COMPUTATIONAL ANALYSIS FOR TIME DEPENDENT TWO DIMENSIONAL FLOW OF AN INCOMPRESSIBLE HOMOGENEOUS NEWTONIAN FLUID (SPIN UP PROBLEM)

by

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ABSTRACT

A digital computer method and a program which employs optimized successive overrelaxation were developed to solve certain problems involving time-dependent two-dimensional flow of an incompressible homogeneous Newtonian fluid.

Bubble sort techniques were used in a boundary field and a vectorization of the inner-most Do-loops of the program was carried out using a CRAY-1 computer. A polygon region was selected for analysis purposes.

As a test program, "Spin-up problem" for a circular region with rotating solid boundary conditions was analyzed. The results of computations using a fine mesh were in good agreement with analytical results.

Finally, a flow contained in an infinite cylinder with part-moving boundaries is examined. It is to be applied to an analysis of a flow in a circular cavity.
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<td>a</td>
<td>radius of curvature, ft.</td>
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<tr>
<td>const.</td>
<td>constant</td>
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<tr>
<td>E</td>
<td>allowable error</td>
</tr>
<tr>
<td>i</td>
<td>spatial location</td>
</tr>
<tr>
<td>j</td>
<td>spatial location</td>
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<tr>
<td>k</td>
<td>a constant</td>
</tr>
<tr>
<td>L</td>
<td>characteristic length of the problem, ft.</td>
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<tr>
<td>m</td>
<td>positive integer</td>
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<tr>
<td>n</td>
<td>time level</td>
</tr>
<tr>
<td>p</td>
<td>fluid pressure, lbf./ft².</td>
</tr>
<tr>
<td>p</td>
<td>a point</td>
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<tr>
<td>q</td>
<td>fluid speed, ft./sec.</td>
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<tr>
<td>r</td>
<td>polar radius, ft.</td>
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<tr>
<td>Re</td>
<td>Reynolds number</td>
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<tr>
<td>s</td>
<td>distance along curve, ft.</td>
</tr>
<tr>
<td>t</td>
<td>time, sec.</td>
</tr>
<tr>
<td>u</td>
<td>fluid velocity, ft./sec.</td>
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<tr>
<td>u</td>
<td>horizontal fluid velocity component, ft./sec.</td>
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<tr>
<td>U</td>
<td>fluid velocity parallel to the wall, ft./sec.</td>
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<tr>
<td>U₀</td>
<td>characteristic velocity of the problem, ft./sec.</td>
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<td>U₀</td>
<td>fluid velocity, U, at the point P, ft./sec.</td>
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<tr>
<td>U₀</td>
<td>fluid velocity, U, at the wall, ft./sec.</td>
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<tr>
<td>v</td>
<td>vertical fluid velocity component, ft./sec.</td>
</tr>
<tr>
<td>x</td>
<td>horizontal coordinate</td>
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<tr>
<td>y</td>
<td>vertical coordinate</td>
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Greek Letters and Special Characters

\( \beta \) aspect ratio

\( \gamma \) unit weight of a fluid, lb./ft\(^3\).

\( \delta \) grid space

\( \nabla \) vector operator \((\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\) for two-dimensional problem

\( \varepsilon \) thin layer of a fluid thickness, ft.

\( \theta \) polar angle in circular flow

\( \theta_1 \) angular horizontal geometric coordinate

\( \lambda \) mutual overlap, ft.

\( \mu \) dynamic (absolute) viscosity of a fluid, lb. sec./ft\(^2\).

\( \nu \) kinematic viscosity of a fluid, ft\(^2\)/sec.

\( \pi \) a polygon

\( \rho \) fluid density, lbm./ft\(^3\).

\( \sigma \) boundary of a region of a two-dimensional flow

\( \psi \) stream function, ft\(^2\)/sec.

\( \psi_p \) stream function at point P, ft\(^2\)/sec.

\( \psi_w \) stream function at the wall, ft\(^2\)/sec.

\( \forall \) "at all points"

\( \xi \) vorticity, sec\(^{-1}\)
ACKNOWLEDGEMENT

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CHAPTER 1

INTRODUCTION

The computational techniques of a computer program which seems to be effective for time-dependent two-dimensional viscous flow problems will be developed. The analysis is applied to a polygon region with solid moving or fixed boundary conditions. The method is outlined in Chapter 3, and more details concerning the test program are given in Chapter 4.

Mathematically, the problem is that of solving numerically a system of nonlinear partial differential equations. It is known that, even without the added complications of time variation or non-linearity, numerical techniques for multi-order equations tend to require substantial computer time. The presence of non-linear terms may not only accentuate the computer time problems, but may also tend to induce computational instabilities.

As might be expected from their potential importance, viscous flow computations have received considerable attention in the literature (Fromm & Harlow (4); Dix (2); Hellums & Churchill (5); Wilkes (15) and Thompson (14); Pearson (9), etc). In general, it appears that there have been difficulties with one or another of computational speed, stability, accuracy, or the ability to handle correct boundary conditions. In a problem of this general difficulty, it seems worth while to
begin with the simplest possible problem possessing realistic boundary conditions. In this research, a rotating flow contained in an infinite cylinder, with moving boundaries (test program) and part-moving boundaries is treated.

In the process of developing computational techniques and a computer program, the treatment of boundary fields, which are very important in recent computational fluid dynamics is utilized. By employing the curvature term in the formula of Thom (12) and the computer programming algorithm (Bubble Sort, etc), the accurate calculations of the vorticities and stream functions are resolved and are in good agreement with those of analytical works in a test program. A vectorization in a relaxation loop which is the most time-consuming operation and in the other inner-most Do-loops results in an increase in speed by a factor of three on the CRAY-1 Computer. As an application, which can be used for the analysis of a flow in a circular cavity with part of its wall moving, the problems for the fluid in a part-rotating infinite cylinder of radius, \( a \), are investigated. These computations are converged and stabilized after reaching the time desired, even though the exact analytical results are unknown. Throughout this research, calculations of pressure are omitted.
CHAPTER 2
GOVERNING EQUATIONS

In terms of vector notations and time $\xi$, the Navier-Stokes equations are derived from Newton's second law of motion in the case of incompressible fluids ($\bar{\rho}$ and $\bar{\mu}$ are constant):

\[ \bar{F} = m \bar{a}, \]  
\[ - \nabla \bar{p} - \gamma \nabla \bar{h} + \bar{\mu} \nabla^2 \bar{u} = \bar{\rho} \frac{D\bar{u}}{D\xi}, \]  

Where $\bar{\rho}$ is density, $\gamma$ is unit weight, $\bar{h}$ is elevation from a datum line and $\bar{\mu}$ is dynamic viscosity; $\frac{D}{D\xi}$ is substantive derivative.

Applying the principle of conservation of mass, the equation of continuity may be written in the following form,

\[ \nabla \cdot \bar{u} = - \frac{1}{\bar{\rho}} \frac{D\bar{\rho}}{D\xi}. \]  

In Cartesian coordinates ($\bar{x}$, $\bar{y}$), the equations relating pressure $\bar{p}$, and velocity components ($\bar{u}$, $\bar{v}$) for two-dimensional incompressible flow are

\[ - \nabla \bar{p} + \bar{\mu} \nabla^2 \bar{u} = \bar{\rho} \frac{D\bar{u}}{D\xi}, \]  

or
\[
\bar{\rho} \left( \bar{u}_t + \bar{u}_x \bar{u} + \bar{u}_y \bar{v} \right) = - \bar{p}_x + \bar{\mu} (\bar{u}_{xx} + \bar{u}_{yy}), \tag{5}
\]

\[
\bar{p} \left( \bar{v}_t + \bar{v}_x \bar{u} + \bar{v}_y \bar{v} \right) = - \bar{p}_y + \bar{\mu} (\bar{v}_{xx} + \bar{v}_{yy}), \tag{6}
\]

For two-dimensional incompressible flow the continuity equation may be written as

\[
\bar{u}_x + \bar{v}_y = 0, \tag{7}
\]

where subscripts denote partial differentiation, and the overbars represent dimensional quantities. A stream function that satisfies equation (4) may be defined in the following manner.

\[
\bar{u} = \bar{\psi}_y, \quad \bar{v} = - \bar{\psi}_x, \tag{8}
\]

where \( \bar{\psi} \) is the stream function. The vorticity \( \bar{\xi} \) is defined by

\[
\bar{\xi} = 2\bar{\omega} = (\bar{u}_y - \bar{v}_x) = (\bar{\psi}_{xx} + \bar{\psi}_{yy}), \tag{9}
\]

where \( \bar{\omega} \) is angular velocity. Then Poisson's equation is

\[
\nabla^2 \bar{\psi}(x, y) = - \bar{\xi}(x, y), \tag{10}
\]

Taking the curl of equation (4) yields
\[ - \text{curl} \, \text{grad} p + \mu \text{curl}^{2} \mathbf{u} = \text{curl} \, \rho \frac{\partial \mathbf{u}}{\partial t} \]  
(11)

\[ \mu \text{curl}^{2} \mathbf{u} = \rho \text{curl} \, \frac{\partial \mathbf{u}}{\partial t}, \]  
(12)

\[ \mu \text{curl}^{2} \mathbf{u} = \rho \frac{\partial}{\partial t} \text{curl} \, \mathbf{u}, \]  
(13)

\[ \bar{\nu} \nabla^{2} \xi = \frac{\partial}{\partial t} \xi \]  
(14)

or

\[ \frac{\partial \xi}{\partial t} = - \mathbf{u} \cdot \left( \nabla \xi \right) + \bar{\nu} \nabla^{2} \xi, \]  
(15)

where \( \bar{\nu} = \frac{\mu}{\rho} \) is the kinematic viscosity. Equation (14), the parabolic vorticity transport equation, can be thought of as the condition that a pressure function \( p \) exist, by taking the divergence of equation (4). An equation for \( p \), the pressure equation, may be obtained from equation (4):

\[ \nabla \cdot \left( - \text{grad} p + \mu \nabla^{2} \mathbf{u} = \rho \frac{\partial \mathbf{u}}{\partial t} \right), \]  
(16)

\[ - \nabla^{2} p + \bar{\nu} \nabla^{2} \left( \mathbf{u}_x + \mathbf{u}_y \right) \]

\[ = \frac{\partial}{\partial t} \left( \mathbf{u}_x + \mathbf{u}_y \right) + \mathbf{u}_x \mathbf{u}_x + \mathbf{u}_y \mathbf{u}_y + \frac{\partial}{\partial x} \left( \mathbf{u}_x + \mathbf{u}_y \right) \]

\[ + \mathbf{v}_x \mathbf{v}_y + \mathbf{v}_y \mathbf{v}_y + \frac{\partial}{\partial y} \left( \mathbf{u}_x + \mathbf{u}_y \right) \]  
(17)
\[
\overline{\nabla^2 p} = -(\overline{u_x^2} + 2\overline{u_y\partial_y u_x} + \overline{v_y^2}).
\]  

(18)

The normalizing system used throughout the analysis is based on the advective time scale \(\overline{L}/U_0\), where \(\overline{L}\) is a characteristic length and \(U_0\) is a characteristic velocity of the problem. The dimensionless quantities are defined as follows,

\[
\begin{align*}
\bar{u} &= \frac{u}{U_0}, \\
\nu &= \frac{v}{U_0} \\
x &= \frac{x}{\overline{L}}, \\
y &= \frac{y}{\overline{L}} \\
\xi &= \frac{\bar{\xi}}{U_0/\overline{L}}, \\
t &= \frac{\bar{t}}{U_0/\overline{L}}.
\end{align*}
\]  

(20)

Combining equations (15) and (10) yields

\[
\frac{\partial \xi}{\partial t} = -\nabla \cdot (\bar{u} \xi) + \frac{1}{\text{Re}} \nabla^2 \xi,
\]  

(21)

\[
\nabla^2 \psi = -\xi,
\]  

(22)

where \(\text{Re}\) is Reynolds number,

\[
\text{Re} = \frac{U_0 \overline{L}}{\nu}.
\]  

(23)
CHAPTER 3
COMPUTATIONAL APPROACH

- Procedure Methodology

Initially the fluid is at rest and the rotating boundary element assumes its rotational speed instantaneously. Thus, for the first time step the values of all variables, $\psi, \xi, u, \text{ and } v$ are zero with the exception of the finite velocities at mesh points on the rotating boundary. This constitutes the initial solution. A small time increment, $\Delta t$, is chosen according to equation (32) (Roach (12)). Then the computational cycle begins as the finite difference equation (FDE) form of the continuum equation for vorticity transport, equation (21), is used to calculate an approximation to $\frac{\partial \xi}{\partial t}$ at all interior points in the computational field. The new values of $\xi$ are calculated at a new time level, increased by an increment $\Delta t$, by "marching" the vorticity transport equation forward in time. The next step in the computational cycle is to solve the FDE form of equation (22), Poisson's equation, for new value of the stream function $\psi$ using the new interior values of $\xi$. Since this Poisson solution for new $\psi$ does not depend on the boundary values for new $\xi$, which are not yet known, the fact that the solution for the new $\psi$ is itself iterative with the optimized successive over-relaxation factor is employed.
At this point, new velocity components can be evaluated by finite difference analogs of equation (8). Then the new values of $\zeta$ on the boundaries of the computational region are calculated as a last step. The computational cycle is repeated until the desired time is reached. The new values of vorticity, $\zeta$, at all boundary points maybe calculated with high accuracy, using the appropriate form of the formula of Thom including the curvature term, and "Bubble Sort" techniques. The calculation of $\psi$ for all interior points at each time step, which required the iteration of equation (10), is the most time consuming operation and in order to reduce computing time to a minimum successive over-relaxation is used to increase the speed of convergence of the iterative procedure, and a vectorization is carried out there. As in the point successive over-relaxation (S.O.R.) iterative process, the new value of $\psi$ at each node is computed from the most recent values available at adjacent nodes, as indicated by equation (28). Schematically, the whole procedure is illustrated in Figure 3.1.1.
Start
- Layout
Finite Different Mesh
Initialize $\psi$ and $\xi$

Time Step Loop

Outer Space Loop

Inner Space Loop
New $\xi$ at interior points from
$$\xi_t = -\nabla \cdot (\bar{u} \xi) + \nabla^2 \xi / \text{Re}$$

SOR Loop

Outer Space Loop

Inner Space Loop
Iterate for new $\psi$ at all points from
$$\nabla^2 \psi = -\xi$$
using new $\xi$ at interior points.
Calculate new $\bar{u}$ from $u = \psi_y$, $v = -\psi_x$.

Calculate new boundary values of $\xi$
using new $\psi$ and $\xi$ values at interior points.

Desired Time Reached
Solution

Figure 3.1.1 Main Program
2 - Finite Difference Equation

In the following analysis the governing equations are written in finite difference form. In finite difference formulations, spatial derivatives are represented by centered differences, whereas time derivatives are represented by forward differences (forward time and centered space method: FTCS) except at the advection terms. A one-step, explicit, two-time-level method which achieves static stability of the advection terms involves the use of one-sided, rather than space-centered, differencing. Backward differences are used when the velocities are positive, and vice-versa (Upwind Differencing Method: Lilly (8)). The governing differential equation (15) with symmetric "legs" is formulated in finite difference form as follows:

\[ \xi^{n+1}(i,j) = \xi^n(i,j) + \Delta t(\nu \nabla^2 \xi - \vec{u} \cdot \nabla \xi) \]

\[ = \xi^n(i,j) + \Delta t(\nu \nabla^2 \xi - \frac{1}{D} u \xi_x - \frac{1}{D} v \xi_y) \]

\[ = \xi^n(i,j) + \frac{\Delta t}{R} \left[ \xi^n(i+1,j) + \xi^n(i-1,j) + \xi^n(i,j+1) \right. \]

\[ + \xi^n(i,j-1) - 4 \xi^n(i,j) \left. \right] - \Delta t(\vec{u} \xi_x + \vec{v} \xi_y) \]

\[ = \xi^n(i,j) + \frac{4 \Delta t}{R} \frac{\xi_{\text{surf}} - \xi^n(i,j)}{\partial_x} \]

\[ - \Delta t(u \xi_x + v \xi_y) \]
\[ n_n n_n n_n = \xi^n (i, j) + H_1 [\xi^n (i+1, j) - \xi^n (i, j)] + \xi^n (i, j-1) \\
- \xi^n (i, j) - \xi^n (i, j) + \xi^n (i+1, j) - \xi^n (i, j) \\
+ \xi^n (i, j+1)] - H_2 \{[\psi^n (i, j-1) - \psi^n (i, j)] \\
+ |\psi^n (i, j-1) - \psi^n (i, j)| (\xi^n (i, j) - \xi^n (i+1, j)) \\
+ [\psi^n (i, j-1) - \psi^n (i, j) \\
- |\psi^n (i, j-1) - \psi^n (i, j)|] (\xi^n (i, j) - \xi^n (i+1, j)) \\
+ [\psi^n (i, j) - \psi^n (i-1, j) \\
+ |\psi^n (i+1, j) - \psi^n (i-1, j)|] (\xi^n (i, j) - \xi^n (i, j+1)) \\
+ [\psi^n (i, j) - \psi^n (i-1, j) - |\psi^n (i+1, j) - \psi^n (i-1, j)|] \\
(\xi^n (i, j-1) - \xi^n (i, j)) \} \]

(24)

where: \( \delta = \) grid space,

\[ H_1 = \frac{\Delta t}{\Delta \xi_R}, \]

\[ H_2 = \frac{\Delta t}{4 \delta^2}, \]
and \( \xi^{n+1} \) denotes the value at time \( t + \Delta t \). The subscripts \( i, j \) refer to points in the region of flow such that in the general co-ordinate system \( X_i = iAX, Y_j = jAY \). Then the governing differential equation (15) with asymmetric "legs" is

\[
\xi^n_{i,j} = \xi^n_{i,j} + \frac{\Delta t}{R} \nabla \psi - \frac{\Delta t}{2} (2u\xi_x + 2v\xi_y),
\]

\[
\xi^n_{i,j} = \xi^n_{i,j} + H5 * ATLED
\]

\[
-H6 \left\{ \frac{\psi_{i,j+1} - \psi_{i,j-1}}{\delta_2 + \delta_4} \right\}
\]

\[
+ \left[ \frac{\psi_{i,j+1} - \psi_{i,j-1}}{\delta_2 + \delta_4} \right] \left[ \frac{\xi_{i,j} - \xi_{i-1,j}}{\delta_3} \right]
\]

\[
+ \frac{\psi_{i+1,j} - \psi_{i-1,j}}{\delta_2 + \delta_4} + \left[ \frac{\psi_{i+1,j} - \psi_{i+1,j}}{\delta_1 + \delta_3} \right] \left[ \frac{\xi_{i+1,j} - \xi_{i,j}}{\delta_1} \right]
\]

\[
+ \left[ \frac{\psi_{i-1,j} - \psi_{i+1,j}}{\delta_1 + \delta_3} \right] + \left[ \frac{\psi_{i-1,j} - \psi_{i+1,j}}{\delta_1 + \delta_3} \right]
\]

\[
\left[ \frac{\psi_{i+1,j} - \psi_{i+1,j}}{\delta_4} \right] + \left[ \frac{\psi_{i+1,j} - \psi_{i+1,j}}{\delta_1 + \delta_3} \right]
\]

\[
+ \left[ \frac{\psi_{i-1,j} - \psi_{i+1,j}}{\delta_1 + \delta_3} \right] \left[ \frac{\xi_{i,j+1} - \xi_{i,j}}{\delta_2} \right]
\]

\[ (25) \]
where: \[ H_5 = \frac{\Delta t}{R}, \]
\[ H_6 = \frac{\Delta t}{2}, \]

\[ \text{ALTED} = \frac{2 \xi (i+1,j)}{\delta_0 (\delta_0 + \delta_3)} + \frac{2 \xi (i,j+1)}{\delta_2 (\delta_2 + \delta_4)} \]
\[ + \frac{2 \xi (i-1,j)}{\delta_3 (\delta_0 + \delta_3)} + \frac{2 \xi (i,j-1)}{\delta_4 (\delta_2 + \delta_4)} \]
\[ - 2 \left[ \frac{1}{\delta_0 (\delta_0 + \delta_3)} + \frac{1}{\delta_2 (\delta_2 + \delta_4)} + \frac{1}{\delta_3 (\delta_0 + \delta_3)} + \frac{1}{\delta_4 (\delta_2 + \delta_4)} \right] \xi^n (i,j) \]

The expression of new or old values of time in the neighborhood of vorticities is omitted, since the scanning of the boundary points is different from that of the interior points. Bubble Sort techniques are used for the computation of vorticities at boundary points, using the formula of Thom (see Chapter 3.5 Boundary Treatment).

For the calculation of the stream functions in the grid centers, the iterative method of successive over-relaxation (SOR) was selected in order to have a very fast asymptotic rate of convergence. A five-point difference equation which represents the Poisson equation (10) can be found by expanding \( \psi(x,y) \) in a Taylor series and substituting back into Poisson's equation. Neglecting terms with coefficients of order \( \delta^2 \) or higher, the finite difference approximation is:
\[
\frac{1}{\delta^2} \psi(i+1,j) + \psi(i,j+1) + \psi(i-1,j) + \psi(i,j-1) \\
- 4\psi(i,j)] = -\xi(i,j)
\] (26)

Rearranging, we have

\[
\psi(i,j) = \langle \psi \rangle_{\text{sur}} + \frac{\xi \delta^2}{4}
\] (27)

where "\langle \rangle_{\text{sur}}" denotes the average of surrounding points.

Hockney (6) describes the method of odd/even SOR, in which the points on the grid mesh are corrected in a particular order:

First to be corrected are all the points for which \(i+j\) are even, followed by all points with \(i+j\) odd.

At each point, the corrected form of the stream function is

\[
\psi^{n+1}(i,j) = \text{ORF} \langle \psi \rangle_{\text{sur}} + \frac{\delta^2}{4} \psi^{n+1}(i,j) + (1 - \text{ORF}) \psi^n(i,j)
\] (28)

ORF is called the over-relaxation factor, \(1 \leq \text{ORF} \leq 2\), and it can be adjusted in order to speed up convergence of the system.

In this analysis, one more program was formulated which may calculate the optimized ORF with the calculations of the minimized error, being defined as

\[
E = \sum_{i=1}^{NX} \sum_{j=1}^{NY} |\psi^{n+1}(i,j) - \psi^n(i,j)|
\] (29)

where:
NX is the number of x divisions, and NY is the number of y divisions.

Since the approach of Frankel (3) for the Dirichlet problem in a rectangular domain of size \((i - 1)\Delta x \times (j - 1)\Delta y\) with constant \(\Delta x\) and \(\Delta y\) it may be shown that

\[
\text{ORF} = 2\left(1 - \sqrt{1 - \eta}\right)
\]  

(30)

where:

\[
\eta = \left[\cos\left(\frac{\pi}{i-1}\right) + \left(\frac{\Delta x}{\Delta y}\right)^2 \cos\left(\frac{\pi}{j-1}\right)\right]^{-1} + \left(\frac{\Delta x}{\Delta y}\right)
\]

The ORF may be approximated as an initial value for the program. Since \(i = j\) and \(\beta = \Delta x/\Delta y = 1\), then

\[
\text{ORF} = 2\left(1 - 1 - \cos\left(\frac{\pi}{i-1}\right)\right)\cos\left(\frac{\pi}{j-1}\right)
\]

(31)

Let \(i-1\) be large. Then, using \(\cos\left(\frac{\pi}{i-1}\right) \approx 1 - \frac{\left(\frac{\pi}{i-1}\right)^2}{2!}\),

The initial guess \((\text{IORF} \approx 1.91\) for 50 by 50 grid system\), which is decreased by small decrements in order to get the minimized error may be thus obtained.

Using this program, the optimum over-relaxation factor was determined for the rectangular coordinates 50 by 50 grid system. The optimum over-relaxation factor was found to be \(\text{ORF} = 1.87\) with the value of error \(E = 8.5 \times 10^{-31}\).
Because of the odd/even fashion of stepping through the grid, the SOR routine can be put into vectorized form for use on the CRAY-1 computer. Vectorizing on the CRAY-1 refers to arranging the FORTRAN code in such a way that the computer can take advantage of its parallel processing capabilities, meaning that calculations normally done sequentially are done concurrently. Vectorizing is possible because the calculation of any single odd/even point does not depend on the previously computed odd/even point on that row. Vectorizing SOR results in an increase in speed by a factor of 3 on the CRAY-1. The program listed in Appendix C is the vectorized version.

A finite-difference equation is said to be stable if its solution remains bounded as \( t \to \infty \). Furthermore, it is said to be convergent if the solution of the difference equation converges to the solution of the corresponding differential equation as the spatial and time increments tend to zero. Lax and Richtmeyer (7) show that stability implies convergence if the difference equations are consistent with the differential equations, i.e. if the truncation errors tend to zero as the spatial and time increments tend to zero. In this project, an analysis of a fluid filled rotating infinite cylinder was resolved with an error of \( E = 0.1 \% \), which is in very good agreement with the computations of analytical work. (see Figure B in Appendix A). The convergence of a flow in a circular cavity with part of wall moving is attained for Reynolds numbers 3.16,
10, 31.6, 100, 316, 1000, and 3160. Considering the application of upwind differencing with diffusion terms in two-dimensional incompressible flow, Roach (12) showed that the time step $\Delta t$, which is required for stability, is

$$\Delta t \leq \frac{1}{2 \frac{\Delta x}{\text{Re}(\Delta x + \Delta y)} + \frac{|u|}{\Delta x} + \frac{|v|}{\Delta y}}. \quad (32)$$

For CRAY-1, the half value of the time step, calculated in equation (32), was used in order to obtain the solution.

3 - Classification of Mesh Points

A polygon, $\Pi$, was setup and periodic meshes covered, with mutual overlap ($\lambda$), by $\Pi$ (see Figure 3.3.1).
Each mesh point is classified according to its relationship to $\pi$. Subroutine CLOSE takes as its input a point $x$ in the plane, and an N sided polygon $\pi$ in the plane, of vertices $[x_i]$. It returns an output ICLOSE of "2" if $\hat{x}$ is inside $\pi$, and an output ICLOSE of "1" if $\hat{x}$ is outside $\pi$ (see Figure 3.3.2).

![Figure 3.3.2 Moving Points Around a Polygon.](image)

$\hat{x}$, shown in Figure 3.3.2 is both the start and the finish point for $p$ as $p$ moves around $\pi$. Subroutine NEAR determines whether a point $p$ is near a polygon $\pi$. By "near" is meant within the small distance, using the matrix $|\Delta x| + |\Delta y| = d^*$ of a vertex, or using the standard matrix $\sqrt{\Delta x^2 + \Delta y^2} = d^*$ and requiring that a vertical line from $p$ intersects the side. With the help of these two subroutines, there may be 4 categories, 1, b, 2, and 3 (see Figure 3.3.1), where:

1 : mesh points outside $\pi$
4 - Treatment of Points According to Category

The mesh points on and outside of $\Gamma$ were skipped. At mesh points inside $\Gamma$ but with a neighbor either on $\Gamma$ or outside a stream function, $\psi$, is computed by asymmetrical finite difference approximation to Poisson's equation (see Figure 3.4.1).

![Figure 3.4.1 Treatment of Points](image)

The length of the shortened "legs" are computed by subroutine POINT. The values of $\psi$ at the "feet" of the shortened legs are given by the boundary conditions, $\psi = \psi(s)$ on $\Gamma$. The computation of the finite difference approximation to Poisson's
equation is accomplished by subroutine SIGMA.

A vorticity, $\xi$, is computed by the use of the no slip boundary condition according to the formula of Thom (Roach(12)). At strictly interior points (interior points with interior neighbors), a stream function is computed from symmetric finite difference approximation to Poisson's equation by the method of relaxation. A vorticity, $\xi$, is computed using weather differenced finite difference approximation to the vorticity transport equation (15).

5 - Boundary Treatment

For each boundary point (3-point) the arc length measured counter-clock-wise from the first vertex of $\Pi$ to the base of the normal from the boundary point to $\Pi$ is computed using the subroutine APROCH, augmented by the subroutine VARC computing the arc length of the vertices (see Figure 3.5.1). In order to apply Bubble Sort technique along the perimeter of a polygon in the counter-clock-wise sense, the subroutine PRANK gives the rank from smallest to largest of the arc lengths. By employing this method, a more accurate analysis may occur because the lengths of $l$ and $n$ in the formula of Thom (12$n$) are differentiated.
Vorticity at $N$ is determined from the formula of Thom. Vorticity at $L$ is determined by linear interpolation between such points as $N$, using the subroutine PERINT. The formula of Thom may be restated for no-slip conditions at a straight wall. Let point $P$ be at a distance $\delta$ from an impermeable wall $w$ which moves at speed $U_w$ (see Figure 3.5.2). Let $y$ be the coordinate normal to the wall ($y = 0$ on $w$).
Let the stream function at the wall be \( \psi_w \) and let the stream function at \( P \) be \( \psi_p \). Let the fluid velocity be \( U(y) \). Due to the no slip condition,

\[
U(0) = U_w \tag{33}
\]

Consider first the case of zero vorticity, in which

\[
U(y) = U_w \quad \forall y, \tag{34}
\]

where \( \forall \) means "at all points."

Eq. (34) states that the fluid between \( w \) and \( P \) is all flowing parallel to the wall at speed \( U_w \) as shown in figure 3.5.3a.

Then the flow between \( w \) and \( P \) is given by

\[
\psi_p - \psi_w = U_w \delta \tag{35}
\]
Consider next the case where \( U(y) \) is not constant but is linear according to

\[
U(y) = U_w + ky
\]  

where \( k \) is constant. Clearly

\[
k = U'(y) = -\xi
\]

where \( \xi \) here is assumed constant. Integrating Eq. (36) gives

\[
\psi_p - \psi_w = \int_0^\delta U(y) \, dy
\]

\[
= U_w \delta - k \frac{\delta^2}{2}
\]

solving for \( \xi \) gives

\[
\xi = \frac{2}{\delta^2} [(U_w \delta) - (\psi_p - \psi_w)]
\]

which is called the formula of Thom. The first term in brackets in equation (39), \( U_w \delta \), is recognized from equation (35) as the flow between \( P \) and \( w \) which would occur if the velocity were constant at \( U_w \). The second term, \( (\psi_p - \psi_w) \), is the flow which actually occurs. The formula of Thom, equation (39), implies that if there is more flow between \( w \) and \( P \) than would be
provided by a constant velocity, then the vorticity will be negative (see Figure 3.5.3b). If there is less flow than $U_w \delta$, then the vorticity will be positive (see Figure 3.5.3c). The formula of Thom may also be derived for a curved wall (see Figure 3.5.4).

From the definition of vorticity, equation (9),

$$\xi = v_x - u_y$$

$$= \frac{U_w d\theta}{dx} - (\psi_y)_y$$

$$= U_w (\frac{d\theta}{dx}) - (\frac{\psi_p - \psi_w}{\delta} - U_w) / \delta / 2.$$

Rearranging:

$$\xi = \frac{U_w}{a} + \frac{2}{\delta^2} [(U_w \delta) - (\psi_p - \psi_w)]$$  \hspace{1cm} (40)
where $a$ is the radius of curvature, or

$$a = \frac{(1+y_x^2)^{3/2}}{y_{xx}}$$

(41)
CHAPTER 4

ANALYTICAL APPROACH FOR A TEST PROBLEM

A cylinder of internal radius, \( a \), and infinite length is filled with a fluid of kinematic viscosity \( \nu \). Initially the fluid is at rest. At time zero the internal wall starts moving at peripheral speed \( U_w \) (Figure 4.4.1).

![Rotating Infinite Cylinder Filled with a Fluid](image)

Figure 4.4.1  Rotating Infinite Cylinder Filled with a Fluid

The overbars are omitted for representing dimensional quantities in this section. The problem shall be formulated in terms of stream function and vorticity. Due to rotational symmetry these may be written, respectively, as \( \psi (r,t) \) and \( \xi (r,t) \), the \( \phi \) being neglected. The differential equations for stream function and vorticity are, respectively

\[
\psi_{rr} + \frac{1}{r} \psi_r = -\xi
\]  

(42)

and
The boundary condition on the peripheral velocity \( q(r,t) \) is

\[
q(a, t) = (-\psi_r)_r = a = U_w
\]  

The initial condition of rest is given by

\[
\psi(r, 0) = 0, \quad (45)
\]

and

\[
\xi(r, 0) = 0. \quad (46)
\]

The problem of spinup is considered solved if \( q(r, t) \) are determined for all \( r \) and \( t \). Starting with the vorticity equation (14) and applying separation of variables in \( \xi(r, t) \), according to

\[
\xi(r, t) = R(r) T(t) \quad (47)
\]

yields, upon substitution into equation (43),

\[
\nu \left( \frac{\xi_{rr}}{r} + \frac{1}{r} \frac{\xi_r}{r} \right) = \xi_t + \vec{u} \cdot \nabla \xi = 0
\]

\[
\nu \left( \frac{R_{rr}}{R} + \frac{1}{r} \frac{R_r}{R} \right) = \frac{T_t}{T} = K = \text{const.} \quad (48)
\]

\[
\vec{u} \cdot \nabla \xi = 0, \quad (49)
\]

\[
(\xi_r, \Theta) = 0
\]
This clearly resulted in two ordinary differential equations,

\[ R_{rr} + R_r - \left(\frac{K}{r}\right)rR = 0 \]  \hspace{1cm} (49)

and

\[ T_t - KT = 0. \]  \hspace{1cm} (50)

Equation (50) has the general solution

\[ T = ce^{kt}. \]  \hspace{1cm} (51)

The vorticity is required to be bounded as \( t \to \infty \) and therefore \( K \leq 0 \). Then,

\[ k = -h; \quad h \geq 0 \]  \hspace{1cm} (52)

Equation (49) is Bessel's equation of order \( m \), which is

\[ xy_{xx} + xy_x + (x^2 - m^2) = 0 \]  \hspace{1cm} (53)

To transform equation (49) into equation (53) \( r \) is nondimensionalized according to

\[ r_n = r/L \]  \hspace{1cm} (54)

Transformation of equation (49) to the dimensionless radius \( r_n \) proceeds as follows.
\[ r_n L \frac{R_n}{r_n} \frac{1}{r_n} + R_n \frac{1}{L} - \frac{k}{\nu} r_n LR = 0 \]

\[ r_n R_n^2 + R_n^2 + (L^2 \frac{h}{\nu}) r_n R = 0 \quad (55) \]

Selecting

\[ L = \sqrt{\frac{\nu}{h}} \quad (56) \]

then

\[ (L^2 \frac{h}{\nu}) = 1 \quad (57) \]

and

\[ r_n^2 R_n^2 + r_n R_n^2 + r_n^2 R = 0 \quad (58) \]

Comparison of equation (58) with equation (53) shows equation (49) to be a Bessel equation of order zero. With the aid of equation (54) and equation (56) solution of equation (49) may be written as

\[ R = J_0 (r_n) = J_0 \left( \sqrt{\frac{h}{\nu} r} \right) \quad (59) \]

The desired solution of the vorticity equation is given by an expansion, in terms of the form (47), as follows.

\[ \xi = \sum_{i=0}^{\infty} c_i e^{-\frac{h_i t}{\nu}} J_0 \left( \sqrt{\frac{h_i}{\nu} r} \right) \quad (60) \]
The constants \([c_i] \) and \([h_i] \) must be chosen so as to satisfy the boundary and initial conditions to determine the constants, an expression for the peripheral verocity \(q = -\psi_r\) is derived. The equation for \(\psi\), Eq. (42), may be written as

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \psi_r) = -\xi
\]  

(61)

The space dependent portion of each term of (60) is the form \(J_o(\alpha_i r)\) so

\[
\frac{1}{r} \frac{d}{dr} (r \psi_r^i) = -J_o(\alpha_i r)
\]

(62)

which is integrated as follows.

\[
r \psi_r^i = -\int_0^r r J_o(\alpha_i r) dr
\]

\[
= -\frac{1}{\alpha_i^2} \int_0^r (\alpha_i r) J_o(\alpha_i r) d(\alpha_i r)
\]

\[
= -\frac{1}{\alpha_i^2} \int_0^{\alpha_i r} J_o(\theta) d\theta
\]

\[
= -\frac{1}{\alpha_i^2} (\alpha_i r) J_1(\alpha_i r)
\]

Thus

\[
\psi_r^i = -\frac{1}{\alpha_i} J_1(\alpha_i r)
\]

(63)

from which
\[ q(r, t) = -\psi_r = \sum_{i=0}^{\infty} c_i e^{-\frac{h_i^t}{h_i}} \sqrt{\frac{\nu}{h_i}} J_1 \left( \sqrt{\frac{h_i}{\nu}} r \right) \] (64)

Now equations (44) and (60) are used to compute the constants \([h_i]\). Near the wall at early times the velocity jumps from 0 to \(U_w\) over a very thin layer of thickness \(\epsilon\) (Figure 4.4.2).

\[ \xi = v_x - u_y = 0 \left( \frac{U_w}{\epsilon} \right) - 0 \left( \frac{U_w}{a} \right) \approx \psi_r \] (65)

so

\[ U_w = \int_{a-\epsilon}^{a} \xi \, dr = \text{const.} \] (66)

where \(0 \left( \frac{U_w}{\epsilon} \right)\) means the order of \(U_w/\epsilon\).

Thus, it can be seen that the vorticity distribution across the friction layer has a constant integral, \(U_w\). The rate of change of that integral with respect to time is given by

\[ 0 = \frac{d}{dt}U_w = \frac{d}{dt} \int_{a-\epsilon}^{a} \xi \, dr = \begin{bmatrix} \text{vorticity rate of} \\ \text{increase in layer} \end{bmatrix} \]
\[
\text{vorticity diffusing into the layer from the wall at } r = a \quad - \quad \text{vorticity diffusing out of the layer at inside edge of layer}
\]

\[
= \nu \left( \frac{\partial \xi}{\partial r} \right)_{r=a} - \nu \left( \frac{\partial \xi}{\partial r} \right)_{r=a-\epsilon}
\]  

For \( r \leq a-\epsilon \) the vorticity and all its derivatives are extremely small by equation (46) so equation (67) becomes

\[
0 = \nu \left( \frac{\partial \xi}{\partial r} \right)_{r=a} - \nu \cdot 0
\]

and

\[
\left( \frac{\partial \xi}{\partial r} \right)_{r=a} = 0 \quad \forall \text{ small } t.
\]  

Equations (50) and (68) can only hold for all small \( t \) if

\[
\frac{d}{dr} J_0 \left( \sqrt{\frac{h_i}{D}} \right) r = 0 \quad \forall r = a
\]

which implies that

\[
J_1 \left( \sqrt{\frac{h_i}{D}} \right) a = 0 \quad \forall i.
\]  

Thus the values of \( \sqrt{\frac{h_i}{D}} a \) are zeros of \( J_1 \):

\[
\sqrt{\frac{h_i}{D}} a = 0; 3.8317; 7.0156; 10.1735; 13.3237; 16.4706; \ldots
\]

* This form of equation (14) is same as that of the energy equation.
From equation (70) the leading term of equation (64) is seen to be indeterminate. That term is evaluated by taking the limit as
\[
\sqrt{\frac{h_o}{\nu}} a \to 0.
\]
thus
\[
\lim_{h_o \to 0} \left[ \frac{J_1 \left( \frac{h_o r}{\sqrt{h_o/\nu}} \right)}{\sqrt{h_o/\nu}} \right] = \lim_{\alpha \to 0} \left[ \frac{J_1 (\alpha r)}{\alpha} \right] = \frac{r}{2}
\] (71)

From (64) and (71),
\[
q(r, t) = \sum_{i=0}^{\infty} c_i e^{-h_i t} \frac{1}{\sqrt{h_i/\nu}} J_1 \left( \sqrt{\frac{h_i}{\nu}} r \right)
\]
or
\[
q(r, t) = c_0 \frac{r}{2} + \sum_{i=1}^{\infty} c_i e^{-h_i t} \sqrt{\frac{\nu}{h_i}} J_1 \left( \sqrt{\frac{h_i}{\nu}} r \right)
\] (72)

At \( r = a \), \( q = U_w \) and \( J_1 \left( \sqrt{\frac{h_i}{\nu}} r \right) = 0 \)
so equation (72) gives
\[
c_0 = \frac{2U_w}{a}.
\] (73)

The dimensionless peripheral velocity \( q' \) is defined as follows.
\[
q' = \frac{q}{U_w}
\] (74)

From equation (72) to equation (74)
\[ q'(r', t') = r' - \sum_{i=1}^{\infty} a_i e^{-\frac{\alpha_i^2 t'}{2}} J_1(\alpha_i r') \]  

(75)

where the \[ \{\alpha_i\} \] are the zeros of \[ J_1 \], \[ a_i = -\frac{c_i}{\sqrt{h_i}} \sqrt{\frac{\nu}{U_w}} \], and the dimensionless radius and time are given by:

\[ r' = \frac{r}{a} \]  

(76)

and

\[ t' = \frac{t \nu}{a^2} \]  

(77)

Clearly as \( t' \to \infty \) equation (75) gives the correct limiting peripheral velocity which is that of rigid body rotation (see curve labeled \( \infty \) in Figure 4.4.3).

\[ \text{Figure 4.4.3 Dimensionless Peripheral Velocity vs. Dimensionless Radius.} \]

There remains the task of computing the coefficients \( \{a_i\} \) so that equation (75) will give the correct peripheral velocity at \( t' = 0 \) (see curve labeled 0 in Figure 4.4.3). Substituting \( t' = 0 \) into equation (75) gives
\[ q'(r', 0) = r' - \sum_{i=1}^{\infty} a_i J_1(\alpha_i r') \quad (78) \]

or

\[ \sum_{i=1}^{\infty} a_i J_1(\alpha_i r') = r' - q'(r', 0) = q'(r', \infty) - q'(r', 0) \]

\[
\begin{cases} 
  r' ; r' < 1 \\
  0 ; r' = 1 
\end{cases} \
(79)
\]

The coefficients \([a_i]\) are obtained classically by convoluting equation (79) with successive components and applying the orthogonality conditions for Bessel functions. Thus both sides of equation (79) are multiplied by \(r'J_1(\alpha_j r')\) and integrated between 0 and 1.

\[
\sum_{i=1}^{\infty} a_i \int_0^1 r'J_1(\alpha_i r')J_1(\alpha_j r') dr' 
\]

\[
\int_0^1 (r')^2 J_1(\alpha_j r') dr' 
(80)
\]

Now applying the orthogonality conditions

\[
\int_0^1 rJ_1(\alpha_i r)J_1(\alpha_j r) dr = \begin{cases} 
  0 & ; i \neq j \\
  \frac{1}{2}(J_1(\alpha_i))^2 & ; i = j 
\end{cases} 
(81)
\]

to equation (80) gives
\[ \frac{1}{2} a_j (J'_1(\alpha_j))^2 = \int_0^1 (r')^2 J_1(\alpha_j r') \, dr' \]  

(82)

The orthogonality equation has effectively isolated a single coefficient \( a_j \) for computation. The derivative \( J'_1(\alpha_j) \) may be evaluated by use of the familiar Bessel function relationship

\[ \frac{d}{dx} (x^m J_m(x)) = x^m J_{m-1}(x) \]

Thus setting \( m = 1 \) yields

\[ \frac{d}{dx} (x J_1(x)) = x J'_1(x) + J_1(x) = x J_0(x) \]

\[ \alpha_j J'_1(\alpha_j) + J_1(\alpha_j) = \alpha_j J_0(\alpha_j). \]

But \( J_1(\alpha_j) = 0 \) so

\[ \alpha_j J'_1(\alpha_j) + 0 = \alpha_j J_0(\alpha_j) \]

and

\[ J'_1(\alpha_j) = J_0(\alpha_j) \]  

(84)

The integral appearing in equation (82) may be evaluated by another application of equation (83). Setting \( m = 2 \) yields

\[ \frac{d}{dx} (x^2 J_2(x)) = x^2 J_1(x) \]
which, upon integration yields

\[ \int_0^\infty x^2 J_1(x) dx = \alpha^2 J_2(\alpha). \]

thus

\[ \int_0^1 (r') J_1(\alpha_j r') dr' = \frac{1}{\alpha_j} \int_0^1 (\alpha_j r')^2 J_1(\alpha_j r') dr' = \frac{1}{\alpha_j^3} \int_0^\infty x J_1(x) dx = \frac{1}{\alpha_j^3} \alpha_j^2 J_2(\alpha_j) = \frac{1}{\alpha_j} J_2(\alpha_j) \] (85)

From equations (82), (84), and (85) we write

\[ a_j = \frac{2}{\alpha_j} \frac{J_2(\alpha_j)}{(J_0(\alpha_j))^2} \] (86)

Manipulation of the formal series for the Bessel function, which is

\[ J_m(x) = \left( \frac{x}{2} \right)^m \frac{1}{m!} \left\{ 1 - \frac{(x/2)^2}{(1)[(m+1)]} + \frac{(x/2)^4}{(1\cdot2)[(m+1)(m+2)]} - \frac{(m+2)^6}{(1\cdot2\cdot3)[(m+1)(m+2)(m+3)]} + \cdots \right\}, \] (87)

easily provides the relationship

\[ J_2(\alpha_i) = - J_0(\alpha_i). \] (88)
Equation (88) simplifies equation (86) to

\[ a_j = \frac{-2}{\alpha_j J_o(\alpha_j)} \]  

(89)

substitution of equation (89) into equation (75) gives

\[ q'(r', t') = r' + \sum_{i=1}^{\infty} \frac{-\alpha_i^2 t'}{i} \frac{J_i(\alpha_i r')}{J_0(\alpha_i)} \]  

(90)

By assembling several of our results we may write the following expressions for the dimensionless vorticity and stream function.

\[ \xi' = \frac{\xi}{(2U_w/a)} = 1 + \sum_{i=1}^{\infty} e^{-\alpha_i^2 t'} \frac{J_i(\alpha_i r')}{J_0(\alpha_i)} \]  

(91)

\[ \psi' = \frac{\psi}{(U_w a/2)} = (1 - r'^2) - 4 \sum_{i=1}^{\infty} \frac{-\alpha_i^2 t'}{\alpha_i^2} \left[ 1 - \frac{J_o(\alpha_i r)}{J_0(\alpha_i)} \right] \]  

(92)

A "Bessel" program is listed in Appendix B, which calculates values of \( q', \xi', \text{ and } \psi' \) at various values of \( r' \) and \( t' \), using three subroutines, SPEED which computes equation (90), VORT which computes equation (91), and FLOW which computes equation (92). The "Bessel" program was run on the Honeywell 66, using IMSL routines.
CHAPTER 5
NUMERICAL RESULTS AND DISCUSSION.

The stream function at center of cylinder in a test problem is plotted with respect to time. A comparison between Bessel function solution and numerical solution is shown in Figure B in Appendix A. Agreement is very good with the approximate error of 0.1%. There was a discrepancy due to omitting the curvature term in Thom's formula since the errors associated with the polygonal approximation propagate into the interior of the flow. The vorticities of the 26th column with 40 vertices are shown in Figure C in Appendix A. The results of Bessel function solution are shown by broken lines. It may be seen that the vorticity lines are becoming horizontal when time is increased, as is expected. The final value of vorticity at center is obtained after approximately 1600 time steps of 0.0002 each with the total dimensionless time of 0.32 secs (see Figure D in Appendix A).

The final values of vorticity at center, with respect to time steps, of a half-moving solid boundary problems (θ = 180°) are shown in Figure E to Figure K in Appendix A for the dimensionless Reynolds numbers, 3.16, 10, 31.6, 100, 316, 1000, and 3160 with the total dimensionless time, being converged. The cases of a quarter-moving solid boundary problems (θ = 90°) are plotted in Figure L to Figure O shown in Appendix A for the dimensionless Reynolds numbers, 100, 316, 1000, and 3160 with
the total dimensionless time, being converged. The duration for convergence with respect to the dimensionless Reynolds numbers in logarithmic scale (see Figure P in Appendix A) may be calculated. Ultimate vorticities at column 26 are resolved in Figure Q, Figure R, and Figure S in Appendix A (Re = 100, 1000, and 3160, $\theta = 180^\circ$). Some final vorticities at center of circle and at center of vortex are shown in Figure T and Figure S in Appendix A. The advection effects may be seen from the equi-vorticity lines in Figure V and Figure W, and furthermore, these data would become valuable in analyzing more delicate and complicated phenomena.
LIST OF REFERENCES


LIST OF REFERENCES (continued)


LIST OF REFERENCE (continued)


Figure A. Set Up 50 Mesh Spaces in Spin Up Problem.
Figure A1. Mesh Layout at Boundary Points with 20 Vertices for Spin-Up Problem (Upper-Half Region in Polygon).
Figure B. Dimensionless Streamfunction vs. Dimensionless Time at Center of Cylinder in Spinup Problem—Comparison of Bessel Function Solution and Numerical Solution.
Figure C. Dimensionless Vorticity vs. Mesh Point at 26th Column at Various Dimensionless Time (Re=1, $\theta=360^\circ$) - Comparison of Bessel Function Solution and Numerical Solution.
Figure D. Dimensionless Vorticity vs. Number of Time Step at Center (Re=1, $\Theta=360^\circ$, 1 Time Step=0.0002).
Figure E. Dimensionless Vorticity vs. Number of Time Step at Center (Re=3.16, $\theta=180^\circ$, 1 Time Step=0.0007).
Figure F. Dimensionless Vorticity vs. Number of Time Step at Center ($Re=10$, $\theta=180^\circ$, 1 Time Step=0.002).
Figure G. Dimensionless Vorticity vs. Number of Time Step at Center ($Re=31.6$, $\theta=180^\circ$, 1 Time Step $=0.004$).
Figure H. Dimensionless Vorticity vs. Number of Time Step at Center (Re=100, θ=180°, 1 Time Step = 0.008).

Dimensionless Time = 32.
Figure I. Dimensionless Vorticity vs. Number of Time Step at Center(Re=316, $\theta=180^\circ$, 1 Time Step=0.008).
Figure J. Dimensionless Vorticity vs. Number of Time Step at Center (Re=1000, θ=180°, 1 Time Step=0.01).
Figure K. Dimensionless Vorticity vs. Number of Time Step at Center (Re=3160, $\theta=180^\circ$, 1 Time Step=0.01).
Figure L. Dimensionless Vorticity vs. Number of Time Step at Center ($Re=100$, $\theta=90^\circ$, 1 Time Step=0.006).
Figure M. Dimensionless Vorticity vs. Number of Time Step at Center (Re=316, θ=90°, 1 Time Step=0.008).
Figure N. Dimensionless Vorticity vs. Number of Time Step at Center (Re=1000, \( \theta = 90^\circ \), 1 Time Step=0.01).
Figure 0. Dimensionless Vorticity vs. Number of Time Step at Center(Re=3160, θ=90°, 1 Time Step=0.009).
\[ \log(T) = \log(R) - 0.2 \]

Figure P. Duration for Convergence (T) vs. Reynolds Number in Logarithmic Scale at $\theta = 90^\circ, 180^\circ, 360^\circ$. 

\[ \log(T) = 0.7 + 0.5 \log(R) \]
Figure Q. Ultimate Dimensionless Vorticity vs. Mesh Points at 26th Column (Re=100, \( \theta = 180^\circ \)).
Figure R. Ultimate Dimensionless Vorticity vs. Mesh Points at 26th Column (Re=1000, θ=180°).
Figure S. Ultimate Dimensionless Vorticity vs. Mesh Points at 26th Column (Re=3160, θ=180°).
Figure T. Ultimate Dimensionless Vorticity at Center vs. Reynolds Number in Logarithmic Scale (θ=90°, 180°).
Figure U. Ultimate Dimensionless Vorticity at Center of Vortex vs. Reynolds Number in Logarithmic Scale ($\theta=90^\circ, 180^\circ$).
Figure V. Dimensionless Equivorticity Line at R=100 and $\theta=180^\circ$. 
Figure W. Dimensionless Equivorticity Line at \( R=316 \) and \( \theta=90^\circ \).
APPENDIX B

COMPUTER PROGRAM FOR ANALYTICAL SOLUTION

The program which calculates the dimensionless stream function and vorticities for the two-dimensional incompressible spin up problem ($\theta = 360^\circ$, Re = 1) by analytical works is listed below. The principal symbols used in the program and their definitions are as follows:

- **ALPHA(100)**: zeros of $J_1$
- **ALPJO(100)**: $J_0(\alpha)$
- **ALPJ0R(100, 20)**: $J_0(\alpha r)$
- **ARGO**: argument of $J_0(\alpha r)$
- **MMBSJO**: Bessel function of order 0
- **MMBSJ1**: Bessel function of order 1
- **PI**: $\pi$, the geometry constant
- **R**: radius
- **XJJ0**: temporary value of $J_0(\alpha r)$
- **XJJ1**: temporary value of $J_1(\alpha r)$
- **XJO**: temporary value of $J_0(\alpha)$
DIMENSION ALPHA(100), R(21), ALPJO(100), ALPJOR(100, 21)
DOUBLE PRECISION ALPHA, ARGO, XJJ1, XJJ0, XJ0, PI, R
&, MMBSJO, MMBSJ1, ARG
DATA PI, T/3.14159D0, 0.0/
EXTERNAL MMBSJ1, MMBSJO, SPEED, VORT, FLOW
DATA ALPHA/3.8317D0, 7.0156D0, 10.1735D0, 13.3237D0,
& 16.4706D0, 19.6159D0, 22.7601D0, 25.9037D0, 29.0468D0,
& 32.1897D0, 35.3323D0, 38.4748D0, 41.6171D0, 44.7593D0,
& 47.9015D0, 51.0435D0, 54.1856D0, 57.3275D0, 60.4695D0,
& 63.6114D0, 66.7532D0, 69.8951D0, 73.0369D0, 76.1787D0,
& 79.3205D0, 82.4623D0, 85.6040D0, 88.7458D0, 91.8875D0,
& 95.0292D0, 98.171D0, 101.3127D0, 104.4544D0, 107.5961D0,
& 110.7378D0, 113.8794D0, 117.0211D0, 120.1628D0, 123.3045D0,
& 126.4461D0, 60*1.00D0/
DO 30 II=1, 60
ALPHA(40+II)=ALPHA(39+II) + PI
30 CONTINUE
R(1)=0.0
DO 20 NNN=1, 20
R(NNN+1)=R(NNN)+.05D0
20 CONTINUE
DO 50 N=1, 100
ARG=ALPHA(N)
XJO=MMBSJO(ARG, IER)
ALPJO(N)=XJO
50 CONTINUE
DO 55 LL=1, 100
DO 55 LLL=1, 21
ARGO=ALPHA(LL)*R(LLL)
XJJ0=MMBSJO(ARGO, IER)
ALPJOR(LL, LLL)=XJJ0
55 CONTINUE
DELTIM=.01
DO 60 IMET=1, 15
CALL SPEED(R, T, ALPJO, ALPHA)
CALL VORT(R, T, ALPJOR, ALPJO, ALPHA)
CALL FLOW(R, T, ALPJOR, ALPJO, ALPHA)
T=T+DELTIM
60 CONTINUE
STOP
END
SUBROUTINE SPEED(R, T, ALPJO, ALPHA)
DIMENSION ALPJO(100), ALPHA(100), R(21)
DOUBLE PRECISION ALPHA, R, XJJ1, ARG1, MMBSJ1
EXTERNAL MMBSJ1
DO 80 K=1, 21
VELSUM=0.0
DO 85 KK=1,100
 ARG1=ALPHA(KK)*R(K)
 XJJ1=MMBSJ1(ARG1,IER)
 EXARG=ALPHA(KK)*ALPHA(KK)*T.
 IF(EXARG.LE.88.0)GO TO 83
 VEL=0.0
 GO TO 84
83 VEL=(2.*EXP(-EXARG)/ALPHA(KK)*(XJJ1/ALPJO(KK))
84 VELSUM=VELSUM+VEL
85 CONTINUE
 VELTOT=VELSUM+R(K)
 WRITE(5,86)T,R(K),VELTOT
86 FORMAT('T=','F8.2','R=','F8.2',' VELOCITY=','F20.8)
80 CONTINUE
 RETURN
END
SUBROUTINE VORT(R,T,ALPJOR,ALPJO,ALPHA)
DIMENSION R(21),ALPJOR(100,21),ALPJO(100),ALPHA(100)
DOUBLE PRECISION ALPHA,R
DO 90 J=1,21
 VORSUM=0.0
 DO 95 JJ=1,100
 AXARG=ALPHA(JJ)*ALPHA(JJ)*T
 IF(AXARG.LE.88.0)GO TO 93
 VORTT=0.0
 GO TO 94
93 VORTT=EXP(-AXARG)*(ALPJOR(JJ,J)/ALPJO(JJ))
94 VORSUM=VORSUM+VORTT
95 CONTINUE
 VORTOT=1.+VORSUM
 WRITE(5,100)T,R(J),VORTOT
100 FORMAT('T=','F8.2',' R=','F8.2',' VORTICITY=','F20.8)
90 CONTINUE
 RETURN
END
SUBROUTINE FLOW(R,T,ALPJO,ALPHA)
DIMENSION R(21),ALPJOR(100,21),ALPJO(100),ALPHA(100)
DOUBLE PRECISION ALPHA,R
DO 110 I=1,21
 PSISUM=0.0
 DO 120 III=1,100
 BXARG=ALPHA(III)*ALPHA(III)*T
 IF(BXARG.LE.88.0)GO TO 103
 PSI=0.0
 GO TO 104
103 PSI=(EXP(-BXARG)/(ALPHA(III))**2.)*(1.-ALPJOR(III,I)
 &/ALPJO(III))
104 PSISUM = PSISUM + PSI
120 CONTINUE
PSITOT = (1. - (R(I))**2.) - 4.*PSISUM
WRITE(5,130) T, R(I), PSITOT
110 CONTINUE
RETURN
END
APPENDIX C

COMPUTER PROGRAM FOR NUMERICAL SOLUTION

The program which calculates the dimensionless stream functions and vorticities for the two-dimensional incompressible spin up problem by computational works is listed below. The principal symbols used in the program and their definitions are as follows:

ARCLEG(K,L) arc length from the 1st vertex to the Lth leg of Kth 3 point
ARCNOR(J) arc length from the 1th vertex to the bases of the normals of the 3 points in computer-clock-wise order around
ARCNOR(K) arc length from the 1st vertex to the Kth normal
ARCVVER(I) arc length from the 1st vertex to the Ith vertex
DCALL(L) 4 distances for calling SUBROUTINE SIGMA
DELTA mesh space
DELTAT time increment
DSIG(K,L) distances to $\pi$ right of, above, left of, and below the Kth point near $\delta$ (boundary)
I grid-point subscript
ICLASS(I,J) classification of mesh points 1-out; 2-in; 3-on $\delta$
IDISPLY(51) mesh display matrix
IPOFR(J) integer position of occurrence in the mesh
scan of the Jth 3-point going C.C.W. around \( \pi \) (position of ranks J)
IROFP(I) ranks of the Ith element of ARCNOR
ISSIG(K,L) side index to right of, above, to left of, and below the Kth point near \( \sigma \) according as whether L is respectively 1, 2, 3 and 4
J grid-point subscript
K boundary point index
L mesh points leg index
NSIG number of points near \( \sigma \)
NTS number of time steps
NV number of vertices
NX number of columns in space mesh
NY number of rows in space mesh
OMEGA(I,J) mesh vorticity
OMNEW(I,J) mesh vorticity storage place
OMWALL(I) vorticity at base of normal of the Ith 3-point C.C.W. around \( \pi \)
PSI(I,J) mesh stream function
R Reynolds number
SCALL(L) 4 stream functions for calling SUBROUTINE SIGMA
SMALL 0.25 DELTA
SPI(I) stream function on \( \pi \)
SRTRN value of stream function returned by
SUBROUTINE SIFMA

SSIG(K,L)  stream function on right of, above, left of, and below Kth point

UPI(I)  velocity on \( \pi \)

VCALL  value of OMEGA(I,J) for calling SIGMA

VOR3L(L)  vorticity at the Lth leg of a 3-point either from an interior point or interpolated at a boundary

VORSUR(L)  surrounding vorticity at the Lth leg of a temporary assymetric spider

WL  mutual overlap \( \lambda \)

XPI(I)  vertex coordinates of \( \pi \)

YMAX  greatest y-coordinates reached by \( \pi \)

YMIN  least y-coordinates reached by \( \pi \)

YPI(I)  vertex coordinates of \( \pi \)
DIMENSION DCALL(4), ISCALL(4), VORSUR(4), SCALL(4)
DIMENSION XPI(200), YPI(200), UPI(200), SPI(200), ARCVVER(200), RO(200)
DIMENSION PSI(51,49), OMEGA(51,49), PSINEW(49)
DIMENSION JSTART(600), JEND(600)
COMMON/COM1/OMNEW(51,49)
COSH(S)=0.5*(EXP(S)+EXP(-S))
SINH(S)=0.5*(EXP(S)-EXP(-S))
5 FORMAT(8I10)
10 FORMAT(8F10.5)
15 FORMAT(1H,125I1)
20 FORMAT(1H-)
25 FORMAT(1H0)
30 FORMAT(1H1)
32 FORMAT(38H STREAM FUNCTION AFTER INITIALIZATION:)
35 FORMAT(22H STREAM FUNCTION AFTER, 15, 12H TIME STEPS:)
37 FORMAT(32H VORTICITY AFTER INITIALIZATION:)
40 FORMAT(16H VORTICITY AFTER, 15, 12H TIME STEPS:)
50 FORMAT(1H , 8I10)
60 FORMAT(1H , 8F12.7)
65 FORMAT(16H WALL VORTICITY:)
70 FORMAT(26H BOUNDARY POINT VORTICITY:)
75 FORMAT(30H PRINCIPAL VORTICITY DIAGONAL:)
80 FORMAT(35H ANTI-PRINCIPAL VORTICITY DIAGONAL:)
TIME1=SECOND(EARLY)
C READ 5, NPI, NX, NY, NTS, NRELAX, INTDIS, IE
READ (5, 5) NPI, NX, NY, NTS, NRELAX, INTDIS, IE
C READ 10, WL, R, YSTART, DELTAT, ORF
READ (5, 10) WL, R, YSTART, DELTAT, ORF
PRINT 50, NPI, NX, NY, NTS, NRELAX, INTDIS, IE
PRINT 60, WL, R, YSTART, DELTAT, ORF
C READ 10, XPI(I), YPI(I), UPI(I), SPI(I), RO(I)
DO 100 I=1, NPI
XPI(I)=1. + COS(6.283185*(I-1)/NPI)
YPI(I)=1. + SIN(6.283185*(I-1)/NPI)
UPI(I)=0.
IF(I.LE.IE) UPI(I)=1.
SPI(I)=0.
RO(I)=1.
100 PRINT 60, XPI(I), YPI(I), UPI(I), SPI(I), RO(I)
CALL VARC(NPI, XPI, YPI, ARCVVER)
PRINT 60, (ARCVVER(I), I=1, NPI)
PERIM=ARCVVER(I)
DELTA=WL/(NX-1)
SMALL = DELTA *.25
H1 = DELTAT/(DELTA*DELTA*R)
H2 = DELTAT/(DELTA*DELTA*4.)
H5 = DELTAT/R
H6 = DELTA *.5
ORFTG = ORF*DELTA*DELTA/4.
ORFD4 = ORF/4.
ORFM1 = ORF-1.
Y = YSTART
DO 720 J = 1, NY
   X = 0.
   DO 710 I = 1, NX
      CALL CLOSE(NPI,XPI,YPI,X,Y,IRTRN)
      IF(IRTRN.EQ.0)GO TO 700
      CALL NEAR(NPI,XPI,YPI,X,Y,SMALL,NYE)
      IF(NYE.NE.0)GO TO 700
      ICLASS(I,J) = 2
   GO TO 710
700 ICLASS(I,J) = 1
710 X = X + DELTA
720 Y = Y - DELTA
DO 730 I = 1, NX
   IF(ICLASS(I,1).EQ.2)ICLASS(I,1) = 3
   IF(ICLASS(I,NY).EQ.2)ICLASS(I,NY) = 3
730 CONTINUE
NYM1 = NY - 1
DO 740 J = 2, NYM1
   DO 740 I = 1, NX
      IL = I - 1
      IF(IL.LT.1)IL = NX
      IR = I + 1
      IF(IR.GT.NX)IR = 1
      IF(ICLASS(I,J).EQ.2.AND.(ICLASS(IR,J).EQ.1.OR.ICLASS(I,J-1).EQ.1.OR.ICLASS(I,J+1).EQ.1)) + ICLASS(I,J) = 3
740 CONTINUE
DO 745 I = 1, NX
   JSTART(I) = 0
   DO 745 J = 1, NY
      IF(ICLASS(I,J-1).NE.2.AND.ICLASS(I,J).EQ.2)JSTART(I) = J
      IF(ICLASS(I,J+1).NE.2.AND.ICLASS(I,J).EQ.2)JEND(I) = J
745 CONTINUE
DO 750 J = 1, NY
   PRINT 15,(ICLASS(I,J),I = 1, NX)
750 CONTINUE
GO TO 758
PTINT 25
JBELOW=NY-10
DO 755 J=JBELOW, NY
PRINT 15, (ICLASS(I,J), I=1, NX)
755 CONTINUE
758 K=0
AA=DELTA/10000.
BB=10000.*DELTA
X=0.0
DO 840 I=1, NX
Y=YSTART
DO 820 J=1, NY
IF( ICLASS(I,J).NE.3) GO TO 810
K=K+1
ISIG(K)=I
JSIG(K)=J
CALL APROCH(Delta, X, Y, NPI, YPI, L, DSQ, ARCN)
H3(K)=2./DSQ
H4(K)=SPI(L)+UPI(L)*(SORT(DSQ)+1./RO(L)*H3(K))
IF(L.NE.1)ARCN+ARCVER(L)
ARCNOR(K)=ARCN
ARCALL(K)=ARCN
CALL POINT(AA, BB, X, Y, NPI, XPI, YPI, DCALL, ISCALL, SCALL)
CALL NEAR(NPI, XPI, YPI, X, Y, AA, NR)
DO 760 L=1, 4
DSIG(K,L)=DCALL(L)
ARCADD=0.
IF(ISCALL(L).NE.1)ARCADD=ARCVER(ISCALL(L))
ARCLEG(K,L)=SCALL(L)+ARCADD
II=ISCALL(L)
ISSIG(K,L)=II
SSIG(K,L)=SPI(II)
760 CONTINUE
IF(NR.EQ.0) GO TO 770
ISSIG(K,1)=-ISSIG(K,1)
GO TO 810
770 II=I+1
IF(II.GT.NX)II=1
IF(ICLASS(II,J).NE.1) ISSIG(K,1)=0
IF(J.EQ.1) GO TO 790
IF(ICLASS(I,J-1).NE.1) ISSIG(K,2)=0
790 I3=I-1
IF(I3.LT.1) I3=NX
IF(ICLASS(I3,J).NE.1) ISSIG(K,3)=0
IF(J.EQ.NY) GO TO 810
IF(ICLASS(I,J+1).NE.1) ISSIG(K,4)=0
810 Y = Y - DELTA
820 CONTINUE
830 X = X + DELTA
840 CONTINUE
NSIG = K
CALL PRANK (NSIG, ARCALL, IPOFR, IROFP)
PRINT 50, (ISIG (IPOFR (K)), K = 1, NSIG)
PRINT 50, (JSIG (IPOFR (K)), K = 1, NSIG)
PRINT 50, (IPOFR (I), I = 1, NSIG), (IROFP (I), I = 1, NSIG)
PRINT 50, NSIG
DO 850 L = 1, 20
PRINT 860, (DSIG (L, J), J = 1, 4), (ISSIG (L, J), J = 1, 4), (SSIG (L, J), 
& J = 1, 4)
PRINT 60, H3 (L), ARCNOR (L)
PRINT 860, (ARCLEG (L, J), J = 1, 4)
850 CONTINUE
860 FORMAT (I1, 4F7.3, 4I3, 4F7.3)
DO 853 K = 1, NSIG
ARCALL (K) = ARCNOR (K)
DO 857 K = 1, NSIG
ARCNOR (K) = ARCALL (IPOFR (K))
Y = YSTART
CPSTI = 0.033333333
DO 880 J = 1, NY
BIGPSI = Y * (-1. + Y * (1.3333333 - 0.44444444 * Y))
S = 3.1622777 * (1. - Y)
HYPSNS = SINH (S)
F = -0.1468 * HYPSNS - 0.03392 * S * COSH (S)
X = 0.0
DO 870 I = 1, NX
PSI (I, J) = 0.0
OMEGA (I, J) = 0.0
OMNEW (I, J) = 0.0
GO TO 870
COSINE = -COS (3.1622777 * X)
XT = X + WL * .5
YT = Y
CALL RETRO (XT, YT)
OMEGA (I, J) = 2.6666667 * (YT - 1.)
PSI (I, J) = BIGPSI + CPSI * F * COSINE
OMNEW (I, J) = PSI (I, J)
870 X = X + DELTA
880 Y = Y - DELTA
GO TO 1031
DO 1030 IRLX = 1, 500
DO 930 I = 1, NX
I1 = 1 + MOD (I, NX)
I3=NX-MOD(NX+1-I,NX)
JS=JSTART(I)
IF(JS.EQ.0)GO TO 930
JE=JEND(I)
JSBLAC=JS
JSRED=JS+1
JEBLAC=JE-MOD(JE-JS,2)
JERED=JE-1+MOD(JE-JS,2)
DO 910 J=JSBLAC,2
910 PSINEW(J)=ORFD4*(PSI(I1,J)+PSI(I,J+1)+PSI(I3,J)+PSI(I,J-1))
++ORFIG*OMEGA(I,J)-ORFM1*PSI(I,J)
DO 920 J=JSBLAC,JEBLAC,2
920 PSI(I,J)=PSINEW(J)
DO 923 J=JSRED,JERED,2
923 PSINEW(J)=ORFD4*(PSI(I1,J)+PSI(I,J+1)+PSI(I3,J)+PSI(I,J-1))
++ORFTG*OMEGA(I,J)-ORFM1*PSI(I,J)
DO 927 J=JSRED,JERED,2
927 PSI(I,J)=PSINEW(J)
930 CONTINUE
DO 1020 K=1,NSIG
I=ISIG(K)
J=JSIG(K)
I1=1+MOD(I,NX)
I3=NX-MOD(NX+1-I,NX)
IF(ISSIG(K,1).GE.0)GO TO 950
PSI(I,J)=SSIG(K,1)
GO TO 1020
950 DO 1010 L=1,4
IF(ISSIG(K,L).EQ.0)GO TO 960
DCALL(L)=DSIG(K,L)
SCALL(L)=SSIG(K,L)
GO TO 1010
960 DCALL(L)=DELTA
GO TO(970,980,990,1000),L
970 SCALL(L)=PSI(I1,J)
GO TO 1010
980 SCALL(L)=PSI(I,J-1)
GO TO 1010
990 SCALL(L)=PSI(I3,J)
GO TO 1010
1000 SCALL(L)=PSI(I,J+1)
1010 CONTINUE
VCALL=OMEGA(I,J)
CALL SIGMA(DCALL,SCALL,VCALL,SRTRN)
PSI(I,J)=SRTRN
1020 CONTINUE
1030 CONTINUE
CONTINUE
DO 1035 J=1,NY
DO 1032 I=1,NX
OMNEW(I,J)=OMEGA(I,J)
1032 CONTINUE
1035 CONTINUE
PRINT 30
PRINT 37
CALL DISCRA(NX,NY,2,1,1,.001)
PRINT 30
PRINT 32
DO 1440 J=1,NY
DO 1445 I=1,NX
OMNEW(I,J)=PSI(I,J)
1445 CONTINUE
1440 CONTINUE
CALL DISCRA(NX,NY,2,1,1,.001)
TIME2=SECOND(LATER)
ELAPSE=TIME2-TIME1
PRINT 1447,ELAPSE
1447 FORMAT(40H TIME FOR INPUT,MESH,AND INITIALIZATION=,E10.3)
DO 3000 ITS=1,NTS
TIME1=SECOND(EARLY)
DO 1532 I=1,NX
I1=I+MOD(I,NX)
I3=NX-MOD(NX+1-I,NX)
JS=JSTART(I)
IF(JS.EQ.O)GO TO 1532
JE=JEND(I)
DO 1530 J=JS,JE
J2=J-1
J4=J+1
U2D=PSI(I,J2)-PSI(I,J4)
V2D=PSI(I3,J)-PSI(I1,J)
OMEGA0=OMEGA(I,J)
OMEGA1=OMEGA(I1,J)
OMEGA2=OMEGA(I,J2)
OMEGA3=OMEGA(I3,J)
OMEGA4=OMEGA(I,J4)
OMXD1=OMEGA1-OMEGA0
OMXD3=OMEGA0-OMEGA3
OMTD2=OMEGA2-OMEGA0
OMYD4=OMEGA0-OMEGA4
U4DP=U2D+ABS(U2D)
U4DM=U2D-ABS(U2D)
V4DP=V2D+ABS(V2D)
V4DM=V2D-ABS(V2D)
OMNEW(I,J)=OMEGA0


++H1*(OMXD1+OMYD2-OMXD3-OMYD4)  
+H2*(U4DP*OMXD3+U4DM*OMXD1+V4DP*OMYD4+V4DM*OMYD2)

1530 CONTINUE
1532 CONTINUE
   DO 1535 K=1,NSIG
      I=ISIG(K)
      J=JSIG
      OMWALL(IROFP(K))=H3(K)*(H4(K)-PSI(I,J))
   1535 CONTINUE
   DO 1780 K=1,NSIG
      I=ISIG(K)
      J=JSIG(K)
      I1=1+MOD(I,NX)
      I3=NX-MOD(NX+1-I,NX)
      IF(ISSIG(K,1).GE.0)GO TO 1550
      OMNEW(I,J)=OMWALL(IROFP(K))
      GO TO 1780
   1550 DO 1610 L=1,4
      IF(ISSIG(K,L).EQ.0)GO TO 1560
     DCALL(L)=DSIG(K,L)
      SLEG=ARCLEG(K,L)
      CALL PERINT(NSIG,PERIM,ARCNOR,OMWALL,SLEG,VLEG)
      VORSUR(L)=VLEG
      SCALL(L)=SSIG(K,L)
      GO TO 1610
   1560 DCALL(L)=DELTA
      GO TO(1570,1580,1590,1600),L
   1570 VORSUR(L)=OMEGA(I1,J)
      SCALL(L)=PSI(I1,J)
      GO TO 1610
   1580 VORSUR(L)=OMEGA(I,J-1)
      SCALL(L)=PSI(I,J-1)
      GO TO 1610
   1590 VORSUR(L)=OMEGA(I3,J)
      SCALL(L)=PSI(I3,J)
      GO TO 1610
   1600 VORSUR(L)=OMEGA(I,J+1)
      SCALL(L)=PSI(I,J+1)
   1610 CONTINUE
   VCENTER=OMEGA(I,J)
   CALL ASLAP(VORSUR,DCALL,VCENTER,ATLED)
   D1PD3=DCALL(1)+DCALL(3)
   D2PD4=DCALL(2)+DCALL(4)
   USIG=(SCALL(2)-SCALL(4)/D2PD4
   VSIG=(SCALL(3)-SCALL(1)/D1PD3
   OMXR=(VORSUR(1)-VCENTER)/DCALL(1)
   OMXL=(VCENTER-VORSUR(3))/DCALL(3)
OMYA = (VORSUR(2) - VCENTER) / DCALL(2)
OMYB = (VCENTER - VORSUR(4)) / DCALL(4)
U2SIGP = USIG + ABS(USIG)
U2SIGM = USIG - ABS(USIG)
V2SIGP = VSIG + ABS(VSIG)
V2SIGM = VSIG - ABS(VSIG)
OMNEW(I, J) = VCENTER + H5*ATLED
+ -H6*(U2SIGP*OMXL + U2SIGM*OMXR + V2SIGP*OMYA + V2SIGM*OMYA)

1780 CONTINUE
DO 1792 J = 1, NY
DO 1790 I = 1, NX
OMEGA(I, J) = OMNEW(I, J) * ORFTG
1790 CONTINUE
1792 CONTINUE
TIME2 = SECOND(LATER)
ELAPSE = TIME2 - TIME1
IF(ITS.NE.NTS) GO TO 1800
PRINT 1795, ELAPSE
1795 FORMAT(37H TIME FOR SOLVING VORTICITY EQUATION=, E10.3)
1800 CONTINUE
TIME1 = SECOND(EARLY)
DO 2020 IRELAX = 1, NRELAX
DO 1930 I = 1, NX
I1 = 1 + MOD(I, NX)
I3 = NX - MOD(NX + 1 - I, NX)
JS = JSTART(I)
IF(JS.EQ.0) GO TO 1930
JE = JEND(I)
JSBLAC = JS
JSRED = JS + 1
JEBLAC = JE - MOD(JE - JS, 2)
JERED = JE - 1 + MOD(JE - JS, 2)
DO 1910 J = JSBLAC, JEBLAC, 2
1910 PSI(J) = ORFD4*(PSI(I1, J) + PSI(I, J + 1) + PSI(I3, J) + PSI(I, J - 1))
+ OMEGA(I, J) - ORFM1*PSI(I, J)
DO 1920 J = JSBLAC, JEBLAC, 2
1920 PSI(I, J) = PSINEW(J)
DO 1923 J = JSRED, JERED, 2
1923 PSI(J) = ORFD4*(PSI(I1, J) + PSI(I, J + 1) + PSI(I3, J) + PSI(I, J - 1))
+ OMEGA(I, J) - ORFM1*PSI(I, J)
DO 1927 J = JSRED, JERED, 2
1927 PSI(I, J) = PSINEW(J)
1930 CONTINUE
DO 2020 K = 1, NSIG
I = ISIG(K)
J = JSIG(K)
II = 1 + MOD(I, NX)
I3 = NX - MOD(NX + 1 - I, NX)
IF(ISSIG(K, 1).GE.0) GO TO 1950
PSI(I, J) = SSIG(K, 1)
GO TO 2020

1950 DO 2010 L = 1, 4
IF(ISSIG(K, L).EQ.0) GO TO 1960
DCALL(L) = DSIG(K, L)
SCALL(L) = SSIG(K, L)
GO TO 2010

1960 DCALL(L) = DELTA
1970 SCALL(L) = PSI(II, J)
GO TO 2010
1980 SCALL(L) = PSI(I, J-1)
GO TO 2010
1990 SCALL(L) = PSI(I3, J)
GO TO 2010
2000 SCALL(L) = PSI(I, J+1)
2010 CONTINUE

VCALL = OMNEW(I, J)
CALL SIGMA(DCALL, SCALL, VCALL, SRTRN)
PSI(I, J) = SRTRN

2020 CONTINUE
TIME2 = SECOND(LATER)
ELAPSE = TIME2 - TIME1
IF(ITS .NE. NTS) GO TO 2027
PRINT 2025, NRELAX, ELAPSE

2025 FORMAT(18H TIME REQUIRED FOR, I4, 13H RELAXATIONS =, E10.3)
2027 DO 2030 J = 1, NY
DO 2035 I = 1, NX
OMEGA(I, J) = OMNEW(I, J)
2030 CONTINUE
2035 CONTINUE

IMPRTR = 0
RINT = (ITS + .25) / INTDIS
IINT = RINT
DIFRI = RINT - IINT
HALFOI = .5 / INTDIS
IF(DIFRI .LT. HALFOI) IMPRTR = 1
IF(IMPRTR .NE. 1) GO TO 3000
PRINT 30
PRINT 40, ITS
CALL DISCRA(NX, NY, 2, 1, 1, .001)
PRINT 30
PRINT 35, ITS
DO 2040 J = 1, NY
DO 2045 I=1,NX
OMNEW(I,J)=PSI(I,J)
2045 CONTINUE
2040 CONTINUE
CALL DISCRA(NX,NY,2,1,1,.001)
PRINT 65
PRINT 60,(OMWALL(K),K=1,NSIG)
PRINT 70
PRINT 60,(OMEGA(ISIG(IPOFR(K)),JSIG(IPOFR(K)),K=1,NSIG)
PRINT 75
PRINT 60,(OMEGA(I,I-1),I=2,50)
PRINT 80
PRINT 60,OMEGA(I,51-I),I=2,50)
3000 CONTINUE
STOP
END
SUBROUTINE POINT(R,B,XP,YP,NV,XV,YV,D,IS,S)
DIMENSION XV(200),YV(200),X(200),Y(200),D(4),IS(4),S(4)
DO 40 L=1,4
D(L)=B
IS(L)=1
S(L)=0
DO 40 I=1,NV
J=MOD(I,NV)+1
XI=XP-XV(I)
XJ=XP-XV(J)
YI=YP-YV(I)
YJ=YP-YV(J)
DO 10 K=1,L
AI=XI
XI=YI
AJ=XJ
XJ=YJ
YI=-AI
YJ=-AJ
10 YJ=-AJ
IF(ABS(XI).LT.R.OR.ABS(XJ).LT.R)GO TO 20
IF(XI*XJ.GT.0)GO TO 40
YINT=YI-(YJ-YI)*XI/(XJ-XI)
IF(YINT.LE.0..OR.YINT.GE.D(L))GO TO 40
D(L)=YINT
S(L)=SORT(XI*XI+(YI-YINT)*(YI-YINT))
GO TO 30
20 IF(YI.LE.0..OR.YI.GE.D(L).OR.ABS(XI).GE.R)GO TO 40
D(L)=YI
S(L)=0
30 IS(L)=I
40 CONTINUE
SUBROUTINE APROCH(DELTA, XP, YP, N, X, Y, IS, DSQ, S)
DIMENSION X(200), Y(200)
DSMIN = DELTA + DELTA
DSQMIN = DMIN * DMIN
DO 100 I = 1, N
   J = I + 1
   IF(J.GT.N) J = 1
   DXIJ = X(J) - X(I)
   DYIJ = Y(J) - Y(I)
   DXIP = XP - X(I)
   DYIP = YP - Y(I)
   DOT = DXIJ * DXIP + DYIJ * DYIP
   IF(DOT.LT.0.) GO TO 50
   SISQ = DXIJ * DXIJ + DYIJ * DYIJ
   IF(DOT.GT.SISQ) GO TO 100
   CROSS = DXIJ * DYIP - DXIP * DYIJ
   DSQ = CROSS * CROSS / SISQ
   IF(DSQ.GE.DSOMIN) GO TO 100
   DSQMIN = DSQ
   SSQ = DOT * DOT / SISQ
   IS = I
   GO TO 100
50 DSQ = DXIP * DXIP + DYIP * DYIP
   IF(DSQ.GE.DSQMIN) GO TO 100
   DSQMIN = DSQ
   SSQ = 0.0
   IS = I
100 CONTINUE
DSQ = DSQMIN
S = SQRT(SSQ)
RETURN
END

SUBROUTINE CLOSE(N, XV, YV, XE, YE, I)
DIMENSION XV(200), YV(200), ANGEL(200)
DO 120 L = 1, N
   X = XV(L) - XE
   Y = YV(L) - YE
   A = ABS(X) + ABS(Y)
   IF(A.LE.1.E-30) GO TO 140
   ANGLE(L) = 1. - X/A
   IF(Y.LT.0.) ANGLE(L) = 4. - ANGLE(L)
120 CONTINUE
SUM = 0.
DO 130 L = 1, N
   LNEXT = L + 1
130 CONTINUE
IF(L.EQ.N) LNEXT=1
CHANGE=ANGLE(LNEXT)-ANGLE(L)
SINGE=1.
IF(CHANGE.LE.0.) SINGE=-1.
SUBTAN=CHANGE*SINGE
IF(SUBTAN.GT.2) CHANGE=-(4.-SUBTAN)*SINGE
SUM=SUM+CHANGE
130 CONTINUE
I=0
IF(ABS(SUM).GT.2.) I=1
RETURN
140 I=0
RETURN
END
SUBROUTINE NEAR(NV,XV,YV,XP,YP,A,NYE)
DIMENSION XV(200),YV(200)
NYE=0
DO 100 I=1,NV
XIP=XP-XV(I)
YIP=YP-YV(I)
DIP=ABS(XIP)+ABS(YIP)
IF(DIP.GT.A) GO TO 10
NYE=I
RETURN
10 J=I+1
IF(J.GT.NV) J=1
XIJ=XV(J)-XV(I)
YIJ=YV(J)-YV(I)
RIPOIJ=XIP*XIJ+YIP*YIJ
IF(RIPOIJ) 100,100,20
20 XJP=YP-YV(J)
RJIOJP=-XIJ*XJP-YIJ*YJP
IF(RJIOJP) 100,100,30
30 DLJ=ABS(XIJ)+ABS(YIJ)
V=ABS(XIP*YIJ-XIJ*YIP)/DIJ
IF(V.GT.A) GO TO 100
NYE=I
RETURN
100 CONTINUE
RETURN
END
SUBROUTINE DISCRA(M,N,KBILD,KTAB,KREICH,SMALL)
DIMENSION IDSPLY(Sl)
COMMON/COM1/A(Sl,49),ICLASS(Sl,49)
DATA KDl/' '/,KD2/'8'/,KD3/'+'/
5 FORMAT(7H COLUMN,I4,2H :)
7 FORMAT(4H ROW,I4,2H :)

10 FORMAT(1H)
20 FORMAT(1H, 7E10.3)
25 FORMAT(50H PICTURE NOT SHOWN BECAUSE FUNCTION REMAINS
&WITHIN,E10.2
+,25H OF ITS' MINIMUM VALUE OF,E10.2)
30 FORMAT(1H, 80A1)
35 FORMAT(1H1)
   IF(KTAB.EQ.0)GO TO 70
   I=26
   PRINT 10
   PRINT 5,1
   PRINT 20,(A(I,J),J=1,N)
40 CONTINUE
   J=25
   PRINT 10
   PRINT 7,J
   PRINT 20,(A(I,J),I=1,M)
50 CONTINUE
   PRINT 10
70 IF(KBILD.EQ.0.AND.KREICH.EQ.0)GO TO 140
   PRINT 35
   AMAX=-100000.
   AMIN=100000.
   DO 80 J=1,N
   DO 80 I=1,M
   IF(ICLASS(I,J).EQ.1)GO TO 80
   AIJ=A(I,J)
   IF(AIJ.LE.AMAX)GO TO 75
   AMAX=AIJ
   IMAX=I
   JMAX=J
75 IF(AIJ.GE.AMIN)GO TO 80
   AMIN=AIJ
   IMIN=I
   JMIN=J
80 CONTINUE
   IF(KREICH.EQ.0)GO TO 90
   PRINT 83,AMIN,IMIN,JMIN,AMAX,IMAX,JMAX
83 FORMAT(5H MIN=,E10.3,2HAT,2I5,4HMAX=,E10.3,2HAT,2I5)
90 IF(KBILD.EQ.0)GO TO 140
   RANGE=AMAX-AMIN
   IF(RANGE.GT.SMALL)GO TO 95
   PRINT 25,SMALL,AMIN
   RETURN
95 DO 130 J=1,N
   DO 120 I=1,M
   STRIPE=KBILD+KBILD

L=STRIPE*(A(I,J)-AMIN)/RANGE+1
IF(L.EQ.2.0R.L.EQ.6.0R.L.EQ.8.0R.L.EQ.10)GO TO 100
IDSPLY(I)=KD2
100 IF(J.EQ.1.OR.J.EQ.N.OR.I.EQ.1.OR.I.EQ.M)GO TO 110
IDSPLY(I)=KD1
GO TO 120
110 IDSPLY(I)=KD3
120 CONTINUE
PRINT 30,(IDSPLY(I),I=1,M)
130 CONTINUE
140 RETURN
END
SUBROUTINE SIGMA(D,S,VOR,PSI)
DIMENSION D(4),S(4)
D1D3=D(1)*D(3)
D2D4=D(2)*D(4)
PSI=(D1D3*(D(2)*S(4)*S(2)+D(4))+D2D4*(D(3)*S(1)+D(1)*S
C(3))/(D(1)+D(3)+VOR*D1D3*D2D4/2.)/(D1D3+D2D4)
RETURN
END
SUBROUTINE RETRO(X,Y)
DIMENSION A(4)
DATA A(1),A(2),A(3),A(4)/2.69324,-.018087,.0001809,
&-.000001855/
DO 10 ITS=1,1500
CALL SPEED(4,A,3.16227701,X,Y,U,V)
X=X-U*.000002
Y=Y-V*.000002
10 CONTINUE
RETURN
END
SUBROUTINE SPEED(N,A,ALPHA,X,Y,U,V)
DIMENSION A(4)
U=0.0
V=0.0
DO 100 I=1,N
U=U-A(I)*ALPHA*COSH(I*ALPHA*(1.-Y))*SIN(I*ALPHA*X)
V=V-A(I)*ALPHA*SINH(I*ALPHA*(1.-Y))*COS(I*ALPHA*X)
100 CONTINUE
RETURN
END
SUBROUTINE PRANK(N,S,K,L)
DO 100 I=1,N
K(I)=1
SMIN=S(1)
DO 50 J=2,N
IF(S(J).GE.SMIN)GO TO 50
K(I)=J
SMIN=S(J)
50 CONTINUE
INDIR=K(I)
L(INDIR)=I
S(INDIR)=1000000000.
100 CONTINUE
RETURN
END
SUBROUTINE VARC(N,X,Y,S)
DIMENSION X(200),Y(200),S(200)
SUM=0.
DO 100 I=1,N
J=MOD(I,N)+1
DXLJ=X(J)-X(I)
DYIJ=Y(J)-Y(I)
SIDEL=SORT(DXLJ*DXLJ+DYIJ*DYIJ)
SUM=SUM+SIDEL
S(J)=SUM
100 CONTINUE
RETURN
END
SUBROUTINE PERINT(N,P,X,Y,YIN,YOUT)
IF(XIN.GE.X(1))GO TO 10
XI=X(N)-P
YI=Y(N)
XJ=X(1)
YJ=Y(1)
GO TO 55
10 IF(XIN.LT.X(N))GO TO 20
XI=X(N)
YI=Y(N)
XJ=P+X(1)
YJ=Y(1)
GO TO 55
20 DO 45 JJ=2,N
XJ=X(JJ)
IF(XIN.GE.XJ)GO TO 45
I=JJ-1
XI=X(I)
YI=Y(I)
YJ=Y(JJ)
GO TO 55
45 CONTINUE
55 YOUT=YI+(YJ-YI)*(XIN-XI)/(XJ-XI)
SUBROUTINE ASLAP(SUR,D,CENTER,ATLED)
DIMENSION SUR(4),D(4)
D1PD3=D(1)+D(3)
D2PD4=D(2)+D(4)
R1=2./(D(1)*D1PD3)
R2=2./(D(2)*D2PD4)
R3=2./(D(3)*D1PD3)
R4=2./(D(4)*D2PD4)
SUMR=R1+R2+R3+R4
ATLED=R1*SUR(1)+R2*SUR(2)+R3*SUR(3)+R4*SUR(4)-SUMR*CENTER
RETURN
END