REPLACEMENT OF A BAND-LIMITED GAUSSIAN WHITE NOISE BY AN INFINITE BANDWIDTH GAUSSIAN WHITE NOISE THROUGH STATISTICS MATCHING

by

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For the department

TABLE OF CONTENTS

INTRODUCTION

It is a well known fact that when the power spectral density of an input noise process is essentially flat over a frequency range considerably greater than the maximum bandwidth of the system, calculation of the output noise process as well as other related analysis can be carried out with a white noise[1] substituted in place of the actual input noise. Fictitious though such a concept as white noise is, the simplicity it offers makes it so attractive that it has been found to be indispensable in most of the stochastic processes analysis. Such rigorous mathematical treatment on stochastic processes as stochastic differential and integral equations is based upon this assumption. As a matter of fact, a great many formulations and derivations in communication and control engineering have been made manageable as well as practicable only through this powerful concept. Fortunately, a great number of noises actually encountered in many problems are of such nature as to make this assumption reasonable. The question still arises, nevertheless, as to the problem of handling those noises that do not lend themselves directly to this assumption.

To answer the whole question satisfactorily is no doubt quite formidable if not impossible. Bearing in mind that the practical usefulness of a method will be greatly reduced unless it can meet this criterion, namely, being mathematically tractable while at the same time maintaining sufficient accuracy, the author of this thesis, working under Professor

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1. Here it is used in the sense of its usual usage, namely, a noise of flat power spectral density with infinite bandwidth. It is also used sometimes to indicate only the property of being flat without regard to the bandwidth. Its meaning, however, is usually clear from the context.

Ronald L. Klein's guidance, has made some progress toward the solution of the problem, although it is not complete in a general sense.

In this study, a Butterworth filter of order n has been chosen as a representative linear time invariant system on which analysis of the problem is based. The first chapter is devoted as a whole to the calculation of correlation function at the output of a Butterworth filter subject to input white noise of arbitrary bandwidth. Both the order of the filter and the noise bandwidth, normalized with respect to the filter bandwidth, are treated as parameters. The results, besides being frequently referred to in later chapters, are so interesting that they may deserve to be studied in their own right. The second chapter examines some possible methods of replacement which are worth consideration. It is followed by Chapter Three on the method actually exploited here. Finally, in Chapter Four, this method is applied to a general all-pole second order filter as an example of its application to system other than a Butterworth filter. The section on conclusion, which actually includes some general observation, discussion, as well as comment, then summarizes the material presented here.

CHAPTER ONE

CORRELATION FUNCTION OF WHITE NOISE PASSING THROUGH A BUTTERWORTH FILTER

Figure 1-1 shows a typical situation in which the replacement of an arbitrary input noise process by a white noise is allowable, as far as the output noise process is concerned. The problem of interest is when

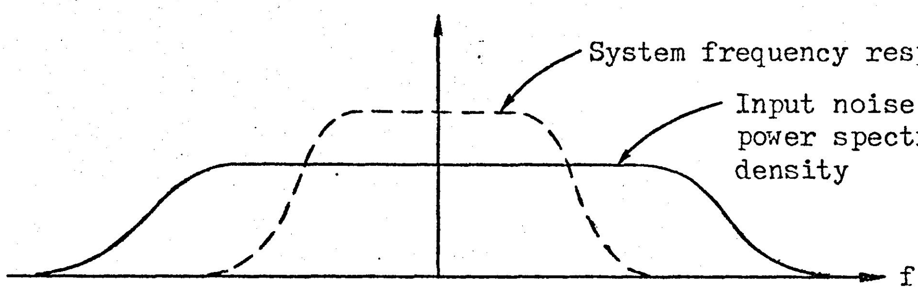

Figure 1-1

the width of the input noise power spectral density curve becomes comparable with, or even smaller than, that of the system frequency response curve. Obviously there can be an infinite number of such power spectral density curves that are of the same bandwidth, under the 3-db bandwidth

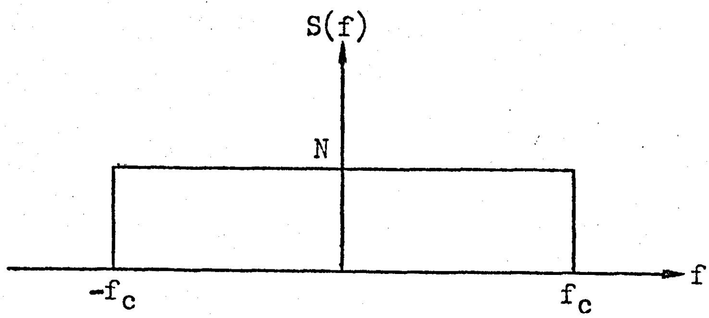

Figure 1-2

basis for example. For the purpose of analysis, an idealized noise process called band-limited white noise is used to represent a noise having a power spectral density curve shown in Figure 1-1. The power spectral density of a band-limited white noise is given by

$$S(f) = \begin{cases} N; & |f| \leq f_c \\ 0; & \text{elsewhere} \end{cases} \qquad (1\text{-}1)$$

where f_c is its bandwidth as shown in Figure 1-2.

The system frequency response curve too can be represented in a great many possible ways. A Butterworth function[2] has been chosen for this analysis partly because of its simplicity. A Butterworth filter, or a maximally flat filter, is a physically realizable low-pass filter the magnitude or gain function of which aside from a scale factor is given by

$$H(\omega) = \frac{1}{\sqrt{1 + \omega^{2n}}} \qquad (1\text{-}2)$$

where ω is the frequency in radians per second and n, a positive integer, is called the order of the filter. With this expression the bandwidth is seen to be unity. It can, however, be modified to represent a filter of any arbitrary bandwidth ω_0 radians per second by writing

$$H(\omega) = \frac{1}{\sqrt{1 + (\frac{\omega}{\omega_0})^{2n}}} \qquad (1\text{-}3)$$

where the functional difference between the H in (1-3) and that in (1-2) is understood. Plots of the Butterworth function of several orders are shown in Figure 1-3. The three sections in this chapter will treat the output correlation function of the model extablished above, and thus provides the reference material for later use.

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2. Norman Balabanian and Theodore A. Bickart, Electrical Network Theory (New York: John Wiley & Sons, Inc., 1969), p. 415-421. Samuel J. Mason and Henry J. Zimmermann, Electronic Circuits, Signals, and System (New York: John Wiley & Sons, Inc., 1965), p. 422-428.

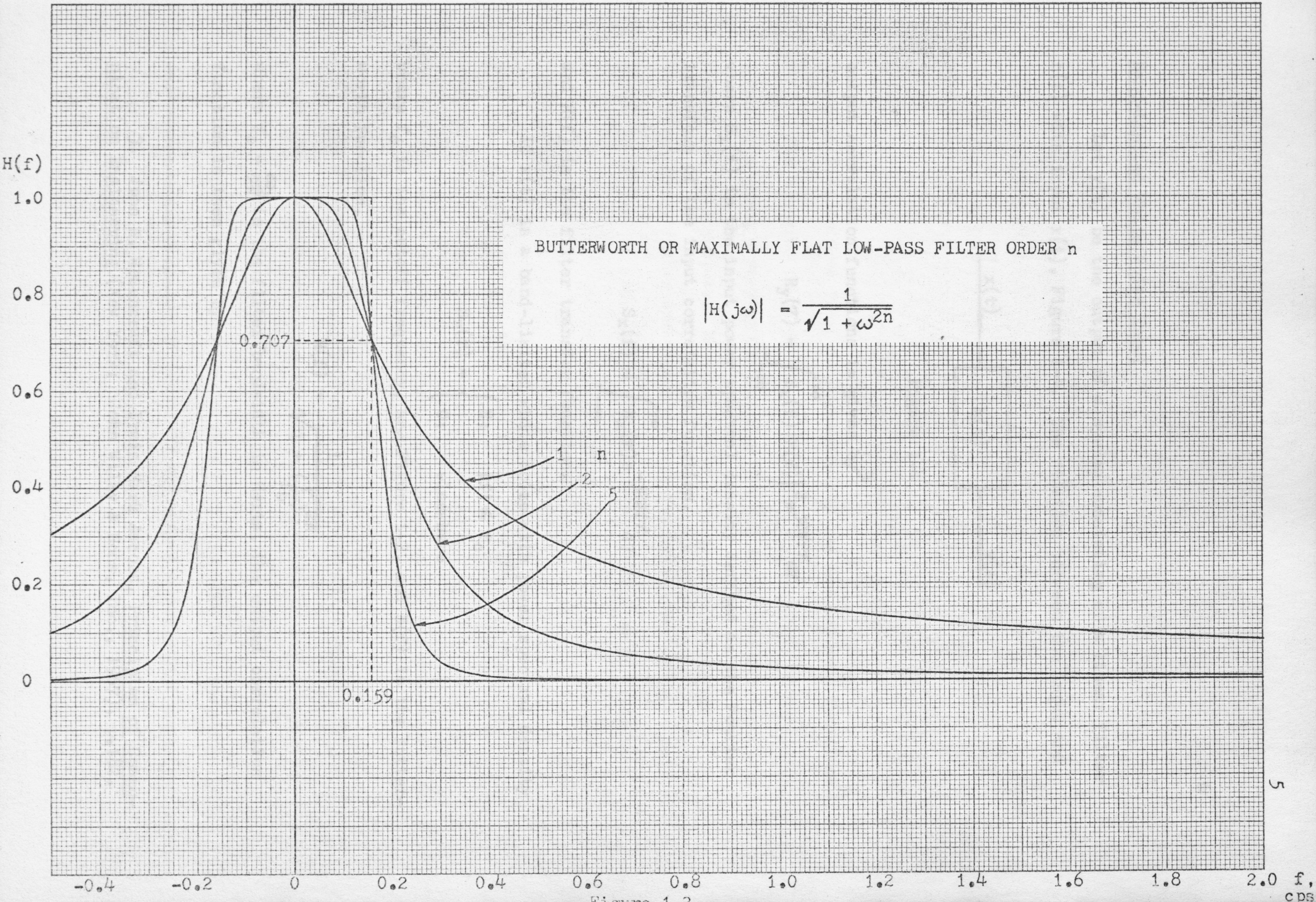
BUTTERWORTH OR MAXIMALLY FLAT LOW-PASS FILTER ORDER n
$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}}$
H(f)
1.0
0.8
0.6
0.4
0.2
0
0.707
0.159
1
2
5
n
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
f, cps

Figure 1-3

Band-Limited White Noise Input

Let $y(t)$ be the output of a system consisting of a linear filter $h(t)$ with input $x(t)$, Figure 1-4. The relation between the input and

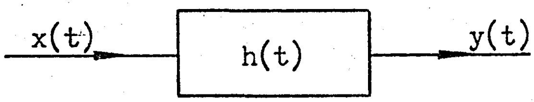

Figure 1-4

output correlation functions is given by[3]

$$R_y(\tau) = \int_{-\infty}^{\infty} S_x(f)\,|H(f)|^2 e^{j2\pi f\tau} df$$

where $S_x(f)$ is the input power spectral density defined as the Fourier transform of the input correlation function,

$$S_x(f) \equiv \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$

and $H(f)$ is the filter transfer function.

If $x(t)$ is a band-limited white noise with power spectral density

$$S_x(f) = \begin{cases} N; & |f| \leq f_c \\ 0; & \text{elsewhere} \end{cases}$$

where f_c is the cutoff frequency, see Figure 1-2, and $H(f)$ is the Butterworth function,

$$H(f) = \frac{1}{\sqrt{1 + (\frac{f}{f_o})^{2n}}}$$

where $f_o = \frac{\omega_o}{2\pi}$ is the filter bandwidth in Hertz, the output correlation function is then given by

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3. John M. Wozencraft and Irwin Mark Jacobs, Principles of Communication Engineering (New York: John Wiley & Sons, Inc., 1967), p.182.

$$R_y(\tau) = \int_{-f_c}^{f_c} \frac{N}{1 + (\frac{f}{f_o})^{2n}} e^{j2\pi f\tau} df$$

$$= \frac{N}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{e^{j\omega\tau}}{1 + (\frac{\omega}{\omega_o})^{2n}} d\omega \qquad (1\text{-}4)$$

where $\omega_c = 2\pi f_c$,

$$= \frac{N}{\pi} \int_{0}^{\omega_c} \frac{\cos\omega\tau}{1 + (\frac{\omega}{\omega_o})^{2n}} d\omega \qquad (1\text{-}5)$$

The last line follows from the properties of the odd and even functions.

The integral in (1-5) can be evaluated by numerical integration. Appendix A shows a FORTRAN IV computer program using Simpson's rule designed for this purpose. Several sample results from this program for the first order filter are also shown, with N and ω_o set equal to unity. There is no loss of generality, however, since we can write (1-5) as

$$R_y(\tau) = \frac{N\omega_o}{\pi} \int_{0}^{\rho} \frac{\cos\xi\omega_o\tau}{1 + \xi^{2n}} d\xi \qquad (1\text{-}6)$$

where $\xi = \frac{\omega}{\omega_o}$ and $\rho = \frac{\omega_c}{\omega_o}$, that is ω and ω_c have now been normalized with respect to ω_o. Then, for any values of N and ω_o, $R_y(\tau)$ is still given by the same curve with <u>the R_y scale multiplied by $N\omega_o$ and the τ scale divided by ω_o</u>.

It is interesting to note that when the cutoff frequency of the input noise is much less than the filter bandwidth, or that the normalized frequency of the input noise is much less than unity, the shape of the output correlation function is very much like that of a sampling function, $\frac{\sin x}{x}$. This is not difficult to understand since under the above condition, the output power spectral density

$$S_y(f) = S_x(f)\,|H(f)|^2 = \begin{cases} \dfrac{N}{1 + (\frac{f}{f_o})^{2n}}; & |f| \leq f_c \\ 0; & \text{elsewhere} \end{cases}$$

does not differ much from that of an ideal low-pass filter subject to a pure white noise input.

The parameters a and b for the approximation

$$R_y(\tau) \approx a\,\frac{\sin b\tau}{\tau} \tag{1-7}$$

can be evaluated as follows. For convenience, let us first set N and ω_o equal to unity so that (1-5) becomes

$$R_y(\tau) = \frac{1}{\pi}\int_0^{\omega_c} \frac{\cos\omega\tau}{1+\omega^{2n}}\,d\omega$$

and think of ω and ω_c as the normalized quantities like ξ and ρ in (1-6). Then, when ω_c is much less than unity we have

$$R_y(\tau) \approx \frac{1}{\pi}\int_0^{\omega_c}\cos\omega\tau\,d\omega = \frac{1}{\pi}\,\frac{\sin\omega_c\tau}{\tau} \tag{1-8}$$

Comparing this result with (1-7) we seethat $a = \frac{1}{\pi}$ and $b = \omega_c$.

It should be noticed that the accuracy of this approximation depends upon ω_c as well as n. Actual calculation shows that an accuracy close to four significant figures can be achieved provided that ω_c^{2n} is less than 0.01. In the case of a second order filter for example, ω_c must be less than 0.316 in order to get such an accuracy. Some values of $R_y(\tau)$ calculated from the computer and those from the approximate formula for $\omega_c = 0.31416$ ($<$ 0.316) are shown below for comparison.

τ	$R_y(\tau)$	$\frac{1}{\pi}\frac{\sin\omega_c\tau}{\tau}$
0	0.0998	0.1000
1	0.0982	0.0984
2	0.0934	0.0936
3	0.0857	0.0858

Infinite Bandwidth White Noise Input

If the input to the unit bandwidth Butterworth filter is an infinite bandwidth white noise with unit power spectral density, (1-4) becomes

$$R_y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\omega\tau}{1 + \omega^{2n}}\, d\omega \qquad (1\text{-}9)$$

The integral in (1-9), unlike that in (1-5), can be evaluated analytically by the method of residues. One formula for evaluating a definite integral of this form is that[4]

$$\int_{-\infty}^{\infty} \cos mx\, Q(x)dx = -2\pi\sum \text{imaginary parts of the residues of } e^{imz}Q(z) \text{ at its poles in the upper half plane} \qquad (1\text{-}10)$$

The poles in our case are given by

$$\omega = e^{j\alpha_k}$$

where

$$\alpha_k = \frac{(2k+1)\pi}{2n}; \qquad k = 0, 1, 2, \ldots, 2n-1$$

and their locations for n = 1, 2, and 3 are shown in Figure 1-5.

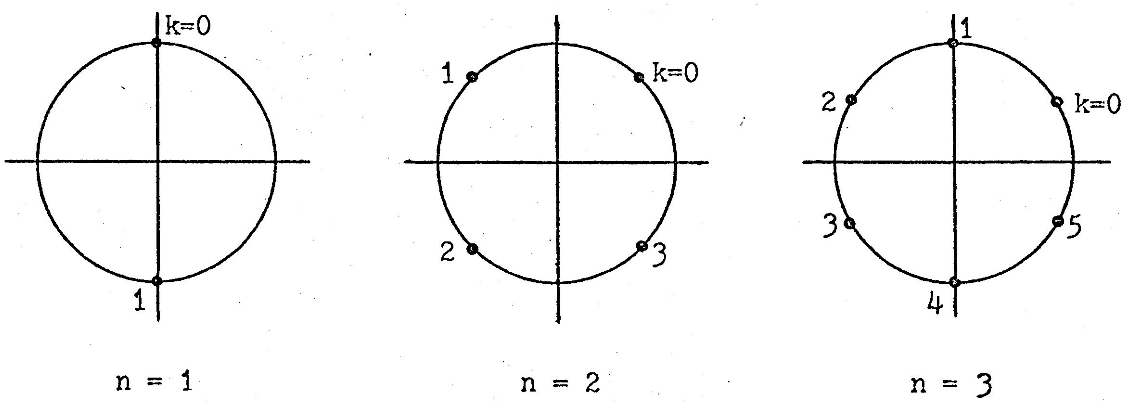


Figure 1-5

4. C. R. Wylie, Jr., Advanced Engineering Mathematics (New York: McGraw-Hill Book Co., Inc., 1960), p. 602.

Using formula (1-10), we have

$$R_y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos\omega\tau}{1+\omega^{2n}}d\omega = -\sum_{k=0}^{n-1}\mathrm{Im}\{A_k\} \tag{1-11}$$

where

$$A_k = \frac{e^{j\tau e^{j\alpha_k}}}{\prod_{\substack{l=0\\ l\neq k}}^{2n-1}(e^{j\alpha_k} - e^{j\alpha_l})} \tag{1-12}$$

is the kth residue of $\frac{e^{j\tau z}}{1+z^{2n}}$.

Let D_k be the denominator of A_k, then, in terms of sine and cosine, we have for its modulus

$$\mathrm{Mod}\, D_k = \prod_{\substack{l=0\\ l\neq k}}^{2n-1}\sqrt{(\cos\alpha_k-\cos\alpha_l)^2+(\sin\alpha_k-\sin\alpha_l)^2} \tag{1-13}$$

and its argument

$$\mathrm{Arg}\, D_k = \sum_{\substack{l=0\\ l\neq k}}^{2n-1}\tan^{-1}\frac{\sin\alpha_k-\sin\alpha_l}{\cos\alpha_k-\cos\alpha_l} \tag{1-14}$$

Using the trigonometric identities for sums to products of sines and cosines, the above expression for the modulus of D_k can readily be simplified to

$$\mathrm{Mod}\, D_k = \prod_{\substack{l=0\\ l\neq k}}^{2n-1}\left|2\sin(k-l)\frac{\pi}{2n}\right| \tag{1-15}$$

with the angle now expressed directly in terms of k, l, and n.

In searching for a simpler expression, let us first write (1-15) as

$$\mathrm{Mod}\, D_k = 2^{2n-1}\prod_{l=0}^{k-1}\sin(k-l)\frac{\pi}{2n}\prod_{l=k+1}^{2n-1}\sin(l-k)\frac{\pi}{2n} \tag{1-16}$$

Then, using the reduction formula $\sin(\pi-\alpha) = \sin\alpha$ the other way round, we have

$$\sin(k-l)\frac{\pi}{2n} = \sin(2n+l-k)\frac{\pi}{2n}$$

whereupon

$$\prod_{l=0}^{k-1}\sin(k-l)\frac{\pi}{2n} = \prod_{l=0}^{k-1}\sin(2n+l-k)\frac{\pi}{2n} = \prod_{l=2n}^{2n+k-1}\sin(l-k)\frac{\pi}{2n}.$$

and (1-16) becomes

$$\text{Mod } D_k = 2^{2n-1}\prod_{l=k+1}^{2n-1}\sin(l-k)\frac{\pi}{2n} = 2^{2n-1}\prod_{m=1}^{2n-1}\sin\frac{m\pi}{2n} \qquad (1\text{-}17)$$

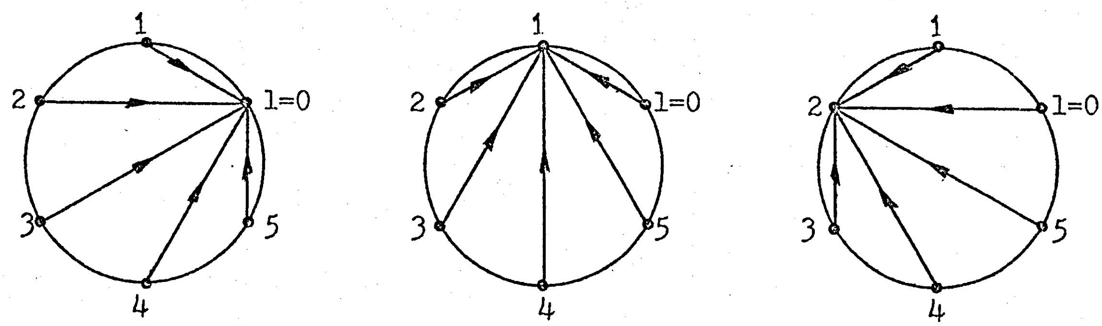
At this point, we note that the modulus of D_k is independent of k for a fixed n. Thus it seems to be more appropriate to write it as Mod D(n) instead of Mod D_k.

It is a simple matter to go from (1-17) to two other equivalent expressions,[5]

$$\text{Mod } D(n) = 2^{2n-1}\prod_{m=1}^{n-1}\sin^2\frac{m\pi}{2n} \qquad (1\text{-}18)$$

and

$$\text{Mod } D(n) = 2^{2n-1}\prod_{m=1}^{n-1}\cos^2\frac{m\pi}{2n} \qquad (1\text{-}19)$$



k = 0 k = 1 k = 2

n = 3

Figure 1-6

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5. We may arrive at expressions (1-18) and (1-19) in a more straightforward manner with the help of the "vector diagram" like those shown in Figure 1-6.

Both of these expressions have the advantage over that of (1-17) in that only about half the number of different factors need be evaluated.

Coming back to (1-17), if we use the double-angle of sine formula, $\sin 2\alpha = 2\sin\alpha\cos\alpha$, we can write (1-17) as

$$\text{Mod } D(n) = 2^{4n-2}\prod_{m=1}^{2n-1}\sin\frac{m\pi}{4n}\prod_{m=1}^{2n-1}\cos\frac{m\pi}{4n} \tag{1-20}$$

whereas if we double the value of n in (1-17), we will get

$$\text{Mod } D(2n) = 2^{4n-1}\prod_{m=1}^{4n-1}\sin\frac{m\pi}{4n} \tag{1-21}$$

It can be shown, by direct expansion and using the reduction formula $\sin(\frac{\pi}{2}+\alpha) = \cos\alpha$, that

$$\prod_{m=1}^{4n-1}\sin\frac{m\pi}{4n} = \prod_{m=1}^{2n-1}\sin\frac{m\pi}{4n}\prod_{m=1}^{2n-1}\cos\frac{m\pi}{4n} \tag{1-22}$$

From (1-20), (1-21), and (1-22) we have

$$\frac{\text{Mod } D(2n)}{\text{Mod } D(n)} = 2 \tag{1-23}$$

For $n = 1$,

$$\text{Mod } D(1) = 2\sin\frac{\pi}{2} = 2$$

for $n = 2$,

$$\text{Mod } D(2) = 2^3\sin\frac{\pi}{4}\,\sin\frac{2\pi}{4}\,\sin\frac{3\pi}{4} = 2^3(\frac{1}{\sqrt{2}})(1)(\frac{1}{\sqrt{2}}) = 4$$

as can be expected in view of (1-23). For $n = 3$, we have, by actual evaluation,

$$\text{Mod } D(3) = 6$$

This inturn may be written as

$$\text{Mod } D(3) = 3(2) = 3 \text{ Mod } D(1)$$

or

$$\text{Mod } D(3) = \tfrac{3}{2}(4) = \tfrac{3}{2} \text{ Mod } D(2)$$

That such relation is true for any two arbitrary n's can be verified by direct substitution. Hence we have the general expression of (1-23),

$$\frac{\text{Mod } D(n_1)}{\text{Mod } D(n_2)} = \frac{n_1}{n_2} \tag{1-24}$$

Since we know that Mod D(1) = 2, it follows immediately that

$$\text{Mod } D(n) = 2n \tag{1-25}$$

which is the kind of thing we have been looking for.

We now turn our attention to the argument of D_k. Let

$$\theta_l = \tan^{-1} \frac{\sin\alpha_k - \sin\alpha_l}{\cos\alpha_k - \cos\alpha_l} \tag{1-26}$$

Again using the identities for sums to products of sines and cosines we get, after expressing the resultant angle directly in terms of k, l, and n,

$$\theta_l = \tan^{-1}\left[-\cot(k+l+1)\frac{\pi}{2n}\right] \tag{1-27}$$

From (1-27) come two possible solutions the proper choice of which is dictated by (1-26) as a reflection in the relative values of the two sines and cosines, and hence those of k and l. The result reads

$$\theta_l = \begin{cases} \frac{\pi}{2} + (k+l+1)\frac{\pi}{2n}; & l < k \\ (k+l+1)\frac{\pi}{2n} - \frac{\pi}{2}; & l > k \end{cases} \tag{1-28}$$

Consequently,

$$\text{Arg } D_k = \sum_{\substack{l=0 \\ l \neq k}}^{2n-1} \theta_l = \sum_{l=0}^{k-1} \theta_l + \sum_{l=k+1}^{2n-1} \theta_l$$

which, upon substituting in each partial sum the proper value of θ_1 from (1-28) and simplifying,

$$= (2n-1)(2k+1)\frac{\pi}{2n}$$

$$= (2n-1)\alpha_k \tag{1-29}$$

This completes the evaluation of D_k.

Putting (1-29) and (1-25) together we have

$$D_k = 2n\, e^{j(2n-1)\alpha_k} \tag{1-30}$$

which is the denominator of the residue A_k given by (1-12). Hence

$$A_k = \frac{e^{j\tau e^{j\alpha_k}}}{2n\, e^{j(2n-1)\alpha_k}} \tag{1-31}$$

It follows that

$$\mathrm{Im}\{A_k\} = \frac{1}{2n}\sin\left[\tau\cos\alpha_k-(2n-1)\alpha_k\right] e^{-\tau\sin\alpha_k} \tag{1-32}$$

Since

$$2n\alpha_k = (2k+1)\pi$$

and that

$$\sin\left[(2k+1)\pi-\alpha\right] = \sin\alpha$$

we can write (1-32) as

$$\mathrm{Im}\{A_k\} = \frac{-1}{2n}\sin(\tau\cos\alpha_k+\alpha_k)\, e^{-\tau\sin\alpha_k} \tag{1-33}$$

Furthermore, we know that an autocorrelation function is an even function of the variable τ, this means that the absolute value of τ should be used. This is justified in view of the cosine term in (1-11). Substituting (1-33) subject to the above reasoning in (1-11) gives

$$R_y(\tau) = \frac{1}{2n}\sum_{k=0}^{n-1}\sin(|\tau|\cos\alpha_k+\alpha_k)\, e^{-|\tau|\sin\alpha_k} \tag{1-34}$$

Applying the reduction formula for sine and cosine to the terms in the second half of the expansion of (1-34) shows that each one of them has a corresponding term in the first half except the middle term, in the case of odd n, which is simply given by $\frac{1}{2n}e^{-|\tau|}$. Thus we have as an alternative of (1-34) the following formula,

$$R_y(\tau) = \begin{cases} \frac{1}{2n}e^{-|\tau|} + \frac{1}{n}\sum_{k=0}^{(n-3)/2} \sin(|\tau|\cos\alpha_k+\alpha_k)e^{-|\tau|\sin\alpha_k} & \text{for odd } n \\ \frac{1}{n}\sum_{k=0}^{(n-2)/2} \sin(|\tau|\cos\alpha_k+\alpha_k)e^{-|\tau|\sin\alpha_k} & \text{for even } n \end{cases} \quad (1\text{-}35)$$

For small values of n, the expressions are relatively simple. Given below are those corresponding to the first three values of n.

For n = 1,

$$R_y(\tau) = \frac{1}{2}e^{-|\tau|}$$

For n = 2,

$$R_y(\tau) = \frac{1}{2}\sin\left(\frac{1}{\sqrt{2}}|\tau| + \frac{\pi}{4}\right)e^{-|\tau|/\sqrt{2}}$$

For n = 3,

$$R_y(\tau) = \frac{1}{6}e^{-|\tau|} + \frac{1}{3}\sin\left(\frac{\sqrt{3}}{2}|\tau| + \frac{\pi}{6}\right)e^{-|\tau|/2}$$

Plots of these functions and some higher order ones are presented in Appendix B.

Butterworth Filter of Infinite Order

Examining the curves of $R_y(\tau)$ reveals an interesting fact that the higher the order of the function is, the more similar to the sampling function the curve will be. This suggests that in the limit as the order goes to infinity, (1-34) may converge to a properly scaled sampling function $a\frac{\sin bx}{x}$. That this is indeed the case can be seen from the following analysis.

From (1-34),

$$\lim_{n\to\infty} R_y(\tau) = \lim_{n\to\infty} \frac{1}{2n}\sum_{k=0}^{n-1} \sin(|\tau|\cos\alpha_k+\alpha_k)\, e^{-|\tau|\sin\alpha_k} \tag{1-36}$$

As n goes to infinity, the angle

$$\alpha_k = \frac{(2k+1)\pi}{2n}$$

will take on every possible value from 0 to π. The increment of the angle

$$\Delta\alpha_k = \frac{\pi}{n}$$

will accordingly become $d\alpha_k$. The sum in (1-36) can now be replaced by an integral,

$$\begin{aligned}\lim_{n\to\infty} R_y(\tau) &= \frac{1}{2\pi}\int_0^{\pi} \sin(|\tau|\cos\alpha_k+\alpha_k)\, e^{-|\tau|\sin\alpha_k}\, d\alpha_k \\ &= \frac{1}{2\pi}\int_0^{\pi} \sin(|\tau|\cos\alpha_k)\cos\alpha_k e^{-|\tau|\sin\alpha_k}\, d\alpha_k \\ &\quad + \frac{1}{2\pi}\int_0^{\pi} \cos(|\tau|\cos\alpha_k)\sin\alpha_k e^{-|\tau|\sin\alpha_k}\, d\alpha_k\end{aligned}$$

In the last expression, the first integral vanishes as can be seen easily by a change of the varible of integration. We now apply the method of integration by parts to the second integral. Let

$$u = e^{-|\tau|\sin\alpha_k}, \qquad dv = \cos(|\tau|\cos\alpha_k)\sin\alpha_k\, d\alpha_k$$

$$du = -|\tau| e^{-|\tau|\sin\alpha_k}\cos\alpha_k\, d\alpha_k, \qquad v = -\frac{\sin(|\tau|\cos\alpha_k)}{|\tau|}$$

Then

$$\begin{aligned}&\int_0^{\pi} \cos(|\tau|\cos\alpha_k)\sin\alpha_k e^{-|\tau|\sin\alpha_k} \\ &= -\frac{\sin(|\tau|\cos\alpha_k)}{|\tau|}\, e^{-|\tau|\sin\alpha_k}\Big|_0^{\pi} - \int_0^{\pi} \sin(|\tau|\cos\alpha_k)\cos\alpha_k e^{-|\tau|\sin\alpha_k}\, d\alpha_k \\ &= 2\,\frac{\sin\tau}{\tau}\end{aligned}$$

The absolute value symbol has been dropped in the last line since its presence is no longer necessary. With this result, (1-37) becomes

$$\lim_{n\to\infty} R_y(\tau) = \frac{1}{2\pi} \cdot 2\frac{\sin\tau}{\tau} = \frac{1}{\pi}\frac{\sin\tau}{\tau} \tag{1-38}$$

The result in (1-38), while it confirms the observation mentioned before, should not be of much surprise to us if we have recognized the fact that as n goes to infinity, the Butterworth filter becomes an ideal low-pass filter, and whence comes the result.

From the preceding section we know that for n = 1, $R_y(\tau)$ is a double-sided exponential function. It can also be shown from (1-9) that $R_y(\tau)$ is an impulse function for n = 0, although a zero order Butterworth filter is not defined, nor physically exists. Recalling that the subscript y in $R_y(\tau)$ is used to indicate its association with the output y(t) in Figure 1-4, let us dissociate it from y(t) for a moment, and rewrite it as $R_n(\tau)$ to indicate explicitly the functional, or rather, the parametrical, dependence of R upon n. In this way, $R_n(\tau)$ may be regarded mathematically as a generating function whereby the impulse function, the exponential function, the sampling function, and those lie between the exponential and the sampling function, are generated as n takes on the value from 0 to ∞. This unified notation for these seemingly different functions will serve to indicate some sort of relation between them as well as their order in this family.

The idea that an ideal low-pass filter is nothing but the limiting case of a realistic Butterworth filter coupled with the fact disclosed in the foregoing analysis that this limit can be taken after the integration process suggests that it may be possible to use this procedure to get an analytic expression for the output correlation function in the case of band-limited white noise input since its power spectral density

$$S_x(\omega) = \begin{cases} 1; & |\omega| \leq \omega_c \\ 0; & \text{elsewhere} \end{cases}$$

may be represented in another way as

$$S_x(\omega) = \lim_{m\to\infty} \frac{1}{1 + \left(\frac{\omega}{\gamma}\right)^{2m}} \tag{1-39}$$

where

$$\gamma = \frac{\omega_c}{(\sqrt{2}-1)^{1/2m}}$$

is a constant being chosen in such a way as to make ω_c the 3-db bandwidth.

Using (1-39) with the assumption that the order of integration and taking the limit is interchangeable, the output correlation function then becomes

$$R_y(\tau) = \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + \left(\frac{\omega}{\gamma}\right)^{2m}} \frac{\cos\omega\tau}{1 + \omega^{2n}}\, d\omega \tag{1-40}$$

Again applying the formula (1-10), and after a procedure similar to that performed in the previous section, we arrive at a result

$$R_y(\tau) = \lim_{m\to\infty} \left\{ \frac{1}{2n} \sum_{k=0}^{n-1} \frac{1}{P_m} \sin(|\tau|\cos\nu_k + \nu_k - S_m)\, e^{-|\tau|\sin\nu_k} \right.$$

$$\left. + \frac{1}{2m} \sum_{k=0}^{m-1} \frac{1}{P_n} \sin(\gamma|\tau|\cos\mu_k + \mu_k - S_n)\, e^{-\gamma|\tau|\sin\mu_k} \right\} \tag{1-41}$$

where

$$P_m = \prod_{q=0}^{2m-1} \sqrt{\frac{1}{\gamma^2} + 1 - \frac{2}{\gamma}\cos(\nu_k - \mu_q)}$$

$$S_m = \sum_{q=0}^{2m-1} \tan^{-1} \frac{\sin\nu_k - \gamma\sin\mu_q}{\cos\nu_k - \gamma\cos\mu_q}$$

$$P_n = \prod_{p=0}^{2n-1} \sqrt{\gamma^2 + 1 - 2\gamma\cos(\mu_k - \nu_p)}$$

$$S_n = \sum_{p=0}^{2n-1} \tan^{-1} \frac{\gamma \sin\mu_k - \sin\nu_p}{\gamma\cos\mu_k - \cos\nu_p}$$

and

$$\mu_k = \frac{(2k+1)\pi}{2m}; \quad \nu_k = \frac{(2k+1)\pi}{2n}$$

Although it can be shown that (1-41) does converge to (1-34) as ω_c, and hence γ, goes to infinity, attempt to take the limit as m approaches infinity fails. For any <u>finite m</u>, however, expression (1-41) is valid except when $\gamma = 1$ and m = n, since in this case the integrand in (1-40) contains second order poles which will lead to an expression somewhat different from (1-41). Taking some finite m, of course, just amounts to taking a power spectral density curve like the one shown in Figure 1-1 instead of the rectangular shape curve in Figure 1-2. If data concerning the power spectral density curve under investigation is available, the curve may be better approximated than a mere rectangular shape curve by selecting some suitable value of m. On the other hand, the value of m may be made sufficiently large to give a close approximation to the rectangular shape curve of Figure 1-2 for analysis in general where knowledge of the exact curve is not cared for.

CHAPTER TWO

REPLACEMENT OF A BAND-LIMITED WHITE NOISE BY AN INFINITE BANDWIDTH WHITE NOISE

A random process is characterized to a certain extent by some statistical averages, or simply, statistics, but it can never be completely specified except in the special case of a Gaussian process. To replace one process by another is no more than a matter of matching some of these statistics in one way or another in order to gain some analytical advantages, usually at the expense of some accuracy.

In an attempt to find a method that will suit our purpose, that is to replace an input noise process of arbitrary bandwidth by a white noise, several approaches have been considered. One of such approaches will be discussed in detail in Chapter Three, with the other being described briefly here.

Generation of an Arbitrary Noise Process by White Noise

Let us refer again to Figure 1-4. We have a system consisting of a unit bandwidth Butterworth filter of order n the transfer function of which is given by

$$H(\omega) = \frac{1}{\sqrt{1 + \omega^{2n}}} \tag{2-1}$$

The input $x(t)$ is assumed to be a band-limited white noise with cutoff frequency ω_c, and hence a power spectral density

$$S_x(\omega) = \begin{cases} 1; & |\omega| \leq \omega_c \\ 0; & \text{elsewhere} \end{cases} \tag{2-2}$$

The correlation function of the output $y(t)$ is found in terms of $H(\omega)$ and $S_x(\omega)$ to be

$$R_y(\tau) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \frac{\cos\omega\tau}{1+\omega^{2n}}\,d\omega \tag{2-3}$$

If the band-limited noise is to be replaced by a white noise and matching of the statistics given by (2-3) would suffice as a criterion for such replacement, a scheme like that depicted in Figure 2-1 can easi-

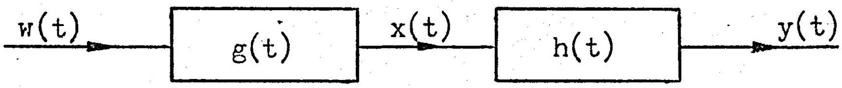

Figure 2-1

ly be conceived for this purpose, using the idea that an arbitrary noise process may be regarded as generated by a white noise passing through a suitable filter.[6] In this particular case, the "suitable" filter g(t) turns out to be an ideal low-pass filter of which the transfer function is

$$G(\omega) = \begin{cases} 1; & |\omega| \leq \omega_c \\ 0; & \text{elsewhere} \end{cases} \tag{2-4}$$

Apparently the original problem of replacing an arbitrary noise by a white noise becomes a problem of curve fitting. Neither the added transfer function $G(\omega)$ nor the composite transfer function

$$G(\omega)H(\omega) = \begin{cases} \dfrac{1}{\sqrt{1+\omega^{2n}}}; & |\omega| \leq \omega_c \\ 0; & \text{elsewhere} \end{cases} \tag{2-5}$$

as shown in Figure 2-2 can be synthesized exactly. However, the mean-square error criterion for example may be used in order to approximate

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6. J. Halcombe Laning, Jr., and Richard H. Battin, Random Processes in Automatic Control (New York: McGraw-Hill Book Co., Inc., 1956), p. 143.

the desired curve, (2-4) or (2-5).

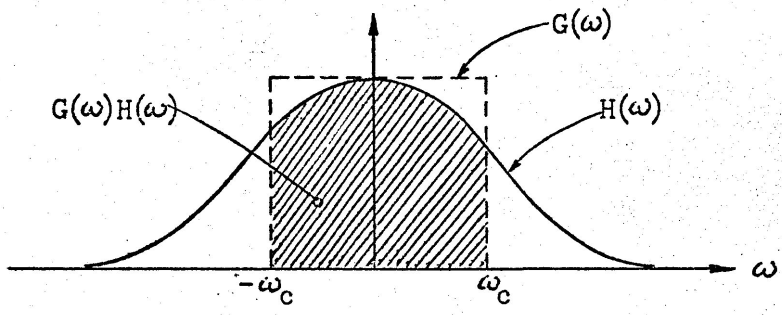

Figure 2-2

Matching of Intensity Coefficient

Instead of matching the output correlation function (2-3), consider just matching the area under the curve. This approach where it is applicable would no doubt greatly simplify the analysis.

The area under a correlation function curve is given by the quantity

$$K = \int_{-\infty}^{\infty} R(\tau)\, d\tau \tag{2-6}$$

which is called the intensity coefficient of the process.[7] Since the power spectral density is, by definition,

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi f\tau} d\tau$$

it follows immediately that

$$K = S(0) \tag{2-7}$$

If we try to apply this method of replacement to our case, we will find to our disappointment that the result turns out to be a trivial one

7. R. L. Stratonovich, Topics in the Theory of Random Noise, Vol. I (New York: Gordon and Breach Science Publishers, Inc., 1963), p. 22.

since, for a power spectral density

$$S_y(\omega) = \begin{cases} \dfrac{1}{1 + \omega^{2n}}; & |\omega| \leq \omega_c \\ 0; & \text{elsewhere} \end{cases}$$

the corresponding intensity coefficient is

$$K_y = S_y(0) = 1 \tag{2-8}$$

for all values of ω_c except zero. On the other hand, it can be seen that this is just the logical result to be expected if we are considering only those cases where the values of ω_c are much greater than the filter bandwidth.

It should be remarked here that extensive use of the concept of intensity coefficient has been made in the so-called Fokker-Plank equation by means of which we can make two equations stochastically equivalent,[8] and hence obtain another basis for the replacement of one process by another.

8. Ibid., p. 97.

CHAPTER THREE

REPLACEMENT OF A BAND-LIMITED GAUSSIAN WHITE NOISE BY AN INFINITE BANDWIDTH GAUSSIAN WHITE NOISE

In this chapter, we will be more specific in regard to certain aspects of the nature of the random process. In particular, we will deal exclusively with stationary Gaussian noise, making use of some of its important properties such that our derivations will be more meaningful.

Another point which is of minor importance yet should be made clear at this time is that we will choose to work only with zero-mean processes. In spite of the fact that a white noise is defined as a stationary zero-mean process,[9] we can if desired devise a white noise with a non-zero mean. But nothing essential is added since the mean can always be treated separately as a deterministic signal. With these assumptions we now proceed to the derivations.

Matching of First Order Statistics

Let us refer to Figure 1-4 once again. Since we are dealing with stationary zero-mean Gaussian processes, we can write the first order probability density function, or simply the density function, of the band-limited input white noise x(t) as

$$p_x(\alpha) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\alpha^2/2\sigma_x^2} \tag{3-1}$$

where σ_x^2 is the variance of x(t). The system assumed linear and time invariant will produce, by virtue of one of the Gaussian properties,[10]

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9. Wozencraft and Jacobs, Op. cit., p. 189.

10. Ibid., p. 178.

at its output another Gaussian noise the first order density function of which can be written down immediately as

$$p_y(\alpha) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\alpha^2/2\sigma_y^2} \tag{3-2}$$

where σ_y^2 is the variance of y(t) and is given by

$$\sigma_y^2 = \overline{y^2(t)} = R_y(0) \tag{3-3}$$

This as well as the density function (3-2) follows from the fact that the output mean

$$m_y = \int_{-\infty}^{\infty} m_x\, h(t-\xi)\, d\xi$$

is zero since the input mean m_x is zero by assumption.

If only the first order statistics are to be of importance in our problem, the variance σ_y^2 will be the key to the solution. Now our problem is to find a means of replacing an input noise process of arbitrary bandwidth by a white noise so as to yield the same statistics of interest at the output, in view of the foregoing statement this goal can be achieved simply by matching the output variances.

To formulate the problem concisely, let z(t) be the result of passing a Gaussian white noise w(t) through the system h(t). As far as the output first order statistics are concerned, w(t) and x(t) will be equivalent if

$$\sigma_z = \sigma_y \tag{3-4}$$

or equivalently

$$R_z(0) = R_y(0) \tag{3-5}$$

since we are dealing with zero-mean processes. Condition (3-4) or

(3-5) together with the knowledge of the system and the noise power spectral density are needed for solving the problem.

For a unit bandwidth Butterworth filter

$$H(\omega) = \frac{1}{\sqrt{1+\omega^{2n}}} \tag{3-6}$$

having as its input a band-limited white noise, $x(t)$, with power spectral density

$$S_x(\omega) = \begin{cases} N_x & \text{for } |\omega| \leq \omega_c \\ 0 & \text{for } |\omega| > \omega_c \end{cases} \tag{3-7}$$

The output correlation function is

$$R_y(\tau) = \frac{N_x}{\pi} \int_0^{\omega_c} \frac{\cos \omega\tau}{1+\omega^{2n}}\, d\omega$$

Hence,

$$R_y(0) = \frac{N_x}{\pi} \int_0^{\omega_c} \frac{d\omega}{1+\omega^{2n}} \tag{3-8}$$

The output correlation function in the case of infinite bandwidth white noise $w(t)$ is

$$R_z(\tau) = \frac{N_w}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \omega\tau}{1+\omega^{2n}}\, d\omega \tag{3-9}$$

where N_w is the power spectral density of $w(t)$. Using (1-11) and (1-34), we can write (3-9) as

$$R_z(\tau) = \frac{N_w}{2n} \sum_{k=0}^{n-1} \sin(|\tau| \cos\alpha_k + \alpha_k)\, e^{-|\tau| \sin\alpha_k}$$

where

$$\alpha_k = \frac{(2k+1)\pi}{2n} \tag{3-10}$$

From here,

$$R_z(0) = \frac{N_w}{2n}\sum_{k=0}^{n-1}\sin\alpha_k \tag{3-11}$$

The sum in (3-11) can be found from the formula[11]

$$\sum_{k=1}^{m}\sin(2k-1)x = \frac{\sin^2 mx}{\sin x}, \quad \sin x \neq 0 \tag{3-12}$$

As a result, we have

$$R_z(0) = \frac{N_w}{2n}\,\frac{1}{\sin\frac{\pi}{2n}} \tag{3-13}$$

To find $R_y(0)$, let us first rewrite (3-8) as

$$R_y(0) = \frac{N_x}{\pi}\, I(n) \tag{3-14}$$

where

$$I(n) = \int_0^{\omega_c}\frac{d\omega}{1+\omega^{2n}} \tag{3-15}$$

The integrand in (3-15) can be expanded into partial fractions,

$$\frac{1}{1+\omega^{2n}} = \sum_{k=0}^{2n-1}\frac{A_k}{\omega - e^{j\alpha_k}} \tag{3-16}$$

where α_k is given by (3-10). The coefficient A_k is just the inverse of D_k given in (1-30), that is

$$A_k = \frac{1}{2ne^{j(2n-1)\alpha_k}} \tag{3-17}$$

The indefinite integral of (3-15), using (3-16) and (3-17), is found to be

$$\int\frac{d}{1+\omega^{2n}} = \frac{1}{2n}\sum_{k=0}^{2n-1}\left[\frac{-1}{2}\cos\alpha_k\ln(\omega^2-2\omega\cos\alpha_k+1) + \sin\alpha_k\tan^{-1}\frac{-\sin\alpha_k}{\omega-\cos\alpha_k}\right] \tag{3-18}$$

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11. C. R. Wylie, Jr., Plane Trigonometry (New York: McGraw-Hill Book Co., Inc., 1955), p. 239.

thereafter the definite integral is evaluated as

$$I(n) = \frac{1}{2n}\sum_{k=0}^{2n-1}\left[\frac{-1}{2}\cos\alpha_k \ln(\omega_c^2 - 2\omega_c\cos\alpha_k + 1) + \sin\alpha_k\left(\tan^{-1}\frac{-\sin\alpha_k}{\omega_c - \cos\alpha_k} - \frac{k\pi}{n}\right)\right] \tag{3-19}$$

The following two expressions for n = 1,and n = 2 may serve to give some idea as to the extent of complexity with respect to the order of a Butterworth filter.

For n = 1,

$$I(1) = \int_0^{\omega_c} \frac{d\omega}{1+\omega^2} = \tan^{-1}\omega_c$$

For n = 2,

$$I(2) = \int_0^{\omega_c} \frac{d\omega}{1+\omega^4} = \frac{1}{4\sqrt{2}}\ln\left(\frac{\omega_c^2+\sqrt{2}\omega_c+1}{\omega_c^2-\sqrt{2}\omega_c+1}\right) - \frac{1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}\omega_c}{\omega_c^2-1} + \frac{\pi}{2\sqrt{2}}$$

Equating (3-13) and (3-14) and rearranging, we get a ratio

$$r = \frac{N_w}{N_x} = \frac{2n}{\pi} I(n)\sin\frac{\pi}{2n} \tag{3-20}$$

where I(n) is given by either (3-15) or (3-19). The message conveyed by this expression is that in order to retain the same first order statistics at the output, the white noise w(t) must be so chosen as to possess a power spectral density r times that of the original noise. Being simple in form and yet quite typical in nature, the ratio for the case of a first order Butterworth filter,

$$r = \frac{2}{\pi}\tan^{-1}\omega_c \tag{3-21}$$

deserves our attention. Note that here ω_c, though not stated explicitly, will be regarded as a normalized quantity, being the noise cutoff frequency normalized with respect to the filter bandwidth. Table 3-1 gives the

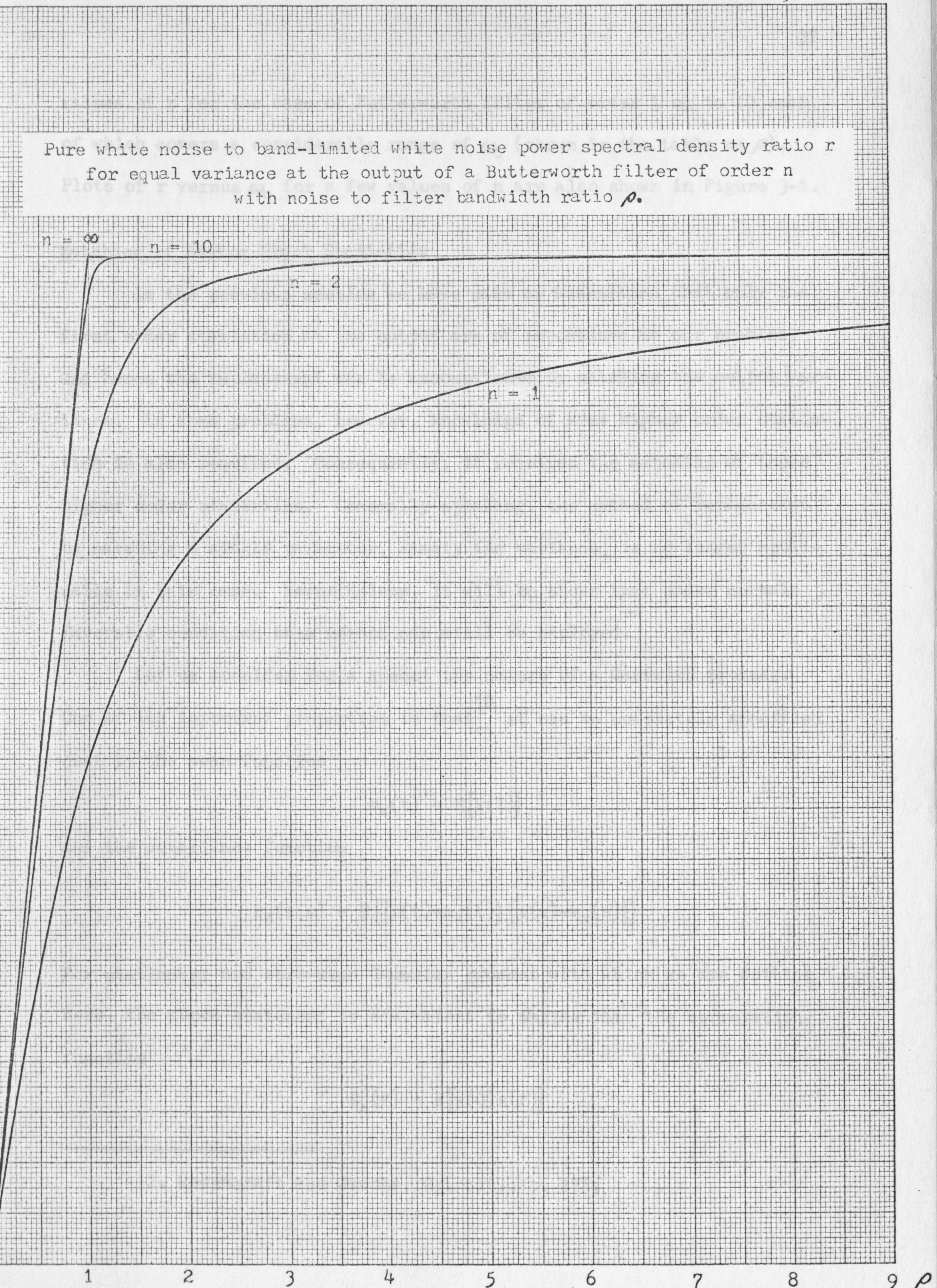
Table 3-1

Pure white noise to band-limited white noise power spectral density ratio r for equal variance at the output of a Butterworth filter of order n with noise to filter bandwidth ratio ρ.

ρ	n									
	1	2	3	4	5	6	7	8	9	10
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.1	0,0635	0,0900	0,0955	0,0974	0,0984	0,0989	0,0992	0,0994	0,0995	0,0996
0.2	0,1257	0,1800	0,1910	0,1949	0,1967	0,1977	0,1983	0,1987	0,1990	0,1992
0.3	0,1855	0,2697	0,2364	0,2923	0,2951	0,2966	0,2975	0,2981	0,2985	0,2988
0.4	0,2422	0,3583	0,3317	0,3898	0,3934	0,3954	0,3967	0,3974	0,3980	0,3984
0.5	0,2952	0,4447	0,4764	0,4870	0,4918	0,4943	0,4958	0,4968	0,4975	0,4979
0.6	0,3440	0,5271	0,5592	0,5836	0,5899	0,5931	0,5949	0,5961	0,5970	0,5975
0.7	0,3888	0,6034	0,6579	0,6779	0,6868	0,6913	0,6938	0,6954	0,6964	0,6971
0.8	0,4296	0,6718	0,7388	0,7662	0,7796	0,7869	0,7910	0,7936	0,7952	0,7963
0.9	0,4665	0,7309	0,8080	0,8425	0,8614	0,8728	0,8803	0,8853	0,8889	0,8914
1.0	0,5000	0,7805	0,8530	0,9011	0,9227	0,9367	0,9464	0,9535	0,9590	0,9633
1.1	0,5303	0,8212	0,9039	0,9406	0,9604	0,9725	0,9803	0,9856	0,9893	0,9920
1.2	0,5577	0,8540	0,9329	0,9648	0,9803	0,9885	0,9931	0,9958	0,9974	0,9984
1.3	0,5826	0,8803	0,9528	0,9790	0,9900	0,9951	0,9975	0,9987	0,9993	0,9996
1.4	0,6051	0,9011	0,9565	0,9872	0,9948	0,9978	0,9990	0,9996	0,9998	0,9999
1.5	0,6257	0,9178	0,9758	0,9920	0,9972	0,9990	0,9996	0,9998	0,9999	1,0000
1.6	0,6444	0,9311	0,9323	0,9949	0,9984	0,9995	0,9998	0,9999	1,0000	1,0000
1.7	0,6615	0,9418	0,9368	0,9966	0,9991	0,9997	0,9999	1,0000	1,0000	1,0000
1.8	0,6772	0,9505	0,9900	0,9977	0,9994	0,9999	1,0000	1,0000	1,0000	1,0000
1.9	0,6916	0,9576	0,9924	0,9984	0,9997	0,9999	1,0000	1,0000	1,0000	1,0000
2.0	0,7048	0,9635	0,9941	0,9989	0,9998	1,0000	1,0000	1,0000	1,0000	1,0000
2.1	0,7171	0,9683	0,9953	0,9992	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000
2.2	0,7284	0,9723	0,9963	0,9994	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000
2.3	0,7369	0,9757	0,9970	0,9996	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000
2.4	0,7487	0,9786	0,9976	0,9997	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
2.5	0,7578	0,9810	0,9980	0,9998	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
2.6	0,7663	0,9831	0,9984	0,9998	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
2.7	0,7742	0,9849	0,9987	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
2.8	0,7816	0,9864	0,9989	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
2.9	0,7886	0,9878	0,9991	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3.0	0,7952	0,9889	0,9992	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000

Pure white noise to band-limited white noise power spectral density ratio r, continued.

ρ	n									
	1	2	3	4	5	6	7	8	9	10
3,1	0,8013	0,9900	0,9993	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3,2	0,8072	0,9909	0,9994	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3,3	0,8127	0,9917	0,9995	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3,4	0,8179	0,9924	0,9996	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3,5	0,8228	0,9930	0,9996	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3,6	0,8275	0,9936	0,9997	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3,7	0,8320	0,9941	0,9997	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3,8	0,8362	0,9945	0,9998	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
3,9	0,8402	0,9950	0,9998	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,0	0,8440	0,9953	0,9998	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,1	0,8477	0,9957	0,9998	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,2	0,8512	0,9960	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,3	0,8545	0,9962	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,4	0,8577	0,9965	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,5	0,8608	0,9967	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,6	0,8637	0,9969	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,7	0,8665	0,9971	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,8	0,8692	0,9973	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
4,9	0,8718	0,9975	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,0	0,8743	0,9976	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,1	0,8767	0,9977	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,2	0,8790	0,9979	0,9999	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,3	0,8813	0,9980	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,4	0,8834	0,9981	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,5	0,8855	0,9982	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,6	0,8875	0,9983	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,7	0,8894	0,9984	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,8	0,8913	0,9985	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
5,9	0,8931	0,9985	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
6,0	0,8949	0,9986	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000



Pure white noise to band-limited white noise power spectral density ratio r for equal variance at the output of a Butterworth filter of order n with noise to filter bandwidth ratio ρ.

values of r for the case of Butterworth filter of order 1 up to 10 each of which covers a considerable range of ω_c (shown in the table as ρ). Plots of r versus ω_c for a few values of n are also shown in Figure 3-1.

Matching of Higher Order Statistics

In the previous section we have made an assumption that only the first order statistics at the output are of importance to our problem, and hence the replacement can be accomplished by matching the output variance. In some problems, however, knowledge of some higher order statistics is also required. Consequently, it requires the matching of those higher order statistics. Generally speaking, the method of replacement by matching a single parameter, namely the variance, is no longer applicable in this case. Nevertheless, it will be shown that under certain favorable condition this method can still be employed.

Let us consider for a moment the nature of a Gaussian process. One of its important properties is that[12] it can be completely specified just by the mean function

$$m_x(t) = E\left[x(t)\right]$$

and the covariance function

$$P_x(t,s) = E\left[\{x(t)-m_x(t)\}\{x(s)-m_x(s)\}\right]$$

For stationary and zero-mean Gaussian process with which we are working here, the above statement is equivalent to saying that the correlation function

$$R_x(\tau) = E\left[x(0)x(\tau)\right]$$

12. Wozencraft and Jacobs, Op. cit., p. 172.

is the only thing needed for its complete specification. In either case it always requires a knowledge of the second order density function whereby density functions of any other order can be obtained.

It can be seen that if we use the method outlined in the first section of the previous chapter to match the output correlation function, the problem is completely solved, at least theorectically, but without enjoying the ease we have in the last section.

It is not quite out of the question to imagine that there may be some circumstance that will permit us to use the advantage of the method of replacement by matching of the variance, or the second moment. If for example we are interested only in the second order statistics of random variables $y(t_1)$ and $y(t_2)$ where t_1 and t_2 are any two instants of time that are farther apart than the correlation time[13] of $y(t)$, then the joint density function of $y(t_1)$ and $y(t_2)$ can be written down immediately since in this case $y(t_1)$ and $y(t_2)$ can be regarded as independent, again using one of the Gaussian properties that being uncorrelated implies independence. The joint density function written down in this way is just the product of the first order density functions of $y(t_1)$ and $y(t_2)$ and hence requires a knowledge of no more than that of the variance. In this way the requirement for matching of the second order statistics can be satisfied using the same procedure as described in the preceding section. By similar reasoning, the replacement by matching of higher order statistics can also be accomplished under this particular condition, namely when the separation between any two variables is greater than the

13. Stratonovich, Op. cit., p. 22.

correlation time, which is the subject to be treated in the next section.

Correlation Time

From the previous argument we have seen how important the part played by the correlation time is in justifying the extension of the method developed in the first section to the case of higher order statistics. In this section the idea will be consolidated by actual evaluation based on the same example that has been used so far, that is the Butterworth filter.

The correlation time is defined by the expression

$$\tau_{cor} = \frac{1}{R(0)} \int_0^{\infty} |R(\tau)| \, d\tau \tag{3-22}$$

Unfortunately, this simple looking formula is, at least in our case, far from simple to use. The only exception is the case of first order filter with infinite bandwidth white noise input where the correlation function is given by

$$R_y(\tau) = \frac{1}{2} e^{-|\tau|}$$

which is always positive, and

$$R_y(0) = \frac{1}{2}$$

in this case, we have

$$\tau_{cor} = \int_0^{\infty} e^{-\tau} d\tau = 1 \tag{3-23}$$

Obviously, a simpler, or rather, more manageable, formula for the correlation time is in order. One of such formula is given as [14]

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14. Ibid., p. 88.

$$\tau_{cor} = \frac{\int_0^\infty \tau R(\tau)\, d\tau}{\int_0^\infty R(\tau)\, d\tau} = \frac{2}{K}\int_0^\infty \tau R(\tau)\, d\tau \tag{3-24}$$

where K is the intensity coefficient. The values of τ_{cor} given by (3-24) and (3-22) will be somewhat different, but of the same order of magnitude.

Substituting the formula

$$R_y(\tau) = \frac{1}{2n}\sum_{k=0}^{n-1} \sin(|\tau| \cos\alpha_k + \alpha_k)\, e^{-|\tau|\sin\alpha_k} \tag{3-25}$$

where

$$\alpha_k = \frac{(2k+1)\pi}{2n}$$

and the value of K found in (2-8), which is one, into (3-24) gives the correlation time in the case of white noise input as

$$\tau_{cor} = \frac{1}{n}\sum_{k=0}^{n-1} I(k) \tag{3-26}$$

where

$$I(k) = \int_0^\infty \tau \sin(\tau\cos\alpha_k + \alpha_k)\, e^{-\tau\sin\alpha_k} d\tau \tag{3-27}$$

the absolute value symbol for τ being dropped in view of the integration range. Applying the integration by parts technique to (3-27) several times, we eventually arrive at the result

$$I(k) = \sin\alpha_k \tag{3-28}$$

From here, we have

$$\tau_{cor} = \frac{1}{n}\sum_{k=0}^{n-1} \sin\alpha_k = \frac{1}{n \sin\frac{\pi}{2n}} \tag{3-29}$$

The result on the right follows from formula (3-12).

The values of τ_{cor} for the first few values of n are shown below.

n	τ_{cor}
1	1.000
2	0.707
3	0.667

Since we know from Chapter One that as n goes to infinity the limit of a Butterworth function as well as that of its corresponding output correlation function exists, we would naturally inquire about such a limit for the correlation time. If we take the limit in (3-29) directly, the result will be an indeterminate form. We have to resort to the formula

$$\lim_{x \to a} \frac{\varphi(x)}{\psi(x)} = \frac{\varphi'(a)}{\psi'(a)}$$

which will lead to an answer[15]

$$\lim_{n \to \infty} \tau_{cor} = \frac{2}{\pi} = 0.637 \qquad (3\text{-}30)$$

While the correlation time in the case of band-limited noise input cannot be treated exactly in general, several formulas do exist for its approximation. From the second section in Chapter One, we know that for small value of ω_c, particularly when ω_c is much less than unity, the output correlation funcion can be approximated accurately as

$$R_y(\tau) \approx \frac{1}{\pi} \frac{\sin \omega_c \tau}{\tau} \qquad (3\text{-}31)$$

With this formula and the fact that K = 1 for all values of ω_c, we get, from (3-24)

$$\tau_{cor} \approx \frac{2}{\pi} \left[\frac{-\cos \omega_c \tau}{\omega_c} \right] \Bigg|_0^{\infty} \qquad (3\text{-}32)$$

---

15. The same result may be obtained simply by using the fact that for small angle, $\sin \alpha \approx \alpha$.

Strictly speaking, this expression cannot be evaluated since $\cos\omega_c\tau$ may take on any value between 1 and -1 as τ goes to infinity. The values of τ_{cor} corresponding to these two extreme values of $\cos\omega_c\tau$ are respectively 0 and $\frac{4}{\pi\omega_c}$. The former seems to be out of the question in our case. But instead of taking the latter directly, we may think of taking the average of these two to make

$$\tau_{cor} \approx \frac{2}{\pi\omega_c} \tag{3-33}$$

since it is still of the same order of magnitude as the latter.

The above result which seems to be taken arbitrarily can, as a matter of fact, be evaluated exactly although indirectly. To see this we note that if we take the filter bandwidth ω_o into account explicitly in (3-25), we would arrive at (3-29) as

$$\tau_{cor} = \frac{1}{n\omega_o \sin\frac{\pi}{2n}} \tag{3-34}$$

whereupon (3-30) becomes

$$\lim_{n\to\infty} \tau_{cor} = \frac{2}{\pi\omega_o} \tag{3-35}$$

similarly, the formula (1-38) for ω_o other than unity would become

$$\lim_{n\to\infty} R_y(\tau) = \frac{1}{\pi}\,\frac{\sin\omega_o\tau}{\tau} \tag{3-36}$$

Hence to get the solution for (3-32) is just a matter of replacing ω_o by ω_c in (3-35) and (3-36).

Finally, we note that when the evaluation of the integral is too involved, the formula

$$\tau_{cor} \approx \frac{1}{\omega_c} \tag{3-37}$$

can be used as a first approximation.

CHAPTER FOUR

REPLACEMENT IN THE CASE OF A GENERAL

ALL-POLE SECOND ORDER FILTER

The method presented in the preceding chapter has been applied with detailed demonstration to the case of a Butterworth filter. Although it is just a specific class of filter, the results derived therefrom will, nevertheless, be of some significance in general. Another class of filter that is of much interest is an all-pole second order filter. Thus we will use it as a further example to demonstrate how the suggested method can be applied to other cases.

The transfer function of a general all-pole second order filter is

$$H(s) = \frac{1}{s^2+2\xi\omega_n s+\omega_n^2} \tag{4-1}$$

Replacing s by $j\omega$, we get, after rearranging,

$$H(j\omega) = \frac{1}{(\omega_n^2-\omega^2)+j(2\xi\omega_n\omega)} \tag{4-2}$$

The magnitude-square function is then given by

$$\left|H(j\omega)\right|^2 = \frac{1}{\omega^4+2\omega_n^2(2\xi^2-1)\omega^2+\omega_n^4} \tag{4-3}$$

For an input Gaussian noise, x(t), with power spectral density

$$S_x(\omega) = \begin{cases} N_x & \text{for } |\omega| \leq \omega_c \\ 0 & \text{for } |\omega| > \omega_c \end{cases}$$

the output correlation function is given by

$$R_y(\tau) = \frac{N_x}{\pi}\int_0^{\infty} \frac{\cos\omega\tau}{\omega^4+2\omega_n^2(2\xi^2-1)\omega^2+\omega_n^4}\,d\omega$$

and the variance is accordingly

$$\sigma_y^2 = R_y(0) = \frac{N_x}{\pi}\int_0^{\infty} \frac{d\omega}{\omega^4+2\omega_n^2(2\zeta^2-1)\omega^2+\omega_n^4} \tag{4-4}$$

Similarly, for an input white Gaussian noise, w(t), with power spectral density

$$S_w(\omega) = N_w, \; -\infty < \omega < \infty$$

the variance at the output will be

$$\sigma_z^2 = R_z(0) = \frac{N_w}{\pi}\int_0^{\infty} \frac{d\omega}{\omega^4+2\omega_n^2(2\zeta^2-1)\omega^2+\omega_n^4} \tag{4-5}$$

To evaluate the integral in (4-4), let us first define

$$\eta = \frac{\omega}{\omega_n} \text{ and } \beta = \frac{\omega_c}{\omega_n} \tag{4-6}$$

such that

$$R_y(0) = \frac{N_x}{\pi\omega^3}\, I(\zeta) \tag{4-7}$$

where

$$I(\zeta) = \int_0^{\beta} \frac{d\eta}{\eta^4+2(2\zeta^2-1)\eta^2+1} \tag{4-8}$$

Setting

$$\eta^4+2(2\zeta^2-1)\eta^2+1 = 0$$

we get

$$\eta^2 = (1-2\zeta^2) \pm 2\zeta\sqrt{\zeta^2-1} \tag{4-9}$$

For $\zeta = 0$, the two values of η^2 in (4-9) are 1, the indefinite integral[16] in (4-8) becomes

$$\int \frac{d\eta}{(\eta^2-1)^2} = \frac{\eta}{2(1-\eta^2)} + \frac{1}{4}\ln\left|\frac{1+\eta}{1-\eta}\right|$$

Hence

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16. Herbert Bristol Dwight, *Tables of Integrals and Other Mathematical Data* (New York: The Macmillan Co., 1934), p. 27.

$$I(\xi) = \frac{\beta}{2(1-\beta^2)} + \frac{1}{4}\ln\left|\frac{1+\beta}{1-\beta}\right| \tag{4-10}$$

For $0<\xi<1$, write (4-9) as

$$\eta^2 = (1-2\xi^2) \pm j2\xi\sqrt{1-\xi^2}$$

or, in the exponential form,

$$= e^{\pm j\theta} \tag{4-11}$$

where

$$\theta = \tan^{-1}\frac{2\xi\sqrt{1-\xi^2}}{1-2\xi^2} \tag{4-12}$$

Using (4-11), the integrand in (4-8) can be written as

$$\frac{1}{\eta^4+2(2\xi^2-1)\eta^2+1} = \frac{1}{2j\sin\theta}\left[\frac{1}{\eta^2-e^{j\theta}} - \frac{1}{\eta^2-e^{-j\theta}}\right]$$

which can be further expanded as

$$\frac{1}{2j\sin\theta}\left[\frac{1}{2e^{j\theta/2}}\left\{\frac{1}{\eta-e^{j\theta/2}} - \frac{1}{\eta+e^{j\theta/2}}\right\} - \frac{1}{2e^{-j\theta/2}}\left\{\frac{1}{\eta-e^{-j\theta/2}} - \frac{1}{\eta+e^{-j\theta/2}}\right\}\right]$$

The indefinite integral of $I(\xi)$ in this case is found to be

$$\int\frac{d\eta}{\eta^4+2(2\xi^2-1)\eta^2+1} = \frac{1}{4j\sin\theta}\left[\frac{1}{e^{j\theta/2}}\ln\frac{\eta-e^{j\theta/2}}{\eta+e^{j\theta/2}} - \frac{1}{e^{-j\theta/2}}\ln\frac{\eta-e^{-j\theta/2}}{\eta+e^{-j\theta/2}}\right]$$

which after simplifying,

$$= \frac{\sec\theta/2}{8}\ln\frac{\eta^2+2\eta\cos\theta/2+1}{\eta^2-2\eta\cos\theta/2+1} - \frac{\operatorname{cosec}\theta/2}{4}\tan^{-1}\frac{2\eta\sin\theta/2}{\eta^2-1}$$

where θ is given by (4-12). It follows that

$$I(\xi) = \frac{\sec\theta/2}{8}\ln\frac{\beta^2+2\beta\cos\theta/2+1}{\beta^2-2\beta\cos\theta/2+1} + \frac{\operatorname{cosec}\theta/2}{4}\left[\pi-\tan^{-1}\frac{2\beta\sin\theta/2}{\beta^2-1}\right] \tag{4-13}$$

For $\xi = 1$, we have two equal roots for η^2, namely, -1 and -1. Consequently, the indefinite integral reduces to

$$\int \frac{d\eta}{(\eta^2+1)^2} = \frac{\eta}{2(\eta^2+1)} + \frac{1}{2}\tan^{-1}\eta$$

the result being found directly from the table of integrals.[17] The corresponding definite integral is

$$I(\xi) = \frac{\beta}{2(\beta^2+1)} + \frac{1}{2}\tan^{-1}\beta \tag{4-14}$$

Finally, for $\xi > 1$, write

$$\eta^2 = -u \pm v \tag{4-15}$$

where

$$u = (2\xi^2-1) \text{ and } v = 2\xi\sqrt{\xi^2-1} \tag{4-16}$$

Again, the integrand in (4-8) can be expanded into partial fractions,

$$\frac{1}{\eta^4+2(2\xi^2-1)\eta^2+1} = \frac{1}{2v}\left[\frac{1}{\eta^2+(u-v)} - \frac{1}{\eta^2+(u+v)}\right]$$

which will permit the use of table of integrals to obtain the result,[18]

$$\int \frac{d\eta}{\eta^4+2(2\xi^2-1)\eta^2+1} = \frac{1}{2v}\left[\frac{1}{\sqrt{u-v}}\tan^{-1}\frac{\eta}{\sqrt{u-v}} - \frac{1}{\sqrt{u+v}}\tan^{-1}\frac{\eta}{\sqrt{u+v}}\right]$$

from where

$$I(\xi) = \frac{1}{2v}\left[\frac{1}{\sqrt{u-v}}\tan^{-1}\frac{\beta}{\sqrt{u-v}} - \frac{1}{\sqrt{u+v}}\tan^{-1}\frac{\beta}{\sqrt{u+v}}\right] \tag{4-17}$$

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17. Ibid., p. 22.

18. Ibid., p. 22.

Using (4-12) and (4-16), formulas (4-13) and (4-17) respectively can be expressed back in terms of the parameter ζ as

$$I(\zeta) = \frac{1}{8\sqrt{1-\zeta^2}} \ln \frac{\beta^2+2\beta\sqrt{1-\zeta^2}+1}{\beta^2-2\beta\sqrt{1-\zeta^2}+1} + \frac{1}{4\zeta}\left[\pi - \tan^{-1}\frac{2\beta\zeta}{\beta^2-1}\right] \tag{4-18}$$

for $0<\zeta<1$, and

$$I(\zeta) = \frac{1}{\zeta}\left[\frac{1}{1-(\zeta-\sqrt{\zeta^2-1})^2}\tan^{-1}\frac{\beta}{\zeta-\sqrt{\zeta^2-1}} + \frac{1}{1-(\zeta+\sqrt{\zeta^2-1})^2}\tan^{-1}\frac{\beta}{\zeta+\sqrt{\zeta^2-1}}\right] \tag{4-19}$$

for $\zeta>1$.

In view of all of these formulas for $I(\zeta)$, our task of evaluating (4-4), or equivalently (4-7), is now complete. The next step is to solve (4-5). This can be done easily by first taking the limit of the formulas for $I(\zeta)$ as β goes to infinity. The limit is found to be

$$\lim_{\beta\to\infty} I(\zeta) = \frac{\pi}{4\zeta} \tag{4-20}$$

for all cases except for $\zeta = 0$. From here we can write (4-5) as

$$R_z(0) = \frac{N_W}{4\zeta\omega^3} \tag{4-21}$$

Equating (4-21) and (4-7) and rearranging, we get

$$r = \frac{N_W}{N_X} = \frac{4\zeta}{\pi} I(\zeta) \tag{4-22}$$

Values of r for various β's and ζ's have been tabulated as shown in Table 4-1. Figure 4-1 gives some of the curves plotted from the Table.

Table 4-1

Pure white noise to band-limited white noise power spectral density ratio r for equal variance at the output of an all-pole second order filter with damping factor ζ and noise bandwidth to filter natural frequency ratio β.

β	ζ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.5	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.1	0.0256	0.0512	0.0765	0.1017	0.1265	0.1509	0.1749	0.1984	0.2213	0.2437
0.2	0.0522	0.1037	0.1539	0.2022	0.2481	0.2914	0.3320	0.3698	0.4048	0.4371
0.3	0.0809	0.1592	0.2328	0.3001	0.3608	0.4147	0.4622	0.5041	0.5409	0.5734
0.4	0.1133	0.2193	0.3135	0.3941	0.4618	0.5183	0.5654	0.6051	0.6386	0.6673
0.5	0.1515	0.2859	0.3958	0.4822	0.5498	0.6031	0.6458	0.6806	0.7093	0.7334
0.6	0.1994	0.3609	0.4785	0.5630	0.6249	0.6718	0.7084	0.7376	0.7615	0.7814
0.7	0.2633	0.4452	0.5595	0.6349	0.6879	0.7271	0.7574	0.7815	0.8011	0.8175
0.8	0.3545	0.5373	0.6357	0.6973	0.7401	0.7717	0.7962	0.8158	0.8319	0.8453
0.9	0.4866	0.6313	0.7039	0.7502	0.7831	0.8079	0.8274	0.8431	0.8562	0.8672
1.0	0.6490	0.7177	0.7523	0.7942	0.8183	0.8373	0.8526	0.8652	0.8758	0.8849
1.1	0.7842	0.7887	0.8101	0.8302	0.8472	0.8613	0.8732	0.8833	0.8919	0.8994
1.2	0.8658	0.8424	0.8482	0.8594	0.8708	0.8812	0.8903	0.8982	0.9052	0.9114
1.3	0.9115	0.8813	0.8781	0.8831	0.8902	0.8976	0.9045	0.9107	0.9164	0.9214
1.4	0.9382	0.9090	0.9013	0.9022	0.9062	0.9113	0.9164	0.9213	0.9258	0.9300
1.5	0.9548	0.9290	0.9193	0.9176	0.9195	0.9228	0.9265	0.9303	0.9339	0.9373
1.6	0.9657	0.9436	0.9334	0.9302	0.9305	0.9324	0.9351	0.9379	0.9408	0.9436
1.7	0.9732	0.9545	0.9445	0.9405	0.9397	0.9406	0.9424	0.9446	0.9468	0.9491
1.8	0.9786	0.9627	0.9534	0.9489	0.9474	0.9476	0.9487	0.9503	0.9520	0.9539
1.9	0.9826	0.9691	0.9605	0.9559	0.9540	0.9536	0.9542	0.9553	0.9566	0.9581
2.0	0.9856	0.9740	0.9662	0.9617	0.9595	0.9587	0.9589	0.9596	0.9606	0.9618
2.1	0.9879	0.9780	0.9709	0.9666	0.9642	0.9632	0.9630	0.9634	0.9642	0.9651
2.2	0.9898	0.9811	0.9748	0.9706	0.9682	0.9670	0.9667	0.9668	0.9673	0.9680
2.3	0.9913	0.9837	0.9780	0.9741	0.9717	0.9704	0.9698	0.9698	0.9701	0.9706
2.4	0.9925	0.9859	0.9807	0.9770	0.9747	0.9733	0.9726	0.9724	0.9726	0.9730
2.5	0.9934	0.9876	0.9830	0.9796	0.9773	0.9759	0.9751	0.9748	0.9748	0.9751
2.6	0.9943	0.9891	0.9849	0.9817	0.9796	0.9781	0.9773	0.9769	0.9768	0.9770
2.7	0.9949	0.9904	0.9865	0.9836	0.9815	0.9801	0.9793	0.9788	0.9786	0.9787
2.8	0.9955	0.9914	0.9880	0.9853	0.9833	0.9819	0.9810	0.9805	0.9803	0.9802
2.9	0.9960	0.9923	0.9892	0.9867	0.9848	0.9835	0.9826	0.9820	0.9817	0.9816
3.0	0.9964	0.9931	0.9903	0.9879	0.9862	0.9849	0.9840	0.9834	0.9831	0.9829

Pure white noise to band-limited white noise power spectral density ratio, continued.

β	ζ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
3.1	0.9968	0.9938	0.9912	0.9890	0.9874	0.9861	0.9852	0.9846	0.9843	0.9841
3.2	0.9971	0.9944	0.9920	0.9900	0.9884	0.9872	0.9863	0.9857	0.9854	0.9852
3.3	0.9974	0.9949	0.9927	0.9909	0.9894	0.9882	0.9874	0.9868	0.9864	0.9861
3.4	0.9976	0.9954	0.9933	0.9916	0.9902	0.9891	0.9883	0.9877	0.9873	0.9870
3.5	0.9978	0.9958	0.9939	0.9923	0.9910	0.9899	0.9891	0.9885	0.9881	0.9878
3.6	0.9980	0.9961	0.9944	0.9929	0.9917	0.9907	0.9899	0.9893	0.9889	0.9886
3.7	0.9982	0.9964	0.9949	0.9935	0.9923	0.9913	0.9906	0.9900	0.9896	0.9893
3.8	0.9983	0.9967	0.9953	0.9940	0.9929	0.9920	0.9912	0.9907	0.9903	0.9899
3.9	0.9985	0.9970	0.9956	0.9944	0.9934	0.9925	0.9918	0.9913	0.9908	0.9905
4.0	0.9986	0.9972	0.9959	0.9948	0.9938	0.9930	0.9923	0.9918	0.9914	0.9911
4.1	0.9987	0.9974	0.9962	0.9952	0.9943	0.9935	0.9928	0.9923	0.9919	0.9916
4.2	0.9988	0.9976	0.9965	0.9955	0.9946	0.9939	0.9933	0.9928	0.9924	0.9921
4.3	0.9989	0.9978	0.9967	0.9958	0.9950	0.9943	0.9937	0.9932	0.9928	0.9925
4.4	0.9989	0.9979	0.9970	0.9961	0.9953	0.9946	0.9941	0.9936	0.9932	0.9929
4.5	0.9990	0.9981	0.9972	0.9963	0.9956	0.9950	0.9944	0.9940	0.9936	0.9933
4.6	0.9991	0.9982	0.9973	0.9966	0.9959	0.9953	0.9947	0.9943	0.9939	0.9937
4.7	0.9991	0.9983	0.9975	0.9968	0.9961	0.9955	0.9950	0.9946	0.9943	0.9940
4.8	0.9992	0.9984	0.9977	0.9970	0.9964	0.9958	0.9953	0.9949	0.9946	0.9943
4.9	0.9992	0.9985	0.9978	0.9972	0.9966	0.9960	0.9956	0.9952	0.9949	0.9946
5.0	0.9993	0.9986	0.9979	0.9973	0.9968	0.9963	0.9958	0.9954	0.9951	0.9949
5.1	0.9993	0.9987	0.9981	0.9975	0.9969	0.9965	0.9960	0.9957	0.9954	0.9951
5.2	0.9994	0.9988	0.9982	0.9976	0.9971	0.9967	0.9963	0.9959	0.9956	0.9953
5.3	0.9994	0.9988	0.9983	0.9977	0.9973	0.9968	0.9964	0.9961	0.9958	0.9956
5.4	0.9994	0.9989	0.9984	0.9979	0.9974	0.9970	0.9966	0.9963	0.9960	0.9958
5.5	0.9995	0.9990	0.9985	0.9980	0.9975	0.9972	0.9968	0.9965	0.9962	0.9960
5.6	0.9995	0.9990	0.9985	0.9981	0.9977	0.9973	0.9970	0.9966	0.9964	0.9962
5.7	0.9995	0.9991	0.9986	0.9982	0.9978	0.9974	0.9971	0.9968	0.9966	0.9963
5.8	0.9996	0.9991	0.9987	0.9983	0.9979	0.9976	0.9972	0.9970	0.9967	0.9965
5.9	0.9996	0.9992	0.9987	0.9984	0.9980	0.9977	0.9974	0.9971	0.9969	0.9966
6.0	0.9996	0.9992	0.9988	0.9984	0.9981	0.9978	0.9975	0.9972	0.9970	0.9968
6.1	0.9996	0.9992	0.9989	0.9985	0.9982	0.9979	0.9976	0.9974	0.9971	0.9969
6.2	0.9996	0.9993	0.9989	0.9986	0.9983	0.9980	0.9977	0.9975	0.9973	0.9971
6.3	0.9997	0.9993	0.9990	0.9987	0.9984	0.9981	0.9978	0.9976	0.9974	0.9972
6.4	0.9997	0.9993	0.9990	0.9987	0.9984	0.9982	0.9979	0.9977	0.9975	0.9973
6.5	0.9997	0.9994	0.9991	0.9988	0.9985	0.9982	0.9980	0.9978	0.9976	0.9974

Pure white noise to band-limited white noise power spectral density ratio, continued.

β	ζ									
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
6.6	0.9997	0.9994	0.9991	0.9988	0.9986	0.9983	0.9981	0.9979	0.9977	0.9975
6.7	0.9997	0.9994	0.9991	0.9989	0.9986	0.9984	0.9982	0.9980	0.9978	0.9976
6.8	0.9997	0.9995	0.9992	0.9989	0.9987	0.9985	0.9982	0.9980	0.9979	0.9977
6.9	0.9997	0.9995	0.9992	0.9990	0.9987	0.9985	0.9983	0.9981	0.9980	0.9978
7.0	0.9997	0.9995	0.9993	0.9990	0.9988	0.9986	0.9984	0.9982	0.9980	0.9979
7.1	0.9998	0.9995	0.9993	0.9991	0.9988	0.9986	0.9984	0.9983	0.9981	0.9980
7.2	0.9998	0.9995	0.9993	0.9991	0.9989	0.9987	0.9985	0.9983	0.9982	0.9980
7.3	0.9998	0.9996	0.9993	0.9991	0.9989	0.9987	0.9986	0.9984	0.9982	0.9981
7.4	0.9998	0.9996	0.9994	0.9992	0.9990	0.9988	0.9986	0.9985	0.9983	0.9982
7.5	0.9998	0.9996	0.9994	0.9992	0.9990	0.9988	0.9987	0.9985	0.9984	0.9982
7.6	0.9998	0.9996	0.9994	0.9992	0.9991	0.9989	0.9987	0.9986	0.9984	0.9983
7.7	0.9998	0.9996	0.9994	0.9993	0.9991	0.9989	0.9988	0.9986	0.9985	0.9984
7.8	0.9998	0.9996	0.9995	0.9993	0.9991	0.9990	0.9988	0.9987	0.9985	0.9984
7.9	0.9998	0.9997	0.9995	0.9993	0.9992	0.9990	0.9989	0.9987	0.9986	0.9985
8.0	0.9998	0.9997	0.9995	0.9993	0.9992	0.9990	0.9989	0.9988	0.9986	0.9985
8.1	0.9998	0.9997	0.9995	0.9994	0.9992	0.9991	0.9989	0.9988	0.9987	0.9986
8.2	0.9998	0.9997	0.9995	0.9994	0.9992	0.9991	0.9990	0.9989	0.9987	0.9986
8.3	0.9998	0.9997	0.9996	0.9994	0.9993	0.9991	0.9990	0.9989	0.9988	0.9987
8.4	0.9999	0.9997	0.9996	0.9994	0.9993	0.9992	0.9990	0.9989	0.9988	0.9987
8.5	0.9999	0.9997	0.9996	0.9994	0.9993	0.9992	0.9991	0.9990	0.9989	0.9988
8.6	0.9999	0.9997	0.9996	0.9995	0.9993	0.9992	0.9991	0.9990	0.9989	0.9988
8.7	0.9999	0.9997	0.9996	0.9995	0.9994	0.9992	0.9991	0.9990	0.9989	0.9988
8.8	0.9999	0.9997	0.9996	0.9995	0.9994	0.9993	0.9992	0.9991	0.9990	0.9989
8.9	0.9999	0.9998	0.9996	0.9995	0.9994	0.9993	0.9992	0.9991	0.9990	0.9989
9.0	0.9999	0.9998	0.9996	0.9995	0.9994	0.9993	0.9992	0.9991	0.9990	0.9989
9.1	0.9999	0.9998	0.9997	0.9996	0.9994	0.9993	0.9992	0.9991	0.9991	0.9990
9.2	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9993	0.9992	0.9991	0.9990
9.3	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9993	0.9992	0.9991	0.9990
9.4	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9993	0.9992	0.9991	0.9991
9.5	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9993	0.9992	0.9992	0.9991
9.6	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9994	0.9993	0.9992	0.9991
9.7	0.9999	0.9998	0.9997	0.9996	0.9995	0.9995	0.9994	0.9993	0.9992	0.9991
9.8	0.9999	0.9998	0.9997	0.9996	0.9996	0.9995	0.9994	0.9993	0.9992	0.9992
9.9	0.9999	0.9998	0.9997	0.9997	0.9996	0.9995	0.9994	0.9993	0.9993	0.9992
10.0	0.9999	0.9998	0.9997	0.9997	0.9996	0.9995	0.9994	0.9994	0.9993	0.9992

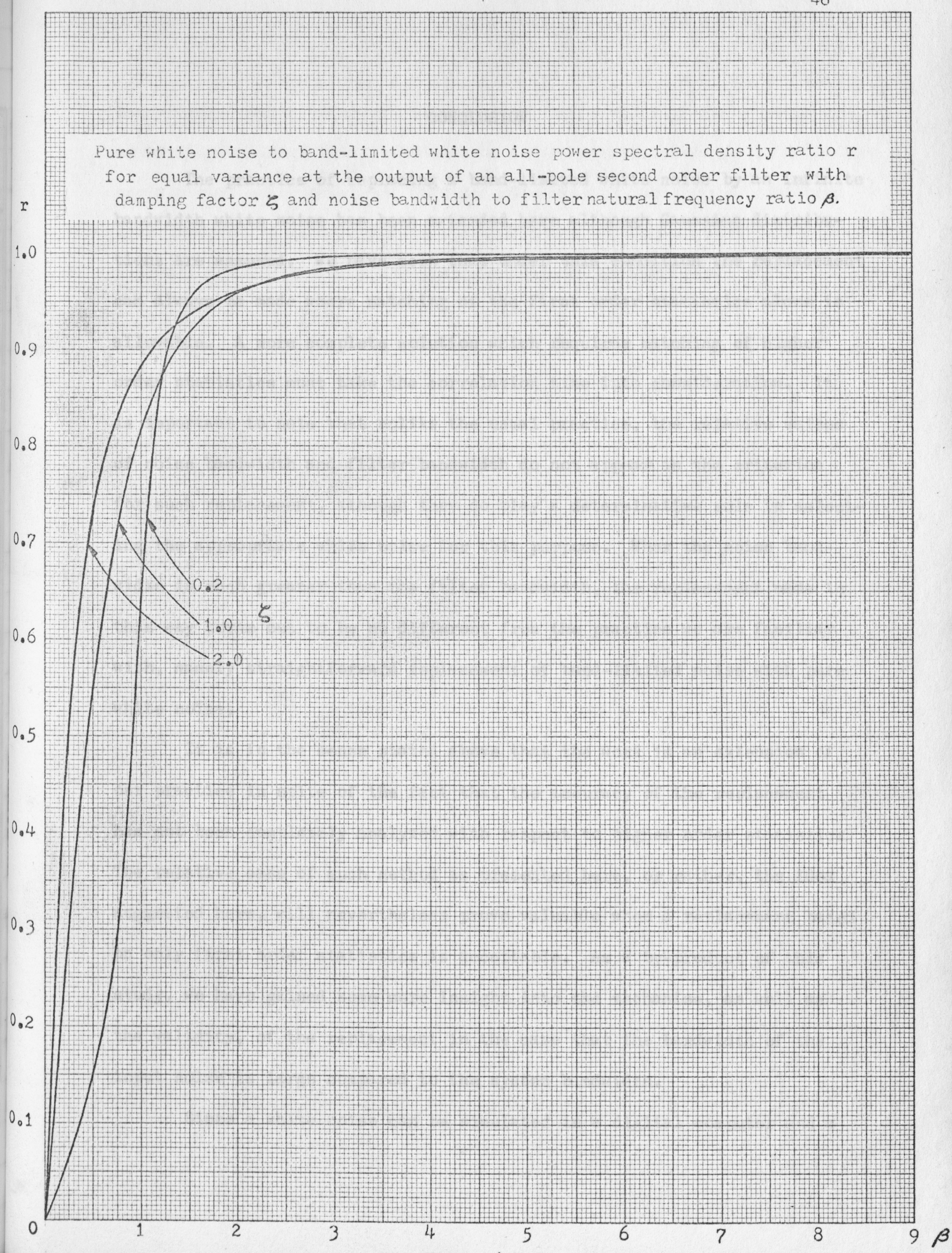

Pure white noise to band-limited white noise power spectral density ratio r for equal variance at the output of an all-pole second order filter with damping factor ζ and noise bandwidth to filter natural frequency ratio β.
r
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.2
1.0
2.0
ζ
1
2
3
4
5
6
7
8
9
β

Figure 4-1

CONCLUSION

The practice of replacing a band-limited white noise by an infinite bandwidth white noise has been extended here although Gaussian distributions are assumed. We have seen that under this condition the replacement can always be made where matching of the first order statistics alone is allowable. A more complete solution which requires matching of higher order statistics must take the correlation time into consideration. It is important to note that unlike the usual practice, the relative values of noise bandwidth and filter bandwidth do not appear in the criterion for such replacement, although they do play a determinative role in selecting the appropriate white noise for this purpose. When the noise bandwidth is much greater than the filter bandwidth, the replacement made in this way turns out to be no different from the practice we are familiar with, namely straightforward replacement of band-limited noise with pure white noise.

It is in the sense stated above that the work is an extension of our practice in spite of the fact that the accuracy of the replacement has not been completely analyzed with respect to higher order statistics. The relative ease of such method as presented here, or rather, the idea suggested here, will nevertheless prove valuable when a replacement based on only first order statistics is acceptable. As a by-product of this study, we have gained some more insight into the mechanism, as well as the validity, of the replacement in the case that the bandwidth of the input noise is large compared to the system bandwidth.

Since a white noise can be simulated in a digital computer and a band-limited white noise can be generated from a white noise by using an appropriate filter, this method can be tested experimentally. It is quite certain that some useful and more definite results can thus be obtained. This prospect is recommended as a continuation of this work in order to evaluate the quality of the replacement scheme proposed here.

APPENDIX A

COMPUTER PROGRAM AND SAMPLE RESULTS

The correlation function at the output of a unit bandwidth Butterworth filter order n subject to a band-limited white noise input with normalized cutoff frequency ω_c is, from Chapter One,

$$R_y(\tau) = \frac{1}{\pi}\int_0^{\omega_c} \frac{\cos\omega\tau}{1+\omega^{2n}}\,d\omega \qquad \text{(A-1)}$$

where ω is the frequency normalized with respect to the filter bandwidth. With the exception of infinite ω_c, we have to resort to numerical methods for the evaluation of (A-1). To accomplish this, a computer program written in FORTRAN IV language has been developed, the numerical integration being performed by using the Simpson's rule.

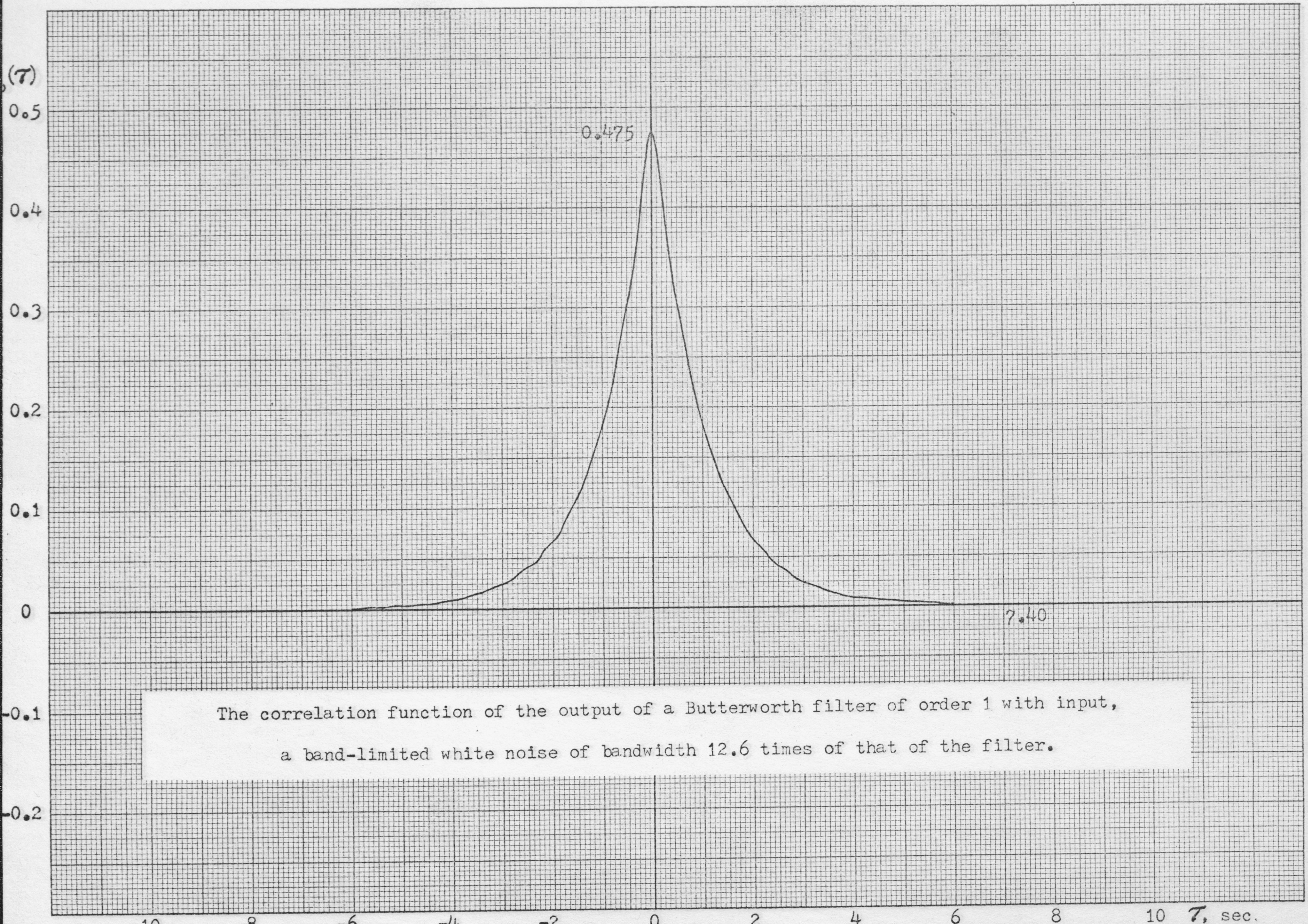
The data format for this particular usage is described in the comments of the program, and will not be repeated here. Adaptation for use with other filter transfer functions can readily be made through appropriate changes of the FUNCTION subprogram HSQ(W), the COMMON statement, and the variable N which in our case is the parameter of the filter denoting filter order.

Plots of $R_y(\tau)$ for many values of ω_c'as contained in this appendix are exclusively of the first order Butterworth filter. The program and these sample results can be found in the following pages.

Computer Program

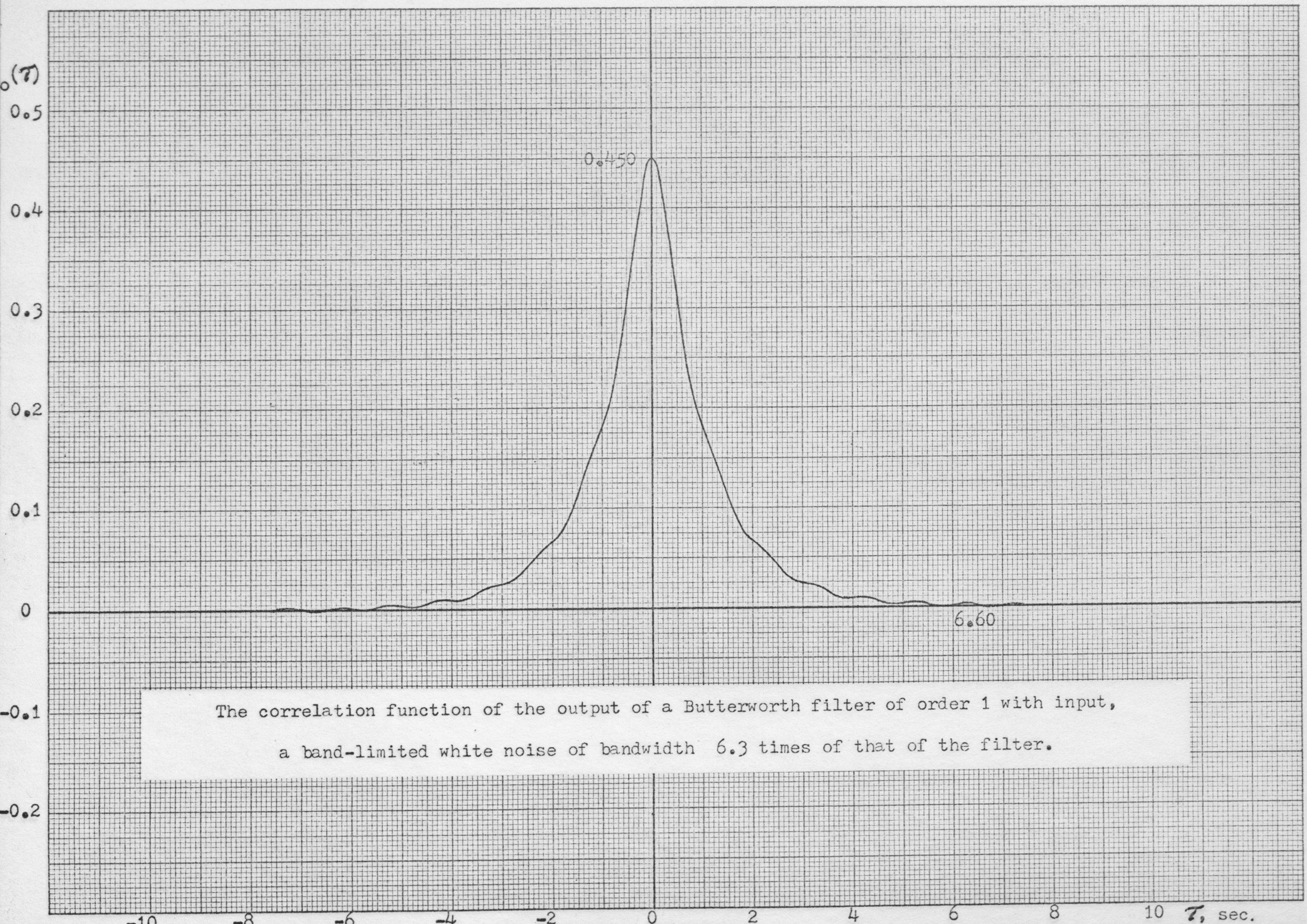
```
C           CALCULATION OF OUTPUT CORRELATION FUNCTION
C           FOR BAND-LIMITED WHITE NOISE INPUT
C     N = BUTTERWORTH FILTER ORDER
C     W = NORMALIZED FREQUENCY (WITH RESPECT TO FILTER BANDWIDTH)
C     WC = NORMALIZED NOISE CUTOFF FREQUENCY
C     FOR INPUT FORMAT, SEE STATEMENTS 101 AND 501
      REAL N
      COMMON TN
  501 FORMAT(2F10.4)
  601 FORMAT(1H1,3HN =,F4.0,5X,4HWC =,F10.4//6X,3HTAU,10X,2HRY//)
  602 FORMAT(2F12.4)
  101 READ(5,501)N,WC
      IF(N.EQ.0.0)STOP
      TN=N+N
      WRITE(6,601)N,WC
      DW=WC/90.0
      TAU=0.0
      DO 201 J=1,201
      W=0.0
      Y=HSQ(W)*COS(W*TAU)
      DO 202 I=1,44
      W=W+DW
      Y=Y+4.0*HSQ(W)*COS(W*TAU)
      W=W+DW
      Y=Y+2.0*HSQ(W)*COS(W*TAU)
  202 CONTINUE
      W=W+DW
      Y=Y+4.0*HSQ(W)*COS(W*TAU)
      W=W+DW
      Y=Y+HSQ(W)*COS(W*TAU)
      RY=Y*DW/(3.0*3.141592)
      WRITE(6,602)TAU,RY
      TAU=TAU+0.1
  201 CONTINUE
      GO TO 101
      END

C     FILTER TRANSFER FUNCTION SQUARED SUBPROGRAM
      FUNCTION HSQ(W)
      COMMON TN
      HSQ=1.0/(1.0+W**TN)
      RETURN
      END
```



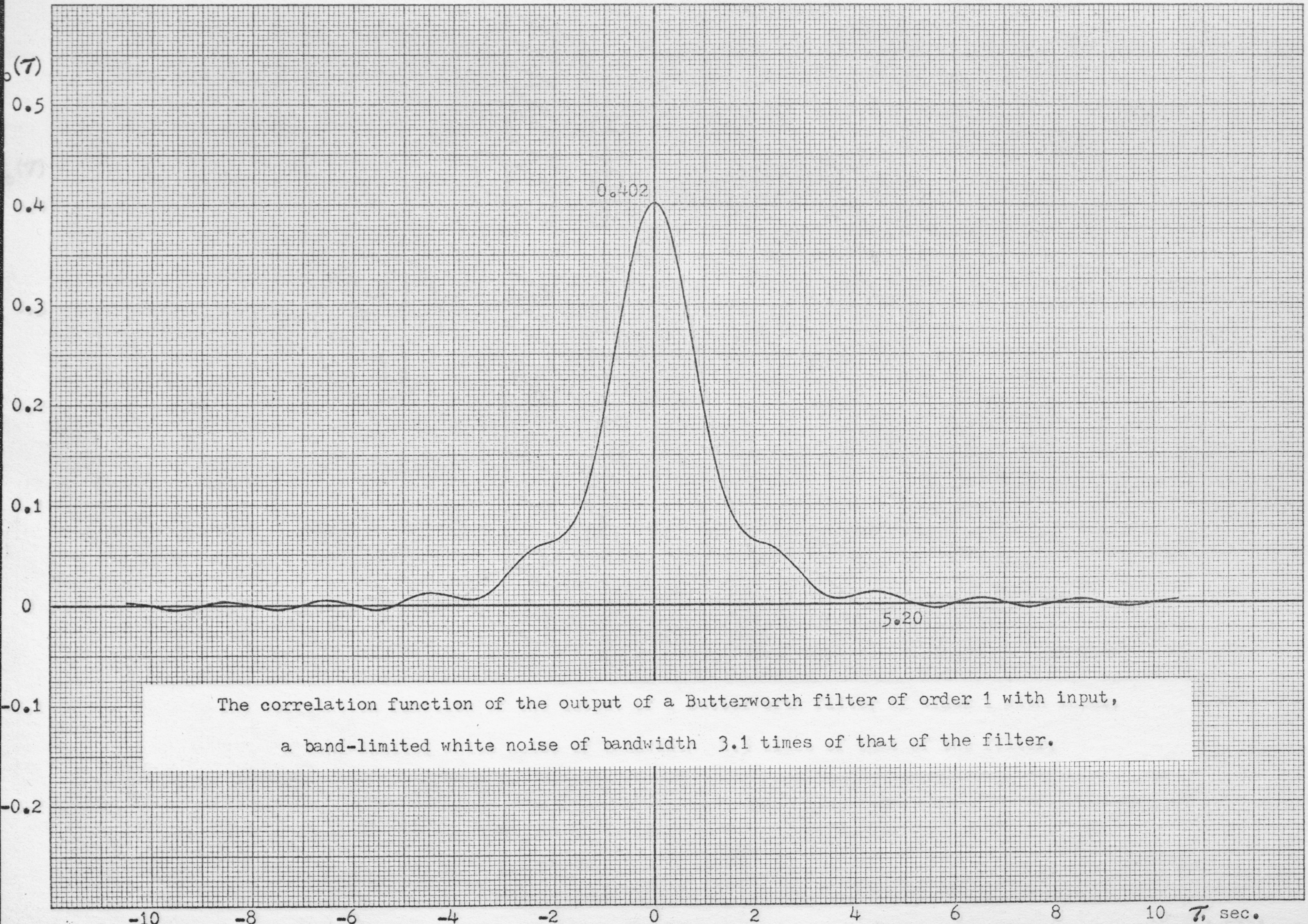
(τ)
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0.475
7.40
The correlation function of the output of a Butterworth filter of order 1 with input,
a band-limited white noise of bandwidth 12.6 times of that of the filter.
-10
-8
-6
-4
-2
0
2
4
6
8
10
τ, sec.

Figure A-1



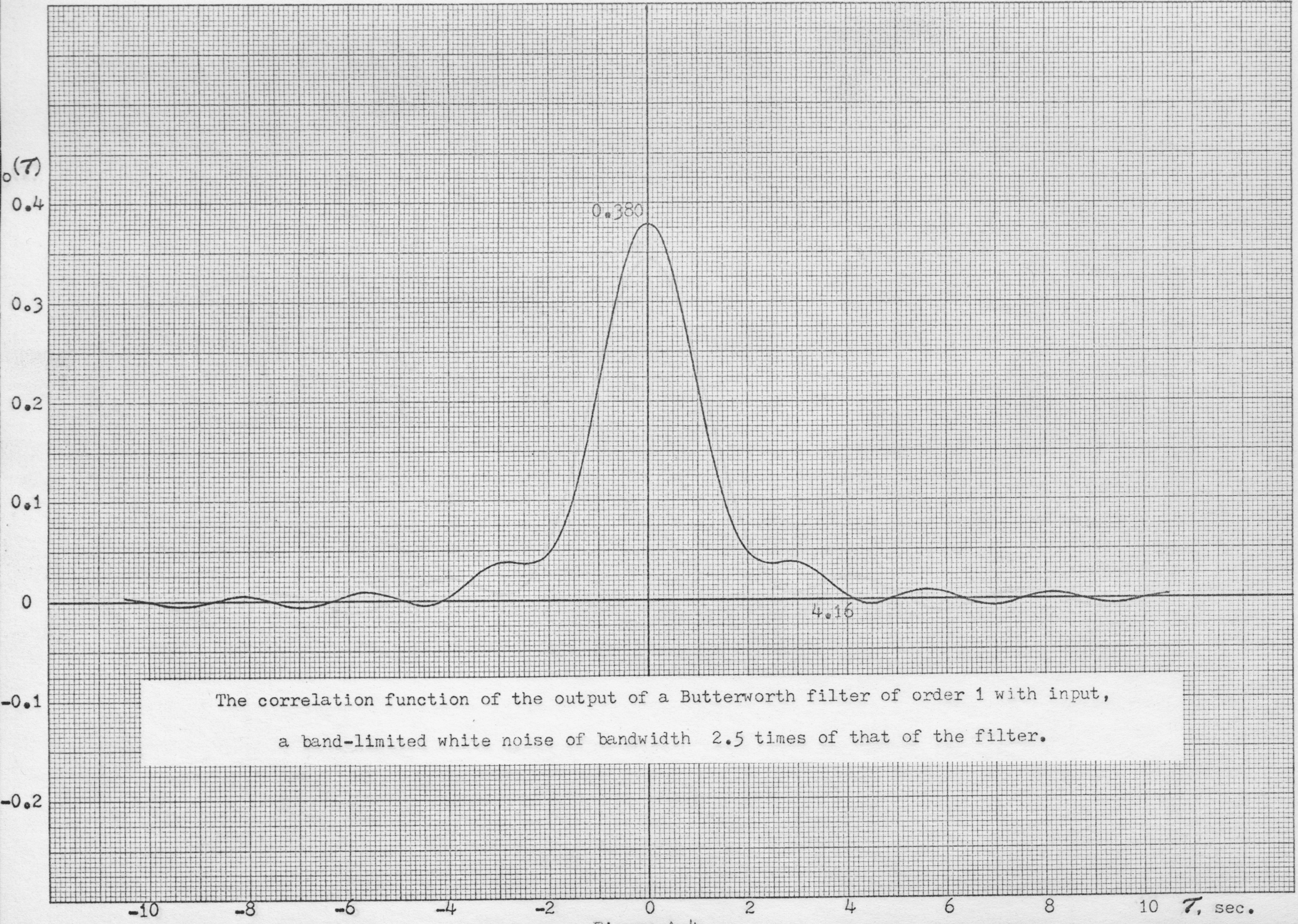
The correlation function of the output of a Butterworth filter of order 1 with input, a band-limited white noise of bandwidth 6.3 times of that of the filter.

Figure A-2



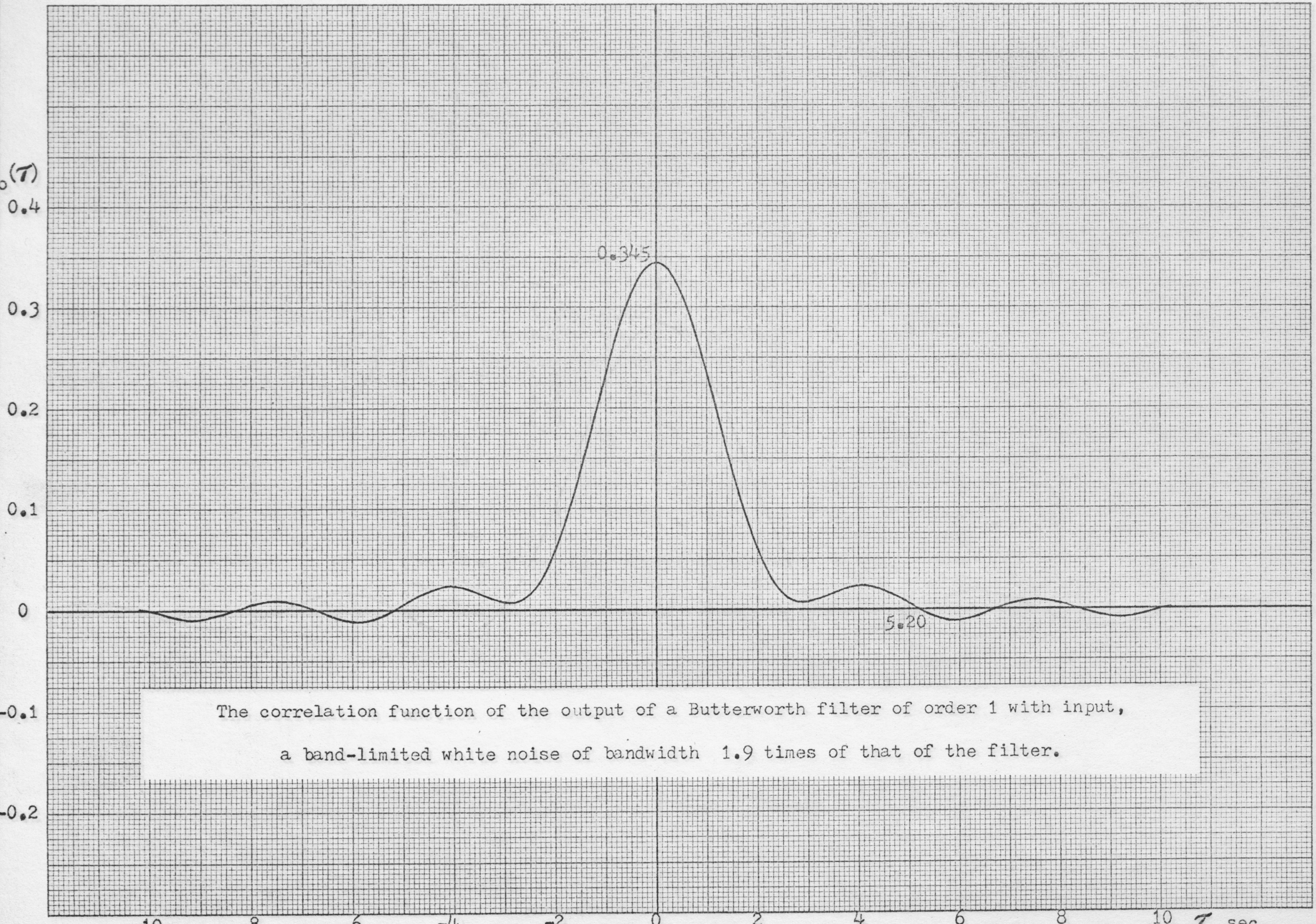
The correlation function of the output of a Butterworth filter of order 1 with input, a band-limited white noise of bandwidth 3.1 times of that of the filter.

Figure A-3



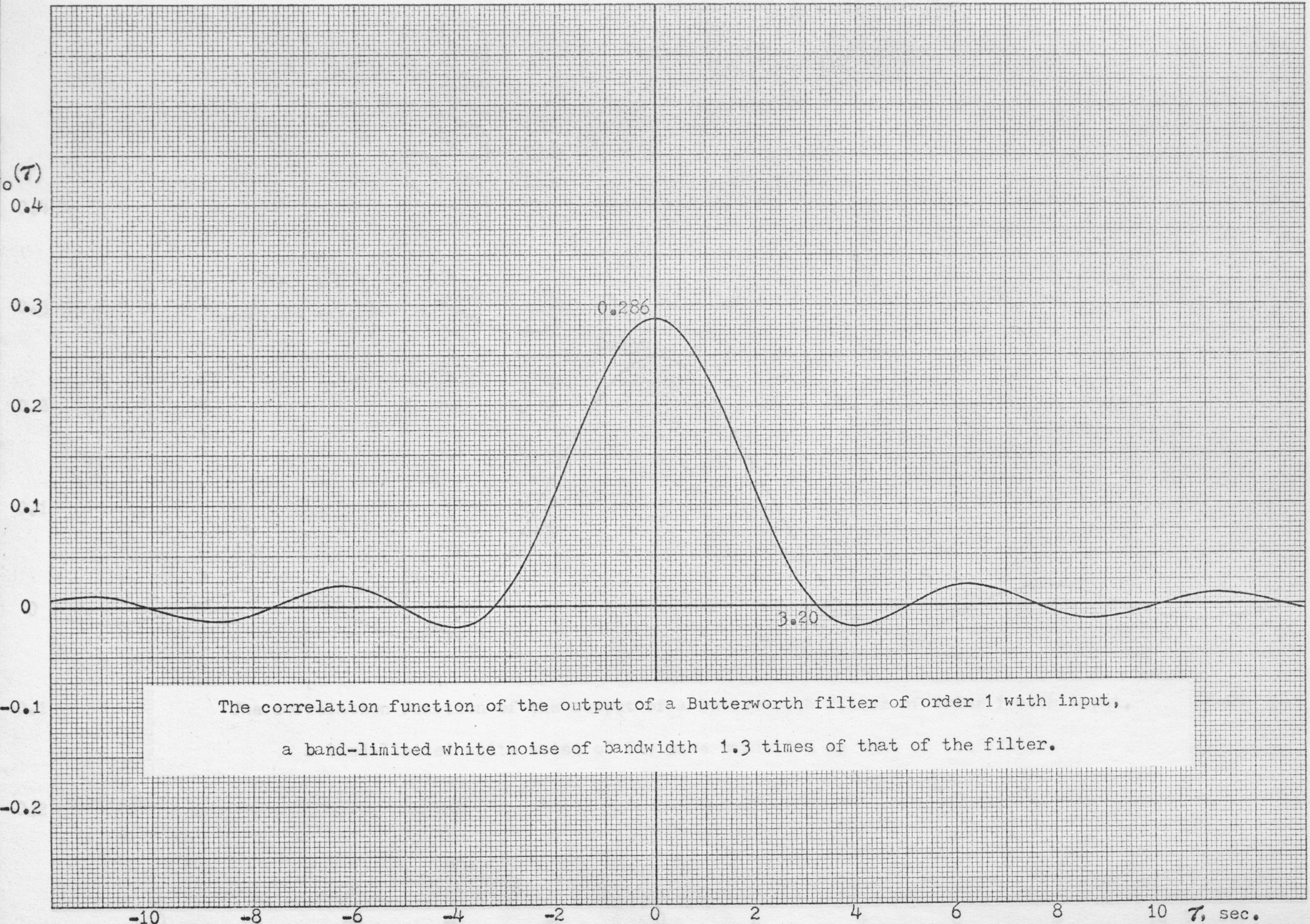
The correlation function of the output of a Butterworth filter of order 1 with input, a band-limited white noise of bandwidth 2.5 times of that of the filter.

Figure A-4



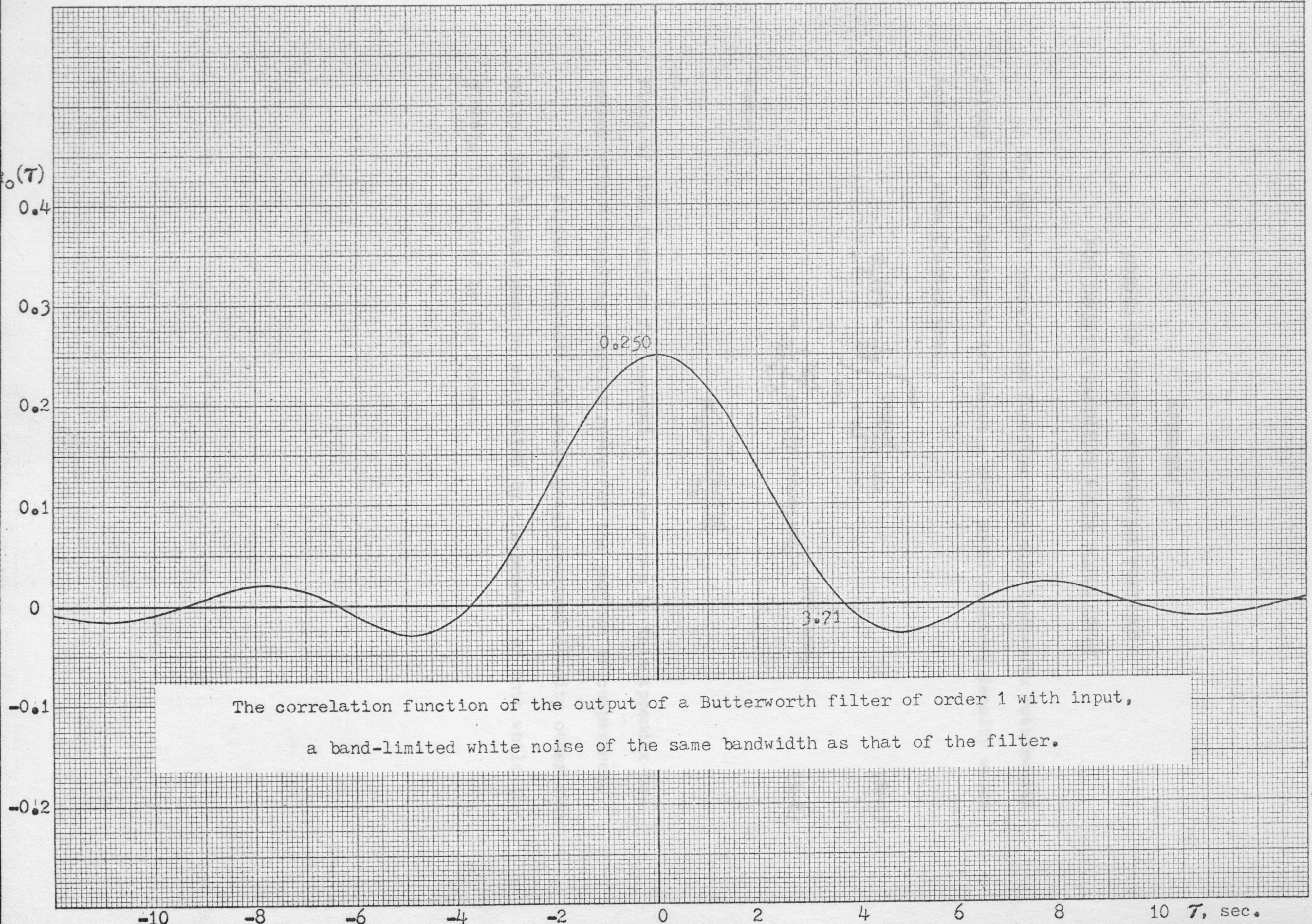
The correlation function of the output of a Butterworth filter of order 1 with input, a band-limited white noise of bandwidth 1.9 times of that of the filter.

Figure A-5



The correlation function of the output of a Butterworth filter of order 1 with input, a band-limited white noise of bandwidth 1.3 times of that of the filter.

Figure A-6



The correlation function of the output of a Butterworth filter of order 1 with input, a band-limited white noise of the same bandwidth as that of the filter.

Figure A-7

APPENDIX B

CURVES OF OUTPUT CORRELATION FUNCTION FOR INFINITE BANDWIDTH WHITE NOISE INPUT

The output correlation function of a unit bandwidth Butterworth filter order n having as it's input a unit power spectral density white noise is, from Chapter One,

$$R_y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\omega\tau}{1+\omega^{2n}}\, d\omega \tag{B-1}$$

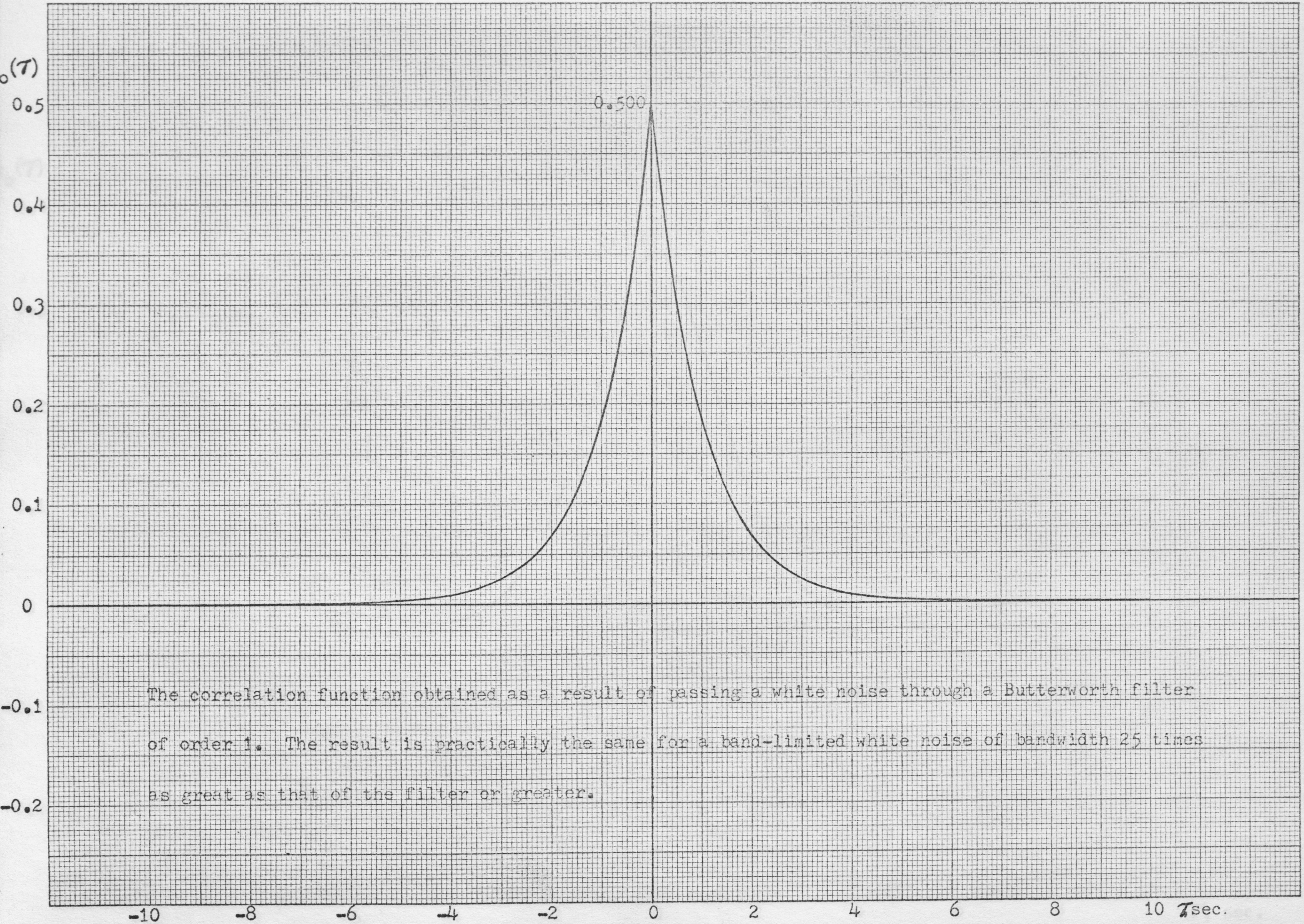
$$= \frac{1}{2n}\sum_{k=0}^{n-1} \sin(|\tau| \cos\alpha_k+\alpha_k)\, e^{-|\tau|\sin\alpha_k} \tag{B-2}$$

where

$$\alpha_k = \frac{(2k+1)\pi}{2n}$$

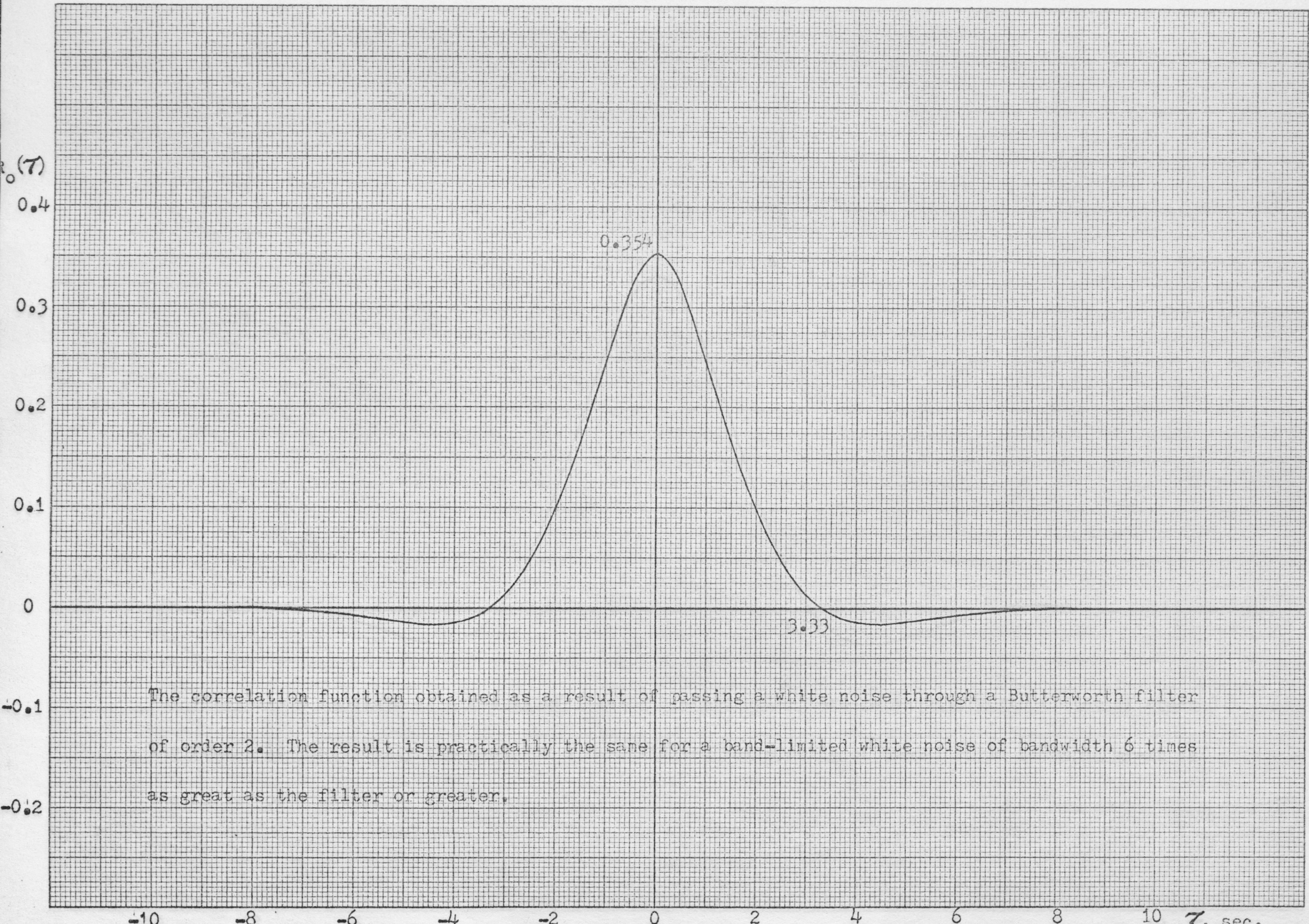
Plots of this function for several n's are shown in this appendix together with a sampling function curve for the purpose of comparison.

For filter bandwidth and noise power spectral density other than unity, the R scale must be multiplied by the product of both while the τ scale is divided by the former.



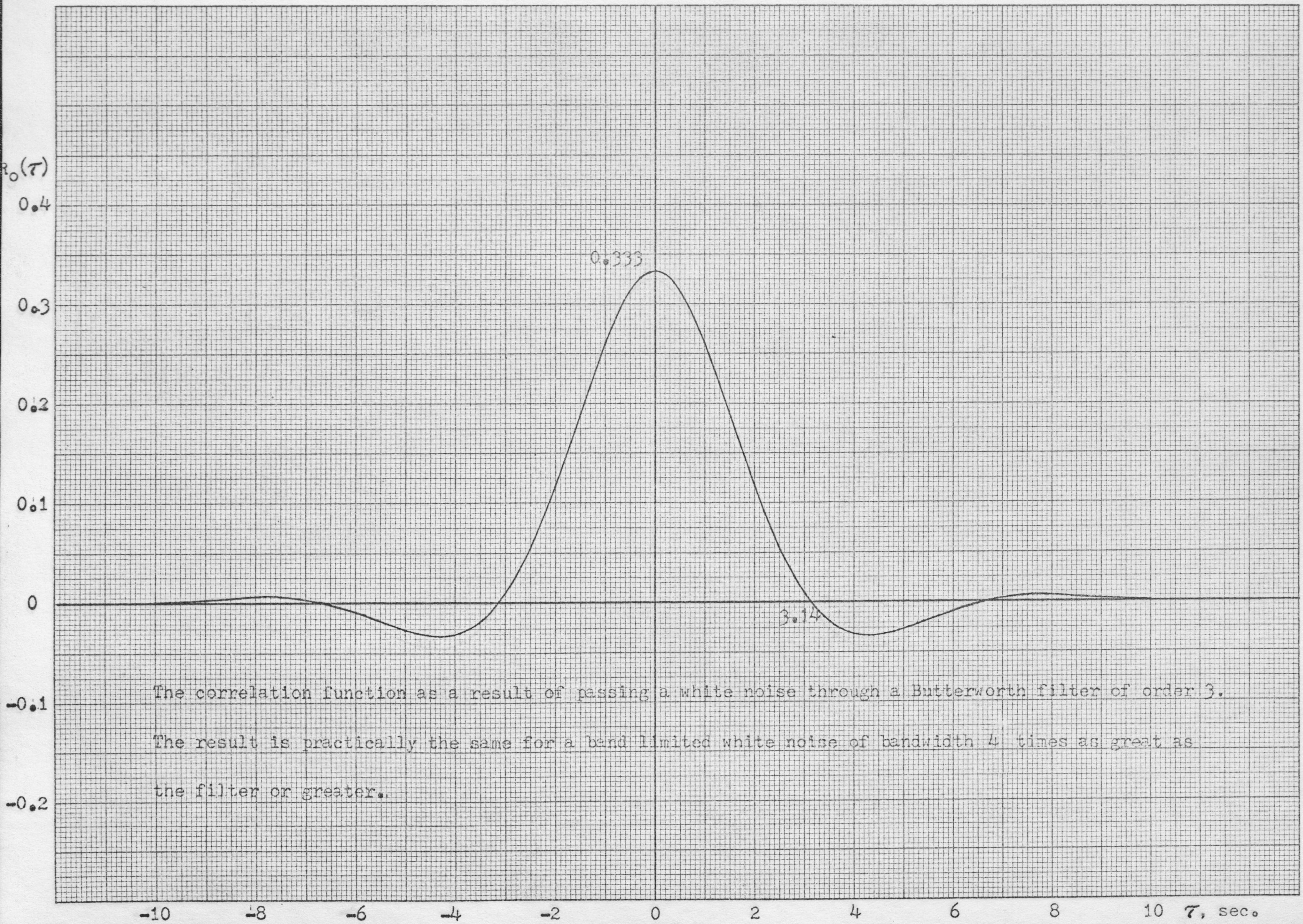
The correlation function obtained as a result of passing a white noise through a Butterworth filter of order 1. The result is practically the same for a band-limited white noise of bandwidth 25 times as great as that of the filter or greater.

Figure B-1



The correlation function obtained as a result of passing a white noise through a Butterworth filter of order 2. The result is practically the same for a band-limited white noise of bandwidth 6 times as great as the filter or greater.

Figure B-2



The correlation function as a result of passing a white noise through a Butterworth filter of order 3.

The result is practically the same for a band limited white noise of bandwidth 4 times as great as the filter or greater.

Figure B-3

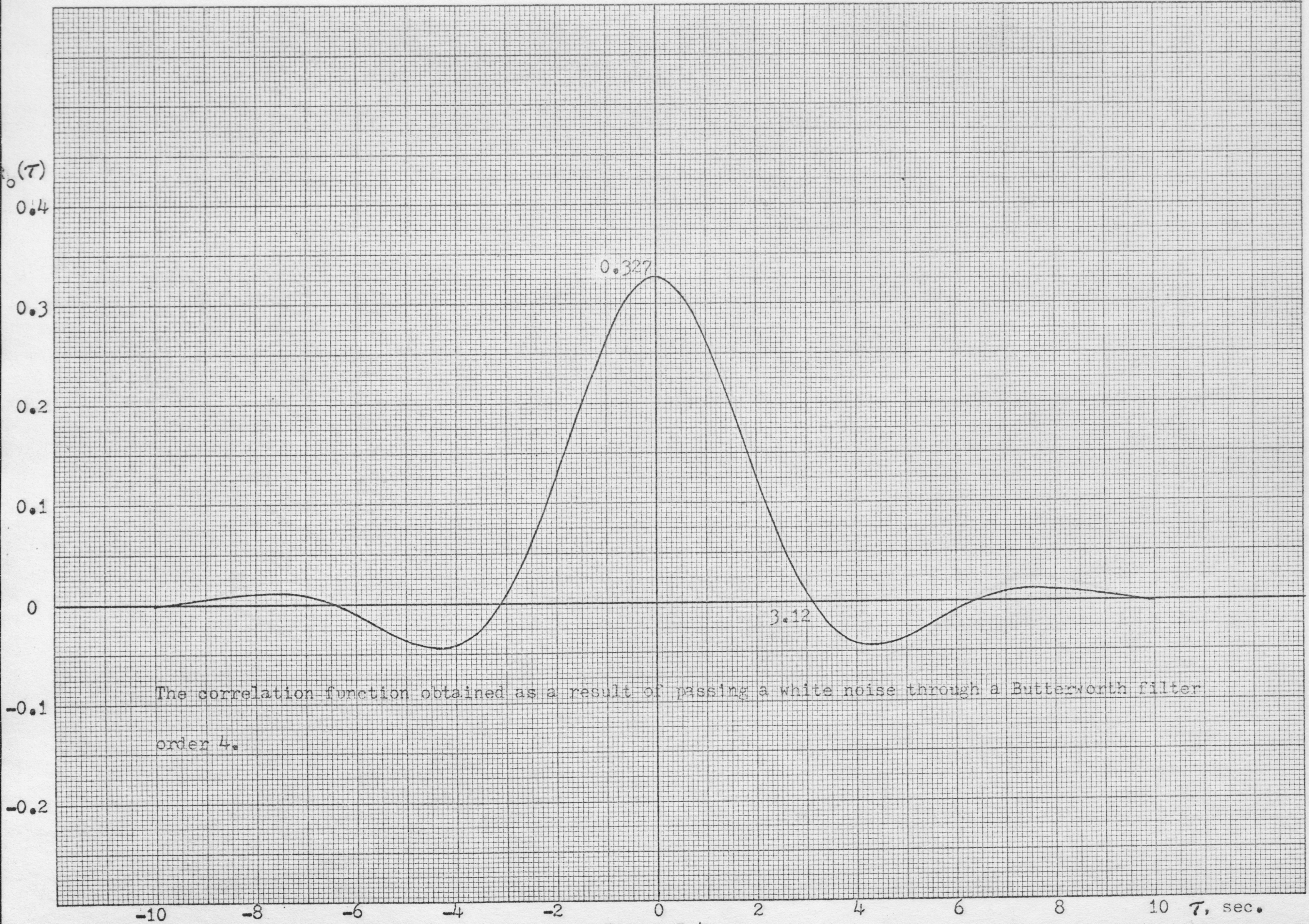

$R_o(\tau)$
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0.327
3.12
-10
-8
-6
-4
-2
0
2
4
6
8
10
τ, sec.
The correlation function obtained as a result of passing a white noise through a Butterworth filter order 4.

Figure B-4

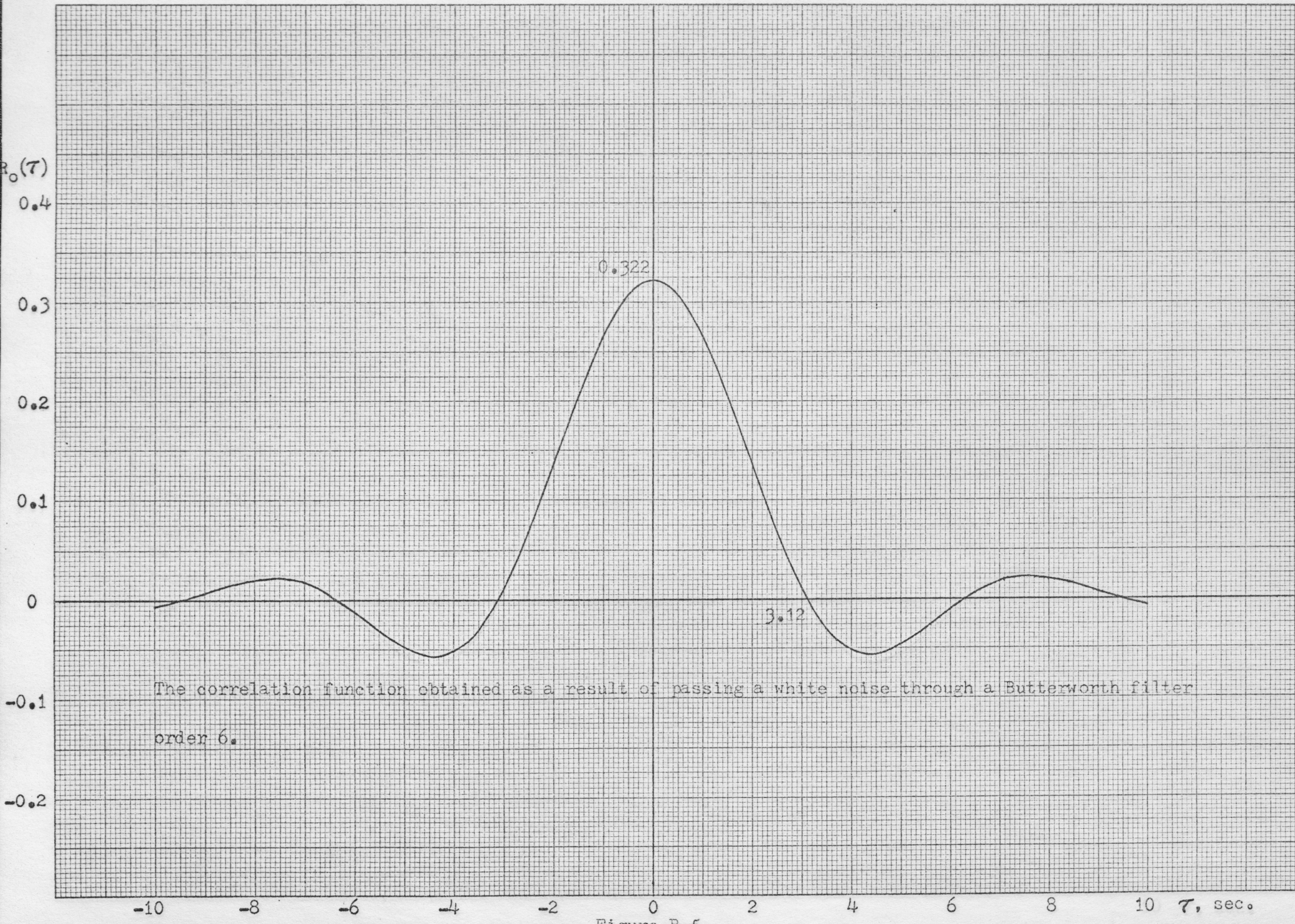

$R_o(\tau)$
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0.322
3.12
The correlation function obtained as a result of passing a white noise through a Butterworth filter order 6.
-10
-8
-6
-4
-2
0
2
4
6
8
10
τ, sec.

Figure B-5

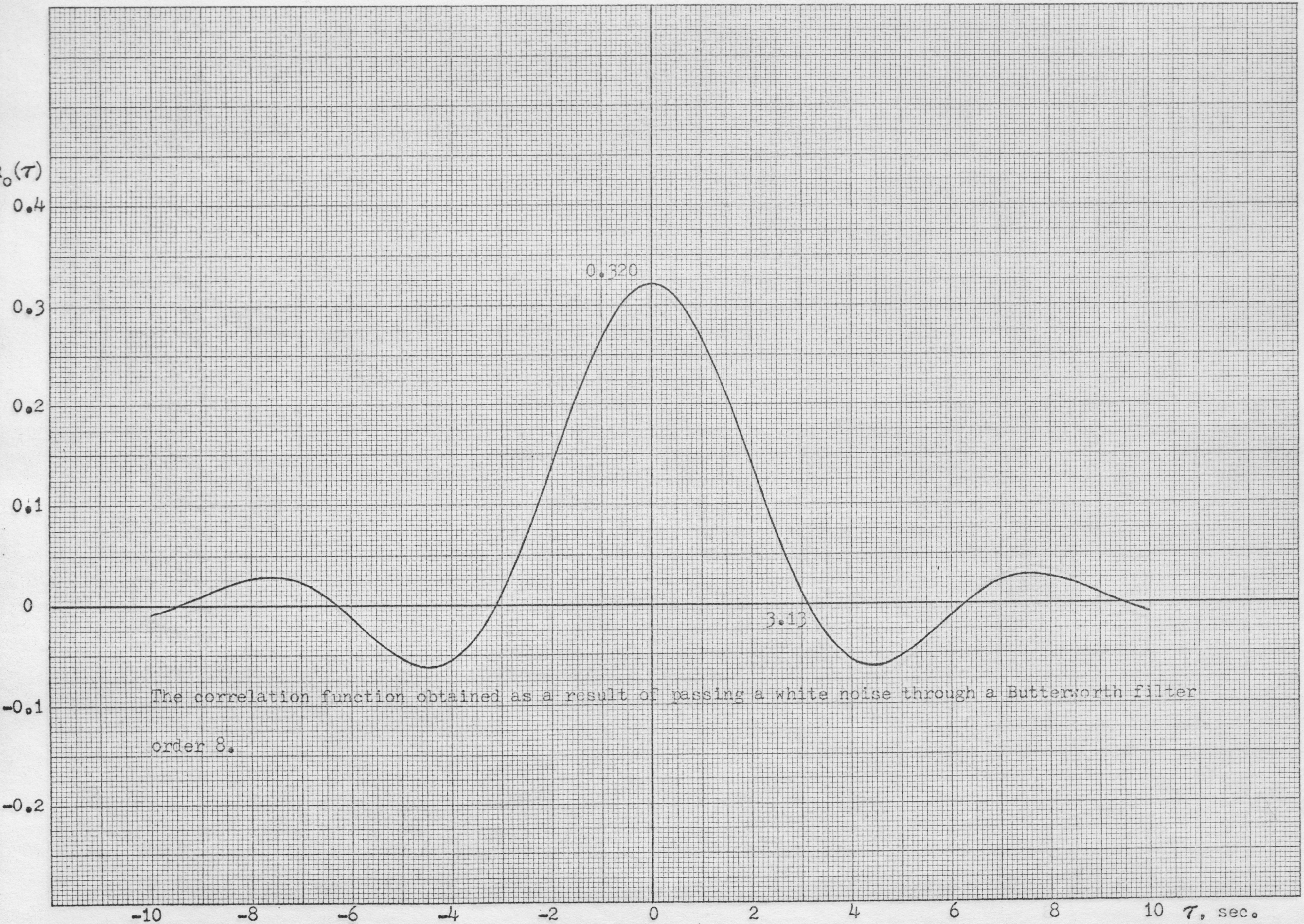
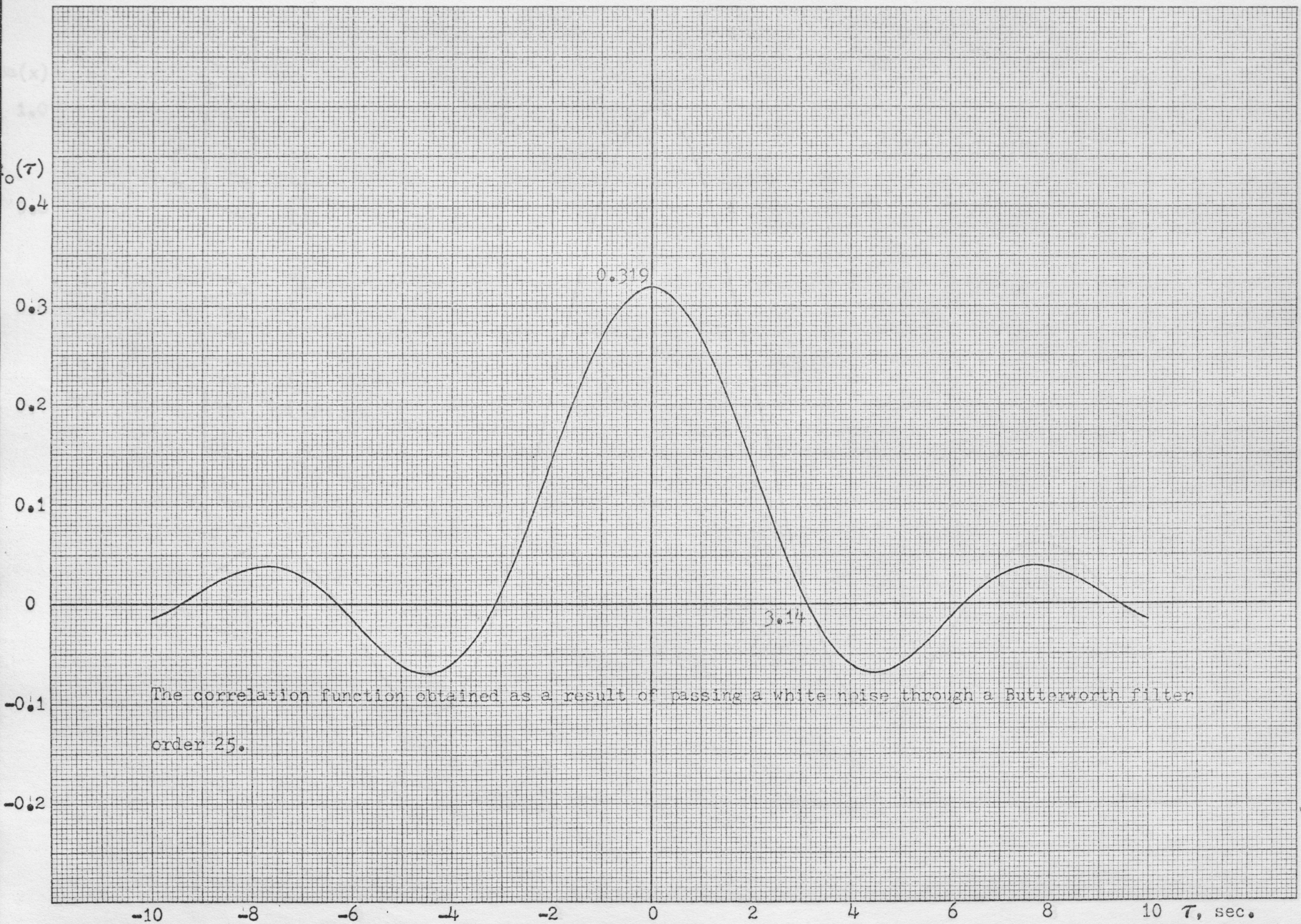

o(τ)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0.320
3.13
The correlation function obtained as a result of passing a white noise through a Butterworth filter order 8.
-10
-8
-6
-4
-2
0
2
4
6
8
10
τ, sec.

Figure B-6



The correlation function obtained as a result of passing a white noise through a Butterworth filter order 25.

Figure B-7

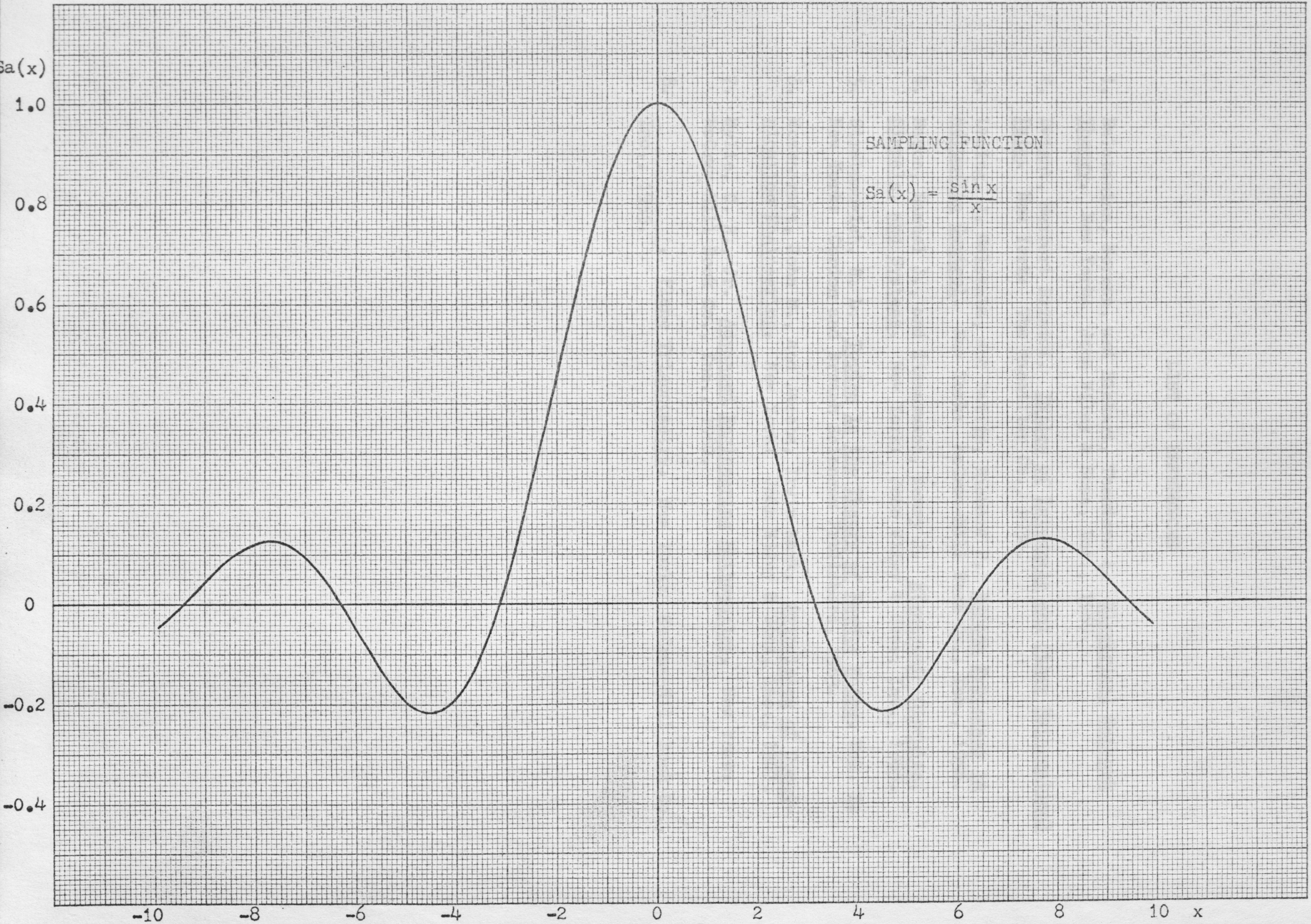

Sa(x)
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-10
-8
-6
-4
-2
0
2
4
6
8
10
x
SAMPLING FUNCTION
$Sa(x) = \frac{\sin x}{x}$

Figure B-8

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