Regularity of Stochastic Burgers'-Type Equations

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# Abstract

In classical partial differential equations (PDEs), it is well known that the solution to Burgers' equation in one spatial dimension with positive viscosity can be solved by the so called Hopf-Cole transformation, which linearizes the PDE. In particular, this converts Burgers' equation to the linear heat equation, which can be solved explicitly. On the other hand, the Feynman-Kac formula is a tool that can be used to solve the heat equation probabilistically. An interesting and perhaps surprising result which we prove is that one can still make sense of these approaches to Burgers' equation in the presence of space-time white noise, which is very rough. After proving that a suitable Feynman-Kac representation solves stochastic Burgers' equation under a Hopf-Cole transformation, we study some regularity properties of this solution. In particular, we prove moment estimates and Hölder continuity, which can be thought of as how "big" the solution gets in time and space, and how "rough" this solution can be. From this, we then obtain sub-exponential moments and bounds on the tails of the probability distribution for the solution. Prior to this work, no results about any kinds of moment estimates or tails of distributions for stochastic Burgers'-type equations had been established. Furthermore, only one publication on Burgers' equation ([3]) contains a discussion of Hölder regularity[1].

[1]The reference does contain Hölder regularity, but the paper has some flaws.

Given the solution to a stochastic partial differential equation (SPDE), it is natural to ask whether this stochastic process has a well-behaved probability law. For example, does the solution have a smooth probability density function or just an absolutely continuous one? Using some powerful tools from Malliavin calculus, we answer this question for stochastic Burgers' equation with our Hopf-Cole solution.

Finally, we study regularity of the probability law of the solution to a more general class of semilinear SPDEs which contain Burgers' equation as an example. These results take a less tangible approach since there is no explicit representation for solutions to these equations. However, as we will see, there are some clever techniques and interesting results that can be used to establish such properties. For example, we prove a comparison theorem for this class of SPDEs which, interestingly enough, will be instrumental in obtaining regularity of the probability density function of the solution at fixed points in time and space.

The projects in this thesis are joint work of the author and David Nualart. The second chapter of this thesis corresponds to work done by the author and David Nualart in [19].

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# Contents

# Chapter 1

# Introduction

## 1.1 Introduction

It is well known that partial differential equations (PDEs) provide mathematical descriptions of many natural phenomena. However, these models exist in a vacuum in the sense that nature can be quite noisy or unpredictable. As such, stochastic PDEs provide a mathematical framework for inserting "noise" into a system.

The study of Burgers' equation, given by

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}\frac{\partial}{\partial x}u(t,x)^2,$$

dates back to the middle of the 20th century and provides a simplified model for turbulence and fluid mechanics ([4]). Around 1950, Hopf and Cole introduced a method, known as the Hopf-Cole transformation, to solve this equation ([15]). Naturally, turbulence is not a completely deterministic process, so it makes sense to insert noise into this system. It is common practice to use *space-time white noise* in this sort of situation to observe how randomness can affect the behavior of solutions to these space-time-dependent models.

In classical PDEs, the solution to Burgers' equation in one spatial dimension with positive viscosity can be solved by the so called Hopf-Cole transformation, which linearizes the PDE. In particular, this transformation converts Burgers' equation to the linear heat equation, which can be solved explicitly. Furthermore, the Feynman-Kac formula is a tool that can be used to solve the heat equation probabilistically (see [16]). An interesting and perhaps surprising result which we prove is that one can still make sense of these approaches to Burgers' equation on the real line in the presence of space-time white noise, which is very rough.

We will also consider the more general class of semilinear SPDEs

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(t,x,u(t,x)) + \frac{\partial}{\partial x}g(t,x,u(t,x)) + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x},$$

again driven by space-time white noise on the real line. The typical conditions we impose are that $f$ and $\sigma$ grow linearly with the solution, $u$, and $g$ grows quadratically. So, for example, if $f = 0$ and $g = -\frac{1}{2}u^2$, we recover Burgers' equation.

There are many papers which study the stochastic Burgers' equation on the spatial domain $[0,1]$. In this paragraph, we list a few such publications. For example, the authors of [18] give an explicit representation of the solution to Burgers' equation with multiplicative space-time white noise by defining a process via a Feynman-Kac representation such that its Hopf-Cole transformation solves Burgers' equation. Using this representation, the authors prove the existence of a smooth density function for the solution to Burgers' equation using Malliavin calculus. This paper is the main inspiration for our work. Other published results include existence, uniqueness, and a comparison theorem for a more general class of semilinear stochastic equations which contains Burgers' equation ([12]), existence to a Burgers' equation with random initial conditions using some technical Malliavin calculus tools ([21]), rates of convergence of numerical schemes for Burgers' equation with space-time white noise ([1]), Feller properties of an appropriate semigroup and the existence of an invariant measure for Burgers' equation perturbed by correlated multiplicative noise ([8]), and existence and uniqueness for a more general class of stochastic PDEs with polynomial nonlinearities ([14]).

On the other hand, very few results regarding stochastic Burgers' equation on the real line have been obtained. To our knowledge, the only such papers are the following. In [3], the authors give a Hopf-Cole solution to Burgers' equation on $\mathbb{R}$ with $\sigma \equiv 1$ in a similar way as in [18]. Well-posedness for Burgers' type equations is studied in [13], [17], and [25].

The first aim of this thesis is to construct a solution to the stochastic Burgers' equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}\frac{\partial}{\partial x}u(t,x)^2 + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x},$$

similar to what is done in [3] and [18], by defining and transforming a process with a Feynman-Kac representation. We then obtain uniqueness for free from [13]. Then, we prove Hölder regularity, two types of moment estimates, and an upper bound on the tails of the probability distribution of the solution. Despite the interest of many who study SPDEs, the only situation in which Hölder regularity for Burgers' type equations has been studied is in the case of additive noise on $\mathbb{R}$ ([3]). Furthermore, to our knowledge, estimates on moments and tails of distributions have not been established for any Burgers' type equation.

Since solutions to SPDEs at fixed parameter values are random variables, it is natural to investigate the probability law of such a random variable. In fact, this is a topic in SPDEs which has garnered much attention over the last twenty years. In particular, many have studied is the existence and regularity of density functions for solutions to SPDEs using the tools of Malliavin calculus, a branch of mathematics referred to as a stochastic calculus of variations. We prove some such regularity results in chapter 3.

## 1.2 Setup

Here, we provide some of the framework that will be common throughout this document. We start by fixing a complete[1] probability space $\mathbb{X} = (\Omega, \mathcal{F}, P)$. We follow standard practice and suppress the dependence on the $\omega \in \Omega$ parameter (the "random" component). For example, instead of denoting a time-evolving stochastic process by $B(t,\omega)$, we simply write $B(t)$ or $B_t$.

With this in mind, let $W = \{W(t,x), t \in \mathbb{R}_+, x \in \mathbb{R}\}$ be a zero-mean Gaussian random field defined on $\mathbb{X}$, with covariance given by

$$E[W(s,x)W(t,y)] = (s \wedge t)(|x| \wedge |y|)\mathbf{1}_{[0,\infty)}(xy)$$

for $s,t \geq 0$, $x,y \in \mathbb{R}$. In other words, $W$ is a *Brownian sheet* on $\mathbb{R}^2$. For any $t \geq 0$, we denote by $\mathcal{F}_t$ the $\sigma$-field generated by the random variables $\{W(s,x), s \in [0,t], x \in \mathbb{R}\}$ and the sets of probability zero[2]. We use the notation $E(\cdot)$ to represent expectation with respect to $W$, and denote its corresponding norm by $\|\cdot\|_p = E(|\cdot|^p)^{1/p}$.

Space-time white noise is the formal space-time derivative of the Brownian sheet, $\dot{W} = \partial_{t,x} W$. However, the Brownian sheet is almost surely Hölder continuous, in time and in space, of order $\alpha$, only for $\alpha < 1/2$. So, the Brownian sheet has no classical

[1]We don't make a fuss about the completeness requirement, but remark that it is a necessary technicality for many results from stochastic calculus to hold, such as the well-definedness of our integrals.

[2]Another technicality which is necessary but does not appear explicitly in our presentation.

derivatives. Thus, a first order derivative of the Brownian sheet must be interpreted in the distributional sense. Hence, some authors present space-time white noise as a generalized Gaussian process with covariance

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y),$$

where $\delta$ is the Dirac delta function. Since the delta function lives in the Sobolev space $H^s$ only when $s < -1/2$, this follows the intuition of a derivative as an operation that reduces the order of regularity by one.

Another technical challenge with the Brownian sheet, as with Brownian motion, is that it has unbounded variation on every interval, which means the classical Lebesgue-Stieltjes integration theory does not apply. To construct stochastic integrals with respect to $W$, one uses probabilistic tools in a similar way as the Itô integral. Such details are given in John Walsh's seminal work on SPDEs [26]. As such, this integration theory is commonly referred to as the Walsh theory of stochastic integration. The Walsh integral has some generalizations, but we only need it in a real-valued context. Furthermore, a major luxury is that the study of solutions of SPDEs in a Walsh-type framework turns out to be equivalent to the Hilbert space valued solutions à la Da Prato and Zabzcyck in many reasonable situations[3], and the choice is mainly for mathematical convenience.

As with the Itô integral, the Walsh integral is a martingale and enjoys the following $L^2$ isometry

$$E\left[\left(\int_0^t \int_{\mathbb{R}} f(s,y)W(ds,dy)\right)^2\right] = E\int_0^t \int_{\mathbb{R}} f(s,y)^2 dyds.$$

[3]See [9] for the theory and [10] for a thorough presentation of this equivalence.

Note that $\iint f\ W(ds,dy)$ denotes the Walsh integral of $f$, integrated with respect to $y$ then $s$. Although this ordering of the differentials is seemingly unconventional for those who do not study SPDEs, it is common practice to write it this way.

One of the most useful tools when dealing with Walsh integrals is the Burkholder-Davis-Gundy (BDG) inequality.

**Lemma 1.2.1.** *Let $M_t$ be a continuous local martingale with $M_0 = 0$ a.s. Then, for any $p \geq 2$ and any finite stopping time $\tau$, we have*

$$E\Big[\Big(\sup_{t\leq\tau} M_t\Big)^p\Big] \leq (4p)^{p/2} E\Big[\langle M\rangle_\tau^{p/2}\Big],$$

*where $\langle M\rangle_\cdot$ is the quadratic variation process of $M$.*

Note that this result is equivalent to to following inequality

$$\Big\|\sup_{t\leq\tau} M_t\Big\|_p \leq (4p)^{1/2}\Big\|\langle M\rangle_\tau\Big\|_{p/2}^{1/2}.$$

We remark that the constant in the BDG inequality above is sharp. The use of this inequality is that Walsh integrals are local martingales with quadratic variation

$$\Big\langle \int_0^\cdot \int_{\mathbb{R}} f(s,y)W(ds,dy)\Big\rangle_t = \int_0^t \int_{\mathbb{R}} f(s,y)^2 dyds.$$

Hence, we can apply the BDG inequality to control moments of Walsh integrals.

To study stochastic Burgers' equation with space-time white noise

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}\frac{\partial}{\partial x}u(t,x)^2 + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t\partial x}, \tag{1.1}$$

we must interpret this as an integral equation since $W$ has no classical derivatives[4]. As with classical PDEs, we define weaker notions of solutions. For example, we say that $u$ is a *weak solution* to (1.1) if for any test function $\phi \in C_c(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}} u(t,x)\phi(x)dx = &\int_{\mathbb{R}} u_0(x)\phi(x)dx + \int_0^t \int_{\mathbb{R}} u(s,x)\phi''(x)dxds \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}} u(s,x)^2\phi'(x)dxds \\ &+ \int_0^t \int_{\mathbb{R}} \sigma(s,x,u(s,x))\phi(x)W(ds,dx),\end{aligned}$$

almost surely for all $t \in [0,T]$, where the last term is a Walsh integral. On the other hand, we say $u$ is a *mild solution* to (1.1) if

$$\begin{aligned}u(t,x) = &\int_{\mathbb{R}} G(t,x-y)u_0(y)dy + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y}G(t-s,x-y)u(s,y)^2dyds \\ &+ \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\sigma_s(y)W(ds,dy),\end{aligned}$$

where $\sigma_t(x) \equiv \sigma(t,x,u(t,x))$ is used for shorthand, $G$ is the heat kernel

$$G(t,x) = (4\pi t)^{-1/2}e^{-x^2/4t}.$$

This should look familiar to those with a PDE background. If the Brownian sheet above is replaced by a sigma-finite measure, this is what's classically known as *Duhamel's principle* in PDE literature. With SPDEs, Duhamel's principle works in the same way, once the integrals are well-defined. This thesis primarily focuses on mild solutions to SPDEs.

[4]We could try to proceed in a distributional context, but the $u^2$ term immediately poses obvious ambiguity.

## 1.3 Overview of Results

This work is organized as follows. In Chapter 2, we define a process via a Feynman-Kac formula in Section 2.2, then show that its Hopf-Cole transformation solves the stochastic Burgers' equation in one spatial dimension in Section 2.3. With this, we establish regularity properties of the solution to Burgers' equation in Section 2.4.

In Section 3.1, we review some basics of Malliavin Calculus and the relevant tools for establishing regularity of density functions for random variables. Then, in Section 3.2, we prove that the solution to stochastic Burgers' equation has a density function which is smooth. Finally, in Section 3.3, we study more general equations which contain Burgers' equation, and prove some results regarding regularity of densities for solutions.

# Chapter 2

# Stochastic Burgers' Equation

## 2.1 Preliminaries

We are concerned with the following version of Burgers' equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}\frac{\partial}{\partial x}u(t,x)^2 + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}$$

indexed by $(t,x) \in [0,T] \times \mathbb{R}$, given a nonrandom initial condition $u_0$ and a Brownian sheet $W$. To study this equation rigorously, we understand the above in its *mild form*; that is, as an integral equation:

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} G(t,x-y)u_0(y)dy + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s,x-y)u(s,y)^2 dyds \\ &+ \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\sigma_s(y)W(ds,dy), \end{aligned} \tag{2.1}$$

where $\sigma_t(x) \equiv \sigma(t,x,u(t,x))$ is used for shorthand, $G$ is the heat kernel

$$G(t,x) = (4\pi t)^{-1/2} e^{-x^2/4t},$$

and the stochastic integral is understood in the Walsh sense.

This chapter is organized as follows. First, we define a process, $\psi$, via a kind of Feynman-Kac representation. Then, we establish several properties of $\psi$, such as moment bounds, Hölder regularity, and differentiability. Next, we show that the Hopf-Cole transformation of $\psi$, which (formally at the moment) is

$$u(t,x) = -2\frac{\partial}{\partial x}\log\psi(t,x),$$

solves (2.1). Appealing to the uniqueness result in [13], our solution is unique. Lastly, we obtain Hölder regularity and an upper bound on moments of the solution to Burgers' equation using properties of the process $\psi$.

Throughout much of the project, we follow similar steps as in [18], but have to adjust almost all of the arguments to handle the challenges posed by an unbounded domain. As such, due to difficulties with integrability, many of our assumptions differ from those in [18], though they are consistent with [13].

Throughout the chapter we assume the following conditions:

(A1) The initial condition $u_0$ is a deterministic, continuous, and bounded function such that $u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

(A2) $\sigma : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a Borel function satisfying the following Lipschitz and growth properties

$$|\sigma(t,x,r) - \sigma(t,x,v)| \leq L|r-v| \tag{2.2}$$

$$|\sigma(t,x,r)| \leq f(x) \tag{2.3}$$

for all $t \geq 0$, $x,r,v \in \mathbb{R}$ and for some constant $L > 0$ and some non-negative function $f \in L^2(\mathbb{R}) \cap L^q(\mathbb{R})$, where $q > 2$.

Under these conditions, it is proved by Gyöngy and Nualart in [13] that there exists a unique $L^2(\mathbb{R})$-valued $\mathcal{F}_t$-adapted continuous stochastic process $u = \{u(t), t \geq 0\}$, which satisfies the integral equation (2.1). Furthermore, the process $u$ has a continuous version in $(t, x)$.

Before our discussion of the Feynman-Kac representation, we prove a technical lemma regarding regularity of the heat kernel $G(t,x) = (4\pi t)^{-1/2} e^{-x^2/4t}$ that will be used several times.

**Lemma 2.1.1.** *Let $\theta_1 > 0$, $\theta_2 \geq 0$ and $\beta > 0$ be such that*

$$\beta(\theta_1 - \theta_2 - 1) < 2 < \beta(3\theta_1 - \theta_2 - 1). \tag{2.4}$$

*Then, for any $0 < t_1 < t_2$, we have*

$$\int_0^{t_1} \left( \int_{\mathbb{R}} |G(t_2 - s, x) - G(t_1 - s, x)|^{\theta_1} |x|^{\theta_2} dx \right)^{\beta} ds \leq C (t_2 - t_1)^{1 - \beta(\theta_1 - \theta_2 - 1)/2},$$

*for some constant $C$ depending on $\theta_1$, $\theta_2$ and $\beta$.*

*Proof.* Set $\tau = t_2 - t_1$. Making the change of variables $x = \sqrt{s} y$ and $s = \tau/\sigma$, yields

$$\begin{aligned}
&\int_0^{t_1} \left( \int_{\mathbb{R}} |G(t_2 - s, x) - G(t_1 - s, x)|^{\theta_1} |x|^{\theta_2} dx \right)^{\beta} ds \\
&\quad = \int_0^{t_1} \left( \int_{\mathbb{R}} \left| \frac{1}{\sqrt{4\pi(\tau + s)}} e^{-\frac{x^2}{4(\tau+s)}} - \frac{1}{\sqrt{4\pi s}} e^{-\frac{x^2}{4s}} \right|^{\theta_1} |x|^{\theta_2} dx \right)^{\beta} ds \\
&\quad \leq C (t_2 - t_1)^{1 - \beta(\theta_1 - \theta_2 - 1)/2} \int_0^{\infty} \sigma^{-2 + \beta(\theta_1 - \theta_2 - 1)/2} \\
&\qquad \times \left( \int_{\mathbb{R}} \left| \frac{1}{\sqrt{\sigma + 1}} e^{-\frac{y^2}{4(\sigma+1)}} - e^{-\frac{y^2}{4}} \right|^{\theta_1} |y|^{\theta_2} dy \right)^{\beta} d\sigma.
\end{aligned}$$

Then, condition (2.4) implies that the above integral in $d\sigma$ is finite, and we get the desired estimate. $\square$

Throughout the chapter we will denote by $C$ a generic constant that might depend on $\sigma$, $f$, $u_0$, $T$ and the exponent $p$ we are considering. The value of this constant may be different from line to line. However, we will specify dependence where we feel it may be relevant.

## 2.2 Feynman-Kac Representation

We now define a process via a kind of Feynman-Kac formula that will be the main focus of this chapter. Given $u_0$, set

$$\psi_0(x) := \exp\left\{-\frac{1}{2}\int_0^x u_0(y)dy\right\}.$$

Let $\beta = \{\beta_s, s \in [0,t]\}$ be a backward Brownian motion (BWBM) that is independent of the Brownian sheet $W$, starting at $x \in \mathbb{R}$ at time $t$ and with variance $2(t-s)$. We use the notation $\mathbb{E}^{\beta}_{x,t}$ to denote the expectation with respect to the law of the BWBM. Let $u$ be the mild solution to Burgers' equation. That is, $u$ satisfies (2.1). We will make use of the notation $\sigma_s(y) := \sigma(s, y, u(s,y))$. Set

$$M^{\beta}_t := \int_0^t \int_{\mathbb{R}} \sigma_s(y)\mathbf{1}_{[0,\beta_s]}(y)W(ds, dy),$$

with the convention that $\mathbf{1}_{[0,\beta_s]}(y)$ is $-\mathbf{1}_{[\beta_s,0]}(y)$ if the BWBM is negative at time $s$. Observe that this stochastic integral is a well-defined martingale due to the square-integrability assumption (2.3) on $\sigma$. With the above notation in mind, define the two-parameter stochastic process $\psi$ by

$$\psi(t,x) := \mathbb{E}^{\beta}_{x,t}\left[\psi_0(\beta_0)e^{-\frac{1}{2}M^{\beta}_t}\right]. \tag{2.5}$$

We first establish some estimates of moments of the process $\psi$, then show that it satisfies a certain integral equation.

**Proposition 2.2.1.** For all $t \geq 0$, $x \in \mathbb{R}$, and integers $p \geq 2$, we have moment estimates of the form

$$\|\psi(t,x)\|_p \leq \exp\left(\frac{1}{4}\|f\|^2_{L^2(\mathbb{R})}tp + \frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}\right). \tag{2.6}$$

*Proof.* Let $\vec{\beta} = \{\beta^i\}_{i=1}^p$ be $p$ independent backward Brownian motions on $[0,t]$ starting at $x \in \mathbb{R}$ at time $t$, with variance $2(t-s)$. By independence and Fubini's theorem, we have

$$\begin{aligned}
\|\psi(t,x)\|_p^p = \mathbb{E}\big(|\psi(t,x)|^p\big) &= \mathbb{E}\left[\prod_{i=1}^p \mathbb{E}^{\beta^i}_{x,t}\left(\psi_0(\beta^i_0)e^{-\frac{1}{2}M^{\beta^i}_t}\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}^{\vec{\beta}}_{x,t}\left(\prod_{i=1}^p \psi_0(\beta^i_0)e^{-\frac{1}{2}M^{\beta^i}_t}\right)\right] \\
&= \mathbb{E}^{\vec{\beta}}_{x,t}\left[\left(\prod_{i=1}^p \psi_0(\beta^i_0)\right)\mathbb{E}\left(\exp\Big\{-\frac{1}{2}\sum_{j=1}^p M^{\beta^j}_t\Big\}\right)\right].
\end{aligned}$$

Now, by the multivariate Itô formula,

$$\begin{aligned}
e^{-\frac{1}{2}\sum_{j=1}^p M^{\beta^j}_t} = 1 &- \frac{1}{2}\int_0^t \sum_{i=1}^p e^{-\frac{1}{2}\sum_{j=1}^p M^{\beta^j}_s}\, dM^{\beta^i}_s \\
&+ \int_0^t \sum_{i,j=1}^p \frac{1}{8} e^{-\frac{1}{2}\sum_{k=1}^p M^{\beta^k}_s}\, d\langle M^{\beta^i}, M^{\beta^j}\rangle_s.
\end{aligned}$$

Since the quadratic covariation of these martingales is

$$\langle M^{\beta^i}, M^{\beta^j}\rangle_t = \int_0^t ds \int_{\mathbb{R}} dy\, \sigma_s^2(y)\mathbf{1}_{[0,\beta_s^i]}(y)\mathbf{1}_{[0,\beta_s^j]}(y),$$

taking the expectation of the above Itô expansion yields

$$\begin{aligned}\mathbb{E}\Big(e^{-\frac{1}{2}\sum_{j=1}^p M_t^{\beta^j}}\Big) = 1+\frac{1}{8}\int_0^t \mathbb{E}\Big(e^{-\frac{1}{2}\sum_{k=1}^p M_s^{\beta^k}} \\ \times \sum_{i,j=1}^p \mathbf{1}_{\beta_s^i\beta_s^j>0}\int_0^{|\beta^i|\wedge|\beta^j|}[\sigma_s^2(y)+\sigma_s^2(-y)]dy\Big)ds \\ \le 1+\frac{p^2}{4}\|f\|_{L^2(\mathbb{R})}^2\int_0^t \mathbb{E}\Big(e^{-\frac{1}{2}\sum_{j=1}^p M_s^{\beta^j}}\Big)ds.\end{aligned}$$

Recall a version of Gronwall's lemma which states that if a function $g$ satisfies $g(t) \le a(t)+\int_0^t b(s)g(s)ds$, where $a$ is non-decreasing and $b$ is non-negative, then $g$ satisfies $g(t) \le a(t)e^{\int_0^t b(s)ds}$. Hence, we have

$$\mathbb{E}\Big(e^{-\frac{1}{2}\sum_{j=1}^p M_t^{\beta^j}}\Big) \le \exp\Big(\frac{\|f\|_{L^2(\mathbb{R})}^2}{4}tp^2\Big).$$

Therefore,

$$\|\psi(t,x)\|_p^p = \mathbb{E}_{x,t}^{\vec{\beta}}\left[\left(\prod_{i=1}^p \psi_0(\beta_0^i)\right)\mathbb{E}\left(\exp\Big\{-\frac{1}{2}\sum_{j=1}^p M_t^{\beta^j}\Big\}\right)\right] \le a^p e^{btp^2},$$

where $a = e^{\frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}}$ and $b = \frac{1}{4}\|f\|_{L^2(\mathbb{R})}^2$. □

**Remark 2.2.2.** Using Jensen's inequality we can show, in the same way as before, that for all integers $p \ge 2$,

$$\|\psi(t,x)^{-1}\|_p^p \le \exp\left(\frac{1}{4}\|f\|_{L^2(\mathbb{R})}^2 tp^2 + \frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}p\right). \tag{2.7}$$

In fact,

$$\psi(t,x)^{-1} \leq \mathbb{E}^{\beta}_{x,t}\left[\exp\left\{\frac{1}{2}\int_0^{\beta_0} u_0(y)dy + \frac{1}{2}M_t^{\beta}\right\}\right].$$

Proposition 2.2.1 implies that for any $T > 0$

$$M_{p,T} := \sup_{t\in[0,T],x\in\mathbb{R}} \|\psi(t,x)\|_p < \infty \tag{2.8}$$

and

$$\sup_{t\in[0,T],x\in\mathbb{R}} \|\psi(t,x)^{-1}\|_p < \infty \tag{2.9}$$

for all real numbers $p \geq 2$.

Next, we show that $\psi$ satisfies a particular integral equation.

**Proposition 2.2.3.** Let $\psi$ be the process defined in (2.5) and let $G(t,x)$ be the heat kernel as before. Then, for $t \geq 0$, $x \in \mathbb{R}$, $\psi(t,x)$ satisfies

$$\begin{aligned}\psi(t,x) = &\int_{\mathbb{R}} G(t,x-y)\psi_0(y)dy \\ &- \frac{1}{2}\int_0^t \int_S \operatorname{sign}(y) G(t-s,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy) \\ &+ \frac{1}{8}\int_0^t \int_S G(t-s,x-z)\psi(s,z)\sigma_s^2(y)dzdyds,\end{aligned} \tag{2.10}$$

where

$$S := \big\{(y,z) \in \mathbb{R}^2 : |z| \geq |y|, \text{ and } \ yz \geq 0\big\}.$$

*Proof.* The proof of this result follows from the same arguments as in [18]. We briefly explain the main idea. First, observe that $\beta_0$ satisfies $\mathbb{E}^{\beta}_{x,t}\big(\psi_0(\beta_0)\big) = \int_{\mathbb{R}} G(t,x-y)\psi_0(y)dy$ since $y \mapsto G(t,x-y)$ is the density of $\beta_0$. Now, apply Itô's formula to get

$$e^{-\frac{1}{2}M_t^\beta} = 1 - \frac{1}{2}\int_0^t \int_{\mathbb{R}} e^{-\frac{1}{2}M_s^\beta} \sigma_s(y)\mathbf{1}_{[0,\beta_s]}(y)W(ds,dy)$$
$$+ \frac{1}{8}\int_0^t \int_{\mathbb{R}} e^{-\frac{1}{2}M_s^\beta} \sigma_s(y)^2|\mathbf{1}_{[0,\beta_s]}(y)|dyds.$$

Multiply by $\psi_0(\beta_0)$ and take the expectation with respect to the BWBM to see that

$$\begin{aligned}\psi(t,x) = &\int_{\mathbb{R}} G(t,x-y)\psi_0(y)dy \\ &- \frac{1}{2}\int_0^t \int_{\mathbb{R}} \sigma_s(y)\mathbb{E}_{x,t}^\beta\Big(\psi_0(\beta_0)e^{-\frac{1}{2}M_s^\beta}\mathbf{1}_{[0,\beta_s]}(y)\Big)W(ds,dy) \\ &+ \frac{1}{8}\int_0^t \int_{\mathbb{R}} \sigma_s(y)^2\,\mathbb{E}_{x,t}^\beta\Big(\psi_0(\beta_0)e^{-\frac{1}{2}M_s^\beta}|\mathbf{1}_{[0,\beta_s]}(y)|\Big)dyds.\end{aligned}$$

Finally, apply the Markov property to get

$$\begin{aligned}\mathbb{E}_{x,t}^\beta\Big(\psi_0(\beta_0)e^{-\frac{1}{2}M_s^\beta}\mathbf{1}_{[0,\beta_s]}(y)\Big) &= \mathbb{E}_{x,t}^\beta\Big[\mathbb{E}\Big(\psi_0(\beta_0)e^{-\frac{1}{2}M_s^\beta}\mathbf{1}_{[0,\beta_s]}(y)\Big|\beta_r, s\le r\le t\Big)\Big] \\ &= \mathbb{E}_{x,t}^\beta\Big[\mathbf{1}_{[0,\beta_s]}(y)\mathbb{E}\Big(\psi_0(\beta_0)e^{-\frac{1}{2}M_s^\beta}\Big|\beta_s\Big)\Big] \\ &= \int_0^\infty G(t-s,x-z)\mathbb{E}_{z,s}^\beta\Big(\psi_0(\beta_0)e^{-\frac{1}{2}M_s^\beta}\Big)\mathbf{1}_{[0,z]}(y)dz \\ &- \int_{-\infty}^0 G(t-s,x-z)\mathbb{E}_{z,s}^\beta\Big(\psi_0(\beta_0)e^{-\frac{1}{2}M_s^\beta}\Big)\mathbf{1}_{[z,0]}(y)dz \\ &= \begin{cases} \displaystyle\int_y^\infty G(t-s,x-z)\psi(s,z)dz & \text{if } y\ge 0 \\ \displaystyle-\int_{-\infty}^y G(t-s,x-z)\psi(s,z)dz & \text{if } y<0.\end{cases}\end{aligned}$$

Similarly,

$$\mathbb{E}^{\beta}_{x,t}\Big(\psi_0(\beta_0)e^{-\frac{1}{2}M^{\beta}_s}|\mathbf{1}_{[0,\beta_s]}(y)|\Big) = \begin{cases} \displaystyle\int_y^{\infty} G(t-s,x-z)\psi(s,z)dz & \text{if } y \geq 0 \\ \displaystyle\int_{-\infty}^{y} G(t-s,x-z)\psi(s,z)dz & \text{if } y < 0. \end{cases}$$

Hence, we have the desired result. $\square$

Next, we establish a Hölder regularity property for $\psi$.

**Proposition 2.2.4.** For $p \geq 2$ and $T > 0$, there exists some constant $C$, depending on $p$, $T$, $\|u_0\|_\infty$, $\|u_0\|_{L^1(\mathbb{R})}$, and $\|f\|_{L^2(\mathbb{R})}$, such that for all $s,t \in [0,T]$, and $x,y \in \mathbb{R}$,

$$\|\psi(t,x) - \psi(s,y)\|_p \leq C\left(|t-s|^{1/2} + |x-y|^{1/2}\right).$$

*Proof.* First we prove the Hölder continuity in the space variable. Let $x_1$ and $x_2$ be such that $|x_1 - x_2| = \delta$. Because $\|\psi(t,x)\|_p$ is uniformly bounded on $[0,T]\times\mathbb{R}$, we can assume that $\delta \leq 1$. We have

$$\begin{aligned}
&\psi(t,x_1) - \psi(t,x_2) \\
&= \int_{\mathbb{R}} [G(t,x_1-y) - G(t,x_2-y)]\psi_0(y)dy \\
&\quad -\frac{1}{2}\int_0^t \int_S \mathrm{sign}(y)[G(t-s,x_1-z) - G(t-s,x_2-z)]\psi(s,z)\sigma_s(y)dzW(ds,dy) \\
&\quad +\frac{1}{8}\int_0^t \int_S [G(t-s,x_1-z) - G(t-s,x_2-z)]\psi(s,z)\sigma_s^2(y)dzdyds \\
&=: I_1(x_1,x_2) - \frac{1}{2}I_2(x_1,x_2) + \frac{1}{8}I_3(x_1,x_2).
\end{aligned}$$

We make a change of variables to get

$$|I_1(x_1,x_2)| \leq \int_{\mathbb{R}} G(t,u)|\psi_0(x_1-u) - \psi_0(x_2-u)|du.$$

By Hypothesis (A1) the function $\psi_0$ has a bounded derivative:

$$|\psi'(x)| \leq \|u_0\|_\infty e^{\frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}}.$$

Therefore, it is Lipschitz and we obtain

$$|I_1(x_1,x_2)| \leq C|x_1 - x_2| = C\delta. \tag{2.11}$$

Consider the decomposition

$$I_2(x_1,x_2) = I_{2,+}(x_1,x_2) + I_{2,-}(x_1,x_2),$$

where

$$I_{2,+}(x_1,x_2) = \int_0^t \int_0^\infty \sigma_s(y) \int_y^\infty [G(t-s,x_1-z) - G(t-s,x_2-z)]\psi(s,z)dzW(ds,dy)$$

and

$$I_{2,-}(x_1,x_2) = -\int_0^t \int_{-\infty}^0 \sigma_s(y) \int_{-\infty}^y [G(t-s,x_1-z) - G(t-s,x_2-z)]\psi(s,z)dzW(ds,dy).$$

Applying Burkholder's and Minkowski's inequalities, we get

$$\begin{aligned}\|I_{2,+}(x_1,x_2)\|_p \le c_p \Big\| \int_0^t \int_0^\infty \sigma_s^2(y) \Big( \int_y^\infty \psi(s,z) \\ \times \big[G(t-s,x_1-z) - G(t-s,x_2-z)\big] dz \Big)^2 dyds \Big\|_{p/2}^{1/2} \\ \le c_p \Big( \int_0^t \int_0^\infty f^2(y) \Big\| \int_y^\infty \psi(s,z) \\ \times \big[G(t-s,x_1-z) - G(t-s,x_2-z)\big] dz \Big\|_p^2 dyds \Big)^{1/2}. \end{aligned} \tag{2.12}$$

Making a change of variables we can write

$$\begin{aligned}&\int_y^\infty \psi(s,z)\big[G(t-s,x_1-z) - G(t-s,x_2-z)\big] dz \\ &= \int_{-\infty}^{x_1-y} \psi(s,x_1-u) G(t-s,u) du - \int_{-\infty}^{x_2-y} \psi(s,x_2-u) G(t-s,u) du \\ &= \int_{-\infty}^{x_1-y} [\psi(s,x_1-u) - \psi(x_2-u)] G(t-s,u) du + \int_{x_2-y}^{x_1-y} \psi(s,x_2-u) G(t-s,u) du.\end{aligned}$$

This leads to the estimate

$$\begin{aligned}\Big\| \int_y^\infty \psi(s,z)\big[G(t-s,x_1-z) - G(t-s,x_2-z)\big] dz \Big\|_p \\ \le \int_{\mathbb{R}} \|\psi(s,x_1-u) - \psi(s,x_2-u)\|_p G(t-s,u) du \\ + \int_{x_2-y}^{x_1-y} \|\psi(s,x_2-u)\|_p G(t-s,u) du.\end{aligned}$$

Let $M_{p,T}$ be the constant introduced in (2.8) and set

$$V_s := \sup_{|x-y|=\delta} \|\psi(s,x) - \psi(s,y)\|_p.$$

Then, by Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\left\| \int_y^\infty \psi(s,z)\big[G(t-s,x_1-z)-G(t-s,x_2-z)\big]\,dz \right\|_p \\
&\qquad\qquad\qquad \le V_s + M_{p,T}|x_1-x_2|^{1/2}\left(\int_{\mathbb{R}} G^2(t-s,u)du\right)^{1/2} \\
&\qquad\qquad\qquad = V_s + M_{p,T}\sqrt{\delta}[8(t-s)]^{-1/4}. \qquad (2.13)
\end{aligned}
$$

Substituting the estimate (2.13) into (2.12) yields

$$
\begin{aligned}
\|I_{2,+}(x_1,x_2)\|_p^2 &\le 2c_p^2\|f\|^2_{L^2(\mathbb{R})}\int_0^t (V_s^2 + 8^{-1/2}M_{p,T}^2\delta(t-s)^{-1/2})ds \\
&\le 2c_p^2\|f\|^2_{L^2(\mathbb{R})}\left(\int_0^t V_s^2 ds + \sqrt{\tfrac{T}{2}}M_{p,T}^2\delta\right). \qquad (2.14)
\end{aligned}
$$

An analogous upper bound can be obtained for $\|I_{2,-}(x_1,x_2)\|_p^2$ in the same way. Similarly, decompose $I_3$ as

$$
I_3(x_1,x_2) = I_{3,+}(x_1,x_2) + I_{3,-}(x_1,x_2),
$$

where

$$
I_{3,+}(x_1,x_2) = \int_0^t \int_0^\infty \sigma_s^2(y)\int_y^\infty [G(t-s,x_1-z)-G(t-s,x_2-z)]\psi(s,z)dzdyds
$$

and

$$
I_{3,-}(x_1,x_2) = \int_0^t \int_{-\infty}^0 \sigma_s^2(y)\int_{-\infty}^y [G(t-s,x_1-z)-G(t-s,x_2-z)]\psi(s,z)dzdyds.
$$

By Minkowsky inequality,

$$\|I_{3,+}(x_1,x_2)\|_p \le \int_0^t \int_0^\infty f^2(y) \left\| \int_y^\infty [G(t-s,x_1-z)-G(t-s,x_2-z)] \right.$$
$$\left. \times \psi(s,z)dz \right\|_p dyds.$$

and the estimate (2.13) leads to

$$\|I_{3,+}(x_1,x_2)\|_p \le \|f\|^2_{L^2(\mathbb{R})} \left( \int_0^t V_s ds + \frac{4}{3} T^{3/4} M_{p,T} 8^{-1/4} \sqrt{\delta} \right). \tag{2.15}$$

We can derive an analogous estimate for $\|I_{3,-}(x_1,x_2)\|_p$. Finally, from (2.11), (2.14), (2.15), and the similar estimates for $I_{2,-}$ and $I_{3,-}$, we deduce

$$V_t^2 \le C_1\delta + C_2 \int_0^t V_s^2 ds$$

for some constants $C_1$ and $C_2$ depending on $p$, $T$, $\|u_0\|_\infty$, $\|u_0\|_{L^1(\mathbb{R})}$ and $\|f\|_{L^2(\mathbb{R})}$. By Gronwall's lemma, $V_t \le C\sqrt{\delta}$, which implies the desired Hölder continuity in the space variable.

For time regularity, let $0 \le t_1 < t_2 \le T$ and consider each of the decomposition

$$\psi(t_2,x) - \psi(t_1,x) = J_1(t_1,t_2) - \frac{1}{2} J_2(t_1,t_2) + \frac{1}{8} J_3(t_1,t_2),$$

where

$$J_1(t_1,t_2) = \int_{\mathbb{R}} \big( G(t_2,x-y) - G(t_1,x-y) \big) \psi_0(y) dy,$$

$$
\begin{aligned}
J_2(t_1,t_2) &= \int_0^{t_2}\int_S \mathrm{sign}(y)G(t_2-s,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy) \\
&\quad -\int_0^{t_1}\int_S \mathrm{sign}(y)G(t_1-s,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy),
\end{aligned}
$$

and

$$
\begin{aligned}
J_3(t_1,t_2) &= \int_0^{t_2}\int_S [G(t_2-s,x-z)-G(t-s,x-z)]\psi(s,z)\sigma_s^2(y)dzdyds \\
&\quad -\int_0^{t_1}\int_S [G(t_1-s,x-z)-G(t-s,x-z)]\psi(s,z)\sigma_s^2(y)dzdyds.
\end{aligned}
$$

Apply the semigroup property and the Lipschitz property of $\psi_0$ to get

$$
\begin{aligned}
|J_1(t_1,t_2)| &= \Big|\int_{\mathbb{R}} G(t_1,x-y)\Big(\int_{\mathbb{R}} G(t_2-t_1,y-z)\big[\psi_0(z)-\psi_0(y)\big]dz\Big)dy\Big| \\
&\leq C\int_{\mathbb{R}} G(t_1,x-y)\Big(\int_{\mathbb{R}} G(t_2-t_1,y-z)|z-y|dz\Big)dy \\
&= C(t_2-t_1)^{1/2}.
\end{aligned}
$$

For the stochastic integral term, we again decompose $J_2$ as

$$
J_2(t_1,t_2) = J_{2,+}(t_1,t_2) - J_{2,-}(t_1,t_2),
$$

where

$$
\begin{aligned}
J_{2,+}(t_1,t_2) = &\int_0^{t_2}\int_0^{\infty}\int_y^{\infty} G(t_2-s,x-z)\psi(s,z)\sigma_s(y)W(ds,dy) \\
&-\int_0^{t_1}\int_0^{\infty}\int_y^{\infty} G(t_1-s,x-z)\psi(s,z)\sigma_s(y)W(ds,dy)
\end{aligned}
$$

and

$$
\begin{aligned}
J_{2,-}(t_1,t_2) = &\int_0^{t_2}\int_{-\infty}^{0}\int_{-\infty}^{y} G(t_2-s,x-z)\psi(s,z)\sigma_s(y)W(ds,dy)\\
&-\int_0^{t_1}\int_{-\infty}^{0}\int_{-\infty}^{y} G(t_1-s,x-z)\psi(s,z)\sigma_s(y)W(ds,dy).
\end{aligned}
$$

Splitting $J_{2,+}$ into two pieces, we can write

$$
\begin{aligned}
\|J_{2,+}(t_1,t_2)\|_p \le &\Big\|\int_0^{t_1}\int_0^{\infty}\sigma_s(y)\Big(\int_y^{\infty}\psi(s,z)\\
&\qquad\times[G(t_2-s,x-z)-G(t_1-s,x-z)]dz\Big)W(ds,dy)\Big\|_p\\
&+\Big\|\int_{t_1}^{t_2}\int_0^{\infty}\sigma_s(y)\Big(\int_y^{\infty}\psi(s,z)G(t_2-s,x-z)dz\Big)W(ds,dy)\Big\|_p\\
=:&\, A_1(t_1,t_2)+A_2(t_1,t_2).
\end{aligned}
$$

Applying Burkholder's inequality and Minkowski's inequality, yields

$$
\begin{aligned}
A_1(t_1,t_2) \le c_p\Big(&\int_0^{t_1}\int_0^{\infty} f(y)^2\Big\|\int_y^{\infty}\psi(s,z)\\
&\times[G(t_2-s,x-z)-G(t_1-s,x-z)]dz\Big\|_p^2 dyds\Big)^{1/2}.
\end{aligned}
$$

Adding and subtracting $\psi(s,x)$ and using the spatial regularity of $\psi$, we obtain

$$
\begin{aligned}
&\Big\|\int_y^{\infty}\psi(s,z)[G(t_2-s,x-z)-G(t_1-s,x-z)]dz\Big\|_p^2\\
&\le 2C\Big(\int_y^{\infty}|G(t_2-s,x-z)-G(t_1-s,x-z)|\,|x-z|^{1/2}dz\Big)^2\\
&\quad +2\|\psi(s,x)\|_p^2\Big(\int_y^{\infty}\big[G(t_2-s,x-z)-G(t_1-s,x-z)\big]dz\Big)^2. \qquad (2.16)
\end{aligned}
$$

By Lemma 2.1.1, with $\beta = 2$, $\theta_1 = 1$ and $\theta_2 = 1/2$, yields

$$\int_0^{t_1} \left( \int_y^{\infty} |G(t_2 - s, x - z) - G(t_1 - s, x - z)| \, |x - z|^{1/2} dz \right)^2 ds \le C(t_2 - t_2)^{3/2}. \tag{2.17}$$

Applying Lemma 2.1.1 again, with $\beta = 2$, $\theta_1 = 1$ and $\theta_2 = 0$, we obtain

$$\int_0^{t_1} \left( \int_y^{\infty} \big[ G(t_2 - s, x - z) - G(t_1 - s, x - z) \big] dz \right)^2 ds \le C(t_2 - t_1). \tag{2.18}$$

Substituting (2.17) and (2.18) into (2.16), we get

$$A_1(t_1, t_2) \le C(t_2 - t_1)^{1/2}.$$

We control the term $A_2(t_1, t_2)$ using a rough estimate as follows

$$\begin{aligned} A_2(t_1, t_2) &\le c_p \Big\| \int_{t_1}^{t_2} \int_0^{\infty} f(y)^2 \Big( \int_y^{\infty} \psi(s, z) G(t_2 - s, x - z) dz \Big)^2 dy ds \Big\|_{p/2}^{1/2} \\ &\le c_p \|f\|_{L^2(\mathbb{R})} \left( \int_{t_1}^{t_2} \Big\| \int_{\mathbb{R}} \psi(s, z) G(t_2 - s, x - z) dz \Big\|_p^2 ds \right)^{1/2} \\ &\le C(t_2 - t_1)^{1/2}. \end{aligned}$$

We can bound $J_{2,-}$ in the same way and get

$$\|J_2(t_1, t_2)\|_p \le C(t_2 - t_1)^{1/2}.$$

Once again, we decompose $J_3$ as $J_3 = J_{3,+} + J_{3,-}$, where

$$\begin{aligned} J_{3,+}(t_1,t_2) = &\int_0^{t_2}\int_0^{\infty}\int_y^{\infty} G(t_2-s,s,z)\psi(s,z)\sigma_s^2(y)dzdyds \\ &- \int_0^{t_1}\int_0^{\infty}\int_y^{\infty} G(t_1-s,s,z)\psi(s,z)\sigma_s^2(y)dzdyds \end{aligned}$$

and

$$\begin{aligned} J_{3,-}(t_1,t_2) = &\int_0^{t_2}\int_{-\infty}^{0}\int_{-\infty}^{y} G(t_2-s,s,z)\psi(s,z)\sigma_s^2(y)dzdyds \\ &- \int_0^{t_1}\int_{-\infty}^{0}\int_{-\infty}^{y} G(t_1-s,s,z)\psi(s,z)\sigma_s^2(y)dzdyds. \end{aligned}$$

We control $J_{3,+}$ in the same way as $J_{2,+}$ to get

$$\begin{aligned} \|J_{3,+}(t_1,t_2)\|_p \le &\Big\| \int_0^{t_1}\int_0^{\infty} \sigma_s(y)^2 \int_y^{\infty} \psi(s,z) \\ &\times [G(t_2-s,x-z) - G(t_1-s,x-z)]dzdyds \Big\|_p \\ &+ \Big\| \int_{t_1}^{t_2}\int_0^{\infty} \sigma_s(y)^2 \int_y^{\infty} \psi(s,z)G(t_2-s,x-z)dzdyds \Big\|_p. \end{aligned}$$

We bound the second term roughly as

$$\begin{aligned} \Big\| \int_{t_1}^{t_2}\int_0^{\infty} \sigma_s(y)^2 \int_y^{\infty} \psi(s,z)G(t_2-s,x-z)dzdyds \Big\|_p \\ \le C\int_{t_1}^{t_2}\int_{\mathbb{R}} G(t_2-s,x-z)dzds \\ = C(t_2-t_1). \end{aligned}$$

Then, notice that for any $\epsilon \in (0,1)$ the first term can be bounded as

$$\begin{aligned}
&\int_0^{t_1}\int_{\mathbb{R}} |G(t_2-s,x-z)-G(t_1-s,x-z)|dzds \\
&\quad \leq C\int_0^{t_1}\Big(\int_{\mathbb{R}} |G(t_2-s,x)-G(t_1-s,x)|^{p_1(1-\epsilon)}dx\Big)^{1/p_1} \\
&\qquad\quad \times\Big[\Big(\int_{\mathbb{R}} G(t_2-s,x)^{p_2\epsilon}dx\Big)^{1/p_2}+\Big(\int_{\mathbb{R}} G(t_1-s,x)^{p_2\epsilon}dx\Big)^{1/p_2}\Big]ds \\
&\quad \leq C\int_0^{t_1}\Big(\int_{\mathbb{R}} |G(t_2-s,x)-G(t_1-s,x)|^{p_1(1-\epsilon)}dx\Big)^{1/p_1},
\end{aligned}$$

for any Hölder conjugates $p_1, p_2$. Notice that if $\beta = 1/p_1$, $\theta_1 = p_1(1-\epsilon)$, and $\theta_2 = 0$, then condition (2.4) is satisfied when, for example, $\epsilon = 1/p_1$ and $p_1 > 4$. Hence, using Lemma 2.1.1 with these parameters yields

$$\int_0^{t_1}\int_{\mathbb{R}} |G(t_2-s,x-z)-G(t_1-s,x-z)|dzds \leq C(t_2-t_1)^{1/2+1/p_1}.$$

Control $J_{3,-}$ in an identical way to obtain

$$\|J_3(t_1,t_2)\|_p \leq C(t_2-t_1)^{1/2}.$$

Combining the above estimates yields

$$\|\psi(t_2,x)-\psi(t_1,x)\|_p \leq C(t_2-t_1)^{1/2}.$$

□

Next we use the established Hölder regularity of the process $\psi$ to study its spatial differentiability.

**Proposition 2.2.5.** The process $\psi(t,\cdot)$ is differentiable in $L^p(\Omega)$ for any $p \geq 2$ and satisfies

$$\frac{\partial\psi}{\partial x}(t,x) = \int_{\mathbb{R}} \frac{\partial}{\partial x} G(t,x-y)\psi_0(y)dy$$

$$-\frac{1}{2}\int_0^t \int_S \operatorname{sign}(y)\frac{\partial}{\partial x} G(t-s,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy) \tag{2.19}$$

$$+\frac{1}{8}\int_0^t \int_S \frac{\partial}{\partial x} G(t-s,x-z)\psi(s,z)\sigma_s(y)^2 dzdyds. \tag{2.20}$$

*Proof.* It is clear that the spatial derivative of the first integral in the expression of $\psi$ equals the first integral above by Leibniz's rule.

To take care of the stochastic integral term, by the Burkholder-Davis-Gundy inequality and the symmetry of $S$, it suffices to show the convergence to zero in $L^{p/2}(\Omega)$, as $h$ tends to zero, of the term

$$I_h(t,x) := \int_0^t \int_0^\infty \Big( \int_y^\infty \Delta_h G(t-s,x-z)\psi(s,z)dz \Big)^2 \sigma_s(y)^2 dy\, ds,$$

where

$$\Delta_h G(t-s,x-z) := \frac{G(t-s,x+h-z) - G(t-s,x-z)}{h} - \frac{\partial}{\partial x} G(t-s,x-z).$$

By Minkowski's inequality, we obtain

$$\|I_h(t,x)\|_{p/2} \leq \int_0^t \int_0^\infty \left\| \int_y^\infty \Delta_h G(t-s,x-z)\psi(s,z)dz \right\|_p^2 f(y)^2 dyds.$$

We show first the convergence to zero of

$$I_h(t,x,s) := \int_0^\infty \left\| \int_y^\infty \Delta_h G(t-s,x-z)\psi(s,z)dz \right\|_p^2 f(y)^2 dy$$

as $h$ tends to zero, for each fixed $s \in [0,t)$. Rough estimates of $I_h(t,x,s)$ lead to

$$I_h(t,x,s) \le \sup_{t,x} \|\psi(t,x)\|_p^2 \|f\|_{L^2(\mathbb{R})}^2 \left( \int_{\mathbb{R}} |\Delta_h G(t-s,z)| dz \right)^2.$$

Apply the mean value theorem twice to see that

$$\Delta_h G(t-s,z) = \frac{1}{h} \int_0^h \int_0^u \frac{\partial^2}{\partial x^2} G(t-s, z+\eta) d\eta du.$$

Finally, by applying Fubini's theorem, we obtain

$$I_h(t,x,s) \le \; C_s |h|^2.$$

Hence, we have that, for each $s \in [0,t)$, $I_h(t,x,s) \to 0$ as $h \to 0$. By the dominated convergence theorem, it now suffices to show that $I_h(t,x,s)$ is bounded by a $ds$-integrable function which is independent of $h$. Again, by the mean value theorem, we can write

$$\begin{aligned} I_h(t,x,s) = \int_0^\infty \Bigg\| \int_y^\infty \frac{1}{h} \int_0^h \bigg( & \frac{\partial}{\partial x} G(t-s, x+\xi-z) \\ & - \frac{\partial}{\partial x} G(t-s, x-z) \bigg) d\xi \; \psi(s,z) dz \Bigg\|_p^2 f(y)^2 dy. \end{aligned}$$

We split up this quantity by adding and subtracting appropriate terms as follows

$$I_h(t,x,s) = \int_0^\infty \left\| \int_y^\infty \left[ \phi_1(s,x,z,h) + \phi_2(s,x,z,h) \right] dz \right\|_p^2 f(y)^2 dy,$$

where

$$\phi_1(s,x,z,h) := \frac{1}{h}\int_0^h \frac{\partial}{\partial x} G(t-s,x+\xi-z)\,[\psi(s,z)-\psi(s,x+\xi)]\,d\xi$$

$$-\frac{\partial}{\partial x}G(t-s,x-z)\,[\psi(s,z)-\psi(s,x)]$$

and

$$\phi_2(s,x,z,h) := \frac{1}{h}\int_0^h \Big[\frac{\partial}{\partial x}G(t-s,x+\xi-z)\psi(s,x+\xi) - \frac{\partial}{\partial x}G(t-s,x-z)\psi(s,x)\Big]\,d\xi.$$

Let us first consider the two terms of $\phi_1$, one at a time. For the first one, we can write, using Minkowski inequality and the Hölder continuity in $L^p(\Omega)$ of $\psi$

$$\begin{aligned}
&\int_0^\infty \Big\| \int_y^\infty \frac{1}{h}\int_0^h \frac{\partial}{\partial x}G(t-s,x+\xi-z)\big[\psi(s,z)-\psi(s,x+\xi)\big]\,d\xi\;dz\Big\|_p^2 f(y)^2 dy \\
&\quad \le C\|f\|^2_{L^2(\mathbb{R})}\frac{1}{h^2}\Big(\int_0^h\int_{\mathbb{R}}\Big|\frac{\partial}{\partial x}G(t-s,x+\xi-z)\Big||x+\xi-z|^{1/2}dz\;d\xi\Big)^2 \\
&\quad = C\|f\|^2_{L^2(\mathbb{R})}(t-s)^{-1/2},
\end{aligned}$$

which is $ds$-integrable. Now, to see that the second term is also bounded by a $ds$-integrable function not depending on $h$, we bound in the same way to get

$$\begin{aligned}
&\int_0^\infty \Big\| \int_y^\infty \frac{\partial}{\partial x}G(t-s,x-z)\big[\psi(s,z)-\psi(s,x)\big]\,dz\Big\|_p^2 f(y)^2 dy \\
&\quad \le C\int_0^\infty \Big(\int_y^\infty \Big|\frac{\partial}{\partial x}G(t-s,x-z)\Big||z-x|^{1/2}dz\Big)^2 f(y)^2 dy \\
&\quad \le C\|f\|^2_{L^2(\mathbb{R})}(t-s)^{-1/2}.
\end{aligned}$$

Let us now control the term $\phi_2$ by first interchanging the $d\xi$ and $dz$ integrals to get

$$\int_0^\infty \Big\| \int_y^\infty \phi_2(s,x,z,h)dz \Big\|_p^2 f(y)^2 dy$$
$$= \int_0^\infty \Big\| \frac{1}{h}\int_0^h \Big[ G(t-s,x+\xi-y)\psi(s,x+\xi) - G(t-s,x-y)\psi(s,x) \Big] d\xi \Big\|_p^2 f(y)^2 dy.$$

Now, add and subtract $G(t-s,x+\xi-y)\psi(s,x)$ to get

$$\begin{aligned} &\int_0^\infty \Big\| \int_y^\infty \phi_2(s,x,z,h)dz \Big\|_p^2 f(y)^2 dy \\ &\le 2\int_0^\infty \Big\| \psi(s,x)\frac{1}{h}\int_0^h \Big[ G(t-s,x+\xi-y) - G(t-s,x-y)\Big] d\xi \Big\|_p^2 f(y)^2 dy \\ &\quad + 2\int_0^\infty \Big\| \frac{1}{h}\int_0^h G(t-s,x+\xi-y)\Big[\psi(s,x+\xi)-\psi(s,x)\Big] d\xi \Big\|_p^2 f(y)^2 dy \\ &=: J_{1,h} + J_{2,h}. \end{aligned}$$

The second term can easily be bounded as follows

$$J_{2,h} \le C\int_{\mathbb{R}} f(y)^2 \Big| \frac{1}{h}\int_0^h G(t-s,x+\xi-y)|\xi|^{1/2} d\xi \Big|^2 dy.$$

We now use the assumption $f \in L^q(\mathbb{R})$ for some $q > 2$ and choose $p_1$ such that $\frac{1}{p_1} + \frac{2}{q} = 1$. Then, by Hölder's inequality, we can write

$$J_{2,h} \le C\|f\|_{L^q(\mathbb{R})}^2 \Big\| \frac{1}{h}\int_0^h G(t-s,x+\xi-\cdot)|\xi|^{1/2} d\xi \Big\|_{L^{2p_1}(\mathbb{R})}^2.$$

Now, by Minkowski's inequality, we have

$$\Big\| \frac{1}{h}\int_0^h G(t-s,x+\xi-\cdot)|\xi|^{1/2} d\xi \Big\|_{L^{2p_1}(\mathbb{R})}^2 \le C(t-s)^{-1+1/2p_1} \Big( \frac{1}{h}\int_0^h |\xi|^{1/2} d\xi \Big)^2,$$

which is $ds$-integrable and independent of $h$ since we can assume $|h| \leq 1$ without loss of generality. Finally, to control $J_{1,h}$, we proceed by again choosing the same value of $p_1$:

$$J_{1,h} \leq C\|f\|^2_{L^q(\mathbb{R})}\Big(\frac{1}{h}\int_0^h \big\|G(t-s,x+\xi-\cdot)-G(t-s,x,\cdot)\big\|_{L^{2p_1}(\mathbb{R})}d\xi\Big)^2$$
$$\leq C\|f\|^2_{L^q(\mathbb{R})}(t-s)^{-1+1/(2p_1)},$$

which is $ds$-integrable.

For the third integral in the expression of $\partial_x\psi$, we use an identical argument to obtain pointwise convergence to zero. Furthermore, it is easy to bound the $ds$ integrand by an integrable function which is independent of $h$ since

$$\Big\|\int_0^\infty\int_y^\infty \Delta_h G(t-s,x-z)\psi(s,z)\sigma_s(y)^2dzdy\Big\|_p$$
$$\leq C\|f\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}}|\Delta_h G(t-s,x-z)|dz$$
$$\leq C\int_{\mathbb{R}}\Big(\Big|\frac{\partial}{\partial x}G(t-s,x+\xi-z)\Big|+\Big|\frac{\partial}{\partial x}G(t-s,x-z)\Big|\Big)dz$$
$$= C(t-s)^{-1/2},$$

where the second inequality follows from the mean value theorem and triangle inequality.

□

In order to obtain a continuity result for the derivative process given above, we first establish uniform moment bounds.

**Proposition 2.2.6.** For all integers $p \geq 2$, we have for any $t \geq 0$,

$$\sup_{x\in\mathbb{R}} \left\| \frac{\partial \psi}{\partial x}(t,x) \right\|_p \leq K\sqrt{p}(t \vee 1)^{1-1/q} \exp\left(8p\|f\|^2_{L^2(\mathbb{R})}t + \|f\|^4_{L^2(\mathbb{R})}t^2\right),$$

where $K$ is a constant depending on $M_{p,T}$ (as in 2.8), $q$, $\|f\|_{L^q(\mathbb{R})}$, $\|f\|_{L^2(\mathbb{R})}$, $\|u_0\|_{L^1(\mathbb{R})}$, and $\|u_0\|_\infty$.

*Proof.* From the integral equation (2.20) satisfied by $\frac{\partial\psi}{\partial x}(t,x)$, we get the decomposition

$$\frac{\partial \psi}{\partial x}(t,x) = \mathcal{I}_1(t,x) - \mathcal{I}_2(t,x) + \mathcal{I}_3(t,x), \tag{2.21}$$

where

$$\mathcal{I}_1(t,x) = \int_{\mathbb{R}} \frac{\partial}{\partial x} G(t,x-y)\psi_0(y)dy$$

$$\mathcal{I}_2(t,x) = \frac{1}{2}\int_0^t \int_S \mathrm{sign}(y) \frac{\partial}{\partial x} G(t-s,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy)$$

and

$$\mathcal{I}_3(t,x) = \frac{1}{8}\int_0^t \int_S \frac{\partial}{\partial x} G(t-s,x-z)\psi(s,z)\sigma_s(y)^2 dzdyds.$$

First observe that integrating by parts yields

$$|\mathcal{I}_1(t,x)| \leq \int_{\mathbb{R}} G(t,x-y)\left|\frac{\partial\psi_0}{\partial y}(y)\right|dy \leq \frac{1}{2}\|u_0\|_\infty e^{\frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}}.$$

Now, decompose $\mathcal{I}_2$ as $\mathcal{I}_2(t,x) = \mathcal{I}_{2,+}(t,x) + \mathcal{I}_{2,-}(t,x)$, where

$$\mathcal{I}_{2,+}(t,x) = \int_0^t \int_0^\infty \int_y^\infty \frac{\partial}{\partial x} G(t-s,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy) \tag{2.22}$$

and

$$\mathcal{I}_{2,-}(t,x) = -\int_0^t \int_{-\infty}^0 \int_{-\infty}^y \frac{\partial}{\partial x} G(t-s,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy). \qquad (2.23)$$

Using the optimal constant in Burkholder's inequality, we get

$$\|\mathcal{I}_{2,+}(t,x)\|_p^2 \le 4p \int_0^t \int_0^\infty \Big\| \int_y^\infty \frac{\partial}{\partial x} G(t-s,x-z)\psi(s,z)dz \Big\|_p^2 f(y)^2 dyds.$$

Integrate by parts, use the triangle inequality, and the uniform bounds on moments of $\psi$ to obtain

$$\begin{aligned}\Big\| \int_y^\infty \frac{\partial}{\partial x} G(t-s,x-z)\psi(s,z)dz \Big\|_p^2 &\le 2M_{p,t}^2 G^2(t-s,x-y) \\ &\quad + 2\Big( \int_{\mathbb{R}} G(t-s,x-z) \Big\| \frac{\partial \psi}{\partial z}(s,z) \Big\|_p dz \Big)^2,\end{aligned}$$

where $M_{p,t} = \sup_{x\in\mathbb{R}} \|\psi(t,x)\|_p$. By Hölder's inequality, if $\frac{1}{q_1} + \frac{2}{q} = 1$, then

$$\begin{aligned}\int_{\mathbb{R}} f(y)^2 G^2(t-s,x-y)dy &\le \Big( \int_{\mathbb{R}} G(t-s,x-y)^{2q_1} dy \Big)^{1/q_1} \|f\|_{L^q(\mathbb{R})}^2 \\ &= k_q \|f\|_{L^q(\mathbb{R})}^2 (t-s)^{-1+1/(2q_1)},\end{aligned}$$

where $k_q$ is a constant depending on $q$. Let

$$U_t := \sup_{x\in\mathbb{R}} \Big\| \frac{\partial \psi}{\partial x}(t,x) \Big\|_p^2.$$

The above estimates yield

$$\begin{aligned}
\|\mathcal{I}_{2,+}(t,x)\|_p^2 &\le 16k_q\frac{pq}{q-2}c_pk\|f\|_{L^q(\mathbb{R})}^2M_{p,t}^2t^{\frac{1}{2}-\frac{1}{q}}+8p\|f\|_{L^2(\mathbb{R})}^2\int_0^t U_s ds\\
&= c_{p,t}^{(1)}+8p\|f\|_{L^2(\mathbb{R})}^2\int_0^t U_s ds,
\end{aligned}$$

where $c_{p,t}^{(1)}$ is a positive constant depending on $p$, $q$, $t$, $\|f\|_{L^q(\mathbb{R})}$, and $\|u_0\|_{L^1(\mathbb{R})}$. We obtain the same bound on $\|\mathcal{I}_{2,-}(t,x)\|_p$ in an identical way. Similarly, $\mathcal{I}_3(t,x) = \mathcal{I}_{3,+}(t,x)+\mathcal{I}_{3,-}(t,x)$ where

$$\mathcal{I}_{3,+}(t,x) = \int_0^t\int_0^\infty\int_y^\infty \frac{\partial}{\partial x}G(t-s,x-z)\psi(s,z)\sigma_s(y)^2dzdyds \tag{2.24}$$

and

$$\mathcal{I}_{3,-}(t,x) = \int_0^t\int_{-\infty}^0\int_{-\infty}^y \frac{\partial}{\partial x}G(t-s,x-z)\psi(s,z)\sigma_s(y)^2dzdyds. \tag{2.25}$$

Again, integrating by parts, using Minkowski's inequality, and Proposition 3.1, we obtain

$$\begin{aligned}
&\|\mathcal{I}_{3,+}(t,x)\|_p\\
&\le \int_0^t\int_0^\infty f(y)^2\Big(M_{p,t}G(t-s,x-y)+\int_y^\infty G(t-s,x-z)\Big\|\frac{\partial\psi}{\partial z}(s,z)\Big\|_p dz\Big)dyds\\
&\le M_{p,t}\|f\|_{L^q(\mathbb{R})}^2\int_0^t\|G(t-s,\cdot)\|_{L^{q_1}(\mathbb{R})}ds+\|f\|_{L^2(\mathbb{R})}^2\int_0^t\sup_{x\in\mathbb{R}}\Big\|\frac{\partial\psi}{\partial x}(s,x)\Big\|_p ds.
\end{aligned}$$

Hence, we have

$$\|\mathcal{I}_{3,+}(t,x)\|_p^2 \le k_q' M_{p,t}^2 \|f\|_{L^q(\mathbb{R})}^4 \Big(\frac{q}{q-1}\Big)^2 t^{2-2/q} + 2t\|f\|_{L^2(\mathbb{R})}^4 \int_0^t U_s ds$$
$$= c_{p,t}^{(3)} + 2t\|f\|_{L^2(\mathbb{R})}^4 \int_0^t U_s ds,$$

for some constant $c_{p,t}^{(3)}$. We can bound $\mathcal{I}_{3,-}$ in the same way. Putting each bound from above together and applying Gronwall's inequality, we obtain the desired result. □

**Proposition 2.2.7.** Suppose that in addition to condition (A1), the initial condition $u_0$ is Hölder continuous of order $\alpha \in [0,1]$. Then, for any $p \ge 2$ and any $T > 0$, there exists some constant $C$, depending on $p$, $T$, $u_0$, and $f$, such that for all $s,t \in [0,T]$, and $x,y \in \mathbb{R}$,

$$\Big\|\frac{\partial \psi}{\partial x}(t,x) - \frac{\partial \psi}{\partial y}(s,y)\Big\|_p \le C\big(|t-s|^{\frac{\alpha}{2}\wedge(\frac{1}{4}-\frac{1}{2q})} + |x-y|^{\alpha\wedge(\frac{1}{2}-\frac{1}{q})}\big),$$

where $q$ is the exponent appearing in Assumption (A2).

*Proof.* We first study Hölder continuity in the space variable. Fix $t \in [0,T]$, let $x_1, x_2 \in \mathbb{R}$ be given, and set $\delta = |x_1 - x_2|$. Without loss of generality we can assume that $\delta \le 1$. We consider spatial increments of each term in (2.21) one at a time. The first term is easily controlled integrating by parts and using the fact that $\psi_0'$ is Hölder continuous of order $\alpha$:

$$|\mathcal{I}_1(t,x_1) - \mathcal{I}_1(t,x_2)| \le \int_{\mathbb{R}} G(t,y)|\psi_0'(x_1-y) - \psi_0'(x_2-y)|dy \le C\delta^\alpha.$$

For the second term, we again use the decomposition $\mathcal{I}_2(t,x) = \mathcal{I}_{2,+}(t,x) + \mathcal{I}_{2,-}(t,x)$, where $\mathcal{I}_{2,+}$ and $\mathcal{I}_{2,-}$ have been introduced in (2.22) and (2.23), respectively. Integrating by parts, we obtain

$$
\begin{aligned}
&\int_y^\infty \Big[\frac{\partial}{\partial x}G(t-s,x_1-z)-\frac{\partial}{\partial x}G(t-s,x_2-z)\Big]\psi(s,z)dz \\
&\quad= \big[G(t-s,x_1-y)-G(t-s,x_2-y)\big]\psi(s,y) \\
&\qquad+\int_y^\infty \big[G(t-s,x_1-z)-G(t-s,x_2-z)\big]\frac{\partial\psi}{\partial z}(s,z)dz \\
&\quad=: \mathcal{I}^A_{2,+}(t-s,x_1,x_2,y)+\mathcal{I}^B_{2,+}(t-s,x_1,x_2,y).
\end{aligned}
$$

Applying Burkholder's inequality, (2.3), Minkowski's inequality, and the uniform bounds on moments, we get

$$
\begin{aligned}
&\Big\|\int_0^t\int_0^\infty \mathcal{I}^A_{2,+}(t-s,x_1,x_2,y)\sigma_s(y)W(ds,dy)\Big\|_p^2 \\
&\quad\le C\int_0^t\int_{\mathbb{R}}\big|G(t-s,x_1-y)-G(t-s,x_2-y)\big|^2 f(y)^2dyds \\
&\quad\le C\|f\|^2_{L^q(\mathbb{R})}\int_0^t\Big(\int_{\mathbb{R}}\big|G(t-s,x_1-y)-G(t-s,x_2-y)\big|^{2q_1}dy\Big)^{1/q_1}ds,
\end{aligned}
\tag{2.26}
$$

where $\frac{2}{q}+\frac{1}{q_1}=1$. Making the substitutions $y=\delta z$ and $t-s=\delta^2 v$, yields

$$
\begin{aligned}
&\int_0^t\Big\|\big[G(t-s,x_1-\cdot)-G(t-s,x_2-\cdot)\big]^2\Big\|_{L^{q_1}(\mathbb{R})}ds \\
&\quad\le C\delta^{1/q_1}\int_0^\infty v^{-1}\Big(\int_{\mathbb{R}}\Big|\exp(-(1+z)^2/4v)-\exp(-z^2/4v)\Big|^{2q_1}dz\Big)^{1/q_1}dv \\
&\quad= C\delta^{1/q_1}.
\end{aligned}
\tag{2.27}
$$

Therefore, from (2.26) and (2.27), we obtain

$$
\Big\|\int_0^t\int_0^\infty \mathcal{I}^A_{2,+}(t-s,x_1,x_2,y)\sigma_s(y)W(ds,dy)\Big\|_p^2 \le C\|f\|^2_{L^q(\mathbb{R})}\delta^{1/q_1}. \tag{2.28}
$$

To handle $\mathcal{I}^B_{2,+}$, we use the same techniques as in the proof of Proposition 3.4 to first write

$$\mathcal{I}^B_{2,+}(t-s,x_1,x_2,y) = \int_{-\infty}^{x_1-y} G(t-s,u)\Big[\frac{\partial\psi}{\partial z}(s,x_1-u) - \frac{\partial\psi}{\partial z}(s,x_2-u)\Big]du + \int_{x_2-y}^{x_1-y}\frac{\partial\psi}{\partial z}(s,x_2-u)G(t-s,u)du.$$

Let

$$\widetilde{V}_s := \sup_{|x-y|=\delta}\Big\|\frac{\partial\psi}{\partial x}(s,x) - \frac{\partial\psi}{\partial x}(s,y)\Big\|_p$$

and

$$N_p := \sup_{t,x}\Big\|\frac{\partial\psi}{\partial x}(t,x)\Big\|_p.$$

Then, we can write

$$\begin{aligned}\big\|\mathcal{I}^B_{2,+}(t-s,x_1,x_2,y)\big\|_p &\le \widetilde{V}_s + N_p\int_{x_2-y}^{x_1-y} G(t-s,u)du \\ &\le \widetilde{V}_s + N_p\sqrt{\delta}[8(t-s)]^{-1/4}.\end{aligned}$$

Hence,

$$\Big\|\int_0^t\int_0^\infty \mathcal{I}^B_{2,+}(t-s,x_1,x_2,y)\sigma_s(y)W(ds,dy)\Big\|_p^2 \le C\|f\|^2_{L^2(\mathbb{R})}\Big(\int_0^t \widetilde{V}_s^2 ds + \delta N_p^2\Big). \tag{2.29}$$

Therefore, from (2.28) and (2.29), we get

$$\big\|\mathcal{I}_{2,+}(t,x_1) - \mathcal{I}_{2,+}(t,x_2)\big\|_p^2 \le C\|f\|^2_{L^q(\mathbb{R})}\delta^{1/q_1} + C\|f\|^2_{L^2(\mathbb{R})}\Big(\delta N_p^2 + \int_0^t \widetilde{V}_s^2 ds\Big).$$

We can get the same bounds on increments of $\mathcal{I}_{2,-}$ in an identical way. Once again, write $\mathcal{I}_3 = \mathcal{I}_{3,+} + \mathcal{I}_{3,-}$, as in (2.24) and (2.25). Integrate by parts, and use the same techniques as above to get

$$\begin{aligned}\big\|\mathcal{I}_{3,+}(t,x_1)-\mathcal{I}_{3,+}(t,x_2)\big\|_p &\le C\bigg(\int_0^t\int_{\mathbb{R}}\big|G(t-s,x_1-y)-G(t-s,x_2-y)\big|f(y)^2dyds \\ &\qquad + C\|f\|^2_{L^2(\mathbb{R})}\sqrt{\delta}N_p+\|f\|^2_{L^2(\mathbb{R})}\int_0^t\widetilde{V}_s ds\bigg) \\ &\le C\Big(\|f\|^2_{L^q(\mathbb{R})}+\|f\|^2_{L^2(\mathbb{R})}\sqrt{\delta}N_p+\|f\|^2_{L^2(\mathbb{R})}\int_0^t\widetilde{V}_s ds\Big).\end{aligned}$$

The same bounds for increments of $\mathcal{I}_{3,-}$ are obtained the same way. Put all of these pieces together by taking the smallest power of $\delta$ to get

$$\widetilde{V}_t^2 \le C\left(\delta^{2\alpha}+\delta^{1-2/q}+\int_0^t\widetilde{V}_s^2 ds\right).$$

Thus, Gronwall's inequality implies that $x\mapsto \frac{\partial\psi}{\partial x}(t,x)$ is Hölder continuous in $L^p(\Omega)$, uniformly in $t$, with order of regularity $\alpha\wedge(1/2-1/q)$.

To establish regularity in time, fix $0\le t_1<t_2\le T$ and write

$$\begin{aligned}|\mathcal{I}_1(t_2,x)-\mathcal{I}_1(t_1,x)| &= \left|\int_{\mathbb{R}}G(t_1,x-y)\left(\int_{\mathbb{R}}G(t_2-t_1,y-z)[\psi_0'(z)-\psi_0'(y)]dz\right)dy\right| \\ &\le C\int_{\mathbb{R}}G(t_1,x-y)\left(\int_{\mathbb{R}}G(t_2-t_1,y-z)|y-z|^\alpha dz\right)dy \\ &= C(t_2-t_1)^{\alpha/2}.\end{aligned}$$

Then, we again split up $\|\mathcal{I}_{2,+}(t_2,x)-\mathcal{I}_{2,+}(t_1,x)\|_p$ into two terms as

$$
\begin{aligned}
&\|\mathcal{I}_{2,+}(t_2,x)-\mathcal{I}_{2,+}(t_1,x)\|_p \\
&\quad\le \Big\| \int_0^{t_1}\int_0^\infty \sigma_s(y)\Big(\int_y^\infty \psi(s,z)\frac{\partial}{\partial x}[G(t_2-s,x-z) \\
&\qquad\qquad\qquad\qquad -G(t_1-s,x-z)]dz\Big)W(ds,dy)\Big\|_p \\
&\qquad + \Big\| \int_{t_1}^{t_2}\int_0^\infty \sigma_s(y)\Big(\int_y^\infty \psi(s,z)\frac{\partial}{\partial x}G(t_2-s,x-z)dz\Big)W(ds,dy)\Big\|_p \\
&\quad =: \widetilde{J}_1+\widetilde{J}_2.
\end{aligned}
$$

Integrate by parts, and apply Burkholder's and Minkowski's inequalities to get

$$
\begin{aligned}
\widetilde{J}_2 &\le c_p\Big(\int_{t_1}^{t_2}\int_0^\infty \Big\|G(t_2-s,x-y)\psi(s,y)+\int_y^\infty G(t_2-s,x-z) \\
&\qquad\qquad\qquad \times\frac{\partial\psi}{\partial z}(s,z)dz\Big\|_p^2 f(y)^2dyds\Big)^{1/2} \\
&\le C\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}} G(t_2-s,x-y)^2f(y)^2dyds \\
&\qquad + \int_{t_1}^{t_2}\int_0^\infty f(y)^2\Big\|\int_y^\infty G(t_2-s,x-z)\frac{\partial\psi}{\partial z}(s,z)dz\Big\|_p^2dyds\Big)^{1/2}.
\end{aligned}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_{\mathbb{R}} G(t_2-s,x-y)^2f(y)^2dyds &\le \int_{t_1}^{t_2}\|G(t_2-s,\cdot)\|^2_{L^{2q_1}(\mathbb{R})}\|f\|^2_{L^q(\mathbb{R})}ds \\
&= C(t_2-t_1)^{1/(2q_1)}
\end{aligned}
$$

where $\frac{1}{q_1}+\frac{2}{q}=1$. For the other term, we make use of the uniform bounds on moments of the derivative of $\psi$ to get

$$\int_{t_1}^{t_2}\int_0^\infty f(y)^2\Big\|\int_y^\infty G(t_2-s,x-z)\frac{\partial\psi}{\partial z}(s,z)dz\Big\|_p^2 dyds\le C(t_2-t_1)$$

for some constant $C$. Hence,

$$\widetilde{J}_2\le C|t_2-t_1|^{1/(4q_1)}.$$

For the term $\widetilde{J}_1$, we first apply Burkholder's inequality and integrate by parts to get

$$\begin{aligned}\widetilde{J}_1\le C\Bigg(&\int_0^{t_1}\int_0^\infty f(y)^2\|\psi(s,y)\|_p^2\big[G(t_2-s,x-y)-G(t_1-s,x-y)\big]^2 dyds\\ &+\int_0^{t_1}\int_0^\infty\Big\|\int_y^\infty\big[G(t_2-s,x-z)-G(t_1-s,x-z)\big]\frac{\partial\psi}{\partial z}(s,z)dz\Big\|_p^2 f(y)^2dyds\Bigg)^{1/2}\\ &=: C(\widetilde{J}_{1,1}+\widetilde{J}_{1,2})^{\frac{1}{2}}.\end{aligned}$$

Using the uniform bounds on $\psi$, choosing $q_1$ such that $\frac{1}{q_1}+\frac{2}{q}=1$, and applying Lemma 2.1.1 with $\beta=1/q_1$, $\theta_1=2q_1$ and $\theta_2=0$, we can write

$$\begin{aligned}\widetilde{J}_{1,1}\ &\le\ C\|f\|^2_{L^q(\mathbb{R})}\int_0^{t_1}\left(\int_{\mathbb{R}}|G(t_2-s,y)-G(t_1-s,y)|^{2q_1}dy\right)^{\frac{1}{q_1}}ds\\ &\le\ C\|f\|^2_{L^q(\mathbb{R})}(t_2-t_1)^{1/(2q_1)}.\end{aligned}$$

For the term $\widetilde{J}_{1,2}$, we same techniques as in the proof of the Hölder regularity in time of $\psi$ by first adding and subtracting $\dfrac{\partial\psi}{\partial x}(s,x)$ and applying the spatial regularity of the derivative of $\psi$ to get

$$\widetilde{J}_{1,2} \le 2\|f\|^2_{L^2(\mathbb{R})} \int_0^{t_1} \Big( \int_{\mathbb{R}} |G(t_2-s,x-z) - G(t_1-s,x-z)||x-z|^{\alpha\wedge(\frac{1}{2}-\frac{1}{q})} dz \Big)^2 ds$$
$$+ 2\int_0^{t_1}\int_0^{\infty} \Big\|\frac{\partial\psi}{\partial x}(s,x)\Big\|_p^2 \Big( \int_y^{\infty} \big[G(t_2-s,x-z)$$
$$- G(t_1-s,x-z)\big] dz\Big)^2 f(y)^2 dy ds.$$

We apply Lemma 2.1.1 with $\beta = 2, \theta_1 = 1$, and $\theta_2 = \alpha \wedge (\frac{1}{2} - \frac{1}{q})$ to get

$$\int_0^{t_1} \Big( \int_{\mathbb{R}} |G(t_2-s,x-z) - G(t_1-s,x-z)||x-z|^{\alpha\wedge(\frac{1}{2}-\frac{1}{q})} dz \Big)^2 ds$$
$$\le C(t_2-t_1)^{1+\alpha\wedge(\frac{1}{2}-\frac{1}{q})}.$$

Another application of Lemma 2.1.1 with $\beta = 2$, $\theta_1 = 1$ and $\theta_2 = 0$, yields

$$\int_0^{t_1} \Big( \int_y^{\infty} [G(t_2-s,x-z) - G(t_1-s,x-z)] dz \Big)^2 ds \le C(t_2-t_1).$$

Hence,

$$\widetilde{J}_1 \le C(t_2-t_1)^{1/(4q_1)}.$$

Put these together to get

$$\|\mathcal{I}_{2,+}(t_2,x) - \mathcal{I}_{2,+}(t_1,x)\|_p \le C(t_2-t_1)^{1/4-1/(2q)}.$$

We can obtain the same upper bound for $I_{2,-}$ and hence

$$\|\mathcal{I}_2(t_2,x) - \mathcal{I}_2(t_1,x)\|_p \le C(t_2-t_1)^{1/4-1/(2q)}.$$

For the third term, we apply the same techniques we used for $\mathcal{I}_2$ to get

$$\|\mathcal{I}_3(t_2,x)-\mathcal{I}_3(t_1,x)\|_p \leq C(t_2-t_1)^{1/4}.$$

Hence, we have the desired result. $\square$

**Remark 2.2.8.** If we do not assume the Hölder continuity of $u_0$, then $\psi_0'$ is only continuous. Then, avoiding the integration by parts in the proof of the Hölder continuity of the first term, we have a result of the form

$$\left\|\frac{\partial\psi}{\partial x}(t,x)-\frac{\partial\psi}{\partial y}(s,y)\right\|_p \leq C(t\wedge s)^{-1/2}\big(|t-s|^{\frac{1}{4}-\frac{1}{2q}}+|x-y|^{\frac{1}{2}-\frac{1}{q}}\big),$$

where the factor $t^{-1/2}$, assuming $t\leq s$, comes from the integral $\int_{\mathbb{R}}\left|\frac{\partial}{\partial t}G(t,y)\right|dy$. That is, the Hölder continuity blows up at $t=0$. However, $\frac{\partial\psi}{\partial x}(t,x)$ is continuous in $L^p(\Omega)$ on $\mathbb{R}_+\times\mathbb{R}$ for all $p\geq 2$, because $\psi_0'$ is continuous.

## 2.3 Hopf-Cole Transformation

In this section, we construct a solution to Burgers' equation (2.1) using the Hopf-Cole transformation and the results of the previous section. Notice first that the process

$$v(t,x) \quad := \quad -2\frac{\partial}{\partial x}\log\psi(t,x) = -\frac{2}{\psi(t,x)}\frac{\partial\psi}{\partial x}(t,x)$$

is well defined and has uniformly bounded moments of order $p$ for all $p\geq 2$, due to Proposition 2.2.6 and Remark 2.2.2. We now establish the main result of the project which asserts that the process $v(t,x)$ is the solution to the Burgers' equation (2.1). Again, uniqueness follows for free from [13].

The main idea of the proof is to introduce the regularized process

$$\psi_\epsilon(t,x) := \int_{\mathbb{R}} G(\epsilon, x-y)\psi(t,y)dy,$$

for $\epsilon \in (0,1]$ and to find the equation satisfied by $u_\epsilon(t,x) := -2\frac{\partial}{\partial x}\log\psi_\epsilon(t,x)$. Based on previous results, it is easy to see that $\psi_\epsilon$ satisfies the following property.

**Lemma 2.3.1.** *For any $p \geq 2$ and $T > 0$, we have*

$$\sup_{x\in\mathbb{R},\epsilon\in(0,1],t\in[0,T]} \left( \|\psi_\epsilon(t,x)\|_p + \|\psi_\epsilon(t,x)^{-1}\|_p + \left\| \frac{\partial\psi_\epsilon}{\partial x}(t,x) \right\|_p \right) < \infty. \tag{2.30}$$

*For any $p \geq 2$, $x \in \mathbb{R}$, and $t \in (0,T]$, we have*

$$\|\psi(t,x) - \psi_\epsilon(t,x)\|_p \leq C\epsilon^{1/4} \tag{2.31}$$

*and*

$$\left\| \frac{\partial\psi}{\partial x}(t,x) - \frac{\partial\psi_\epsilon}{\partial x}(t,x) \right\|_p \leq Ct^{-1/2}\epsilon^{1/4-1/(2q)}. \tag{2.32}$$

*Proof.* Inequality (2.30) follows form Jensen's inequality, Propositions 2.2.1 and 2.2.6, and Remark 2.2.2. Inequalities (2.31) and (2.32) are consequences of Proposition 2.2.4 and Remark 2.2.8. □

**Theorem 2.3.2.** *The process $v(t,x) = -2\frac{\partial}{\partial x}\log\psi(t,x)$ is a solution to (2.1).*

*Proof.* From Proposition 2.2.3, we have that $\psi_\epsilon$ satisfies

$$\begin{aligned}\psi_\epsilon(t,x) = &\int_{\mathbb{R}} G(t+\epsilon, x-y)\psi_0(y)dy \\ &- \frac{1}{2}\int_0^t \int_S \operatorname{sign}(y) G(t+\epsilon-s, x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy) \\ &+ \frac{1}{8}\int_0^t \int_S G(t+\epsilon-s, x-z)\psi(s,z)\sigma_s(y)^2 dzdyds.\end{aligned}$$

Next, apply the semigroup property of the heat kernel to get

$$\begin{aligned}
\psi_\epsilon(t,x) &= \int_{\mathbb{R}} G(t,x-z)\Big(\int_{\mathbb{R}} G(\epsilon,z-y)\psi_0(y)dy\Big)dz \\
&\quad -\frac{1}{2}\int_0^t\int_S\int_{\mathbb{R}} \mathrm{sign}(v)G(t-s,x-y)G(\epsilon,y-z)\psi(s,z)\sigma_s(v)dydzW(ds,dv) \\
&\quad +\frac{1}{8}\int_0^t\int_S\int_{\mathbb{R}} G(t-s,x-y)G(\epsilon,y-z)\psi(s,z)\sigma_s(v)^2 dydzdvds.
\end{aligned}$$

Note that this is the mild formulation of the following stochastic heat equation

$$\begin{aligned}
\psi_\epsilon(t,x) &= \int_{\mathbb{R}} G(\epsilon,x-y)\psi_0(y)dy + \int_0^t \frac{\partial^2\psi_\epsilon}{\partial x^2}(s,x)ds \\
&\quad -\frac{1}{2}\int_0^t\int_S \mathrm{sign}(y)G(\epsilon,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy) \\
&\quad +\frac{1}{8}\int_0^t\int_S G(\epsilon,x-z)\psi(s,z)\sigma_s(y)^2 dzdyds.
\end{aligned}$$

The process $t \to \psi_\epsilon(t,x)$ is a semimartingale and applying Itô's formula to $\log\psi_\epsilon(t,x)$ yields

$$\begin{aligned}
\log\psi_\epsilon(t,x) &= \log\Big(\int_{\mathbb{R}} G(\epsilon,x-y)\psi_0(y)dy\Big) + \int_0^t \frac{1}{\psi_\epsilon(s,x)}\frac{\partial^2\psi_\epsilon}{\partial x^2}(s,x)ds \\
&\quad -\frac{1}{2}\int_0^t\int_S \mathrm{sign}(y)\frac{1}{\psi_\epsilon(s,x)}G(\epsilon,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy) \\
&\quad +\frac{1}{8}\int_0^t\int_S \frac{1}{\psi_\epsilon(s,x)}G(\epsilon,x-z)\psi(s,z)\sigma_s(y)^2 dzdyds \\
&\quad -\frac{1}{8}\int_0^t\int_{\mathbb{R}} \frac{1}{\psi_\epsilon(s,x)^2}\Psi_\epsilon(s,x,y)^2\sigma_s(y)^2 dyds,
\end{aligned}$$

where

$$\Psi_\epsilon(s,x,y) := \mathbf{1}_{(y\geq 0)}\int_y^\infty G(\epsilon,x-z)\psi(s,z)dz + \mathbf{1}_{(y<0)}\int_{-\infty}^y G(\epsilon,x-z)\psi(s,z)dz.$$

Now, noting that basic calculus gives $\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2}(\log f) + (\frac{\partial}{\partial x}\log f)^2$, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\log\psi_\epsilon(t,x) = {} & \frac{\partial}{\partial x}\log\Big(\int_{\mathbb{R}} G(\epsilon, x-y)\psi_0(y)dy\Big) \\
& + \int_0^t \frac{\partial^2}{\partial x^2}\Big(\frac{\partial}{\partial x}\log\psi_\epsilon(s,x)\Big)ds + \int_0^t \frac{\partial}{\partial x}\Big(\Big(\frac{\partial}{\partial x}\log\psi_\epsilon(s,x)\Big)^2\Big)ds \\
& - \frac{1}{2}\int_0^t\int_S \operatorname{sign}(y)\frac{\partial}{\partial x}\Big(\frac{1}{\psi_\epsilon(s,x)}G(\epsilon,x-z)\Big)\psi(s,z)\sigma_s(y)dzW(ds,dy) \\
& + \frac{1}{8}\int_0^t\int_S \frac{\partial}{\partial x}\Big(\frac{1}{\psi_\epsilon(s,x)}G(\epsilon,x-z)\Big)\psi(s,z)\sigma_s(y)^2dzdyds \\
& - \frac{1}{8}\int_0^t\int_{\mathbb{R}} \frac{\partial}{\partial x}\Big(\frac{1}{\psi_\epsilon(s,x)}\Psi_\epsilon(s,x,y)\Big)^2\sigma_s(y)^2dyds.
\end{aligned}$$

So, the process $u_\epsilon(t,x) := -2\frac{\partial}{\partial x}\log\psi_\epsilon(t,x)$ satisfies the following integral equation

$$\begin{aligned}
u_\epsilon(t,x) = {} & \int_{\mathbb{R}} G(t,x-y)u_\epsilon(0,y)dy - \frac{1}{2}\int_0^t\int_{\mathbb{R}} G(t-s,x-y)\frac{\partial}{\partial y}u_\epsilon(s,y)^2dyds \\
& + \int_0^t\int_S\int_{\mathbb{R}} \operatorname{sign}(y)G(t-s,x-v) \\
& \qquad \times \frac{\partial}{\partial v}\Big(\frac{1}{\psi_\epsilon(s,v)}G(\epsilon,v-z)\Big)\psi(s,z)\sigma_s(y)dvdzW(ds,dy) \\
& - \frac{1}{4}\int_0^t\int_S\int_{\mathbb{R}} G(t-s,x-v)\frac{\partial}{\partial v}\Big(\frac{1}{\psi_\epsilon(s,v)}G(\epsilon,v-z)\Big)\psi(s,z)\sigma_s(y)^2dvdydzds \\
& + \frac{1}{4}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} G(t-s,x-v)\frac{\partial}{\partial v}\left(\frac{1}{\psi_\epsilon(s,v)}\Psi_\epsilon(s,v,y)\right)^2\sigma_s(y)^2dvdyds.
\end{aligned}$$

Finally, integration by parts yields

$$
\begin{aligned}
u_\epsilon(t,x) &= \int_{\mathbb{R}} G(t,x-y)u_\epsilon(0,y)dy + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s,x-y)u_\epsilon(s,y)^2 dyds \\
&\quad - \int_0^t \int_S \int_{\mathbb{R}} \mathrm{sign}(y) \frac{\partial}{\partial v} G(t-s,x-v) \\
&\qquad\qquad\qquad \times \frac{1}{\psi_\epsilon(s,v)} G(\epsilon, v-z)\psi(s,z)\sigma_s(y) dvdzW(ds,dy) \\
&\quad + \frac{1}{4}\int_0^t \int_S \int_{\mathbb{R}} \frac{\partial}{\partial v} G(t-s,x-v)\frac{1}{\psi_\epsilon(s,v)} G(\epsilon,v-z)\psi(s,z)\sigma_s(y)^2 dvdzdyds \\
&\quad - \frac{1}{4}\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial}{\partial v} G(t-s,x-v)\frac{1}{\psi_\epsilon^2(s,v)} \Psi_\epsilon(x,v,y)^2 \sigma_s(y)^2 dvdyds \\
&= \sum_{i=1}^5 A_{i,\epsilon}.
\end{aligned}
$$

We will study the convergence of each term in the above expression. This will be done in several steps:

*Step 1.* For the term $A_{1,\epsilon}$, taking into account that

$$u_\epsilon(0,x) = -2\frac{(\psi_0' * G(\epsilon,\cdot))(x)}{(\psi_0 * G(\epsilon,\cdot))(x)}$$

and $\psi_0'$ is continuous and bounded, it is easy to show that

$$A_{1,\epsilon} \to \int_{\mathbb{R}} G(t,x-y)u_0(y)dy,$$

as $\epsilon$ tends to zero.

*Step 2.* From Lemma 2.3.1 it follows that

$$\|u_\epsilon(t,x) - v(t,x)\|_p \le Ct^{-1/2}\epsilon^{\frac{1}{4}-\frac{1}{2q}}.$$

With this, it is easy to see that

$$A_{2,\epsilon} \to \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s, x-y) v(s,y)^2 dy ds,$$

as $\epsilon \to 0$, in $L^p(\Omega)$ for all $p \geq 2$.

*Step 3.* We now show the convergence of the stochastic integral term $A_{3,\epsilon}$. Integrating by parts, first with respect to $v$, then with respect to $z$, we get for $y > 0$,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\partial}{\partial v} G(t-s, x-v) &\frac{1}{\psi_\epsilon(s,v)} \Big( \int_y^\infty G(\epsilon, v-z)\psi(s,z)dz \Big) dv \\
&= \int_{\mathbb{R}} G(t-s, x-v) \frac{1}{\psi_\epsilon(s,v)^2} \frac{\partial \psi_\epsilon}{\partial v}(s,v) \Big( \int_y^\infty G(\epsilon, v-z)\psi(s-z)dz \Big) dv \\
&\quad - \int_{\mathbb{R}} G(t-s, x-v) \frac{1}{\psi_\epsilon(s,v)} \Big( \int_y^\infty \frac{\partial}{\partial v} G(\epsilon, v-z)\psi(s,z)dz \Big) dv \\
&= \int_{\mathbb{R}} G(t-s, x-v) \frac{1}{\psi_\epsilon(s,v)^2} \frac{\partial \psi_\epsilon}{\partial v}(s,v) \Big( \int_y^\infty G(\epsilon, v-z)\psi(s,z)dz \Big) dv \\
&\quad - \int_{\mathbb{R}} G(t-s, x-v) \frac{1}{\psi_\epsilon(s,v)} \Big( \int_y^\infty G(\epsilon, v-z) \frac{\partial \psi}{\partial z}(s,z)dz \Big) dv \\
&\quad - \psi(s,y) \int_{\mathbb{R}} G(t-s, x-v) \frac{1}{\psi_\epsilon(s,v)} G(\epsilon, v-y) dv \\
&\quad =: G_{1,+,\epsilon}(s,y) - G_{2,+,\epsilon}(s,y) - G_{3,\epsilon}(y,s).
\end{aligned}$$

In a similar way, for $y < 0$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\partial}{\partial v} G(t-s, x-v) \frac{1}{\psi_\epsilon(s,v)} &\Big( \int_{-\infty}^y G(\epsilon, v-z)\psi(s,z)dz \Big) dv \\
&= G_{1,-,\epsilon}(s,y) - G_{2,-,\epsilon}(s,y) + G_{3,\epsilon}(y,s),
\end{aligned}$$

where the terms $G_{1,-,\epsilon}(s,y)$ and $G_{2,-,\epsilon}(s,y)$ are analogous to $G_{1,+,\epsilon}(s,y)$ and $G_{2,+,\epsilon}(s,y)$, respectively, by just replacing the integral $\int_y^\infty$ by $\int_{-\infty}^y$.

We claim that the following convergences hold in $L^p(\Omega)$, for any $p \geq 2$, as $\epsilon \to 0$:

$$\int_0^t \int_{\mathbb{R}} G_{3,\epsilon}(s,y)\sigma_s(y)W(ds,dy) \to \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\sigma_s(y)W(ds,dy), \tag{2.33}$$

$$\int_0^t \int_{\mathbb{R}} [G_{1,+,\epsilon}(s,y) - G_{2,+,\epsilon}(s,y)]\sigma_s(y)W(ds,dy) \to 0. \tag{2.34}$$

and

$$\int_0^t \int_{\mathbb{R}} [G_{1,-,\epsilon}(s,y) - G_{2,-,\epsilon}(s,y)]\sigma_s(y)W(ds,dy) \to 0. \tag{2.35}$$

*Proof of (2.33):* Applying Burkholder's inequality and Minkowski's inequality, we can write

$$\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}^2} G(\epsilon, v-y)\Big(\frac{\psi(s,y)}{\psi_\epsilon(s,v)} G(t-s,x-v) - G(t-s,x-y)\Big)\sigma_s(y)dvW(ds,dy)\right\|_p^2 \\
&\leq C \int_0^t \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} G(\epsilon,v-y)\frac{\psi(s,y)}{\psi_\epsilon(s,v)} G(t-s,x-v)dv - G(t-s,x-y)\right\|_p^2 f(y)^2 dyds \\
&\leq C(B_{1,\epsilon} + B_{2,\epsilon}),
\end{aligned}$$

where

$$B_{1,\epsilon} = \int_0^t \int_{\mathbb{R}} \left( \int_{\mathbb{R}} G(\epsilon,v-y)G(t-s,x-v) \left\| \frac{\psi(s,y)}{\psi_\epsilon(s,v)} - 1 \right\|_p dv \right)^2 f(y)^2 dyds$$

and

$$B_{2,\epsilon} = \int_0^t \int_{\mathbb{R}} (G(t-s+\epsilon,x-y) - G(t-s,x-y))^2 f(y)^2 dyds.$$

Using the definition of $\psi_\epsilon$ and 2.2.4, it is not difficult to see that $\psi_\epsilon$ is Hölder continuous of order 1/2 in the spatial variable. With this and Lemma 2.3.1, we have

$$\left\| \frac{\psi(s,y)}{\psi_\epsilon(s,v)} - 1 \right\|_p \leq C(\epsilon^{1/4} + |y-v|^{1/2}).$$

Therefore,

$$B_{1,\epsilon} \leq C\epsilon^{1/2} \int_0^t \int_{\mathbb{R}} G^2(t+\epsilon-s, x-y) f(y)^2 dy ds$$
$$+ C \int_0^t \int_{\mathbb{R}} \Big( \int_{\mathbb{R}} G(\epsilon, v-y) G(t-s, x-v) |v-y|^{1/2} dv \Big)^2 f(y)^2 dy ds.$$

Clearly,

$$\int_0^t \int_{\mathbb{R}} G^2(t+\epsilon-s, x-y) f(y)^2 dy ds \leq C,$$

by Hölder's inequality and assumption (A.2). Next, make the change of variables $v - y = z$ and choose $q_1 > 1$ such that $\frac{1}{q_1} + \frac{2}{q} = 1$, to get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \Big( \int_{\mathbb{R}} G(\epsilon, v-y) G(t-s, x-v) |v-y|^{1/2} dv \Big)^2 f(y)^2 dy ds \\
&\quad \leq \|f\|^2_{L^q(\mathbb{R})} \int_0^t \Big( \int_{\mathbb{R}} G(\epsilon, z) |z|^{1/2} dz \Big)^2 \|G(t-s, \cdot)\|^2_{L^{2q_1}(\mathbb{R})} ds \\
&\quad \leq C\epsilon^{1/2}.
\end{aligned}$$

Hence, $B_{1,\epsilon} \to 0$ as $\epsilon \to 0$. On the other hand, again using Lemma 2.1.1, yields

$$B_{2,\epsilon} \leq C \|f\|^2_{L^q(\mathbb{R})} \epsilon^{1/2 - 1/q}.$$

*Proof of (2.34):* Adding and subtracting $\psi(s,v)$ and $\frac{\partial \psi}{\partial v}(s,v)$ in the $dz$ integrals of the first and second terms, respectively, we get the decomposition

$$\int_0^t \int_{\mathbb{R}} [G_{1,+,\epsilon}(s,y) - G_{2,+,\epsilon}(s,y)] \sigma_s(y) W(ds,dy) = J_{1,\epsilon} + J_{2,\epsilon} + J_{3,\epsilon},$$

where

$$J_{1,\epsilon} = \int_0^t \int_0^\infty \int_{\mathbb{R}} \frac{G(t-s,x-v)}{\psi_\epsilon^2(s,v)} \frac{\partial \psi_\epsilon}{\partial v}(s,v) \\ \times \left( \int_y^\infty G(\epsilon, v-z) \big[ \psi(s,z) - \psi(s,v) \big] dz \right) dv \sigma_s(y) W(ds,dy),$$

$$J_{2,\epsilon} = \int_0^t \int_0^\infty \int_{\mathbb{R}} \frac{G(t-s,x-v)}{\psi_\epsilon(s,v)} \\ \times \left( \int_y^\infty G(\epsilon, v-z) \Big[ \frac{\partial \psi}{\partial z}(s,z) - \frac{\partial \psi}{\partial v}(s,v) \Big] dz \right) dv \sigma_s(y) W(ds,dy)$$

and

$$J_{3,\epsilon} = \int_0^t \int_0^\infty \int_{\mathbb{R}} \frac{G(t-s,x-v)}{\psi_\epsilon^2(s,v)} \Big[ \psi(s,v) \frac{\partial \psi_\epsilon}{\partial v}(s,v) - \psi_\epsilon(s,v) \frac{\partial \psi}{\partial v}(s,v) \Big] \\ \times \Big( \int_y^\infty G(\epsilon, v-z) dz \Big) dv \; \sigma_s(y) W(ds,dy).$$

Applying Burkholder and Minkowski inequalities yields, for any $p \geq 2$,

$$\begin{aligned} \|J_{1,\epsilon}\|_p^2 \;\; &\leq \;\; c_p \int_0^t \int_0^\infty \left( \int_{\mathbb{R}} G(t-s,x-v) \left\| \frac{1}{\psi_\epsilon^2(s,v)} \frac{\partial \psi_\epsilon}{\partial v}(s,v) \right\|_{2p} \right. \\ &\qquad \left. \times \int_{\mathbb{R}} G(\epsilon, v-z) \left\| \psi(s,z) - \psi(s,v) \right\|_{2p} dz dv \right)^2 f^2(y) dy ds. \end{aligned}$$

By Lemma 2.3.1 and Proposition 2.2.4, we obtain

$$\begin{aligned} \|J_{1,\epsilon}\|_p^2 \;\; &\leq \;\; c_p \|f\|_{L^2(\mathbb{R})}^2 \int_0^t \left( \int_{\mathbb{R}} G(t-s,x-v) \int_{\mathbb{R}} G(\epsilon, v-z) |z-v|^{1/2} dz dv \right)^2 ds \\ &\leq \;\; C\epsilon^{1/2}. \end{aligned}$$

For the term $J_{2,\epsilon}$ we can write, using Burkholder and Minkowski inequalities and applying Lemma 2.3.1

$$
\begin{aligned}
\|J_{2,\epsilon}\|_p^2 &\leq c_p \int_0^t \int_0^\infty \left( \int_{\mathbb{R}} G(t-s,x-v) \left\| \psi_\epsilon^{-1}(s,v) \right\|_{2p} \right. \\
&\quad \times \left. \int_{\mathbb{R}} G(\epsilon, v-z) \left\| \frac{\partial \psi}{\partial z}(s,z) - \frac{\partial \psi}{\partial v}(s,v) \right\|_{2p} dz dv \right)^2 f^2(y) dy ds \\
&\leq c_p \|f\|_{L^2(\mathbb{R})}^2 \int_0^t \left( \int_{\mathbb{R}^2} G(t-s,x-v) G(\epsilon, v-z) \right. \\
&\quad \times \left. \left\| \frac{\partial \psi}{\partial z}(s,z) - \frac{\partial \psi}{\partial v}(s,v) \right\|_{2p} dz dv \right)^2 ds.
\end{aligned}
$$

By the continuity of $(s,z) \to \frac{\partial \psi}{\partial z}(s,z)$ in $L^p$, for any $p \geq 2$, in $[0,t] \times \mathbb{R}$, established in Remark 2.2.8, it follows that the integrand of the above integral on $[0,t]$ converges to zero for any $s \in [0,t]$. On the other hand, the integrand is bounded by an integrable function, which does not depend on $\epsilon$. Therefore, by the dominated convergence theorem, we conclude that $\|J_{2,\epsilon}\|_p^2$ converges to zero as $\epsilon$ tends to zero.

Finally for $J_{3,\epsilon}$, using Burkholder and Minkowski inequalities and applying Lemma 2.3.1, we have

$$
\begin{aligned}
\|J_{3,\epsilon}\|_p^2 &\leq c_p \|f\|_{L^2(\mathbb{R})}^2 \int_0^t \left( \int_{\mathbb{R}} G(t-s,x-v) \left\| \psi_\epsilon^{-2}(s,v) \right\|_{2p} \right. \\
&\quad \times \left. \left\| \psi(s,v) \frac{\partial \psi_\epsilon}{\partial v}(s,v) - \psi_\epsilon(s,v) \frac{\partial \psi}{\partial v}(s,v) \right\|_{2p} dv \right)^2 ds.
\end{aligned}
$$

For $(s,v) \in (0,t) \times \mathbb{R}$, the term $\left\| \psi(s,v) \frac{\partial \psi_\epsilon}{\partial v}(s,v) - \psi_\epsilon(s,v) \frac{\partial \psi}{\partial v}(s,v) \right\|_{2p}$ converges to zero as $\epsilon$ tends to zero, due to the estimates (2.31) and (2.32). Therefore, by the dominated convergence theorem we conclude that $\|J_{3,\epsilon}\|_p^2$ tends to zero as $\epsilon$ tends to zero. The proof of (2.35) is similar and omitted.

*Step 4.* Finally, we show that $A_{4,\epsilon} + A_{5,\epsilon}$ converges to zero in $L^p(\Omega)$ for all $p \geq 2$, as $\epsilon$ tends to zero. Once again, we show convergence of the terms when $z \geq y \geq 0$. When $z \leq y \leq 0$, the proof follows in the same way. The contribution of $\{y > 0\}$ can be expressed as follows

$$H_\epsilon := \int_0^t \int_0^\infty \int_{\mathbb{R}} \frac{\partial}{\partial v} G(t-s, x-v) \frac{1}{\psi_\epsilon^2(s,v)} \Big( \int_y^\infty G(\epsilon, v-z)\psi(s,z)dz \Big) \\ \times \Big( \psi_\epsilon(s,v) - \int_y^\infty G(\epsilon, v-z)\psi(s,z)dz \Big) \sigma_s(y)^2 dv dy ds.$$

Adding and subtracting $\psi(s,v)$ in the second $dz$ integral, we get

$$H_\epsilon = \sum_{i=1}^3 H_{i,\epsilon},$$

where

$$H_{i,\epsilon} = \int_0^t \int_0^\infty \int_{\mathbb{R}} \frac{\partial}{\partial v} G(t-s, x-v) \\ \times \frac{1}{\psi_\epsilon^2(s,v)} \Big( \int_y^\infty G(\epsilon, v-z)\psi(s,z)dz \Big) F_i \sigma_s(y)^2 dv dy ds,$$

where

$$F_1 := \int_{-\infty}^y G(\epsilon, v-z)\psi_\epsilon(s,v)dz,$$
$$F_2 := \int_y^\infty G(\epsilon, v-z)\big[\psi_\epsilon(s,v) - \psi(s,v)\big]dz,$$
$$F_3 := \int_y^\infty G(\epsilon, v-z)\big[\psi(s,v) - \psi(s,z)\big]dz.$$

We show convergence of each of these three terms, one at a time. To control the term $H_{1,\epsilon}$, apply Minkowski's inequality, Hölder's inequality, and Lemma 2.3.1, to get, for anu $p \geq 2$

$$\|H_{1,\epsilon}\|_p \leq C \int_0^t \int_0^\infty \int_{\mathbb{R}} \frac{\partial}{\partial v} G(t-s, x-v) \\ \times \Big( \int_y^\infty G(\epsilon, v-z) dz \Big) \Big( \int_{-\infty}^y G(\epsilon, v-z) dz \Big) f(y)^2 dv dy ds.$$

Notice that, for any fixed $s, y, v$, we have

$$\frac{\partial}{\partial v} G(t-s, x-v) \Big( \int_y^\infty G(\epsilon, v-z) dz \Big) \Big( \int_{-\infty}^y G(\epsilon, v-z) dz \Big) f(y)^2 \to 0$$

as $\epsilon \to 0$. Furthermore, we can trivially bound this integrand by

$$\Big| \frac{\partial}{\partial v} G(t-s, x-v) \Big| f(y)^2,$$

which is independent of $\epsilon$, and $(dv \otimes dy \otimes ds)$-integrable on $\mathbb{R} \times (0, \infty) \times [0, t]$. Hence, by dominated convergence, $\|H_{1,\epsilon}\|_p \to 0$ as $\epsilon \to 0$.

We bound the term with $H_{2,\epsilon}$ as follows

$$\|H_{2,\epsilon}\|_p \leq C \sup_{s,v} \|\psi_\epsilon(s,v) - \psi(s,v)\|_{2p} \int_0^t \int_0^\infty \int_{\mathbb{R}} \Big| \frac{\partial}{\partial v} G(t-s, x-v) \Big| f(y)^2 dv dy ds$$

for some positive constant $C > 0$ and all $p \geq 2$. This quantity converges to zero as $\epsilon \to 0$ by Lemma 2.3.1. Lastly, apply the same techniques to get

$$\|H_{3,\epsilon}\|_p \leq C \int_0^t \int_0^\infty \int_{\mathbb{R}} \Big| \frac{\partial}{\partial v} G(t-s, x-v) \Big| \Big( \int_y^\infty G(\epsilon, v-z) |v-z|^{1/2} dz \Big) f(y)^2 dv dy ds \\ \leq C \epsilon^{1/4}.$$

which converges to zero as $\epsilon \to 0$. Therefore, $A_{4,\epsilon} + A_{5,\epsilon}$ converges to zero in $L^p(\Omega)$ as $\epsilon \to 0$, for all $p \geq 2$.

*Step 5.* As a conclusion, we deduce that the process $v(t,x)$ satisfies

$$v(t,x) = \int_{\mathbb{R}} G(t,x-y)u_0(y)dy + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s,x-y)v(s,y)^2 dyds \\ + \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\sigma_s(y)W(ds,dy).$$

Since $u$ also satisfies this equation, we have $u \equiv v$ by uniqueness of solutions. $\square$

## 2.4 Regularity

We start with an easy, yet interesting, consequence of some of our results about $\psi$ and its regularity.

**Proposition 2.4.1.** Let $u(t,x)$ denote the solution to Burgers' equation (2.1). Assume that the initial condition $u_0$ is $\alpha$ Hölder continuous for some $\alpha \in (0,1)$. Then, for all $t,s \in [0,T]$, $x,y \in \mathbb{R}$, and $p \geq 2$, we have

$$\|u(t,x) - u(s,y)\|_p \leq C(|t-s|^{\frac{\alpha}{2}\wedge(\frac{1}{4}-\frac{1}{2q})} + |x-y|^{\alpha\wedge(\frac{1}{2}-\frac{1}{q})}).$$

*Proof.* Indeed, by adding and subtracting an appropriate term, we have

$$\begin{aligned} \|u(t,x) - u(t,y)\|_p &= 2\Big\| \frac{1}{\psi(s,y)} \frac{\partial \psi}{\partial y}(s,y) - \frac{1}{\psi(s,x)} \frac{\partial \psi}{\partial x}(s,x) \Big\|_p \\ &\leq 2\Big\| \frac{\partial \psi}{\partial y}(s,y) \frac{\psi(s,x) - \psi(s,y)}{\psi(s,x)\psi(s,y)} \Big\|_p \\ &\qquad + 2\Big\| \frac{1}{\psi(s,x)} \Big[ \frac{\partial \psi}{\partial y}(s,y) - \frac{\partial \psi}{\partial x}(s,x) \Big] \Big\|_p \\ &\leq C\left( |x-y|^{1/2} + |x-y|^{\alpha\wedge(\frac{1}{2}-\frac{1}{q})} \right), \end{aligned}$$

where the last inequality follows from Cauchy-Schwarz inequality, (2.7), and Propositions 2.2.4, 2.2.6, and 2.2.7. Using the same technique of adding and subtracting appropriate terms yields the desired regularity in $t$. $\square$

**Remark 2.4.2.** Regarding the assumptions on the initial condition.

(i) From Remark 2.2.8 it follows that if we do not assume the Hölder continuity of $u_0$, then we have

$$\|u(t,x)-u(s,y)\|_p \leq C(t\wedge s)^{-1/2}\left(|t-s|^{\frac{1}{4}-\frac{1}{2q}}+|x-y|^{\frac{1}{2}-\frac{1}{q}}\right),$$

Moreover, $u(t,x)$ is continuous in $L^p(\Omega)$ on $[0,T]\times\mathbb{R}$ for all $p\geq 2$.

(ii) Proposition (2.4.1) allows us to deduce the existence of a version of $u(t,x)$, which is locally Hölder continuous in space of order $\alpha\wedge(\frac{1}{2}-\frac{1}{q})$ and in time of order $\frac{\alpha}{2}\wedge(\frac{1}{4}-\frac{1}{2q})$.

The next proposition provides some moment estimates for the solution to Burgers equation.

**Proposition 2.4.3.** Let $u(t,x)$ denote the solution to Burgers' equation (2.1). Then, for all $t\in[0,T]$ and $x\in\mathbb{R}$, and $p\geq 2$, we have

$$\sup_{x\in\mathbb{R}}\|u(t,x)\|_p \leq K\sqrt{p}\,(t\vee 1)^{1-\frac{1}{q}}\exp\left(\frac{33}{2}\|f\|^2_{L^2(\mathbb{R})}tp+\frac{1}{2}\|f\|^4_{L^2(\mathbb{R})}t^2+\frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}\right),$$

where $K$ is a constant depending on $\|f\|_{L^q(\mathbb{R})}$ and $\|u_0\|_\infty$.

*Proof.* By Hölder's inequality, we can write

$$\|u(t,x)\|_p = 2\left\|\psi(t,x)^{-1}\frac{\partial\psi}{\partial x}(t,x)\right\|_p \leq 2\left\|\psi(t,x)^{-1}\right\|_{2p}\left\|\frac{\partial\psi}{\partial x}(t,x)\right\|_{2p}.$$

Then, the result follows from Remark 2.2.2 and Proposition 2.2.6. □

**Corollary 2.4.4.** Let $u(t,x)$ be as above. Then, we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E\left[e^{\alpha(\log_+ u(t,x))^2}\right] < \infty$$

for all $0 < \alpha < \frac{1}{4a}$ , where $a = \frac{33}{2}\|f\|^2_{L^2(\mathbb{R})}T$, and $\log_+ X := \log(X \vee 1)$.

*Proof.* Using the above result regarding moment estimates, we see that $u$ satisfies

$$E\left(|u(t,x)|^p\right) \le C^p e^{ap^2}$$

for some constant $C = C(\|f\|_{L^q(\mathbb{R})}, \|f\|_{L^2(\mathbb{R})}, \|u_0\|_\infty, \|u_0\|_{L^1(\mathbb{R})}, T)$. Then, apply Lemma A.10 to obtain the result. □

From this, we can extract information about the tail of the probability distribution of the solution to Burgers' equation.

**Corollary 2.4.5.** Arbitrarily fix $\alpha \in (0, \frac{1}{4a})$ as above. Then for all $\lambda > e$, we have

$$\limsup_{\lambda\uparrow\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \frac{\log P(u(t,x) > \lambda)}{(\log\lambda)^2} \le -\alpha.$$

*Proof.* For any $t$, $x$, and $\lambda$, we see that

$$P(u(t,x) > \lambda) \le e^{-\alpha(\log_+\lambda)^2} E\left[e^{\alpha(\log_+ u(t,x))^2}\right].$$

Then, by Corollary 2.4.4, the result follows easily. □

# Chapter 3

# Existence and Regularity of Densities

## 3.1 Malliavin Calculus

In this chapter, we present some new results regarding the existence and regularity of density functions for solutions to stochastic Burgers'-type equations. There are powerful tools from Malliavin calculus which allow us to obtain such results. As such, we begin by reviewing some of the standard Malliavin calculus machinery that will be used in what follows. For a more thorough presentation of the subject refer to [23]. For some techniques which are now standard in the study of SPDEs, refer to [2]. Let $\mathcal{H} := L^2([0,T] \times \mathbb{R})$, and $\mathcal{S}$ be the set of smooth cylindrical random variables

$$\mathcal{S} := \{F = f(W(h_1), \ldots, W(h_n)) \mid f \in C_p^\infty(\mathbb{R}^n),\ h_i \in \mathcal{H}\}.$$

The subscript $p$ above is to denote polynomial growth of the derivatives. Given such a random variable, $F \in \mathcal{S}$, the Malliavin derivative of $F$ is the $\mathcal{H}$-valued stochastic process $\{D_{t,x}F, (t,x) \in [0,T] \times \mathbb{R}\}$ defined by

$$D_{t,x}F := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i(t,x).$$

Define the higher order derivative operator, $D^k$, iteratively. This iterated derivative operator is closable from $\mathcal{S}$ into $L^p(\Omega;\mathcal{H}^{\otimes k})$ for each $k,p \geq 1$. Let $\mathbb{D}^{k,p} := \overline{\mathcal{S}}^{\|\cdot\|_{k,p}}$ be the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p := E(|F|^p) + E\sum_{i=1}^{n} \|D^i F\|_{\mathcal{H}^{\otimes i}}^p,$$

and define $\mathbb{D}^\infty := \bigcap_{k,p} \mathbb{D}^{k,p}$.

Let $\mathbb{D}_{loc}^{k,p}$ denote the set of all random variables which are locally Malliavin differentiable of order $k$. That is, the set of $F$ for which there exists a sequence $(\Omega_n, F_n) \subset \mathcal{F} \times \mathbb{D}^{k,p}$ such that $\Omega_n \nearrow \Omega$ and $F_n = F$ on $\Omega_n$ with probability one. The following is a useful result for establishing regularity of densities.

**Proposition 3.1.1.** If $F \in \mathbb{D}_{loc}^{1,1}$ and $\|DF\|_{\mathcal{H}} > 0$ a.s., then $F$ has a density function which is absolutely continuous with respect to the Lebesgue measure.

Next is a fundamental result of Malliavin calculus and provides sufficient criteria for existence of a smooth density for (one dimensional) random variables.

**Theorem 3.1.2.** *Let $F$ be a random variable. Then, the following criteria are sufficient conditions on the existence and regularity of a density funciton.*

*(i) If $F \in \mathbb{D}^{1,1}$ and $\|DF\|_{\mathcal{H}} \overset{a.s.}{>} 0$, then $F$ has an absolutely continuous density.*

*(ii) If $F \in \mathbb{D}^\infty$ and $E(\|DF\|_{\mathcal{H}}^{-p}) < \infty$ for all $p \geq 2$, then $F$ has a $C^\infty$ density.*

This result can be generalized to random vectors by replacing the conditions on the norms by the same conditions on the determinant of the Malliavin matrix[1]. The following fact is a standard and useful tool in establishing Malliavin differentiability by an approximating procedure (see Lemma 1.2.3 in [23]).

[1] The $i,j$ entry of the Malliavin matrix is defined by the inner product of the derivatives of the $i$th and $j$th components of the random vector.

**Lemma 3.1.3.** *Suppose $\{u^{(k)}\}_k$ is a sequence of processes such that $u^{(k)} \in \mathbb{D}^{1,2}$ for all $k$, $\sup_k \sup_{t,x} E\big(\|Du^{(k)}(t,x)\|_{\mathcal{H}}^2\big) < \infty$, and $u^{(k)} \to u$ in $L^2(\Omega)$. Then $u \in \mathbb{D}^{1,2}$ and $Du^{(k)}$ converges to $Du$ in the weak topology of $L^2(\Omega, \mathcal{H})$.*

## 3.2 Density via Feynman-Kac

In this section, we study existence and regularity of densities for stochastic Burgers' equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}\frac{\partial}{\partial x}u(t,x)^2 + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}$$

indexed by $(t,x) \in [0,T] \times \mathbb{R}$, given a nonrandom initial condition $u_0$ and a Brownian sheet $W$. Using what we proved in chapter 2, we follow similar steps as in [18], but have to adjust some arguments due to the fact that we are working in an unbounded spatial domain. Let

$$u(t,x) = -2\frac{\partial}{\partial x}\log\psi(t,x) = -2\frac{1}{\psi(t,x)}\frac{\partial \psi}{\partial x}(t,x)$$

be the mild solution to Burgers' equation (2.1). Recall that $\psi$ is defined by a Feynman-Kac formula

$$\psi(t,x) := \mathbb{E}^{\beta}_{x,t}\left[\psi_0(\beta_0)e^{-\frac{1}{2}M_t^{\beta}}\right], \tag{3.1}$$

where $\psi_0(x) := \exp\left\{-\frac{1}{2}\int_0^x u_0(y)dy\right\}$ and $M_t^{\beta} := \int_0^t \int_{\mathbb{R}} \sigma_s(y)\mathbf{1}_{[0,\beta_s]}(y)W(ds,dy)$.

**Theorem 3.2.1.** *Suppose that for all $(t,x) \in [0,T] \times \mathbb{R}$, and some $x_0 \in \mathbb{R}$, we have*

$$\begin{cases} \sigma(t,x,r) \equiv \sigma(t,x) \\ \sigma(0,x_0) \neq 0 \end{cases} \tag{H}$$

*and that $\sigma$ is continuous. Then, for fixed $(t,x) \in (0,T] \times \mathbb{R}$, the solution, $u(t,x)$, to Burgers' equation (2.1) has a smooth probability density function.*

We remark that the assumption that $\sigma$ does not depend on the third component is not a significant jump due to the strong integrability assumption (2.3).

We prove Theorem 3.2.1 by establishing each the hypotheses of part *(ii)* of Theorem 3.1.2, the first of which is the following.

**Proposition 3.2.2.** Suppose that $\sigma$ satisfies (H) and let $u$ be the solution to (2.1). Then, for fixed $t, x$, $u(t,x) \in \mathbb{D}^\infty$.

*Proof.* Due to the chain rule for Malliavin calculus, it suffices to prove differentiability of $\psi$ and $\dfrac{\partial \psi}{\partial x}$. We establish these using the explicit Feynman-Kac representation of $\psi$, and a Picard iteration scheme for $\dfrac{\partial \psi}{\partial x}$.

Using the definition of $\psi$, we see that

$$D_{r,v}\psi(t,x) = -\frac{1}{2}\mathbb{E}^{\beta}_{x,t}\Big[\psi_0(\beta_0)\sigma_r(v)\mathbf{1}_{[0,\beta_r]}(v)e^{-\frac{1}{2}M^{\beta}_t}\Big].$$

The integrability assumption on $\sigma$ implies that

$$E\Big[\Big(\int_0^t\int_{\mathbb{R}}\big|D_{r,v}\psi(t,x)\big|^2 dv dr\Big)^{p/2}\Big] < \infty$$

for all $p$. Hence, $\psi(t,x) \in \mathbb{D}^{1,p}$. Furthermore, we can take higher order derivatives and obtain $\psi(t,x) \in \mathbb{D}^\infty$ in the same way .

Next, we study Malliavin-differentiability of the process $\partial_x\psi$. Consider the corresponding integral equation

$$\begin{aligned}\frac{\partial\psi}{\partial x}(t,x) = &\int_{\mathbb{R}} \frac{\partial G}{\partial x}(t,x-y)\psi_0(y)dy \\ &- \frac{1}{2}\int_0^t \int_S \mathrm{sign}(y)\frac{\partial G}{\partial x}(t-s,x-z)\psi(s,z)\sigma_s(y)dzW(ds,dy) \\ &+ \frac{1}{8}\int_0^t \int_S \frac{\partial G}{\partial x}(t-s,x-z)\psi(s,z)\sigma_s(y)^2 dzdyds. \end{aligned} \tag{3.2}$$

Integrate by parts to get

$$\begin{aligned}\frac{\partial\psi}{\partial x}(t,x) = &\int_{\mathbb{R}} G(t,x-y)\psi_0'(y)dy \\ &- \frac{1}{2}\int_0^t \int_{\mathbb{R}} \mathrm{sign}(y)G(t-s,x-y)\psi(s,y)\sigma_s(y)W(ds,dy) \\ &+ \frac{1}{2}\int_0^t \int_S \mathrm{sign}(y)G(t-s,x-z)\frac{\partial\psi}{\partial z}(t,z)\sigma_s(y)dzW(ds,dy) \\ &+ \frac{1}{8}\int_0^t \int_{\mathbb{R}} G(t-s,x-y)\psi(s,y)\sigma_s(y)^2 dyds \\ &- \frac{1}{8}\int_0^t \int_S G(t-s,x-z)\frac{\partial\psi}{\partial z}(s,z)\sigma_s(y)^2 dzdyds. \end{aligned} \tag{3.3}$$

Set $\phi_t^{(0)}(x) := (G(t,\cdot)*\psi_0')(x)$, and define the Picard iteration by

$$\begin{aligned}\phi_t^{(n)}(x) = \phi_t^{(0)}(x) &- \frac{1}{2}\int_0^t \int_{\mathbb{R}} \mathrm{sign}(y)G(t-s,x-y)\psi(s,y)\sigma_s(y)W(ds,dy) \\ &+ \frac{1}{2}\int_0^t \int_S \mathrm{sign}(y)G(t-s,x-z)\phi_s^{(n-1)}(z)\sigma_s(y)dzW(ds,dy) \\ &+ \frac{1}{8}\int_0^t \int_{\mathbb{R}} G(t-s,x-y)\psi(s,y)\sigma_s(y)^2 dyds \\ &- \frac{1}{8}\int_0^t \int_S G(t-s,x-z)\phi_s^{(n-1)}(z)\sigma_s(y)^2 dzdyds. \end{aligned} \tag{3.4}$$

We prove

$$\phi_t^{(n)}(x) \xrightarrow[n\to\infty]{L^p(\Omega)} \frac{\partial \psi}{\partial x}(t,x) \tag{3.5}$$

by showing that

$$\sum_{n\geq 1} \sup_{t,x} \left\| \Delta\phi_t^{(n)}(x) \right\|_p^2 < \infty, \tag{3.6}$$

where

$$\Delta\phi_t^{(n)}(x) := \phi_t^{(n)}(x) - \phi_t^{(n-1)}(x).$$

For all $n \geq 0$, we have

$$\begin{aligned}
\Delta\phi_t^{(n+1)}(x) = \frac{1}{2}\int_0^t \int_S & sign(y) G(t-s, x-z) \Delta\phi_s^{(n)}(z)\sigma_s(y) W(ds,dy) \\
& - \frac{1}{8}\int_0^t \int_S G(t-s,x-z)\Delta\phi_s^{(n)}(z)\sigma_s(y)^2 dyds. \\
=: \frac{1}{2}F_1(t,x) - \frac{1}{8}F_2(t,x).
\end{aligned}$$

Decompose these terms into

$$F_1(t,x) = F_{1,A}(t,x) - F_{1,B}(t,x),$$

$$F_2(t,x) = F_{2,A}(t,x) + F_{2,B}(t,x),$$

where

$$F_{1,A}(t,x) := \int_0^t \int_0^\infty \int_y^\infty G(t-s,x-z)\Delta\phi_s^{(n)}(z)\sigma_s(y)dzW(ds,dy),$$
$$F_{1,B}(t,x) := \int_0^t \int_{-\infty}^0 \int_{-\infty}^y G(t-s,x-z)\Delta\phi_s^{(n)}(z)\sigma_s(y)dzW(ds,dy),$$
$$F_{2,A}(t,x) := \int_0^t \int_0^\infty \int_y^\infty G(t-s,x-z)\Delta\phi_s^{(n)}(z)\sigma_s(y)^2dzdyds,$$
$$F_{2,B}(t,x) := \int_0^t \int_{-\infty}^0 \int_{-\infty}^y G(t-s,x-z)\Delta\phi_s^{(n)}(z)\sigma_s(y)^2dzdyds.$$

Apply Burkholder's inequality and the bounds on $\sigma$ to get

$$\|F_{1,A}(t,x)\|_p^2 \le C\int_0^t \Big(\int_{\mathbb{R}} G(t-s,x-z)\|\Delta\phi_s^{(n)}(z)\|_p dz\Big)^2 ds.$$

Apply the same techniques on $F_{1,B}$ to get

$$\|F_1(t,x)\|_p^2 \le C\int_0^t \Big(\int_{\mathbb{R}} G(t-s,x-z)\|\Delta\phi_s^{(n)}(z)\|_p dz\Big)^2 ds. \tag{3.7}$$

In a similar way, we obtain

$$\|F_2(t,x)\|_p^2 \le C\Big(\int_0^t \int_{\mathbb{R}} G(t-s,x-z)\|\Delta\phi_s^{(n)}(z)\|_p dzds\Big)^2. \tag{3.8}$$

Putting (3.7) and (3.8) together yields

$$\sup_x \|\Delta\phi_t^{(n+1)}(x)\|_p^2 \le C_{p,T,\sigma}\int_0^t \sup_x \|\Delta\phi_s^{(n)}(x)\|_p^2 ds.$$

Apply Lemma A.7 to get (3.6). Hence, (3.5) holds as desired. Next, we show that $\phi_t^{(n)}(x) \in \mathbb{D}^{1,p}$, using induction. Suppose that $\phi_t^{(n-1)}(x) \in \mathbb{D}^{1,p}$ and set

$$\Phi_{t,x}^{(n-1)} := E\Big(\big\|D\phi_t^{(n-1)}(x)\big\|_{L^2([0,t]\times\mathbb{R})}^p\Big).$$

Using (3.4), we see that

$$D_{r,v}\phi_t^{(n)}(x) = -\frac{1}{2}\big[G_1(t,x,r,v) - G_2(t,x,r,v) - G_3(t,x,r,v)\big] + \frac{1}{8}G_4(t,x,r,v), \tag{3.9}$$

where

$$\begin{aligned}
G_1(t,x,r,v) &= \operatorname{sign}(v)G(t-r,x-v)\psi(r,v)\sigma_r(v) \\
&\qquad + \int_r^t \int_{\mathbb{R}} G(t-s,x-y)D_{r,v}\psi(s,y)\sigma_s(y)W(ds,dy), \\
G_2(t,x,r,v) &= \sigma_r(v)\Big(\mathbf{1}_{v\geq 0}\int_v^\infty G(t-r,x-z)\phi_r^{(n-1)}(z)dz \\
&\qquad + \mathbf{1}_{v<0}\int_{-\infty}^v G(t-r,x-z)\phi_r^{(n-1)}(z)dz\Big), \\
G_3(t,x,r,v) &= \int_r^t \int_S G(t-s,x-z)D_{r,v}\phi_s^{(n-1)}(z)\sigma_s(y)dzW(ds,dy), \\
G_4(t,x,r,v) &= \int_r^t \int_{\mathbb{R}} G(t-s,x-y)D_{r,v}\psi(s,y)\sigma_s(y)^2dyds \\
&\qquad - \int_r^t \int_S G(t-s,x-z)D_{r,v}\phi_s^{(n-1)}(z)\sigma_s(y)^2dzdyds.
\end{aligned}$$

holds for $(r,v) \in (0,t) \times \mathbb{R}$. Let's consider the two terms of $G_1$ first, one at a time. Observe that Minkowski's and Hölder's inequalities yield

$$\begin{aligned}
&\Big\| \int_0^t \int_{\mathbb{R}} G^2(t-r,x-v)\psi(r,v)^2 \sigma_r(v)^2 dv dr \Big\|_{p/2} \\
&\qquad\qquad \le \int_0^t \int_{\mathbb{R}} G^2(t-r,x-v) \|\psi(r,v)\|_p^2 f(v)^2 dv dr \\
&\qquad\qquad \le C
\end{aligned}$$

for some constant, $C$, depending on $p$, $f$, and $T$. Similarly, apply Burkholder's inequality and the fact that $\psi \in \mathbb{D}^{1,p}$ to get

$$\begin{aligned}
&\Big\| \int_0^t \int_{\mathbb{R}} \Big( \int_r^t \int_{\mathbb{R}} G(t-s,x-y)\psi(s,y)\sigma_s(y) W(ds,dy) \Big)^2 dv dr \Big\|_{p/2} \\
&\qquad \le \int_0^t \int_{\mathbb{R}} \Big( \int_0^s \int_{\mathbb{R}} \|D_{r,v}\psi(s,y)\|_p^2 dv dr \Big) G^2(t-s,x-y) f(y)^2 dy ds \\
&\qquad \le C
\end{aligned}$$

for some constant $C$. Using Minkowski's inequality, we obtain

$$\begin{aligned}
&\Big\| \int_0^t \int_{\mathbb{R}} \sigma_r(v)^2 \Big( \int_v^\infty G(t-r,x-z) \phi_r^{(n-1)}(z) dz \Big)^2 dv dr \Big\|_{p/2} \\
&\qquad\qquad \le C \int_0^t \Big( \int_{\mathbb{R}} G(t-r,x-z) \|\phi_r^{(n-1)}(z)\|_p dz \Big)^2 dr \\
&\qquad\qquad \le C'
\end{aligned}$$

for some positive constant $C'$. By symmetry, we have the same bound for $G_2$. To handle $G_3$, we perform similar computations after applying Burkholder's inequality to see that

$$
\begin{aligned}
&E\Big[\Big(\int_0^t\int_{\mathbb{R}}\Big(\int_r^t\int_0^\infty\int_y^\infty G(t-s,x-z)D_{r,v}\phi_s^{(n-1)}(z)\sigma_s(y)dzW(ds,dy)\Big)^2dvdr\Big)^{p/2}\Big]\\
&\quad\leq c_pE\Big[\Big(\int_0^t\int_{\mathbb{R}}\int_r^t\int_0^\infty\Big(\int_y^\infty G(t-s,x-z)D_{r,v}\phi_s^{(n-1)}(z)dz\Big)^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\sigma_s(y)^2dyds\Big)^2dvdr\Big)^{p/2}\Big]\\
&\quad\leq CE\Big[\Big(\int_0^t\int_{\mathbb{R}}\int_r^t\int_{\mathbb{R}}G(t-s,x-z)\big(D_{r,v}\phi_s^{(n-1)}(z)\big)^2dzdsdvdr\Big)^{p/2}\Big]\\
&\quad\leq C'\int_0^t\int_{\mathbb{R}}G(t-s,x-z)E\Big(\|D\phi_s^{(n-1)}(z)\|^p_{L^2([0,s]\times\mathbb{R})}\Big)dzds\\
&\quad=C'\int_0^t\int_{\mathbb{R}}G(t-s,x-z)\Phi_{s,z}^{(n-1)}dzds
\end{aligned}
$$

for some positive constant $C'$ depending on $p,t.$ and $f$. Again, by symmetry, we obtain the same upper bound on $G_3$. Use the same arguments to obtain an upper bound for $G_4$ of the same form. Putting these pieces together yields

$$\sup_{x\in\mathbb{R}}\Phi_{t,x}^{(n)}\leq C_1+C_2\int_0^t\sup_{x\in\mathbb{R}}\Phi_{s,x}^{(n-1)}ds.$$

Therefore, by Lemma A.7, we have

$$\sup_{n,t,x}\Phi_{t,x}^{(n)}<\infty.$$

That is, $\phi_t^{(n)}(x)\in\mathbb{D}^{1,p}$. Hence, $\frac{\partial\psi}{\partial x}(t,x)\in\mathbb{D}^{1,p}$ and

$$\sup_{t,x}E\left(\left\|D\frac{\partial\psi}{\partial x}(t,x)\right\|^p_{L^2([0,t]\times\mathbb{R})}\right)<\infty.$$

Using the same arguments as above, we can show that $\frac{\partial\psi}{\partial x}(t,x) \in \mathbb{D}^{k,p}$ for all $k$. That is, we have that $\frac{\partial\psi}{\partial x}(t,x) \in \mathbb{D}^{\infty}$.

$\square$

We now prove a fact regarding the norm of the Malliavin derivative of $u$, from which we obtain the main result of the section.

**Proposition 3.2.3.** Suppose that $\sigma$ satisfies (H) and let $u$ be the solution to (2.1). Then, for fixed $t, x$, and all $p \geq 2$, we have

$$E\big(\|Du(t,x)\|_{\mathcal{H}}^{-p}\big) < \infty.$$

*Proof.* Fix $a \in (\frac{1}{2}, 1)$. It suffices to show that for all $q \geq 2$, there exists $\epsilon_0$ depending on $a$ and $q$ such that

$$P\left(\int_0^t \int_{\mathbb{R}} \big(D_{r,v}u(t,x)\big)^2 dv dr \leq \epsilon^a\right) \leq \epsilon^{(1-a)q} \tag{3.10}$$

holds for $\epsilon \leq \epsilon_0$. To see why this is sufficient, set $X := \|Du(t,x)\|_{\mathcal{H}}^2$ and notice that

$$\begin{aligned} E(X^{-p}) &= \int_0^\infty P(X^{-p} > y) dy \\ &\leq 1 + \int_1^\infty P(X < y^{-1/p}) dy. \end{aligned}$$

Then, make the change variables $y = \epsilon^{-ap}$ to get

$$\int_1^\infty P(X < y^{-1/p}) dy = ap \int_0^1 P(X < \epsilon^a) \epsilon^{-ap-1} d\epsilon.$$

Since $1 - a > 0$, we can pick a $q$ large enough so that $(1-a)q > ap$. Hence, by making appropriate choices of $\epsilon_0$ and $q$, we apply (3.10) and see that this integral is bounded.

Choose $\epsilon > 0$ such that $\sigma_t(x)^2 \geq C > 0$ on $(t,x) \in [0, \sqrt{\epsilon}\,] \times [x_0 - \sqrt{\epsilon}, x_0 + \sqrt{\epsilon}]$. Now using the mild formulation, (2.1), we get

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \big(D_{r,v}u(t,x)\big)^2 dvdr \geq & \frac{1}{2}\int_0^{\sqrt{\epsilon}} \int_{x_0-\sqrt{\epsilon}}^{x_0+\sqrt{\epsilon}} \sigma_r(v)^2 G(t-r, x-v)^2 dvdr \\ & - \int_{t-\epsilon}^t \int_{x_0-\sqrt{\epsilon}}^{x_0+\sqrt{\epsilon}} \Big(\int_r^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s, x-y) u(s,y) \\ & \times D_{r,v}u(s,y) dyds\Big)^2 drdv,\end{aligned}$$

where the $x_0$ is what appears in (H). Using this hypothesis yields

$$\frac{1}{2}\int_0^{\sqrt{\epsilon}} \int_{x_0-\sqrt{\epsilon}}^{x_0+\sqrt{\epsilon}} \sigma_r(v)^2 G(t-r, x-v)^2 dvdr \geq C\sqrt{\epsilon}$$

for some constant $C$. Now, pick $\epsilon_0 > 0$ small enough such that $2\epsilon^a \leq C\sqrt{\epsilon}$ holds for $0 < \epsilon \leq \epsilon_0$. Hence,

$$\begin{aligned}& P\Big(\int_0^t \int_{\mathbb{R}} \big(D_{r,v}u(t,x)\big)^2 dvdr \leq \epsilon^a\Big) \\ & \leq P\Big(\int_{t-\epsilon}^t \int_{x_0-\sqrt{\epsilon}}^{x_0+\sqrt{\epsilon}} \Big(\int_r^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s, x-y) u(s,y) D_{r,v}u(s,y) dyds\Big)^2 drdv \geq \epsilon^a\Big) \\ & \leq \epsilon^{-\frac{ap}{2}} E\Big[\Big(\int_{t-\epsilon}^t \int_{x_0-\sqrt{\epsilon}}^{x_0+\sqrt{\epsilon}} \Big(\int_r^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s, x-y) \\ & \qquad \times u(s,y) D_{r,v}u(s,y) dyds\Big)^2 drdv\Big)^{p/2}\Big].\end{aligned}$$

Apply Cauchy Schwarz with respect to the measure given by $\frac{\partial}{\partial y}G$ to get

$$\begin{aligned}& \Big(\int_r^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s, x-y) u(s,y) D_{r,v}u(s,y) dyds\Big)^2 \\ & \qquad \leq C_{t,x,p} \int_r^t \int_{\mathbb{R}} \Big|\frac{\partial}{\partial y} G(t-s, x-y)\Big| \Big(D_{r,v}u(s,y)\Big)^2 dyds\end{aligned}$$

for some constant $C_{t,x,p}$ depending on $t, x, p$. Hence,

$$P\left(\int_0^t \int_{\mathbb{R}} \left(D_{r,v}u(t,x)\right)^2 dvdr \leq \epsilon^a\right)^{2/p}$$
$$\leq C\epsilon^{-a} \int_{t-\epsilon}^{t} \int_{x_0-\sqrt{\epsilon}}^{x_0+\sqrt{\epsilon}} \int_r^t \int_{\mathbb{R}} \left|\frac{\partial}{\partial y} G(t-s, x-y)\right| \cdot \left\|D_{r,v}u(s,y)\right\|_p^2 dydsdrdv.$$

Exchange the orders of integration to get

$$P\left(\int_0^t \int_{\mathbb{R}} \left(D_{r,v}u(t,x)\right)^2 dvdr \leq \epsilon^a\right) \leq C\epsilon^{-\frac{ap}{2}+\frac{p}{2}} = C\epsilon^{\frac{p}{2}(1-a)}.$$

□

## 3.3 More General Equations

We consider the class of semilinear stochastic partial differential equations given by

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + f(t,x,u(t,x)) + \frac{\partial}{\partial x} g(t,x,u(t,x)) + \sigma(t,x,u(t,x)) \frac{\partial^2 W}{\partial t \partial x} \tag{3.11}$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, where $W$ is a Brownian sheet, and $f$ and $\sigma$ have some appropriate growth conditions. When $f = 0 = \sigma$, and $g = -\frac{1}{2}u^2$, this is the classical (viscous) Burgers' equation. Furthermore, the case when the spatial domain is $[0,1]$, rather than $\mathbb{R}$, has been widely studied. For example, see [12] for existence and uniqueness results, [18],[22],[27] for results regarding the probability density of solutions, and [12] for comparison theorems. There are many more results regarding these equations on the unit spatial interval. However, there are very few results on the unbounded spatial domain. The primary goal of this section is to establish the existence and regularity of a probability density for the solution at fixed points in time and space. To accomplish this, we use the standard tools from Malliavin calculus. Along the way, we prove a comparison theorem for the solution which will aid in the proof of our main result of the section. The approach we take is inspired by that in [12].

We say that an $L^2(\mathbb{R})$-valued stochastic process $u = \{u(t) : t \in [0,T]\}$ which is continuous and $\mathcal{F}_t$-adapted is a *weak solution* to (3.11) if for any test function $\phi \in C_c(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}} u(t,x)\phi(x)dx = &\int_{\mathbb{R}} u_0(x)\phi(x)dx + \int_0^t \int_{\mathbb{R}} u(s,x)\phi''(x)dxds \\ &+ \int_0^t \int_{\mathbb{R}} f(s,x,u(s,x))\phi(x)dxds \\ &- \int_0^t \int_{\mathbb{R}} g(s,x,u(s,x))\phi'(x)dxds \\ &+ \int_0^t \int_{\mathbb{R}} \sigma(s,x,u(s,x))\phi(x)W(ds,dx),\end{aligned}$$

almost surely for all $t \in [0,T]$, where the last integral is understood in the Walsh sense (see [26] for a detailed treatment of this). On the other hand, we say that $u$ is a *mild solution* of (3.11) if for all $(t,x)$, $u$ almost surely satisfies the following integral equation

$$\begin{aligned}u(t,x) = &\int_{\mathbb{R}} G(t,x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}} G(t-s,x-y)f(s,y,u(s,y))dyds \\ &- \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s,x-y)g(s,y,u(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\sigma(s,y,u(s,y))W(ds,dy),\end{aligned}$$

where $G$ is the heat kernel on the real line

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Under suitable conditions, which we state below, Gyöngy and Nualart proved the equivalence of weak and mild solutions for this class of equations. Furthermore, they proved global in time existence and uniqueness for a solution to (3.11), and that the solution has a continuous version. We impose the following conditions on the coefficients in (3.11):

(A1) The initial condition $u(0,x) = u_0(x)$ is nonrandom with $u_0 \in L^2(\mathbb{R})$.

(A2) $f, \sigma : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are Borel functions satisfying the following linear growth and Lipschitz conditions

$$\begin{aligned}
|f(t,x,r)| &\leq f_1(x) + C|r| \\
|f(t,x,r) - f(t,x,s)| &\leq L|r-s| \\
|g(t,x,r)| &\leq g_1(x) + g_2(x)|r| + C|r|^2 \\
|g(t,x,r) - g(t,x,s)| &\leq (g_3(x) + L|r| + L|s|)|r-s| \\
|\sigma(t,x,r)| &\leq h(x) + C|r| \\
|\sigma(t,x,r) - \sigma(t,x,s)| &\leq L|r-s|,
\end{aligned}$$

for all $t \in [0,T]$, $x, r, s \in \mathbb{R}$, and for some positive constants $C, L$ and some non-negative functions $f_1, g_1, g_3 \in L^2(\mathbb{R})$, and $g_2, h \in L^2(\mathbb{R}) \cap L^q(\mathbb{R})$ for some $q > 2$. Additionally, we again assume $\sigma$ is continuous and impose the following nondegeneracy condition:

$$\text{There exists } x_0 \in \mathbb{R} \text{ such that } \sigma(0, x_0, r) \neq 0 \text{ for all } r \in \mathbb{R}. \tag{3.12}$$

### 3.3.1 Comparison Theorem

In this section, we prove a comparison theorem which will aid in establishing regularity of a density function for the solution to this class of SPDEs. To obtain the comparison theorem, we approximate the coefficients and of the noise in (3.11), use Itô's formula on an appropriate functional of the difference of solutions to show that the positive part of this difference is zero, then prove convergence of the approximations to the solution of (3.11). This approach is similar to the method that was originally developed by Donati-Martin and Pardoux in [11], which has since been implemented in other contexts (e.g. [12]).

**Theorem 3.3.1.** *Suppose $u_0, v_0 \in L^2(\mathbb{R})$ are such that $u_0(x) \le v_0(x)$ for all $x$. If $u$ and $v$ are the solution to (3.11) with initial conditions $u_0$ and $v_0$, respectively, then $u(t,x) \le v(t,x)$ a.s. for all $(t,x) \in [0,T] \times \mathbb{R}$.*

*Proof of Theorem 3.3.1.* Let $u_0, v_0 \in L^2(\mathbb{R})$ be such that $u_0(x) \le v_0(x)$ for all $x \in \mathbb{R}$. Define the following approximating sequence of functions which are globally Lipschitz continuous in the third argument

$$f_n(t,x,r) = \begin{cases} f(t,x,r) & \text{if } |r| \le n \\ 0 & \text{if } |r| > n \end{cases}.$$

Define $g_n$ in terms of $g$ in an identical way. Next, fix an orthonormal basis $\{\phi_k\}$ of $L^2(\mathbb{R})$ such that this sequence is uniformly bounded in $k$, and define

$$W_k(t) := \int_0^t \int_{\mathbb{R}} \phi_k(x) W(ds,dx).$$

The collection $\{W_k(t), t \ge 0\}_k$ consists of mutually independent $\mathcal{F}_t$-Wiener processes. Now, with $n$ fixed, we consider the following evolution equation

$$\begin{cases} du_n(t) = A_n(t,u_n(t))dt + \sum_{k=1}^{n} B_k(t,u_n(t))dW_k(t) \\ u_n(0) = u_0 \end{cases} \tag{3.13}$$

in the Gelfand triple $H^1 \hookrightarrow L^2(\mathbb{R}) \equiv L^2(\mathbb{R})^* \hookrightarrow H^{-1}$, where $A_n(t) : H^1 \to H^{-1}$ and $B_n(t) : H^1 \to L^2(\mathbb{R})$ are nonlinear operators defined by the following actions

$$\begin{aligned}
\Big\langle A_n(t,\psi),\phi\Big\rangle_A &:= -\int_{\mathbb{R}} \psi'(x)\phi'(x)dx + \int_{\mathbb{R}} \Big(f_n(t,x,\psi(x))\phi(x) - g_n(t,x,\psi(x))\phi'(x)\Big)dx \\
\Big\langle B_k(t,\psi),h\Big\rangle_B &:= \int_{\mathbb{R}} \sigma(t,x,\psi(x))\phi_k(x)h(x)dx
\end{aligned}$$

for any $\phi, \psi \in H^1$ and $h \in L^2(\mathbb{R})$. It is known that (3.13) has a unique solution $u_n \in C([0,T];L^2(\mathbb{R}))$ that satisfies

$$\int_0^T \|u_n(t)\|_{H^1}^2 dt < \infty$$

almost surely (see chapter 7 of [9] or chapter 5 of [20]). Let $w_n := u_n - v_n$, where $u_n$ and $v_n$ are solutions to (3.13) with initial conditions $u_0$ and $v_0$, respectively, such that $u_0(x) \le v_0(x)$ for all $x$. We show that

$$\big|w_n(t,x)\big|_+ = 0, \text{ for } dx - \text{a.e. } x \in \mathbb{R} \tag{3.14}$$

almost surely for all $t \in [0,T]$, where $|\theta|_+ = \max(\theta, 0)$ is the positive part of $\theta$. For each $k \in \mathbb{N}$, let $\rho_k : \mathbb{R} \to \mathbb{R}$ be defined by

$$\rho_k(z) := \begin{cases} 2kz & \text{for } z \in [0,\frac{1}{k}] \\ 2\mathbf{1}_{z\ge 0} & \text{for } z \notin [0,\frac{1}{k}] \end{cases}.$$

Then, set

$$\psi_k(h) := \mathbf{1}_{h\ge 0}\int_0^h \int_0^y \rho_k(z)dzdy.$$

It's easy to see that $\psi_k \in C^2(\mathbb{R})$, and satisfies $0 \le \psi_k'(x) \le 2|x|_+$, $0 \le \psi_k''(x) \le 2\mathbf{1}_{x\ge 0}$, and $\psi_k(x) \nearrow |x|_+^2$ as $k \to \infty$. Now, define the functional $\Phi_k : L^2(\mathbb{R}) \to \mathbb{R}$ by

$$\Phi_k(h) := \int_{\mathbb{R}} \psi_k(h(x))dx.$$

One can show that $\Phi_k$ is twice Frechet differentiable at every $h \in L^2(\mathbb{R})$, that $\Phi_k'(h) \equiv \Phi_{k,h}'$ is a continuous linear functional on $L^2(\mathbb{R})$, and that $\Phi_k''(h) \equiv \Phi_{k,h}''$ is a continuous bilinear form on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Furthermore, these Frechet derivatives are given by the following

$$\Phi_{k,h}'(h_1) = \int_{\mathbb{R}} \psi_k'(h(x))h_1(x)dx$$

$$\Phi_{k,h}''(h_1,h_2) = \int_{\mathbb{R}} \psi_k''(h(x))h_1(x)h_2(x)dx$$

for $h_1, h_2 \in L^2(\mathbb{R})$. With this, we apply Itô's formula[2] to get

$$\begin{aligned}\Phi_k(w_n(t)) = M_{n,k}(t) + \Phi_k(w_n(0)) + \int_0^t \Big\langle A_n(s,u_n(s)) - A_n(s,v_n(s)), \psi_k'(w_n(s)) \Big\rangle_A ds \\ + \frac{1}{2}\int_0^t \sum_{i=1}^n \Big\langle \psi_k''(w_n(s))\big[\sigma(u_n(s)) - \sigma(v_n(s))\big]\phi_i, \big[\sigma(u_n(s)) - \sigma(v_n(s))\big]\phi_i \Big\rangle_B ds\end{aligned}$$

for some continuous local martingale $M_{n,k}$ satisfying $M_{n,k}(0) = 0$ a.s. We control each of the three terms of $A$ one at a time. First, using the chain rule yields

$$\int_{\mathbb{R}} \frac{\partial}{\partial x} w_n(s,x) \cdot \frac{\partial}{\partial x}\psi_k'(w_n(s,x))dx = \int_{\mathbb{R}} \Big(\frac{\partial}{\partial x} w_n(s,x)\Big)^2 \psi_k''(w_n(s,x))dx \ge 0$$

[2]See, for example, [9] or [20] for this generalized version of Itô's formula for Hilbert space valued processes.

since $\psi_k''(\cdot) \geq 0$. Furthermore, since $f_n$ is globally Lipschitz and $\psi_k'(\cdot) \leq 2|\cdot|_+$, we have

$$\int_{\mathbb{R}} \left[ f_n(s,x,u_n(s,x)) - f_n(s,x,v_n(s,x)) \right] \psi_k'(w_n(s,x))dx \leq L \|w_n(s,\cdot)_+\|^2_{L^2(\mathbb{R})}$$

for some positive constant $L$. Lastly, since $g_n$ is Lipschitz, apply the basic inequality $2ab \leq a^2 + b^2$ to get

$$\int_{\mathbb{R}} \left[ g_n(s,x,u_n(s,x) - g_n(s,x,v_n(s,x)) \right] \frac{\partial}{\partial x} \psi_k'(w_n(s,x))dx \\ \leq C\|w_n(s,\cdot)_+\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\int_{\mathbb{R}} \left( \frac{\partial}{\partial x} w_n(s,x) \right)^2 dx.$$

Finally, using the fact that $\sigma$ is assumed to be Lipschitz, and the boundedness of our basis functions, we easily obtain

$$E\left[\Phi_k(w_n(t \wedge \tau))\right] \leq C \int_0^t \|w_n(s \wedge \tau, \cdot)_+\|^2_{L^2(\mathbb{R})} ds$$

for some positive constant $C$ and any stopping time $\tau$. Finally, choose an appropriate sequence of stopping times $\{\tau_k\}_{k\geq 1}$, let $k$ tend to infinity, and apply Gronwall's inequality to get $\|w_n(t,\cdot)_+\|_{L^2(\mathbb{R})} = 0$. Hence, $u_n(t,x) \leq v_n(t,x)$. Now, we just need to show that these approximate solutions converge to the solution of (3.11). Indeed, for any $\phi \in H^1$, it is easy to show that

$$\int_0^t \sup_{x\in\mathbb{R}} E|u_n(s,x) - u(s,x)|^2 ds \leq C \int_0^t \sup_{x\in\mathbb{R}} E|u_n(s,x) - u(s,x)|^2 ds$$

since the coefficients of (3.13) are Lipschitz. From this, we have convergence of the approximating sequence of solutions.

□

### 3.3.2 Density by Truncating Solutions

In this section we assume that

$$g(t,x,u(t,x)) = -\frac{1}{2}u(t,x)^2.$$

The idea of the approximating procedure in this section is that we truncate the solution operator. In particular, we restrict our function space to an $L^2(\mathbb{R})$ ball of radius $N$, and let $N$ tend to infinity. Fortunately for us, the proofs in this section work the same way as the related proofs found in [27] due to the standard bounds on space-time convolutions with the heat kernel found in [13] and [27]. That said, we still provide most of the details for the sake of completeness.

We again assume (A1) and (A2) as before. We will use Theorem 3.3.1 in our proof of the following theorem, which is the main result of the section:

**Theorem 3.3.2.** *Suppose (A1) and (A2) hold. Then, the solution, $u(t,x)$, to (3.11) has a density which is absolutely continuous with respect to the Lebesgue measure.*

Now, we perform the following truncation on solutions which will serve as a sequence of processes which localize the solution to (3.11). For fixed $N$, define the map $\pi_N : L^2(\mathbb{R}) \to B_N$ by

$$\pi_N(u) = \begin{cases} u, & \text{if } \|u\|_{L^2(\mathbb{R})} \leq N, \\ \frac{N}{\|u\|_{L^2(\mathbb{R})}}u, & \text{if } \|u\|_{L^2(\mathbb{R})} > N. \end{cases}$$

Consider the operator $\mathcal{A}$ defined by

$$\mathcal{A}u(t,x) = (G(t,\cdot) * u_0)(x) + \big(G \star f_N\big)(t,x) - \big(\partial_x G \star u_N^2\big)(t,x) + \big(G \circledast \sigma_N\big)(t,x), \tag{3.15}$$

where the above notations are shorthand for the following convolutions

$$
\begin{aligned}
\big(G(t,\cdot)*u_0\big)(x) &= \int_{\mathbb{R}} G(t,x-y)u_0(y)dy \\
\big(G\star f_N\big)(t,x) &= \int_{(0,t)\times\mathbb{R}} G(t-s,x-y)f(s,y,(\pi_N u)(s,y))dyds, \\
\big(\partial_x G\star u_N^2\big)(t,x) &= \int_{(0,t)\times\mathbb{R}} \frac{\partial}{\partial y}G(t-s,x-y)\big((\pi_N u)(s,y)\big)^2 dyds, \\
\big(G\circledast\sigma_N\big)(t,x) &= \int_{(0,t)\times\mathbb{R}} G(t-s,x-y)\sigma(s,y,(\pi_N u)(s,y))W(ds,dy).
\end{aligned}
$$

**Proposition 3.3.3.** Let $u$ be the solution to (3.11). Then, for $t,x$ fixed, we have $u(t,x)\in\mathbb{D}^{1,1}_{loc}$.

*Proof.* Let $u^{(0)} := u_0$ and define the Picard iteration scheme $u^{(k)} := \mathcal{A}u^{(k-1)}$ for $k\geq 1$. Using identical arguments as in [13], one can show that this sequence converges converges to the solution of (3.11) in a suitable Banach space. Hence, it suffices to show that $u^{(k)}(t,x)\in\mathbb{D}^{1,1}$ for all $k$. In fact, we actually prove $u^{(k)}(t,x)\in\mathbb{D}^{1,2}$.

First, observe that for $F_n(\theta) := \mathbf{1}_{[0,n^2]}(\theta)+\frac{n}{\theta^{1/2}}\mathbf{1}_{(n^2,\infty)}(\theta)$, we can write

$$
\pi_N(w) = wF_N(\|w\|^2_{L^2(\mathbb{R})}).
$$

Moreover, it is clear that $|F_n(\theta)|\leq 1$ and $|F_n'(\theta)|\leq\frac{1}{2\theta}\mathbf{1}_{(n^2,\infty)}(\theta)$. Hence, if $w=\{w(\xi),\ \xi\in\mathbb{R}\}$ is a Malliavin differentiable process which is suitably integrable, we can see that the derivative of $\pi_N w$ satisfies

$$
\begin{aligned}
D(\pi_N(w(\xi))) =& D(w(\xi))F_N(\|w\|^2_{L^2(\mathbb{R})}) \\
&+ w(\xi)F_N'(\|w\|^2_{L^2(\mathbb{R})})\int_{\mathbb{R}} 2|w(\xi)|sign\big(w(\xi)\big)D(w(\xi))d\xi.
\end{aligned}
$$

Therefore, by Hölder's inequality, it follows that

$$\|D(\pi_N(w))\|_{L^2(\mathbb{R};H)} \leq 2\|Dw\|_{L^2(\mathbb{R};H)}. \tag{3.16}$$

Next, we show that if $u(t,x) \in \mathbb{D}^{1,2}$, it follows that $\mathcal{A}u(t,x) \in \mathbb{D}^{1,2}$, where $\mathcal{A}$ is the truncated solution operator defined in (3.15). Indeed, by the chain rule, it is easy to see that

$$D\big(G \star f_N\big)(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s,x-y)L_N^{(1)}(s,y)D(\pi_N(u(s,y)))dyds,$$

where $L_N^{(1)}$ is an adapted process which is bounded by the Lipschitz constant $L$. Hence, using (3.16) and (A.9), we have

$$\big\|D\big(G \star f_N\big)(t,\cdot)\big\|_{L^2(\mathbb{R};H)} \leq C \int_0^t \big\|Du(s,\cdot)\big\|_{L^2(\mathbb{R};H)} ds.$$

In a similar way, it is easy to see that

$$\big\|D\big(\partial_x G \star u_N^2\big)(t,\cdot)\big\|_{L^2(\mathbb{R};H)} \leq CN \int_0^t (t-s)^{-\frac{3}{4}} \big\|Du(s,\cdot)\big\|_{L^2(\mathbb{R};H)} ds.$$

Calculate the Malliavin derivative of the stochastic integral to get

$$\begin{aligned} D_{r,v}\Big(\big(G \circledast \sigma_N\big)(t,x)\Big) &= G(t-r,x-v)\sigma(r,v,(\pi_N u)(r,v)) \\ &\quad + \int_r^t \int_{\mathbb{R}} G(t-s,x-y)L_N^{(2)}(s,y)D_{r,v}\big(\pi_N u(s,y)\big)W(ds,dy), \end{aligned}$$

where $L_N^{(2)}$ is an adapted process which is bounded by the Lipschitz constant $L$. Hence, in the same way, we get

$$\left\|D\left(G\circledast\sigma_N\right)(t,\cdot)\right\|_{L^2(\mathbb{R};H)}\leq C_1+C_2\int_0^t(t-s)^{-\frac{3}{4}}\left\|Du(s,\cdot)\right\|_{L^2(\mathbb{R};H)}ds.$$

Thus, $\mathcal{A}u(t,x)\in\mathbb{D}^{1,2}$ by integrating $t$ on $[0,T]$. Furthermore, it is clear that this derivative satisfies the following equation

$$\begin{aligned}
D_{r,v}\mathcal{A}u(t,x)=&\int_r^t\int_{\mathbb{R}}G(t-s,x-y)L_N^{(1)}(s,y)D(\pi_N(u(s,y)))dyds\\
&-\int_r^t\int_{\mathbb{R}}\frac{\partial}{\partial y}G(t-s,x-y)\pi_N\big(u(s,y)\big)D_{r,v}u(s,y)dyds\\
&+G(t-r,x-v)\sigma(r,v,(\pi_N u)(r,v))\\
&+\int_r^t\int_{\mathbb{R}}G(t-s,x-y)L_N^{(2)}(s,y)D_{r,v}\big(\pi_N u(s,y)\big)W(ds,dy).
\end{aligned}$$

□

**Proposition 3.3.4.** Let $u$ be the solution to (3.11) with $g(u)=-u^2/2$. Then, for any $t\in[0,T]$, $x\in\mathbb{R}$, we have

$$\|Du(t,x)\|_{\mathcal{H}}>0\quad a.s.$$

*Proof.* Here we employ a clever technique which can be found in [24] and [27] which allows us to maintain the multiplicative noise framework. Fix $t,x\in[0,T]\times\mathbb{R}$. Let $u_n$ be the solution to (3.15) and define the sequence of stopping times

$$\tau_n:=\inf\{t\geq 0:\|u_n(t,\cdot)\|_{L^2(\mathbb{R})}\geq n\}\wedge T.$$

Since $\sigma$ is assumed to be continuous, there exists a neighborhood $(a_0, b_0)$ of $x_0$ and a stopping time $\tau > 0$ such that

$$\sigma(t, x, u(t, x)) \geq \delta_0 > 0$$

on $\Gamma := [0, \tau] \times [a_0, b_0]$, for some constant $\delta_0$. Pick another neighborhood $(a, b)$ of $x_0$ such that $[a, b] \subset (a_0, b_0)$, and define $\delta := \delta_0(b - a)$. It is enough to show that

$$\xi(s, x) := \int_a^b D_{r,v} u(s, x) dv > 0 \quad a.s.$$

on $\{r \in [0, T] : r \leq \tau_n \wedge \tau\}$. The process $\xi = \{\xi(s, x), s \in [r, T]\}$ satisfies the following integral equation

$$\begin{aligned}
\xi(s, x) = &\int_a^b G(s - r, x - v) \sigma(r, v, \pi_n u(r, v)) dv \\
&+ \int_r^s \int_{\mathbb{R}} \Big[ G(s - \theta, x - y) L_1^{(n)}(\theta, y) - \frac{\partial}{\partial y} G(s - \theta, x - y) \pi_n u(\theta, y) \Big] \xi(\theta, y) dy d\theta \\
&+ \int_r^s \int_{\mathbb{R}} G(s - \theta, x - y) L_2^{(n)}(\theta, y) W(d\theta, dy).
\end{aligned}$$

We show $\xi(t, x) > 0$ a.s. on $\Gamma$. Uniformly partition the interval $[r, T]$ into $m$ subintervals with endpoints given by $r_i^{(m)} := r + \frac{i}{m}(T - r)$ for $i = 0, 1, \dots, m$, and define

$$\alpha := \frac{1}{2} \inf_{\substack{1 \leq i \leq m \\ y \in [a, b + \frac{i}{m} d]}} \int_a^{b + \frac{i-1}{m} d} G\Big(\frac{t}{m}, y - v\Big) dv > 0.$$

Then, for $1 \leq i \leq m$, define the sequence of sets

$$E_i := \{\xi(r_i^{(m)}, y) \geq \delta \alpha^i \mathbf{1}_{[a, b + \frac{i}{m} d]}(y), \ \forall \, y \in \mathbb{R}\} \cap \Gamma.$$

Once we prove the following, we are done:

*Claim:* Let $\epsilon > 0$ be given. Then, for all $m$ large enough and $i = 1, \dots m-1$, we have

$$P(E_{i+1}^c \big| E_1 \cap \cdots \cap E_i) \leq \frac{\epsilon}{m}. \tag{3.17}$$

To see why this claim implies the result, observe that for any $m \geq 1$, we have

$$\begin{aligned} P(\{\xi(t,x) > 0\} \cap \Gamma) &\geq P(E_m) \\ &\geq P(E_1 \cap \cdots \cap E_m) \\ &= P(E_m \big| E_1 \cap \cdots \cap E_{m-1}) P(E_{m-1} \big| E_1 \cap \cdots \cap E_{m-2}) \cdots P(E_1). \end{aligned}$$

Then, if (3.17) holds for all $m$ large enough, we see that

$$P(\{\xi(t,x) > 0\} \cap \Gamma) \geq \left(1 - \frac{\epsilon}{m}\right)^m \geq 1 - \epsilon.$$

Hence, since $\epsilon$ is arbitrary, we may conclude that $\xi(t,x) > 0$ a.s. on $\Gamma$. We proceed with the proof of the claim. Consider the time evolution of the process $\xi$ on the interval $[r_i^{(m)}, r_{i+1}^{(m)}]$, starting at $\xi(r_i^{(m)}, \cdot)$. This process satisfies

$$\begin{aligned} \xi(s,x) = &\int_{\mathbb{R}} G(s - r_i^{(m)}, x - v) \xi(r_i^{(m)}, v) dv \\ &+ \int_{r_i^{(m)}}^s \int_{\mathbb{R}} \left[ G(s-\theta, x-y) L_1^{(n)}(\theta, y) - \frac{\partial}{\partial y} G(s-\theta, x-y) \pi_n u(\theta, y) \right] \xi(\theta, y) dy d\theta \\ &+ \int_{r_i^{(m)}}^s \int_{\mathbb{R}} G(s-\theta, x-y) L_2^{(n)}(\theta, y) W(d\theta, dy). \end{aligned}$$

Next, let $\zeta$ be the process which solves the same equation, but with initial condition $\delta\alpha^i \mathbf{1}_{[a,b+\frac{i}{m}d]}$. That is, $\zeta$ solves

$$\begin{aligned}\zeta(s,x) = \delta\alpha^i \int_{\mathbb{R}} G(s-r_i^{(m)}, x-v)\mathbf{1}_{[a,b+\frac{i}{m}d]}(v)dv \\ + \int_{r_i^{(m)}}^{s}\int_{\mathbb{R}} \Big[G(s-\theta,x-y)L_1^{(n)}(\theta,y) - \frac{\partial}{\partial y}G(s-\theta,x-y)\pi_n u(\theta,y)\Big]\zeta(\theta,y)dyd\theta \\ + \int_{r_i^{(m)}}^{s}\int_{\mathbb{R}} G(s-\theta,x-y)L_2^{(n)}(\theta,y)W(d\theta,dy)\end{aligned}$$

for $s$ in $[r_i^{(m)}, r_{i+1}^{(m)}]$. Then, on the set $E_1 \cap \cdots \cap E_i$, for $i = 1\dots, m-1$, the comparison theorem (3.3.1) implies that

$$\xi(s,x) \geq \zeta(s,x) \geq 0$$

with probability one.

Now, note that

$$\begin{aligned}\zeta(r_i^{(m+1)}, x) \geq 2\delta\alpha^{i+1} + \int_{r_i^{(m)}}^{r_i^{(m+1)}}\int_{\mathbb{R}} \Big[G(r_i^{(m+1)} - \theta, x-y)L_1^{(n)}(\theta,y) \\ - \frac{\partial}{\partial y}G(r_i^{(m+1)} - \theta, x-y)\pi_n u(\theta,y)\Big]\zeta(\theta,y)dyd\theta \\ + \int_{r_i^{(m)}}^{r_i^{(m+1)}}\int_{\mathbb{R}} G(r_i^{(m+1)} - \theta, x-y)L_2^{(n)}(\theta,y)W(d\theta,dy).\end{aligned}$$

Furthermore, on $E_{i+1}^c \cap \Gamma$, there exists $y \in [a, b+\frac{i+1}{m}d]$ such that $\zeta(r_i^{(m+1)}, y) < \delta\alpha^{i+1}$. Hence,

$$P(E_{i+1}^c \cap \Gamma \big| E_1 \cap \cdots \cap E_i) \leq \frac{1}{(\delta\alpha)^p} E\Big(\sup_y \big|\mathcal{J}_1 - \mathcal{J}_2 + \mathcal{J}_3\big|^p \;\Big|\; E_1 \cap \cdots \cap E_i\Big),$$

where

$$\mathcal{J}_1 = \int_{r_i^{(m)}}^{r_i^{(m+1)}} \int_{\mathbb{R}} G(r_i^{(m+1)} - \theta, x-y) L_1^{(n)}(\theta, y)\zeta(\theta, y) dy d\theta$$

$$\mathcal{J}_2 = \int_{r_i^{(m)}}^{r_i^{(m+1)}} \int_{\mathbb{R}} \frac{\partial}{\partial y} G(r_i^{(m+1)} - \theta, x-y) \pi_n u(\theta, y)\zeta(\theta, y) dy d\theta$$

$$\mathcal{J}_3 = \int_{r_i^{(m)}}^{r_i^{(m+1)}} \int_{\mathbb{R}} G(r_i^{(m+1)} - \theta, x-y) L_2^{(n)}(\theta, y) W(d\theta, dy).$$

Using the uniform bounds in Lemma A.9 leads to

$$P(E_{i+1}^c \cap \Gamma \big| E_1 \cap \cdots \cap E_i) \leq \frac{\left(r_i^{(m+1)} - r_i^{(m)}\right)^\gamma}{(\delta\alpha)^p} \int_r^t E\Big( \|\zeta(\theta, \cdot)\|_{L^p(\mathbb{R})}^p \Big| E_1 \cap \cdots \cap E_i \Big) d\theta$$

for some $\gamma > 0$. By Lemma A.9 and Gronwall's inequality, this integral is bounded by a constant. Furthermore, since

$$r_i^{(m+1)} - r_i^{(m)} = \frac{k+1}{m}(T-r) - \frac{k}{m}(T-r) = \frac{T-r}{m},$$

we have

$$P(E_{i+1}^c \cap \Gamma \big| E_1 \cap \cdots \cap E_i) \leq \frac{C}{m^\gamma},$$

which yields the claim.

□

### 3.3.3 Density by Truncating Coefficients

In this section, we consider the same set of equations as in (3.11), assuming the coefficients depend only on the solution

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(u(t,x)) + \frac{\partial}{\partial x}g(u(t,x)) + \sigma(u(t,x))\frac{\partial^2 W}{\partial t \partial x}. \tag{3.18}$$

Once again, $f$ and $\sigma$ have linear growth, and $g$ has quadratic growth. The main idea of this section is that we perform an approximation by truncating the coefficients. Assuming the coefficients are "nice," we are able to prove Malliavin differentiability, which in turn implies local Malliavin differentiability for the solution to (3.18). We then prove positivity of the norm of the Malliavin derivative of the solution to (3.18) at fixed points. This will yield the existence of a density which is absolutely continuous (with respect to Lebesgue). Throughout the section, we assume $f$ and $g$ are locally Lipschitz and that $\sigma$ is globally Lipschitz. Furthermore, we impose the following growth conditions on the coefficients:

$$(H) \qquad \begin{aligned} &f, g, \sigma \in C^1(\mathbb{R}) \\ &f', \sigma' \in L^\infty(\mathbb{R}) \\ &|g(u)| \leq C(1+u^2) \end{aligned}$$

To achieve local differentiability of the solution to (3.18), we will assume that $g'$ is bounded, which will serve as our approximating procedure.

**Proposition 3.3.5.** In addition to $(H)$, assume that $g'$ is bounded. Then it follows that $u(t,x) \in \mathbb{D}^{1,p}$ and its derivative satisfies

$$\begin{aligned} D_{r,v}u(t,x) = &\int_r^t \int_{\mathbb{R}} G(t-s,x-y)f'(u(s,y))D_{r,v}u(s,y)dyds \\ &- \int_r^t \int_{\mathbb{R}} \frac{\partial}{\partial y}G(t-s,x-y)g'(u(s,y))D_{r,v}u(s,y)dyds \\ &+ \int_r^t \int_{\mathbb{R}} G(t-s,x-y)\sigma'(u(s,y))D_{r,v}u(s,y)W(ds,dy) \\ &+ G(t-r,x-v)\sigma(u(r,v)). \end{aligned}$$

*Proof.* We proceed by using a Picard iteration scheme in the same way as we did in the proof of Proposition 3.2.2. Set $u_0(t,x) = (G(t,\cdot) * u_0)(x)$ and for $n \geq 1$, define

$$\begin{aligned} u_{n+1}(t,x) = &u_0(t,x) + \int_0^t \int_{\mathbb{R}} G(t-s,x-y)f(u_n(s,y))dyds \\ &- \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial y}G(t-s,x-y)g(u_n(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\sigma(u_n(s,y))W(ds,dy). \end{aligned}$$

Then, due to the restrictive hypotheses on the coefficients, it is easy to see that

$$\sup_{x\in\mathbb{R}} E\big(|u_{n+1}(t,x) - u_n(t,x)|^p\big) \leq C \int_0^t \sup_{x\in\mathbb{R}} E\big(|u_n(s,x) - u_{n-1}(s,x)|^p\big)ds$$

which leads to

$$\sum_n \sup_{t,x} E\big(|u_{n+1}(t,x) - u_n(t,x)|^p\big) < \infty$$

in the same way as before. Therefore it is enough to prove differentiability of the process $u_n(t,x)$. Once again, we prove this by induction. Since the coefficients are assumed to have bounded first derivatives, it follows that

$$
\begin{aligned}
&E\Big(\|Du_{n+1}(t,x)\|^p_{L^2([0,T]\times\mathbb{R})}\Big) \\
&\qquad\qquad \leq C_1 + C_2\int_0^t\int_{\mathbb{R}}\bigg(G(t-s,x-y)^2 + \Big|\frac{\partial}{\partial y}G(t-s,x-y)\Big|\bigg) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times E\Big(\|Du_n(t,x)\|^p_{L^2([0,T]\times\mathbb{R})}\Big)dyds.
\end{aligned}
$$

Using the induction hypothesis gives us $u_n(t,x) \in \mathbb{D}^{1,p}$. $\square$

Now, set $g(u) = -\frac{1}{2}u^2$ for the remainder of the section.

**Proposition 3.3.6.** Suppose that $\sigma$ satisfies (H) and let $u$ be the mild solution to (3.11). Then, for fixed $t, x$, and all $p \geq 2$, we have

$$
E\big(\|Du(t,x)\|^{-p}_{\mathcal{H}}\big) < \infty.
$$

*Proof.* In a similar way as in Proposition 3.2.3, we show that for $a \in (\frac{1}{8}, \frac{1}{4})$ fixed, and all $q \geq 2$, there exists $\epsilon_0$ depending on $a$ and $q$ such that

$$
P\bigg(\int_0^t\int_{\mathbb{R}}\big(D_{r,v}u(t,x)\big)^2 dvdr \leq \epsilon^a\bigg) \leq \epsilon^{(1-4a)q} \tag{3.19}
$$

Since $\sigma$ is assumed to be continuous, there exists $\epsilon > 0$ such that $\sigma(t,x)^2 \geq C > 0$ on $(t,x) \in [0, \sqrt[8]{\epsilon}] \times [x_0 - \sqrt[8]{\epsilon}, x_0 + \sqrt[8]{\epsilon}]$. Now, note the following

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}}\Big(D_{r,v}u(t,x)\Big)^2 dvdr &\geq \frac{1}{2}\int_0^{\sqrt[8]{\epsilon}}\int_{x_0-\sqrt[8]{\epsilon}}^{x_0+\sqrt[8]{\epsilon}}\sigma_r(v)^2 G(t-r,x-v)^2 dvdr \\
&\quad - \int_{t-\epsilon}^t\int_{x_0-\sqrt[8]{\epsilon}}^{x_0+\sqrt[8]{\epsilon}}\Big(\mathcal{K}_1(t,x,r,v)+\mathcal{K}_2(t,x,r,v)\Big)^2 dvdr,
\end{aligned}
$$

where

$$\mathcal{K}_1(t,x,r,v) = \int_r^t \int_{\mathbb{R}} G(t-s,x-y)f'(u(s,y))D_{r,v}u(s,y)dyds$$

and

$$\mathcal{K}_2(t,x,r,v) = \int_r^t \int_{\mathbb{R}} \frac{\partial}{\partial y} G(t-s,x-y)u(s,y)D_{r,v}u(s,y)dyds.$$

Again, pick $\epsilon_0 > 0$ small enough so that

$$\int_0^{\sqrt[8]{\epsilon}} \int_{x_0-\sqrt[8]{\epsilon}}^{x_0+\sqrt[8]{\epsilon}} \sigma_r(v)^2 G(t-r,x-v)^2 dvdr \geq C\sqrt[8]{\epsilon} \geq 2\epsilon^a$$

holds for all $0 < \epsilon \leq \epsilon_0$. Hence, for all such $\epsilon$, we have

$$\begin{aligned} P\bigg(\int_0^t \int_{\mathbb{R}} \big(D_{r,v}u(t,x)\big)^2 dvdr \leq \epsilon^a\bigg) \\ \leq \frac{1}{\epsilon^{ap/2}} E\bigg[\bigg(\int_{t-\epsilon}^t \int_{x_0-\sqrt[8]{\epsilon}}^{x_0+\sqrt[8]{\epsilon}} \Big(\mathcal{K}_1(t,x,r,v) + \mathcal{K}_2(t,x,r,v)\Big)^2 dvdr\bigg)^{p/2}\bigg]. \end{aligned}$$

We apply Minkowski's inequality and bound each term, one at a time.

$$\begin{aligned} &\int_{t-\epsilon}^t \int_{x_0-\sqrt[8]{\epsilon}}^{x_0+\sqrt[8]{\epsilon}} \big\|\mathcal{K}_1(t,x,r,v)\big\|_p^2 dvdr \\ &\leq C \int_{t-\epsilon}^t \int_{x_0-\sqrt[8]{\epsilon}}^{x_0+\sqrt[8]{\epsilon}} \Big(\int_r^t \int_{\mathbb{R}} G(t-s,x-y)\big\|D_{r,v}u(s,y)\big\|_p^2 dyds\Big)^2 dvdr \\ &\leq C \int_{t-\epsilon}^t \int_{\mathbb{R}} G(t-s,x-y) \int_{t-\epsilon}^s \int_{\mathbb{R}} (t-r)\big\|D_{r,v}u(s,y)\big\|_p^2 dvdrdyds \\ &\leq C\epsilon \end{aligned}$$

for some positive constant $C$ depending on $\sup_{t,x} E\|Du(t,x)\|^p_{\mathcal{H}}$. The other term can be controlled as follows.

$$\begin{aligned}
&\int_{t-\epsilon}^{t}\int_{x_0-\sqrt[8]{\epsilon}}^{x_0+\sqrt[8]{\epsilon}} \left\|\mathcal{K}_2(t,x,r,v)\right\|_p^2 dvdr \\
&\quad \leq \int_{t-\epsilon}^{t}\int_{\mathbb{R}}\Bigg(\int_r^t \Big(\int_{\mathbb{R}}\Big|\frac{\partial G}{\partial y}(t-s,x-y)\Big|\|u(s,y)^2\|_p dy\Big)^{1/2} \\
&\qquad\qquad \times \Big(\int_{\mathbb{R}}\Big|\frac{\partial G}{\partial y}(t-s,x-y)\Big|\|\big(D_{r,v}u(s,y)\big)^2\|_p dy\Big)^{1/2} ds\Bigg)^2 dvdr \\
&\quad \leq C\int_{t-\epsilon}^{t}\int_{\mathbb{R}}(t-r)^{7/4}\int_r^t (t-s)^{-1/4}\int_{\mathbb{R}}\Big|\frac{\partial G}{\partial y}(t-s,x-y)\Big|\|\big(D_{r,v}u(s,y)\big)^2\|_p \\
&\quad \leq C\int_{t-\epsilon}^{t}(t-s)^{-3/4}ds \\
&\quad = C\epsilon^{1/4},
\end{aligned}$$

where $C$ is a constant depending on $\sup_{t,x} E\|Du(t,x)\|^p_{\mathcal{H}}$. Finally, with these bounds in hand, we get

$$\begin{aligned}
&P\Big(\int_0^t\int_{\mathbb{R}}\big(D_{r,v}u(t,x)\big)^2 dvdr \leq \epsilon^a\Big) \\
&\qquad \leq \frac{1}{\epsilon^{ap/2}}E\Bigg[\Bigg(\int_{t-\epsilon}^{t}\int_{x_0-\sqrt[8]{\epsilon}}^{x_0+\sqrt[8]{\epsilon}}\Big(\mathcal{K}_1(t,x,r,v)+\mathcal{K}_2(t,x,r,v)\Big)^2 dvdr\Bigg)^{p/2}\Bigg] \\
&\qquad \leq C\epsilon^{-ap/2}\big(\epsilon+\epsilon^{1/4}\big)^{p/2} \\
&\qquad \leq C\epsilon^{\frac{p}{8}(1-4a)}.
\end{aligned}$$

Hence, we have the result.

□

Now, define the sequence of sets

$$\Omega_n := \{\omega \in \Omega \big| \sup_{t,x} |u(t,x)| \leq n\}.$$

Clearly, we have $\Omega_n \nearrow \Omega$ a $n \to \infty$. Then, let $g_n$ be a sequence of functions of class $C^1$ with bounded first derivatives such that

$$g_n(u) := \begin{cases} g(u) & \text{if } |u| \leq n \\ 0 & \text{if } |u| > n \end{cases}.$$

Now, let $u_n$ denote the solution to (3.18) with $g_n$ instead of $g$. Then, $u_n(t,x) = u(t,x)|_{\Omega_n}$. Hence, since $u_n$ is Malliavin differentiable, it follows that $u$ is locally Malliavin differentiable. Thus, by Proposition 3.3.6, it follows that $u(t,x)$ has a density function which is absolutely continuous with respect to the Lebesgue measure.

# Appendix

Here we state a couple results that are used in the paper. The following Gronwall-type lemma is very useful and not too difficult to prove by iteration.

**Lemma A.7.** *Let $\{h_n(t), t \in [0,T]\}_{n\geq 1}$, $h_n : \mathbb{R}_+ \to \mathbb{R}_+$ be a sequence of nonnegative, Borel-measurable functions. Suppose that for all $n \geq 1$, we have*

$$h_{n+1}(t) \leq C \int_0^t h_n(s)ds$$

*for some positive constant $C$. Then, it follows that*

$$\sum_{n\geq 1} \sup_t h_n(t) < \infty.$$

The following calculation is used many times in this work, but not explicitly referred to as it is straightforward.

**Lemma A.8.** *Let $G(t,x) = (4\pi t)^{-1/2} e^{-x^2/4t}$ be the heat kernel on the real line. Then, for any $a > 0$, we have*

$$\int_{\mathbb{R}} |x|^a G(t,x)dx \leq Ct^{a/2}.$$

The following useful estimates on space-time convolutions with the heat kernel and its derivative can be found in [13] and [27] for example:

**Lemma A.9.** *Let $G(t,x)$ be the heat kernel as above.*

*(i) For any $p \geq 1$, we have*

$$\left\| \int_0^t \int_{\mathbb{R}} G(t-s, \cdot - y)\eta(s,y)dyds \right\|_{L^p(\mathbb{R})} \leq C \int_0^t \left\|\eta(s,\cdot)\right\|_{L^p(\mathbb{R})} ds$$

$$\left\| \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s, \cdot - y)\eta(s,y)dyds \right\|_{L^{2p}(\mathbb{R})} \leq C \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{4p}} \left\|\eta(s,\cdot)\right\|_{L^p(\mathbb{R})} ds.$$

*(ii) For any $p > 2q > 2$, we have*

$$E \sup_{t\in[0,T]} \left\| \int_0^t \int_{\mathbb{R}} G(t-s,x-\cdot)\eta(s,y)W(ds,dy) \right\|_{L^p(\mathbb{R})}^p \le E\int_0^T \|\eta(s,\cdot)\|_{L^{2q}(\mathbb{R})}^p ds.$$

*(iii) For any $p > 1$, we have the following uniform bounds*

$$\left| \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\eta(s,y)dyds \right| \le C\int_0^t (t-s)^{-\frac{1}{2p}} \|\eta(s,\cdot)\|_{L^p(\mathbb{R})} ds$$
$$\left| \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(t-s,x-y)\eta(s,y)dyds \right| \le C\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2p}} \|\eta(s,\cdot)\|_{L^p(\mathbb{R})} ds.$$

*(iv) For any $q > 1$ and $p > \max\left(2q, \dfrac{4q}{q-1}\right)$, we have*

$$E \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\eta(s,y)W(ds,dy) \right|^p \le CE\int_0^T \|\eta(s,\cdot)\|_{L^{2q}(\mathbb{R})}^{pq} ds.$$

The proofs of these results are interesting in their own right and employ a "factorization" technique which was developed in [7], and can be found in [27] and [13] for example.

Lastly, the following result can be used when one has moment estimates of an exponential type. Its proof and some applications to stochastic heat equations can be found in [5] and [6].

**Lemma A.10.** *Let $X$ be a nonnegative random variable that satisfies*

$$E(X^p) \le C^p e^{ap^b}$$

*for all $p$, and some constants $a, C > 0$, $b > 1$. Then, for all $\alpha \in \left(0, \frac{b-1}{b} \cdot \frac{1}{(ab)^{1/(b-1)}}\right)$, it follows that*

$$E\left[e^{\alpha(\log_+ X)^{b/(b-1)}}\right] < \infty,$$

*where $\log_+ X := \log(X \vee 1)$ and $\log$ denotes the natural logarithm.*

# Bibliography


[1] Aureli Alabert and István Gyöngy. On numerical approximation of stochastic Burgers' equation. In *From stochastic calculus to mathematical finance*, pages 1–15. Springer, Berlin, 2006. Cited on 3

[2] Vlad Bally and Etienne Pardoux. Malliavin calculus for white noise driven parabolic SPDEs. *Potential Anal.*, 9(1):27–64, 1998. Cited on 57

[3] L. Bertini, N. Cancrini, and G. Jona-Lasinio. The stochastic Burgers equation. *Comm. Math. Phys.*, 165(2):211–232, 1994. Cited on iii, 3

[4] J. M. Burgers. Statistical problems connected with the solution of a nonlinear partial differential equation. In *Nonlinear Problems of Engineering*, pages 123–137. Academic Press, New York, 1964. Cited on 1

[5] Daniel Conus, Mathew Joseph, and Davar Khoshnevisan. On the chaotic character of the stochastic heat equation, before the onset of intermitttency. *Ann. Probab.*, 41(3B):2225–2260, 2013. Cited on 91

[6] Daniel Conus, Mathew Joseph, Davar Khoshnevisan, and Shang-Yuan Shiu. On the chaotic character of the stochastic heat equation, II. *Probab. Theory Related Fields*, 156(3-4):483–533, 2013. Cited on 91

[7] G. Da Prato, S. Kwapień, and J. Zabczyk. Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics*, 23(1):1–23, 1987. Cited on 91

[8] Giuseppe Da Prato and Dariusz Gatarek. Stochastic Burgers equation with correlated noise. *Stochastics Stochastics Rep.*, 52(1-2):29–41, 1995. Cited on 3

[9] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014. Cited on 5, 73, 74

[10] Robert C. Dalang and Lluí s Quer-Sardanyons. Stochastic integrals for spde's: a comparison. *Expo. Math.*, 29(1):67–109, 2011. Cited on 5

[11] C. Donati-Martin and É. Pardoux. White noise driven SPDEs with reflection. *Probab. Theory Related Fields*, 95(1):1–24, 1993. Cited on 72

[12] István Gyöngy. Existence and uniqueness results for semilinear stochastic partial differential equations. *Stochastic Process. Appl.*, 73(2):271–299, 1998. Cited on 2, 69, 70, 72

[13] István Gyöngy and David Nualart. On the stochastic Burgers' equation in the real line. *Ann. Probab.*, 27(2):782–802, 1999. Cited on 3, 10, 11, 42, 76, 77, 90, 91

[14] István Gyöngy and Carles Rovira. On stochastic partial differential equations with polynomial nonlinearities. *Stochastics Stochastics Rep.*, 67(1-2):123–146, 1999. Cited on 3

[15] Eberhard Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.*, 3:201–230, 1950. Cited on 1

[16] Mark Kac. *Mark Kac: probability, number theory, and statistical physics*, volume 14 of *Mathematicians of Our Time*. MIT Press, Cambridge, Mass.-London, 1979. Selected papers, Edited by K. Baclawski and M. D. Donsker. Cited on 2

[17] Jong Uhn Kim. On the stochastic Burgers equation with a polynomial nonlinearity in the real line. *Discrete Contin. Dyn. Syst. Ser. B*, 6(4):835–866, 2006. Cited on 3

[18] Jorge A. León, D. Nualart, and Roger Pettersson. The stochastic Burgers equation: finite moments and smoothness of the density. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 3(3):363–385, 2000. Cited on 2, 3, 10, 15, 59, 69

[19] P. Lewis and D. Nualart. Stochastic Burgers' Equation on the Real Line: Regularity and Moment Estimates. *ArXiv e-prints*, September 2017. Cited on iv

[20] Wei Liu and Michael Röckner. *Stochastic partial differential equations: an introduction*. Universitext. Springer, Cham, 2015. Cited on 73, 74

[21] Salah-Eldin A. Mohammed and Tusheng Zhang. Stochastic Burgers equation with random initial velocities: a Malliavin calculus approach. *SIAM J. Math. Anal.*, 45(4):2396–2420, 2013. Cited on 2

[22] Pierre-Luc Morien. On the density for the solution of a Burgers-type SPDE. *Ann. Inst. H. Poincaré Probab. Statist.*, 35(4):459–482, 1999. Cited on 69

[23] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006. Cited on 57, 58

[24] Étienne Pardoux and Tu Sheng Zhang. Absolute continuity of the law of the solution of a parabolic SPDE. *J. Funct. Anal.*, 112(2):447–458, 1993. Cited on 79

[25] Aubrey Truman and Jiang-Lun Wu. Fractal Burgers' equation driven by Lévy noise. In *Stochastic partial differential equations and applications—VII*, volume 245 of *Lect. Notes Pure Appl. Math.*, pages 295–310. Chapman & Hall/CRC, Boca Raton, FL, 2006. Cited on 3

[26] John B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986. Cited on 5, 70

[27] N. Lanjri Zaidi and D. Nualart. Burgers equation driven by a space-time white noise: absolute continuity of the solution. *Stochastics Stochastics Rep.*, 66(3-4):273–292, 1999. Cited on 69, 76, 79, 90, 91