A REDUCED SYSTEM OF DIFFERENTIAL EQUATIONS FOR THE INVARIANTS OF TERNARY FORMS

by

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INTRODUCTION

Let us consider a ternary form of order $n$ in the variables $x_1, x_2, x_3$, with coefficients $a_i, b_i, c_i, \ldots$.

Under the general linear transformation

$$
T = \begin{cases} 
    x_1 = \alpha_1 s_1 + \alpha_{12} s_2 + \alpha_{13} s_3 , \\
    x_2 = \alpha_{21} s_1 + \alpha_{22} s_2 + \alpha_{23} s_3 , \\
    x_3 = \alpha_{31} s_1 + \alpha_{32} s_2 + \alpha_{33} s_3 ,
\end{cases}
$$

the determinant $\Delta$ of the coefficients not being zero, the form is replaced by a new form of the same order in the variables $s_1, s_2, s_3$, and with coefficients which may be denoted by $H_i, B_i, C_i, \ldots$. If for every such transformation a function $F(H_i, B_j, C_k, \ldots)$ of the new coefficients is equal to $\Delta$ times the same function $F(a_i, b_j, c_k, \ldots)$ of the original coefficients, then $F$ is called an invariant of the form under transformation $T$, or is said to be invariant under $T$.

In what follows we shall be interested in the fact that the necessary and sufficient condition that a homogeneous function $F$ of the coefficients of a ternary form be an invariant under $T$, is that $F$ be a solution of a certain complete system of nine linear partial differential equations of the first order.*

*(E.B. Elliott, Algebra of Quantics, p. 379)
By the use of the special transformation

\[
\begin{align*}
X_1 &= S_1 + \alpha_{12} S_2 + \alpha_{13} S_3, \\
X_2 &= S_2, \\
X_3 &= S_3,
\end{align*}
\]

rather than the general transformation \( T \), Junker** has found solutions for two equations of the general system of partial differential equations satisfied by an invariant of the ternary form. He has shown how, by the introduction of these solutions as new variables, the number of differential equations which the invariants must satisfy may be reduced by two.

The purpose of this paper is to obtain the solutions of three equations of the general set and, by their introduction as new variables, to reduce the number of differential equations in the complete system by three. We shall find that the invariants under the special transformation

\[
T_1 = \left\{ \begin{array}{l}
X_1 = S_1 + \alpha_{12} S_2 + \alpha_{13} S_3, \\
X_2 = S_2 + \alpha_{23} S_3, \\
X_3 = S_3,
\end{array} \right.
\]

(1)

are the desired solutions. In that which follows we shall define a seminvariant of the ternary form as any homogeneous function of the coefficients which is invariant under the

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special transformation $T_1$.

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The general ternary \( n-i \), \( C_n \) may be expressed as a determinant with linear elements in the form:

\[
C_n = \begin{vmatrix}
 x_1 + a_{11} x_2 + b_{11} x_3 & a_{12} x_2 + b_{12} x_3 & \cdots & a_{1n} x_2 + b_{1n} x_3 \\
 a_{21} x_2 + b_{21} x_3 & x_1 + a_{22} x_2 + b_{22} x_3 & \cdots & \cdots \\
 \cdots & \cdots & \cdots & \cdots \\
 a_{n1} x_2 + b_{n1} x_3 & a_{n2} x_2 + b_{n2} x_3 & \cdots & x_1 + a_{nn} x_2 + b_{nn} x_3 
\end{vmatrix}
\]

When the determinant is expanded the general term of \( C_n \) reads

\[
I^{i\ell k} x_1^i x_2^j x_3^k \quad (i + j + k = n)
\]

where by \( I^{r,0} \) \((r < n)\) we denote the sum of all principal minors of order \( n-r \) of the determinant

\[
\mathcal{A} = \begin{vmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 \cdots & \cdots & \cdots & \cdots \\
 a_{n1} & \cdots & \cdots & a_{nn} 
\end{vmatrix}
\]

and by \( I^{r,s} \) \((r+s \leq n)\) the sum of all possible determinants formed by replacing the columns \( S \) at a time, on each of the principal minors of \( I^{r,0} \) by the corresponding columns from the minors of

\[
\mathcal{B} = \begin{vmatrix}
 b_{11} & b_{12} & \cdots & b_{1n} \\
 \cdots & \cdots & \cdots & \cdots \\
 b_{n1} & \cdots & \cdots & b_{nn} 
\end{vmatrix}
\]

It is evident that \( I^{n,0} = 1 \).

---

Thus \( C_n \) may be written
\[
\begin{align*}
I_n^0 x_n &+ x_n^{n-1} \left( I_{n-1}^0 x_2 + I_{n-1}^0 x_3 \right) \\
&+ x_n^{n-2} \left( I_{n-2}^0 x_2 + I_{n-2}^0 x_3 + I_{n-2}^0 x_3^2 \right) \\
&+ x_n^{n-3} \left( I_{n-3}^0 x_2^3 + I_{n-3}^0 x_3^2 + I_{n-3}^0 x_3 + I_{n-3}^0 x_3^3 \right) \\
&+ \cdots
\end{align*}
\]
(3)

The coefficients \( I^{rs} \) may be obtained from \( \theta = I_0^0 \) by means of two differential operators,
\[
D_a = \sum_{i=1}^{n} \frac{\partial}{\partial a_{ii}} \quad \text{and} \quad D_{ab} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \frac{\partial}{\partial a_{ij}}.
\]
If for the ternary cubic, for example, we apply the operator \( D_a = \sum_{i=1}^{3} \frac{\partial}{\partial a_{ii}} \) to
\[
\theta = \begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = I_{1,0}^0,
\]
we have
\[
D_a I_{1,0}^0 = \begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{vmatrix} + \begin{vmatrix}
a_{11} & a_{13} \\
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{vmatrix} + \begin{vmatrix}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{vmatrix} = I_{1,0}^1.
\]
The application of \( D_a \) to \( I_{1,0}^0 \), in turn, gives
\[
D_a I_{1,0}^1 = 2 (a_{11} + a_{22} + a_{33}) = 2 I_{2,0}^1.
\]
In general, for any ternary form,
\[
D_a I_{r,s}^{rs} = (r+1) I_{r+s}^{rs}.
\]
If we represent the result of \( k \) successive applications of \( D_a \) upon \( I_{r,s}^{rs} \) by \( D_a I_{r,s}^{k} = \sum_{i=1}^{n} \left( \frac{\partial}{\partial a_{ii}} \right)^k I_{r,s}^{rs} \), we have \( D_a I_{r,s}^{k} = \frac{(r+k)!}{r!} I_{r+k,s}^{rs} \).

The application of the operator
\[
D_{ab} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \frac{\partial}{\partial a_{ij}}
\]
to the determinant \( I_{opp} \) gives for the ternary cubic

*(*df., Stouffer, loc. cit., p. 363)
\[ D_{ab} I^{ao} = \begin{vmatrix} b_{11} a_{12} a_{13} \\ b_{21} a_{22} a_{23} \\ b_{31} a_{32} a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} b_{12} a_{13} \\ a_{21} b_{22} a_{23} \\ a_{31} b_{32} a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} a_{12} b_{13} \\ a_{21} a_{22} b_{23} \\ a_{31} a_{32} b_{33} \end{vmatrix} = I^{ao}. \]

Similarly, we find
\[ D_{ab} I^{ao} = 2 \begin{vmatrix} b_{11} a_{12} b_{13} \\ b_{21} a_{22} b_{23} \\ b_{31} a_{32} b_{33} \end{vmatrix} + 2 \begin{vmatrix} a_{11} b_{12} a_{13} \\ a_{21} b_{22} a_{23} \\ a_{31} b_{32} a_{33} \end{vmatrix} + 2 \begin{vmatrix} a_{11} a_{12} b_{13} \\ a_{21} a_{22} b_{23} \\ a_{31} a_{32} b_{33} \end{vmatrix} = 2 I^{ao}, \]

and
\[ D_{ab} I^{br} = \begin{vmatrix} a_{11} b_{12} b_{13} \\ a_{21} b_{22} b_{23} \\ a_{31} b_{32} b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} a_{12} a_{13} \\ b_{21} a_{22} a_{23} \\ b_{31} a_{32} a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} b_{12} a_{13} \\ b_{21} b_{22} a_{23} \\ b_{31} b_{32} a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} a_{12} b_{13} \\ b_{21} a_{22} b_{23} \\ b_{31} a_{32} b_{33} \end{vmatrix} = I^{br}. \]

In general, for any ternary form,
\[ D_{ab} I^{rs} = (s+1) I^{rs+1}, \]

and
\[ D_{ab}^{k} I^{rs} = \frac{(s+k)!}{s!} I^{rs+k}, \]

where \( D_{ab}^{k} I^{rs} \) represents the result of \( k \) successive applications of \( D_{ab} \) upon \( I^{rs} \).

When expressed in terms of \( I^{rs} \) the set of differential equations* which must be satisfied by an invariant of the ternary \( n \)-ic may be written in the following form. The summations extend over the coefficients of \( C_n \).

*(Elliott, loc. cit., p. 379)
\[
\begin{align*}
E_1 &= \sum \sum (i+1) I_{i+1, k} \frac{\partial f}{\partial I_{i+1, k}} = 0, \\
E_2 &= \sum \sum (i+1) I_{i+1, k-1} \frac{\partial f}{\partial I_{i+1, k}} = 0, \\
E_3 &= \sum \sum (n-i+1-k) I_{n-i+1-k} \frac{\partial f}{\partial I_{n-i+1-k}} = 0, \\
E_4 &= \sum \sum (k+1) I_{i+1, k+1} \frac{\partial f}{\partial I_{i+1, k+1}} = 0, \\
E_5 &= \sum \sum (n-i-k+1) I_{i+1, k-1} \frac{\partial f}{\partial I_{i+1, k-1}} = 0, \\
E_6 &= \sum \sum (k+1) I_{n-i-k+1} \frac{\partial f}{\partial I_{n-i-k+1}} = 0, \\
E_7 &= \sum \sum (i-1) I_{i+1, k} \frac{\partial f}{\partial I_{i+1, k}} = 0, \\
E_8 &= \sum \sum (n-i-2k) I_{i+1, k} \frac{\partial f}{\partial I_{i+1, k}} = 0, \\
E_9 &= \sum \sum (n-2i-k) I_{n-i-k} \frac{\partial f}{\partial I_{n-i-k}} = 0.
\end{align*}
\]

The above nine equations form a complete system of differential equations in that the alternant of any pair of them gives either zero or one of the set.

In order to find the seminvariants of the ternary n-ic we must first determine which equations of set (4) the seminvariants must satisfy. Since our results are to be valid for every n, it will be sufficient to identify these equations by means of some convenient value for n. We shall consider the case for the ternary cubic.
After transformation (1) has been applied to $C_3$ in form (5), the new coefficients $I_{r,s}^{r,s}$ may be calculated. The finite transformations of the coefficients take the form

$$I_0^0 = I_0^0 + a_{12} I_1^0 + \ldots,$$
$$I_1^0 = I_1^0 + 2a_{12} I_2^0 + \ldots,$$
$$I_0^2 = I_0^2 + 2a_{13} I_1^1 + a_{13} I_0^1 + a_{12} I_1^2 + \ldots.$$ 

From these finite transformations we may build the infinitesimal transformations of the coefficients. If $\delta t$ is an infinitesimal, $a_{12}$, $a_{13}$, and $a_{23}$ are the values of the parameters which reduce the finite transformations to identities, and $\phi_{i,k}$ are arbitrary constants, then the infinitesimal transformations are obtained from the corresponding finite transformations by replacing the parameters $a_{i,k}$ by $(a_{i,k} + \phi_{i,k} \delta t)$. 

Denoting by $\delta I_{r,s}^{r,s}$ the increment taken by $I_{r,s}^{r,s}$ under the infinitesimal transformation, we have the following infinitesimal transformations of the coefficients,

$$\delta I_0^0 = \phi_{12} I_1^0 \delta t,$$
$$\delta I_1^0 = 2\phi_{12} I_2^0 \delta t,$$
$$\delta I_0^2 = 2\phi_{23} I_1^1 \delta t + \phi_{13} I_0^1 \delta t + \phi_{12} I_1^2 \delta t,$$
$$\delta I_1^1 = 2\phi_{23} I_2^1 \delta t + 2\phi_{13} I_2^0 \delta t + 2\phi_{12} I_1^0 \delta t + \phi_{13} I_1^1 \delta t + \phi_{12} I_1^2 \delta t,$$

The condition that a function $f(I_0^0, I_1^0, \ldots, I_{r,s}^{r,s})$ of the coefficients be invariant under transformation (1), is obtained by equating to zero the increment $\delta f$ taken
by that function under the infinitesimal transformations of the coefficients.

We have

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial I_0^0} \delta I_0^0 + \frac{\partial f}{\partial I_0^1} \delta I_0^1 + \frac{\partial f}{\partial I_0^2} \delta I_0^2 + \cdots + \frac{\partial f}{\partial I_0^n} \delta I_0^n = 0,
\]

\[
= \phi_{12}(I_0^0 \frac{\partial f}{\partial I_0^0} + 2I_1^0 \frac{\partial f}{\partial I_0^1} + I_{12} \frac{\partial f}{\partial I_0^2} + \cdots )
\]

\[
+ \phi_{13}(I_{11} \frac{\partial f}{\partial I_1^1} + 2I_{12} \frac{\partial f}{\partial I_1^2} + \cdots )
\]

\[
+ \phi_{23}(2I_0^0 \frac{\partial f}{\partial I_0^0} + 2I_{10} \frac{\partial f}{\partial I_1^0} + \cdots ) = 0.
\]

If this is to be true for arbitrary values of \( \phi_{ik} \) it follows that

\[
I_0^0 \frac{\partial f}{\partial I_0^0} + 2I_{10} \frac{\partial f}{\partial I_1^0} + I_{12} \frac{\partial f}{\partial I_0^2} + \cdots = 0,
\]

\[
I_{11} \frac{\partial f}{\partial I_1^1} + 2I_{12} \frac{\partial f}{\partial I_1^2} + \cdots = 0,
\]

\[
2I_0^0 \frac{\partial f}{\partial I_0^0} + 2I_{10} \frac{\partial f}{\partial I_1^0} + \cdots = 0.
\]

These are seen to be equations \( E_1, E_2, \) and \( E_3 \) of (4) for the case \( n = 3 \). It follows that the seminvariants of the ternary \( n \)-ic must be solutions of the corresponding three equations \( E_1, E_2, \) and \( E_3 \).

A case which is of more interest to us than the preceding is the effect of the transformation (1) upon the determinant (2) before expansion. Here we see the transformations of the elements of the determinant in contrast to the transformations of the coefficients \( I_0^5 \) as a whole. Denoting the transforms of the coefficients \( a_{ij} \) and \( b_{ij} \) by \( \overline{a}_{ij} \) and \( \overline{b}_{ij} \) we have the following finite transformations of these elements,
\[ a_{ii} = a_{ii} + \alpha_{12}, \]
\[ \bar{a}_{ij} = a_{ij} \quad (i \neq j), \]
\[ b_{ii} = b_{ii} + \alpha_{23} a_{ii} + \alpha_{13}, \]
\[ b_{ij} = b_{ij} + \alpha_{23} a_{ij} \quad (i \neq j). \]

From these the infinitesimal transformations of elements are seen to be

\[
\begin{cases}
\delta a_{ii} = \phi_{i2} \delta t, \\
\delta a_{ij} = 0 \quad (i \neq j), \\
\delta b_{ii} = (\phi_{13} + \phi_{23} a_{ii}) \delta t, \\
\delta b_{ij} = (\phi_{23} a_{ij}) \delta t \quad (i \neq j).
\end{cases}
\]

From the above transformations it follows that

\[ \delta I^{r,0} = \sum_{i,k} \frac{\partial I^{r,0}}{\partial a_{ik}} \delta a_{ik} = \sum_{a_{ii}} \frac{\partial I^{r,0}}{\partial a_{ii}} \phi_{i2} \delta t, \]

or

\[ \delta I^{r,0} = (r + 1) \phi_{i2} I^{r+1,0} \delta t. \]

Thus the differential equation which any function of \( I^{r,0} \) must satisfy in order to be invariant under the particular transformation \( T_1 \), is

\[
\sum_{i=0}^{n-1} (i+1) I^{i+1,0} \frac{\partial f}{\partial I^{i,0}} = 0.
\]

Equation (6) is the form which \( E_1 \) would take if applied only to \( I^{r,0} \). Any function of \( I^{r,0} \) is a solution of equations \( E_2 \) and \( E_3 \) since no partial derivatives with respect to \( I^{r,0} \) appear in them.

We now may apply almost directly the results
obtained by Stouffer* in two papers on seminvariants and
invariants of linear homogeneous differential equations.
In order to establish the connections with our problem we
shall give here a brief outline of some of the results in
these two papers.

The differential equations considered are of the form

\[ y''_i + \sum_{k=1}^{n} \left( 2p_{i,k} y'_k + q_{i,k} y_k \right) = 0 \quad (i = 1, 2, \ldots, n), \]

where \( p_{i,k} \) and \( q_{i,k} \) are functions of the independent variable
\( x \). A seminvariant with respect to the system of equations
(7) is defined as any function of the coefficients and their
derivatives which is invariant under the transformation of
the dependent variables.

\[ y_k = \sum_{\lambda=1}^{n} \alpha_k^\lambda (x) y_\lambda \quad (k = 1, 2, \ldots, n). \]

If this function is also invariant under the transformation
of the independent variable, \( S = S(x) \), it is called an
invariant.

It is shown by the use of infinitesimal transformations that

\[ I^{0,0} = \begin{vmatrix} u_{11} & u_{12} \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{vmatrix} \]

is a seminvariant, where \( u_{i,k} \) are certain functions of \( p_{i,k} \)

*"On Seminvariants of Linear Homogeneous Differential Equa-
Linear Homogeneous Differential Equations," Proceedings of
and \( q_{ik} \) and the first derivatives \( p'_{ik} \) of \( p_{ik} \) with respect to \( x \), and also that the operator
\[
D_u = \sum_{i=1}^{n} \frac{\partial}{\partial u_{ii}}
\]
transforms any seminvariant into a seminvariant. Furthermore all independent seminvariants involving only \( p_{ik}, p'_{ik}, \) and \( q_{ik} \), are given by \( I^{r_0} = D_u I^{r_0} \) \( (r = 0, 1, \ldots, n-1) \).

In extending the search for seminvariants, certain auxiliary functions \( V_{ik} \) are defined in terms of \( p_{ik} \) and \( q_{ik} \) and their derivatives \( p'_{ik}, p''_{ik}, \) and \( q'_{ik} \).

It is proved that
\[
D_{uv} = \sum_{i=1}^{n} \sum_{k=1}^{n} V_{ik} \frac{\partial}{\partial u_{ik}}
\]
operating on the seminvariants \( I^{r_0} \) \( (r = 0, 1, \ldots, n-1) \), transforms them into seminvariants \( I^{r_s} \) \( (s = 0, 1, \ldots, r_s \leq n) \).

The process of finding the seminvariants \( I^{r_s} \) may be continued in order to include higher derivatives of \( p_{ik} \) and \( q_{ik} \). However, it is our purpose only to establish the analogy between the seminvariants \( I^{r_s} \) of this problem and the coefficients \( I^{r_s} \) of equation (3).

Let us next consider Stouffer's method of finding the invariants which depend only on \( I^{r_s} \). The infinitesimal transformations taken by the elements of \( I^{r_s} \) under the transformation \( \xi = \xi(x) \) are given by
\[
\left\{ \begin{array}{l}
\delta u_{ii} = ( - 2 \phi' u_{ii} + \frac{1}{2} \phi^{(2)} ) \delta t, \\
\delta u_{ik} = - 2 \phi' u_{ik} \delta t \quad (i \neq k), \\
\delta V_{ik} = ( - 3 \phi' v_{ik} - 2 \phi'' u_{ik} + \frac{1}{2} \phi^{(3)} ) \delta t, \\
\delta V_{i} = ( - 3 \phi' v_{i} - 2 \phi'' u_{i} ) \delta t \quad (i \neq k), \\
\end{array} \right.
\]
where \( \phi, \phi', \ldots, \phi^{(n)} \) represent an arbitrary function of \( x \) and the first to the fourth derivatives of that function with respect to \( x \). From these values of \( S u_{\alpha} \), it is shown that an invariant \( F(I_0^0, I_1^0, \ldots, I_r^0) \) must satisfy the differential equation

\[
\sum_{j=0}^{n-1} (j+1) I_j^0 \frac{\partial f}{\partial I_j^0} = 0.
\]

The most general solutions of (9) are found to be

\[
\Theta_\alpha = (I^{n-\alpha})^{-\alpha} q! \sum_{\ell=2}^{d-\alpha} \frac{(I^{n-\alpha})^{d-\ell}}{(d-\ell)! \ell!} \Theta_\ell - \frac{\alpha! n^{n-\alpha}}{(n-1)(n-2) \cdots (n-\alpha+1)} \quad (\alpha = 2, 3, \ldots, n).
\]

From the infinitesimal transformations \( S u_{\alpha} \) and \( S v_{\alpha} \), the invariants involving \( I_r^s \) \((S > 0)\) are developed by means of the operators \( D_u \) and \( D_u v \). The infinitesimal transformation under \( \tau = \tau (x) \), of any relative invariant \( \Theta_{\alpha+\beta} \), homogeneous of degree \( \alpha \) in \( u_{iK} \) and of degree \( \beta \) in \( v_{iK} \), is shown to be

\[
\delta \Theta_{\alpha+\beta} = -(2\alpha + 3\beta) \phi' \Theta_{\alpha+\beta} \delta \tau.
\]

From this it follows that

\[
\delta(D_u v \Theta_{\alpha+\beta}) = \sum_{i=1}^{n} \sum_{K=1}^{N} v_{iK} \frac{\partial}{\partial u_{iK}} \left( \delta \Theta_{\alpha+\beta} - S u_{iK} \frac{\partial}{\partial u_{iK}} \Theta_{\alpha+\beta} \right)
+ \sum_{i=1}^{n} \sum_{K=1}^{N} S v_{iK} \frac{\partial}{\partial u_{iK}} \Theta_{\alpha+\beta},
\]

\[
= \left[ - (2\alpha + 3\beta - 2) \phi' D_u v \Theta_{\alpha+\beta} \right. \\
\left. + (-3\phi' D_u v \Theta_{\alpha+\beta} - 2\phi'' \Theta_{\alpha+\beta} + \frac{1}{2} \phi''' D_u \Theta_{\alpha+\beta}) \right] \delta \tau,
\]
or

\[ s(D_{uv} \Theta_{\alpha+\beta}) = \begin{bmatrix} -(2d+3\beta+1) \phi' D_{uv} \Theta_{\alpha+\beta} \\
-2d \phi'' \Theta_{\alpha+\beta} + \frac{1}{2} \phi^b D_u \Theta_{\alpha+\beta} \end{bmatrix} \eta. \]

The fact that \( D_u \Theta_2 = 0 \), gives by induction \( D_u \Theta_q = 0 \)
\((q=2, 3, \ldots, n)\), which results in

\[ s(D_{uv} \Theta_d) = \begin{bmatrix} -(2d+1) \phi' D_{uv} \Theta_d - 2d \phi'' \Theta_d \end{bmatrix} \eta. \]

The infinitesimal increments taken by the functions

\[ \Theta_{d,\beta} = 2 \Theta_2 D_{uv} \Theta_{d,\beta-1} - (d+\beta-1) \Theta_{d,\beta-1} D_{uv} \Theta_2 \]
\((q=3, 4, \ldots, n; \beta = 1, 2, \ldots, \alpha)\),

may now be found immediately and the functions \( \Theta_{\omega,\beta} \) are thus proved to be the desired invariants.

It is proved that there is only one more independent invariant involving \( I_{r,s}^{r,s} \) \((s > 0)\), and this is found by the method of undetermined coefficients to be

\[ s_{2x} = 4 \Theta_2 (I_n^{n-1})^2 - \frac{3n}{n-1} I_n^{n-2} \Theta_2 - (\Theta_2')^2, \]

where \( \Theta_2' \) is the derivative of \( \Theta_2 \) with respect to \( x \).

A comparison of our problem with Stouffer's, as outlined above, makes it clear that:

1. The seminvariants \( I_{r,s}^{r,s} \) of his problem are the same functions of \( u_{ik} \) and \( v_{ik} \) as the coefficients \( I_{r,s}^{r,s} \) of
our problem are of $a_{ik}$ and $b_{ik}$.

(2) The operators $D_u$ and $D_{uv}$ correspond exactly to the operators $D_a$ and $D_{ab}$.

(3) The infinitesimal transformations (8) of the seminvariants $I_{rs}^r$ are identical in form with the infinitesimal transformations (5) of the coefficients $I_{rs}^r$. This becomes evident if we replace the coefficients $\phi_1', \frac{1}{2} \phi_2^{(3)}, \frac{1}{2} \phi_3^{(4)}$, and $-2 \phi''$ of (8) by $0, \phi_2, \phi_3, \text{ and } \phi_{23}$ respectively.

(4) Differential equation (9) of his work is exactly our equation (6).

It follows as a consequence of these facts that, with certain minor changes in constants, the seminvariants of our problem are the same functions of the coefficients $I_{rs}^r$ as the invariants $\Theta_\alpha, \Theta_\beta, \text{ and } O_{02}$ of Stouffer's problem are of his seminvariants $I_{rs}^r$.

Returning then to the ternary n-1c and our own notation we have the following functions as the general solutions of equation (6) and thus as seminvariants involving $I_{rs}^r$,

$$ R_\alpha = (I_{n-i}^0)^{\alpha} - \alpha! \sum_{i=2}^{d-1} \frac{1}{(q-i)!} \frac{1}{i!} (I_{n-i}^0)^{d-i} R_i $$

$$ - \frac{\alpha!}{(n-1)(n-2)\cdots(n-q+1)} \int_{n-q}^{n} (I_{n}^0)^{d-1} \int_{n-q+1}^{n} (I_{n}^0)^{d-1} \cdots (n-1) $$

We are now interested in the seminvariants involving $I_{rs}^r$ ($s > 0$) and shall follow the method outlined above in obtaining them. We shall assume $R_{\alpha,\beta}$ to be such a
seminvariant which is homogeneous in $a_{ij}$ of degree $\alpha$ and in $b_{ij}$ of degree $\beta$. If we represent the increment taken by a function $F$ under the infinitesimal transformations (5) by $\delta F$, we have

$$\delta A_{\alpha\beta} = 0,$$

and

$$\delta (D_{ab} A_{\alpha\beta}) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} \frac{\partial}{\partial a_{ik}} \left( \delta a_{ik} \frac{\partial}{\partial a_{ik}} A_{\alpha\beta} \right),$$

\begin{equation}
(11)
\end{equation}

$$+ \sum_{i=1}^{n} \sum_{k=1}^{n} \delta b_{ik} \frac{\partial}{\partial a_{ik}} A_{\alpha\beta},$$

$$= (\phi_{13} D_{a} A_{\alpha\beta} + \alpha \phi_{23} A_{\alpha\beta}) \delta t.$$  

Since $D_{a} A_{\alpha} = 0 \ (\alpha = 2, 3, \ldots, n)$, we have for $\beta = 0$ in equation (11),

$$\delta (D_{ab} A_{\alpha}) = \alpha \phi_{23} A_{\alpha} \delta t.$$  

The expression

$$R_{\alpha, 1} = 2 R_{2} D_{ab} A_{\alpha} - \alpha R_{\alpha} D_{ab} A_{2} \ (\alpha = 3, 4, \ldots, n),$$

is therefore a seminvariant.

Similarly

$$R_{\alpha, 2} = 2 R_{2} D_{ab} A_{\alpha, 1} - \alpha R_{\alpha, 1} D_{ab} A_{2} \ (\alpha = 3, 4, \ldots, n),$$

and in general

\begin{equation}
(12)
R_{\alpha\beta} = 2 R_{2} D_{ab} A_{\alpha, \beta - 1} - \alpha R_{\alpha, \beta - 1} D_{ab} A_{2} \ (\alpha = 3, 4, \ldots, n; \beta = 1, 2, \ldots, d),
\end{equation}

are invariant under transformation (1) or are seminvariants of our ternary n-ic.
We find easily that we may substitute $D_{ab} R_2$ for $\Theta_2'$ in $\mathcal{O}_{o2}$ of the above outline, and can write the last seminvariant involving $I^{r,s}$ $(s > 0)$ as follows

$$B = 4R_2(I^{h, i})^2 - \frac{8h}{h-1} I^{h-2, i} I^{n, 0} R_2 - (D_{ab} R_2)^2.$$  

Since the seminvariants $R_{\alpha \beta}$ and $B$ are defined in terms of $R_2$, the resulting system is valid only under the assumption $R_2 \neq 0$. Consideration of the case for $R_2 = 0$ will be left until a later time. In this paper we shall assume $R_2 \neq 0$.

Our seminvariants $R_\alpha, R_{\alpha \beta}$, and $B$ may be proved independent just as were $\Theta_\alpha, \Theta_{\alpha \beta}$, and $\mathcal{O}_{o2}$, for if they are arranged in the order $R_\alpha, B,$ and $R_{\alpha \beta}$, and according to ascending values of $\alpha$ and $\beta$ each contains a coefficient $I^{r,s}$ not in those which precede.

From the above we have

**THEOREM I.** The functions $R_\alpha, R_{\alpha \beta}$, and $B, R_2 \neq 0$, defined in equations (10), (12), (13), are solutions of the three equations $E_1, E_2$ and $E_5$ of the nine differential equations of system (4) which the invariants of a ternary n-ic must satisfy, or, are seminvariants of our ternary n-ic.

**II**

**THE DIFFERENTIAL EQUATIONS IN TERMS OF THE SEMINVARIANTS**

The expressions $R_\alpha, R_{\alpha \beta}, B,$ and $I^{n, 0}, R_2 \neq 0$, may
be introduced as new variables into the set of nine differential equations (4), and the new coefficients expressed in terms of these semi-invariants. The equations \( E_1, E_2, \) and \( E_5 \) will be satisfied identically by these new variables and thus the number of differential equations in the set will be reduced by three.

By the proper choice of the three parameters \( a_{12}, a_{13}, \) and \( a_{23} \) of transformation (1), we may in general reduce to zero the three transformed coefficients \( \overline{I}^{n-1,0}, \overline{I}^{n-1,1}, \) and \( \overline{I}^{n-2,1} \) of the resulting ternary n-ic. If transformation (1) is applied to the n-ic in form (3) we have

\[
\overline{I}^{n-1,0} = n \overline{I}^{n,0} a_{12} + \overline{I}^{n-1,0}, \\
\overline{I}^{n-1,1} = n \overline{I}^{n,0} a_{13} + \overline{I}^{n-1,0} a_{23} + \overline{I}^{n-1,1}, \\
\overline{I}^{n-2,1} = n(n-1) \overline{I}^{n,0} a_{12} a_{13} + (n-1) \overline{I}^{n,0} a_{12} a_{23} \\
\quad + (n-1) \overline{I}^{n-1,0} a_{12} + (n-1) \overline{I}^{n-1,0} a_{13} + 2 \overline{I}^{n-2,0} a_{23} + \overline{I}^{n-2,1}.
\]

If then we choose

\[
a_{12} = \frac{\overline{I}^{n-1,0}}{n \overline{I}^{n,0}}, \\
a_{13} = \frac{2 \overline{I}^{n-1,1} \overline{I}^{n-2,0} - \overline{I}^{n-2,1} \overline{I}^{n-1,0}}{(n-1)(\overline{I}^{n,0})^2 - 2n \overline{I}^{n-2,0} \overline{I}^{n,0}}, \\
a_{23} = \frac{n \overline{I}^{n-2,1} \overline{I}^{n,0} - (n-1) \overline{I}^{n-1,0} \overline{I}^{n-1,1}}{(n-1)(\overline{I}^{n,0})^2 - 2n \overline{I}^{n-2,0} \overline{I}^{n,0}},
\]

the transformed coefficients \( \overline{I}^{n-1,0}, \overline{I}^{n-1,1}, \) and \( \overline{I}^{n-2,1} \) will be zero. This choice for \( a_{12}, a_{13}, \)
and \( \alpha_{23} \) is always possible since \( I^{n,0} = 1 \), and since we have assumed \( (I^{n-1,0})^2 - \frac{2n}{n-1} \frac{\partial}{\partial I^{n-2,0}} I^{n,0} = \alpha_{23} \), not equal to zero. Not only \( \alpha_{d}, \alpha_{d\beta}, B \), and \( I^{n,0} \), but any functions of these seminvariants will be invariant under transformation (1), and therefore unchanged by the various choices of the parameters \( \alpha_{12}, \alpha_{13}, \text{ and } \alpha_{23} \).

Thus when working with these functions we may, as a means of simplification and without loss of generality, set

\[ I^{n-1,0} = I^{n-1,1} = I^{n-2,1} = 0. \]

In order to introduce the new variables \( \alpha_{d}, \alpha_{d\beta}, B \), and \( I^{n,0} \) into a partial differential equation \( E_{i} = 0 \), we operate with \( E_{i} \) on \( \alpha_{d}, \alpha_{d\beta}, B \), and \( I^{n,0} \).

The resulting equation will read

\[ \sum E_{i}(\alpha_{d}) \frac{\partial f}{\partial \alpha_{d}} + E_{i}(\alpha_{d\beta}) \frac{\partial f}{\partial \alpha_{d\beta}} + E_{i}(B) \frac{\partial f}{\partial B} + E_{i}(I^{n,0}) \frac{\partial f}{\partial I^{n,0}} = 0, \]

where by \( E_{i}(F) \) we represent the results obtained by operating on \( F \) by \( E_{i} \).

We shall assume in that which follows that after having operated with \( E_{i} \) upon the seminvariants we have set

\[ I^{n-1,0} = I^{n-1,1} = I^{n-2,1} = 0. \]

First consider

\[ E_{3} = \sum \sum (n-i+1-k) I^{i-1,k} \frac{\partial f}{\partial I^{i,k}} = 0. \]

If we operate upon the seminvariants with \( E_{3} \) and express

*(Horn, Einführung in die Theorie der partiellen Differentialgleichungen.)*
the results in terms of the seminvariants, we have

\[ E_3 (I^n) = 0, \]

\[ E_3 (R^d) = \frac{\alpha(n-1)}{n} R_2 R_{d-1} + \frac{(n-2)}{n} R_{d+1}, \]

\[ E_3 (B) = \frac{(n-2)}{6n} R_{32} + \frac{3(n-2)}{2n} R_3 B, \]

(14) \[ E_3 (R_{d\beta}) = \frac{(n-2)}{n} \frac{R_3 R_{\alpha \beta}}{R_2} - \frac{\alpha(n-2)}{3n} \frac{R_{d,\beta-1} R_{31}}{R_2} + 2 R_2 E_3 (D_{ab} R_{\alpha, \beta-1}). \]

If \( \beta = 1 \) in (14), the last term also may be expressed directly in terms of the new variables and we have

\[ E_3 (R_{\alpha, 1}) = \frac{(n-2)}{n} \frac{R_3 R_{d1}}{R_2} + \frac{\alpha(n-1)}{n} R_2 R_{d-1, 1} + \frac{\alpha(n-2)}{n(n+1)} R_{d+1, 1} - \frac{\alpha(n-2)}{3n} \frac{R_{d3} R_{31}}{R_2}, \]

and if \( \beta = 2 \) we have

\[ E_3 (R_{d2}) = \frac{(n-2) a(n-2)}{3n} \frac{R_{31} R_{d1}}{R_2} + \frac{2(n-2)}{n} \frac{R_3 R_{d2}}{R_2} + \frac{\alpha(n-2)}{an} \frac{R_3 R_{\alpha, B}}{R_2} \]

\[ + \frac{(a-1)(n-\alpha)}{n(n+1)} R_{d+1, 2} - \frac{(n-\alpha)}{n} R_{\alpha, 1} B + \frac{\alpha(n-2)}{n} R_{\alpha-1, 2} R_2 \]

\[ - \frac{\alpha(n-1)}{n} R_{d-1} R_2 B - \frac{\alpha(n-2)}{6n} \frac{R_{32} R_{\alpha}}{R_2}. \]

In the general case it is necessary to prove that \( E_3 (D_{ab} R_{\alpha, \beta-1}) \) is a term which for any particular value of \( \beta \) may be expressed in terms of the new variables. We shall give this proof in three parts.

(I) We shall first show that \( D_{ab} (R_{d\beta}) \) may be expressed as a function of \( R_2, R_d, D_{ab} R_2, D_{ab} R_2, D_{ab} R_\alpha, D_{ab} R_\alpha, \ldots, D_{ab} R_d. \)
If we can find a value of $\gamma$ such that $R_{\alpha}\gamma$ is a function of

(15) $R_{\alpha}, R_{\alpha}, D_{ab} R_{\alpha}, D_{ab}^2 R_{\alpha}, D_{ab} R_{\alpha}, \ldots, D_{ab}^r R_{\alpha}, (a > 2)$,

then from (12) and the fact that $D_{ab}^3 (R_{\alpha}) = 0$, it follows that $R_{\alpha, \gamma + 1}$ is a function of (15) and $D_{ab}^r R_{\alpha}$.

But we may write $R_{\alpha, 2}$ in the form

$R_{\alpha, 2} = 2 R_2 D_{ab} (2 R_2 D_{ab} R_{\alpha} - \alpha R_{\alpha} D_{ab} R_2)$

$- \alpha (2 R_2 D_{ab} R_{\alpha} - \alpha R_{\alpha} D_{ab} R_2) D_{ab} R_2$.

It now follows that $R_{\alpha, 3}$, $R_{\alpha, 4}$, , , , and finally that $R_{\alpha, \beta}$ may be expressed as a function of (15). Thus (I) follows immediately since the operation by $D_{ab}$ on $R_{\alpha, \beta}$ written as above can introduce at most only one new term $D_{ab}^r R_{\alpha}$.

(II) We shall next show that

$$E_3 [D_{ab}^q (R_{\alpha})] = \frac{(\gamma - q)(n - q)}{n (n + 1)} D_{ab}^q R_{\alpha + 1} - \frac{\alpha (n - 1)}{n} R_2 D_{ab}^q R_{\alpha - 1}$$

$(q = 1, 2, \ldots, \alpha)$.

By direct operation with $D_{ab}$ upon (10) we obtain

$$D_{ab} (R_{\alpha}) = \alpha (I^{n - 1})^{q-1} I^{n - 1} - \alpha! \sum_{i=2}^{q-2} \frac{1}{(q-1)!} (I^{n - 1})^{q-1} I^{n - 1} R_i$$

$$- \alpha! \sum_{i=2}^{q-2} \frac{1}{(q-1)!} (I^{n - 1})^{q-1} D_{ab} (R_i) - \alpha I^{n - 1} R_{\alpha - 1}$$

$$- \alpha I^{n - 1} D_{ab} R_{\alpha - 1} - \frac{\alpha!}{(n-1)(n-2) \ldots (n-q+1)} (I^{n - 1})^{q-1}$$

A second operation gives us

$$D_{ab}^2 (R_{\alpha}) = \ldots \ldots \text{a series of terms of 2nd or higher degree in } I^{n-h} \text{ and } I^{h-n} \text{ plus}$$

$$-2 \alpha I^{n-h} D_{ab} R_{\alpha - 1} - \alpha I^{n-h} D_{ab}^2 R_{\alpha - 1}$$

$$- \frac{2 \alpha!}{(n-1)(n-2) \ldots (n-q+1)} (I^{n - 1})^{q-1}$$
By induction we may show that $D_{ab}^q (A_\alpha) = (---)$-terms of
2nd or higher degree in $I^{n-\ell,0}$ and $I^{n-\ell,1}$) plus
$$ -q \alpha I^{n-\ell,1} D_{ab} A_{\alpha-1} - \alpha I^{n-\ell,0} D_{ab} A_{\alpha-1} $$
$$ - \frac{q! \alpha! (n-\ell-1) I^{n-\ell,q} (I^{n,0})^{d-1}}{(n-1)(n-2)} \frac{1}{(n-q+1)} $$

If we operate on the above by $E_3$ and set $I^{n-\ell,0}=I^{n-\ell,1}=I^{n-2,1}=0$
we have (II).

(III) We shall show finally that
$$ D_{ab}^q (A_\alpha) \quad (q = 1, 2, \ldots, d) $$
may be expressed as a function
of the invariants $A_\alpha, A_{\alpha \beta}, \ldots$ and $B$.

If we apply the operator $D_{ab}$ to $A_\alpha$ and $A_{\alpha \beta}$ and set
$I^{n-\ell,0}=I^{n-\ell,1}=I^{n-2,1}=0$, we have

$$ D_{ab} (A_2) = 0, $$
$$ D_{ab}^2 (A_2) = \frac{B}{2 A_2}, $$
$$ D_{ab}^3 (A_2) = 0, $$
$$ D_{ab} (A_d) = \frac{A_{d1}}{2 A_2} \quad (d = 3, \ldots, n), $$
$$ D_{ab}^2 (A_d) = \frac{1}{4 A_2} (A_{d2} + 2 (d-1) A_{d1} D_{ab} A_2 + d(d-2) A_{d} (D_{ab} A_2)^2 ) $$
$$ \uparrow 2 \alpha A_2 A_{\alpha} D_{ab}^2 A_2), $$
$$ D_{ab} (A_{d\beta}) = \frac{A_{d\beta+1}}{2 A_2} \quad (\beta = 1, \ldots, (d-1)). $$

If we can find a value for $r$ such that $D_{ab}^r (A_d)$ may be
expressed as a function of

$$ A_2, A_{d1}, A_{d2}, \ldots, A_{d \alpha \gamma}, D_{ab} A_2, D_{ab}^2 A_2, $$
then by means of the above results $D_{ab}^r (A_d) \quad (r = 1, 2, \ldots, d-1)$,
may in turn be expressed as a function of $A_2, A_d, A_d\beta$, and $B$. In (16) we have $D_{ab}(A_d)$ defined in terms of functions (17). Thus by induction (III) follows.

By the three steps (I), (II), and (III), it now follows that $E_3(D_{ab} R_{d,\beta-1})$ may be expressed in terms of the seminvariants, thus completing the proof that $E_3(R_{d,\beta})$ may be expressed in terms of them as new variables.

The method of introduction of the new variables $A_d, A_d\beta, B,$ and $I^{n'}$ into the remaining differential equations of the set (4) parallels very closely the method used for $E_3$.

In the case of

$$E_4 = \sum \sum (k+l) I^{-l,k+1} \frac{\partial f}{\partial I^{-l,k}} = 0,$$

we have

$$E_4(I^{n'}) = 0,$$

$$E_4(A_d) = \frac{(n-q)}{2(n+1)} \frac{R_{d+1,1}}{A_2},$$

$$E_4(B) = \frac{(n-2)}{12n} \frac{R_{33}}{A_2^2} + \frac{3(n-2)}{4n} \frac{R_{31}B}{A_2^2},$$

$$E_4(A_d\beta) = \frac{(n-2)}{6n} \frac{R_{31}A_d}{A_2^2} \frac{\alpha(n-2)}{12n} \frac{R_{32}R_{d,\beta-1}}{A_2^2}$$

$$- \frac{\alpha(n-2)}{4n} \frac{R_{31}B A_d\beta - 1}{A_2^2} + 2 A_2 E_4(D_{ab} R_{d,\beta-1}).$$

We find thru the operation upon $D_{ab}^y(A_d)$ by $E_4$ that
Thus from (I and III) in the discussion of \( E_3(R_{d\beta}) \)
it follows that \( E_4(R_{d\beta}) \) can be expressed in terms
of \( R_d, R_{d\beta}, B, \) and \( I^{\eta,\rho} \).

In the case of
\[
E_\omega = \sum \sum (k+1) I^{i,k+1} \frac{\partial f}{\partial I^{ik}} = 0,
\]
we have
\[
E_\omega (I^{\eta,\rho}) = 0,
E_\omega (R_d) = \frac{R_{d1}}{2 R_2},
E_\omega (B) = 0,
E_\omega (R_{d\beta}) = D_{ab} (R_{d\beta}).
\]
It has been shown earlier that \( D_{ab} (R_{d\beta}) \) is a function
of \( R_d, R_{d\beta}, \) and \( B \).

In the case of
\[
E_\eta = \sum \sum (i-k) I^{ik} \frac{\partial f}{\partial I^{ik}} = 0,
\]
we have
\[
E_\eta (I^{\eta,\rho}) = \eta I^{\eta,\rho},
E_\eta (R_d) = \alpha (n-1) R_d,
E_\eta (B) = 2 (2n-3) B,
E_\eta (R_{d\beta}) = 2 (n-1) R_{d\beta} + R_2 E_\eta (D_{ab} R_{d\beta} R_{\alpha\beta}).
\]
The fact that
\[ E_q(D_{ab} A_d) = (n \alpha - \alpha - q) D_{ab} A_d, \]
shows as above by means of (I) and (III) that \( E_q(A_{\alpha \beta}) \)
can be expressed in terms of \( A_{\alpha}, A_{\alpha \beta}, B, \) and \( I^{h\rho}. \)

In the case of

\[ E_q = \sum \sum (n - i - 2 k) I^{i k} \frac{\partial f}{\partial I^{i k}} = 0, \]

we have

\[ E_q(I^{h\rho}) = 0, \]
\[ E_q(A_{\alpha}) = \alpha A_{\alpha}, \]
\[ E_q(B) = 0, \]
\[ E_q(A_{\alpha \beta}) = 2 A_{\alpha \beta} + 2 \mu_2 E_q(D_{ab} A_{\alpha \beta - 1}). \]

We find that

\[ E_q(D_{ab} A_d) = (\alpha - 2 q) D_{ab} A_d, \]
which assures by means of (I) and (III) that \( E_q(A_{\alpha \beta}) \)
can be expressed in terms of \( A_{\alpha}, A_{\alpha \beta}, B, \) and \( I^{h\rho}. \)

In the case of

\[ E_q = \sum \sum (n - 2 i - k) I^{i k} \frac{\partial f}{\partial I^{i k}} = 0, \]

we have

\[ E_q(I^{h\rho}) = -n I^{h\rho}, \]
\[ E_q(A_{\alpha}) = \alpha (2 - n) A_{\alpha}, \]
\[ E_q(B) = 2 (3 - 2 n) B, \]
\[ E_q(A_{\alpha \beta}) = (n - 2 n) A_{\alpha \beta} + 2 A_{\alpha \beta} E_q(D_{ab} A_{\alpha \beta - 1}). \]
From the fact that
\[ E_q(D_{ab} R_\alpha) = (2d - n\rho - q) D_{ab} R_\alpha, \]
it follows as above from (I) and (III) that \( E_q(R_{\alpha \beta}) \)
can also be expressed in terms of \( R_\alpha, R_{\alpha \beta}, B, \) and \( I^{n_0}. \)

The seminvariants \( I^{n_0}, R_\alpha, R_{\alpha \beta}, \) and \( B, \) have been
introduced as new variables into the set of differential
equations \( E_1, E_2, \ldots, E_q. \) As a result
we have

**THEOREM II.** The following six partial differential equations
must be satisfied by the invariants of a ternary n-ic,

\[ (R_2 \neq 0). \]

\[ E_3 = \sum \left[ \frac{(n-1)}{n} R_2 R_{d-1} + \frac{(n-d)}{n} R_{d+1} \right] \frac{\partial f}{\partial R_2} + \left[ \frac{(n-2) R_{d-1} + 3(n-2) R_{d-1} B}{R_2} \right] \frac{\partial f}{\partial B} \]

\[ + \sum \sum \left[ \frac{(n-2) R_3 R_{d-1} - d(n-2) R_{d-1} R_{d+1} + 2 R_{d-1} E_3(D_{ab} R_{\alpha \beta})} \right] \frac{\partial f}{\partial R_{d-1}} = 0, \]

\[ E_4 = \sum \left[ \frac{(n-2)}{2(n+1)} R_{d+1} R_2 \right] \frac{\partial f}{\partial R_2} + \left[ \frac{(n-2) R_3 R_{d-1} + 3(n-2) R_3 B}{2 n} \right] \frac{\partial f}{\partial B} \]

\[ + \sum \sum \left[ \frac{(n-2) R_3 R_{d-1}}{n} - \frac{d(n-2) R_2 R_{d-1}}{12 n} - \frac{d(n-2) R_3 B R_{d-1}}{4 n} \right] \frac{\partial f}{\partial R_{d-1}} \]

\[ + 2 R_2 E_4(D_{ab} R_{\alpha \beta}) \] \( \frac{\partial f}{\partial R_{d-1}} = 0, \]
The summations extend over all subscripts \( \alpha \) and \( \beta \), where
\( I^{n,0}_{n}, A_{\alpha}, A_{\alpha \beta}, \) and \( B \), are the functions of the coefficients of the ternary \( n \)-ic as defined in equations (3), (10), (12) and (13).

In the case of the ternary cubic we have

**THEOREM III.** The following six differential equations must be satisfied by the invariants of a ternary cubic \( f(A_3 \neq 0) \).

\[
E_6 = \sum \left[ \frac{A_{\alpha \beta}}{2 A_{\alpha}} \right] \frac{\partial f}{\partial A_{\alpha}} + \sum \sum \left[ D_{ab}(A_{\alpha \beta}) \right] \frac{\partial f}{\partial A_{\alpha \beta}} = 0,
\]
\[
E_7 = n I^{n,0} \frac{\partial f}{\partial I^{n,0}} + \sum \left[ (n-1) A_{\alpha} \right] \frac{\partial f}{\partial A_{\alpha}} + (4n-6) B \frac{\partial f}{\partial B}
+ \sum \sum \left[ 2(n-1) A_{\alpha \beta} + 2 A_{\alpha} E_6 (D_{ab} A_{\alpha \beta}) \right] \frac{\partial f}{\partial A_{\alpha \beta}} = 0,
\]
\[
E_8 = \sum \alpha A_{\alpha} \frac{\partial f}{\partial A_{\alpha}}
+ \sum \sum \left[ 2 A_{\alpha \beta} + 2 A_{\alpha} E_8 (D_{ab} A_{\alpha \beta}) \right] \frac{\partial f}{\partial A_{\alpha \beta}} = 0,
\]
\[
E_9 = \sum \left[ (2 \alpha - 3n) A_{\alpha} \right] \frac{\partial f}{\partial A_{\alpha}} - n I^{n,0} \frac{\partial f}{\partial I^{n,0}} + (6-4n) B \frac{\partial f}{\partial B}
+ \sum \sum \left[ (4-2n) A_{\alpha \beta} + 2 A_{\alpha} E_9 (D_{ab} A_{\alpha \beta}) \right] \frac{\partial f}{\partial A_{\alpha \beta}} = 0.
\]
\[ E_4 = 36 A_2 A_{31} \frac{\partial f}{\partial A_2} + (36 A_{31}^2 - \frac{2}{3} A_2^2 B + \frac{2}{3} A_3^2 B - 4 A_3 A_{32}) \frac{\partial f}{\partial A_{31}} \]
\[ + (27 A_3 A_{31} B - 2 A_3 A_{33}) \frac{\partial f}{\partial A_{32}} + (162 A_{31} B - 12 A_{33}) \frac{\partial f}{\partial B} \]
\[ + (-4 A_2^2 B^2 + 945 A_{31}^2 B - 18 A_{31} A_{33} + 4 A_3^2 B^2) \frac{\partial f}{\partial A_{32}} B - 108 A_{32}^2 \frac{\partial f}{\partial A_{33}} = 0, \]
\[ E_6 = 27 A_{31} \frac{\partial f}{\partial A_3} + 2 A_{32} \frac{\partial f}{\partial A_{31}} + A_{33} \frac{\partial f}{\partial A_{32}} \]
\[ + (30 A_{32} B - \frac{3 A_3 B^2}{2}) \frac{\partial f}{\partial A_{33}} = 0, \]
\[ E_7 = 3 I^{30} \frac{\partial f}{\partial I^{30}} + 4 A_2 \frac{\partial f}{\partial A_2} + 6 A_3 \frac{\partial f}{\partial A_3} + 6 B \frac{\partial f}{\partial B} \]
\[ + 9 A_{31} \frac{\partial f}{\partial A_{31}} + 12 A_{32} \frac{\partial f}{\partial A_{32}} + 15 A_{33} \frac{\partial f}{\partial A_{33}} = 0, \]
\[ E_8 = 2 A_2 \frac{\partial f}{\partial A_2} + 3 A_3 \frac{\partial f}{\partial A_3} + 3 A_{31} \frac{\partial f}{\partial A_{31}} + 3 A_{32} \frac{\partial f}{\partial A_{32}} + 3 A_{33} \frac{\partial f}{\partial A_{33}} = 0, \]
\[ E_9 = 3 I^{30} \frac{\partial f}{\partial I^{30}} + 2 A_2 \frac{\partial f}{\partial A_2} + 3 A_3 \frac{\partial f}{\partial A_3} + 6 A_{31} \frac{\partial f}{\partial A_{31}} \]
\[ + 6 B \frac{\partial f}{\partial B} + 9 A_{32} \frac{\partial f}{\partial A_{32}} + 12 A_{33} \frac{\partial f}{\partial A_{33}} = 0. \]

As a matter of convenience the factors 27, 54, 54, and (-3), were divided out of \( A_{31}, A_{32}, A_{33}, \) and \( B, \) respectively.