). In addition, $J^{-1} : Ker[L] \to Ker[L]^{\perp}$.

We will take advantage of a simple version of the index counting theorem:

$n_{\text{unstable}}(JL) + \text{even number} = n(L) - n(D)$.

Then we need to compute $n(D)$.

$$D = \begin{pmatrix} \langle L^{-1}J^{-1}\phi_1, J^{-1}\phi_1\rangle & \langle L^{-1}J^{-1}\phi_1, J^{-1}\phi_2\rangle & \langle L^{-1}J^{-1}\phi_1, J^{-1}\phi_3\rangle & \langle L^{-1}J^{-1}\phi_1, J^{-1}\phi_4\rangle \\ \langle L^{-1}J^{-1}\phi_2, J^{-1}\phi_1\rangle & \langle L^{-1}J^{-1}\phi_2, J^{-1}\phi_2\rangle & \langle L^{-1}J^{-1}\phi_2, J^{-1}\phi_3\rangle & \langle L^{-1}J^{-1}\phi_2, J^{-1}\phi_4\rangle \\ \langle L^{-1}J^{-1}\phi_3, J^{-1}\phi_2\rangle & \langle L^{-1}J^{-1}\phi_3, J^{-1}\phi_2\rangle & \langle L^{-1}J^{-1}\phi_3, J^{-1}\phi_3\rangle & \langle L^{-1}J^{-1}\phi_3, J^{-1}\phi_4\rangle \\ \langle L^{-1}J^{-1}\phi_4, J^{-1}\phi_1\rangle & \langle L^{-1}J^{-1}\phi_4, J^{-1}\phi_1\rangle & \langle L^{-1}J^{-1}\phi_4, J^{-1}\phi_3\rangle & \langle L^{-1}J^{-1}\phi_4, J^{-1}\phi_4\rangle \end{pmatrix}$$

where $\phi_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \phi' \end{pmatrix}$, $\phi_2 = \begin{pmatrix} \phi' \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\phi_3 = \begin{pmatrix} 0 \\ \phi \\ 0 \\ 0 \end{pmatrix}$, $\phi_4 = \begin{pmatrix} 0 \\ 0 \\ \phi \\ 0 \end{pmatrix}$

$$D_{11} = \left\langle \begin{pmatrix} L_+^{-1} & 0 & 0 & 0 \\ 0 & L_-^{-1} & 0 & 0 \\ 0 & 0 & L_-^{-1} & 0 \\ 0 & 0 & 0 & L_+^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \phi' \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi' \\ 0 \\ 0 \end{pmatrix} \right\rangle = \langle L_-^{-1}\phi', \phi'\rangle > 0.$$

$$D_{22} = \left\langle \begin{pmatrix} L_+^{-1} & 0 & 0 & 0 \\ 0 & L_-^{-1} & 0 & 0 \\ 0 & 0 & L_-^{-1} & 0 \\ 0 & 0 & 0 & L_+^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\phi' \\ 0 \end{pmatrix} \right\rangle = \langle L_-^{-1}\phi', \phi'\rangle > 0.$$

$$D_{33} = \left\langle \begin{pmatrix} L_+^{-1} & 0 & 0 & 0 \\ 0 & L_-^{-1} & 0 & 0 \\ 0 & 0 & L_-^{-1} & 0 \\ 0 & 0 & 0 & L_+^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\phi \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\phi \end{pmatrix} \right\rangle = \langle L_+^{-1}\phi, \phi\rangle < 0.$$

$$D_{44} = \left\langle \begin{pmatrix} L_+^{-1} & 0 & 0 & 0 \\ 0 & L_-^{-1} & 0 & 0 \\ 0 & 0 & L_-^{-1} & 0 \\ 0 & 0 & 0 & L_+^{-1} \end{pmatrix} \begin{pmatrix} \phi \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \phi \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \langle L_+^{-1}\phi, \phi\rangle < 0.$$

And $D_{ij} = 0$, where $i \neq j$, and $i, j \in \{1,2,3,4\}$.

Thus $n(D) = 2$ and $n(L) = 2$, $n_{\text{unstable}}(JL) = 0$, thus the waves are spectrally stable.

We have the following result.

Theorem 4.1.1. *The standing wave solutions* $e^{-iwt}\sqrt{w}\,\text{sech}(\sqrt{w}x)$ *of the nonlocal NLS equation (4.1.1) are spectrally stable.*

4.2 Reverse time nonlocal NLS

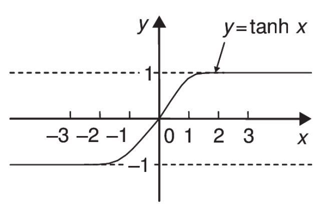
In this section, we consider the following reverse time nonlocal NLS equation [2]

$$iq_t(x,t) = q_{xx}(x,t) - 2q^2(x,t)q(x,-t), \tag{4.2.1}$$

Consider the standing waves in the form $q(x,t) = e^{iwt}\phi(x)$, Thus ϕ satisfies the second order ODE

$$\phi'' + w\phi - 2\phi^3 = 0 \tag{4.2.2}$$

Thus it can be multiplied on both sides by ϕ' to get $\phi''\phi' + w\phi\phi' - 2\phi^3\phi' = 0$. Integrate once to get $\frac{1}{2}(\phi')^2 + \frac{1}{2}w\phi^2 - \frac{1}{2}\phi^4 = \frac{1}{2}A \Rightarrow \phi' = \pm\sqrt{\phi^4 - w\phi^2 + A}$, where A is a constant. In the special case, by taking $A = \frac{w^2}{4}$, we get the explicit form of the wave $\phi(x) = \sqrt{\frac{w}{2}}\tanh(\sqrt{\frac{w}{2}}x)$.



To study the linear stability, we linearize around ϕ. Set $q = e^{iwt}(\phi(x) + u(x,t))$.

Then it follows that $-we^{iwt}(\phi+u) + ie^{iwt}u_t - e^{iwt}(\phi'' + u_{xx}) + 2e^{iwt}(\phi^2 + 2\phi u)(\phi(x) + u(x,-t)) = 0$ and we obtain the equation for a complex solution $u(x,t)$

$$-wu + iu_t - u_{xx} + 2\phi^2 u(x,-t) + 4\phi^2 u = 0.$$

Let $u = u_1 + iu_2$ and separate the real and imaginary parts,

$$-w(u_1+iu_2)+i(u_1+iu_2)_t-(u_1+iu_2)_{xx}+2\phi^2(u_1(x,-t)+iu_2(x,-t))+4\phi^2(u_1+iu_2)=0.$$

The resulting 2×2 system looks like:

$$\begin{cases} -wu_1-(u_2)_t-(u_1)_{xx}+4\phi^2u_1+2\phi^2u_1(x,-t)=0 \\ -wu_2+(u_1)_t-(u_2)_{xx}+4\phi^2u_2+2\phi^2u_2(x,-t)=0 \end{cases}$$

we introduce new variables to formally get rid of the nonlocality of the system, consider variables $U_1,\ V_1,\ U_2$ and V_2 as follows:

$$U_1=\frac{u_1(x,t)+u_1(x,-t)}{2},\ \text{even in } t,$$

$$V_1=\frac{u_1(x,t)-u_1(x,-t)}{2},\ \text{ odd in } t$$

$$U_2=\frac{u_2(x,t)+u_2(x,-t)}{2},\ \text{ even in } t$$

$$V_2=\frac{u_2(x,t)-u_2(x,-t)}{2},\ \text{odd in } t,$$

Further, $u_1(x,t)=U_1+V_1$, $u_2(x,t)=U_2+V_2$, $u_1(x,-t)=U_1-V_1$ and $u_2(x,-t)=U_2-V_2$. The system becomes

$$\begin{cases} (U_2+V_2)_t+w(U_1+V_1)+(U_1+V_1)_{xx}-4\phi^2(U_1+V_1)-2\phi^2(U_1-V_1)=0 \\ (U_1+V_1)_t-w(U_2+V_2)-(U_2+V_2)_{xx}+4\phi^2(U_2+V_2)+2\phi^2(U_2-V_2)=0 \end{cases}$$

Since U_1,U_2 are even and V_1,V_2 are odd. This can be written as a system of four equations

$$\begin{cases} (U_2)_t + wU_1 + (U_1)_{xx} - 4\phi^2 U_1 - 2\phi^2 U_1 = 0 \\ (V_2)_t + wV_1 + (V_1)_{xx} - 4\phi^2 V_1 + 2\phi^2 V_1 = 0 \\ (U_1)_t - wU_2 - (U_2)_{xx} + 4\phi^2 U_2 + 2\phi^2 U_2 = 0 \\ (V_1)_t - wV_2 - (V_2)_{xx} + 4\phi^2 V_2 - 2\phi^2 V_2 = 0 \end{cases}$$

which will become

$$\begin{cases} (U_2)_t = -(U_1)_{xx} - wU_1 + 6\phi^2 U_1 \\ (V_2)_t = -(V_1)_{xx} - wV_1 + 2\phi^2 V_1 \\ (U_1)_t = (U_2)_{xx} + wU_2 - 6\phi^2 U_2 \\ (V_1)_t = (V_2)_{xx} + wV_2 - 2\phi^2 V_2 \end{cases}$$

for $(U_1, V_1, U_2, V_2) \in L^2_{\text{even}} \times L^2_{\text{odd}} \times L^2_{\text{even}} \times L^2_{\text{odd}}$.

Introduce the operators $L_+ = -\partial_{xx} - w + 6\phi^2$, $L_- = -\partial_{xx} - w + 2\phi^2$ acting on H^2_{even} or H^2_{odd}.

Then the system can be written in the form

$$\begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}_t = \begin{pmatrix} -L_+ U_2 \\ -L_- V_2 \\ L_+ U_1 \\ L_- V_1 \end{pmatrix}, \text{after transformations} \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}_t \to e^{\lambda t} \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix}$$

We have the eigenvalue problem

$$\lambda \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} -L_+ & 0 \\ 0 & -L_- \end{pmatrix} \begin{pmatrix} U_2 \\ V_2 \end{pmatrix}, \quad \lambda \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} \begin{pmatrix} U_1 \\ V_1 \end{pmatrix}$$

or directly

$$\lambda^2 \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} -L_+^2 & 0 \\ 0 & -L_-^2 \end{pmatrix} \begin{pmatrix} U_1 \\ V_1 \end{pmatrix}$$

We obtain $L_+^2 U_1 = -\lambda^2 U_1$ and $L_+^2 V_1 = -\lambda^2 V_1$, it follows that λ is pure imaginary. So there are no eigenvalues such that $\Re\lambda > 0$. Hence the waves are stable in the sense that the eigenvalue is pure imaginary.

Theorem 4.2.1. *The standing wave solutions* $e^{iwt}\sqrt{\frac{w}{2}}\tanh(\sqrt{\frac{w}{2}}x)$ *of the reverse time nonlocal NLS equation (4.2.1) are stable in the sense that the eigenvalue is pure imaginary.*

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