CONTRIBUTIONS TO THE THEORY OF FUNCTIONS OF A BICOMPLEX VARIABLE

by

James D. Riley
A. B., Park College, 1942
A. M., University of Kansas, 1948

Submitted to the Department of Mathematics and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Advisory Committee:

Redacted Signature

Chairman

Redacted Signature

Redacted Signature

TABLE OF CONTENTS

INTRODUCTION

The customary axiomatic definition of the system of ordinary complex numbers may be given as follows: (see Dickson,(1)*)

Let a,b,c,d be real numbers. Two couples (a,b) and (c,d) are called equal if and only if a=c and b=d. Addition and multiplication of two couples are defined by the formulas:

$$(a,b)+(c,d) = (a+c,b+d)$$

$$(a,b)(c,d) = (ac-bd,ad+bc)$$

Addition and multiplication are commutative and associative, and the distributive law holds.

Subtraction is defined as the operation inverse to addition. It is always possible and unique.

Division is defined as the operation inverse to multiplication. Division, except by (0,0) is possible and unique:

$$\frac{(c,d)}{(a,b)} = \frac{ac+bd}{a^2+b^2}, \frac{ad-bc}{a^2+b^2}$$

Now let (a,0) be a, and (0,1) be i. Then

$$i^2 = (0,1)(0,1) = (-1,0) = -1$$

$$(a,b) = (a,0)+(0,b) = (a,0)+(b,0)(0,1) = a+bi$$

Thus the set of all real couples, with the above definitions, becomes the field of all complex numbers. The theory of complex-valued analytic functions of a complex variable has been extensively developed.

---

* Numbers refer to the bibliography at the end of the paper.

The question next arises as to what occurs if the above definitions are applied to couples of complex numbers, and the corresponding function theory investigated. This new system permits the same definition of the four fundamental operations, except that division will not be possible by the couple (a,b) if $a^2+b^2=0$. This occurs if $b=\pm ai$, and the system is therefore not a field. Furthermore, the product of the couples (a,ai) and (a,-ai) is $(a^2-a^2, -a^2i+a^2i)=(0,0)$. Thus nil-factors or divisors of zero occur. However, the system is a linear algebra.

The question in the above paragraph recently occurred to Prof. G. B. Price of the University of Kansas, independently of others. It was subsequently discovered that Futagawa, (2) and (3), has published two articles on the theory of functions of quadruples, which are equivalent to the couples of complex numbers considered above. Scorza (4) and Spampinato (5) each have presented results concerning a system equivalent to these couples of complex numbers except for notation. From the extensive literature concerning analytic functions on linear algebras in general, mention is made here only of the papers by Scheffers (6) and Ringleb (7) and several sections in a book by Hille (8). Takasu (9) has presented a theory of functions on an algebra which is a generalization of, and includes as a special case, the algebra presently being discussed. An article by Ward (10) includes an extensive bibliography which eliminates the necessity of including a complete set of references here.

A simplified notation is obtained by introducing a new unit $j=(0,1)$. Then

$$j^2=(0,1)(0,1)=(-1,0)=-1.$$

The couple $(a,b)=(a,0)+(0,b)=(a,0)+(b,0)(0,1)=a+bj$ will be termed a bicomplex number. This number may also be written as a real linear combination of the four units 1, i, j, ij. A geometric interpretation is afforded by the four-dimensional Euclidean space.

By squaring the numbers $\frac{1}{2}(1+ij)$ and $\frac{1}{2}(1-ij)$ it is found that they are idempotent elements. A result of Scheffers (see Dickson (1) p. 26-27) then states that this system is reducible. In fact the numbers $e_1=\frac{1}{2}(1+ij), e_2=ie_1$, $e_3=\frac{1}{2}(1-ij), e_4=ie_3$ form a basis if real coefficients are used, and $e_1e_3=e_1e_4=e_2e_3=e_2e_4=0$. Then if complex coefficients are permitted, e_1 and e_3 alone form the basis. The bicomplex number $a+bj$ is uniquely represented as $(a-bi)e_1+(a+bi)e_3$.

Now consider the bicomplex variable $z=x+jy$, x and y complex. Then $z=(x-iy)e_1+(x+iy)e_3$. For convenience let $x-iy=z_1$ and $x+iy=z_3$. Then $z=z_1e_1+z_3e_3$. A fundamental result of Ringleb (7) (which he proves for reducible linear algebras in general) then states that an analytic function $f(z)$ (the analyticity of a function of a bicomplex variable will be defined in section I.) can be decomposed uniquely into the sum of functions analytic in the separate sub-algebras, i.e., $f(z)=g(z_1)e_1+h(z_3)e_3$, where $g(z_1)$ is an analytic function of z_1 and $h(z_3)$ is an analytic function of z_3, and that conversely if $g(z_1)$ is an analytic function of z_1 and $h(z_3)$ is an analytic function of z_3, then $f(z)=g(z_1)e_1+h(z_3)e_3$ is an analytic function of z. Here $f(z)$ takes bicomplex values, while $g(z_1)$ and $h(z_3)$ take only complex values. Prof. Price independently discovered and orally communicated this result for the bicomplex case. His proof uses power series and integration. A proof directly based on the definition of an analytic function and not using differentials, as Ringleb's proof does, will be given in section I. This provides a powerful method for the study of analytic functions of a bicomplex variable. This decomposition actually occurs in Futagawa's work for the special case where $f(z)$ is sin z or cos z, but he makes no use of the decomposition.

Let $z=z_1e_1+z_3e_3$ and $w=w_1e_1+w_3e_3$. Since $e_1e_3=0$, $e_1^2=e_1$, and $e_3^2=e_3$, $zw=z_1w_1e_1+z_3w_3e_3$. Thus $zw=0$ if and only if $z_1w_1=0$, and $z_3w_3=0$. Thus the product of two non-zero numbers is zero if and only if one of them is a complex multiple of e_1 and the other is a complex multiple of e_3. Note also that $ze_1=z_1e_1$ and $ze_3=z_3e_3$. Thus it will suffice to say multiple instead of complex multiple. The set of numbers which are multiples of e_1 will be termed the first nil-plane. Similarly the set of numbers which are multiples of e_3 will be termed the second nil-plane.* A non-zero number which is a multiple of e_1 will be termed a first nil-factor and a non-zero number which is a multiple of e_3 will be termed a second nil-factor. By these conventions, the origin belongs to both nil-planes, but is not a nil-factor.

The elementary functions have been discussed by Futagawa. However they may well be defined by the formula $f(z)=f(z_1)e_1+f(z_3)e_3$,

---

* See Futagawa (2) for a geometric interpretation of the nil-planes.

where, in the right member, f denotes the elementary function whose generalization to bicomplex values is desired, since for z complex, $z = ze_1 + ze_3$, and thus $z_1 = z_3 = z$. In fact, this formula provides a natural way of extending every complex-valued function of a complex variable into the bicomplex space.

Two immediate generalizations to the bicomplex case of the concept of absolute value of a complex number will be employed extensively. They are the norm of $z = x + jy$, denoted by $\|z\|$, and defined as $\|z\| \equiv \sqrt{|x|^2 + |y|^2}$ and the absolute value of z, denoted by $|z|$, and defined as $|z| \equiv \sqrt{|x^2+y^2|}$. The norm of z is readily seen to be the Euclidean distance norm, and thus satisfies the properties required of a norm. The absolute value of z does not satisfy the triangle inequality and is zero for the class of numbers by which division is not permitted, i,e., when z is zero or a nil-factor. (This absolute value is the first modulus in Futagawa's polar representation of z.) It is frequently convenient to express $|z|$ and $\|z\|$ in terms of $|z_1|$ and $|z_3|$. Thus

$$|z| = \sqrt{|x^2+y^2|} = \sqrt{(x-iy)(x+iy)} = \sqrt{|z_1 z_3|} = \sqrt{|z_1| \cdot |z_3|}$$

Then $|z| \neq 0$ and division by z is possible if and only if z_1 and z_3 are both non-zero. In the representation of the bicomplex number system based on its reducibility, division by z takes a particularly simple form, since

$$\frac{1}{z} = \frac{1}{z_1 e_1 + z_3 e_3} = \frac{1}{z_1 e_1 + z_3 e_3} \cdot \frac{\frac{1}{z_1} e_1 + \frac{1}{z_3} e_3}{\frac{1}{z_1} e_1 + \frac{1}{z_3} e_3} = \frac{\frac{1}{z_1} e_1 + \frac{1}{z_3} e_3}{e_1 + e_3} = \frac{1}{z_1} e_1 + \frac{1}{z_3} e_3$$

The identity

$$\|z\| \equiv \frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |z_3|^2}$$

is easily verified by expressing both members in terms of the four real components of z, or somewhat more conveniently by employing polar coordinates in the z_1- and z_3- planes.

Note that if z is in a nil-plane, say the first, then $\|z\| = \frac{1}{\sqrt{2}}|z_1|$, so that distance in a nil-plane, if measured by the absolute value of the complex number z_1, differs from distance in the bicomplex space by a constant factor.

Note also that if $z = x + jy$ is a complex number, so that $y = 0$ and $z = x$, then both $|z|$ and $\|z\|$ are equal to $|x|$.

The first step in the main part of this paper will be a proof of the previously mentioned decomposition theorem due to Ringleb, specialized to the bicomplex case.

The second section will concern power series and their region of convergence. A new norm,

$$N(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z|^4}}$$

which is especially suited to describe the region of convergence for a power series, after appropriate normalization, will be presented. An improved form of the generalized Taylor's theorem will be developed.

The third section will discuss singularities and zeros. The implications from inequalities resembling Cauchy's inequality in the complex case and involving the norms and absolute values will be investigated, with results which in certain cases differ in an interesting way from what might be expected on the basis oa a purely formal analogy with the complex case.

Section four will generalize Cauchy's theorem and Cauchy's integral formula to the bicomplex case. Cauchy's theorem occurs in Futagawa (2) and in Prof. Price's work, as well as in the literature on more general algebras. Cauchy's integral formula likewise occurs in Futagawa (2) for a curve in a plane parallel to the plane of complex numbers. Prof. Price has also established Cauchy's formula for a class of space curves. In some cases the formula has $\frac{1}{2\pi i}$ as the coefficient and in other cases $\frac{1}{2\pi j}$. The present paper will present a criterion to distinguish between these cases for plane curves.

The fifth section answers in the negative, by a simple application of the decomposition theorem, the following question: Given in the complex plane a complex-valued analytic function $f(z)$, whose natural boundary is a simple closed curve Γ, is it possible, by extending the definition of $f(z)$ to bicomplex values of z, to obtain an analytic continuation of $f(z)$, along a path in the bicomplex space, to points in the complex plane outside of Γ? An affirmative answer to what appears to be the same question is given by Futagawa (3), pp. 80-120.

The sixth section will present some rather direct extensions of well-known results from the complex function theory, pointing out some differences that occur.

The last section will consider a generalized bicomplex variable defined by Takasu (9), to show that some of the included cases are interestingly different and that a special study of them is needed to supplement the unified treatment, especially since the decomposition theorem of Ringleb is not available in a theory including irreducible cases.

This paper was written as a Ph. D. thesis at the University of Kansas under the supervision of Prof. V. Wolontis and many of the problems and numerous changes have been suggested by him.

I. ANALYTIC FUNCTIONS - DECOMPOSITION

Analyticity will now be defined and the decomposition theorem proved. The definition and the first part of the proof bear considerable similarity to the corresponding definition and the derivation of the Cauchy-Riemann equations in the theory of functions of a complex variable. It will also be discovered that differentiation of an analytic function with respect to z will be equivalent to differentiation of the separate components with respect to their respective variables, z_1 and z_3.

Throughout this paper, the topological concepts employed for sets of bicomplex numbers will be those of four-dimensional Euclidean space. For example, a set of points S will be called open if for every z_0 in S there exists a $K>0$ such that every z for which $\|z-z_0\|<K$ is also in S. An open connected set will be called a region. The set of all bicomplex numbers with this topology will be called the bicomplex space. If T is a region, and if each z in T is written in the form $z=z_1e_1+z_3e_3$, (where $e_1=\frac{1}{2}(1+ij), e_3=\frac{1}{2}(1-ij)$, see pages 3-4), then the set T_1 of values of z_1 is a region in the z_1- plane (in the topology of that plane) and the set T_3 of values of z_3 is a region in the z_3- plane. These regions T_1 and T_3 will be termed the component regions of T. If the regions T_1 and T_3 are given, the largest region T whose component regions are T_1 and T_3 will be termed the product-region of T_1 and T_3.

It should be observed that for convenience the regions T_1 and T_3 have been chosen in the complex z_1- and z_3- planes, which are not planes of the bicomplex space. If component-regions in the space itself are desired, the components z_1e_1 and z_3e_3 of the number z, located in the first and second nil-planes, respectively, should be considered.

Let $z_0 = x_0 + jy_0$ be a bicomplex number. The bicomplex variable $z = x + jy$ will be said to approach z_0, and z_0 will be termed the limit of z if $\|z - z_0\|$ approaches zero. It may be verified that z approaches z_0 if and only if x approaches x_0 and y approaches y_0.

Definition: Let f(z) be a bicomplex-valued function of the bicomplex variable $z = x + jy$, defined in a region T. Let z_0 be a point in T. Then f(z) will be termed analytic at z_0 if and only if there exists a bicomplex number $f'(z_0)$ such that for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that

$$\left\| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right\| < \varepsilon$$

whenever $\|z - z_0\| < \delta_\varepsilon$ and $|z - z_0| \neq 0$.

Definition: A function f(z) will be termed analytic in a region T if it is analytic at each point of T.

Theorem: (Decompostion theorem of Ringleb)

Let f(z) be analytic in a region T, and let T_1 and T_3 be the component regions of T, in the z_1- and z_3- planes, respectively. Then there exists a unique pair of complex-valued analytic functions, $g(z_1)$ and $h(z_3)$, defined in T_1 and T_3, respectively, such that

$$\text{(A)} \qquad f(z) = g(z_1)e_1 + h(z_3)e_3$$

for all z in T. Conversely, if $g(z_1)$ is any complex-valued analytic function in a region T_1 and $h(z_3)$ any complex-valued analytic function in a region T_3, then the bicomplex-valued

<u>function $f(z)$ defined by the formula (A) is an analytic function of the bicomplex variable z in the product-region T of T_1 and T_3.</u>

Proof: Let $f(z)=u(x,y)+jv(x,y)$. Let $z_0=x_0+jy_0$ be an arbitrary point in T, and let z approach z_0 in such a way that y is always equal to y_0, i.e., $z=x+jy_0$. Then $z-z_0=x-x_0$, hence the assumption that $|z-z_0|\neq 0$ is satisfied for all $x\neq x_0$, and

$$\frac{f(z)-f(z_0)}{z-z_0}=\frac{[u(x,y_0)+jv(x,y_0)]-[u(x_0,y_0)+jv(x_0,y_0)]}{x-x_0}=$$

$$=\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0}+j\,\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}.$$

This tends to a limit if and only if each term tends separately to a limit. But this means simply that the complex-valued functions u and v of the two complex variables x and y possess partial derivatives with respect to x, $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$, at $x=x_0, y=y_0$, and that

$$f'(z_0)=\left(\frac{\partial u}{\partial x}+j\frac{\partial v}{\partial x}\right)_{x=x_0,\,y=y_0}$$

Similarly if z approaches z_0 so that $z-z_0=(y-y_0)j$, i.e., $z=x_0+jy$, it is found that $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ exist at $x=x_0$, $y=y_0$ and

$$f'(z_0)=\left(\frac{\partial v}{\partial y}-j\frac{\partial u}{\partial y}\right)_{x=x_0,\,y=y_0}$$

Comparison of the two expressions for $f'(z_0)$ gives

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

for any point $z_0=x_0+jy_0$ of T. These are the generalized Cauchy-Riemann equations. Now, using the representation mentioned on pages 3-4

$$w = f(z) = u + jv = (u - iv)e_1 + (u + iv)e_3$$

$$z = x + jy = (x - iy)e_1 + (x + iy)e_3$$

and with the notation

$$w_1 = u - iv \qquad w_3 = u + iv$$

$$z_1 = x - iy \qquad z_3 = x + iy$$

then $w = w_1 e_1 + w_3 e_3$ and $z = z_1 e_1 + z_3 e_3$.

Then since the partial derivatives of u and v with respect to x and y exist in T, the partial derivatives of w_1 and w_3 with respect to x and y exist in T and

$$\frac{\partial w_1}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}, \quad \frac{\partial w_1}{\partial y} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}$$

$$\frac{\partial w_3}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \frac{\partial w_3}{\partial y} \quad \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Using the generalized Cauchy-Riemann equations

$$\frac{\partial w_1}{\partial y} = -i\frac{\partial w_1}{\partial x}; \quad \frac{\partial w_3}{\partial y} = i\frac{\partial w_3}{\partial x}$$

Also since $x = \frac{1}{2}(z_1 + z_3)$ and $y = \frac{i}{2}(z_1 - z_3)$

$$\frac{\partial x}{\partial z_1} = \frac{\partial x}{\partial z_3} = \frac{1}{2}, \quad \frac{\partial y}{\partial z_1} = \frac{i}{2}, \quad \frac{\partial y}{\partial z_3} = -\frac{i}{2}$$

Since w_1 and w_3 are analytic functions of x and y, and x and y are analytic functions of z_1 and z_3, for any point $z_0 = x_0 + jy_0$ of T, w_1 and w_3 are analytic functions of z_1 and z_3. Further

$$\frac{\partial w_1}{\partial z} = \frac{\partial w_1}{\partial x}\frac{\partial x}{\partial z_1} + \frac{\partial w_1}{\partial y}\frac{\partial y}{\partial z_1} \quad \frac{\partial w_1}{\partial x}\frac{1}{2} - i\frac{\partial w_1}{\partial x}\frac{i}{2} = \frac{\partial w_1}{\partial x}$$

$$\frac{\partial w_1}{\partial z_3} = \frac{\partial w_1}{\partial x}\frac{\partial x}{\partial z_3} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z_3} = \frac{\partial w_1}{\partial x}\frac{1}{2} - i\frac{\partial w_1}{\partial x}\left(-\frac{i}{2}\right) = 0$$

$$\frac{\partial w_3}{\partial z_1} = \frac{\partial w_3}{\partial x}\frac{\partial x}{\partial z_1} + \frac{\partial w_3}{\partial y}\frac{\partial y}{\partial z_1} = \frac{\partial w_3}{\partial x}\frac{1}{2} + i\frac{\partial w_3}{\partial x}\cdot\frac{i}{2} = 0$$

$$\frac{\partial w_3}{\partial z_3} = \frac{\partial w_3}{\partial x}\frac{\partial x}{\partial z_3} + \frac{\partial w_3}{\partial y}\frac{\partial y}{\partial z_3} = \frac{\partial w_3}{\partial x}\frac{1}{2} + i\frac{\partial w_3}{\partial x}\left(-\frac{i}{2}\right) = \frac{\partial w_3}{\partial x}$$

Since these equations hold at all points of the region T, it follows that w_1 is an analytic function of z_1 alone and w_3 is an analytic function of z_3 alone, in the region T. Then placing $w_1(z_1)=g(z_1)$ for z_1 in T_1 and $w_3(z_3)=h(z_3)$ for z_3 in T_3,

$$f(z)=g(z_1)e_1+h(z_3)e_3$$

and the representation is unique, since w_1 is uniquely determined as u-iv and w_3 is uniquely determined as u+iv.

Conversely if $g(z_1)$ is an analytic function of z_1 in a region T_1 of the z_1- plane and $h(z_3)$ is an analytic function of z_3 in a region T_3 of the z_3- plane, then $g(z_1)e_1+h(z_3)e_3$ is defined as a function of $z=z_1e_1+z_3e_3$ in the product-region of the bicomplex space having the components T_1 and T_3. Denoting this function by f(z) then, if $z_0=z_1^\circ e_1+z_3^\circ e_3$ is a point of T, and z a point of T for which $|z-z_0|\neq 0$,

$$\frac{f(z)-f(z_0)}{z-z_0}=\frac{g(z_1)-g(z_1^\circ)}{z_1-z_1^\circ}e_1+\frac{h(z_3)-h(z_3^\circ)}{z_3-z_3^\circ}e_3$$

Since the right side approaches a limit as $z_1\rightarrow z_1^\circ$, $z_1\neq z_1^\circ$, and $z_3\rightarrow z_3^\circ$, $z_3\neq z_3^\circ$, then the left side approaches a limit as $z\rightarrow z_0$ and $|z-z_0|\neq 0$, since $|z-z_0|=0$ if and only if $z_1=z_1^\circ$ or $z_3=z_3^\circ$.

<u>Corollary 1: Let f(z) be analytic in a region T which intersects the complex plane. Let S be a set of points in the intersection of T and the complex plane and let S have a limit point in this intersection. Suppose that for all z in S, f(z) assumes complex values. Then f(z) assumes complex values for every z in the intersection of T and the complex plane and f(z) may be defined for every value z in the components T_1 and T_3 of T</u>

so that the Ringleb decomposition formula becomes

$$f(z)=f(z_1)e_1+f(z_3)e_3$$

for all $z=z_1e_1+z_3e_3$ in T.

Proof: By the Ringleb decomposition theorem $f(z)=g(z_1)e_1+h(z_3)e_3$ for z in T. For $z=x+jy$ in S, z is complex and $y=0$. Then $z_1=x-iy=x$, $z_3=x+iy=x$, and $z=z_1=z_3=t$, where t is a new complex variable introduced for convenience. For each z in S, f(z) is a complex number, and thus $g(z_1)=f(z)$ and $h(z_3)=f(z)$, or $f(t)=g(t)=h(t)$. Now S has a limit point in the complex plane. Thus $g(t)\equiv h(t)$. Thus $g(z_1)=h(z_3)$ whenever $z_1=z_3$. But $z_1=z_3$ for all z in the complex plane. Thus if z has a value t in the intersection of T and the complex plane, $f(t)=$ $=g(t)e_1+g(t)e_3=g(t)$ or $f(z)=f(z)e_1+f(z)e_3$, and f(z) assumes complex values there.

Now for every value of z_1 in T_1 for which $f(z_1)$ is not already defined (recall that the complex values for which f(z) is defined are those of T; compare pages 9-10), define $f(z_1)=g(z_1)$; and for every value of z_3 in T_3 for which $f(z_3)$ is not already defined, define $f(z_3)=h(z_3)$. Then $f(z_1)$ is an analytic function of z_1 in T_1 and $f(z_3)$ is an analytic function of z_3 in T_3. Thus $f(z)=f(z_1)e_1+f(z_3)e_3$ for z in T.

Corollary 2.

$$\frac{df}{dz}=\frac{dg}{dz_1}e_1+\frac{dh}{dz_3}e_3$$

Proof: By direct substitution of results obtained in the course of proving the theorem

$$\frac{df}{dz} = f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)e_1 + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)e_3 =$$

$$= \frac{\partial w_1}{\partial x}e_1 + \frac{\partial w_3}{\partial x}e_3 = \frac{\partial w_1}{\partial z_1}e_1 + \frac{\partial w_3}{\partial z_3}e_3 = \frac{dg}{dz_1}e_1 + \frac{dh}{dz_3}e_3$$

Remark: If $f(z)$ is analytic in a region T, then the decomposition formula $f(z) = g(z_1)e_1 + h(z_3)e_3$ will automatically define an analytic function, coinciding with $f(z)$ in T, at all points of the product region of T_1 and T_3, which in general will include points not in T. This trait of the bicomplex function theory has no counterpart in the theory of functions of a complex variable.

The decomposability of a function is in itself a rather strong requirement, as shown by the following.

Remark: Let $f(z)$ be any bicomplex-valued function of a complex variable, i.e., a bicomplex-valued function defined if and only if z is in the complex plane. Then if it be required that $f(z)$ be extended into the bicomplex space in such a way that $f(z)$ is decomposable as $f(z) = g(z_1)e_1 + h(z_3)e_3$, then the extension is already uniquely determined. This follows from the fact that for z in the complex plane, $z = z_1 = z_3$ and the definition of $f(z)$ for these values determines $g(z_1)$ and $h(z_3)$ in their entire domains of definition, the complex z_1- and z_3- planes, respectively. Thus $f(z)$ is determined in the entire bicomplex space. Of course if $f(z)$ is analytic the continuation will be analytic.

Corollary: If $f(z)$ and $F(z)$ are two analytic functions of the bicomplex variable z which are equal for all complex values of z, the functions are equal for all bicomplex values of z.

The above remark and its corollary could be generalized in various ways.

II. POWER SERIES AND TAYLOR'S THEOREM

<u>Definition: Let $\sum_{n=0}^{\infty} a_n$ be a series of bicomplex terms, and let $s_k = \sum_{n=0}^{k} a_n$. The series will be said to converge if for $\varepsilon > 0$ there exists an integer N such that for all $m,n > N$, $\|s_m - s_n\| < \varepsilon$.</u>

Let $a_n = b_n e_1 + c_n e_1$, where b_n and c_n are complex. It will be useful to show that $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge in the ordinary sense. Therefore let $\sum_{n=0}^{\infty} a_n$ converge. Let $s_k = p_k e_1 + q_k e_3$. Since $a_n = b_n e_1 + c_n e_3$, $s_k = \sum_{n=0}^{k} a_n = \left(\sum_{n=0}^{k} b_n\right) e_1 + \left(\sum_{n=0}^{k} c_n\right) e_3$. Thus $p_k = \sum_{n=0}^{k} b_n$ and $q_k = \sum_{n=0}^{k} c_n$. Now $s_m - s_n = (p_m - p_n) e_1 + (q_m - q_n) e_3$ and $\|s_m - s_n\| = \frac{1}{\sqrt{2}}\sqrt{|p_m - p_n|^2 + |q_m - q_n|^2} < \varepsilon$ for $m,n > N$. Thus $|p_m - p_n| < \varepsilon\sqrt{2}$ and $|q_m - q_n| < \varepsilon\sqrt{2}$. Therefore $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge.

Conversely let $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge. Then for $\varepsilon > 0$, there exists N_1 such that $|p_m - p_n| < \varepsilon$ for $m,n > N_1$ and N_2 such that $|q_m - q_n| < \varepsilon$ for $m,n > N_2$. Then for $N = \max(N_1, N_2)$, these inequalities both hold for $m,n > N$. Thus $\|s_m - s_n\| < \varepsilon$ for $m,n > N$ and $\sum_{n=0}^{\infty} a_n$ converges.

<u>Definition: Let $\sum_{n=0}^{\infty} a_n$ be a series of bicomplex terms, and let $s_k = \sum_{n=0}^{k} a_n$. The series will be said to converge to the sum S if for $\varepsilon > 0$ there exists an integer N such that for $k > N$, $\|s_k - S\| < \varepsilon$.</u>

It is easily verified that a series is convergent if and only if the series converges to a sum S, and that $S = Pe_1 + Qe_3$, where $P = \sum_{n=0}^{\infty} b_n$, $Q = \sum_{n=0}^{\infty} c_n$.

Now let $\sum_{n=0}^{\infty} a_n z^n$ denote a power series. It converges if and only if $\sum_{n=0}^{\infty} b_n z_1{}^n$ and $\sum_{n=0}^{\infty} c_n z_3{}^n$ converge, from the above analysis, since $a_n z^n = (b_n e_1 + c_n e_3)(z_1 e_1 + z_3 e_3)^n = (b_n z_1{}^n) e_1 + (c_n z_3{}^n) e_3$.

<u>Definition: The set of all interior points of the set of points at which a power series is convergent will be termed the region of convergence of the power series.</u>

It follows from the Ringleb decomposition theorem that a convergent power series represents an analytic function in its region of convergence.

Since $\sum_{n=0}^{\infty} b_n z_1^n$ and $\sum_{n=0}^{\infty} c_n z_3^n$ are complex power series they will have radii of convergence, which may be zero or infinite. Let the radius of convergence of $\sum_{n=0}^{\infty} b_n z_1^n$ be R_1 and the radius of convergence of $\sum_{n=0}^{\infty} c_n z_3^n$ be R_3. If $R_1 = 0$ then the set of points of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is restricted to the second nil-plane and if $R_3 = 0$ to the first nil-plane. In these two cases the region of convergence as defined above will be empty. If $R_1, R_3 > 0$ however, then the set of points of convergence contains a hypersphere, $\|z\| < k$ for some $k > 0$, centered at the origin, and the region of convergence is not empty. Let

$$\xi = \left(\frac{e_1}{R_1} + \frac{e_3}{R_3}\right) z = \frac{z_1}{R_1} e_1 + \frac{z_3}{R_3} e_3 = \xi_1 e_1 + \xi_3 e_3$$

or

$$z = (R_1 e_1 + R_3 e_3)\xi = R_1 \xi_1 e_1 + R_3 \xi_3 e_3.$$

Then the series $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (R_1^n e_1 + R_3^n e_3)\xi^n$ converges for $|\xi_1| < 1$ and $|\xi_3| < 1$. Thus a power series may be normalized to unit radii of convergence of both component power series.

It seems desirable to be able to describe the region of convergence in terms of z itself, particularly in terms of a norm such that the series converges when the norm is less than some constant and diverges when the norm is greater than this constant.

Since $\|z\| = \frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |z_3|^2}$, it is seen that this norm fails to describe the region of convergence for the normalized power series. For if $|z_1| < 1$ and $|z_3| < 1$, then $\|z\| < 1$, and if $|z_1| = |z_3| = 1$, then $\|z\| = 1$. But if $|z_3| = 0$ and $1 < |z_1| < \sqrt{2}$, then $\|z\| < 1$. The problem is solved by the following theorem, which is applicable when $R_1 = R_3 = R$, which is the case when the power series has been normalized, and which is in particular the case whenever the series represents a function which is complex whenever z is complex, for then $f(z) = f(z_1)e_1 + f(z_3)e_3$ and $a_n = b_n = c_n$ for all n.

<u>Theorem:</u> <u>Let</u>

$$N(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z|^4}}$$

<u>Then $N(z)$ is a norm, and if $\sum_{n=0}^{\infty} a_n z^n$ is a power series whose component series $\sum_{n=0}^{\infty} b_n z_1^n$ and $\sum_{n=0}^{\infty} c_n z_3^n$ $(a_n = b_n e_1 + c_n e_3)$ both have the same radius of convergence $R>0$, then $\sum_{n=0}^{\infty} a_n z^n$ converges for $N(z) < R$ and diverges for $N(z) > R$.</u>

Proof: Substituting $\|z\| = \frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |z_3|^2}$ and $|z| = \sqrt{|z_1| \cdot |z_3|}$ in the expression for $N(z)$ gives

$$N(z) = \sqrt{\tfrac{1}{2}(|z_1|^2 + |z_3|^2) + \sqrt{\tfrac{1}{4}(|z_1|^2 + |z_3|^2)^2 - |z_1|^2 |z_3|^2}} =$$

$$= \sqrt{\tfrac{1}{2}\left[(|z_1|^2 + |z_3|^2) + \sqrt{(|z_1|^2 - |z_3|^2)^2}\right]} = \max(|z_1|, |z_3|)$$

Hence, if $N(z) < R$, both $|z_1| < R$ and $|z_3| < R$ and $\sum_{n=0}^{\infty} a_n z^n$ converges, whereas if $N(z) > R$, either $|z_1|$ or $|z_3|$ will be greater than R, hence the series must diverge.

Using the above representation of $N(z)$, it is immediately verified that $N(z)$ is a norm and also that $N(zw) \leq N(z)N(w)$. Also the following inequalities may be verified

$$\|z\| \leq N(z) \leq \|z\|\sqrt{2} \quad \text{and} \quad |z| \leq N(z)$$

Each of the equations $\|z\| = N(z)$ and $|z| = N(z)$ is satisfied if and only if $|z_1| = |z_3|$ (which is then also equal to $|z|$), hence in particular when z is complex. The equation $N(z) = \|z\|\sqrt{2}$ is satisfied if and only if z is a nil-factor (i.e., $z_1 = 0$ or $z_3 = 0$).

The existence of a Taylor series is demonstrated by Futagawa (3), without the use of the Ringleb decomposition. His conclusion is that the series is absolutely convergent in the hypersphere of radius one-fourth the distance from the point of expansion (the center of the hypersphere) to the boundary of the region T of analyticity of the function. Prof. Price has verified the existence of a Taylor series in a neighborhood of a point of T using Cauchy's formula for a curve lying in a plane through the point and having the point in its interior. Taylor series in more general systems are discussed by several authors. With the use of the decomposition theorem and the above norm, N(z), it is possible to show that the region of convergence not only contains the hypersphere of radius equal to the distance from the point of expansion to the boundary of the region T but actually extends outside of this hypersphere in certain directions.

The actual process of expanding an analytic function as a power series may be carried out without employing the decomposition directly, as is evidenced by

Taylor's Theorem: Let f(z) be analytic in a four-dimensional region T, and let α be a point of T. Then f(z) may be expanded as a generalized Taylor series about the point α:

$$f(z) = f(\alpha) + \frac{z-\alpha}{1!} f'(\alpha) + \frac{(z-\alpha)^2}{2!} f''(\alpha) + \cdots + \frac{(z-\alpha)^n}{n!} f^{(n)}(\alpha) + \cdots,$$

wherever f(z) is defined and the series is convergent. If d is the greatest lower bound of $\|z-\alpha\|$ for z a boundary point of T,

<u>then the above series for $f(z)$ converges for $N(z) < d\sqrt{2}$.</u>
<u>In particular this implies convergence in the hypersphere $\|z-\alpha\| < d$.</u>

Proof: By the Ringleb decomposition theorem

$$f(z) = g(z_1)e_1 + h(z_3)e_3$$

and also

$$f'(z) = g'(z_1)e_1 + h'(z_3)e_3$$

in T. Then

$$f^{(n)}(z) = g^{(n)}(z_1)e_1 + h^{(n)}(z_3)e_3$$

in T. Let $\alpha = \alpha_1 e_1 + \alpha_3 e_3$ be a point in T. Then, by Taylor's theorem for the complex case,

$$\text{(A)} \quad g(z_1) = g(\alpha_1) + \frac{z_1-\alpha_1}{1!} g'(\alpha_1) + \cdots + \frac{(z_1-\alpha_1)^n}{n!} g^{(n)}(\alpha_1) + \cdots$$

and

$$\text{(B)} \quad h(z_3) = h(\alpha_3) + \frac{z_3-\alpha_3}{1!} h'(\alpha_3) + \cdots + \frac{(z_3-\alpha_3)^n}{n!} h^{(n)}(\alpha_3) + \cdots$$

for $|z_1-\alpha_1| < R_1$ and $|z_3-\alpha_3| < R_3$, where the radii of convergence, R_1 and R_3, are not zero because of the openness of T. There exists a point β_1 in the z_1- plane such that $|\beta_1-\alpha_1| = R_1$ and $g(z_1)$ has a singularity for $z_1 = \beta_1$. Then $f(z)$ is singular for $z = \beta_1 e_1 + \alpha_3 e_3$. Thus $\|(\beta_1 e_1 + \alpha_3 e_3) - \alpha\| \geq d$ or $\|(\beta_1 - \alpha_1)e_1\| \geq d$. But $\|(\beta_1-\alpha_1)e_1\| = \frac{1}{\sqrt{2}}|\beta_1 - \alpha_1| = \frac{R_1}{\sqrt{2}}$. Therefore $\frac{R_1}{\sqrt{2}} \geq d$ or $R_1 \geq d\sqrt{2}$. By similar reasoning, $R_3 \geq d\sqrt{2}$.

Now consider the series

$$\text{(C)} \quad f(\alpha) + \frac{z-\alpha}{1!} f'(\alpha) + \cdots + \frac{(z-\alpha)^n}{n!} f^{(n)}(\alpha) + \cdots$$

The series (C) has the component series (A) and (B), and thus (C) converges for $z = z_1 e_1 + z_3 e_3$ if and only if (A) and (B) both converge. Then (A), (B), and hence (C), certainly converge for $N(z) < d\sqrt{2}$, and for those points of the set $N(z) < d\sqrt{2}$ which belong to T, the sum of the series (C) is $g(z_1)e_1 + h(z_3)e_3 = f(z)$. Moreover, since $N(z) \leq \|z\|\sqrt{2}$, the inequality $N(z) < d\sqrt{2}$ holds in particular for $\|z\| < d$.

Remarks: This conclusion does not mean that (C) diverges for $N(z) > d\sqrt{2}$. This is the case, however, if $R_1 = R_3 = d\sqrt{2}$, which shows that no better general conclusion is possible. Convergence for $\|z\| < d$ is not the best possible conclusion, which brings out the fact that a power series with a bounded region of convergence, hence representing a function which cannot be analytic for all values of z, always converges for some points which are more distant from the point of expansion than the nearest singularity, if distance is measured in the sense of the Euclidean norm.

<u>Definition: The series $\sum_{n=0}^{\infty} a_n$, where the a_n are bicomplex numbers, is termed absolutely convergent if $\sum_{n=0}^{\infty}\|a_n\|$ is convergent.</u>

It may be verified that a necessary and sufficient condition for absolute convergence of the series $\sum_{n=0}^{\infty} a_n$ is the absolute convergence of the component series, $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$, since

$$|b_n| \leq \sqrt{|b_n|^2+|c_n|^2} = \|a_n\|\sqrt{2} \text{ and } |c_n| \leq \sqrt{|b_n|^2+|c_n|^2} = \|a_n\|\sqrt{2},$$

and conversely

$$\|a_n\| < \sqrt{|b_n|^2+|c_n|^2} \leq |b_n|+|c_n|.$$

A power series is absolutely convergent in its region of convergence.

<u>Definition: Let $\sum_{n=0}^{\infty} a_n$ be a series of bicomplex terms. The series will be said to be quasi-absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ converges.</u>

If for some integer N all a_n for $n > N$ are nil-factors, then the series $\sum_{n=0}^{\infty} a_n$ will be quasi-absolutely convergent even though the series itself may diverge. However, if for some N there exists $k \geq 0$ such that $\|a_n\| \leq k\,|a_n|$ for $n > N$, then the quasi-absolute convergence will imply absolute convergence. This condition will be shown to hold if the a_n are in a plane through the origin (for definition of a plane see page 37) which does not have any other point in common with a nil-plane. Of course if the plane intersects a nil-plane in a line, then the a_n may lie along the line and quasi-absolute convergence will not imply convergence.

The proof will use the following:

Lemma. Let a,b,c,d be complex numbers and let cx+dy be different from zero for all real $(x,y)\neq(0,0)$. Then $F(x,y)=\frac{ax+by}{cx+dy}$ is bounded for all real $(x,y)\neq(0,0)$

Proof: If $x=0$, $F=\frac{b}{d}$. For $x\neq 0$, let $\frac{y}{x}=\lambda$. Then $F=\frac{a+b\lambda}{c+d\lambda}$, where λ is real. Consider λ as a complex variable. Then F is a linear fractional transformation which takes the real axis into a straight line or a circle. Since $c+d\lambda\neq 0$ for real λ, the real axis transforms into a circle. Thus F is bounded for $(x,y)\neq(0,0)$.

Now let the points a_n lie in a plane through the origin which has no other point in common with a nil-plane. Then each a_n may be expressed in the form $l\alpha+m\beta$, where l and m are real, α and β are fixed bicomplex numbers which are not zero or a nil-factor, and $l\alpha+m\beta$ is not zero or a nil-factor except for $l=m=0$. Let $\alpha=\alpha_1e_1+\alpha_3e_3$, and $\beta=\beta_1e_1+\beta_3e_3$. Then

$$\|l\alpha+m\beta\| = \frac{1}{\sqrt{2}}\sqrt{|l\alpha_1+m\beta_1|^2+|l\alpha_3+m\beta_3|^2}$$

and

$$|l\alpha+m\beta| = \sqrt{|l\alpha_1+m\beta_1|\cdot|l\alpha_3+m\beta_3|}.$$

Then

$$\frac{\|l\alpha+m\beta\|}{|l\alpha+m\beta|}=\frac{\frac{1}{\sqrt{2}}\sqrt{|l\alpha_1+m\beta_1|^2+|l\alpha_3+m\beta_3|^2}}{\sqrt{|l\alpha_1+m\beta_1|\cdot|l\alpha_3+m\beta_3|}}$$

This will be bounded if and only if

$$\frac{|l\alpha_1+m\beta_1|}{|l\alpha_3+m\beta_3|}+\frac{|l\alpha_3+m\beta_3|}{|l\alpha_1+m\beta_1|}$$

is bounded. Since $l\alpha+m\beta$ is never a nil-factor or zero except when

$l = m = 0$, each term is bounded by the lemma for either l or m different from zero. Let the bound be k. If $l = m = 0$, then $\|l\alpha + m\beta\| = |l\alpha + m\beta| = 0$. Thus in all cases $\|l\alpha + m\beta\| \leq k\,|l\alpha + m\beta|$ and quasi-absolute convergence implies absolute convergence, which in turn implies convergence.

III. SINGULARITIES AND ZEROS

Definition: A point z_0 will be called a singularity of a function $f(z)$ if z_0 is a boundary point of a region T in which $f(z)$ is analytic and if there does not exist a region T' including T and containing z_0 and a function $g(z)$ analytic in T' and coinciding with $f(z)$ in T. The point z_0 will be called a removable singularity if the described T' and $g(z)$ do exist.

Note that a removable singularity is not a singularity.

The decomposition theorem of Ringleb leads at once to the result that $f(z) = g(z_1)e_1 + h(z_3)e_3$ can have a singularity at $z = z_0 = z_1^0\, e_1 + z_3^0\, e_3$ if and only if $g(z_1)$ has a singularity at $z_1 = z_1^0$ or $h(z_3)$ has a singularity at $z_3 = z_3^0$. But then it follows that $f(z)$ has a singularity at every point of the intersection of the closure of the region of analyticity of $f(z)$ and one of the nil-planes with respect to z_0, i.e., the set of points which are of the form $z_0 + \alpha$ for α in a nil-plane.

Thus there are no isolated singularities.

Since a function $f(z)$ with a singularity at the origin can have no point of analyticity in one of the nil-planes, $f(z)$ will be said to be singular in that nil-plane, even though all points of the nil-plane will in general not be boundary points of the region T in which $f(z)$ is analytic.

A hypersphere $\|z - z_0\| < k$, for some $k > 0$, will be referred to as a neighborhood P' of the point z_0.

Definition: A neighborhood P' of a point minus the nil-planes with respect to the point will be called a pseudo-neighborhood P of the point. The neighborhood P' will be called the associated neighborhood of P.

<u>Definition: Let $f(z)$ be analytic in a pseudo-neighborhood of $z = z_0 = z_1^0 e_1 + z_2^0 e_2$. Then $f(z)$ will be said to have a pole of order at most n, where n is a non-negative integer, in the nil-planes with respect to z_0 if both $g(z_1)$ has a pole of order at most n at $z_1 = z_1^0$ and $h(z_2)$ has a pole of order at most n at $z_2 = z_2^0$. If both $g(z_1)$ and $h(z_2)$ have a pole of order n at these points, then $f(z)$ will be said to have a pole of order n!</u>

Here a point of analyticity in the complex planes, and hence also in the bicomplex space, has been referred to as a pole of order zero.

In the theory of functions of a complex variable, the inequalities $|f(z)| < M$, $|f(z)|\cdot|z|^n < M$, and $|f(z)| \leq M|z|^n$ holding in a neighborhood of the origin were shown to imply that $f(z)$ has at the origin a removable singularity, a pole, and a zero, respectively.

These results motivate using the norm and the absolute value in assuming similar inequalities in the bicomplex space to hold in a pseudo-neighborbood of the origin and investigating the effect on the function. Because $\|z\| \leq N(z) \leq \|z\|\sqrt{2}$, the norm $N(z)$ may be substituted in any of these inequalities for z and the conclusion will be unchanged. This of course includes substituting $N[f(z)]$ for $\|f(z)\|$.

The first step might be to assume that $f(z)$ is analytic in a pseudo-neighborhood P of the origin and that $\|f(z)\| < M$ in P, then prove that $g(z_1)$ and $h(z_2)$ have removable singularities at their respective origins, and that therefore $f(z)$ has a removable singularity at all points of the intersection of the nil-planes with the associated neighborhood P. This can be done. However it is included in the following perhaps unexpected more general theorem, since for $\|z\| < 1$, then $\|f(z)\| \cdot \|z\|^n < \|f(z)\| < M$.

Theorem: Let $f(z)$ be analytic in a pseudo-neighborhood P of the origin and let

$$\|f(z)\| \cdot \|z\|^n < M$$

in P for a positive integer n, where M is a positive constant. Then $f(z)$ has a removable singularity at all points of the intersection of the nil-planes with the associated neighborhood P'.

This theorem may be proved directly, but will be proved as a corallary of the next theorem.

Since the boundedness of $\|f(z)\| \cdot \|z\|^n$ in a pseudo-neighborhood of the origin is too strong a restriction on $f(z)$ to permit singularities, the next step might be to weaken this assumption by replacing one of the norms or both by the absolute value, first checking the restriction on $f(z)$ of the boundedness of $|f(z)|$ in a pseudo-neighborhood of the origin. Again this is included in a more general

Theorem: Let $f(z)$ be analytic in a pseudo-neighborhood P of the origin and let

$$|f(z)| \cdot \|z\|^n < M$$

in P, where n is a positive integer and M is a positive constant. Then there are two cases:

Case 1. $|f(z)| \equiv 0$ in P, but $f(z)$ may be singular in a nil-plane.

<u>Case 2. f(z) has a removable singularity at all points of the intersection of the nil-planes with the associated neighborhood P'.</u>

Proof: If $f(z)\not\equiv 0$ in P, then there exists a point $z = ae_1 + be_1$ in P such that $a \neq 0, b \neq 0, h(b) = c \neq 0$. Then for $z_3 = b$ and $0 < |z_1| < |a|$,

$$|f(z)| \cdot \|z\|^n = \sqrt{|g(z_1)| \cdot |c|}\left[\frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |b|^2}\right]^n < M$$

or

$$|g(z_1)| < \frac{M^2 2^n}{|c| \left[|z_1|^2 + |b|^2\right]^n} \leq \frac{M^2 2^n}{|c| \cdot |b|^{2n}}$$

for $0 < |z_1| < |a|$. Therefore $g(z_1)$ has a removable singularity at $z_1 = 0$. Similarly $h(z_3)$ has a removable singularity at $z_3 = 0$. Therefore f(z) has a removable singularity at all points of the intersection of the nil-planes with the associated neighborhood P'.

Proof of the preceding theorem: Since $|f(z)| \cdot \|z\|^n \leq \|f(z)\| \cdot \|z\|^n < M$, either the conclusion is already proved or else $|f(z)| \equiv 0$ in P. But then $f(z) \equiv g(z_1)e_1$ or $f(z) \equiv h(z_3)e_3$, say $g(z_1)e_1$. Then for $z_3 = b \neq 0$

$$|g(z_1)| \cdot \|b\|^n = \sqrt{2}\|f(z)\| \cdot \|b\|^n < M\sqrt{2}$$

or

$$|g(z_1)| < \frac{M\sqrt{2}}{\|b\|^n}$$

Therefore $g(z_1)$ has a removable singularity at the origin and the conclusion is proved in any case.

<u>Theorem: Let f(z) be analytic in a pseudo-neighborhood P of the origin and let</u>

$$\|f(z)\| \cdot |z|^n < M$$

<u>in P, where n is a positive integer and M is a positive constant. Then f(z) has a pole of order at most $\left[\frac{n}{2}\right]$ in the nil-planes,</u>

<u>where</u> $\left[\frac{n}{2}\right] = \frac{n}{2}$ <u>if n is even and</u> $\left[\frac{n}{2}\right] = \frac{n-1}{2}$ <u>if n is odd. Further, if n is odd,</u>

$$\|f(z)\| \cdot |z|^{n-1}$$

<u>is also bounded in some neighborhood of the origin.</u>

Proof:

$$\|f(z)\| \cdot |z|^n = \frac{1}{\sqrt{2}}\sqrt{|g(z_1)|^2 + |h(z_3)|^2}\ \sqrt{|z_1|^n\ |z_3|^n} < M$$

or

$$\left[|g(z_1)|^2 + |h(z_3)|^2\right] \cdot |z_1|^n |z_3|^n < 2M^2$$

for z in P. Let $z = ae_1 + be_3$ be a point in P. Then $a \neq 0$ and $b \neq 0$.

Let $z_3 = b$ and $0 < |z_1| < \frac{1}{2}|a|$, then $z = z_1 e_1 + z_3 e_3$ is in P. Then

$$\left|\left[g(z_1)\right]^2\right| \cdot |z_1|^n = |g(z_1)|^2\ |z_1|^n \leq \left[|g(z_1)|^2 + |h(b)|^2\right]\ |z_1|^n < \frac{2M^2}{|b|^n}$$

for $0 < |z_1| < \frac{1}{2}|a|$. Thus $\left[g(z_1)\right]^2$ has a pole of order at most n at $z_1 = 0$. Therefore $g(z_1)$ has a pole of order at most $\left[\frac{n}{2}\right]$ at $z_1 = 0$. Similarly $h(z_3)$ has a pole of order at most $\left[\frac{n}{2}\right]$ at $z_3 = 0$. Thus $f(z)$ has a pole of order at most $\left[\frac{n}{2}\right]$ in the nil-planes.

Further if $n = 2r+1$, where r is an integer, then $\left[\frac{n}{2}\right] = r$ and $|g(z_1)| \cdot |z_1|^r < N$ and $|h(z_3)| \cdot |z_3|^r < P$ in some neighborhood of their respective origins, where N and P are positive constants. Let $|z_1| < 1$ and $|z_3| < 1$. Then

$$\|f(z)\| \cdot |z|^{n-1} = \|f(z)\| \cdot\ |z|^{2r} = \frac{1}{\sqrt{2}}\sqrt{|g(z_1)|^2 + |h(z_3)|^2}\ \left[\sqrt{|z_1| \cdot |z_3|}\right]^{2r} =$$

$$= \frac{1}{\sqrt{2}}\sqrt{|g(z_1)|^2\ |z_1 z_3|^{2r} + |h(z_3)|^2\ |z_1 z_3|^{2r}} < \frac{1}{\sqrt{2}}\sqrt{|g(z_1)|^2\ |z_1|^{2r} + |h(z_3)|^2\ |z_3|^{2r}} <$$

$$< \frac{1}{\sqrt{2}}\sqrt{N^2 + P^2}$$

This last part of the proof also serves to show that the conclusion of the theorem is the best possible.

<u>Theorem:</u> <u>Let f(z) be analytic in a pseudo-neighborhood P of the origin and let</u>

$$|f(z)| \cdot |z|^n < M$$

in P, where n is a positive integer and M is a positive constant. Then there are two cases.

Case 1. *$|f(z)| \equiv 0$ in P, but f(z) may be singular in one nil-plane.*

Case 2. *f(z) has a pole of order at most n in the nil-planes in the associated neighborhood P'.*

Proof: If $|f(z)| \equiv 0$, then $|f(z)| \cdot |z|^n \equiv 0 < M$.

If $|f(z)| \not\equiv 0$, then $g(z_1) \not\equiv 0$, and $h(z_3) \not\equiv 0$. Then there exists $z = ae_1 + be_3$ in P such that $a \neq 0, b \neq 0$, and $h(b) = c \neq 0$. Let $z_3 = b$ and $0 < |z_1| < \frac{1}{2}|a|$, then $z = z_1 e_1 + z_3 e_3$ is in P. Now

$$|f(z)| \cdot |z|^n = \sqrt{|g(z_1)| \cdot |h(z_3)| \cdot |z_1|^n \cdot |z_3|^n} < M$$

for z in P, or

$$|g(z_1)| \cdot |z_1|^n < \frac{M^2}{|c| \cdot |b|^n}$$

$0 < |z_1| < \frac{1}{2}|a|$. Thus $g(z_1)$ has a pole of order at most n at $z_1 = 0$. Similarly $h(z_3)$ has a pole of order at most n at $z_3 = 0$. Therefore f(z) has a pole of order at most n in the nil-planes.

Remark: If $f(z) = \frac{1}{z^n}$, then $|f(z)| \cdot |z|^n = 1$ for all z in the pseudo-neighborhood of the origin. Thus the conclusion of the theorem is the best possible.

Theorem: Let f(z) be analytic in a pseudo-neighborhood P of the origin. Then f(z) has a pole of order n in the nil-planes if and only if $|f(z)z^n|$ approaches a limit $K \neq 0$ as z approaches the origin.

Proof: In P, $|f(z)z^n| = \sqrt{|g(z_1)| \cdot |z_1|^n \cdot |h(z_3)| \cdot |z_3|^n}$. As z approaches zero, z_1 and z_3 approach zero independently. Thus $|f(z)z^n|$ approaches a finite non-zero limit if and only if

$|g(z_1)| \cdot |z_1|^n$ and $|h(z_3)| \cdot |z_3|^n$ approach finite non-zero limits. This means $g(z_1)$ has a pole of order n at $z_1 = 0$ and $h(z_3)$ has a pole of order n at $z_3 = 0$, or $f(z)$ has a pole of order n in the nil-planes.

<u>Definition: Let $f(z)$ be analytic in a neighborhood of the origin. Then $f(z)$ will be said to have a zero of order at least n, where n is a positive integer, at the origin if and only if both $g(z_1)$ has a zero of order at least n at $z_1 = 0$ and $h(z_3)$ has a zero of order at least n at $z_3 = 0$.</u>

<u>Theorem: Let $f(z)$ be analytic in a pseudo-neighborhood P of the origin, and let</u>

$$\|f(z)\| \leq M\|z\|^n$$

<u>in P, where n is a positive integer and M is a positive constant. Then $f(z)$ has a zero of order at least n at the origin.</u>

Proof: Since for $\|z\| < 1$, $\|f(z)\| \leq M\|z\|^n < M$, $f(z)$ has a removable singularity in the nil-planes. As z approaches zero, $f(z)$ approaches zero. Therefore redefine $f(0) = g(0) = h(0) = 0$, and $f(z)$ is then analytic in the associated neighborhood P'. Let $z_3 = 0$ and $z = z_1 e_1$ be in P. Then $f(z) = g(z_1)e_1$, and $\|z\| = \sqrt{2}|z_1|$. Therefore $\|f(z)\| = \sqrt{2}|g(z_1)| \leq M\|z\|^n = 2^{\frac{n}{2}}M|z_1|^n$ or $|g(z_1)| \leq 2^{\frac{n-1}{2}}M|z_1|^n$. Thus $g(z_1)$ has a zero of order at least n at $z_1 = 0$. Similarly $h(z_3)$ has a zero of order at least n at $z_3 = 0$. Thus $f(z)$ has a zero of order at least n at $z = 0$.

The conclusion is best possible, for if $f(z) = z^n$, then $\|f(z)\| = \frac{1}{\sqrt{2}}\sqrt{|z_1|^{2n} + |z_3|^{2n}} \leq M\|z\|^n = M\left[\frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |z_3|^2}\right]^n$ for $M = (\sqrt{2})^{n-1}$, since $|z_1|^{2n} + |z_3|^{2n} \leq \left[|z_1|^2 + |z_3|^2\right]^n$.

Theorem: Let $f(z)$ be analytic in a neighborhood P of the origin and let

$$\|f(z)\| \leq M|z|^n$$

in P, where n is a positive integer and M is a positive constant. Then $f(z) \equiv 0$ for z in P.

Proof: Let $z_3 = 0$ and $z_1 e_1$ be in P, then $|z| = 0$ and thus $\|f(z)\| = 0$. Therefore $h(0) = 0$ and $g(z_1) \equiv 0$ for $z_1 e_1$ in P. Similarly $h(z_3) \equiv 0$ for $z_3 e_3$ in P. Thus $f(z) \equiv 0$ for z in P.

Theorem: Let $f(z)$ be analytic in a neighborhood P of the origin and let

$$|f(z)| \leq M|z|^n$$

in P, where n is a positive integer and M is a positive constant. Then there are two cases:

Case 1. $|f(z)| \equiv 0$

Case 2. $f(z)$ has a zero of order at least n at the origin.

Proof: If $|f(z)| \equiv 0$, then $|f(z)| \leq M|z|^n$.

If $|f(z)| \not\equiv 0$, then $g(z_1) \not\equiv 0$ and $h(z_3) \not\equiv 0$. Then for some $z = ae_1 + ce_3$ in P; $a \neq 0$, $c \neq 0$, $g(a) = b \neq 0$ and $h(c) = d \neq 0$. Since for z in P

$$|f(z)| = \sqrt{|g(z_1)| \cdot |h(z_3)|} \leq M\sqrt{|z_1|^n \cdot |z_3|^n}$$

or

$$|g(z_1)| \cdot |h(z_3)| \leq M^2 |z_1|^n \cdot |z_3|^n$$

Now let $z_3 = c$ and $|z_1| < \frac{1}{2}|a|$, then $z = z_1 e_1 + z_3 e_3$ is in P and $|g(z_1)| \cdot |d| \leq M^2 |z_1|^n \cdot |c|^n$ or $|g(z_1)| \leq \frac{M^2 |c|^n}{|d|} |z_1|^n$. Thus $g(z_1)$ has a zero of order at least n at $z_1 = 0$. Similarly $h(z_3)$ has a zero of order at least n at $z_3 = 0$. Then the conclusion follows.

The conclusion is best possible, for if $f(z) = z^n$, then

$$|f(z)| = \sqrt{|z_1|^n \, |z_3|^n} = \left[\sqrt{|z_1| \cdot |z_3|}\right]^n = |z|^n$$

Theorem: Let $f(z)$ be analytic in a neighborhood P of the origin and let

$$|f(z)| \leq M\|z\|^n$$

in P, where n is a positive integer and M is a positive constant. Then $|f(0)| = 0$ and the sum of the orders of the zeros of the component functions at their respective origins is at least $2n$.

Proof: Suppose first that $g(0) = b \neq 0$. Let $z_1 = 0$ and $z = z_3e_3$ be in P. Then

$$|f(z)| = \sqrt{|g(z_1)| \cdot |h(z_3)|} \leq M\left[\frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |z_3|^2}\right]^n$$

or

$$|h\ (z_3)| \leq \frac{M^2}{2^n|b|}|z_3|^{2n}$$

Thus $h(z_3)$ has a zero of order at least $2n$ at $z_3=0$.

Now suppose $g(z_1)$ has a zero of order $m<2n$. Then $\lim_{z_1\to 0}\left|\frac{g(z_1)}{z_1^m}\right| = b > 0$. Now let $z_1 = z_3$ for $z = z_1e_1 + z_3e_3$ in P. Then $\lim_{z_3\to 0}\left|\frac{g(z_3)}{z_3^m}\right| = b > 0$, and there exists a $\delta > 0$ such that for $|z_3| < \delta$, $\left|\frac{g(z_3)}{z_3^m}\right| \geq \frac{b}{2}$. Since $\sqrt{|g(z_3)h(z_3)|} \leq M\left[\frac{1}{\sqrt{2}}\sqrt{|z_3|^2+|z_3|^2}\right]^n = M|z_3|^n$, then $|h(z_3)| \leq \frac{M^2}{|g(z_3)/z_3^m|}|z_3|^{2n-m} \leq \frac{2M^2}{b}|z_3|^{2n-m}$ for $|z_3| < \delta$. Thus $h(z_3)$ has a zero of order at least $2n-m$ at $z_1=0$. In any case the sum of the orders is at least $2n$.

The conclusion is best possible, for if $f(z) = z_1^m e_1 + z_3^{2n-m}e_3$, then $|f(z)| = \sqrt{|z_1|^m|z_3|^{2n-m}} \leq \sqrt{[\|z\|\sqrt{2}]^m[\|z\|\sqrt{2}]^{2n-m}} = [\sqrt{2}\|z\|]^n$, since $|z_1| \leq \|z\|\sqrt{2}$ and $|z_3| \leq \|z\|\sqrt{2}$.

Remark: A function $f(z)$ which has a zero at the origin has the further interesting property that for z in the intersection of the first nil-plane and the region of analyticity of $f(z)$, the function $f(z)$ has a value which is a first nil-factor, and correspondingly for the second nil-plane. For if $f(0)=0$, then $g(0)=h(0)=0$ and for z in the first nil-plane $z_3=0$, so that $f(z)=g(z_1)e_1$.

IV INTEGRATION

Let C be a rectifiable curve connecting the two distinct points a and b of the bicomplex space. Let C_1 be the set of values which z_1 takes for all z on C, and let C_3 be the set of values which z_3 takes for z on C. C_1 will be called the projection of C on the z_1- plane. If C is contained in a region T, then C_1 is contained in the component region T_1 and C_3 in the component region T_3.

A straight line in the bicomplex space is defined as the set of points $z = k\alpha + (1-k)\beta$, where k is real and α and β are two distinct points. Decomposing the defining formula into its components, one verifies that a straight line projects into a straight line or a point. Therefore if C is a rectifiable curve, C_1 and C_3 are also rectifiable curves.

Let further $z_i, i = 0,1,2,\cdots,n$ be n+1 distinct points on the curve C, where $z_0 = a, z_n = b$ and z_i is situated on C between z_{i-1} and z_{i+1} as C is traced from a to b; finally, let f(z) be a function analytic at all points of C, including its end points a and b.

<u>Definition: Consider the expression</u>

$$S_n = \sum_{i=1}^{n} f(\xi_i)(z_i - z_{i-1})$$

<u>where ξ_i is an arbitrary point on the section of C which connects z_{i-1} and z_i and denote by Δ_n the $\max_{i=1,\dots,n} \|z_i - z_{i-1}\|$. Let the number of points z_i on C tend to infinity in such a way that Δ_n tends to zero. Then $\lim_{n\to\infty} S_n$, (shown below to exist and be the same for all sequences of subdivisions) will be called the integral of f(z) along C from a to b and denoted by $\int_C f(z)dz$.</u>

Let $\xi_i = \xi_1^i e_1 + \xi_3^i e_3$; $z_i = z_1^i e_1 + z_3^i e_3$; $\Delta_1^n = \max_{i=1,\dots,n} |z_1^i - z_1^{i-1}|$; $\Delta_3^n = \max_{i=1,\dots,n} |z_3^i - z_3^{i-1}|$; $S_n = S_1^n e_1 + S_3^n e_3$. Then

$$S_n = \left[\sum_{i=1}^n g(\xi_1^i)(z_1^i - z_1^{i-1})\right] e_1 + \left[\sum_{i=1}^n h(\xi_3^i)(z_3^i - z_3^{i-1})\right] e_3$$

and

$$S_1^n = \sum_{i=1}^n g(\xi_1^i)(z_1^i - z_1^{i-1}), \quad S_3^n = \sum_{i=1}^n h(\xi_3^i)(z_3^i - z_3^{i-1})$$

Δ_n tends to zero if and only if Δ_1^n and Δ_3^n tend to zero. From the theory of functions of a complex variable,

$$\lim_{n\to\infty} S_1^n = \int_{C_1} g(z_1)dz_1 \; ; \quad \lim_{n\to\infty} S_3^n = \int_{C_3} h(z_3)dz_3$$

Since these limits exist, $\lim_{n\to\infty} S_n$ exists and

$$\int_C f(z)dz = \left[\int_{C_1} g(z_1)dz_1\right] e_1 + \left[\int_{C_3} h(z_3)dz_3\right] e_3.$$

A number of elementary properties of the integral may be verified from the definition. Among them that if C is divided at a point of C into two curves C′ and C″, then $\int_C f(z)dz = \int_{C'} f(z)dz + \int_{C''} f(z)dz$. Then the integral over a closed rectifiable curve in a given direction may be defined by taking two distinct points on C and combining the integrals over the separate parts of C in the given direction. If the closed curve is traced in the opposite direction, the value of the integral will of course be the negative of the previous.

<u>Cauchy's Integral Theorem</u>: <u>Let C be a closed rectifiable curve in a simply connected region T, and let f(z) be analytic in T. Then</u>

$$\int_C f(z)dz = 0.$$

Proof: Let C_1 and C_3 be the projections of C on the z_1- and z_3-plane, respectively. Since T is simply connected the component regions T_1 and T_3 are simply connected. Now

$$\int_C f(z)dz = \left[\int_{C_1} g(z_1)dz_1\right] e_1 + \left[\int_{C_3} h(z_3)dz_3\right] e_3$$

The quantities in brackets are zero. (See Bieberbach (11), page 108, for an argument showing that Cauchy's theorem holds for rectifiable curves which, as C_1 and C_3 may intersect themselves.)

Definition: Let C be a rectifiable curve in the bicomplex space whose projections C_1 and C_3 in the z_1- and z_3-planes, respectively, are simple closed curves. A curve C in this class will be called a P-curve.

Since C_1 and C_3 are simple closed curves, a P-curve is a simple closed curve.

Definition: Let C be a P-curve and let $z = z_1 e_1 + z_3 e_3$ be a point such that z_1 is in the interior of C_1 and z_3 is in the interior of C_3. Define the interior I of the P-curve C as the totality of all points z satisfying this condition.

This last definition has been given by Prof. Price.

Definition: Let C be a P-curve. As C_3 is traced in the positive direction in the z_3-plane, which is an ordinary complex plane, C will be traced in a certain direction. Designate this direction as the principal direction on C.

Definition: Let C be a P-curve. As C is traced in the principal direction, C_1 will be traced in a certain direction in the z_1-plane. If this direction is positive, designate C as a P-plus curve. If it is negative, designate C as a P-minus curve.

The proof of the next theorem requires the following

Lemma: $ij\ z = z_1 e_1 - z_3 e_3$

Proof: $ij\ z = ij(x+jy) = -iy+ix\ j = [-iy-i(ix)]\ e_1 + [-iy+i(ix)]\ e_3 =$

$= (x-iy)e_1 - (x+iy)e_3 = z_1 e_1 - z_3 e_3$

Then, of course, $ij\ f(z) = g(z_1)e_1 - h(z_3)e_3$

Theorem: (Cauchy's Integral Formula) Let C be a P-curve and let $f(z)$ be analytic in its interior I and continuous on the closure of I. Let z be any point in the interior I of C. Then

Case 1. If C is a P-plus curve,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w - z}$$

where C is traced in the principal direction.

Case 2. If C is a P-minus curve

$$f(z) = \frac{1}{2\pi j}\int_C \frac{f(w)dw}{w - z}$$

where C is traced in the principal direction.

Proof: $f(z) = g(z_1)e_1+h(z_3)e_3$ in I

By Cauchy's integral formula in the complex case

$$g(z_1) = \frac{1}{2\pi i}\int_{C_1^+} \frac{g(w_1)dw_1}{w_1 - z_1}; \quad h(z_3) = \frac{1}{2\pi i}\int_{C_3^+} \frac{h(w_3)dw}{w_3 - z_3}$$

Case 1.

$$\frac{1}{2\pi i}\int_C \frac{f(w)dw}{w - z} = \left[\frac{1}{2\pi i}\int_{C_1^+} \frac{g(w_1)dw_1}{w_1 - z_1}\right]e_1 + \left[\frac{1}{2\pi i}\int_{C_3^+} \frac{h(w_3)dw_3}{w_3 - z_3}\right]e_3 =$$

$$= g(z_1)e_1+h(z_3)e_3 = f(z)$$

Case 2.

$$\frac{1}{2\pi j}\int_C \frac{f(w)dw}{w - z} = \left[\frac{1}{2\pi j}\int_{C_1^-} \frac{g(w_1)dw_1}{w_1 - z_1}\right]e_1 + \left[\frac{1}{2\pi j}\int_{C_3^+} \frac{h(w_3)dw_3}{w_3 - z_3}\right]e_3 =$$

$$= \left[\frac{-1}{2\pi j}\int_{C_1^+} \frac{g(w_1)dw_1}{w_1 - z_1}\right]e_1 + \left[\frac{1}{2\pi j}\int_{C_3^+} \frac{h(w_3)dw_3}{w_3 - z_3}\right]e_3 =$$

$$= -\frac{i}{j}\left[\frac{1}{2\pi i}\int_{C_1^+} \frac{g(w_1)dw_1}{w_1 - z_1}\right]e_1 + \frac{i}{j}\left[\frac{1}{2\pi i}\int_{C_3^+} \frac{h(w_3)dw_3}{w_3 - z_3}\right]e_3 =$$

$$= -\frac{i}{j}\, g(z_1)e_1 + \frac{i}{j}\, h(z_3)e_3 = -\frac{i}{j}\left[g(z_1)e_1 - h(z_3)e_3\right] =$$

$$= -\frac{i}{j}\, ij\, f(z) = f(z), \text{ by the lemma.}$$

In order to apply Cauchy's integral formula, a criterion is needed to determine if the curve is a P-plus curve or a P-minus curve. The remainder of this section will be devoted to establishing such a criterion for plane curves. A plane is defined as the set of points of the form $z = n\alpha + m\beta + (1-n)(1-m)\gamma$ where n and m are real and $\alpha, \beta,$ and γ are three distinct points not on a line. For convenience the point z about which the integration is performed will be translated to the origin. Then the question asked becomes the following:

Let P be a plane through the origin, and let C be a simple closed rectifiable curve in p. For which planes P will the curve be a P-curve, and in which of those will the curve be a P-plus curve and in which will the curve be a P-minus curve?

It appears at once that a simple closed rectifiable curve in the complex plane is a P-plus curve, for there the result in case 1 holds. By analogy it would be expected that the plane determined by the line $z=k$ and the line $z=kj$ (k real) contains P-minus curves. This is correct, as can be verified by the promised criterion.

For the proof of the validity of this criterion, a few elementary results from the geometry of four-dimensional Euclidean space will be needed. They are undoubtedly well-known. For the sake of completeness, these results will be formulated and proved.

For the remainder of this section, advantage will be taken of the isomorphisms $z_1 e_1 \leftrightarrow z_1$ and $z_3 e_3 \leftrightarrow z_3$ to establish results in the nil-planes by actually performing the computation for the z_1- and z_3- planes.

The first and second nil-planes will be referred to as conjugate to each other.

Lemma 1. Let P be a plane through the origin which intersects one of the nil-planes in a straight line. Then every point of P projects into the same straight line in the conjugate nil-plane.

Proof: In the defining formula of a plane, let γ be the origin and β a point on the line of intersection, thus a nil-factor, say $\beta_1 e_1$. Then a point in P is of the form $l\alpha + m\beta_1 e_1$. The projection in the z_3- plane is of the form $l\alpha_3$. The set of such points lie on a line through the origin of the z_3- plane.

Lemma 2. Let P be a plane through the origin which has no other point in common with a nil-plane. Let C be a simple closed curve in P. Then the projection of C in the conjugate nil-plane is also a simple closed curve.

Proof: The projections C_1 and C_3 of the curve C in the first and second nil-planes, respectively, are clearly closed. Suppose the second nil-plane is the one with which P is assumed to have only the origin in common, and that C_1 is not simple. Then for two different points α and β on C, $\alpha_1 \neq \beta_1$. Therefore $\alpha_3 \neq \beta_3$. But then $\alpha - \beta$ is in P and is a second nil-factor. This is a contradiction, and the lemma follows.

In the remainder of this section there will be a change of notation. A bicomplex point z will sometimes be represented as $x+iy+jz+iju$, where x,y,z,u are real. It will always be clear which interpretation is meant. This will be useful in referring to the geometry of the four-dimensional space.

By separating the defining equation of a plane into its four components in this notation, letting γ be the origin, and eliminating l and m in the resulting four equations, it is verified that a plane P through the origin may be represented by two homogeneous independent real linear equations:

$$P: \begin{cases} Ax + By + Cz + Du = 0 \\ ax + by + cz + du = 0 \end{cases}$$

The equation of a line may be represented by three homogeneous independent real linear wquations and a point by four such equations. The intersection of two planes may be a line or a point.

Lemma 3. Let P be a plane through the origin, other than a nil-plane. Let P be represented by the system of equations.

$$P: \begin{cases} Ax+By+Cz+Du = 0 \\ ax+by+cz+du = 0 \end{cases}$$

Then P intersects the first nil-plane in a line if and only if

$$\Delta_1(P) = \begin{vmatrix} A+D & B-C \\ a+d & b-c \end{vmatrix} = 0$$

P intersects the second nil-plane in a line if and only if

$$\Delta_2(P) = \begin{vmatrix} A-D & B+C \\ a-d & b+c \end{vmatrix} = 0$$

Proof: The equations of the first nil-plane may be taken as

(A) $\qquad x - u = 0, \qquad y + z = 0$

The equations of the second nil-plane may be taken as

(B) $\qquad x + u = 0, \qquad y - z = 0$

The intersection of P and the first nil-plane is determined by solving the equations (A) with the equations P. These equations can have solutions other than (0,0,0,0) if and only if the determinant of the system vanishes. This determinant is

$$\begin{vmatrix} A & B & C & D \\ a & b & c & d \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{vmatrix}$$

and is easily reduced to the determinant $\Delta_1(P)$. Similarly if the system consisting of equations P and (B) are solved simultaneously, the resulting determinant of the system reduces to $\Delta_2(P)$.

Lemma 4. Let P be a plane through the origin which does not intersect a nil-plane except at the origin. Let C be a circle in P with center at O. Then the projection of C into the conjugate nil-plane is an ellipse. Here a circle is considered an ellipse, but a straight line segment is not.

Proof: Let P be represented by the pair of linear equations:

$$(P)\quad \begin{cases} Ax + By + Cz + Du = 0 \\ ax + by + cz + du = 0 \end{cases}$$

Then the circle will be the intersection of this plane with the hypersphere $x^2+y^2+z^2+u^2=r^2$. The projection of a point $x+iy+jz+iju$ in the z_1- plane is $z_1=(x+u)+i(y-z)$ and in the z_3- plane the projection is $z_3=(x-u)+i(y+z)$. Then if in the z_1- and z_3- plane, the usual rectangular cartesian coordinates are denoted here by X_1, Y_1 and X_3, Y_3,

$$(\bar{A})\qquad x+u=X_1, \qquad y-z=Y_1$$

and

$$(\bar{B})\qquad x-u=X_3, \qquad y+z=Y_3$$

The system of the four equations $(\bar{A})$ and (P) may be solved for x,y,z,u as linear combinations of X_1, Y_1 if and only if the determinant of the system is non-zero. By comparing equations $(\bar{A})$ with equations (B) of lemma 3, it is seen that this determinant is $\Delta_2(P)$. Then if P does not intersect the second nil-plane in a line, the linear combinations of X_1 and Y_1 may be substituted in the equation of the hypersphere to obtain the equation of the projection of C in the z_1- plane. This equation is quadratic. The curve is bounded. By lemma 2 it is not degenerate. Therefore it is an ellipse.

Similarly if P does not intersect the first nil-plane in a line, the projection of C in the z_3- plane is an ellipse.

Theorem: Let a plane P through the origin be represented by the system of equations:

$$P: \begin{cases} Ax + By + Cz + Du = 0 \\ ax + by + cz + du = 0 \end{cases}$$

Let

$$\Delta_1(P) = \begin{vmatrix} A+D & B-C \\ a+d & b-c \end{vmatrix} \qquad \Delta_2(P) = \begin{vmatrix} A-D & B+C \\ a-d & b+c \end{vmatrix}$$

Let C be a simple closed rectifiable curve in P containing the origin in its interior (in the topology of the plane P). C is a P-curve if and only if $\Delta_1(P) \neq 0$ and $\Delta_2(P) \neq 0$. Further C is a P-plus curve if $\Delta_1(P)$ and $\Delta_2(P)$ are of the same sign, and C is a P-minus curve if $\Delta_1(P)$ and $\Delta_2(P)$ are of different signs.

Proof: If P is a nil-plane, then one of the determinants is zero, and the curve C is not a P- curve. Otherwise, by lemmas 2 and 3, C is a P- curve if $\Delta_1(P) \neq 0$ and $\Delta_2(P) \neq 0$. By lemmas 1 and 3, C is not a P- curve if either $\Delta_1(P)$ or $\Delta_2(P)$ is zero.

Now let $\Delta_1(P)$ and $\Delta_2(P)$ be different from zero. Since C is a simple closed curve in P containing the origin in its interior (in the topology of the plane P), the curve C contains in its interior a circle $\bar{C}$ in P with center at the origin. The projections of C will contain in their respective interiors the respective projections of $\bar{C}$. The curve C will be a P-plus curve if and only if $\bar{C}$ is a P-plus curve. Thus without loss of generality it may be assumed that C is a circle.

Because the projections C_1 and C_3 of the circle C are ellipses by lemma 4, it will be found sufficient to consider only two points on C which are distinct and not at opposite ends of a diameter. From the relative positions of these two points and their respective projections, it is possible to determine the directions in which C_1 and C_3 are traced when C is traced in its principal direction. This will then determine if C is a P-plus or a P-minus curve.

Let α: (a_1, b_1, c_1, d_1) and β: (a_2, b_2, c_2, d_2) be two distinct points on the circle C not at opposite ends of a diameter. Then

$$\alpha = a_1 + b_1 i + c_1 j + d_1 ij = [(a_1+d_1)+(b_1-c_1)i]\,e_1 + [(a_1-d_1)+(b_1+c_1)i]\,e_3$$

$$\beta = a_2 + b_2 i + c_2 j + d_2 ij = [(a_2+d_2)+(b_2-c_2)i]\,e_1 + [(a_2-d_2)+(b_2+c_2)i]\,e_3$$

Since $\Delta_1(P)$ and $\Delta_2(P)$ are not zero, α and β cannot be nil-factors. Therefore let

$$(a_1+d_1)+(b_1-c_1)i = r_1 e^{i\theta_1}; \quad (a_1-d_1)+(b_1+c_1)i = r_3 e^{i\theta_3}$$

$$(a_2+d_2)+(b_2-c_2)i = R_1 e^{i\phi_1}; \quad (a_2-d_2)+(b_2+c_2)i = R_3 e^{i\phi_3};$$

then $\alpha_1 = r_1 e^{i\theta_1}$, $\alpha_3 = r_3 e^{i\theta_3}$, $\beta_1 = R_1 e^{i\phi_1}$, $\beta_3 = R_3 e^{i\phi_3}$. Now the direction from α_1 to β_1 on C_1, not passing through $-\alpha_1$ or $-\beta_1$, will be positive if

$$\pi < \theta_1 - \phi_1 < 2\pi \text{ or } 0 < \phi_1 - \theta_1 < \pi$$

and negative if

$$0 < \theta_1 - \phi_1 < \pi \text{ or } \pi < \phi_1 - \theta_1 < 2\pi$$

(Because of the symmetry of the projections and the assumption that α and β are distinct and not at opposite ends of a diameter, $\phi_1 - \theta_1$ cannot be an integral multiple of π.) Thus the direction will possess the same sign as $\sin(\phi_1 - \theta_1)$. Now

$$\phi_1 - \theta_1 = \arg\frac{\beta_1}{\alpha_1} = \arg\frac{(a_2+d_2)+(b_2-c_2)i}{(a_1+d_1)+(b_1-c_1)\,i} =$$

$$= \arg\frac{[(a_1+d_1)(a_2+d_2)+(b_1-c_1)(b_2-c_2)] + [(a_1+d_1)(b_2-c_2)-(a_2+d_2)(b_1-c_1)]\,i}{(a_1+d_1)^2+(b_1-c_1)^2}$$

Thus sign $\left[\sin(\phi_1-\theta_1)\right] = \text{sign}\left[(a_1+d_1)(b_2-c_2)-(a_2+d_2)(b_1-c_1)\right] =$

$$= \text{sign}\left[\begin{vmatrix} a_1+d_1 & b_1-c_1 \\ a_2+d_2 & b_2-c_2 \end{vmatrix}\right]$$

Similarly sign $\begin{bmatrix}\text{corresponding} \\ \text{direction} \\ \text{on } C_3\end{bmatrix} = \text{sign}\left[\sin(\phi_3-\theta_2)\right] = \text{sign}\left[\begin{vmatrix} a_1-d_1 & a_2-d_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}\right]$

Now by multiplication of determinants, using the fact that the points satisfy the equations of P

$$\begin{vmatrix} A & B & C & D \\ a & b & c & d \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & 1 & 0 \\ b_1 & b_2 & 0 & 1 \\ c_1 & c_2 & 0 & 1 \\ d_1 & d_2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & A-D & B+C \\ 0 & 0 & a-d & b+c \\ a_1-d_1 & a_2-d_2 & 1 & 0 \\ b_1+c_1 & b_2+c_2 & 0 & 1 \end{vmatrix}$$

By reducing the determinants on the left to the second order and making the obvious LaPlace expansion by second order minors on the right

$$\begin{vmatrix} A+D & B-C \\ a+d & b-c \end{vmatrix} \cdot \begin{vmatrix} a_1+d_1 & b_1-c_1 \\ a_2+d_2 & b_2-c_2 \end{vmatrix} = \begin{vmatrix} A-D & B+C \\ a-d & b+c \end{vmatrix} \cdot \begin{vmatrix} a_1-d_1 & a_2-d_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}$$

or

$$\Delta_1(P) \times \begin{vmatrix} a_1+d_1 & b_1-c_1 \\ a_2+d_2 & b_2-c_2 \end{vmatrix} = \Delta_2(P) \times \begin{vmatrix} a_1-d_1 & a_2-d_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}$$

Thus

$$\text{sign } \Delta_1(P) \times \text{sign}\begin{bmatrix}\text{direction} \\ \text{on } C_1\end{bmatrix} = \text{sign } \Delta_2(P) \times \text{sign}\begin{bmatrix}\text{direction} \\ \text{on } C_3\end{bmatrix}$$

Thus if $\Delta_1(P)$ and $\Delta_2(P)$ have like signs, C is a P-plus curve; and if $\Delta_1(P)$ and $\Delta_2(P)$ have unlike signs, C is a P-minus curve.

V. ANALYTIC CONTINUATION

Definition: Let $f(z)$ be analytic in a region T. If there exists a region T' such that $T \subset T'$ and a function $g(z)$ such that $g(z)$ is analytic in T' and $f(z) = g(z)$ for z in T, then the function $g(z)$ will be said to continue the function $f(z)$ analytically into the region T.

If $f(z)$ is a complex-valued function analytic in a region S of the complex plane (considered as a subset of the bicomplex space), it has previously been pointed out on page 5 that the expression $f(z_1)e_1 + f(z_3)e_3$ defines a bicomplex-valued analytic function of a bicomplex variable in the product region T of the component regions of S. Any analytic continuation of this function will also be referred to as an analytic continuation of $f(z)$.

It has also been pointed out previously that if $f(z)$ is analytic in T and the component-regions are T_1 and T_3, then $f(z)$ is automatically continued analytically into the product region of T_1 and T_3 by the formula $f(z) = g(z_1)e_1 + h(z_3)e_3$ for z_1 in T_1 and z_3 in T_3.

The usual method of analytical continuation by power series is applicable to the bicomplex space (see Futagawa (3)), and theorems similar to those in the complex case can be established.

The following question, already mentioned in the introduction, has been raised by Futagawa (3):

Given a complex-valued function $f(z)$ of a complex variable z; and a simple closed curve Γ in the complex plane. Suppose that $f(z)$ is analytic in the interior of Γ, but that singularities of $f(z)$ are everywhere dense on Γ. In other words, Γ is a natural boundary of $f(z)$. Is it ever possible to continue $f(z)$ into the bicomplex space along a continuous path which intersects the complex plane outside of Γ?

Futagawa appears to answer this question in the affirmative (Futagawa (5), pages 80-120). However if $f(z)$ is continued into the bicomplex space, then $f(z)=f(z_1)e_1+f(z_3)e_3$, and since for z complex, $z_1=z_3=z$, the projections Γ_1 and Γ_3 of the curve in the z_1- and z_3- planes, respectively, are curves congruent to Γ, and are hence the natural boundaries of the component functions $f(z_1)$ and $f(z_3)$, respectively. The projections of a continuous curve C in the bicomplex space joining a point inside Γ in the complex plane to a point outside Γ in the complex plane are continuous curves joining pairs of points similarly situated with respect to Γ_1 and Γ_3, hence crossing Γ_1 and Γ_3, respectively. Hence analytic continuation along C is impossible and the above question is answered in the negative.

VI. EXTENSION OF VARIOUS THEOREMS TO THE BICOMPLEX SPACE

Many theorems from the theory of analytic functions of a complex variable can be extended with little or no change to the bicomplex space and proved by the Ringleb decomposition. Some examples chosen rather arbitrarily will be presented here. The maximum-minimum principle will be found to hold for $\|f(z)\|$, $|f(z)|$ and $L[f(z)]$. Schwarz's lemma holds if the norm, $\|z\|$, is used. The condition for equality has a modification, however.

In theorems where the assumptions involve the behavior of an analytic function on a set of points S having a limit point interior to the region T of analyticity, such as Vitali's theorem and the uniqueness theorem for power series, the additional assumption must be made in the bicomplex case that the set of points does not lie in the nil-planes with respect to a finite number of points, and that the closure of the set of points is in T. The purpose of this last assumption is to prevent S from consisting of the sum of two sets A and B such that A is contained in the nil-planes with respect to a finite number of points and yet has a limit point in T, while B is not contained in the nil-planes with respect to a finite number of points but has all its limit points on the boundary of T.

Periodic functions generalize immediately. If the period is not a nil-factor, then the component functions are both periodic.

The property of conformal mapping in the complex plane can of course not be expected to extend to the bicomplex case, as an example will confirm.

Maximum-minimum principle: Let $f(z)$ be analytic in a region T. Then none of the expressions $\|f(z)\|$, $L[f(z)]$, or $|f(z)|$ can assume a maximum or a non-zero minimum at a point z_0 interior to T.

Proof: 1. Suppose $\|f(z)\| = \frac{1}{\sqrt{2}}\sqrt{|g(z_1)|^2 + |h(z_3)|^2}$ assumes a maximum at an interior point $z_0 = z_1^\circ e_1 + z_3^\circ e_3$ of T. Then $|g(z_1)|^2 + |h(z_3^\circ)|^2$ assumes a maximum at $z_1 = z_1^\circ$, an interior point of the component region T. This contradicts the maximum modulus principle in the complex theory. Thus $\|f(z)\|$ cannot assume a maximum at an interior point of T.

Suppose now that $\|f(z)\|$ assumes a minimum at $z = z_0$. Then $|g(z_1)|^2 + |h(z_3^\circ)|^2$, and therefore $|g(z_1)|$, assumes a minimum at $z_1 = z_1^\circ$. Thus $|g(z_1^\circ)| = 0$. Similarly $|h(z_3^\circ)| = 0$. Therefore $\|f(z)\|$ is zero at $z = z_0$.

2. Let $L[f(z)]$ have a maximum at $z = z_0 = z_1^\circ e_1 + z_3^\circ e_3$ interior to T. Recall that $L[f(z)] = \max[|g(z_1)|, |h(z_3)|]$. Suppose, for instance, that $|g(z_1^\circ)| \geq |h(z_3^\circ)|$. Then $|g(z_1)|$ has a maximum at $z_1 = z_1^\circ$. This cannot happen. Therefore $L[f(z)]$ cannot have a maximum at an interior point of T. Now suppose $L[f(z)]$ has a minimum other than zero at $z_0 = z_1^\circ e_1 + z_3^\circ e_3$ in T and $|g(z_1^\circ)| > |h(z_3^\circ)|$. Then again $|g(z_1)|$ has a non-zero minimum, which cannot happen. If $|g(z_1^\circ)| = |h(z_3^\circ)|$ then either $|g(z_1)|$ or $|h(z_3)|$ has a minimum, which cannot happen. Thus $L[f(z)]$ cannot have a non-zero minimum.

3. Suppose $|f(z)|$ has a maximum at $z = z_0 = z_1^\circ e_1 + z_3^\circ e_3$ interior to T. Since $|f(z)| = \sqrt{|g(z_1)| \cdot |h(z_3)|}$, then $|g(z_1)| \cdot |h(z_3^\circ)|$ has a maximum at $z_1 = z_1^\circ$. This cannot happen. Similarly $|f(z)|$ cannot have a non-zero minimum.

<u>Schwarz's lemma: Let $f(z)$ be analytic in the hypersphere $\|z\| < R$. Let $\|f(z)\| < M$ for $\|z\| < R$, and let $f(0) = 0$. Then</u>

$$\|f(z)\| \leq \frac{M\|z\|}{R}$$

<u>for $\|z\| < R$, where equality can hold if and only if $f(z) \equiv \frac{M}{R} Kz$, where K is a bicomplex constant such that $\|K\| = |K| = 1$.</u>

Proof: Since $f(0) = 0$, $g(0) = h(0) = 0$. Let $|z_1| < R\sqrt{2}$ and $z_3 = 0$. Then $\|z\| < R$, and $\|f(z)\| < M$, by hypothesis. Therefore $|g(z_1)| < M\sqrt{2}$ for $|z_1| < R\sqrt{2}$. By Schwarz's lemma for the complex case

$$|g(z_1)| \leq \frac{M}{R}|z_1|$$

for $|z_1| < R\sqrt{2}$ and equality holds only if $g(z_1) \equiv \frac{M}{R} K_1 z_1$, where $|K_1| = 1$. Similarly

$$|h(z_3)| \leq \frac{M}{R}|z_3|$$

for $|z_3| < R\sqrt{2}$ and equality holds only if $h(z_3) \equiv \frac{M}{R} K_3 z_3$, where $|K_3| = 1$. Then for $\|z\| < R$,

$$\|f(z)\| = \frac{1}{\sqrt{2}}\sqrt{|g(z_1)|^2 + |h(z_3)|^2} \leq \frac{1}{\sqrt{2}}\sqrt{\frac{M^2}{R^2}|z_1|^2 + \frac{M^2}{R^2}|z_3|^2} =$$

$$= \frac{M}{R}\frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |z_3|^2} = \frac{M\|z\|}{R}$$

and equality holds only if

$$f(z) \equiv \frac{M}{R} K_1 z_1 e_1 + \frac{M}{R} K_3 z_3 e_3 = \frac{M}{R} Kz,$$

where $K = K_1 e_1 + K_3 e_3$ with $|K_1| = 1$ and $|K_3| = 1$. This is the case if and only if $\|K\| = |K| = 1$.

<u>Definition: Let $f_n(z)$ be an infinite sequence of bicomplex-valued functions, defined on a set S. The sequence shall be termed uniformly convergent if for every $\varepsilon > 0$ there exists an integer $N(\varepsilon) > 0$ such that $\|f_m(z) - f_n(z)\| < \varepsilon$ for every $m, n \geq N(\varepsilon)$ and for every z in S.</u>

If the $f_n(z)$ are analytic for all n it is easily verified that a necessary and sufficient condition for $f_n(z)$ to converge uniformly on S is that $g_n(z_1)$ converges uniformly on S_1 and $h_n(z_3)$ converges uniformly on S_3.

Vitali's Convergence Theorem: Let $f_n(z)$ be a sequence of functions, each analytic in a region T. Let

$$\|f_n(z)\| \leq M$$

for every n and for every z in T, and let $f_n(z)$ tend to a limit, as $n \to \infty$, at a set S of points that is not contained in the nil-planes with respect to a finite number of points, and such that the closure of S is in T. Then $f_n(z)$ tends uniformly to a limit on any closed subset of T, the limit being therefore an analytic function of z in T.

Proof: Since $\|f_n(z)\| \leq M$, $|g(z_1)| \leq M\sqrt{2}$ and $|h(z_3)| \leq M\sqrt{2}$. The projections S_1 and S_3 of the set S are infinite point sets having a limit point in T_1 and T_3, respectively, since the closure of S is in T, and S is not contained in the nil-planes with respect to a finite number of points. Therefore Vitali's convergence theorem applies to the $g_n(z_1)$ and $h_n(z_3)$, which are uniformly convergent in every closed subset of T_1 and T_3, respectively. Thus the sequence $f_n(z)$ is uniformly convergent in every closed subset of T.

This section will be concluded by an example to show that even if an analytic function f(z) is assumed to have a derivative different from zero or a nil-factor at a point, the mapping performed by f(z) need not preserve angles in the bicomplex space. The example will be based on the following:

Lemma: The transformation $w = az+b$, where a is not equal to zero or a nil-factor, takes every straight line into a straight line.

Proof: Let α and β be two distinct points on the line. Then $a\alpha+b \neq a\beta+b$. On the line, z is of the form $k\alpha+(1-k)\beta$, where k is real. Then $az+b = a[k\alpha+(1-k)\beta]+b = k(a\alpha+b)+(1-k)(a\beta+b)$, which is a straight line through the distinct points $a\alpha+b$ and $a\beta+b$.

Example: Consider the transformation $w = (2e_1+e_3)z$. This function is analytic in the entire space. The derivative has the constant value $2e_1+e_3$, which is not zero or a nil-factor. Now let the bicomplex variable z be denoted by $x+iy+jz+iju$ where x,y,z,u are real. In the plane $x=0$, $u=0$, consider the lines $y=0$ and $y=z$. These lines intersect at the origin at an angle of $\pi/4$. By the lemma the transformation takes these lines into lines again. But a point on the line $y=z$ is a nil-factor and the transformation leaves the line fixed; while a point on the line $y=0$ is of the form kj, where k is real. Then

$$(2e_1+e_3)kj = -\tfrac{1}{2}\,ki+\tfrac{3}{2}kj$$

which determines the line $x=0$, $u=0$, $3y+z=0$ and this line does not make an angle of $\pi/4$ with the line $y=z$.

VII. TAKASU'S ALGEBRA

Takasu (9) has considered the theory of functions of a generalized bicomplex variable

$$z = x_1 + jx_2 + j'(x_3 + jx_4)$$

where $j^2 = \mu + \nu j$, $j'^2 = \mu' + \nu' j'$, and μ, ν, μ', ν', are real constants and x_1, x_2, x_3, x_4 are real variables. The fundamental operations are defined by requiring the usual formal laws of operation to hold. The system of such numbers z is seen to be an associative commutative linear algebra with the modulus $1+0j+j'(0+0j)$, denoted by 1 (see Dickson (1), pages 4-7).

In view of Ringleb's decomposition theorem one might ask: For what values of μ, ν, μ', ν' is this system reducible? Scheffers has given the following criterion: (See Dickson (1), page 27.)

A linear associative algebra A with a modulus is reducible if and only if it contains an element $x \neq 0,1$ such that $x^2 = x$ and $xz = zx$ for every element z of A. An equivalent condition is that there exist in A an element $y \neq \pm 1$ such that $y^2 = 1$ and $yz = zy$ for every z in A.

Proof of equivalence: Assume that there exists $x \neq 0,1$ such that $x^2 = x$. Then $(2x-1)^2 = 4x^2 - 4x + 1 = 4(x^2 - x) + 1 = 1$. Since $x \neq 0$, then $2x-1 \neq -1$; since $x \neq 1$, then $2x-1 \neq 1$.

Conversely, assume that there exists y such that $y^2 = 1$, and $y \neq \pm 1$. Then $\left[\frac{1}{2}(y+1)\right]^2 = \frac{1}{4}(y^2+2y+1) = \frac{1}{4}(1+2y+1) = \frac{1}{4}(2y+2) = \frac{1}{2}(y+1)$. Since $y \neq +1$, then $\frac{1}{2}(y+1) \neq 1$; since $y \neq -1$, then $\frac{1}{2}(y+1) \neq 0$.

Clearly y commutes with every element z of A if and only if x commutes with z.

Further $\left[\frac{1}{2}(1-y)\right]^2 = \frac{1}{4}(y^2-2y+1) = \frac{1}{4}(2-2y) = \frac{1}{2}(1-y)$. And $\frac{1}{2}(1+y) \cdot \frac{1}{2}(1-y) = \frac{1}{2}(1-y^2) = 0$. Thus $\frac{1}{2}(1+y)$ and $\frac{1}{2}(1-y)$ are idempotent divisors of zero (nil-factors).

To simplify the computation through which it will be determined for what values of μ, ν, μ', ν' the system is reducible, the definitions of j^2 and j'^2 will be transformed in the following way.

Since $j^2 = \nu j + \mu$, then $(2j-v)^2 = 4j^2 - 4\nu j + \nu^2 = 4\nu j + 4\mu - 4\nu j + \nu^2 = 4\mu + \nu^2$. If $\nu^2 + 4\mu > 0$, then $\sqrt{\nu^2+4\mu}$ is real, and

$$\left(\frac{2j-\nu}{\sqrt{\nu^2+4\mu}}\right)^2 = 1$$

If $\nu^2+4\mu < 0$, then $\sqrt{-(\nu^2+4\mu)}$ is real, and

$$\left(\frac{2j-\nu}{\sqrt{-(\nu^2+4\mu)}}\right)^2 = -1$$

If $\nu^2+4\mu = 0.$, then

$$(2j-v)^2 = 0$$

Similarly from the equation defining j'^2,

$$\left(\frac{2j'-\nu'}{\sqrt{\nu'^2+4\mu'}}\right)^2 = 1 \quad \text{if} \quad \nu'^2+4\mu' > 0$$

$$\left(\frac{2j'-\nu'}{\sqrt{-(\nu'^2+4\mu')}}\right)^2 = -1 \quad \text{if} \quad \nu'^2+4\mu' < 0$$

$$(2j-\nu')^2 = 0 \quad \text{if} \quad \nu'^2+4\mu' = 0$$

These relations divide the algebra into nine cases, which may be reduced to five by isomorphisms. If the relations are represented briefly as $K^2=1$, $K^2=-1$, $K^2=0$, and $K'^2=1$, $K'^2=-1$, $K'^2=0$, then the cases may be tabulated as follows:

$K'^2 \backslash K^2$	1	-1	0
1	A	B	C
-1	D	E	F
0	G	H	J

The cases B and D are seen to be isomorphic simply by interchanging K and K'. Case E is seen to be isomorphic to casses B and D, since $(KK')^2$ 1, and the elements K and KK' of case E can be made to correspond to the elements K and K' of case B. By interchanging K and K' it is also seen that cases C and G are isomorphic and that cases H and F are isomorphic. These facts summarized in tabular form become:

$K'^2 \backslash K^2$	1	-1	0
1	I	II	III
-1	II	II	IV
0	III	IV	V

Cases I, II, and III are reducible since they contain an element whose square is unity, and the system is commutative. Case II is, of course, that of the ordinary bicomplex variable discussed in the previous sections of this paper.

In case IV, let $K^2 = -1$, and $K'^2 = 0$. The elements $e_1 = 1$, $e_2 = K$, $e_3 = K'$, $e_3 = KK'$ form a basis with the multiplication table

e_1	e_2	e_3	e_4
e_2	$-e_1$	e_4	$-e_3$
e_3	e_4	0	0
e_4	$-e_3$	0	0

Then if a,b,c,d are real,

$$(ae_1+be_2+ce_3+de_4)^2 = (a^2-b^2)e_1+2abe_2+2(ac-bd)e_3+2(ad+bc)e_4$$

This is equal to one if and only if

$$\begin{cases} a^2 - b^2 = 1 \\ ab = 0 \\ ac-bd = 0 \\ ad+bc = 0 \end{cases}$$

This system of equations has only two solutions, $a=\pm1$, $b=c=d=0$. Thus the algebra in case IV is irreducible.

In case V, $K^2=0$ and $K'^2=0$. The elements $e_1=1$, $e_2=K$, $e_3=K'$, $e_4=KK'$ form a basis with the multiplication table

e_1	e_2	e_3	e_4
e_2	0	e_4	0
e_3	e_4	0	0
e_4	0	0	0

Then if a,b,c,d are real

$$(ae_1+be_2+ce_3+de_4)^2 = a^2e_1+2abe_2+2ace_3+2(ad+bc)e_4$$

This is equal to one if and only if

$$\begin{cases} a^2=1 \\ ab=0 \\ ac=0 \\ ad+bc=0 \end{cases}$$

This system of equations has only two solutions, $a=\pm1$, $b=c=d=0$. Thus the algebra in case V is irreducible.

Many questions about the function theory in the separate cases can be raised. Case I decomposes into four separate sub-algebras, which are each isomorphic to the algebra of real numbers. Then to what extent will the theory of functions in case I resemble that of a complex or ordinary bicomplex variable?

Cases IV and V have nil-potent elements, for which all power series would terminate. Case IV contains the complex number system as a sub-algebra. Then is it possible, or again impossible (see section V), to continue a complex-valued analytic function of a complex variable in this space beyond its natural boundary in the complex plane?

BIBLIOGRAPHY

1. Dickson, L. E. Linear Algebras, Cambridge: at the University Press, 1914.

2. Futagawa, Michiji. "On the Theory of Functions of a Quaternary Variable," Tôhoku Math. Journal, vol. 29 (1928), pp. 175-222.

3. Futagawa, Michiji. "On the Theory of Functions of a Quaternary Variable (Part II)," Tôhoku Math. Journal, vol. 35 (1932) pp. 69-120.

4. Scorza, Dragoni G. "Sulle funzioni olomorfe di una variabile bicomplessa," Reale Accademia d'Italia, Memorie della classe di scienze fisiche, matematiche e naturali, vol. 5 (1934), pp. 597-665.

5. Spampinato, N. "Sulla Rappresentaziene delle funzioni di variabile bicomplessa totalmente derivabili," Annali di Matematica pura ed applicata, (4), vol. 14 (1936) pp. 305-325.

6. Scheffers, G. "Verallgemeinerung der Grundlagen der gewöhnlich complexen Funktionen," I, II, Berichte uber die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-physikalische Klasse, vol. 45 (1893), pp. 828-848; vol. 46 (1894), pp. 120-134.

7. Ringleb, Friedrich. "Beiträge zur Funktionentheorie in hyperkomplexen Systemen," I. Rendiconti del Circolo Matematico di Palermo, vol. 57 (1933), pp. 311-340.

8. Hille, Einar. Functional Analysis and Semi-Groups, American Mathematical Society (New York City) Colloquium Publications, vol. 31, 1948.

9. Takasu, Tsurusaburo. "Theorie der Funktionen einer allgemeinen bikomplexen Veranderlichen, I," Tohoku Science Reports, vol. 32. p. 1, 1943.

10. Ward, James A. "A Theory of Analytic Functions in Linear Associative Algebras," Duke Math Journal, vol. 7, (1940), p. 233.

11. Bieberbach, Ludwig. Lehrbuch der Funktionentheorie, vol. I, B. G. Teubner, Leipzig and Berlin, 1921.