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ITÔ-WIENER CHAOS EXPANSION WITH EXACT RESIDUAL AND CORRELATION, VARIANCE INEQUALITIES *

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ABSTRACT We give a formula of expanding the solution of a stochastic differential equation (abbreviated as SDE below) into a finite Itô-Wiener chaos with explicit residual. And then we apply this formula to obtain several inequalities for diffusions such as FKG type inequality and variance inequality and a correlation inequality for Gaussian measure. A simple proof for Houdré-Kagan's variance inequality for Gaussian measure is also given this way.

1. Introduction. Let $(\Omega, \mathcal{F}, P)$ be the canonical Wiener space on $\Re_+ = [0, \infty)$ and let B be the standard Wiener process on $(\Omega, \mathcal{F}, P)$. It is well-known (see [It]) that any square integrable real random variable F on $(\Omega, \mathcal{F}, P)$ can be expanded as

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} J_n(f_n), \tag{1.1}$$

where $\mathbb{E}$ is the expectation with respect to P, $f_n : \Re_+^n \to \Re$, $n = 1, 2, \cdots$ (called the Itô-Wiener coefficients of F. We omit their explicit dependence on F) are square integrable with respect to the Lebesgue measure on $\Re_+^n$ and $J_n(f_n)$ is the multiple Itô-Wiener integral (of order n),

$$J_n(f_n) = \int_{0<s_1<\cdots<s_n<\infty} f_n(s_1, \cdots, s_n) dB_{s_1} \cdots dB_{s_n}.$$

In the following, however, we will use $J_n(f_n)$ to represent the integration till time 1 instead of ∞! The convergence of the series in (1.1) is in $L^2(\Omega, \mathcal{F}, P)$ and this result is called the Itô-Wiener expansion theorem.

Generally, the solution of an SDE is square integrable and then admits an Itô-Wiener chaos expansion. The explicit formula for such an expansion is known as

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the Isobe-Sato formula, see for instance, [IS], [HM], [BL] etc. In this paper, using the transition probability, we obtain in a very simple way the explicit expression of the Itô-Wiener coefficients (for F being the solution of an SDE) and in particular the explicit expression for the residual, i.e., $F - \mathbb{E}F - \sum_{k=1}^{n} J_n(f_n)$, expressed by an $n+1$ multiple integral with a random coefficient. This result implies the Isobe-Sato formula easily. As application of our formula, we first establish an FKG type inequality and a variance inequality for diffusions and then we obtain a correlation inequality for Gaussian (measure). With an additional simple technique of integration by parts, we give a simple proof of a variance inequality for Gaussian measure, recently obtained by Houdré and Kagan [HK].

This paper mainly uses a technique (see (2.6) below) emploied first by Prof. J. Neveu in his simplest probabilistic proof for Nelson's hypercontractivity which I learnt from Prof. P.A. Meyer (see also [DMM]). It is then a great pleasure for me to dedicate this paper to their 60 birthday.

2. Itô-Wiener chaos expansion with exact residual. For simplicity of notation, we consider a one dimensional SDE:

$$dx_t = \sigma(x_t)dB_t + b(x_t)dt, \quad 0 \le t < \infty, \quad x_0 = \xi \in \Re. \tag{2.1}$$

Let C_b^∞ be the set of all C^∞ functions $f : \Re \to \Re$ with bounded derivatives. Throughout this paper, we assume that *the coefficients* $\sigma, b : \Re \to \Re$ *are in* C_b^∞ *and that the transition probability* $P_t(x, dy)$ *associated with the equation (2.1) exists and* $\int_\Re f(y)P_t(x,dy) \in C_b^\infty$ *for* $f \in C_b^\infty$. $P_t(x, dy)$ satisfies then the following parabolic equation:

$$\frac{\partial P_t}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 P_t}{\partial x^2} + b\frac{\partial P_t}{\partial x} \tag{2.2}$$

with initial condition $P_0(x, dy) = \delta(dy - x)$. The study of above regularity of the transition density is one of the objectives of the Malliavin calculus. We will not concern with it here. We will say f is a nice function if $f \in C_b^\infty$. Of course we could replace C_b^∞ by some other class of functions, for instance, the class of functions of polynomial growth (not for σ, b). However, to keep the paper simple it is not our attention to state our results in their most generality.

Under the above assumption, (2.1) has a unique solution which is belong to $L^p(\Omega, \mathcal{F}, P)$ for any $p \ge 1$. So it admit an Itô-Wiener's expansion. We will concern with the explicit form of the coefficients and the explicit residual for the expansion of $f(x_1)$ for some nice function f.

Denote $P_t f(x) = \int_\Re f(y)P_t(x,dy)$, $\nabla_\sigma f(x) = \sigma(x)\frac{d}{dx}f(x)$ and $P_t'(x,dy) = \frac{d}{dx}P_t(x,dy)$. In what follows P_t and ∇_σ are considered as two operators acting the function space C_b^∞. These two operators do not commute. The following examples will expalin our notation

$$\nabla_\sigma \nabla_\sigma f(x) = \sigma(x)\sigma'(x)f'(x) + (\sigma(x))^2 f''(x);$$

$$\nabla_\sigma P_s \nabla_\sigma P_t f(x) = \sigma(x) \int_{\Re^2} P'_s(x, dy_1)\sigma(y_1)P'_t(y_1, dy_2)f(y_2).$$

We will only discuss x_1. All formulas below are valid for any x_t $(t > 0)$ with a slight change. Our first result is

Theorem 2.1: Let the notation be as above. For any nice function f,

$$f(x_1) = \mathbb{E}f(x_1) + \sum_{k=1}^{n} J_n(f_n) + J_{n+1}(g_{n+1}), \tag{2.3}$$

where

$$g_n(s_1, \cdots, s_n) = \nabla_\sigma P_{s_2 - s_1} \cdots \nabla_\sigma P_{1-s_n} f(x_{s_1}); \tag{2.4}$$

$$f_n = \mathbb{E}g_n(s_1, \cdots, s_n) = P_{s_1} \nabla_\sigma P_{s_2 - s_1} \cdots \nabla_\sigma P_{1-s_n} f(\xi). \tag{2.5}$$

Proof: Applying the Itô formula to the process $P_{t-s}f(x_s)$, $0 \le s \le t$ and noting $P_0 f = f$ and equation (2.2), we obtain

$$f(x_t) = P_t f(\xi) + \int_0^t \nabla_\sigma P_{t-u} f(x_u) dB_u. \tag{2.6}$$

Applying (2.6) for $t = 1$, we have

$$\begin{aligned} f(x_1) &= P_1 f(\xi) + \int_0^1 \nabla_\sigma P_{1-s} f(x_s) dB_s \\ &= \mathbb{E}f(x_1) + \int_0^1 \nabla_\sigma P_{1-s} f(x_s) dB_s. \end{aligned}$$

Consider the integrand of the above integral. Applying (2.6) to $t = s$ and $\nabla_\sigma P_{1-s} f$, we obtain

$$\nabla_\sigma P_{1-s} f(x_s) = P_s \nabla_\sigma P_{1-s} f(\xi) + \int_0^s \nabla_\sigma P_{s-u} \nabla_\sigma P_{1-s} f(x_u) dB_u.$$

So

$$\begin{aligned} f(x_1) = \mathbb{E}f(x_1) &+ \int_0^1 P_s \nabla_\sigma P_{1-s} f(\xi) dB_s \\ &+ \int_0^1 \int_0^s \nabla_\sigma P_{s-u} \nabla_\sigma P_{1-s} f(x_u) dB_u dB_s. \end{aligned}$$

Continuing to use (2.6), we proves the theorem. ∎

Remark 1: We can also obtain a similar formula for a multi-dimensional diffusion this way (see section 6 below for a multi-dimensional Brownian motion case).

Remark 2: One can verify easily that $J_{n+1}(g_{n+1})$ is orthogonal to all Itô-Woener chaos of order less or equal to n. According to Itô-Wiener chaos expansion theorem, we have

$$f(x_1) = \mathbb{E}f(x_1) + \sum_{n=1}^{\infty} J_n(f_n), \tag{2.7}$$

where f_n is given by (2.5) and the series is convergent in $L^2(\Omega, \mathcal{F}, P)$. The formula (2.7) with (2.5) is called the Isobe-Sato formula. Prof. J. Potthoff told me that it might be difficult to prove the $L^2(\Omega, \mathcal{F}, P)$ convergence of (2.7) directly from an L^2 estimate of (2.5) (see [HM]) without using the Itô-Wiener expansion theorem.

The case $\sigma = 1$, $b = 0$ and $\xi = 0$, i.e., the case x_t is the Wiener process B_t is of particular interest. In this case the transition density $P_t(x,y) = \frac{P_t(x,dy)}{dy}$ is given by

$$\Phi_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}}. \tag{2.8}$$

It is not difficult to show

Corollary 2.2: Let f be a nice function and let $\mu(dx) = \Phi_1(x,0)dx$. Then

$$f(B_1) = \int_{\Re} f(x)\mu(dx) + \sum_{k=1}^{n} f_n H_n(x) + J_{n+1}(g^B_{n+1}), \tag{2.9}$$

where H_n is the n-th Hermite polynomial, $H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{\frac{-x^2}{2}}$,

$$f_n = \int_{\Re} f(x) H_n(x) \mu(dx) \tag{2.10}$$

and

$$g^B_{n+1}(s_1, \cdots, s_{n+1}) = \Phi_{1-s_1} f^{(n+1)}(B_{s_1}). \tag{2.11}$$

Proof: From the fact that $\Phi_s \Phi_t = \Phi_{s+t}$ and $\frac{d^n}{dx^n}\Phi_t f = \Phi_t f^{(n)}$ (We also use the same notation Φ_t to represent the operator acting on C_b^∞ whose kernel is Φ_t), we have by (2.5), integration by parts and the definition of H_n,

$$f_n = \Phi_1 f^{(n)}(0) = \int_{\Re} \Phi_1(x) f^{(n)}(x) dx = \int_{\Re} f(x) H_n(x) \mu(dx).$$

Note that $J_n(1) = H_n(B_1)$. This proves the corollary. ∎

Remark : The formula (2.10) is well-known since long time. But the formula (2.11) might be new though simple.

3. FKG inequality. Using the formula (2.3) in the case $n = 0$, we can obtain an FKG type inequality for diffusions. We need a simple lemma.

Lemma 3.1: Let $P_t(x, dy)$ be the transition probability of (2.1). Let f be continuously differentiable and $f' \geq 0$. Then

$$P_t' f(x) \equiv \frac{d}{dx} P_t f(x) \geq 0, \quad x \in \Re. \tag{3.1}$$

Proof : Denote the solution of (2.1) by $x_t(x)$. From the well-known formula $P_t f(x) = \mathbb{E} f(x_t(x))$, we have

$$\frac{d}{dx} P_t f(x) = \mathbb{E}\{f'(x_t(x)) \frac{d}{dx} x_t(x)\}. \tag{3.2}$$

Differentiating (2.1) with respect to x and letting $z_t = \frac{d}{dx} x_t(x)$, we obtain

$$\frac{d}{dx} x_t(x) \equiv z_t = 1 + \int_0^t \sigma'(x_s) z_s dB_s + \int_0^t b'(x_s) z_s ds.$$

The solution z_t of this equation has the following explicit form

$$z_t = \exp\{\int_0^t \sigma'(x_s) dB_s + \int_0^t [b'(x_s) - \frac{1}{2} |\sigma'(x_s)|^2] ds\}$$

which is almost surely positive. Combining this fact with (3.2), we prove the lemma. ∎

Theorem 3.2: Let the assumption in the beginning of section 2 be satisfied and let x_t be the solution of (2.1). Let $f' \geq 0$ and $g' \geq 0$. Then

$$\mathbb{E}[f(x_1) g(x_1)] \geq \mathbb{E} f(x_1) \mathbb{E} g(x_1). \tag{3.3}$$

Proof : Taking $n = 1$ in formula (2.3), we have

$$f(x_1) = \mathbb{E} f(x_1) + \int_0^1 \sigma(x_t) P_{1-t}' f(x_t) dB_t. \tag{3.4}$$

A similar formula holds for g. So

$$\begin{aligned} \mathbb{E} f(x_1) g(x_1) &= \mathbb{E} f(x_1) \mathbb{E} g(x_1) + \int_0^1 \mathbb{E}\{|\sigma(x_t)|^2 P_{1-t}' f(x_t) P_{1-t}' g(x_t)\} dt \\ &\geq \mathbb{E} f(x_1) \mathbb{E} g(x_1), \end{aligned}$$

where the last inequality follows from the fact that from lemma 3.1, $f' \geq 0$ implies $P_{1-t}' f \geq 0$ and $g' \geq 0$ implies $P_{1-t}' g \geq 0$. ∎

Remark: This result may have some connection with those of [BM].

4. Variance inequality. We are going to deduce a variance inequality for diffusions. Let the assumption in the beginning of section 2 be satisfied. Denote by A the operator

$$Af(x)=\frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2}(x)+b(x)\frac{\partial f}{\partial x}(x)=\frac{1}{2}\sigma^2 f''+bf',\quad f\in C^2(\Re)$$

and $\mathcal{L}g=\frac{\partial g}{\partial s}g+Ag$ for $g\in C^{1,2}([0,1]\times\Re;\Re)$. We will establish a variance inequality for the measure determined by x_1. Precisely, we have

Theorem 4.1: If $b'\le\frac{A\sigma}{\sigma}$, then

$$\mathbb{E}|f(x_1)-\mathbb{E}f(x_1)|^2\le\mathbb{E}|\sigma(x_1)f'(x_1)|^2, \tag{4.1}$$

where we understand $b'\le\frac{A\sigma}{\sigma}$ as multiplying by σ in the case $\sigma=0$.

Proof: From (3.4), we obtain

$$\mathbb{E}|f(x_1)-\mathbb{E}f(x_1)|^2=\int_0^1\mathbb{E}|\sigma(x_t)P'_{1-t}f(x_t)|^2dt. \tag{4.2}$$

Set $g_s(x)=\sigma(x)P'_{1-s}f(x)$ and apply the Itô formula to the process $[g_s(x_s)]^2$, $t\le s\le 1$. Then

$$\mathbb{E}|\sigma(x_1)f'(x_1)|^2=\mathbb{E}|g_t(x_t)|^2+\int_t^1\mathbb{E}[\mathcal{L}(g_s^2)(x_s)]ds.$$

Using the simple identity $\mathcal{L}(fg)=f\mathcal{L}g+g\mathcal{L}f+\sigma^2f'g'$, we have

$$\mathbb{E}|\sigma(x_1)f'(x_1)|^2\ge\mathbb{E}|g_t(x_t)|^2+2\int_t^1\mathbb{E}[g_s(x_s)\mathcal{L}(g_s)(x_s)]ds \tag{4.3}$$

and

$$\mathcal{L}(g_s)=(A\sigma)P'_{1-s}f+\sigma\mathcal{L}P'_{1-s}f+\sigma^2\sigma'P''_{1-s}f. \tag{4.4}$$

Differentiating then the equation $\mathcal{L}P_{1-s}f=0$ with respect to x, we obtain

$$\mathcal{L}P'_{1-s}f=-\sigma\sigma'P''_{1-s}f-b'P'_{1-s}f. \tag{4.5}$$

Combining (4.4) and (4.5),

$$\mathcal{L}g_s=[(A\sigma)-\sigma b']P'_{1-s}f=[\frac{A\sigma}{\sigma}-b']g_s. \tag{4.6}$$

Inserting (4.6) into (4.3), we have

$$\mathbb{E}|\sigma(x_1)f'(x_1)|^2\ge\mathbb{E}|g_t(x_t)|^2+2\int_t^1\mathbb{E}\{\sigma(x_s)[(A\sigma)-\sigma b'](x_s)[P'_{1-s}f(x_s)]^2\}ds.$$

When the condition of the theorem is satisfied, we have

$$\mathbb{E}|\sigma(x_1)f'(x_1)|^2 \geq \mathbb{E}|\sigma(x_t)P'_{1-t}f(x_t)|^2.$$

Together with (4.2), we have been proved the theorem. ∎

Remark 1: M. Ledoux [Le1] and [Le2] obtained an expansion of the variance for a class of general Markov processes using iterated gradients. A simple consequence is that he also obtained a variance inequality of type (4.1). But his method works for the process which has an invariant measure. We do not assume this condition for diffusion processes. Moreover, his variance inequality may be different from ours.

Remark 2: The condition of the theorem is equivalent to $\Gamma_2 \geq 0$ in [BE]. This condition and the logarithmic Sobolev type inequalities under this condition have been studied extensively.

5. Variance identity and inequality for Gaussian. Using the same idea as in the preceding sections we will give a simple proof of a variance inequality for Gaussian (measure) obtained recently by Houdré and Kagan. Denote

$$\nabla^n\Phi_t(x,y) = \frac{\partial^n}{\partial x^n}\Phi_t(x,y), \qquad \nabla^n\Phi_t f(x) = \int_R \nabla^n\Phi_t(x,y)f(y)dy,$$

where Φ_t is given by (2.8).

Lemma 5.1 : Let B_t be the standard Brownian motion starting at 0, $n \geq 1$ and let $f \in C_b^\infty$. Then

$$\begin{aligned}&\int_0^1 \frac{t^{n-1}}{(n-1)!}\mathbb{E}[\nabla^n\Phi_{1-t}f(B_t)]^2 dt \\ &= \frac{1}{n!}\mathbb{E}[f^{(n)}(B_1)]^2 - \int_0^1 \frac{t^n}{n!}\mathbb{E}[\nabla^{n+1}\Phi_{1-t}f(B_t)]^2 dt. \qquad (5.1)\end{aligned}$$

Proof : Applying the integration by parts formula to the left hand side of (5.1), we obtain

$$\begin{aligned}\int_0^1 \frac{t^{n-1}}{(n-1)!}\mathbb{E}[\nabla^n\Phi_{1-t}f(B_t)]^2 dt &= \frac{t^n}{n!}\mathbb{E}[\nabla^n\Phi_{1-t}f(B_t)]^2\Big|_0^1 \\ &\quad - \int_0^1 \frac{t^n}{n!}\frac{d}{dt}\{\mathbb{E}[\nabla^n\Phi_{1-t}f(B_t)]^2\}dt. \qquad (5.2)\end{aligned}$$

To calculate the above differential, we use (2.6) for $f(x)$ to be $\nabla^n\Phi_{1-t}f(x)$ and $X_t = B_t$ (Note that $B_0 = 0$). Thus we have

$$\nabla^n\Phi_{1-t}f(B_t) = \nabla^n\Phi_1 f(0) + \int_0^t \nabla^{n+1}\Phi_{1-s}f(B_s)dB_s.$$

Consequently

$$\mathbb{E}[\nabla^n \Phi_{1-t} f(B_t)]^2 = [\nabla^n \Phi_1 f(0)]^2 + \int_0^t \mathbb{E}[\nabla^{n+1}\Phi_{1-s} f(B_s)]^2 ds.$$

Then

$$\frac{d}{dt}\mathbb{E}[\nabla^n \Phi_{1-t} f(B_t)]^2 = \mathbb{E}[\nabla^{n+1}\Phi_{1-t} f(B_t)]^2. \tag{5.3}$$

Inserting (5.3) into (5.2) we obtain (5.1). ∎

Theorem 5.2: Let X be a centered normalized Gaussian random variable and $f \in C_b^\infty$. Then

$$\mathbb{E}|f(X) - \mathbb{E}f(X)|^2 = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}\mathbb{E}|f^{(k)}(X)|^2 + (-1)^{k+2}\int_0^1 \frac{t^n}{n!}\mathbb{E}[\nabla^{n+1}\Phi_{1-t} f(B_t)]^2 dt. \tag{5.4}$$

Proof : Obviously we can put $X = B_1$. We prove this theorem by recurrence on n. When $n = 1$, (2.9) gives

$$f(B_1) = \mathbb{E}f(X) + \int_0^1 \nabla \Phi_{1-t} f(B_t) dB_t.$$

So

$$\mathbb{E}|f(B_1) - \mathbb{E}f(B_1)|^2 = \int_0^1 \mathbb{E}|\nabla \Phi_{1-t} f(B_t)|^2 dt.$$

This is (5.4) for $n = 1$.

To pass from n to $n+1$, we need only a simple application of lemma 5.1 (formula (5.1)). ∎

Now the following variance inequality due to Houdré and Kagan [HK] is an easy consequence of theorem 5.2.

Theorem 5.3: Let X be a centered normalized Gaussian random variable and let f be a sufficiently differentiable function from $\Re^d$ to $\Re$ with bounded derivatives. Then

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!}\mathbb{E}|f^{(k)}(X)|^2 \le \mathbb{E}|f(X) - \mathbb{E}f(X)|^2 \le \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!}\mathbb{E}|f^{(k)}(X)|^2. \tag{5.5}$$

Remark : M. Ledoux [Le1], [Le2] gave an another simple proof of (5.5). He also obtains a similar expansion for entropy, see also [Ho].

6. Correlation inequality. In contrast with the preceding sections, we will work on an arbitrary Euclidean space $\Re^d$. We use $<\cdot,\cdot>$ to denote the Euclidean scalar product and $|\cdot|$ the Euclidean norm on $\Re^d$. Let μ be the standard Gaussian measure on $\Re^d$ with the density $\Phi(x)=(2\pi)^{-d/2}\exp(-|x|^2/2)$.

Recall that $H_n(y)=(-1)^n\phi_1(y)^{-1}\frac{d^n}{dy^n}\phi_1(y), y\in\Re$ is the real Hermite polynomial of degree n. If $f\in L^2(\Re^d;d\mu)$, then f can be expanded according to the Hermite polynomials

$$f=\sum_{n=0}^{\infty}f_n,\qquad f_n=\sum_{n_1+\cdots+n_d=n}a_{n_1,\cdots,n_d}H_{n_1}(x_1)\cdots H_{n_d}(x_d),$$

where $a_{n_1,\cdots,n_d}$ are some real numbers. We say that f is degenerate if $f_1=0$. Our main result of this section is

Theorem 6.1: If $f,g\in L^2(\Re^d;d\mu)$ are convex and one of them is degenerate, then

$$\int_{\Re^d}f(x)g(x)\mu(dx)\geq\int_{\Re^d}f(x)\mu(dx)\int_{\Re^d}g(x)\mu(dx). \tag{6.1}$$

Proof: Let now $B_t=(B_t^1,\cdots,B_t^d), 0\leq t\leq 1$ be the d-dimensional Brownian motion starting at 0 on the interval $[0,1]$. Let

$$\Phi_t(x,y)=(2\pi t)^{-d/2}\exp[-|y-x|^2/(2t)],\quad x,y\in\Re^d.$$

Let ∇f be the gradient and $\nabla^2 f$ be the Hessian of f, i.e.,

$$\nabla f(x)=(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_d})^*,\quad \nabla^2 f(x)=(\frac{\partial^2 f}{\partial x_i\partial x_j})_{1\leq i,j\leq d}.$$

Lemma 6.2: If $f\in L^2(\Re^d;d\mu)$ is a convex function, then for any $t>0$, $\Phi_t f$ is convex and smooth (as a function of x).

Proof: Using $\Phi_t f(x)=\mathbb{E}f(B_t+x), x\in\Re^d$, we can prove the lemma easily. ∎

We return to the proof of the therem. Analogue to (2.6), we have

$$f(B_t)=\mathbb{E}f(B_t)+\sum_{i=1}^{d}\int_0^t[\frac{\partial}{\partial x_i}\Phi_{t-s}f](B_s)dB_s^i. \tag{6.2}$$

Applying the above formula to $t=1$, we obtain

$$f(B_1)=\mathbb{E}f(B_1)+\sum_{i=1}^{d}\int_0^1[\frac{\partial}{\partial x_i}\Phi_{1-t}f](B_t)dB_t^i. \tag{6.3}$$

In (6.2), replacing f by $\frac{\partial}{\partial x_i}\Phi_{1-t}f$, and using the fact that $\Phi_t\frac{\partial h}{\partial x_i}(x) = \frac{\partial}{\partial x_i}\Phi_t h$ for any differentiable function h (this follows simply from an integration by parts), we obtain

$$\begin{aligned}\frac{\partial}{\partial x_i}\Phi_{1-t}f(B_t) &= \mathbb{E}[\frac{\partial}{\partial x_i}\Phi_{1-t}f(B_t)] + \sum_{j=1}^{d}\int_0^t \{\frac{\partial}{\partial x_j}\Phi_{t-s}[\frac{\partial}{\partial x_i}\Phi_{1-t}f]\}(B_s)dB_s^j \\ &= \mathbb{E}[\frac{\partial}{\partial x_i}\Phi_{1-t}f(B_t)] + \sum_{j=1}^{d}\int_0^t [\frac{\partial^2}{\partial x_i \partial x_j}\Phi_{1-s}f](B_s)dB_s^j. \qquad (6.4)\end{aligned}$$

We also used the fact that $\Phi_{t-s}\Phi_{1-t}f = \Phi_{1-t}f$ in the obtention of (6.4). Inserting (6.4) into (6.3) we obtain

$$\begin{aligned}f(B_1) = \mathbb{E}f(B_1) &+ \sum_{i=1}^{d}\int_0^1 \mathbb{E}\{\frac{\partial}{\partial x_i}\Phi_{1-t}f(B_t)\}dB_t^i \\ &+ \sum_{i,j=1}^{d}\int_{0\le s<t\le 1}\{\frac{\partial^2}{\partial x_i\partial x_j}\Phi_{1-s}f(B_s)\}dB_s^j dB_t^i. \qquad (6.5)\end{aligned}$$

A formula similar to (6.5) also holds for g. Now for $1 \le i \le d$,

$$\begin{aligned}&\mathbb{E}\{\frac{\partial}{\partial x_i}\Phi_{1-t}f(B_t)\} = \Phi_t[\frac{\partial}{\partial x_i}\Phi_{1-t}]f(0) \\ &= [\frac{\partial}{\partial x_i}P_1 f](0) = \int_{\Re^d}\frac{\partial}{\partial x_i}\phi(x)f(x)dx \\ &= \int_{\Re^d}\phi(x)^{-1}\frac{\partial}{\partial x_i}\phi(x)\mu(dx) = \int_{\Re^d}H_1(x_i)f(x)\mu(dx).\end{aligned}$$

So the second term of (6.5) is f_1 in the expansion of f, which is zero in the case f is degenerate. By the orthogonality of the expansion (6.5), if one of f and g is degenerate, then

$$\begin{aligned}&\int_{\Re^d} f(x)g(x)\mu(dx) = \mathbb{E}f(B_1)g(B_1) \\ &\quad = \mathbb{E}f(B_1)\mathbb{E}g(B_1) + \sum_{1\le i,j\le d}\int_{0\le s<t\le 1}\mathbb{E}\{\frac{\partial^2}{\partial x_i\partial x_j}\Phi_{1-s}f(B_s) \\ &\qquad\qquad \frac{\partial^2}{\partial x_i\partial x_j}\Phi_{1-s}g(B_s)\}dsdt \\ &\quad = \int_{\Re^d} f(x)\mu(dx)\int_{\Re^d} g(x)\mu(dx) \\ &\qquad + \int_{0\le s<t\le 1}\mathbb{E}\mathrm{Tr}\{\nabla^2\Phi_{1-s}f(B_s)\nabla^2\Phi_{1-s}g(B_s)\}dsdt,\end{aligned}$$

where Tr means the trace of a matrix. By lemma 6.2, when f and g are convex, $\Phi_{1-t}f$ and $\Phi_{1-t}g$ are also convex and smooth. So, for any $x \in \Re^d$, $\nabla^2\Phi_{1-s}f(x)$ and $\nabla^2\Phi_{1-s}f(x)$ are positive definitive. Thus, $\mathrm{Tr}\{\nabla^2\Phi_{1-s}f(B_s)\nabla^2\Phi_{1-s}g(B_s)\} \geq 0$ almost everywere. Combining this with the above last equality we conclude the proof of theorem 6.1. ∎

Remark The formula (6.5) is a mulit-dimensional version of (2.9) when $n = 1$.

Since an even function is always degenerate, we have

Corollary 6.3: If $f, g \in L^2(\Re^d; d\mu)$ are convex and one of them is even, i.e. $f(-x) = f(x)$, then (6.1) holds.

Remark: The inequalities (3.3) and (6.1) are of the same type but under different assumptions. In the Gaussian case, the convexity type condition is studied by L. D. Pitt [Pi] in the case $d = 2$ and f, g are indicator functions of convex, balanced set. C. Borell [Bo] and H. Sugita [Su] have extended this result to any finite dimension but for more restrictive sets. This is also the reason that we have been worked on any dimension in this last section. we refer to [GJ] and [FFS] for the correlation inequality for non-Gaussian measures.

We should mention that our result does not imply the result of L.Pitt. Since the indicator function f of a convex set is quasiconcave (defined by $f(\alpha x + (1-\lambda)y) \geq \min(f(x), f(y))$), a result such as theorem 6.1 but replacing "convex" by "quasiconcave" (or equivalently "quasiconvex") is very strong and implies the result of Pitt. It is conjectured that for any $A, B \in \Re^d$ $(d \geq 3)$, convex and symmetric,

$$\mu(A \cap B) \geq \mu(A)\mu(B).$$

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