Non-Classical Continuum Theories for Solid and Fluent Continua

By

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Abstract

This dissertation presents non-classical continuum theories for solid and fluent continua. In these theories additional physics due to internal rotations and rotation rates arising from the Jacobian of deformation and the velocity gradient tensor as well as Cosserat rotations and rotation rates are considered. While the internal rotations and rotation rates are completely defined by the deformation physics, the Cosserat rotations and Cosserat rotation rates are additional degrees of freedom at a material point. The non-classical theories that only consider the internal rotations and the internal rotation rates are referred to as internal polar theories, while those that consider both are called polar or non-classical theories.

The conservation and balance laws and the constitutive theories are derived for non-classical continuum theories. It is shown that these non-classical theories require modifications of the balance laws used in classical continuum theories. In the presence of additional rotation and rotation rate physics in non-classical theories, the modifications of the balance laws used in classical continuum theories are not sufficient to ensure equilibrium of the deforming matter. It is shown that these theories require the *balance of moments of moments* as an additional balance law. The constitutive theories for solid and fluent continua are derived using the conditions resulting from the entropy inequality and the representation theorem. Use of integrity in their derivations ensures completeness of the resulting constitutive theories. Specific derivations and details of the constitutive theories for thermoelastic and thermoviscoelastic solids with and without memory are presented for small deformation, small strain physics. Detailed derivations of the constitutive theories for compressible as well as incompressible thermoviscous and thermoviscoelastic fluent continua are also presented. Retardation and/or memory moduli are derived for polymeric solids and fluids.

The present theories are compared with published works, particularly with the micropolar theories of Eringen, to highlight the significance and the thermodynamic consistency of the present work, as well as to contrast the differences.

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Contents

Chapter 1

Introduction

1.1 Introduction

The classical continuum theories have been effective tools for describing the deformation physics of continuous media for more than a century. In recent decades, however, materials that exhibit complex behavior that is difficult to describe using the classical theories have become increasingly common, and the need for better descriptions of their deformation physics is growing. The classical theory of continuous media consists of conservation and balance laws using fundamental principles of mechanics and thermodynamics based on the following assumptions:

(1) The matter is always in thermodynamic equilibrium.

(2) The matter is continuous.

(3) The matter and all disturbances are in an inertial frame.

(4) The matter is homogeneous and isotropic.

(5) Physics at a smaller scale is neglected.

(6) A material particle has only translational degrees of freedom.

In general, a theory of continuous media that deviates from these assumptions can be classified as a non-classical continuum theory. Many everyday engineering applications require consideration of physics that is over and beyond assumptions (1) – (6). Composite materials, ceramics impregnated with metal fibers, and heterogeneous media in general encompassing a large variety of solids and fluids require further considerations in the formulation of continuum theories beyond the assumptions (1) – (6). The other motivation for non-classical theories is to closely examine the basic underlying assumptions in the classical continuum theories and to possibly undertake the derivation of new theories in which these assumptions could be avoided. Such theories will naturally describe more enhanced and more precise physics compared to classical theories. The non-classical continuum theories incorporating internal rotations and internal rotation rates are such theories (called internal polar theories). The development of these theories suggests that perhaps further advancement can be made if additional degrees of freedom over and beyond translational degrees of freedom are endowed at each material point. This is done by assuming the existence of a triad at each material point with its axes parallel to the x-frame and assuming the existence of three rotations or rotation rates about the axes of this triad as additional degrees of freedom. These additional degrees of freedom are called Cosserat rotations or Cosserat rotation rates. The non-classical theories that consider only Cosserat rotations and rotation rates are called Cosserat theories.

A closer examination of the deformation physics of solid and fluent continua reveals that the internal rotations or the internal rotation rates always exist in all deforming solid and fluent continua as these are due to the Jacobian of deformation and the velocity gradient tensor, hence their consideration in the continuum theory is essential. The Cosserat rotations or Cosserat rotation rates on the other hand are assumed to be additional degrees of freedom at a material point with the objective of further enhancing the internal polar theory. In view of the deformation physics, the internal rotations and the internal rotation rates arising from the Jacobian of deformation and the velocity gradient tensor must always be an intrinsic part of the non-classical continuum theory as these always exist in every deforming matter. Thus, non-classical theories with internal rotations or rotation rates are a meritorious approach to consider for a new continuum theory. This theory will correct deficiencies inherent in the classical continuum theories (due to lack of consideration of internal rotations or rotation rates) and provide further enhancement due to Cosserat rotations or rotation rates that will perhaps permit more precise kinematic considerations.

1.2 Literature Review

In their book [1], the Cosserat brothers discuss the existence of a triad of vectors, called directors, associated with each material particle. The directors give particles in these *Cosserat continua* orientation and therefore allow rotations. These rotations occur in addition to the material point translations/displacements. Resistance to these Cosserat rotations by the deforming matter results in conjugate moments. The influence of Cosserat rotations at a material point and the conjugate moments are fully accounted for in Reference [1] using what is called *Euclidean action* and variation of the internal energy density. This work was largely ignored until its revival and expansion in 1958 by Günther [2]. Following this, several authors including Toupin [3], Mindlin and Tiersten [4], Koiter [5], and Mindlin [6] used the concept of a Cosserat continuum to develop the so-called couple-stress theories in which a non-symmetric stress tensor generates moments.

In one of the first papers following the revival of interest in Cosserat theories, Aero and Kuvshinskii [7] in 1960 pointed out that classical mechanics is inadequate for short acoustic waves in crystals and for laws of piezoelectric phenomena. The authors presented what they call "a phenomenological theory of continuous media by taking into account rotational interaction among the particles." By assuming the existence of the moment tensor on the oblique plane of the tetrahedron, use of Cauchy principle, and balance of angular momenta, the derivation shows that Cauchy stress tensor must be non-symmetric and that the gradients of the Cauchy moment tensors are balanced by the antisymmetric components of the Cauchy stress tensor.

Eringen used the concept of the Cosserat continuum to develop his *micropolar theories* [8–20]. He describes the material point and its associated directors as existing and deforming within the elementary tetrahedron. In Reference [14], he treats the directors as rigid fibers embedded in the elementary tetrahedron that rotate relative to the tetrahedron's center of mass as well as displacing with the tetrahedron. These rotations are additional degrees of freedom that are external to the Jacobian of deformation. Eringen defines additional stress and strain measures to account for the new physics. These micropolar theories in general do not have closure and require additional compatibility equations to reduce the system to a unique solution. In the following decades, Lakes, et al. have applied Eringen's micropolar theories to various models and experiments for materials such as anisotropic bone [21–23], linear isotropic micropolar materials [24], negative Poisson's ratio materials [25], foams [26, 27], and piezoelectric solids [28]. Eringen's work on micropolar theories [14] is of special interest in context with the present work. Extensive work has been done in this area following Mindlin and Eringen. Readers may see References [29–68] for additional information.

The vast majority of work in this area uses the Lagrangian description for elastic Cosserat solids; fluids are very rarely considered. Furthermore, the strain energy density function is typically used to derive constitutive theories, limiting their use to materials without dissipation.

Eringen in 1964 [9] presented conservation and balance laws, as well as constitutive theories, for what he referred to as *simple microfluids*. These fluids are a generalization of Stokes fluids in which local micromotion is taken into account that influences the macro-behavior of the fluid. It is advocated in this paper that microfluids possess local inertia, hence new principles such as (i) conservation of micro-inertia moments and (ii) balance of first stress moments must be added. In another paper in 1966 [10], Eringen presented a theory of micropolar fluids that are a subcategory of the general microfluids and are much simpler in their mathematical formulation of the deformation physics. These fluids can support couple-stresses and body couples only. Such theories may adequately represent fluids consisting of bar-like elements. In Reference [17], Eringen presents details of conservation and balance laws as well as constitutive theories for various types of microfluids. The fundamental idea behind microfluids is that in a subvolume of an infinitesimal (a representative volume or elementary tetrahedron in continuum theories), there is assumed to be a triad of directors at a point that can rotate and/or elongate during the motion or deformation of the fluent continua, thus giving rise to different types of micro-theories. The aggregate influence of the physics of sub-scale motion and deformation on the tetrahedron is established. The additional physics of deformation of the directors naturally requires some restrictions so that the entire body (or volume) of deforming fluid remains in equilibrium. These additional principles are Eringen's conservation of micro-inertia moments and balance of first stress moments [17].

Recently, Surana, et al. [69–76] published fundamental work establishing *internal polar* theories for solid and fluent continua. In these theories, the polar decomposition of the Jacobian of deformation $\boldsymbol{J}$ into left or right stretch tensor $\boldsymbol{S}_l$ or $\boldsymbol{S}_r$ and orthogonal rotation tensor $\boldsymbol{R}$ and subsequent discarding of $\boldsymbol{R}$ do not occur. Instead, $\boldsymbol{J}$ is used in its entirety to derive conservation and balance laws. The rotations are defined by the antisymmetric part of $\boldsymbol{J}$ and are therefore completely internal to the description of the deforming matter, hence there is no need to introduce additional rotation degrees of freedom. Furthermore, there is no concept of directors or sub-continuum scale phenomena that would drive these rotations. In References [69–74], small strains and small strain rates are considered to derive the conservation of mass, balance of linear momenta, balance of angular momenta, balance of energy, and entropy inequality. In References [75, 76], these theories are extended to solids experiencing finite deformation and finite strain. A new balance law, the balance of moments of moments, is introduced to account for the additional physics due to rotations and rotation rates. This balance law, first proposed by Yang, et al. [77], leads to symmetry of the Cauchy moment tensor and has not been widely used, but is essential for equilibrium of the deforming matter. Constitutive theories provide closure to the mathematical model resulting from the conservation and balance laws.

1.3 Scope of Work

Classical continuum theories for isotropic, homogeneous continua are founded on the assumption that a material point has only translational degrees of freedom. Thus, for solid continua in the

Lagrangian description, displacements are the only degrees of freedom at a material point, and for fluent continua in the Eulerian description, velocities are the only degrees of freedom at a material point or a location. The origin of the present work proposed here is due to Surana, et al. [69–76], who showed that in case of solid continua if the Jacobian of deformation ($\boldsymbol{J}$) is a complete measure of the kinematics of deformation, then the Jacobian of deformation in its entirety must form the basis for the derivations of the conservation and balance laws. Polar decomposition of the Jacobian of deformation shows that the deformation physics in $\boldsymbol{J}$ can be decomposed into pure stretch and pure rotation. Classical continuum theories are based entirely on pure stretch (related to strain measures), and the rotation tensor (internal rotations due to $\boldsymbol{J}$) is completely ignored in these theories.

Surana, et al. [69, 71, 74–76] presented non-classical continuum theories for solid matter incorporating $\boldsymbol{J}$ in its entirety, thus incorporating the stretch tensor as well as internal rotations due to $\boldsymbol{J}$ in the conservation and balance laws. Surana, et al. have also presented constitutive theories for thermoelastic and thermoviscoelastic solids without memory for non-classical internal polar solids. In these non-classical theories, the degrees of freedom at a material point remain translations but the physics of internal rotations due to $\boldsymbol{J}$ are incorporated in the conservation and balance laws.

Likewise Surana, et al. [70, 72] have presented non-classical continuum theories for fluent continua that are based on the entirety of the velocity gradient tensor ($\bar{\boldsymbol{L}}$), hence these theories incorporate internal rotations due to $\bar{\boldsymbol{L}}$ that are neglected in the currently used classical continuum theories. There are also three degrees of freedom (velocities) at a material point in these non-classical theories, but the physics of internal rotation rates due to $\bar{\boldsymbol{L}}$ are incorporated in the thermodynamic framework. Surana, et al. [73] also presented constitutive theories for thermoviscous internal polar fluids.

In the work proposed here, the internal polar non-classical continuum theories of Surana, et al. [69–76] are extended for solid and fluid continua by including Cosserat rotations as an additional three unknown degrees of freedom at each material point. Both internal and Cosserat rotations and their rates are assumed to be about the axes of a triad parallel to the x-frame at each material point.

It is assumed that when the varying rotations and the rotation rates are resisted by the deforming matter, conjugate moments are created. Rotations, rotation rates, and the conjugate moments result in additional energy storage and/or dissipation.

The purpose of this work is to derive the conservation and balance laws for non-classical continuum theories for solid and fluid continua and the associated constitutive theories

(1) Non-classical continuum theory for solids in Lagrangian description incorporating internal and Cosserat rotations.

(2) Non-classical continuum theory for fluent continua incorporating internal and Cosserat rotation rates.

(3) Constitutive theories for solid continua in Lagrangian description based on small strain assumption:

   (a) Thermoelastic solids

   (b) Thermoviscoelastic solids without memory

   (c) Thermoviscoelastic solids with memory

(4) Constitutive theories for fluid continua:

   (a) Thermoviscous fluids

   (b) Thermoviscoelastic fluids with memory

Complete derivations of the conservation and balance laws are presented for both non-classical continuum theories. Derivations of the constitutive theories are based on the entropy inequality in conjunction with the representation theorem. Material coefficients are established in all constitutive theories.

Chapter 2

Notations and Preliminary Considerations

In this chapter some preliminary considerations are discussed that are important to the development and understanding of the theories presented in this dissertation.

2.1 Internal Polar Non-Classical Continuum Theories

The concepts used in References [69–76] are summarized in the following. If the Jacobian of deformation $[J]$ (finite deformation) or the displacement gradient tensor $[{}^dJ]$ (infinitesimal deformation) and the velocity gradient tensor $[\bar{L}]$ are measures of deformation in solid and fluent continua, then the thermodynamic frameworks for solid and fluent continua must incorporate $[J]$, $[{}^dJ]$, and $[\bar{L}]$ in their entirety. First, consider solid continua (using $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}$ as coordinates in the deformed and undeformed configurations, respectively). The Jacobian of deformation (or deformation gradient) $[J]$ and displacement gradient $[{}^dJ]$ are given by

$$[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = \left[\frac{\partial\{u\}}{\partial\{x\}}\right] + [I] = [{}^dJ] + [I] \tag{2.1}$$

Polar decompositions of $[J]$ and $[{}^dJ]$ give

$$[J] = [R][S_r] = [S_l][R] \tag{2.2}$$

$$[{}^dJ] = [{}^dR][{}^dS_r] = [{}^dS_l][{}^dR] \tag{2.3}$$

where $[S_r]$ and $[S_l]$ are the right and left symmetric and positive-definite stretch tensors and $[R]$ is an orthogonal rotation tensor. Decomposition of $[J]$ and $[{}^dJ]$ into symmetric ($[{}_sJ]$ and $[{}^d_sJ]$) and antisymmetric ($[{}_aJ]$ and $[{}^d_aJ]$) tensors gives

$$[J] = [{}_sJ] + [{}_aJ] \tag{2.4}$$

$$[{}_sJ] = \frac{1}{2}\left([J] + [J]^T\right) \tag{2.5}$$

$$[{}_aJ] = \frac{1}{2}\left([J] - [J]^T\right) \tag{2.6}$$

and

$$[{}^dJ] = [{}^d_sJ] + [{}^d_aJ] \tag{2.7}$$

$$[{}^d_sJ] = \frac{1}{2}\left([{}^dJ] + [{}^dJ]^T\right) \tag{2.8}$$

$$[{}^d_aJ] = \frac{1}{2}\left([{}^dJ] - [{}^dJ]^T\right) \tag{2.9}$$

It is clear that

$$[{}_aJ] = [{}^d_aJ] \tag{2.10}$$

Note that $[{}^d_sJ]$ is a measure of infinitesimal strain (based on the linearization of the Green strain tensor) and $[{}_aJ] = [{}^d_aJ]$ can only be used to determine the rotation angles. Thus, for finite deformation, consideration of $[J]$ in its entirety implies consideration of $[R]$ and $[S_r]$ (or $[S_l]$) or alternatively consideration of $[{}_aJ]$ and $[S_r]$ (or $[S_l]$); $[S_r]$ (or $[S_l]$) is a measure of stretches (strains) whereas $[R]$ and $[{}_aJ]$ are measures of the same rotation physics in $[J]$ but in two different forms: $[R]$ is a rotation matrix whereas $[{}_aJ]$ contains rotation angles corresponding to the rotation matrix $[R]$.

For infinitesimal deformation consider $[{}^dJ]$; hence, the strain measure $[{}^d_sJ]$ and the rotation measure $[{}^d_aJ]$ both must be considered in the thermodynamic framework. Thus, consideration of $[J]$ in its entirety implies incorporating $[R]$ or $[{}_aJ]$ in the presently used classical theories, as $[S_r]$ is already considered in the form of strain measures. When $[J]$ varies between neighboring material points, so does $[R]$ or $[_aJ]$, and if it is resisted by deforming solid continua, conjugate moments are created. Varying rotations and their rates and the conjugate moments clearly result in additional resistance to motion (i.e., additional energy storage and/or dissipation as well as possible additional modes of relaxation or memory).

In the case of small strains, consider $[{}^dJ]$. Therefore, the infinitesimal strain tensor $[{}^d_sJ]$ and the rotations in $[{}^d_aJ]$ (same as $[_aJ]$) must be considered in their entirety. The remaining physics of resistance offered by the continua to $[{}^d_aJ]$ and the resulting conjugate moments remain the same as described in the case of finite deformation.

This physics of internal rotations that vary between neighboring material points and the resulting conjugate moments has been incorporated in the derivation of the conservation and balance laws as well as the associated constitutive theories by Surana, et al. [69, 71, 74–76]. The resulting theories are referred to as internal polar non-classical continuum theories for solid continua.

In the case of fluent continua, the velocity gradient tensor $\bar{\boldsymbol{L}}$ must be considered in its entirety in the derivation of the conservation and balance laws. Polar decomposition of $\bar{\boldsymbol{L}}$ into rates of stretch and rates of rotation or its decomposition into symmetric and antisymmetric tensors clearly shows that the current thermodynamic framework for fluent continua neglects varying rotation rates between the neighboring material points. Surana, et al. [70, 72, 73] showed that when rotation rates are incorporated in the thermodynamic framework, the Cauchy stress tensor is not symmetric, the existence of moments is due to resistance to rotation rates by the fluent continua, the Cauchy principle and Cauchy moment tensor are also realized as in the case of solid continua, and the discussion of deformation measures described above for solid continua hold here as well except that in fluent continua it is $\bar{\boldsymbol{L}}$ rather than $\boldsymbol{J}$ or ${}^d\boldsymbol{J}$. Surana, et al. [70, 72, 73] also presented constitutive theories for the symmetric part of the Cauchy stress tensor, the symmetric Cauchy moment tensor, and the heat vector using the conditions resulting from the entropy inequality in conjunction with the theory of generators and invariants (i.e., the representation theorem) [78–99].

2.2 Notations and Measures

Quantities with an over-bar are quantities in the current (deformed) configuration (all quantities with an over-bar are functions of coordinates $\bar{x}_i$ and time t – the Eulerian description). Quantities without an over-bar are quantities referred to the reference configuration (these are functions of undeformed coordinates x_i and time t – Lagrangian description). The configuration at time $t = t_0 = 0$, commencement of the evolution, is considered as the reference configuration. Thus, x_i and $\bar{x}_i$ are coordinates of the same material point in the reference and current configurations, respectively, both measured in a fixed Cartesian x-frame. Quantities with a subscript comma and index are differentiated with respect to the indexed spatial coordinate (e.g. $Q_{,m} = \frac{\partial Q}{\partial x_m}$).

2.2.1 Solid Continua

Consider the Jacobian of deformation defined by $\boldsymbol{J} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial \bar{x}_j}{\partial x_i}$. The rows are covariant base vectors, whereas in Murnaghan's notation $[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = \left[\frac{\bar{x}_1, \bar{x}_2, \bar{x}_3}{x_1, x_2, x_3}\right]$, the columns are the covariant base vectors (i.e., in this definition $[J]$ is the transpose of $\boldsymbol{J}$ of the first definition). Both definitions are obviously covariant measures in the Lagrangian description. Likewise, $\bar{\boldsymbol{J}} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial x_j}{\partial \bar{x}_i}$ and $[\bar{J}] = \left[\frac{\partial\{x\}}{\partial\{\bar{x}\}}\right] = \left[\frac{x_1, x_2, x_3}{\bar{x}_1, \bar{x}_2, \bar{x}_3}\right]$ are also Jacobians of deformation but they are contravariant measures in the Eulerian description. Columns of $\bar{\boldsymbol{J}}$ are the contravariant base vectors whereas in the case of $[\bar{J}]$, its rows are the contravariant base vectors (i.e., $\bar{\boldsymbol{J}}$ is transpose of $[\bar{J}]$). In this work, the corresponding Jacobians are defined by Murnaghan's notation (whether $[J]$ or $\boldsymbol{J}$ symbol is used).

Since the work presented in this dissertation considers small strains and small rotations, the distinction between covariant and contravariant measures disappears as $\bar{x}_i \approx x_i$ (i.e., the deformed configuration is not substantially different from the undeformed configuration). For such deformation, det$[J]$ = det$[\bar{J}] \approx 1$, hence in the development of the theory there is a need to separate displacements from the deformed coordinates. The displacement gradient $[{}^dJ]$ in (2.1) is defined as

$$[^dJ] = \left[\frac{\partial\{u\}}{\partial\{x\}}\right] = \left[\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{2.11}$$

The Cauchy stress tensor is used as a measure of stress because the deformed and undeformed tetrahedron can be treated the same for small deformation. Hence, the conservation and balance laws must be based on the entirety of $[^dJ]$ (i.e., $[^d_sJ]$ and $[^d_aJ]$ both must be considered in the conservation and balance laws).

The displacement gradient $[^dJ]$ can be written in component form as

$$^dJ_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = {}^d_sJ_{ij} + {}^d_aJ_{ij} \tag{2.12}$$

in which

$$[^d_aJ] = \begin{bmatrix} 0 & {}_i\Theta_{x_3} & -{}_i\Theta_{x_2} \\ -{}_i\Theta_{x_3} & 0 & {}_i\Theta_{x_1} \\ {}_i\Theta_{x_2} & -{}_i\Theta_{x_1} & 0 \end{bmatrix} \tag{2.13}$$

$${}_i\Theta_{x_1} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \; ; \quad {}_i\Theta_{x_2} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) \; ; \quad {}_i\Theta_{x_3} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \tag{2.14}$$

Alternatively (2.14) can be derived as

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_i \times \boldsymbol{e}_j \frac{\partial u_j}{\partial x_i} = \epsilon_{ijk} \boldsymbol{e}_k \frac{\partial u_j}{\partial x_i} \tag{2.15}$$

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1 \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \boldsymbol{e}_2 \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \boldsymbol{e}_3 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{2.16}$$

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1 \left(-2{}_i\Theta_{x_1}\right) + \boldsymbol{e}_2 \left(-2{}_i\Theta_{x_2}\right) + \boldsymbol{e}_3 \left(-2{}_i\Theta_{x_3}\right) \tag{2.17}$$

The sign difference in (2.14) and (2.17) is due to the fact that rotations in (2.14) are clockwise, whereas quantities in (2.16) are twice the magnitude compared to those in (2.14) and are counterclockwise. In this work, (2.14) is considered as the definition of rotations (clockwise). The rotations defined in (2.14) exist at every material point in the deforming solid.

The right and left stretch tensors $[{}^dS_r]$ and $[{}^dS_l]$ are symmetric and positive-definite, and $[{}^dR]$ is an orthogonal rotation tensor, a rotation matrix corresponding to the rotation angles defined in (2.14). $[{}_a^dJ]$ and $[{}^dR]$ contain the same physics as these are both derived from $[{}^dJ]$, but in different forms. $[{}_a^dJ]$ contains rotation angles while $[{}^dR]$ is the corresponding rotation matrix or tensor. Both in their forms given here can be used in derivations as needed. The same holds true for $[R]$ and $[{}_aJ]$ derived from $[J]$. However, deriving $[{}^dR]$ from $[{}_a^dJ]$ (or $[R]$ from $[{}_aJ]$) or vice versa in general in $\mathbb{R}^3$ may not be possible or unique [67, 68, 100]. Fortunately there is no need for this here.

Incorporating $[{}^dJ]$ in its entirety in the derivation of conservation and balance laws implies incorporating $[{}_s^dJ]$ and $[{}_a^dJ]$; that is, both displacement gradients and rotations ${}_i\Theta_{x_1}$, ${}_i\Theta_{x_2}$, and ${}_i\Theta_{x_3}$ about the axes of a triad located at each material point are considered. Rotations in $[{}_a^dJ]$ are internal and are completely defined by the antisymmetric part of $[{}^dJ]$.

Let ${}_e\Theta_{x_1}$, ${}_e\Theta_{x_2}$, and ${}_e\Theta_{x_3}$ be the additional Cosserat rotations (unknown) about the same triad as used for internal rotations ${}_i\boldsymbol{\Theta}$, considered positive counterclockwise for consistency with published works. Let $[{}_a^e\gamma]$ be the antisymmetric matrix of rotation angles defined using rotations ${}_e\Theta_i$, then

$$[{}_a^e\gamma] = \begin{bmatrix} 0 & {}_e\Theta_{x_3} & -{}_e\Theta_{x_2} \\ -{}_e\Theta_{x_3} & 0 & {}_e\Theta_{x_1} \\ {}_e\Theta_{x_2} & -{}_e\Theta_{x_1} & 0 \end{bmatrix} \tag{2.18}$$

Let

$$[\mathbb{J}] = [{}_s^dJ] + [{}_a^dJ] - [{}_a^e\gamma] \tag{2.19}$$

$$[\mathbb{J}] = [{}_s^dJ] + [{}_ar] \tag{2.20}$$

$[\mathbb{J}]$ is referred to as the total deformation tensor.

$$[{}_a r] = [{}^d_a J] - [{}^e_a \gamma] = \begin{bmatrix} 0 & {}_t\Theta_{x_3} & -{}_t\Theta_{x_2} \\ -{}_t\Theta_{x_3} & 0 & {}_t\Theta_{x_1} \\ {}_t\Theta_{x_2} & -{}_t\Theta_{x_1} & 0 \end{bmatrix} \tag{2.21}$$

in which $[{}_a r]$ is the antisymmetric matrix containing total rotations ${}_t\Theta_{x_1}$, ${}_t\Theta_{x_2}$, and ${}_t\Theta_{x_3}$ about the axes of the triad at a material point, considered positive in the clockwise sense. Obviously,

$$\begin{aligned} {}_t\Theta_{x_1} &= {}_i\Theta_{x_1} - {}_e\Theta_{x_1} \\ {}_t\Theta_{x_2} &= {}_i\Theta_{x_2} - {}_e\Theta_{x_2} \\ {}_t\Theta_{x_3} &= {}_i\Theta_{x_3} - {}_e\Theta_{x_3} \end{aligned} \tag{2.22}$$

Due to varying $[\mathbb{J}]$ between material points, total rotations ${}_t\boldsymbol{\Theta}$ vary between neighboring material points. When these are resisted by the deforming matter, conjugate moments are generated which, together with ${}_t\boldsymbol{\Theta}$ and their rates, result in additional energy storage and/or dissipation, as well as additional rheology.

Remarks.

(1) $[{}^d_s J]$ represents the usual infinitesimal strain tensor as in the linear theory of elasticity.

(2) $[{}^d_a J]$, $[{}^e_a \gamma]$, and $[{}_a r]$ are antisymmetric tensors containing rotation angles, and are not measures of strain.

(3) Based on (1) and (2), $[\mathbb{J}]$ is not a strain tensor, but rather the addition of the strain tensor $[{}^d_s J]$ and the total rotation angle tensor $[{}_a r]$.

(4) The rotation tensors are tensors of rank two. This is obvious from the definitions of $[{}^d_a J]$, $[{}^e_a \gamma]$, and $[{}_a r]$. For example, definition of ${}^d_a\boldsymbol{J}$ from (2.9) clearly shows that it is a tensor of

rank two (indexes in the following agree with with Murnaghan's notation).

$$ {}_a^d\boldsymbol{J} = \frac{1}{2}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \tag{2.23} $$

The gradient of ${}_a^d\boldsymbol{J}$ in (2.23) can be written as

$$ \boldsymbol{\nabla}{}_a^d\boldsymbol{J} = \boldsymbol{e}_l \frac{\partial}{\partial x_l} \otimes \left(\frac{1}{2}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \right) = \frac{1}{2}\boldsymbol{e}_l \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial}{\partial x_l} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \tag{2.24} $$

Clearly $\boldsymbol{\nabla}{}_a^d\boldsymbol{J}$, i.e., the gradient of ${}_a^d\boldsymbol{J}$, is a tensor of rank three. An alternative presentation of the gradients of the rotations is simpler and easier to incorporate in the further developments. Instead, represent total rotations as a vector.

$$ \{{}_t\Theta\}^T = [{}_t\Theta_{x_1}, \;\; {}_t\Theta_{x_2}, \;\; {}_t\Theta_{x_3}] \tag{2.25} $$

Gradients of ${}_t\boldsymbol{\Theta}$ in (2.25) can be defined using

$$ \left[{}^{t\Theta}J\right] = \left[\frac{\partial\{{}_t\Theta\}}{\partial\{x\}} \right] \quad \text{or} \quad {}^{t\Theta}J_{ij} = \frac{\partial({}_t\Theta_i)}{\partial x_j} \tag{2.26} $$

The gradient tensor $\left[{}^{t\Theta}J\right]$ of total rotations can be decomposed into symmetric and antisymmetric tensors $\left[{}^{t\Theta}_{s}J\right]$ and $\left[{}^{t\Theta}_{a}J\right]$.

$$ \left[{}^{t\Theta}J\right] = \left[{}^{t\Theta}_{s}J\right] + \left[{}^{t\Theta}_{a}J\right] \tag{2.27} $$

$$ \begin{aligned} \left[{}^{t\Theta}_{s}J\right] &= \frac{1}{2}\left(\left[{}^{t\Theta}J\right] + \left[{}^{t\Theta}J\right]^T \right) \\ \left[{}^{t\Theta}_{a}J\right] &= \frac{1}{2}\left(\left[{}^{t\Theta}J\right] - \left[{}^{t\Theta}J\right]^T \right) \end{aligned} \tag{2.28} $$

2.2.1.1 Considerations of $\mathbb{J}$, Stress, and Moment Tensors

When the gradients of displacements vary between neighboring material points, so do the internal rotations ${}_a^d\boldsymbol{J}$. Likewise, the Cosserat rotations ${}_a^e\boldsymbol{\gamma}$ may also vary between the neighboring material points. Hence, the total rotation tensor ${}_a\boldsymbol{r}$ can vary between the material points. When rotations ${}_a\boldsymbol{r}$ are resisted by the deforming matter, conjugate moments are created. ${}_a\boldsymbol{r}$, their rates, and the conjugate moments can result in additional energy storage, dissipation, and rheology compared to classical theories. Thus, in the deforming matter total rotations ${}_a\boldsymbol{r}$ are conjugate to a moment tensor which necessitates that on the boundary of the deformed volume there must exist a resultant moment tensor.

Consider a volume of matter $\underset{\sim}{V}$ in the reference configuration with closed boundary $\partial\underset{\sim}{V}$ (Figure 2.1). The volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in the cut principle of Cauchy. Consider a tetrahedron T_1 such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other and parallel to the planes of the x-frame. Upon deformation, $\underset{\sim}{V}$ and $\partial\underset{\sim}{V}$ occupy $\bar{\underset{\sim}{V}}$ and $\partial\bar{\underset{\sim}{V}}$ and likewise V and ∂V deform into $\bar{V}$ and $\partial\bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$, whose edges (under finite deformation) are non-orthogonal covariant base vectors $\widetilde{\boldsymbol{g}}_i$. The planes of the tetrahedron formed by the covariant base vectors are flat but obviously non-orthogonal to each other. Assume the tetrahedron to be the small neighborhood of material point $\bar{o}$ so that the assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial\bar{V}$ is valid. When the deformed tetrahedron is isolated from the volume $\bar{V}$, it must be in equilibrium under the action of a disturbance on the surface $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrate with the mating faces in volume $\bar{V}$ when the tetrahedron T_1 is placed back in the volume $\bar{V}$.

Consider the deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress per unit area on the plane $\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area on the plane $\bar{A}\bar{B}\bar{C}$ (referred to as *moment* for short), and $\bar{\boldsymbol{n}}$ be the normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, and $\bar{\boldsymbol{n}}$ all have different directions when the deformation is finite. Based on the small deformation assumption, the deformed coordinates $\bar{x}_i$ are approximately the same as the undeformed coordinates x_i, thus the deformed tetrahedron $\bar{T}_1$ in the current configuration is close to its map T_1 in the reference configuration. With this assumption all stress measures (first and second Piola-Kirchhoff stress tensors, Cauchy stress tensor) are

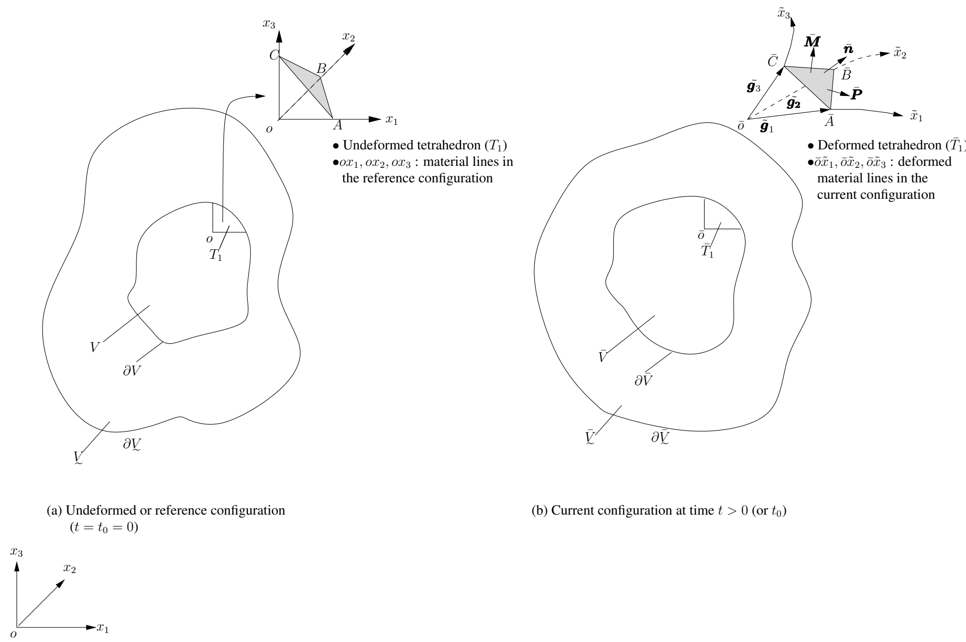

Figure 2.1: Volume of matter in undeformed and deformed configurations

approximately the same. The same holds for the moment tensors. With the assumption $\bar{\boldsymbol{x}} \approx \boldsymbol{x}$,

$$\bar{\boldsymbol{P}} = \boldsymbol{P} \ , \quad \bar{\boldsymbol{M}} = \boldsymbol{M} \tag{2.29}$$

The Cauchy principle for $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{M}}$ leads to the following for $\boldsymbol{P}$ and $\boldsymbol{M}$.

$$\boldsymbol{P} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \ , \quad \boldsymbol{M} = \boldsymbol{m} \cdot \boldsymbol{n} \tag{2.30}$$

in which $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{m}$ is the Cauchy moment tensor (per unit area).

2.2.2 Fluid Continua

In References [70,72,73], Surana, et al. presented some discussion of the mathematical description for fluent continua. The validity of the conservation and balance laws used for fluent continua in Eulerian description was discussed in Reference [101]. In the strict sense the Lagrangian and Eulerian descriptions can be derived from each other due to the fact that the deformed coordinates of a material point and its coordinates in the reference configuration are related through the displacements of the material points. In fluids, transport physics are monitored at a fixed location, hence displacements of the material points (which are not generally possible to measure in the complex motion of fluids) are not available. Fortunately, the displacements of the material points are not needed in the constitutive theories either. In their absence, the correspondence and transparency between the Lagrangian and the Eulerian descriptions is lost. Nonetheless, the conservation and balance laws used for fluent continua are viewed to be in the Eulerian description. In this dissertation this is assumed to be true and the development of the non-classical continuum theory proceeds accordingly. With this in mind, rate terms for fluids are not equivalent to the material derivative of some lower order term in Lagrangian description. For example, the internal rotation rates for fluids are not ${}_i\dot{\boldsymbol{\Theta}} = \frac{D_i\boldsymbol{\Theta}}{Dt}$, but instead a new fundamental quantity ${}_i^t\bar{\boldsymbol{\Theta}}$. Back superscript t is used to identify fundamental rate quantities in the following work.

Considering the lack of transparency between the Lagrangian description and that used for fluids, the velocity gradient tensor $[\bar{L}]$ is perhaps a more meaningful measure of deformation for fluid continua than the Jacobian of deformation $[\bar{J}]$. The velocity gradient tensor is defined as

$$[\bar{L}] = \left[\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}\right] = \left[\frac{\bar{v}_1, \bar{v}_2, \bar{v}_3}{\bar{x}_1, \bar{x}_2, \bar{x}_3}\right] = \begin{bmatrix} \frac{\partial\bar{v}_1}{\partial\bar{x}_1} & \frac{\partial\bar{v}_1}{\partial\bar{x}_2} & \frac{\partial\bar{v}_1}{\partial\bar{x}_3} \\ \frac{\partial\bar{v}_2}{\partial\bar{x}_1} & \frac{\partial\bar{v}_2}{\partial\bar{x}_2} & \frac{\partial\bar{v}_2}{\partial\bar{x}_3} \\ \frac{\partial\bar{v}_3}{\partial\bar{x}_1} & \frac{\partial\bar{v}_3}{\partial\bar{x}_2} & \frac{\partial\bar{v}_3}{\partial\bar{x}_3} \end{bmatrix} \tag{2.31}$$

$[\bar{L}]$ can be decomposed into symmetric and antisymmetric tensors.

$$[\bar{L}] = [{}_s\bar{L}] + [{}_a\bar{L}] = [\bar{D}] + [\bar{W}] \tag{2.32}$$

$$[\bar{D}] = \frac{1}{2}\left([\bar{L}] + [\bar{L}]^T\right) \tag{2.33}$$

$$[\bar{W}] = \frac{1}{2}\left([\bar{L}] - [\bar{L}]^T\right) \tag{2.34}$$

where $[\bar{W}]$ is a matrix containing rotation rates.

$$[\bar{W}] = \begin{bmatrix} 0 & {}^t_i\bar{\Theta}_{x_3} & -{}^t_i\bar{\Theta}_{x_2} \\ -{}^t_i\bar{\Theta}_{x_3} & 0 & {}^t_i\bar{\Theta}_{x_1} \\ {}^t_i\bar{\Theta}_{x_2} & -{}^t_i\bar{\Theta}_{x_1} & 0 \end{bmatrix} \tag{2.35}$$

$${}^t_i\bar{\Theta}_{x_1} = \frac{1}{2}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_3} - \frac{\partial \bar{v}_3}{\partial \bar{x}_2}\right) \;\; ; \;\; {}^t_i\bar{\Theta}_{x_2} = \frac{1}{2}\left(\frac{\partial \bar{v}_3}{\partial \bar{x}_1} - \frac{\partial \bar{v}_1}{\partial \bar{x}_3}\right) \;\; ; \;\; {}^t_i\bar{\Theta}_{x_3} = \frac{1}{2}\left(\frac{\partial \bar{v}_1}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_1}\right) \tag{2.36}$$

${}^t_i\bar{\Theta}_i$ are positive clockwise. Internal rotation rates could alternatively be defined by

$$\begin{aligned} \bar{\boldsymbol{\nabla}} \times \bar{\boldsymbol{v}} &= \boldsymbol{e}_i \times \boldsymbol{e}_j \frac{\partial \bar{v}_j}{\partial \bar{x}_i} = \epsilon_{ijk} \boldsymbol{e}_k \frac{\partial \bar{v}_j}{\partial \bar{x}_i} \\ &= \boldsymbol{e}_1 \left(\frac{\partial \bar{v}_3}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_3}\right) + \boldsymbol{e}_2 \left(\frac{\partial \bar{v}_1}{\partial \bar{x}_3} - \frac{\partial \bar{v}_3}{\partial \bar{x}_1}\right) + \boldsymbol{e}_3 \left(\frac{\partial \bar{v}_2}{\partial \bar{x}_1} - \frac{\partial \bar{v}_1}{\partial \bar{x}_2}\right) \\ &= \boldsymbol{e}_1 \left(-2{}^t_i\bar{\Theta}_{x_1}\right) + \boldsymbol{e}_2 \left(-2{}^t_i\bar{\Theta}_{x_2}\right) + \boldsymbol{e}_3 \left(-2{}^t_i\bar{\Theta}_{x_3}\right) \end{aligned} \tag{2.37}$$

Angles in (2.37) are twice the magnitude of angles in (2.35) and in the counterclockwise sense. The definition of ${}^t_i\bar{\Theta}_i$ in (2.35) and (2.36) is used in the following.

A polar decomposition can also be performed on $[\bar{L}]$.

$$[\bar{L}] = [{}^t\bar{R}][{}^t\bar{S}_r] = [{}^t\bar{S}_l][{}^t\bar{S}_r] \tag{2.38}$$

where $[{}^t\bar{S}_r]$ is the transformation matrix associated with rotation rates and $[{}^t\bar{S}_l]$ and $[{}^t\bar{S}_r]$ are the left and right stretch rate tensors, analogous to the polar decomposition presented for solids in Section 2.2.1. The discussion for $[{}^dR]$, $[{}^dS_l]$, and $[{}^dS_r]$ presented for solids applies here as well. $[\bar{W}]$ is a matrix containing rotation angle rates and $[{}^t\bar{R}]$ is a rotation rate matrix. They contain the same physics in different forms.

The conservation and balance laws must be based on the entirety of $[\bar{L}]$. The models presented in this dissertation are derived using $[\bar{D}]$ and $[\bar{W}]$. Additionally, let ${}^t_e\bar{\Theta}_{x_1}$, ${}^t_e\bar{\Theta}_{x_2}$, and ${}^t_e\bar{\Theta}_{x_3}$ be the Cosserat rotation rates about the same triad of axes as used for internal rotation rates ${}^t_i\bar{\boldsymbol{\Theta}}$, considered positive counterclockwise. Let $[{}_e\bar{W}]$ be the antisymmetric matrix containing rates of rotation angles defined by ${}^{t}_{e}\bar{\boldsymbol{\Theta}}$.

$$[{}_{e}\bar{W}] = \begin{bmatrix} 0 & {}^{t}_{e}\bar{\Theta}_{x_3} & -{}^{t}_{e}\bar{\Theta}_{x_2} \\ -{}^{t}_{e}\bar{\Theta}_{x_3} & 0 & {}^{t}_{e}\bar{\Theta}_{x_1} \\ {}^{t}_{e}\bar{\Theta}_{x_2} & -{}^{t}_{e}\bar{\Theta}_{x_1} & 0 \end{bmatrix} \tag{2.39}$$

Following (2.19) – (2.21) for solids, it is useful to define total deformation rate and rotation rate tensors.

$$[\bar{\mathbb{L}}] = [\bar{L}] - [{}_{e}\bar{W}] = [\bar{D}] + [\bar{W}] - [{}_{e}\bar{W}] \tag{2.40}$$

$$[\bar{\mathbb{L}}] = [\bar{D}] + [{}_{t}\bar{W}] \tag{2.41}$$

$$[{}_{t}\bar{W}] = \begin{bmatrix} 0 & {}^{t}_{t}\bar{\Theta}_{x_3} & -{}^{t}_{t}\bar{\Theta}_{x_2} \\ -{}^{t}_{t}\bar{\Theta}_{x_3} & 0 & {}^{t}_{t}\bar{\Theta}_{x_1} \\ {}^{t}_{t}\bar{\Theta}_{x_2} & -{}^{t}_{t}\bar{\Theta}_{x_1} & 0 \end{bmatrix} \tag{2.42}$$

in which $[{}_{t}\bar{W}]$ is the antisymmetric matrix containing total rotation rates ${}^{t}_{t}\bar{\Theta}_{x_1}$, ${}^{t}_{t}\bar{\Theta}_{x_2}$, and ${}^{t}_{t}\bar{\Theta}_{x_3}$ about the axes of the triad at a material point, considered positive in the clockwise sense. Obviously,

$$\begin{aligned} {}^{t}_{t}\bar{\Theta}_{x_1} &= {}^{t}_{i}\bar{\Theta}_{x_1} - {}^{t}_{e}\bar{\Theta}_{x_1} \\ {}^{t}_{t}\bar{\Theta}_{x_2} &= {}^{t}_{i}\bar{\Theta}_{x_2} - {}^{t}_{e}\bar{\Theta}_{x_2} \\ {}^{t}_{t}\bar{\Theta}_{x_3} &= {}^{t}_{i}\bar{\Theta}_{x_3} - {}^{t}_{e}\bar{\Theta}_{x_3} \end{aligned} \tag{2.43}$$

Remarks.

(1) $[\bar{D}]$ represents the usual velocity gradient or strain rate tensor.

(2) $[\bar{W}]$, $[{}_{e}\bar{W}]$, and $[{}_{t}\bar{W}]$ are antisymmetric tensors containing rotation rates, and are not measures of strain rate.

(3) Based on (1) and (2), $[\bar{\mathbb{L}}]$ is not a strain rate tensor, but rather the addition of the strain rate tensor $[\bar{D}]$ and the total rotation rate tensor $[{}_{t}\bar{W}]$.

(4) The rotation rate tensors are tensors of rank two. This is obvious from the definitions of $[\bar{W}]$, $[{}_e\bar{W}]$, and $[{}_t\bar{W}]$. For example, the definition of $\bar{\boldsymbol{W}}$ from (2.32) or (2.34) clearly shows that it is a tensor of rank two.

$$\bar{\boldsymbol{W}} = \frac{1}{2}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\frac{\partial \bar{v}_j}{\partial \bar{x}_i} - \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \right) \tag{2.44}$$

The gradient of $\bar{\boldsymbol{W}}$ in (2.44) can be written as

$$\bar{\boldsymbol{\nabla}}\bar{\boldsymbol{W}} = \boldsymbol{e}_l \frac{\partial}{\partial \bar{x}_l} \otimes \left(\frac{1}{2}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\frac{\partial \bar{v}_j}{\partial \bar{x}_i} - \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \right) \right) = \frac{1}{2}\boldsymbol{e}_l \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial}{\partial \bar{x}_l} \left(\frac{\partial \bar{v}_j}{\partial \bar{x}_i} - \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \right) \tag{2.45}$$

Clearly $\bar{\boldsymbol{\nabla}}\bar{\boldsymbol{W}}$, i.e., the gradient of $\bar{\boldsymbol{W}}$, is a tensor of rank three. Likewise, the gradients of ${}^{t}_{e}\bar{\boldsymbol{\Theta}}$ and ${}^{t}_{t}\bar{\boldsymbol{\Theta}}$ are also tensors of rank three. An alternative presentation of the gradients of ${}^{t}_{t}\bar{\boldsymbol{\Theta}}$ is simpler and easier to incorporate in the further developments. Instead, represent total rotation rates as a vector.

$$\{{}^{t}_{t}\bar{\Theta}\}^T = \left[{}^{t}_{t}\bar{\Theta}_{x_1},\;\; {}^{t}_{t}\bar{\Theta}_{x_2},\;\; {}^{t}_{t}\bar{\Theta}_{x_3}\right] \tag{2.46}$$

Gradients of ${}^{t}_{t}\bar{\boldsymbol{\Theta}}$ in (2.46) can be defined using

$$\left[{}^{t\bar{\Theta}}_{\;t}\bar{J}\right] = \left[\frac{\partial \{{}^{t}_{t}\bar{\Theta}\}}{\partial \{\bar{x}\}} \right] \quad \text{or} \quad {}^{t\bar{\Theta}}_{\;t}\bar{J}_{ij} = \frac{\partial ({}^{t}_{t}\bar{\Theta}_i)}{\partial \bar{x}_j} \tag{2.47}$$

The gradient tensor ${}^{t\bar{\Theta}}_{\;t}\bar{\boldsymbol{J}}$ of total rotation rates in (2.47) can be decomposed into symmetric and antisymmetric tensors $[{}^{t\bar{\Theta}}_{\;t\,s}\bar{J}]$ and $[{}^{t\bar{\Theta}}_{\;t\,a}\bar{J}]$.

$$\left[{}^{t\bar{\Theta}}_{\;t}\bar{J}\right] = \left[{}^{t\bar{\Theta}}_{\;t\,s}\bar{J}\right] + \left[{}^{t\bar{\Theta}}_{\;t\,a}\bar{J}\right] \tag{2.48}$$

$$\begin{aligned} \left[{}^{t\bar{\Theta}}_{\;t\,s}\bar{J}\right] &= \frac{1}{2}\left(\left[{}^{t\bar{\Theta}}_{\;t}\bar{J}\right] + \left[{}^{t\bar{\Theta}}_{\;t}\bar{J}\right]^T \right) \\ \left[{}^{t\bar{\Theta}}_{\;t\,a}\bar{J}\right] &= \frac{1}{2}\left(\left[{}^{t\bar{\Theta}}_{\;t}\bar{J}\right] - \left[{}^{t\bar{\Theta}}_{\;t}\bar{J}\right]^T \right) \end{aligned} \tag{2.49}$$

2.2.2.1 Covariant and Contravariant Bases

In this case deformation is not infinitesimal, i.e., $\bar{x}_i \neq x_i$, hence it is not basis independent and it becomes necessary to define covariant and contravariant measures. As mentioned previously, the edges of the deformed tetrahedron are covariant base vectors $\widetilde{\boldsymbol{g}}_i$ that are tangent to deformed curvilinear material lines.

$$\widetilde{\boldsymbol{g}}_i = \boldsymbol{e}_k \frac{\partial \bar{x}_k}{\partial x_i} \tag{2.50}$$

and

$$J_{ij} = \frac{\partial \bar{x}_i}{\partial x_j} \tag{2.51}$$

The columns of $\boldsymbol{J}$ are covariant base vectors $\widetilde{\boldsymbol{g}}_i$ that form a non-orthogonal covariant basis. Contravariant base vectors $\widetilde{\boldsymbol{g}}^i$ are normal to the faces of the deformed tetrahedron formed by the covariant base vectors.

$$\widetilde{\boldsymbol{g}}^j = \boldsymbol{e}_l \frac{\partial x_j}{\partial \bar{x}_l} \tag{2.52}$$

and

$$\bar{J}_{ij} = \frac{\partial x_i}{\partial \bar{x}_j} \tag{2.53}$$

The rows of $\bar{\boldsymbol{J}}$ are contravariant base vectors $\widetilde{\boldsymbol{g}}^j$. These form a non-orthogonal contravariant basis. Covariant and contravariant bases are reciprocal to each other [78].

2.2.2.2 Considerations of $\bar{\mathbb{L}}$, Stress, Moment, and Strain Rate Tensors

Similar to the argument made in Section 2.2.1.1, when the gradients of velocities vary between neighboring materials, so do the rates of rotations, both internal and Cosserat. When the rotation rates ${}_t\bar{\boldsymbol{W}}$ are resisted by the fluid, conjugate moments are created. Thus, in the deforming matter, the total rotation rates ${}_t\bar{\boldsymbol{W}}$ are conjugate to the moment tensor, which necessitates that on the boundary of the deformed volume there must exist a resultant moment vector in addition to the resultant force vector present in classical theories. Considering the bases discussed in Section 2.2.2.1, it is possible to define measures for the stress and moment tensors.

Contravariant Cauchy Stress Tensor

The definition of the stresses on the non-oblique faces of the deformed tetrahedron formed by the covariant base vectors $\widetilde{\boldsymbol{g}}_i$ in the contravariant directions orthogonal to the faces of the deformed tetrahedron is the most natural. Let $\bar{\underset{\sim}{\boldsymbol{\sigma}}}^{(0)}$ or $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ be the contravariant stress tensor with components $\bar{\underset{\sim}{\sigma}}_{ij}^{(0)}$ or $\underset{\sim}{\sigma}_{ij}^{(0)}$ and dyads $\widetilde{\boldsymbol{g}}_i \otimes \widetilde{\boldsymbol{g}}_j$. Component $\bar{\underset{\sim}{\sigma}}_{11}^{(0)}$ or $\underset{\sim}{\sigma}_{11}^{(0)}$ is in the $\widetilde{\boldsymbol{g}}^1$ direction on a face of the tetrahedron with unit exterior normal $\widetilde{\boldsymbol{g}}^1$, i.e., on the $\widetilde{\boldsymbol{g}}^1$ face. Likewise $\bar{\underset{\sim}{\sigma}}_{12}^{(0)}$ or $\underset{\sim}{\sigma}_{12}^{(0)}$ and $\bar{\underset{\sim}{\sigma}}_{31}^{(0)}$ or $\underset{\sim}{\sigma}_{31}^{(0)}$ act on the $\widetilde{\boldsymbol{g}}^1$ and $\widetilde{\boldsymbol{g}}^3$ faces in the $\widetilde{\boldsymbol{g}}^2$ and $\widetilde{\boldsymbol{g}}^1$ directions. Using dyads $\widetilde{\boldsymbol{g}}_i \otimes \widetilde{\boldsymbol{g}}_j$ or the contravariant law of transformation,

$$\boldsymbol{\sigma}^{(0)} = \widetilde{\boldsymbol{g}}_i \otimes \widetilde{\boldsymbol{g}}_j \underset{\sim}{\sigma}_{ij}^{(0)} \tag{2.54}$$

Using (2.50) in (2.54),

$$\begin{aligned} \boldsymbol{\sigma}^{(0)} &= \boldsymbol{e}_i \otimes \boldsymbol{e}_j \sigma_{ij}^{(0)} \\ \sigma_{ij}^{(0)} &= J_{ik} \underset{\sim}{\sigma}_{kl}^{(0)} J_{jl} \\ [\sigma^{(0)}] &= [J][\underset{\sim}{\sigma}^{(0)}][J]^T \end{aligned} \tag{2.55}$$

$\boldsymbol{\sigma}^{(0)}$ is the contravariant Cauchy stress tensor (Lagrangian description) from which $\bar{\boldsymbol{\sigma}}^{(0)}$ can be easily obtained by replacing $[J]$ by $[\bar{J}]^{-1}$ and $\boldsymbol{\sigma}^{(0)}$ by $\bar{\boldsymbol{\sigma}}^{(0)}$ in (2.55). Since the dyads of $\boldsymbol{\sigma}^{(0)}$ or $\bar{\boldsymbol{\sigma}}^{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the Cauchy principle holds between $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{\sigma}}^{(0)}$.

$$\bar{\boldsymbol{P}} = (\bar{\boldsymbol{\sigma}}^{(0)})^T \cdot \bar{\boldsymbol{n}} \tag{2.56}$$

Covariant Cauchy Stress Tensor

Instead of using contravariant directions and stress components $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ and covariant basis $\widetilde{\boldsymbol{g}}_i$, covariant stress components $(\underset{\sim}{\boldsymbol{\sigma}}_{(0)})_{ij}$ or $(\bar{\underset{\sim}{\boldsymbol{\sigma}}}_{(0)})_{ij}$ and contravariant basis $\widetilde{\boldsymbol{g}}^i$ is an option. Consideration of $(\underset{\sim}{\boldsymbol{\sigma}}_{(0)})_{ij}$ of course will require a different deformed tetrahedron such that covariant base vectors $\widetilde{\boldsymbol{g}}_i$ are normal to its oblique faces. The adverse consequences of choosing this measure of stress for finite deformation are discussed in References [78, 102]. Here this measure is entertained as an alternative to the contravariant stress measure. Using dyads $\widetilde{\boldsymbol{g}}^i \otimes \widetilde{\boldsymbol{g}}^j$ and components $(\underset{\sim}{\boldsymbol{\sigma}}_{(0)})_{ij}$,

$$\bar{\boldsymbol{\sigma}}_{(0)} = \widetilde{\boldsymbol{g}}^i \otimes \widetilde{\boldsymbol{g}}^j (\underset{\sim}{\boldsymbol{\sigma}}_{(0)})_{ij} \tag{2.57}$$

And using (2.52)

$$\begin{aligned} \bar{\boldsymbol{\sigma}}_{(0)} &= \boldsymbol{e}_i \otimes \boldsymbol{e}_j (\bar{\boldsymbol{\sigma}}_{(0)})_{ij} \\ (\bar{\boldsymbol{\sigma}}_{(0)})_{ij} &= \bar{J}_{ki} (\underset{\sim}{\boldsymbol{\sigma}}_{(0)})_{kl} \bar{J}_{lj} \\ [\bar{\sigma}_{(0)}] &= [\bar{J}]^T [\underset{\sim}{\sigma}_{(0)}][\bar{J}] \end{aligned} \tag{2.58}$$

$\bar{\boldsymbol{\sigma}}_{(0)}$ is the covariant Cauchy stress tensor (Eulerian description) from which $\boldsymbol{\sigma}_{(0)}$ can be obtained by replacing $[\bar{J}]$ with $[J]^{-1}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$ with $\boldsymbol{\sigma}_{(0)}$ in (2.58). Since the dyads of $\bar{\boldsymbol{\sigma}}_{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the Cauchy principle holds between $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$.

$$\bar{\boldsymbol{P}} = (\bar{\boldsymbol{\sigma}}_{(0)})^T \cdot \bar{\boldsymbol{n}} \tag{2.59}$$

Contravariant and Covariant Cauchy Moment Tensors

When the deformed tetrahedron with moment $\bar{\boldsymbol{M}}$ on its oblique face $\bar{A}\bar{B}\bar{C}$ is isolated from the volume $\bar{V}$, moments (per unit area) will exist on its faces. As in the case of stress measures, the contravariant basis is the most natural way to define these. Utilizing notations parallel to those used in the case of Cauchy stress tensors, the following is defined using a contravariant measure of the moment tensor.

$$\boldsymbol{m}^{(0)} = \widetilde{\boldsymbol{g}}_i \otimes \widetilde{\boldsymbol{g}}_j \underset{\sim}{m}_{ij}^{(0)} \tag{2.60}$$

Using (2.50) in (2.60)

$$\begin{aligned}
\boldsymbol{m}^{(0)} &= \boldsymbol{e}_i \otimes \boldsymbol{e}_j m_{ij}^{(0)} \\
m_{ij}^{(0)} &= J_{ik} \underset{\sim}{m}_{kl}^{(0)} J_{jl} \\
[m^{(0)}] &= [J][\underset{\sim}{m}^{(0)}][J]^T \\
[\bar{m}^{(0)}] &= [\bar{J}]^{-1}[\underset{\sim}{\bar{m}}^{(0)}][[\bar{J}]^{-1}]^T
\end{aligned} \tag{2.61}$$

and the Cauchy principle

$$\bar{\boldsymbol{M}} = (\bar{\boldsymbol{m}}^{(0)})^T \cdot \bar{\boldsymbol{n}} \tag{2.62}$$

Likewise when using a covariant measure of the moment tensor

$$\bar{\boldsymbol{m}}_{(0)} = \widetilde{\boldsymbol{g}}^i \otimes \widetilde{\boldsymbol{g}}^j (\underset{\sim}{\boldsymbol{m}}_{(0)})_{ij} \tag{2.63}$$

and using (2.52) in (2.63)

$$\begin{aligned}
\bar{\boldsymbol{m}}_{(0)} &= \boldsymbol{e}_i \otimes \boldsymbol{e}_j (\bar{\boldsymbol{m}}_{(0)})_{ij} \\
(\bar{\boldsymbol{m}}_{(0)})_{ij} &= \bar{J}_{ki} (\underset{\sim}{\boldsymbol{m}}_{(0)})_{kl} \bar{J}_{lj} \\
[\bar{m}_{(0)}] &= [\bar{J}]^T [\underset{\sim}{m}_{(0)}][\bar{J}] \\
[m_{(0)}] &= [[J]^{-1}]^T [\underset{\sim}{m}_{(0)}][J]^{-1}
\end{aligned} \tag{2.64}$$

and the Cauchy principle

$$\bar{\boldsymbol{M}} = (\bar{\boldsymbol{m}}_{(0)})^T \cdot \bar{\boldsymbol{n}} \tag{2.65}$$

Remarks.

(1) At this stage, all four of these measures are non-symmetric tensors until proven otherwise.

(2) The use of covariant stress and moment measures requires additional manipulation of the elementary tetrahedron that is not in compliance with the physics of deformation.

(3) The contravariant Cauchy stress tensor paired with the covariant strain tensor are the most

natural measures [78, 102], however, for the sake of generality, the derivations presented in this dissertation use a basis independent notation:

$$^{(0)}\bar{\boldsymbol{\sigma}}; \quad ^{(0)}\boldsymbol{\gamma}; \quad ^{(0)}\bar{\boldsymbol{m}} \tag{2.66}$$

where the back superscript value denoting the order of convected time derivative of the quantity may be replaced by a forward superscript for contravariant measures or a forward subscript for covariant measures.

2.3 Balance of Moments of Moments Balance Law

The classical conservation and balance laws include conservation of mass, balance of linear momenta, balance of angular momenta, and the first and second laws of thermodynamics. Mechanical equilibrium in classical theories requires only that the forces and moments acting on a material point be balanced. Yang, et al. [77] and Surana, et al. [69–76] argue that these are insufficient when considering theories with higher order kinematic measures such as rotations. In classical mechanics, a moment acting at a material point is a free vector that will produce the same motion regardless of its location. Yang notes that this does not hold when the continuum is considered to be a collection of what he refers to as representative volume elements (i.e., elementary tetrahedra). A moment or rotation applied to one volume element is not free to be applied to another element somewhere else in the continuum. Furthermore, Surana has shown that the moments acting on the oblique face of the elementary tetrahedron are due to and associated with the rotations of that tetrahedron, and therefore are not free vectors. To account for this additional physics, Yang shows that the moments of moments must balance with the moments due to shear stresses.

Surana, et al. [69–76] also make an inductive argument that every higher order kinematic variable requires an additional balance law to maintain mechanical equilibrium of the continuum. For displacement as the only kinematic variable (classical theories), balance of linear momenta and balance of angular momenta (i.e., balance of forces and balance of moments) are enough to maintain equilbrium of the material. When rotations are added as a kinematic variable, the balance of moments of moments must be introduced for the material to be in equilibrium. If another higher order kinematic variable were introduced, another balance law would be necessary.

The result of excluding the balance of moments of moments (as shown in detail in later chapters) is that the Cauchy moment tensor is non-symmetric, so additional constitutive relations are required to describe its antisymmetric components. Surana, et al. [103, 104] have shown, theoretically and numerically, that these non-zero antisymmetric moments are obviously non-physical in light of this balance law. The antisymmetric components negate the symmetric moment tensor and their increased influence actually reduces the resistance of the matter to deformation. Additionally, Yang points out that the concept of directors, or the rigid vectors attached to particles to allow rotations in the work of the Cosserats [1], are required for some models to have closure when the balance of moments of moments is not considered but are not necessary if the balance law is utilized. In fact, the rigid fibers embedded in the material of Eringen's micropolar theories [8–20], which are mathematically directors, may not even have any physical relevance [77].

Chapter 3

Mathematical Models for Solid Continua

The conservation and balance laws are derived for homogeneous, isotropic solids in the Lagrangian description undergoing infinitesimal deformation. From the energy equation and entropy inequality, energy conjugate pairs are determined. Using the conjugate pairs, constitutive theories are derived for elastic materials and compared with those presented by Eringen [14] for micropolar solids. Additionally, constitutive theories for viscoelastic solids with and without memory are also derived.

3.1 Conservation and Balance Laws

The non-classical continuum theory considered here incorporates new physics due to the internal rotations the Cosserat rotations at the material points. This physics is absent in the currently used thermodynamic framework for isotropic, homogeneous solid continua, i.e., classical continuum mechanics. The new physics due to rotations may influence some or all of the conservation and balance laws. In order to determine the precise influence of the new physics (or lack thereof) on the conservation and balance laws, the derivations of the conservation and balance laws must be initiated from the fundamental stage as done in case of classical continuum theories [78, 79] so that the resulting equations in the present theory can be compared with the classical theory to determine how these laws are influenced by the physics due to the internal and Cosserat rotations.

It is entirely possible that some laws are not influenced by this new physics in which case the corresponding equations will obviously be the same as those for the non-polar or classical case.

In a non-classical continuum theory with displacements, displacement gradients, internal and Cosserat rotations, and their gradients as quantities describing the kinematics of deformation, the following conservation and balance laws must be considered based on the assumption of thermodynamic equilibrium during the evolution of the deforming matter: (1) conservation of mass, (2) balance of linear momenta, (3) balance of angular momenta, (4) balance of moments of moments (Yang, et al. [77]), (5) first law of thermodynamics (i.e., balance of energy), and (6) second law of thermodynamics (i.e., entropy inequality).

3.1.1 Conservation of Mass

The continuity equation resulting from the principle of conservation of mass remains the same for the non-classical continuum theory considered here as in case of classical continuum mechanics as long as the matter is treated as homogeneous and isotropic. In the Lagrangian description, the continuity equation [78, 79] can be written as

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x}, t) \tag{3.1}$$

For infinitesimal deformation $|J| \approx 1$, hence $\rho_0(\boldsymbol{x}) \approx \rho(\boldsymbol{x}, t)$, where $\rho_0(\boldsymbol{x})$ is the density of the material point at $\boldsymbol{x}$ in the reference configuration and $\rho(\boldsymbol{x}, t)$ is the density of a material point at $\bar{\boldsymbol{x}}$ in the current configuration.

3.1.2 Balance of Linear Momenta

For a deforming volume of matter the rate of change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. The derivation is identical to that for classical continuum theory. Thus, the following holds (for small deformation) [78, 79].

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0$$

$$\text{or} \tag{3.2}$$

$$\rho_0 \frac{D\{v\}}{Dt} - \rho_0 \{F^b\} - [\sigma]^T \{\nabla\} = 0$$

In the Lagrangian description $\frac{D}{Dt} = \frac{\partial}{\partial t}$, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ are velocities, $\boldsymbol{F}^b$ are body forces per unit mass, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. Equations (3.2) are the momentum equations in the x_1, x_2, and x_3 directions. The Cauchy stress tensor is non-symmetric at this stage as its symmetry has not been established.

3.1.3 Balance of Angular Momenta

The principle of the balance of angular momenta for a non-classical continuum can be stated as follows: *the time rate of change of the total moment of momenta for a non-classical continuum is equal to the vector sum of the moments of external forces and the moments.* Thus, due to the surface stress $\bar{\boldsymbol{P}}$, total surface moment $\bar{\boldsymbol{M}}$ (per unit area) created when the internal and Cosserat rotations are resisted by the deforming continuum, body force $\bar{\boldsymbol{F}}^b$ (per unit mass), and the momentum $\bar{\rho}\bar{\boldsymbol{v}}d\bar{V}$ for an elemental mass $\bar{\rho}d\bar{V}$ in the current configuration (using the Eulerian description), the following holds.

$$\frac{D}{Dt} \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}} \, d\bar{V} = \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}} \right) \, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b \, d\bar{V} \tag{3.3}$$

The negative sign for $\bar{\boldsymbol{M}}$ is due to clockwise rotations being positive. Consider each term in (3.3) individually.

$$\frac{D}{Dt} \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}} d\bar{V} = \frac{D}{Dt} \int\limits_{\bar{V}(t)} \epsilon_{ijk} \bar{x}_i \bar{v}_j \bar{\rho} d\bar{V} \tag{3.4}$$

$$
\begin{aligned}
&= \frac{D}{Dt}\int\limits_V \epsilon_{ijk} x_i v_j \rho_0 dV \\
&= \int\limits_V \rho_0 \epsilon_{ijk} \frac{D}{Dt}\left(x_i v_j\right) dV \\
&= \int\limits_V \rho_0 \epsilon_{ijk}\left(v_i v_j + x_i \frac{Dv_j}{Dt}\right) dV
\end{aligned} \tag{3.4}
$$

Consider the first term on the right side of (3.3).

$$
\begin{aligned}
\int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} &= \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times (\bar{\boldsymbol{\sigma}})^T \cdot \bar{\boldsymbol{n}} - (\bar{\boldsymbol{m}})^T \cdot \bar{\boldsymbol{n}}\right) d\bar{A} \\
&= \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times (\bar{\boldsymbol{\sigma}})^T \cdot \bar{\boldsymbol{n}} d\bar{A} - (\bar{\boldsymbol{m}})^T \cdot \bar{\boldsymbol{n}} d\bar{A}\right) \\
&= \int\limits_{\partial V} \left(\boldsymbol{x} \times (\boldsymbol{\sigma})^T \cdot \boldsymbol{n} dA - (\boldsymbol{m})^T \cdot \boldsymbol{n} dA\right) \\
&= \int\limits_{\partial V} \left(\epsilon_{ijk} x_i \sigma_{mj} n_m - m_{mk} n_m\right) dA
\end{aligned} \tag{3.5}
$$

in which $\bar{\boldsymbol{\sigma}}$ is the *Cauchy stress tensor* and $\bar{\boldsymbol{m}}$ is the *Cauchy moment tensor*. Using the divergence theorem, (3.5) yields

$$
\begin{aligned}
\int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} &= \int\limits_V \left(\epsilon_{ijk}\left(x_i \sigma_{mj}\right)_{,m} - m_{mk,m}\right) dV \\
&= \int\limits_V \left(\epsilon_{ijk}\left(\delta_{im}\sigma_{mj} + x_i(\sigma_{mj})_{,m}\right) - m_{mk,m}\right) dV \\
&= \int\limits_V \left(\epsilon_{ijk}\left(\sigma_{ij} + x_i(\sigma_{mj})_{,m}\right) - m_{mk,m}\right) dV
\end{aligned} \tag{3.6}
$$

Consider the second term on the right side of (3.3).

$$
\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b d\bar{V} = \int\limits_{\bar{V}(t)} \epsilon_{ijk} \bar{x}_i \bar{F}_j^b \bar{\rho} d\bar{V} = \int\limits_V \epsilon_{ijk} x_i F_j^b \rho_0 dV \tag{3.7}
$$

Substituting from (3.4), (3.5), and (3.7) into (3.3)

$$\int_V \rho_0 \epsilon_{ijk} \left(v_i v_j + x_i \frac{Dv_j}{Dt} \right) dV = \int_V \left(\epsilon_{ijk}(\sigma_{ij} + x_i(\sigma_{mj})_{,m}) - m_{mk,m} \right) dV + \int_V \epsilon_{ijk} x_i F_j^b \rho_0 \, dV \tag{3.8}$$

Since

$$\epsilon_{ijk} v_i v_j = 0 \tag{3.9}$$

equation (3.8) reduces to

$$\int_V \epsilon_{ijk} \left(x_i \left(\rho_0 \frac{Dv_j}{Dt} \quad - \rho_0 F_j^b - \sigma_{mj,m} \right) \right) dV + \int_V (m_{mk,m} - \epsilon_{ijk}\sigma_{ij}) \, dV = 0 \tag{3.10}$$

Using the balance of linear momenta (3.2) in (3.10),

$$\int_V (m_{mk,m} - \epsilon_{ijk}\sigma_{ij}) \, dV = 0 \tag{3.11}$$

and since the volume V is arbitrary

$$m_{mk,m} - \epsilon_{ijk}\sigma_{ij} = 0 \tag{3.12}$$

$$\text{or} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.13}$$

$$\text{or} \qquad [m]^T \{\nabla\} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.14}$$

Equation (3.12) represents the balance of angular momenta. *The Cauchy stress tensor $\boldsymbol{\sigma}$ is non-symmetric*.

$$\boldsymbol{\epsilon} : {}_s\boldsymbol{\sigma} = 0 \implies \boldsymbol{\epsilon} : \boldsymbol{\sigma} = \boldsymbol{\epsilon} : {}_a\boldsymbol{\sigma} \tag{3.15}$$

Using (3.15) in (3.12), note that the antisymmetric components of the Cauchy stress tensor are balanced by the gradients of the Cauchy moment tensor.

Remarks.

(1) In the balance of angular momenta, the rate of change of angular momenta is balanced by the vector sum of the moments of the forces. Thus, this balance law naturally contains moments due to components of the stress tensor acting on the faces of the deformed tetrahedron. Normal stress components obviously do not contribute to this. Likewise, symmetric components of the shear stress are self-equilibrating. Hence, the moments contained in this balance law due to stresses are only caused by the shear stresses contained in the antisymmetric part of the Cauchy stress tensor.

(2) In the case of the classical continuum theory, the balance of angular momenta is a statement of self-equilibrating moments due to shear stresses that requires

$$\boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.16}$$

which implies that $\boldsymbol{\sigma}$ is symmetric. An important point to note is that (3.16) accounts for stress couples due to shear stresses.

(3) In the case of non-classical continua, the existence of moments due to the material constitution resisting the rotations, both internal and Cosserat, results in the shear stress couples from the antisymmetric part of the Cauchy stress tensor being balanced by the internal moments. Thus, for non-classical continua, the balance of angular momenta yields (3.14) instead of (3.16).

(4) Varying rotations between neighboring material points, when resisted by the deforming matter, require the existence of the moment tensor $\boldsymbol{m}$. The balance of angular momenta establishes a relationship between $\boldsymbol{m}$ and $\boldsymbol{\sigma}$.

3.1.4 Balance of Moments of Moments

As described in Section 2.3, this is an additional balance law [77, 103, 104] that is required in the case of non-classical continuum theories that incorporate internal rotations [69–76] arising due to the Jacobian of deformation and/or unknown Cosserat rotations at each material point. Both rotations are internal to the deforming volume, but ${}_{i}\boldsymbol{\Theta}$ are known in terms of displacements whereas ${}_{e}\boldsymbol{\Theta}$ are unknown. Regardless of the nature of the rotations, this balance law is necessary.

Consider the current configuration at time t in the Eulerian description. For the deforming volume of matter to be in equilibrium, the moments of moments must vanish. In the balance of moments of moments, $\bar{\boldsymbol{M}}$ and the antisymmetric components of the Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$, i.e., $\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}$, must be considered. Thus, (neglecting inertial terms) in the Eulerian description

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times (\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}})\, d\bar{V} - \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} = 0 \tag{3.17}$$

Expand the second term in (3.17) and then convert the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using the divergence theorem.

$$\begin{aligned}
\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} &= \int_{\partial\bar{V}} \epsilon_{ijk} x_i \bar{M}_j \, d\bar{A} \\
&= \int_{\partial\bar{V}} \epsilon_{ijk} \bar{x}_i \bar{m}_{mj} \bar{n}_m \, d\bar{A} \\
&= \int_{\bar{V}} (\epsilon_{ijk} \bar{x}_i \bar{m}_{mj})_{,m} \, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\bar{x}_{i,m} \bar{m}_{mj} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\delta_{im} \bar{m}_{mj} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\bar{m}_{ij} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V}
\end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&= \int\limits_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} + \int\limits_{\bar{V}} \epsilon_{ijk} \bar{x}_i \bar{m}_{mj,m} \, d\bar{V} \\
&= \int\limits_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} + \int\limits_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} \right) \, d\bar{V}
\end{aligned} \tag{3.18}
$$

Using equation (3.18) in (3.17) and collecting terms

$$
\int\limits_{\bar{V}} \bar{\boldsymbol{x}} \times \left(-\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}} \right) d\bar{V} - \int\limits_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} = 0 \tag{3.19}
$$

The first term in (3.19) vanishes due to the balance of angular momenta (3.12) and (3.19) reduces to

$$
\int\limits_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} = 0 \tag{3.20}
$$

and since $\bar{V}$ is arbitrary, (3.20) implies

$$
\epsilon_{ijk} \bar{m}_{ij} = 0 \qquad \text{and} \qquad \epsilon_{ijk} m_{ij} = 0 \tag{3.21}
$$

Equation (3.21) implies that *the Cauchy moment tensor* $\boldsymbol{m}$ *is symmetric*. In the non-classical continuum theory presented here, the Cauchy moment tensor is symmetric, but the Cauchy stress tensor is non-symmetric, whereas in the corresponding classical theory, the Cauchy stress tensor is symmetric and the Cauchy moment tensor is null as the rotations are ignored in the theory. Symmetry of the Cauchy moment tensor is due to this balance law. It is worth noting that in most other works on non-classical continuum theories except Yang, et al. [77], this balance law is not considered. As a consequence the moment tensor is reported as a non-symmetric tensor, or at least assumed to be symmetric without any support or reasoning. In the absence of this balance law the moment tensor is non-symmetric, hence a constitutive theory for the antisymmetric part the of Cauchy moment tensor is required. The constitutive theory for the symmetric part of the Cauchy moment tensor remains the same regardless of whether the balance of moments of moments is a balance law. Thus, if one considers the balance of moments of moments as a required balance law, then the constitutive theory for the antisymmetric part of the moment tensor represents spurious material response that does not exist in the actual physics of deformation. Based on the discussion presented in Chapter 2, this balance law is included in the development of the non-classical theories presented in this dissertation.

3.1.5 First Law of Thermodynamics

The sum of work and heat added to a deforming volume of matter results in a change of the total energy of the system. This is expressed as a rate equation in the Eulerian description in the following.

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{3.22}$$

$\bar{E}_t$, $\bar{Q}$, and $\bar{W}$ are the total energy, heat added, and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt}\int_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} \tag{3.23}$$

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}(t)} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}\,d\bar{A} \tag{3.24}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}(t)} (\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot{}_t\dot{\bar{\boldsymbol{\Theta}}})\,d\bar{A} \tag{3.25}$$

where $\bar{e}$ is the specific internal energy, $\bar{\boldsymbol{F}}^b$ is the body force vector per unit mass, and $\bar{\boldsymbol{q}}$ is the rate of heat. Note that the additional term $\bar{\boldsymbol{M}}\cdot{}_t\dot{\bar{\boldsymbol{\Theta}}}$ in $\frac{D\bar{W}}{Dt}$ contributes additional rate of work due to the rate of total rotations. Expanding the integrals and following Reference [78], one can show the following.

$$\frac{D}{Dt}\int_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} = \int_{V}\left(\rho_0\frac{De}{Dt} + \rho_0\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt} - \rho_0\boldsymbol{F}^b\cdot\boldsymbol{v}\right)dV \tag{3.26}$$

Using

$$
\begin{aligned}
\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} &= \boldsymbol{q} \cdot \boldsymbol{n} dA \\
\bar{\rho} d\bar{V} &= \rho_0 dV \\
d\bar{V} &= |J| dV
\end{aligned}
\tag{3.27}
$$

and applying the divergence theorem

$$
-\int\limits_{\partial \bar{V}(t)} \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} = -\int\limits_{\partial V} \boldsymbol{q} \cdot \boldsymbol{n}\, dA = -\int\limits_{V} \boldsymbol{\nabla} \cdot \boldsymbol{q}\, dV \tag{3.28}
$$

Using the stress tensor $\boldsymbol{\sigma}$ and the moment tensor $\boldsymbol{m}$ and following Reference [78], it can be shown that

$$
\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}}\, d\bar{A} = \boldsymbol{v} \cdot (\boldsymbol{\sigma})^T \cdot \boldsymbol{n}\, dA = \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) \cdot d\boldsymbol{A} \tag{3.29}
$$

$$
\bar{\boldsymbol{M}} \cdot {}_t\dot{\bar{\boldsymbol{\Theta}}}\, d\bar{A} = \left({}_t\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot \boldsymbol{n}\, dA = \left({}_t\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot d\boldsymbol{A} \tag{3.30}
$$

Thus, (3.22) becomes

$$
\begin{aligned}
&\int\limits_V \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV \\
&\qquad = -\int\limits_V \boldsymbol{\nabla} \cdot \boldsymbol{q}\, dV + \int\limits_{\partial V} \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) \cdot d\boldsymbol{A} + \int\limits_{\partial V} \left({}_t\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot d\boldsymbol{A}
\end{aligned}
\tag{3.31}
$$

and using the divergence theorem for the integrals over ∂V

$$
\begin{aligned}
&\int\limits_V \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV \\
&\qquad = -\int\limits_V \boldsymbol{\nabla} \cdot \boldsymbol{q}\, dV + \int\limits_V \boldsymbol{\nabla} \cdot \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) dV + \int\limits_V \boldsymbol{\nabla} \cdot \left({}_t\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) dV
\end{aligned}
\tag{3.32}
$$

Following Reference [78]

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{v}\boldsymbol{\cdot}(\boldsymbol{\sigma})^T\right)=\boldsymbol{v}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma})+\sigma_{ji}\frac{\partial v_i}{\partial x_j} \tag{3.33}$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\left({}_t\dot{\boldsymbol{\Theta}}\boldsymbol{\cdot}(\boldsymbol{m})^T\right)={}_t\dot{\boldsymbol{\Theta}}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{m})+m_{ji}\frac{\partial_t\dot{\Theta}_i}{\partial x_j} \tag{3.34}$$

and substituting from (3.33) and (3.34) into (3.32)

$$\begin{aligned}&\int\limits_V\left(\rho_0\frac{De}{Dt}+\rho_0\boldsymbol{v}\boldsymbol{\cdot}\frac{D\boldsymbol{v}}{Dt}-\rho_0\boldsymbol{F}^b\boldsymbol{\cdot}\boldsymbol{v}\right)dV\\&\quad=-\int\limits_V\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}\,dV+\int\limits_V\left(\boldsymbol{v}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma})+\sigma_{ji}\frac{\partial v_i}{\partial x_j}+{}_t\dot{\boldsymbol{\Theta}}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{m})+m_{ji}\frac{\partial_t\dot{\Theta}_i}{\partial x_j}\right)dV\end{aligned} \tag{3.35}$$

Moving all terms to the left of the equality and regrouping

$$\begin{aligned}&\int\limits_V\boldsymbol{v}\boldsymbol{\cdot}\left(\rho_0\frac{D\boldsymbol{v}}{Dt}-\rho_0\boldsymbol{F}^b-\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma}\right)dV\\&\quad+\int\limits_V\left(\rho_0\frac{De}{Dt}+\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}-\sigma_{ji}\frac{\partial v_i}{\partial x_j}-m_{ji}\frac{\partial_t\dot{\Theta}_i}{\partial x_j}-{}_t\dot{\boldsymbol{\Theta}}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{m})\right)dV=0\end{aligned} \tag{3.36}$$

Using (3.2) (the balance of linear momenta), (3.36) reduces to

$$\int\limits_V\left(\rho_0\frac{De}{Dt}+\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}-\sigma_{ji}\frac{\partial v_i}{\partial x_j}-m_{ji}\frac{\partial_t\dot{\Theta}_i}{\partial x_j}-{}_t\dot{\boldsymbol{\Theta}}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{m})\right)dV=0 \tag{3.37}$$

Since the volume V is arbitrary, the following holds.

$$\rho_0\frac{De}{Dt}+\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}-\sigma_{ji}\frac{\partial v_i}{\partial x_j}-\left(m_{ji}\frac{\partial_t\dot{\Theta}_i}{\partial x_j}+{}_t\dot{\boldsymbol{\Theta}}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{m})\right)=0 \tag{3.38}$$

Note that in ${}_t\dot{\boldsymbol{\Theta}}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{m})$, the term $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{m}$ can be substituted from (3.13), thereby eliminating the gradient of $\boldsymbol{m}$ but introducing $\boldsymbol{\sigma}$ in its place.

3.1.6 Second Law of Thermodynamics

If $\bar{\eta}$ is the entropy density in volume $\bar{V}(t)$, $\bar{h}$ is the entropy flux between $\bar{V}(t)$ and the volume of matter surrounding it, and $\bar{s}$ is the source of entropy in $\bar{V}(t)$ due to non-contacting bodies, then the rate of increase of entropy in volume $\bar{V}(t)$ is at least equal to that supplied to $\bar{V}(t)$ from all contacting and non-contacting sources [78]. Thus

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho}\, d\bar{V} \geq \int\limits_{\partial\bar{V}(t)} \bar{h}\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\, d\bar{V} \tag{3.39}$$

Assume Cauchy's postulate for $\bar{h}$ holds.

$$\bar{h} = -\bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \tag{3.40}$$

Using (3.40) in (3.39)

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho}\, d\bar{V} \geq -\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\, d\bar{V} \tag{3.41}$$

Inequality (3.41) needs to be transformed to the Lagrangian description. This can be done using

$$\begin{aligned} d\bar{V} &= |J| dV \\ \rho_0 &= |J|\bar{\rho} \\ \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} &= \boldsymbol{\psi} \cdot \boldsymbol{n}\, dA \end{aligned} \tag{3.42}$$

Using (3.42) in (3.41)

$$\frac{D}{Dt}\int\limits_{V} \eta\rho_0\, dV \geq -\int\limits_{\partial V} \boldsymbol{\psi} \cdot \boldsymbol{n}\, dA + \int\limits_{V} s\rho_0\, dV \tag{3.43}$$

Using Gauss's divergence theorem for the terms over ∂V gives (noting that $\boldsymbol{\psi}$ is a tensor of rank one)

$$\frac{D}{Dt}\int_V \eta\rho_0\, dV \geq -\int_V \boldsymbol{\nabla}\cdot\boldsymbol{\psi}\, dV + \int_V s\rho_0\, dV \tag{3.44}$$

or

$$\int_V \left(\rho_0\frac{D\eta}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{\psi} - \rho_0 s\right) dV \geq 0 \tag{3.45}$$

and since volume V is arbitrary

$$\rho_0\frac{D\eta}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{\psi} - \rho_0 s \geq 0 \tag{3.46}$$

Inequality (3.46) is the entropy inequality in the most fundamental form resulting from the second law of thermodynamics. Inequality (3.46) is strictly a statement that contains entropy terms, hence it contains no information regarding reversible deformation processes, such as in case of elastic solids, and therefore provides no information or mechanisms regarding the derivations of the constitutive theories for such solids. Only when the rate of mechanical work results in rate of entropy production will inequality (3.46) have some information regarding the associated conjugate pairs that result in rate of entropy production. Note that (3.46) in its present form also does not provide any information regarding the constitutive theory for heat vector $\boldsymbol{q}$.

Another form of the entropy inequality is possible to derive using the relationship between $\boldsymbol{\psi}$ and $\boldsymbol{q}$ and the energy equation. Since the energy equation has all possible mechanisms that result in energy storage and dissipation, the form of the entropy inequality derived using the energy equation is expected to be helpful in the derivations of the constitutive theories. Using

$$\boldsymbol{\psi} = \frac{\boldsymbol{q}}{\theta} \qquad , \qquad s = \frac{r}{\theta} \tag{3.47}$$

where θ is the absolute temperature, $\boldsymbol{q}$ is the heat vector, and r is a suitable potential, then

$$\boldsymbol{\nabla}\cdot\boldsymbol{\psi} = \psi_{i,i} = \frac{q_{i,i}}{\theta} - \frac{q_i\theta_{,i}}{\theta^2} = \frac{q_{i,i}}{\theta} - \frac{q_i g_i}{\theta^2} = \frac{\boldsymbol{\nabla}\cdot\boldsymbol{q}}{\theta} - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta^2} \tag{3.48}$$

Substituting from (3.48) into (3.46) and multiplying throughout by θ yields

$$\rho_0 \frac{D\eta}{Dt} + (\boldsymbol{\nabla}\cdot\boldsymbol{q} - \rho_0 r) - \frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g} \geq 0 \tag{3.49}$$

From the energy equation (3.38) (after inserting the $\rho_0 r$ term)

$$\boldsymbol{\nabla}\cdot\boldsymbol{q} - \rho_0 r = -\rho_0 \frac{De}{Dt} + \sigma_{ji}\frac{\partial v_i}{\partial x_j} + m_{ji}\frac{\partial_t \dot{\Theta}_i}{\partial x_j} + {}_t\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{m}) \tag{3.50}$$

Substituting from (3.50) into (3.49)

$$\rho_0\theta\frac{D\eta}{Dt} - \rho_0\frac{De}{Dt} + \sigma_{ji}\frac{\partial v_i}{\partial x_j} + m_{ji}\frac{\partial_t \dot{\Theta}_i}{\partial x_j} + {}_t\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{m}) - \frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g} \geq 0 \tag{3.51}$$

or

$$\rho_0\left(\frac{De}{Dt} - \theta\frac{D\eta}{Dt}\right) + \frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} - m_{ji}\frac{\partial_t \dot{\Theta}_i}{\partial x_j} - {}_t\dot{\boldsymbol{\Theta}}\,(\boldsymbol{\nabla}\cdot\boldsymbol{m}) \leq 0 \tag{3.52}$$

Let Φ be the Helmholtz free energy density defined by

$$\Phi = e - \eta\theta \tag{3.53}$$

$$\therefore \qquad \frac{De}{Dt} - \theta\frac{D\eta}{Dt} = \frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt} \tag{3.54}$$

Substituting from (3.54) into (3.52) gives

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} - m_{ji}\frac{\partial_t \dot{\Theta}_i}{\partial x_j} - {}_t\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{m}) \leq 0 \tag{3.55}$$

or

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \mathrm{tr}\left([\sigma]\left[\frac{\partial\{v\}}{\partial\{x\}}\right]\right) - \mathrm{tr}\left([m]\left[\frac{\partial\{{}_t\dot{\Theta}\}}{\partial\{x\}}\right]\right) - {}_t\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{m}) \leq 0 \tag{3.56}$$

or

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \text{tr}\left([\sigma] \, [{}^d\dot{J}] \right) - \text{tr}\left([m] \, [{}^{t\Theta}\dot{J}] \right) - {}_t\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \leq 0 \tag{3.57}$$

in which $[{}^{t\Theta}\dot{J}]$ is the material derivative of $[{}^{t\Theta}J]$ defined in (2.47) and $[{}^d\dot{J}]$ is the material derivative of $[{}^dJ]$. $[{}^d\dot{J}]$ is same as $[\dot{J}]$, but it is beneficial to work with $[{}^d\dot{J}]$ and $[{}^dJ]$ for small deformation and small strain.

Inequality (3.57) is the most fundamental form of the entropy inequality in the Helmholtz free energy density Φ. A slightly more expanded and more useful form of (3.57) for deriving constitutive theories can be derived using (2.19). Recall (2.19):

$$\begin{aligned} [\mathbb{J}] &= [{}^dJ] - [{}^e_a\gamma] = [{}^d_sJ] + [{}^d_aJ] - [{}^e_a\gamma] = [\varepsilon] + [{}_ar] \\ [\dot{\mathbb{J}}] &= [{}^d\dot{J}] - [{}^e_a\dot{\gamma}] = [{}^d_s\dot{J}] + [{}^d_a\dot{J}] - [{}^e_a\dot{\gamma}] = [\dot{\varepsilon}] + [{}_a\dot{r}] \end{aligned} \tag{3.58}$$

where $[\varepsilon]$ is the linearized strain tensor. Also, using (2.47) and (2.43)

$$[{}^{t\Theta}J] = \left[\frac{\partial\{{}_t\Theta\}}{\partial\{x\}} \right] = \left[\frac{\partial\{{}_i\Theta\}}{\partial\{x\}} \right] - \left[\frac{\partial\{{}_e\Theta\}}{\partial\{x\}} \right] = [{}^{i\Theta}J] - [{}^{e\Theta}J] \tag{3.59}$$

Hence

$$[{}^{t\Theta}\dot{J}] = [{}^{i\Theta}\dot{J}] - [{}^{e\Theta}\dot{J}] \tag{3.60}$$

and from the balance of angular momenta (3.12)

$$\boldsymbol{\nabla} \cdot \boldsymbol{m} = \boldsymbol{\epsilon} : \boldsymbol{\sigma} \tag{3.61}$$

and

$${}_t\dot{\boldsymbol{\Theta}} = {}_i\dot{\boldsymbol{\Theta}} - {}_e\dot{\boldsymbol{\Theta}} \tag{3.62}$$

Substituting from (3.58), (3.61), and (3.62) into (3.57)

$$\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\text{tr}\left([\sigma]\left([\dot{\mathbb{J}}]+[{}_a^e\dot{\gamma}]\right)\right)-\text{tr}\left([m][{}^{t\Theta}\dot{J}]\right)-({}_i\dot{\boldsymbol{\Theta}}-{}_e\dot{\boldsymbol{\Theta}})\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})\leq 0 \quad (3.63)$$

or

$$\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\text{tr}\left([\sigma][\dot{\mathbb{J}}]\right)-\text{tr}\left([\sigma][{}_a^e\dot{\gamma}]\right)-\text{tr}\left([m][{}^{t\Theta}\dot{J}]\right)$$
$$-{}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})+{}_e\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})\leq 0 \quad (3.64)$$

A simple calculation shows

$$\text{tr}\left([\sigma][{}_a^e\dot{\gamma}]\right)={}_e\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma}) \quad (3.65)$$

Using (3.65) in (3.64), (3.64) reduces to

$$\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\text{tr}\left([\sigma][\dot{\mathbb{J}}]\right)-\text{tr}\left([m][{}^{t\Theta}\dot{J}]\right)-{}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})\leq 0 \quad (3.66)$$

The entropy inequality (3.66) is the desired form that is useful in deriving constitutive theories. By making similar substitutions and simplifications, the energy equation (3.38) can be written as follows.

$$\rho_0\frac{De}{Dt}+\boldsymbol{\nabla}\cdot\boldsymbol{q}-\text{tr}\left([\sigma][\dot{\mathbb{J}}]\right)-\text{tr}\left([m][{}^{t\Theta}\dot{J}]\right)-{}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})=0 \quad (3.67)$$

3.2 Rate of Work Conjugate Pairs in the Entropy Inequality

In solids, the constitutive theories can be derived once appropriate argument tensors and energy conjugate pairs are known. From (3.67) it appears that $([\sigma],[\dot{\mathbb{J}}])$ and $([m],[{}^{t\Theta}\dot{J}])$ are rate of work conjugate pairs. However, the additional term ${}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})$ also needs to be accounted for. Note that $[\sigma]$, $[\dot{\mathbb{J}}]$ are both non-symmetric tensors whereas $[m]$ is a symmetric tensor (due to balance of moments of moments) but $[{}^{t\Theta}\dot{J}]$ is a non-symmetric tensor. Whether $([\sigma],[\dot{\mathbb{J}}])$ and $([m],[{}^{t\Theta}\dot{J}])$ are true rate of work conjugate pairs requires further consideration.

Consider the entropy inequality (3.66). Decompose $\boldsymbol{\sigma}$ into symmetric (${}_s\boldsymbol{\sigma}$) and antisymmetric (${}_a\boldsymbol{\sigma}$) tensors and use $\dot{\mathbb{J}}$ from (3.58). Also decompose ${}^{t\Theta}\dot{\boldsymbol{J}}$ into symmetric (${}^{t\Theta}_{s}\dot{\boldsymbol{J}}$) and antisymmetric (${}^{t\Theta}_{a}\dot{\boldsymbol{J}}$) tensors and substitute these into the entropy inequality (3.66).

$$\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma} \tag{3.68}$$

$${}^{t\Theta}\dot{\boldsymbol{J}} = {}^{t\Theta}_{s}\dot{\boldsymbol{J}} + {}^{t\Theta}_{a}\dot{\boldsymbol{J}} \tag{3.69}$$

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \text{tr}\left(\left([{}_s\sigma] + [{}_a\sigma]\right)\left([\dot{\varepsilon}] + [{}_a\dot{r}]\right)\right) \\ - \text{tr}\left([m]\left([{}^{t\Theta}_{s}\dot{J}] + [{}^{t\Theta}_{a}\dot{J}]\right)\right) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \tag{3.70}$$

Since

$$\text{tr}\left([{}_s\sigma][{}_a\dot{r}]\right) = 0 \tag{3.71}$$

$$\text{tr}\left([{}_a\sigma][\dot{\varepsilon}]\right) = 0 \tag{3.72}$$

$$\text{tr}\left([m][{}^{t\Theta}_{a}\dot{J}]\right) = 0 \tag{3.73}$$

the entropy inequality (3.70) can be written as

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \text{tr}\left([{}_s\sigma][\dot{\varepsilon}]\right) - \text{tr}\left([{}_a\sigma][{}_a\dot{r}]\right) - \text{tr}\left([m][{}^{t\Theta}_{s}\dot{J}]\right) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \tag{3.74}$$

In (3.74) ($\boldsymbol{q}$,$\boldsymbol{g}$), ($[{}_s\sigma]$,$[\dot{\varepsilon}]$), ($[{}_a\sigma]$,$[{}_a\dot{r}]$), and ($[m]$,$[{}^{t\Theta}_{s}\dot{J}]$) are the rate of work conjugate pairs that are in conformity with the works of Spencer, Wang, Zheng, etc. [80–99], i.e., a symmetric tensor is conjugate with a symmetric tensor and an antisymmetric tensor is conjugate with an antisymmetric tensor. Using (3.68), (3.69), and (3.71) – (3.73), the energy equation can also be written as

$$\rho_0\frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \text{tr}\left([{}_s\sigma][\dot{\varepsilon}]\right) - \text{tr}\left([{}_a\sigma][{}_a\dot{r}]\right) - \text{tr}\left([m][{}^{t\Theta}_{s}\dot{J}]\right) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) = 0 \tag{3.75}$$

It is straighforward to identify the conjugate pairs as

$$\begin{aligned}&({}_s\boldsymbol{\sigma},\,\boldsymbol{\varepsilon})\\&({}_a\boldsymbol{\sigma},\,{}_a\boldsymbol{r})\\&(\boldsymbol{m},\,{}_s^{t\Theta}\boldsymbol{J})\end{aligned} \tag{3.76}$$

3.2.1 Summary of the Conservation and Balance Laws

$$\rho_0(\boldsymbol{x}) = \rho(\boldsymbol{x}, t) \tag{3.77}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla}\boldsymbol{\cdot}{}_s\boldsymbol{\sigma} - \boldsymbol{\nabla}\boldsymbol{\cdot}{}_a\boldsymbol{\sigma} = 0 \tag{3.78}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{m} - \boldsymbol{\epsilon} : {}_a\boldsymbol{\sigma} = 0 \tag{3.79}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{3.80}$$

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \mathrm{tr}\left([{}_s\sigma][\dot{\varepsilon}]\right) - \mathrm{tr}\left([{}_a\sigma][{}_a\dot{r}]\right) - \mathrm{tr}\left([m][{}_s^{t\Theta}J]\right) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) = 0 \tag{3.81}$$

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \mathrm{tr}\left([{}_s\sigma][\dot{\varepsilon}]\right) - \mathrm{tr}\left([{}_a\sigma][{}_a\dot{r}]\right) - \mathrm{tr}\left([m][{}_s^{t\Theta}J]\right) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \tag{3.82}$$

In this mathematical model the dependent variables are (numbers in brackets refer to the number of variables): $\boldsymbol{v}$ (3), ${}_s\boldsymbol{\sigma}$ (6), ${}_a\boldsymbol{\sigma}$ (3), $\boldsymbol{m}$ (6), $\boldsymbol{q}$ (3), θ (1), ${}_e\boldsymbol{\Theta}$ (3), a total of 25. Φ, e, and η are not dependent variables as these can be shown to be deterministic from others. The equations in the model are: linear momentum (3), angular momentum (3), energy (1), constitutive theories for ${}_s\boldsymbol{\sigma}$ (6), ${}_a\boldsymbol{\sigma}$ (3), $\boldsymbol{m}$ (6), $\boldsymbol{q}$ (3), a total of 25, hence the mathematical model has closure.

3.3 Constitutive Theories for Thermoelastic Solids

For thermoelastic solids the mechanical deformation is reversible, i.e., the rate of work gets stored as the rate of strain energy density and there is no rate of entropy production due to the mechanical deformation process. This implies that there is no rate dependence in the constitutive theories. In such solids, the constitutive theories can be derived using the Helmholtz free energy density Φ, provided its argument tensors and the conjugate pairs in the constitutive theories are known.

Once the rate of work conjugate pairs are established from the energy equation or entropy inequality, the constitutive theories can also be derived using these in conjunction with the representation theorem. Note that the dependendent variables and their arguments are pairs of symmetric or antisymmetric tensors as required by the representation theorem [80–99].

3.3.1 Dependent Variables in the Constitutive Theories

It is straightforward to conclude from the conservation and balance laws that Φ, η, ${}_s\boldsymbol{\sigma}$, ${}_a\boldsymbol{\sigma}$, $\boldsymbol{m}$, and $\boldsymbol{q}$ are possible dependent variables in the constitutive theories. From (3.82), note that ${}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$, ${}_a\boldsymbol{\sigma} = {}_a\boldsymbol{\sigma}({}_a\boldsymbol{r})$, $\boldsymbol{m} = \boldsymbol{m}({}^{t\Theta}_{s}\boldsymbol{J})$, and $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g})$ are perfectly admissible due to the conjugate pairs. In addition, temperature θ can be considered as an argument for all of the dependent variables due to thermoelastic behavior. Since for thermoelastic solids the mechanical deformation is reversible, the Helmholtz free energy density must contain a mechanism of stored reversible energy. Thus,

$$\Phi = \Phi(\boldsymbol{\varepsilon}, {}_a\boldsymbol{r}, {}^{t\Theta}_{s}\boldsymbol{J}, \boldsymbol{g}, \theta) \tag{3.83}$$

Due to conjugate pairs, ${}_s\boldsymbol{\sigma}$, ${}_a\boldsymbol{\sigma}$, and $\boldsymbol{m}$ cannot be functions of $\left([{}_a r], [{}^{t\Theta}_{s}J]\right)$, $\left([\varepsilon], [{}^{t\Theta}_{s}J]\right)$, and $([{}_a r], [\varepsilon])$, respectively. Then at this stage

$$\begin{aligned}
\Phi &= \Phi(\boldsymbol{\varepsilon}, {}_a\boldsymbol{r}, {}^{t\Theta}_{s}\boldsymbol{J}, \boldsymbol{g}, \theta) \\
\eta &= \eta(\boldsymbol{\varepsilon}, {}_a\boldsymbol{r}, {}^{t\Theta}_{s}\boldsymbol{J}, \boldsymbol{g}, \theta) \\
{}_s\boldsymbol{\sigma} &= {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \\
{}_a\boldsymbol{\sigma} &= {}_a\boldsymbol{\sigma}({}_a\boldsymbol{r}, \theta) \\
\boldsymbol{m} &= \boldsymbol{m}({}^{t\Theta}_{s}\boldsymbol{J}, \theta) \\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{g}, \theta)
\end{aligned} \tag{3.84}$$

3.3.2 Entropy Inequality: Further Considerations

Using Φ in (3.84), the material derivative of Φ needed in (3.82) is defined.

$$\frac{D\Phi}{Dt} = \dot{\Phi} = \frac{\partial \Phi}{\partial \varepsilon_{ki}} \dot{\varepsilon}_{ki} + \frac{\partial \Phi}{\partial ({}_a r)_{ki}} (\dot{{}_a r})_{ki} + \frac{\partial \Phi}{\partial ({}^{t}_{s}{}^{\Theta} J)_{ki}} (\dot{{}^{t}_{s}{}^{\Theta} J})_{ki} + \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{\partial \Phi}{\partial \theta} \dot{\theta} \tag{3.85}$$

Substituting (3.85) into (3.82) and regrouping terms

$$\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki} \right) \dot{\varepsilon}_{ki} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}_a r)_{ki}} - {}_a\sigma_{ki} \right) (\dot{{}_a r})_{ki} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}^{t}_{s}{}^{\Theta} J)_{ki}} - m_{ki} \right) (\dot{{}^{t}_{s}{}^{\Theta} J})_{ki} \\ + \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) \dot{\theta} + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \tag{3.86}$$

For inequality (3.86) to hold for arbitrary but admissible $\dot{\boldsymbol{\varepsilon}}$, ${}_a\dot{\boldsymbol{r}}$, ${}^{t}_{s}{}^{\Theta}\dot{\boldsymbol{J}}$, $\dot{\boldsymbol{g}}$, and $\dot{\theta}$, the following must hold.

$$\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki} = 0 \implies {}_s\sigma_{ki} = \rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} \tag{3.87}$$

$$\rho_0 \frac{\partial \Phi}{\partial ({}_a r)_{ki}} - {}_a\sigma_{ki} = 0 \implies {}_a\sigma_{ki} = \rho_0 \frac{\partial \Phi}{\partial ({}_a r)_{ki}} \tag{3.88}$$

$$\rho_0 \frac{\partial \Phi}{\partial ({}^{t}_{s}{}^{\Theta} J)_{ki}} - m_{ki} = 0 \implies m_{ki} = \rho_0 \frac{\partial \Phi}{\partial ({}^{t}_{s}{}^{\Theta} J)_{ki}} \tag{3.89}$$

$$\frac{\partial \Phi}{\partial \theta} + \eta = 0 \implies \eta = \frac{\partial \Phi}{\partial \theta} \tag{3.90}$$

$$\frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \tag{3.91}$$

Using (3.87) – (3.89), constitutive theories can be derived for ${}_s\boldsymbol{\sigma}$, ${}_a\boldsymbol{\sigma}$, and $\boldsymbol{m}$. Equation (3.90) implies that η is deterministic from $-\frac{\partial \Phi}{\partial \theta}$, hence η is not a dependent variable in the constitutive theory. To ensure that the entropy inequality (3.91) is satisfied, the following must hold.

$$\frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} \leq 0 \tag{3.92}$$

$${}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) = 0 \tag{3.93}$$

Equation (3.92) can be used to derive the constitutive theory for $\boldsymbol{q}$. Since the sign of ${}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma})$ is uncertain, (3.93) must hold for (3.91) to be unconditionally satisfied. Equation (3.93) serves as a constraint on ${}_i\dot{\boldsymbol{\Theta}}$ and the antisymmetric components of $\boldsymbol{\sigma}$, i.e., (3.93) is a compatibility equation.

3.3.3 Constitutive Theory for ${}_s\boldsymbol{\sigma}$

The constitutive theory for ${}_s\boldsymbol{\sigma}$ can be derived using (3.87) and by considering $\Phi = \Phi(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)$, or by using ${}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta)$ and the representation theorem. It is well-known that the constitutive theories for thermoelastic solids derived through either approach will be identical [78]. Although the representation theorem is more general, both approaches are considered for completeness.

3.3.3.1 Constitutive Theory for ${}_s\boldsymbol{\sigma}$ Using Φ as a Function of the Invariants of $\boldsymbol{\varepsilon}$ and θ

The constitutive theory for ${}_s\boldsymbol{\sigma}$ is derived using (3.87) and by considering Φ as a function of the invariants of $\boldsymbol{\varepsilon}$ and θ.

$$
{}_s\boldsymbol{\sigma} = \rho_0 \frac{\partial \Phi(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)}{\partial \boldsymbol{\varepsilon}} \tag{3.94}
$$

$I_\varepsilon, II_\varepsilon, III_\varepsilon$ are the principal invariants of $\boldsymbol{\varepsilon}$, a symmetric tensor of rank two. Using (3.94) it is straightforward to derive the following for ${}_s\boldsymbol{\sigma}$ [78].

$$
{}_s\boldsymbol{\sigma} = {}^{s\sigma}\alpha^0 \boldsymbol{I} + {}^{s\sigma}\alpha^1 \boldsymbol{\varepsilon} + {}^{s\sigma}\alpha^2 \boldsymbol{\varepsilon}^{-1} \tag{3.95}
$$

in which

$$
\begin{aligned}
{}^{s\sigma}\alpha^0 &= \rho_0 \left(\frac{\partial \Phi}{\partial I_\varepsilon} + \frac{\partial \Phi}{\partial II_\varepsilon} I_\varepsilon \right) \\
{}^{s\sigma}\alpha^1 &= -\rho_0 \frac{\partial \Phi}{\partial II_\varepsilon} \\
{}^{s\sigma}\alpha^2 &= \rho_0 \frac{\partial \Phi}{\partial III_\varepsilon}
\end{aligned} \tag{3.96}
$$

Using the Cayley–Hamilton theorem [78], (3.95) can be written as

$$ {}_s\boldsymbol{\sigma} = {}^{s\sigma}\tilde{\alpha}^0 \boldsymbol{I} + {}^{s\sigma}\tilde{\alpha}^1 \boldsymbol{\varepsilon} + {}^{s\sigma}\tilde{\alpha}^2 \boldsymbol{\varepsilon}^2 \tag{3.97} $$

in which ${}^{s\sigma}\tilde{\alpha}^i$; $i = 0, 1, 2$ are functions of ${}^{s\sigma}\alpha^i$; $i = 0, 1, 2$, invariants I_ε, II_ε, and III_ε, and θ. The form (3.97) is preferred over (3.95) due to the absence of $\boldsymbol{\varepsilon}^{-1}$. This constitutive theory in (3.97) is not usable yet as I_ε, II_ε, III_ε, and θ are in the current configuration, hence are functions of unknown deformation. Nonetheless (3.97) is a fundamental form of the constitutive theory for ${}_s\boldsymbol{\sigma}$.

3.3.3.2 Constitutive Theory for ${}_s\boldsymbol{\sigma}$ Using the Representation Theorem

Consider

$$ {}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \tag{3.98} $$

${}_s\boldsymbol{\sigma}$ is a symmetric tensor of rank two whose arguments are $\boldsymbol{\varepsilon}$, a symmetric tensor of rank two, and θ, a tensor of rank zero. Based on the representation theorem [78–99], ${}_s\boldsymbol{\sigma}$ can be expressed as a linear combination of the combined generators of its arguments that are symmetric tensors of rank two. The combined generators of $\boldsymbol{\varepsilon}$ and θ that are symmetric tensors of rank two are $\boldsymbol{I}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\varepsilon}^2$ [78].

$$ {}_s\boldsymbol{\sigma} = {}^{s\sigma}\tilde{\alpha}^0 \boldsymbol{I} + {}^{s\sigma}\tilde{\alpha}^1 \boldsymbol{\varepsilon} + {}^{s\sigma}\tilde{\alpha}^2 \boldsymbol{\varepsilon}^2 \tag{3.99} $$

${}^{s\sigma}\tilde{\alpha}^i = {}^{s\sigma}\tilde{\alpha}^i(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)$; $i = 0, 1, 2$. The constitutive theory (3.99) is obviously the same as (3.97) derived using the Helmholtz free energy density. Invariants i_ε, $\ddot{u}_\varepsilon$, and $\dddot{u}_\varepsilon$ are an option instead of I_ε, II_ε, and III_ε as the two sets of invariants are related. The final outcome remains unaffected.

3.3.3.3 Material Coefficients in the Constitutive Theory for ${}_s\boldsymbol{\sigma}$

Material coefficients are derived from ${}^{s\sigma}\tilde{\alpha}^i(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta)$; $i = 0, 1, 2$ in (3.99). Expand ${}^{s\sigma}\tilde{\alpha}^i$ in a Taylor series in I_ε, II_ε, III_ε, and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in the invariants of $\boldsymbol{\varepsilon}$ and θ. Usie the following notation for convenience.

$$ {}^{s\sigma}\underset{\sim}{I}^1 = I_\varepsilon \; ; \quad {}^{s\sigma}\underset{\sim}{I}^2 = II_\varepsilon \; ; \quad {}^{s\sigma}\underset{\sim}{I}^3 = III_\varepsilon \tag{3.100} $$

$$ {}^{s\sigma}\tilde{\alpha}^i = {}^{s\sigma}\tilde{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^i}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j - \left({}^{s\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^i}{\partial \theta}\Bigg|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \tag{3.101} $$

Substituting from (3.101) into (3.99)

$$ \begin{aligned} {}_s\boldsymbol{\sigma} = & \left({}^{s\sigma}\tilde{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^0}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j - \left({}^{s\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^0}{\partial \theta}\Bigg|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) \boldsymbol{I} \\ & + \left({}^{s\sigma}\tilde{\alpha}^1\Big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^1}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j - \left({}^{s\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^1}{\partial \theta}\Bigg|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) \boldsymbol{\varepsilon} \\ & + \left({}^{s\sigma}\tilde{\alpha}^2\Big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^2}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j - \left({}^{s\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^2}{\partial \theta}\Bigg|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) \boldsymbol{\varepsilon}^2 \end{aligned} \tag{3.102} $$

Collecting the coefficients (defined in the known configuration $\underline{\Omega}$) of $\boldsymbol{I}$, $\boldsymbol{\varepsilon}$, ${}^{s\sigma}\underset{\sim}{I}^j\boldsymbol{\varepsilon}$; $j = 1, 2, 3$, ${}^{s\sigma}\underset{\sim}{I}^j\boldsymbol{\varepsilon}^2$; $j = 1, 2, 3$, $(\theta - \theta_{\underline{\Omega}})\boldsymbol{I}$, $(\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}$, and $(\theta - \theta_{\underline{\Omega}})(\boldsymbol{\varepsilon})^2$

$$ \begin{aligned} {}_s\boldsymbol{\sigma} = & \left({}^{s\sigma}\tilde{\alpha}^0_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^0}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) \boldsymbol{I} + \left({}^{s\sigma}\tilde{\alpha}^1_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^1}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) \boldsymbol{\varepsilon} \\ & + \left({}^{s\sigma}\tilde{\alpha}^2_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^2}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) (\boldsymbol{\varepsilon})^2 + \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^0}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j \boldsymbol{I} \right) \\ & + \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^1}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j \boldsymbol{\varepsilon} \right) + \sum_{j=1}^{3} \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^2}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j \boldsymbol{\varepsilon}^2 \right) + \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^0}{\partial \theta}\Bigg|_{\underline{\Omega}} ((\theta - \theta_{\underline{\Omega}})\boldsymbol{I}) \\ & + \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^1}{\partial \theta}\Bigg|_{\underline{\Omega}} ((\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}) + \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^2}{\partial \theta}\Bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}^2\right) \end{aligned} \tag{3.103} $$

Define

$$
\begin{aligned}
{}^{0}_{s}\sigma\big|_{\underline{\Omega}} &= {}^{s\sigma}\underline{b}_0 & {}^{s\sigma}\underline{a}_j &= \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^0}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} \; ; \; j = 1,2,3 \\
{}^{s\sigma}\underline{b}_i = {}^{s\sigma}\tilde{\alpha}^i\big|_{\underline{\Omega}} - \sum_{j=1}^{3} &\frac{\partial\, {}^{s\sigma}\tilde{\alpha}^i}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} \; ; \; i = 0,1,2 & {}^{s\sigma}\underset{\sim}{c}_{1j} &= \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^1}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} \; ; \; j = 1,2,3 \\
{}^{s\sigma}\underset{\sim}{c}_{2j} &= \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^2}{\partial\, {}^{s\sigma}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} \; ; \; j = 1,2,3 & {}^{s\sigma}\underset{\sim}{d}_1 &= \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^1}{\partial \theta}\bigg|_{\underline{\Omega}} \\
{}^{s\sigma}\underset{\sim}{d}_2 &= \frac{\partial\, {}^{s\sigma}\tilde{\alpha}^2}{\partial \theta}\bigg|_{\underline{\Omega}} & {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}} &= -\frac{\partial\, {}^{s\sigma}\tilde{\alpha}^0}{\partial \theta}\bigg|_{\underline{\Omega}}
\end{aligned}
\tag{3.104}
$$

Substituting (3.104) into (3.103)

$$
\begin{aligned}
{}_{s}\boldsymbol{\sigma} = {}^{0}_{s}\sigma\big|_{\underline{\Omega}}\boldsymbol{I} &+ {}^{s\sigma}\underline{b}_1\boldsymbol{\varepsilon} + {}^{s\sigma}\underline{b}_2\boldsymbol{\varepsilon}^2 + \sum_{j=1}^{3} {}^{s\sigma}\underline{a}_j\left({}^{s\sigma}\underset{\sim}{I}^j\boldsymbol{I}\right) + \sum_{j=1}^{3} {}^{s\sigma}\underset{\sim}{c}_{1j}\left({}^{s\sigma}\underset{\sim}{I}^j\boldsymbol{\varepsilon}\right) \\
&+ \sum_{j=1}^{3} {}^{s\sigma}\underset{\sim}{c}_{2j}\left({}^{s\sigma}\underset{\sim}{I}^j\boldsymbol{\varepsilon}^2\right) + {}^{s\sigma}\underset{\sim}{d}_1\left((\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}\right) + {}^{s\sigma}\underset{\sim}{d}_2\left((\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}^2\right) \\
&+ {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}\left((\theta - \theta_{\underline{\Omega}})\boldsymbol{I}\right)
\end{aligned}
\tag{3.105}
$$

${}^{0}_{s}\sigma\big|_{\underline{\Omega}}$ is the initial stress in the configuration $\underline{\Omega}$. This constitutive theory requires determination of 14 material coefficients as defined in (3.104) (excluding ${}^{0}_{s}\sigma\big|_{\underline{\Omega}}$), all evaluated in a known configuration $\underline{\Omega}$. The constitutive theory (3.105) for ${}_{s}\boldsymbol{\sigma}$ is the most general form of the constitutive theory for ${}_{s}\boldsymbol{\sigma}$ as a function of $\boldsymbol{\varepsilon}$ and temperature θ resulting from the entropy inequality or the representation theorem. This theory is based on integrity, hence is complete, but it contains too many material coefficients to be determined, experimentally or otherwise. Simplified forms are considered in a later section.

3.3.4 Constitutive Theory for $\boldsymbol{m}$

The constitutive theory for $\boldsymbol{m}$ can be derived using (3.89) and by considering $\Phi = \Phi(I_{({}^{\Theta}_{s}J)}, II_{({}^{\Theta}_{s}J)}, III_{({}^{\Theta}_{s}J)}, \theta)$ or by using $\boldsymbol{m} = \boldsymbol{m}({}^{t}{}^{\Theta}_{s}\boldsymbol{J}, \theta)$ and using the representation theorem. The constitutive theories derived in either approach will be identical for thermoelastic solids. Although the representation theorem is more general, both approaches are considered for completeness.

3.3.4.1 Constitutive Theory for $\boldsymbol{m}$ Using Φ as a Function of the Invariants of ${}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}$ and θ

The constitutive theory for $\boldsymbol{m}$ is derived using (3.89) and by considering Φ as a function of invariants of ${}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}$ and θ.

$$\boldsymbol{m} = \rho_0 \frac{\partial \Phi(I_{(\overset{\Theta}{s}J)}, II_{(\overset{\Theta}{s}J)}, III_{(\overset{\Theta}{s}J)}, \theta)}{\partial ({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}})} \tag{3.106}$$

Using (3.106) it is straightforward to derive the following for $\boldsymbol{m}$ [78].

$$\boldsymbol{m} = {}^{m}\alpha^{0}\boldsymbol{I} + {}^{m}\alpha^{1}({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}) + {}^{m}\alpha^{2}({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}})^{-1} \tag{3.107}$$

in which

$$\begin{aligned} {}^{m}\alpha^{0} &= \rho_0 \left(\frac{\partial \Phi}{\partial I_{(\overset{\Theta}{s}J)}} + \frac{\partial \Phi}{\partial II_{(\overset{\Theta}{s}J)}} I_{(\overset{\Theta}{s}J)} \right) \\ {}^{m}\alpha^{1} &= -\rho_0 \frac{\partial \Phi}{\partial II_{(\overset{\Theta}{s}J)}} \\ {}^{m}\alpha^{2} &= \rho_0 \frac{\partial \Phi}{\partial III_{(\overset{\Theta}{s}J)}} \end{aligned} \tag{3.108}$$

Using Cayley-Hamilton theorem [78], (3.107) can be written as

$$\boldsymbol{m} = {}^{m}\tilde{\alpha}^{0}\boldsymbol{I} + {}^{m}\tilde{\alpha}^{1}({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}) + {}^{m}\tilde{\alpha}^{2}({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}})^{2} \tag{3.109}$$

in which

$$\begin{aligned} {}^{m}\tilde{\alpha}^{i} &= {}^{m}\tilde{\alpha}^{i}({}^{m}\alpha^{0}, {}^{m}\alpha^{1}, {}^{m}\alpha^{2}, I_{(\overset{\Theta}{s}J)}, II_{(\overset{\Theta}{s}J)}, III_{(\overset{\Theta}{s}J)}, \theta) \\ &\text{or} \\ {}^{m}\tilde{\alpha}^{i} &= {}^{m}\tilde{\alpha}^{i}(I_{(\overset{\Theta}{s}J)}, II_{(\overset{\Theta}{s}J)}, III_{(\overset{\Theta}{s}J)}, \theta) \end{aligned} \tag{3.110}$$

Obviously (3.109) and (3.110) are all in the current configuration, hence ${}^{m}\tilde{\alpha}^{i}$; $i = 0, 1, 2$ are functions of unknown deformation.

3.3.4.2 Constitutive Theory for $\boldsymbol{m}$ Using the Representation Theorem

Consider

$$\boldsymbol{m} = \boldsymbol{m}({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}, \theta) \tag{3.111}$$

$\boldsymbol{m}$ and ${}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}$ are symmetric tensors of rank two and θ is a tensor of rank zero. Based on the representation theorem [78–99], $\boldsymbol{m}$ can be expressed as a linear combination of the combined generators of its argument tensors that are symmetric tensors of rank two. $\boldsymbol{I}$, ${}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}$, and $({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}})^2$ are the combined generators of ${}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}$ and θ that are symmetric tensors of rank two [78].

$$\boldsymbol{m} = {}^{m}\tilde{\alpha}^0 \boldsymbol{I} + {}^{m}\tilde{\alpha}^1 ({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}) + {}^{m}\tilde{\alpha}^2 ({}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}})^2 \tag{3.112}$$

${}^{m}\tilde{\alpha}^i$ are functions of the invariants of ${}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}$ and θ. The constitutive theory (3.112) is obviously the same as (3.109) derived using the Helmholtz free energy density.

3.3.4.3 Material Coefficients in the Constitutive Theory for $\boldsymbol{m}$

In order to derive the material coefficients in (3.112), expand each ${}^{m}\tilde{\alpha}^i$; $i = 0, 1, 2$ in a Taylor series in $I_{({}_{s}^{\Theta}J)}$, $II_{({}_{s}^{\Theta}J)}$, $III_{({}_{s}^{\Theta}J)}$, and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in the invariants of ${}^{t}_{s}\!\overset{\Theta}{\boldsymbol{J}}$ and θ. For simplicity let

$$ {}^{m}\underset{\sim}{I}^1 = I_{({}_{s}^{\Theta}J)} \; ; \quad {}^{m}\underset{\sim}{I}^2 = II_{({}_{s}^{\Theta}J)} \; ; \quad {}^{m}\underset{\sim}{I}^3 = III_{({}_{s}^{\Theta}J)} \tag{3.113}$$

Using the notation (3.113), the Taylor series expansion yields

$$ {}^{m}\tilde{\alpha}^i = {}^{m}\tilde{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial\, {}^{m}\tilde{\alpha}^i}{\partial\, {}^{m}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} \left({}^{m}\underset{\sim}{I}^j - ({}^{m}\underset{\sim}{I}^j)_{\underline{\Omega}} \right) + \frac{\partial\, {}^{m}\alpha^i}{\partial \theta}\bigg|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \; ; \; i = 0, 1, 2 \tag{3.114}$$

Substituting (3.114) into (3.112), collecting coefficients (those defined in $\underline{\Omega}$) of $[I]$, ${}^{m}\underset{\sim}{I}^j[I]$; $j = 1, 2, 3$, ${}^{m}\underset{\sim}{I}^j[{}^{t}_{s}\!\overset{\Theta}{J}]$; $j = 1, 2, 3$, ${}^{m}\underset{\sim}{I}^j[{}^{t}_{s}\!\overset{\Theta}{J}]^2$; $j = 1, 2, 3$, $(\theta - \theta_{\underline{\Omega}})[I]$, $(\theta - \theta_{\underline{\Omega}})[{}^{t}_{s}\!\overset{\Theta}{J}]$ and $(\theta - \theta_{\underline{\Omega}})[{}^{t}_{s}\!\overset{\Theta}{J}]^2$, and defining

$$
\begin{aligned}
{}^{0}\bar{m}\big|_{\underline{\Omega}} &= {}^{m}\underset{\sim}{b}_0 & {}^{m}\underset{\sim}{a}_j &= \frac{\partial\, {}^{m}\tilde{\alpha}^0}{\partial\, {}^{m}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} ;\ j = 1,2,3 \\
{}^{m}\underset{\sim}{b}_i = {}^{m}\tilde{\alpha}^i\big|_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{m}\tilde{\alpha}^i}{\partial\, {}^{m}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} &;\ i = 0,1,2 & {}^{m}\underset{\sim}{c}_{1j} &= \frac{\partial\, {}^{m}\tilde{\alpha}^1}{\partial\, {}^{m}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} ;\ j = 1,2,3 \\
{}^{m}\underset{\sim}{c}_{2j} = \frac{\partial\, {}^{m}\tilde{\alpha}^2}{\partial\, {}^{m}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} &;\ j = 1,2,3 & {}^{m}\underset{\sim}{d}_1 &= \frac{\partial\, {}^{m}\tilde{\alpha}^1}{\partial\theta}\bigg|_{\underline{\Omega}} \\
{}^{m}\underset{\sim}{d}_2 &= \frac{\partial\, {}^{m}\tilde{\alpha}^2}{\partial\theta}\bigg|_{\underline{\Omega}} & {}^{m}\underset{\sim}{\alpha}_{\mathrm{tm}} &= -\frac{\partial\, {}^{m}\tilde{\alpha}^0}{\partial\theta}\bigg|_{\underline{\Omega}}
\end{aligned}
\tag{3.115}
$$

the following holds for $\boldsymbol{m}$.

$$
\begin{aligned}
[m] = {}^{0}\bar{m}\big|_{\underline{\Omega}}\,[I] &+ {}^{m}\underset{\sim}{b}_1[{}^{t}_{s}\!\overset{\Theta}{J}] + {}^{m}\underset{\sim}{b}_2[{}^{t}_{s}\!\overset{\Theta}{J}]^2 + \sum_{j=1}^{3} {}^{m}\underset{\sim}{a}_j \left({}^{m}\underset{\sim}{I}^j [I]\right) + \sum_{j=1}^{3} {}^{m}\underset{\sim}{c}_{1j} \left({}^{m}\underset{\sim}{I}^j [{}^{t}_{s}\!\overset{\Theta}{J}]\right) \\
&+ \sum_{j=1}^{3} {}^{m}\underset{\sim}{c}_{2j} \left({}^{m}\underset{\sim}{I}^j [{}^{t}_{s}\!\overset{\Theta}{J}]^2\right) + {}^{m}\underset{\sim}{d}_1 \left((\theta - \theta_{\underline{\Omega}})[{}^{t}_{s}\!\overset{\Theta}{J}]\right) + {}^{m}\underset{\sim}{d}_2 \left((\theta - \theta_{\underline{\Omega}})[{}^{t}_{s}\!\overset{\Theta}{J}]^2\right) \\
&+ {}^{m}\underset{\sim}{\alpha}_{\mathrm{tm}} \left((\theta - \theta_{\underline{\Omega}})[I]\right)
\end{aligned}
\tag{3.116}
$$

This constitutive theory requires determination of 14 material coefficients defined in (3.115), all evaluated in the known configuration $\underline{\Omega}$. The constitutive theory (3.116) is the most general and complete constitutive theory for $[m]$ as it is based on integrity. Simplified constitutive theories for $\boldsymbol{m}$ are presented in a subsequent section.

3.3.5 Constitutive Theory for ${}_{a}\boldsymbol{\sigma}$

The constitutive theory for ${}_{a}\boldsymbol{\sigma}$ can be derived using (3.88) and by considering Φ as a function of the invariants of ${}_{a}\boldsymbol{r}$, i.e., $\Phi = \Phi(I_{({}_ar)}, II_{({}_ar)}, III_{({}_ar)}, \theta)$ or $\Phi = \Phi(i_{({}_ar)}, \ddot{u}_{({}_ar)}, \dddot{m}_{({}_ar)}, \theta)$. Since ${}_{a}\boldsymbol{r}$ is antisymmetric,

$$
I_{({}_ar)} = \mathrm{tr}({}_{a}\boldsymbol{r}) = 0;\ \ II_{({}_ar)} = \frac{1}{2}\left((\mathrm{tr}({}_{a}\boldsymbol{r}))^2 - \mathrm{tr}(({}_{a}\boldsymbol{r})^2)\right) = -\frac{1}{2}\mathrm{tr}(({}_{a}\boldsymbol{r})^2) \neq 0;\ \ III_{({}_ar)} = \det({}_{a}\boldsymbol{r}) = 0
\tag{3.117}
$$

and

$$i_{({}_a r)} = \text{tr}({}_a\boldsymbol{r}) = 0 \;; \quad \ddot{u}_{({}_a r)} = \text{tr}(({}_a\boldsymbol{r})^2) \neq 0 \;; \quad \dddot{u}_{({}_a r)} = \text{tr}(({}_a\boldsymbol{r})^3) = 0 \tag{3.118}$$

Thus the only non-zero invariants in this case are $\ddot{u}_{({}_a r)}$ and $II_{({}_a r)}$. These are obviously related.

$$II_{({}_a r)} = -\frac{1}{2}\ddot{u}_{({}_a r)} \tag{3.119}$$

Regardless of the choice of invariant,

$$\Phi = \Phi({}^{a\sigma}\underset{\sim}{I}^1, \theta) \tag{3.120}$$

in which ${}^{a\sigma}\underset{\sim}{I}^1$ is the only non-zero invariant of ${}_a\boldsymbol{r}$. Since the factor of $-\frac{1}{2}$ between $II_{({}_a r)}$ and $\ddot{u}_{({}_a r)}$ is irrelevant, either is valid. Let

$${}^{a\sigma}\underset{\sim}{I}^1 = \text{tr}(({}_a\boldsymbol{r})^2) \tag{3.121}$$

Thus,

$${}_a\boldsymbol{\sigma} = {}_a\boldsymbol{\sigma}({}^{a\sigma}\underset{\sim}{I}^1, \theta) \tag{3.122}$$

Now it is possible to derive the constitutive theory for ${}_a\boldsymbol{\sigma}$ using (3.122) and (3.88). Alternatively, one can use ${}_a\boldsymbol{\sigma} = {}_a\boldsymbol{\sigma}({}_a\boldsymbol{r}, \theta)$ and then apply the representation theorem. Again, both approaches will yield identical constitutive theories for thermoelastic solids, and both are considered for completeness.

3.3.5.1 Constitutive Theory for ${}_a\boldsymbol{\sigma}$ Using Φ as a Function of the Invariants of ${}_a\boldsymbol{r}$ and θ

Using (3.88),

$${}_a\boldsymbol{\sigma} = \rho_0 \frac{\partial \Phi({}^{a\sigma}\underset{\sim}{I}^1, \theta)}{\partial({}_a\boldsymbol{r})} \tag{3.123}$$

in which

$${}^{a\sigma}\underset{\sim}{I}^1 = \text{tr}\left(({}_a\boldsymbol{r})^2\right) = ({}_a r)_{kl}({}_a r)_{lk} \tag{3.124}$$

Using (3.123)

$$ {}_{a}\boldsymbol{\sigma} = \rho_0 \frac{\partial \Phi}{\partial {}^{a\sigma}\underset{\sim}{I}^1} \frac{\partial {}^{a\sigma}\underset{\sim}{I}^1}{\partial ({}_{a}\boldsymbol{r})} \tag{3.125} $$

Using (3.125) and (3.124)

$$ \begin{aligned} \frac{\partial {}^{a\sigma}\underset{\sim}{I}^1}{\partial ({}_{a}\boldsymbol{r})} = \frac{\partial {}^{a\sigma}\underset{\sim}{I}^1}{\partial ({}_{a}r)_{ij}} &= \frac{\partial}{\partial ({}_{a}r)_{ij}} \left(({}_{a}r)_{kl} ({}_{a}r)_{lk} \right) \\ &= \frac{\partial ({}_{a}r)_{kl}}{\partial ({}_{a}r)_{ij}} ({}_{a}r)_{lk} + ({}_{a}r)_{kl} \frac{\partial ({}_{a}r)_{lk}}{\partial ({}_{a}r)_{ij}} = ({}_{a}r)_{ij} + ({}_{a}r)_{ij} = 2({}_{a}r)_{ij} \end{aligned} \tag{3.126} $$

Substituting from (3.126) into (3.125),

$$ {}_{a}\boldsymbol{\sigma} = \rho_0 \frac{\partial \Phi}{\partial {}^{a\sigma}\underset{\sim}{I}^1} 2({}_{a}\boldsymbol{r}) = {}^{a\sigma}\alpha({}_{a}\boldsymbol{r}) \tag{3.127} $$

in which

$$ {}^{a\sigma}\alpha = 2\rho_0 \frac{\partial \Phi}{\partial {}^{a\sigma}\underset{\sim}{I}^1} = {}^{a\sigma}\alpha({}^{a\sigma}\underset{\sim}{I}^1, \theta) \tag{3.128} $$

3.3.5.2 Constitutive Theory for ${}_{a}\boldsymbol{\sigma}$ Using the Representation Theorem

Consider

$$ {}_{a}\boldsymbol{\sigma} = {}_{a}\boldsymbol{\sigma}({}_{a}\boldsymbol{r}, \theta) \tag{3.129} $$

${}_{a}\boldsymbol{\sigma}$ and ${}_{a}\boldsymbol{r}$ are antisymmetric tensors of rank two and θ is a tensor of rank zero. Combined generators of ${}_{a}\boldsymbol{r}$ and θ that are antisymmetric tensors of rank two include only ${}_{a}\boldsymbol{r}$ itself, which forms the basis of the space containing the tensor ${}_{a}\boldsymbol{\sigma}$, hence ${}_{a}\boldsymbol{\sigma}$ can be expressed as a linear combination of ${}_{a}\boldsymbol{r}$.

$$ {}_{a}\boldsymbol{\sigma} = {}^{a\sigma}\alpha({}_{a}\boldsymbol{r}) \tag{3.130} $$

in which

$$ {}^{a\sigma}\alpha = {}^{a\sigma}\alpha({}^{a\sigma}\underset{\sim}{I}^1, \theta) \tag{3.131} $$

Note that (3.130), (3.131) are the same as (3.127), (3.128) derived using the Helmholtz free energy density.

3.3.5.3 Material Coefficients in the Constitutive Theory for ${}_a\boldsymbol{\sigma}$

Consider (3.127) or (3.130) and (3.128) or (3.131). Expand ${}^{a\sigma}\alpha$ in a Taylor series in ${}^{a\sigma}\underset{\sim}{I}^1$ and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in ${}^{a\sigma}\underset{\sim}{I}^1$ and θ.

$$ {}^{a\sigma}\alpha = {}^{a\sigma}\alpha\big|_{\underline{\Omega}} + \frac{\partial\, {}^{a\sigma}\alpha}{\partial\, {}^{a\sigma}\underset{\sim}{I}^1}\bigg|_{\underline{\Omega}} \left({}^{a\sigma}\underset{\sim}{I}^1 - ({}^{a\sigma}\underset{\sim}{I}^1)_{\underline{\Omega}}\right) - \frac{\partial\, {}^{a\sigma}\alpha}{\partial\theta}\bigg|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \tag{3.132} $$

Substituting (3.132) in (3.130) and collecting coefficients,

$$ {}_a\boldsymbol{\sigma} = \left({}^{a\sigma}\alpha\big|_{\underline{\Omega}} - \frac{\partial\, {}^{a\sigma}\alpha}{\partial\, {}^{a\sigma}\underset{\sim}{I}^1}\bigg|_{\underline{\Omega}} ({}^{a\sigma}\underset{\sim}{I}^1)_{\underline{\Omega}} \right) ({}_a\boldsymbol{r}) + \frac{\partial\, {}^{a\sigma}\alpha}{\partial\, {}^{a\sigma}\underset{\sim}{I}^1}\bigg|_{\underline{\Omega}} ({}^{a\sigma}\underset{\sim}{I}^1)({}_a\boldsymbol{r}) - \frac{\partial\, {}^{a\sigma}\alpha}{\partial\theta}\bigg|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}})\,({}_a\boldsymbol{r}) \tag{3.133} $$

Let

$$ {}^{a\sigma}\underline{b}_1 = \left({}^{a\sigma}\alpha\big|_{\underline{\Omega}} - \frac{\partial\, {}^{a\sigma}\alpha}{\partial\, {}^{a\sigma}\underset{\sim}{I}^1}\bigg|_{\underline{\Omega}} ({}^{a\sigma}\underset{\sim}{I}^1)_{\underline{\Omega}} \right) \; ; \quad {}^{a\sigma}\underset{\sim}{c}_{11} = \frac{\partial\, {}^{a\sigma}\alpha}{\partial\, {}^{a\sigma}\underset{\sim}{I}^1}\bigg|_{\underline{\Omega}} \; ; \quad {}^{a\sigma}\underset{\sim}{d}_2 = \frac{\partial\, {}^{a\sigma}\alpha}{\partial\theta}\bigg|_{\underline{\Omega}} \tag{3.134} $$

Substituting (3.134) in (3.133),

$$ {}_a\boldsymbol{\sigma} = {}^{a\sigma}\underline{b}_1({}_a\boldsymbol{r}) + {}^{a\sigma}\underset{\sim}{c}_{11}({}^{a\sigma}\underset{\sim}{I}^1)({}_a\boldsymbol{r}) - {}^{a\sigma}\underset{\sim}{d}_2\,(\theta - \theta_{\underline{\Omega}})\,({}_a\boldsymbol{r}) \tag{3.135} $$

This constitutive theory requires three material coefficients: ${}^{a\sigma}\underline{b}_1$, ${}^{a\sigma}\underset{\sim}{c}_{11}$, and ${}^{a\sigma}\underset{\sim}{d}_2$. However, if the $(\theta - \theta_{\underline{\Omega}})$ term is neglected then (3.135) requires only two material coefficients, ${}^{a\sigma}\underline{b}_1$ and ${}^{a\sigma}\underset{\sim}{c}_{11}$. The constitutive theory (3.135) contains up to cubic terms in the components of the total rotation tensor ${}_a\boldsymbol{r}$. This constitutive theory is based on integrity, hence is complete.

3.3.6 Linear Constitutive Theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$, and ${}_a\boldsymbol{\sigma}$

The constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$, and ${}_a\boldsymbol{\sigma}$ derived in Sections 3.3.3, 3.3.4, and 3.3.5 are complete as they are based on integrity, but require determination of too many material coefficients.

In this section simplified forms of the theories are considered that require the minimum number of material coefficients, but at the expense of only modeling simplified and compromised physics.

Consider (3.105), (3.116), and (3.135), the complete constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$, and ${}_a\boldsymbol{\sigma}$. The following assumptions are made.

1. The constitutive theory for each variable is linear in its argument tensors and the components of the argument tensor, i.e., quadratic and higher degree terms in the components of the argument tensors and the product terms are neglected.

2. The initial stress field and initial moment field are neglected.

3. All terms containing $(\theta - \underline{\theta_\Omega})$ are neglected.

Based on these assumptions, (3.105), (3.116), and (3.135) reduce to

$$ {}_s\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} \qquad ({}^{s\sigma}\underline{b}_1 = 2\mu \text{ and } {}^{s\sigma}\underline{a}_1 = \lambda) \tag{3.136} $$

$$ \boldsymbol{m} = \alpha({}^{t\Theta}_{s}\boldsymbol{J}) + B\mathrm{tr}({}^{t\Theta}_{s}\boldsymbol{J})\boldsymbol{I} \qquad ({}^{m}\underline{b}_1 = \alpha \text{ and } {}^{m}\underline{a}_1 = B) \tag{3.137} $$

$$ {}_a\boldsymbol{\sigma} = \kappa({}_a\boldsymbol{r}) \qquad ({}^{a\sigma}\underline{b}_1 = \kappa) \tag{3.138} $$

Remarks.

(1) These constitutive theories require only five material coefficients. μ and λ are the usual elasticity material coefficients (such as Lamé's constants). α, B, and κ are three new coefficients. α is required in the presence of rotations (${}_i\boldsymbol{\Theta}$ or ${}_e\boldsymbol{\Theta}$ or both) and κ and B are only needed when Cosserat rotations are present.

(2) In the case of purely internal polar non-classical continuum theories, the constitutive theory for ${}_a\boldsymbol{\sigma}$ is not needed as these are balanced by gradients of the moment tensor in the balance of angular momenta, hence in this case $\kappa = 0$. ${}^{t\Theta}_{s}\boldsymbol{J}$ becomes the symmetric part of the internal rotation gradient tensor, and the coefficient B is no longer required as the trace of the gradient of internal rotations is zero.

(3) The material coefficient κ is necessitated due to the presence of Cosserat rotations at the material points.

(4) If the balance of moments of moments balance law is neglected, then the moment tensor is not symmetric. Following the procedure presented, the constitutive theories in such a case would be identical to when the balance of moments of moments is considered, with the addition of the constitutive theory for the antisymmetric part of the moment tensor.

$$ {}_s\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} \tag{3.139} $$

$$ {}_s\boldsymbol{m} = \alpha({}^{t\Theta}_{s}\boldsymbol{J}) + B\mathrm{tr}({}^{t\Theta}_{s}\boldsymbol{J})\boldsymbol{I} \tag{3.140} $$

$$ {}_a\boldsymbol{\sigma} = \kappa({}_a\boldsymbol{r}) \tag{3.141} $$

$$ {}_a\boldsymbol{m} = \beta({}^{t\Theta}_{a}\boldsymbol{J}) \tag{3.142} $$

3.3.7 Mathematical Model of Eringen

Consider the conservation and balance laws presented by Eringen for micropolar solids [14] and the associated constitutive theories for thermoelastic behavior. The conservation of mass, balance of linear momenta, and balance of angular momenta are identical to those presented in this dissertation. The balance of moments of moments as used in this work and shown necessary by Yang, et al. [77] and Surana, et al. [103, 104] is not used in the work presented by Eringen [14]. The consequence of this is that the Cauchy moment tensor is not symmetric.

Recall that

$$ \mathbb{J} = {}^{d}\boldsymbol{J} - {}^{e}_{a}\boldsymbol{\gamma} = {}^{d}_{s}\boldsymbol{J} + {}^{d}_{a}\boldsymbol{J} - {}^{e}_{a}\boldsymbol{\gamma} \tag{3.143} $$

or

$$ \mathbb{J} = {}^{d}_{s}\boldsymbol{J} + {}_a\boldsymbol{r} \ ; \quad {}_a\boldsymbol{r} = {}^{d}_{a}\boldsymbol{J} - {}^{e}_{a}\boldsymbol{\gamma} \tag{3.144} $$

in which ${}^{d}_{a}\boldsymbol{J}$ contains internal rotations due to $\boldsymbol{J}$ or ${}^{d}\boldsymbol{J}$ and ${}^{e}_{a}\boldsymbol{\gamma}$ contains Cosserat rotations. When these are resisted by the deforming matter, the conjugate Cauchy moment tensor is created, which together with ${}_a\boldsymbol{r}$ results in rate of work as used in the present model. There are several major differences between the work presented here and that in Reference [14].

(1) In Reference [14], only the rate of work due to ${}_e\boldsymbol{\Theta}$ is considered, i.e., the rate of work due to ${}_i\boldsymbol{\Theta}$ (internal rotation rates) is neglected. As a consequence the energy equation and the entropy inequality only contain material derivatives of the gradients of the rotations ${}_e\boldsymbol{\Theta}$ defined by

$$[{}^{e\Theta}\dot{J}] = \frac{D}{Dt}\left[\frac{\partial\{{}_e\Theta\}}{\partial\{x\}}\right] = \frac{\partial}{\partial t}\left[\frac{\partial\{{}_e\Theta\}}{\partial\{x\}}\right] = \left[\frac{\partial\{{}_e\dot{\Theta}\}}{\partial\{x\}}\right] \tag{3.145}$$

(2) Another consequence of this is that the ${}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma})$ term that appears in the energy equation and entropy inequality in equations (3.66) and (3.67) is not present in Reference [14].

(3) Due to remark (1), the conjugate pairs in the entropy inequality are affected, hence the constitutive theories are affected as well.

3.3.7.1 Conservation and Balance Laws

The mathematical model (excluding the constitutive theories) used in Reference [14] is given in the following.

$$\rho_0(\boldsymbol{x}) = \rho(\boldsymbol{x}, t) \tag{3.146}$$

$$\rho_0\frac{D\boldsymbol{v}}{Dt} - \rho_0\boldsymbol{F}^b - \boldsymbol{\nabla}\cdot({}_s\boldsymbol{\sigma}) - \boldsymbol{\nabla}\cdot({}_a\sigma) = 0 \tag{3.147}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.148}$$

(No balance law for balance of moments of moments)

$$\rho_0\frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \text{tr}\left([\sigma][\dot{\mathbb{J}}]\right) - \text{tr}\left([m][{}^{e\Theta}\dot{J}]\right) = 0 \tag{3.149}$$

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \text{tr}\left([\sigma][\dot{\mathbb{J}}]\right) - \text{tr}\left([m][{}^{e\Theta}\dot{J}]\right) \leq 0 \tag{3.150}$$

Remarks.

Comparing (3.149) and (3.150) with (3.66) and (3.67), observe the following.

(1) $\boldsymbol{m}$ is non-symmetric as opposed to symmetric $\boldsymbol{m}$ in (3.66) and (3.67).

(2) The rate of work term tr $\left([m][{}^{i\Theta}\dot{J}]\right)$ is neglected in (3.149) and (3.150). Due to this, the additional term ${}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma})$ in (3.66) and (3.67) is not present in (3.149) and (3.150).

(3) From (3.150) one could possibly conclude that $[\sigma]$ and $[\mathbb{J}]$, $[m]$ and $[{}^{e\Theta}J]$ appear to be work conjugate. This requires further considerations.

3.3.7.2 Constitutive Theories

By assuming $\Phi = \Phi(\mathbb{J}, {}^{e\Theta}\boldsymbol{J}, \theta)$ and using the material derivative of Φ, i.e., $\dot{\Phi}$, in the entropy inequality it is established that

$$\boldsymbol{\sigma} = \rho_0 \frac{\partial \Phi(\mathbb{J}, {}^{e\Theta}\boldsymbol{J}, \theta)}{\partial \mathbb{J}} \tag{3.151}$$

$$\boldsymbol{m} = \rho_0 \frac{\partial \Phi(\mathbb{J}, {}^{e\Theta}\boldsymbol{J}, \theta)}{\partial ({}^{e\Theta}\boldsymbol{J})} \tag{3.152}$$

Φ is assumed to be a quadratic polynomial in $\mathbb{J}$ and ${}^{e\Theta}\boldsymbol{J}$. Using (3.151) and (3.152), constitutive theories for $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ that are linear in $\mathbb{J}$ and ${}^{e\Theta}\boldsymbol{J}$ are derived. After some simplifications and assumptions, the following constitutive theories for $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ are reported.

$$\sigma_{kl} = \lambda \varepsilon_{rr} \delta_{kl} + (2\mu + \kappa)\varepsilon_{kl} + \kappa \epsilon_{klm}({}_i\Theta_m - {}_e\Theta_m) \tag{3.153}$$

$$m_{kl} = \alpha({}^{e\Theta}J)_{rr}\delta_{kl} + \beta({}^{e\Theta}J)_{kl} + \zeta({}^{e\Theta}J)_{lk} \tag{3.154}$$

Remarks.

(1) Considering $\Phi = \Phi(\mathbb{J}, {}^{e\Theta}\boldsymbol{J}, \theta)$, even for linear constitutive theories $\boldsymbol{\sigma}$ can be a function of ${}^{e\Theta}\boldsymbol{J}$ and $\boldsymbol{m}$ can be a function of $\mathbb{J}$ in Eringen's theory, which is not supported by the conjugate pairs in (3.150) (assuming them to be truly conjugate).

(2) By decomposing $\boldsymbol{\sigma}$ into symmetric (${}_s\boldsymbol{\sigma}$) and antisymmetric (${}_a\boldsymbol{\sigma}$) tensors and likewise $\dot{\mathbb{J}}$ into symmetric (${}_s\dot{\mathbb{J}}$) and antisymmetric (${}_a\dot{\mathbb{J}}$) tensors and using them in $\text{tr}([\sigma][\dot{\mathbb{J}}])$, it is straightforward to show that $\text{tr}([{}_a\sigma][{}_s\dot{\mathbb{J}}])$ and $\text{tr}([{}_s\sigma][{}_a\dot{\mathbb{J}}])$ do not result in rate of work and that only $\text{tr}([{}_s\sigma][{}_s\dot{\mathbb{J}}])$ and $\text{tr}([{}_a\sigma][{}_a\dot{\mathbb{J}}])$ produce rate of work, thus the derivation using (3.151) produces terms that do not result in work. Similar arguments hold for (3.152).

(3) If the balance of moments of moments is considered, then the Cauchy moment tensor becomes symmetric, which replaces β and ζ by just one material coefficient as there will not be a constitutive theory for the antisymmetric Cauchy moment tensor. The material coefficients in the constitutive theory for the moment tensor are reduced by one. This yields five material coefficients, three in (3.153) and two in (3.154).

(4) Note that the constitutive theories for ${}_s\boldsymbol{\sigma}$ and ${}_a\boldsymbol{\sigma}$ are completely unrelated. ${}_s\boldsymbol{\sigma}$ is work conjugate with strain $\boldsymbol{\varepsilon}$ and ${}_a\boldsymbol{\sigma}$ is work conjugate to ${}_a\boldsymbol{r}$, hence the material coefficient in the constitutive theory for ${}_a\boldsymbol{\sigma}$ should not appear in the material coefficient in the constitutive theory for ${}_s\boldsymbol{\sigma}$, implying that the $\kappa\varepsilon_{kl}$ term in (3.153) cannot be justified and in fact should be zero. If this term is neglected in (3.153) and the modifications in remark (3) are made, then (3.153) and (3.154) reduce to

$$\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma} = \lambda \text{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} + \kappa_a \boldsymbol{r} \tag{3.155}$$

$$\boldsymbol{m} = \alpha \text{tr}({}^{e\Theta}\boldsymbol{J})\boldsymbol{I} + \beta({}^{e\Theta}\boldsymbol{J}) \tag{3.156}$$

which are identical to the constitutive theories derived in this dissertation when the balance of moments of moments is used as a balance law and internal rotations are neglected.

(5) Some disturbing aspects of the derivation of the constitutive theories based on the polynomial approach used in Reference [14] are that (i) these are limited to linear theories, (ii) the material coefficients have to be constant, and (iii) there is no real basis for this derivation when compared to the two commonly accepted and well-founded approaches: the Helmholtz

free energy density as a function of invariants of the argument tensors and the representation theorem (theory of generators and invariants).

(6) Lastly, (3.151) and (3.152) lead to constitutive theories that can not be supported by the representation theorem when $\boldsymbol{\sigma}$, $\boldsymbol{m}$, $\mathbb{J}$, and ${}^{e\Theta}\boldsymbol{J}$ are non-symmetric tensors. Consider $\text{tr}([\sigma][\dot{\mathbb{J}}])$. $[\sigma]$ and $[\dot{\mathbb{J}}]$ are non-symmetric tensors of rank two and appear to be rate of work conjugate to each other. Thus, if one considers $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbb{J}, \theta)$, then it may be possible to consider Φ as a function of the invariants of $\mathbb{J}$ and possibly derive a constitutive theory for $\boldsymbol{\sigma}$ using

$$[\sigma] = \rho_0 \frac{\partial \Phi(I_{\mathbb{J}}, II_{\mathbb{J}}, III_{\mathbb{J}}, \theta)}{\partial [\mathbb{J}]} \tag{3.157}$$

$I_{\mathbb{J}}$, $II_{\mathbb{J}}$, and $III_{\mathbb{J}}$ are the principal invariants of $\mathbb{J}$. Using (3.157) it is straightforward to derive the following [78].

$$[\sigma] = {}^{\sigma}\underset{\sim}{c}_0[I] + {}^{\sigma}\underset{\sim}{c}_1[\mathbb{J}] + {}^{\sigma}\underset{\sim}{c}_2[\mathbb{J}]^{-1} \tag{3.158}$$

Using Cayley-Hamilton theorem, $[\mathbb{J}]^{-1}$ can be obtained in terms of $[I]$, $[\mathbb{J}]$, and $[\mathbb{J}]^2$ and then substituted in (3.158) to obtain

$$[\sigma] = {}^{\sigma}c_0[I] + {}^{\sigma}c_1[\mathbb{J}] + {}^{\sigma}c_2[\mathbb{J}]^2 \tag{3.159}$$

Expression (3.159), a linear combination of $[I]$, $[\mathbb{J}]$, and $[\mathbb{J}]^2$, suggests that when $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbb{J}, \theta)$, then in (3.159) $[I]$, $[\mathbb{J}]$, and $[\mathbb{J}]^2$ must be the combined generators of the argument tensors $[\mathbb{J}]$ and θ that must constitute the basis of the space of $[\sigma]$. Based on the works of Spencer, Wang, Zheng, etc. [80–99], when $\boldsymbol{\sigma}$ and $\mathbb{J}$ both are non-symmetric and when $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbb{J}, \theta)$ then $[I]$, $[\mathbb{J}]$, and $[\mathbb{J}]^2$ are not the combined generators of $[\mathbb{J}]$, i.e., based on the representation theorem (3.159) is not valid. From (3.157) based on the Helmholtz free energy density that leads to (3.158) and (3.159), there is no concrete argument of its lack of validity. However based on the representation theorem it is conclusive that (3.159) is not valid when $[\sigma]$ and $[\mathbb{J}]$ are both non-symmetric tensors. On the other hand, when both are symmetric, (3.157) – (3.159) are perfectly valid and (3.159) is supported by the representation theorem. That is, in this case $[I]$, $[\mathbb{J}]$, and $[\mathbb{J}]^2$ form the basis of the space of $[\sigma]$.

The details presented here conclusively substantiate that $\boldsymbol{\sigma}$ cannot be expressed in terms of a basis that can be derived from $\mathbb{J}$. This is a major point of departure of the present work from the work presented by Eringen [14]. Similar arguments hold if $[m]$ and $[{}^{e}\Theta J]$ are considered as work conjugate, both of which are non-symmetric tensors.

(7) The constitutive theories derived in this work are supported by the representation theorem and not limited to linear constitutive theories as in Reference [14]. All constitutive theories derived in this dissertation are based on integrity (complete basis), hence any desired nonlinear constitutive theories can be easily extracted from these.

3.4 Constitutive Theories for Thermoviscoelastic Solids without Memory

For thermoviscoelastic solids the mechanical deformation is only partially reversible, i.e., some of the rate of work gets stored as rate of strain energy density, but there is entropy production and energy dissipation due to the mechanical deformation process. Due to the presence of dissipation, constitutive theories cannot be derived using the Helmholtz free energy density as a function of the invariants of the argument tensors.

Once the rate of work conjugate pairs are established from the energy equation or entropy inequality, the constitutive theories can be derived using these in conjunction with the theory of generators and invariants. Note that the dependendent variables and their arguments are pairs of symmetric or antisymmetric tensors as required by the representation theorem [80–99].

3.4.1 Dependent Variables in the Constitutive Theories

It is straightforward to conclude from the conservation and balance laws that Φ, η, ${}_s\boldsymbol{\sigma}$, ${}_a\boldsymbol{\sigma}$, $\boldsymbol{m}$, and $\boldsymbol{q}$ are possible dependent variables in the constitutive theories. From (3.82), note that ${}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$, ${}_a\boldsymbol{\sigma} = {}_a\boldsymbol{\sigma}({}_a\boldsymbol{r})$, $\boldsymbol{m} = \boldsymbol{m}({}^{t}_{s}{}^{\Theta}\boldsymbol{J})$, and $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g})$ are perfectly admissible due to the conjugate pairs. Since the deforming matter has elasticity as well as dissipation, $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ (or $\boldsymbol{\varepsilon}_{[1]}$) both need to be argument tensors of ${}_s\boldsymbol{\sigma}$. If it is assumed that higher order time derivatives of $\boldsymbol{\varepsilon}$ also contribute to dissipation, then $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}_{[k]}$; $k = 1, 2, \dots, n_\varepsilon$ need to be argument tensors of ${}_s\boldsymbol{\sigma}$. Further assume that the dissipation mechanism is also due to $\boldsymbol{m}$, ${}_a\boldsymbol{\sigma}$, and the time derivatives of the corresponding conjugate tensors. In addition, temperature θ can be considered as an argument for all of the dependent variables due to thermoviscoelastic behavior.

$$
\begin{aligned}
{}_s\boldsymbol{\sigma} &= {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n_\varepsilon,\ \theta) \\
{}_a\boldsymbol{\sigma} &= {}_a\boldsymbol{\sigma}({}_a\boldsymbol{r}, {}_a\boldsymbol{r}_{[j]};\ j = 1, 2, \dots, n_{({}_a r)},\ \theta) \\
\boldsymbol{m} &= \boldsymbol{m}({}^{t}_{s}{}^{\Theta}\boldsymbol{J}, {}^{t}_{s}{}^{\Theta}\boldsymbol{J}_{[k]};\ k = 1, 2, \dots, n_{({}^{t}_{s}{}^{\Theta}J)},\ \theta) \\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{g}, \theta) \\
\Phi &= \Phi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n_\varepsilon,\ {}_a\boldsymbol{r}, {}_a\boldsymbol{r}_{[j]};\ j = 1, 2, \dots, n_{({}_a r)}, \\
&\qquad {}^{t}_{s}{}^{\Theta}\boldsymbol{J}, {}^{t}_{s}{}^{\Theta}\boldsymbol{J}_{[k]};\ k = 1, 2, \dots, n_{({}^{t}_{s}{}^{\Theta}J)},\ \boldsymbol{g}, \theta) \\
\eta &= \eta(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n_\varepsilon,\ {}_a\boldsymbol{r}, {}_a\boldsymbol{r}_{[j]};\ j = 1, 2, \dots, n_{({}_a r)}, \\
&\qquad {}^{t}_{s}{}^{\Theta}\boldsymbol{J}, {}^{t}_{s}{}^{\Theta}\boldsymbol{J}_{[k]};\ k = 1, 2, \dots, n_{({}^{t}_{s}{}^{\Theta}J)},\ \boldsymbol{g}, \theta)
\end{aligned}
\tag{3.160}
$$

Note that the argument tensors of Φ and η are the totality of all of the argument tensors of all of the constitutive variables. At this stage (3.160) is the most general choice. During the derivation of constitutive theories, some arguments of some constitutive variables may be ruled out due to some other considerations.

3.4.2 Entropy Inequality: Further Considerations

Using Φ in (3.160), the material derivative of Φ needed in (3.82) is defined.

$$\begin{aligned}\frac{D\Phi}{Dt} = \dot{\Phi} = \frac{\partial \Phi}{\partial \varepsilon_{ki}}\dot{\varepsilon}_{ki} + \sum_{j=1}^{n_\varepsilon} \frac{\partial \Phi}{\partial (\varepsilon_{[j]})_{ki}}(\dot{\varepsilon}_{[j]})_{ki} + \frac{\partial \Phi}{\partial ({}_a r_{ki})}({}_a\dot{r}_{ki}) + \sum_{j=1}^{n_{(ar)}} \frac{\partial \Phi}{\partial ({}_a r_{[j]})_{ki}}({}_a\dot{r}_{[j]})_{ki} \\ + \frac{\partial \Phi}{\partial ({}^{t\Theta}_{s}J_{ki})}({}^{t\Theta}_{s}\dot{J}_{ki}) + \sum_{j=1}^{n_{{}^{t\Theta}_{s}J}} \frac{\partial \Phi}{\partial ({}^{t\Theta}_{s}J_{[j]})_{ki}}({}^{t\Theta}_{s}\dot{J}_{[j]})_{ki} + \frac{\partial \Phi}{\partial g_i}\dot{g}_i + \frac{\partial \Phi}{\partial \theta}\dot{\theta}\end{aligned} \tag{3.161}$$

Substituting (3.161) into the entropy inequality (3.82) and collecting terms

$$\begin{aligned}&\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki}\right)\dot{\varepsilon}_{ki} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}_a r_{ki})} - {}_a\sigma_{ki}\right){}_a\dot{r}_{ki} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}^{t\Theta}_{s}J_{ki})} - m_{ki}\right){}^{t\Theta}_{s}\dot{J}_{ki} \\ &+ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} + \rho_0\left(\eta + \frac{\partial \Phi}{\partial \theta}\right)\dot{\theta} + \frac{\partial \Phi}{\partial g_i}\dot{g}_i + \sum_{j=1}^{n_\varepsilon} \frac{\partial \Phi}{\partial (\varepsilon_{[j]})_{ki}}(\dot{\varepsilon}_{[j]})_{ki} \\ &+ \sum_{j=1}^{n_{(ar)}} \frac{\partial \Phi}{\partial ({}_a r_{[j]})_{ki}}({}_a\dot{r}_{[j]})_{ki} + \sum_{j=1}^{n_{{}^{t\Theta}_{s}J}} \frac{\partial \Phi}{\partial ({}^{t\Theta}_{s}J_{[j]})_{ki}}({}^{t\Theta}_{s}\dot{J}_{[j]})_{ki} - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \le 0\end{aligned} \tag{3.162}$$

For arbitrary but admissible $\dot{\theta}$, $\dot{\boldsymbol{g}}$, $\dot{\boldsymbol{\varepsilon}}_{[j]}$; $j = 1,2,\dots,n_\varepsilon$, ${}_a\dot{\boldsymbol{r}}_{[j]}$; $j = 1,2,\dots,n_{ar}$, and ${}^{t\Theta}_{s}\dot{\boldsymbol{J}}_{[j]}$; $j = 1,2,\dots,n_{{}^{t\Theta}_{s}J}$, the entropy inequality will hold if their coefficients are zero. Hence, the following hold.

$$\rho_0\left(\eta + \frac{\partial \Phi}{\partial \theta}\right) = 0 \implies \eta = -\frac{\partial \Phi}{\partial \theta} \tag{3.163}$$

$$\begin{aligned}\frac{\partial \Phi}{\partial g_i} = 0 &\implies \Phi \ne \Phi(\boldsymbol{g}) \\ \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}_{[j]}} = 0;\ j = 1,2,\dots,n_\varepsilon &\implies \Phi \ne \Phi(\boldsymbol{\varepsilon}_{[j]};\ j = 1,2,\dots,n_\varepsilon) \\ \frac{\partial \Phi}{\partial_a \boldsymbol{r}_{[j]}} = 0;\ j = 1,2,\dots,n_{ar} &\implies \Phi \ne \Phi({}_a\boldsymbol{r}_{[j]};\ j = 1,2,\dots,n_{ar}) \\ \frac{\partial \Phi}{\partial {}^{t\Theta}_{s}\boldsymbol{J}_{[j]}} = 0;\ j = 1,2,\dots,n_{{}^{t\Theta}_{s}J} &\implies \Phi \ne \Phi({}^{t\Theta}_{s}\boldsymbol{J}_{[j]};\ j = 1,2,\dots,n_{{}^{t\Theta}_{s}J})\end{aligned} \tag{3.164}$$

Condition (3.163) implies that η is not a constitutive variable. Using (3.164), the entropy inequality reduces to

$$\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki}\right) \dot{\varepsilon}_{ki} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}_a r_{ki})} - {}_a\sigma_{ki}\right) {}_a\dot{r}_{ki} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}^{t}_{s}\!\overset{\Theta}{}\!J_{ki})} - m_{ki}\right) {}^{t}_{s}\!\overset{\Theta}{}\!\dot{J}_{ki} \\ + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \quad (3.165)$$

and the argument tensors of Φ are modified as well.

$$\Phi = \Phi(\boldsymbol{\varepsilon}, {}_a\boldsymbol{r}, {}^{t}_{s}\!\overset{\Theta}{}\!\boldsymbol{J}, \theta) \quad (3.166)$$

The argument tensors of the remaining constitutive variables remain the same as in (3.160).

The entropy inequality in the form given by (3.165) is essential. For example, for arbitrary but admissible choices of $\dot{\boldsymbol{\varepsilon}}$, ${}_a\dot{\boldsymbol{r}}$, and ${}^{t}_{s}\!\overset{\Theta}{}\!\dot{\boldsymbol{J}}$, if it is assumed that their coefficients in (3.165) are zero, then

$$\begin{aligned} {}_s\boldsymbol{\sigma} &= \rho_0 \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}} \implies {}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \\ {}_a\boldsymbol{\sigma} &= \rho_0 \frac{\partial \Phi}{\partial {}_a\boldsymbol{r}} \implies {}_a\boldsymbol{\sigma} = {}_a\boldsymbol{\sigma}({}_a\boldsymbol{r}, \theta) \\ \boldsymbol{m} &= \rho_0 \frac{\partial \Phi}{\partial {}^{t}_{s}\!\overset{\Theta}{}\!\boldsymbol{J}} \implies \boldsymbol{m} = \boldsymbol{m}({}^{t}_{s}\!\overset{\Theta}{}\!\boldsymbol{J}, \theta) \end{aligned} \quad (3.167)$$

Note that (3.167) are invalid based on (3.160), hence in the entropy inequality the following must hold (leaving the first term as is).

$$\frac{\partial \Phi}{\partial {}_a\boldsymbol{r}} = 0\,; \quad \frac{\partial \Phi}{\partial {}^{t}_{s}\!\overset{\Theta}{}\!\boldsymbol{J}} = 0 \quad (3.168)$$

Using (3.168) in (3.165), the entropy inequality reduces to

$$\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki}\right) \dot{\varepsilon}_{ki} - {}_a\sigma_{ki}({}_a\dot{r}_{ki}) - m_{ki}({}^{t}_{s}\!\overset{\Theta}{}\!\dot{J}_{ki}) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \quad (3.169)$$

and the argument tensors of Φ are modified.

$$\Phi = \Phi(\boldsymbol{\varepsilon}, \theta) \tag{3.170}$$

In order to proceed further, consider the decomposition of the symmetric Cauchy stress tensor ${}_s\boldsymbol{\sigma}$ into equilibrium ${}_e({}_s\boldsymbol{\sigma})$ and deviatoric ${}_d({}_s\boldsymbol{\sigma})$ stress tensors.

$${}_s\boldsymbol{\sigma} = {}_e({}_s\boldsymbol{\sigma}) + {}_d({}_s\boldsymbol{\sigma}) \tag{3.171}$$

Substituting (3.171) in (3.169)

$$\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_e({}_s\sigma)_{ki}\right) \dot{\varepsilon}_{ki} - {}_d({}_s\sigma)_{ki}\dot{\varepsilon}_{ki} - {}_a\sigma_{ki}({}_a\dot{r}_{ki}) - m_{ki}({}^{t}_{s}\dot{\overset{\Theta}{J}}_{ki}) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \tag{3.172}$$

For small deformation, small strain the matter is incompressible ($|J| = 1$), hence

$$\frac{\partial \Phi}{\partial \varepsilon_{ki}} = \frac{\partial \Phi}{\partial |J|} \frac{\partial |J|}{\partial \varepsilon_{ki}} = 0 \qquad \text{as} \qquad \frac{\partial \Phi}{\partial |J|} = 0 \tag{3.173}$$

Thus, the first term in (3.172) cannot be used to derive the constitutive theory for ${}_e({}_s\boldsymbol{\sigma})$. Note that ${}_e({}_s\boldsymbol{\sigma})$ in (3.172) is in fact only valid for compressible matter if the coefficient of $\dot{\varepsilon}_{kl}$ is set to zero. The incompressibility condition must be introduced in (3.172).

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = \text{tr}[\bar{D}] = \text{tr}[\bar{L}] = \text{tr}(\dot{\boldsymbol{J}} \boldsymbol{J}^{-1}) = \dot{J}_{kl}(J^{-1})_{lk} = \dot{J}_{kl}\delta_{lk} = 0 \tag{3.174}$$

Also

$$\text{tr}([\bar{L}]^T) = \text{tr}\big((\boldsymbol{J}^{-1})^T \dot{\boldsymbol{J}}^T\big) = (J^{-1})_{lk}\dot{J}_{lk} = \dot{J}_{lk}\delta_{lk} = 0 \tag{3.175}$$

Since

$$\text{tr}[\bar{L}] = \text{tr}([\bar{L}]^T) \tag{3.176}$$

it follows that

$$\frac{1}{2}\big(\text{tr}[\bar{L}] + \text{tr}([\bar{L}]^T)\big) = \frac{1}{2}(\dot{J}_{kl}\delta_{lk} + \dot{J}_{lk}\delta_{lk}) = \dot{\varepsilon}_{kl}\delta_{kl} = 0 \tag{3.177}$$

Let $p(\theta)$ be an arbitrary Lagrange multiplier. Then the incompressibility condition based on (3.177) becomes

$$p(\theta)\dot{\varepsilon}_{ki}\delta_{ki} = 0 \tag{3.178}$$

Adding (3.178) to the left side of (3.172) and using $\frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}} = 0$,

$$\big(p(\theta)\delta_{ki} - {}_e({}_s\sigma)_{ki}\big)\dot{\varepsilon}_{ki} - {}_d({}_s\sigma)_{ki}\dot{\varepsilon}_{ki} - {}_a\sigma_{ki}({}_a\dot{r}_{ki}) - m_{ki}({}^{t}_{s}\dot{\overset{\Theta}{J}}_{ki}) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \tag{3.179}$$

For arbitrary but admissible $\dot{\boldsymbol{\varepsilon}}$, (3.179) holds if

$$p(\theta)\delta_{ki} - {}_e({}_s\sigma)_{ki} = 0 \tag{3.180}$$

or

$${}_e({}_s\boldsymbol{\sigma}) = p(\theta)\boldsymbol{I} \tag{3.181}$$

This is the constitutive theory for the equilibrium stress for an incompressible solid. $p(\theta)$ is called the mechanical pressure. If compressive pressure is assumed to be positive, then $p(\theta)$ in (3.181) can be replaced by $-p(\theta)$. The entropy inequality now reduces to

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_d({}_s\sigma)_{ki}\dot{\varepsilon}_{ki} - {}_a\sigma_{ki}({}_a\dot{r}_{ki}) - m_{ki}({}^{t}_{s}\dot{\overset{\Theta}{J}}_{ki}) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0 \tag{3.182}$$

The corresponding energy equation becomes

$$\rho_0\frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \text{tr}\big([{}_d({}_s\sigma)][\dot{\varepsilon}]\big) - \text{tr}\big([{}_a\sigma][{}_a\dot{r}]\big) - \text{tr}\big([m][{}^{t}_{s}\dot{\overset{\Theta}{J}}]\big) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) = 0 \tag{3.183}$$

The entropy inequality (3.182) is satisfied if

$$
\begin{aligned}
{}^{s\sigma}\Psi &= \operatorname{tr}\big([{}_d({}_s\sigma)][\dot{\varepsilon}]\big) \geq 0 \\
{}^{a\sigma}\Psi &= \operatorname{tr}\big([{}_a\sigma][{}_a\dot{r}]\big) \geq 0 \\
{}^{sm}\Psi &= \operatorname{tr}\big([m][{}^{t}_{s}\dot{\Theta}\!J]\big) \geq 0
\end{aligned}
\tag{3.184}
$$

$$
\frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} \leq 0 \tag{3.185}
$$

and

$$
{}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) = 0 \tag{3.186}
$$

The conditions (3.184) imply that the rate of work due to ${}_d({}_s\boldsymbol{\sigma})$, ${}_a\boldsymbol{\sigma}$, and $\boldsymbol{m}$ must be positive. Inequality (3.185) can be used to derive the constitutive theory for $\boldsymbol{q}$. Equation (3.186) serves as a constraint (compatibility condition) on ${}_i\boldsymbol{\Theta}$ and the antisymmetric components of the Cauchy stress tensor $\boldsymbol{\sigma}$. The rate of work or the work conjugate pairs in (3.182) are in conformity with (3.160). The argument tensors of the constitutive variables in (3.160) can now be revised.

$$
\begin{aligned}
{}_d({}_s\boldsymbol{\sigma}) &= {}_d({}_s\boldsymbol{\sigma})(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n_{\varepsilon},\ \theta) \\
{}_a\boldsymbol{\sigma} &= {}_a\boldsymbol{\sigma}({}_a\boldsymbol{r}, {}_a\boldsymbol{r}_{[j]};\ j = 1, 2, \dots, n_{({}_a r)},\ \theta) \\
\boldsymbol{m} &= \boldsymbol{m}({}^{t}_{s}\Theta\boldsymbol{J}, {}^{t}_{s}\Theta\boldsymbol{J}_{[k]};\ k = 1, 2, \dots, n_{({}^{t}_{s}\Theta J)},\ \theta) \\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{g}, \theta) \\
{}_e({}_s\boldsymbol{\sigma}) &= p(\theta)\boldsymbol{I} \\
\Phi &= \Phi(\theta)
\end{aligned}
\tag{3.187}
$$

and

$$
{}_s\boldsymbol{\sigma} = {}_e({}_s\boldsymbol{\sigma}) + {}_d({}_s\boldsymbol{\sigma}) \tag{3.188}
$$

3.4.3 Constitutive Theory for ${}_d({}_s\boldsymbol{\sigma})$

Consider the argument tensors of ${}_d({}_s\boldsymbol{\sigma})$ in (3.187). Let ${}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i$; $i = 1, 2, \dots, N_{s\sigma}$ be the combined generators of the argument tensors of ${}_d({}_s\boldsymbol{\sigma})$ that are symmetric tensors of rank two and let ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{s\sigma}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration based on the representation theorem [80–99].

$$
{}_d({}_s\boldsymbol{\sigma}) = {}^{s\sigma}\underset{\sim}{\alpha}^0 \boldsymbol{I} + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underset{\sim}{\alpha}^i ({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i) \tag{3.189}
$$

in which

$$
{}^{s\sigma}\underset{\sim}{\alpha}^i = {}^{s\sigma}\underset{\sim}{\alpha}^i({}^{s\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_{s\sigma},\ \theta); \quad i = 1, 2, \dots, N_{s\sigma} \tag{3.190}
$$

3.4.3.1 Material Coefficients in the Constitutive Theory for ${}_d({}_s\boldsymbol{\sigma})$

To determine the material coefficients in (3.189), expand each ${}^{s\sigma}\underset{\sim}{\alpha}^i$ in a Taylor series in ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{s\sigma}$ and θ about a known configuration $\underline{\Omega}$, retain only up to linear terms in ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{s\sigma}$ and θ, and then substitute these ${}^{s\sigma}\underset{\sim}{\alpha}^i$ in (3.189). Collect coefficients of those terms that are defined in the current configuration to obtain

$$
\begin{aligned}
{}_d({}_s\boldsymbol{\sigma}) = {}^0_s\sigma\big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{j=1}^{M_{s\sigma}} {}^{s\sigma}\underline{a}_j \big({}^{s\sigma}\underset{\sim}{I}^j\big) \boldsymbol{I} - {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}} (\theta - \theta_{\underline{\Omega}}) \boldsymbol{I} \\
+ \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underline{b}_i \big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) + \sum_{i=1}^{N_{s\sigma}} \sum_{j=1}^{M_{s\sigma}} {}^{s\sigma}\underset{\sim}{c}_{ij} \big({}^{s\sigma}\underset{\sim}{I}^j\big) \big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underset{\sim}{d}_i (\theta - \theta_{\underline{\Omega}}) \big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big)
\end{aligned} \tag{3.191}
$$

in which

$$
\begin{aligned}
{}^0_s\sigma\big|_{\underline{\Omega}} &= {}^{s\sigma}\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} - \sum_{j=1}^{M_{s\sigma}} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}} \big({}^{s\sigma}\underset{\sim}{I}^j\big)_{\underline{\Omega}} \\
{}^{s\sigma}\underline{a}_j &= \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}}; \quad j = 1, 2, \dots, M_{s\sigma} \\
{}^{s\sigma}\underline{b}_i &= {}^{s\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} - \sum_{j=1}^{M_{s\sigma}} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}} \big({}^{s\sigma}\underset{\sim}{I}^j\big)_{\underline{\Omega}}; \quad i = 1, 2, \dots, N_{s\sigma}
\end{aligned} \tag{3.192}
$$

$$
\begin{aligned}
{}^{s\sigma}\underset{\sim}{c}_{ij} &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} ; \quad \begin{array}{l} i = 1, 2, \dots, N_{s\sigma} \\ j = 1, 2, \dots, M_{s\sigma} \end{array} \\
{}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}} &= -\left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial\theta}\right|_{\underline{\Omega}} \\
{}^{s\sigma}\underset{\sim}{d}_i &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial\theta}\right|_{\underline{\Omega}} ; \quad i = 1, 2, \dots, N_{s\sigma}
\end{aligned}
\tag{3.192}
$$

${}^{s\sigma}\underline{a}_j$, ${}^{s\sigma}\underline{b}_i$, ${}^{s\sigma}\underset{\sim}{c}_{ij}$, ${}^{s\sigma}\underset{\sim}{d}_i$, and ${}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}$ are the material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_{s\sigma} + N_{s\sigma} + M_{s\sigma}N_{s\sigma} + N_{s\sigma} + 1)$ material coefficients. The material coefficients defined in (3.191) are functions of $({}^{s\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\theta_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{s\sigma}\underset{\sim}{\alpha}^i;\ i = 0, 1, \dots, N_{s\sigma}$. The complete theory contains too many material coefficients to be determined, experimentally or otherwise. Simplified forms are considered in a later section.

3.4.4 Constitutive Theory for $\boldsymbol{m}$

Consider the argument tensors of $\boldsymbol{m}$ in (3.187). Let ${}^m\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \dots, N_m$ be the combined generators of the argument tensors of $\boldsymbol{m}$ that are symmetric tensors of rank two and let ${}^m\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_m$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration.

$$
\boldsymbol{m} = {}^m\underset{\sim}{\alpha}^0 \boldsymbol{I} + \sum_{i=1}^{N_m} {}^m\underset{\sim}{\alpha}^i ({}^m\underset{\sim}{\boldsymbol{G}}^i)
\tag{3.193}
$$

in which

$$
{}^m\underset{\sim}{\alpha}^i = {}^m\underset{\sim}{\alpha}^i({}^m\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_m,\ \theta); \quad i = 1, 2, \dots, N_m
\tag{3.194}
$$

3.4.4.1 Material Coefficients in the Constitutive Theory for $\boldsymbol{m}$

To determine the material coefficients in (3.193), expand each ${}^m\underset{\sim}{\alpha}^i$ in a Taylor series in ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_m$ and θ about a known configuration $\underline{\Omega}$, retain only up to linear terms in ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_m$ and θ, and then substitute these ${}^m\underset{\sim}{\alpha}^i$ in (3.193). After collecting coefficients of those terms that are defined in the current configuration, the following is obtained.

$$
\begin{aligned}
\boldsymbol{m} = {}^0m|_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M_m} {}^m\underline{a}_j \left({}^m\underset{\sim}{I}^j\right)\boldsymbol{I} - {}^m\underset{\sim}{\alpha}_{\text{tm}}\left(\theta - \theta_{\underline{\Omega}}\right)\boldsymbol{I} \\
+ \sum_{i=1}^{N_m} {}^m\underset{\sim}{b}_i \left({}^m\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_m}\sum_{j=1}^{M_m} {}^m\underset{\sim}{c}_{ij}\left({}^m\underset{\sim}{I}^j\right)\left({}^m\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_m} {}^m\underset{\sim}{d}_i\left(\theta - \theta_{\underline{\Omega}}\right)\left({}^m\underset{\sim}{\boldsymbol{G}}^i\right)
\end{aligned}
\tag{3.195}
$$

${}^m\underline{a}_j$, ${}^m\underset{\sim}{b}_i$, ${}^m\underset{\sim}{c}_{ij}$, ${}^m\underset{\sim}{d}_i$, and ${}^m\underset{\sim}{\alpha}_{\text{tm}}$ are material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_m + N_m + M_m N_m + N_m + 1)$ material coefficients. The material coefficients are functions of $({}^m\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^m\underset{\sim}{\alpha}^i;\ i = 0, 1, \dots, N_m$. Explicit forms of the material coefficients can be obtained from (3.192) by simply replacing the back superscript ${}_s\sigma$ with m and ${}^0_s\sigma|_{\underline{\Omega}}$ by ${}^0m|_{\underline{\Omega}}$. This complete theory contains too many material coefficients to be determined, experimentally or otherwise. Simplified forms are considered in a later section.

3.4.5 Constitutive Theory for ${}_a\boldsymbol{\sigma}$

Consider the argument tensors of ${}_a\boldsymbol{\sigma}$ in (3.187). Let ${}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \dots, N_{a\sigma}$ be the combined generators of the argument tensors of ${}_a\boldsymbol{\sigma}$ that are antisymmetric tensors of rank two and let ${}^{a\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_{a\sigma}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration based on the representation theorem [80–99] (note that the identity tensor is not an antisymmetric tensor of rank two, hence is not a generator of ${}_a\boldsymbol{\sigma}$).

$$
{}_a\boldsymbol{\sigma} = \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underset{\sim}{\alpha}^i \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \tag{3.196}
$$

in which

$$
{}^{a\sigma}\underset{\sim}{\alpha}^i = {}^{a\sigma}\underset{\sim}{\alpha}^i\left({}^{a\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_{a\sigma},\ \theta\right); \quad i = 1, 2, \dots, N_{a\sigma} \tag{3.197}
$$

3.4.5.1 Material Coefficients in the Constitutive Theory for ${}_a\boldsymbol{\sigma}$

To determine the material coefficients in (3.196), expand each ${}^{a\sigma}\underset{\sim}{\alpha}^i$ in a Taylor series in ${}^{a\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M_{a\sigma}$ and θ about a known configuration $\underline{\Omega}$, retain only up to linear terms in ${}^{a\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M_{a\sigma}$ and θ, and then substitute these ${}^{a\sigma}\underset{\sim}{\alpha}^i$ in (3.196). After collecting coefficients of those terms that are defined in the current configuration, the following is obtained.

$$ {}_a\boldsymbol{\sigma} = \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underset{\sim}{\underline{b}}_i \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_{a\sigma}} \sum_{j=1}^{M_{a\sigma}} {}^{a\sigma}\underset{\sim}{\underline{c}}_{ij} \left({}^{a\sigma}\underset{\sim}{I}^j\right) \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underset{\sim}{\underline{d}}_i \left(\theta - \theta_{\underline{\Omega}}\right) \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \tag{3.198} $$

${}^{a\sigma}\underset{\sim}{\underline{b}}_i$, ${}^{a\sigma}\underset{\sim}{\underline{c}}_{ij}$, and ${}^{a\sigma}\underset{\sim}{\underline{d}}_i$ are material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(N_{a\sigma} + M_{a\sigma}N_{a\sigma} + N_{a\sigma})$ material coefficients. The material coefficients are functions of $({}^{a\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{a\sigma}\underset{\sim}{\alpha}^i$; $i = 0, 1, \ldots, N_{a\sigma}$. Explicit forms of the material coefficients can be obtained from (3.192) by simply replacing the back superscript s with a. This complete theory contains too many material coefficients to be determined, experimentally or otherwise. Simplified forms are considered in the following section.

3.4.6 Linear Constitutive Theories for ${}_d({}_s\boldsymbol{\sigma})$, $\boldsymbol{m}$, and ${}_a\boldsymbol{\sigma}$

The constitutive theories for ${}_d({}_s\boldsymbol{\sigma})$, $\boldsymbol{m}$, and ${}_a\boldsymbol{\sigma}$ derived in Sections 3.4.3, 3.4.4, and 3.4.5 are complete as they are based on integrity, but require determination of too many material coefficients. In this section simplified forms of the theories are considered that require the minimum number of material coefficients, but at the expense of only modeling simplified and compromised physics.

3.4.6.1 Simplified Constitutive Theory for ${}_d({}_s\boldsymbol{\sigma})$

The constitutive theory (3.191) obviously requires determination of too many material coefficients. If n_ε is limited to 1, then

$$ {}_d({}_s\boldsymbol{\sigma}) = {}_d({}_s\boldsymbol{\sigma})(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{[1]}, \theta) = {}_d({}_s\boldsymbol{\sigma})(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, \theta) \tag{3.199} $$

If the constitutive theory is further limited to be linear in $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ and product terms containing $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ are neglected, then (3.191) simplifies to (after neglecting initial stress and temperature terms without loss of generality)

$$_{d}({}_{s}\boldsymbol{\sigma}) = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu_1\dot{\boldsymbol{\varepsilon}} + \lambda_1 \mathrm{tr}(\dot{\boldsymbol{\varepsilon}})\boldsymbol{I} \tag{3.200}$$

This constitutive theory requires only four material coefficients. μ and λ are Lamé's constants for the strain terms. μ_1 and λ_1 are the corresponding material coefficients related to strain rates.

3.4.6.2 Simplified Constitutive Theory for $\boldsymbol{m}$

A much simplified constitutive theory for $\boldsymbol{m}$ can be obtained if $n_{{}^{t\Theta}_{s}J}$ is limited to 1.

$$\boldsymbol{m} = \boldsymbol{m}({}^{t\Theta}_{s}\boldsymbol{J}, {}^{t\Theta}_{s}\dot{\boldsymbol{J}}, \theta) \tag{3.201}$$

If the constitutive theory is further limited to be linear in ${}^{t\Theta}_{s}\boldsymbol{J}$ and ${}^{t\Theta}_{s}\dot{\boldsymbol{J}}$ and the product terms containing ${}^{t\Theta}_{s}\boldsymbol{J}$ and ${}^{t\Theta}_{s}\dot{\boldsymbol{J}}$ are neglected, then (3.195) simplifies to (after neglecting initial moment and temperature terms without loss of generality)

$$_{s}\boldsymbol{m} = \alpha({}^{t\Theta}_{s}\boldsymbol{J}) + B\mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}\big)\boldsymbol{I} + \alpha_1({}^{t\Theta}_{s}\dot{\boldsymbol{J}}) + B_1\mathrm{tr}\big({}^{t\Theta}_{s}\dot{\boldsymbol{J}}\big)\boldsymbol{I} \tag{3.202}$$

The material coefficients α, B, α_1, and B_1 can be functions of the invariants and the temperature θ.

3.4.6.3 Simplified Constitutive Theory for ${}_{a}\boldsymbol{\sigma}$

The constitutive theory (3.198) based on integrity can be simplified by choosing $n_{{}_{a}r} = 1$.

$$_{a}\boldsymbol{\sigma} = {}_{a}\boldsymbol{\sigma}({}_{a}\boldsymbol{r}, {}_{a}\dot{\boldsymbol{r}}, \theta) \tag{3.203}$$

In this case ${}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^1 = {}_a\boldsymbol{r}$, ${}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^2 = {}_a\dot{\boldsymbol{r}}$, and ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^3 = [{}_ar][{}_a\dot{r}] - [{}_a\dot{r}][{}_ar]$ are the only combined generators and ${}^{a\sigma}\underset{\sim}{I}^1 = \text{tr}\big(({}_a\boldsymbol{r})^2\big)$, ${}^{a\sigma}\underset{\sim}{I}^2 = \text{tr}\big(({}_a\dot{\boldsymbol{r}})^2\big)$, and ${}^{a\sigma}\underset{\sim}{I}^3 = \text{tr}\big([{}_ar][{}_a\dot{r}]\big)$ are the only invariants, giving rise to a constitutive theory with 19 material coefficients ($N_{{}_a\sigma} = 3$, $M_{{}_a\sigma} = 3$). A linear constitutive theory for ${}_a\boldsymbol{\sigma}$ (neglecting initial stress and θ terms and excluding products of ${}_a\boldsymbol{r}$ and ${}_a\dot{\boldsymbol{r}}$) will be

$$ {}_a\boldsymbol{\sigma} = \kappa({}_a\boldsymbol{r}) + \kappa_1({}_a\dot{\boldsymbol{r}}) \tag{3.204} $$

The material coefficients κ and κ_1 can be functions of the invariants and the temperature θ.

Remarks.

(1) These constitutive theories require ten material coefficients. μ and λ are Lamé's constants for the strain terms, and μ_1 and λ_1 are corresponding material coefficients related to strain rates, all present in classical theories. α and α_1 are required in the presence of rotations and rotation rates (internal or Cosserat). κ, κ_1, B, and B_1 are only needed when Cosserat rotations and rotation rates are present.

(2) In the case of purely internal polar non-classical continuum theories, the constitutive theory for ${}_a\boldsymbol{\sigma}$ is not needed as these are balanced by gradients of the moment tensor in the balance of angular momenta, hence in this case κ and κ_1 are zero. ${}^{t}{}^{\Theta}_{s}\boldsymbol{J}$ and its rates become the symmetric part of the internal rotation gradient tensor and its rates, and the coefficients B and B_1 are no longer required as the trace of the gradient of internal rotations is zero.

(3) If the balance of moments of moments balance law is neglected, then the moment tensor is not symmetric. Following the procedure presented, the constitutive theories in such a case would be identical to when the balance of moments of moments is considered, with the addition of the constitutive theory for the antisymmetric part of the moment tensor.

$$ {}_d({}_s\boldsymbol{\sigma}) = 2\mu\boldsymbol{\varepsilon} + \lambda\text{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu_1\dot{\boldsymbol{\varepsilon}} + \lambda_1\text{tr}(\dot{\boldsymbol{\varepsilon}})\boldsymbol{I} \tag{3.205} $$

$$ {}_s\boldsymbol{m} = \alpha({}^{t}{}_{s}^{\Theta}\boldsymbol{J}) + B\text{tr}\big({}^{t}{}_{s}^{\Theta}\boldsymbol{J}\big)\boldsymbol{I} + \alpha_1({}^{t}{}_{s}^{\Theta}\dot{\boldsymbol{J}}) + B_1\text{tr}\big({}^{t}{}_{s}^{\Theta}\dot{\boldsymbol{J}}\big)\boldsymbol{I} \tag{3.206} $$

$$ {}_a\boldsymbol{\sigma} = \kappa({}_a\boldsymbol{r}) + \kappa_1({}_a\dot{\boldsymbol{r}}) \tag{3.207} $$

$$ {}_a\boldsymbol{m} = \beta({}^{t}{}_{a}^{\Theta}\boldsymbol{J}) + \beta_1({}^{t}{}_{a}^{\Theta}\dot{\boldsymbol{J}}) \tag{3.208} $$

3.5 Constitutive Theories for Thermoviscoelastic Solids with Memory

For thermoviscoelastic solids the mechanical deformation is only partially reversible, i.e., some of the rate of work gets stored as rate of strain energy density, but there is entropy production and energy dissipation due to the mechanical deformation process. Due to the presence of dissipation, constitutive theories cannot be derived using the Helmholtz free energy density as a function of the invariants of the argument tensors. Because the material also has memory (or rheology), its physical response, i.e., stresses and moments, exhibits relaxation phenomena.

Once the rate of work conjugate pairs are established from the energy equation or entropy inequality, the constitutive theories can be derived using these in conjunction with the theory of generators and invariants. Details are considered in the following. Note that the dependent variables and their arguments are pairs of symmetric or antisymmetric tensors as required by the representation theorem [80–99].

3.5.1 Dependent Variables in the Constitutive Theories

It is straightforward to conclude from the conservation and balance laws that Φ, η, ${}_s\boldsymbol{\sigma}$, ${}_a\boldsymbol{\sigma}$, $\boldsymbol{m}$, and $\boldsymbol{q}$ are possible dependent variables in the constitutive theories. From (3.82), note that ${}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$, ${}_a\boldsymbol{\sigma} = {}_a\boldsymbol{\sigma}({}_a\boldsymbol{r})$, $\boldsymbol{m} = \boldsymbol{m}({}^{t}{}_{s}^{\Theta}\boldsymbol{J})$, and $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g})$ are perfectly admissible due to the conjugate pairs. Since the deforming matter has elasticity as well as dissipation, $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ (or $\boldsymbol{\varepsilon}_{[1]}$) both need to be argument tensors of ${}_s\boldsymbol{\sigma}$. If it is assumed that higher order time derivatives of $\boldsymbol{\varepsilon}$ also contribute to dissipation, then $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}_{[k]}$; $k = 1, 2, \ldots, n_\varepsilon$ need to be argument tensors of ${}_s\boldsymbol{\sigma}$.

Furthermore, the presence of memory (or rheology) in the material requires that the stress be a differential equation in time, that is, ${}_s\boldsymbol{\sigma}^{[1]} = {}_s\boldsymbol{\sigma}^{[1]}({}_s\boldsymbol{\sigma}^{[0]})$. Again, if it is assumed that higher order time derivatives of ${}_s\boldsymbol{\sigma}$ also contribute to the memory mechanism, then ${}_s\boldsymbol{\sigma}^{[m_{s\sigma}]}$ must be a constitutive variable with ${}_s\boldsymbol{\sigma}^{[i]};\, i = 0, 1, \dots, (m_{{}_s\sigma} - 1)$ as its argument tensors.

Further assume that the dissipation and memory are also due to $\boldsymbol{m}$, ${}_a\boldsymbol{\sigma}$, and the time derivatives of the corresponding conjugate tensors. In addition, temperature θ can be considered as an argument for all of the dependent variables due to thermoviscoelastic behavior.

$$
\begin{aligned}
&{}_s\boldsymbol{\sigma}^{[m_{s\sigma}]} = {}_s\boldsymbol{\sigma}^{[m_{s\sigma}]}\big(\boldsymbol{\varepsilon}_{[i]};\, i = 0,1,\dots,n_{\varepsilon},\ {}_s\boldsymbol{\sigma}^{[j]};\, j = 0,1,\dots,(m_{{}_s\sigma}-1),\ \theta\big) \\
&{}_a\boldsymbol{\sigma}^{[m_{a\sigma}]} = {}_a\boldsymbol{\sigma}^{[m_{a\sigma}]}\big({}_a\boldsymbol{r}_{[i]};\, i = 0,1,\dots,n_{{}_a r},\ {}_a\boldsymbol{\sigma}^{[j]};\, j = 0,1,\dots,(m_{{}_a\sigma}-1),\ \theta\big) \\
&\boldsymbol{m}^{[m_m]} = \boldsymbol{m}^{[m_m]}\big({}^{t\Theta}_{\ s}\boldsymbol{J}_{[i]};\, i = 0,1,\dots,n_{{}^{t\Theta}_{s}J},\ \boldsymbol{m}^{[j]};\, j = 0,1,\dots,(m_m-1),\ \theta\big) \\
&\Phi = \Phi\big(\boldsymbol{\varepsilon}_{[i]};\, i = 0,1,\dots,n_{\varepsilon},\ {}_a\boldsymbol{r}_{[j]};\, j = 0,1,\dots,n_{{}_a r},\ {}^{t\Theta}_{\ s}\boldsymbol{J}_{[k]};\, k = 0,1,\dots,n_{{}^{t\Theta}_{s}J}, \\
&\qquad {}_s\boldsymbol{\sigma}^{[i]};\, i = 0,1,\dots,(m_{{}_s\sigma}-1),\ {}_a\boldsymbol{\sigma}^{[j]};\, j = 0,1,\dots,(m_{{}_a\sigma}-1), \\
&\qquad \boldsymbol{m}^{[k]};\, k = 0,1,\dots,(m_m-1),\ \theta\big) \\
&\eta = \eta\big(\boldsymbol{\varepsilon}_{[i]};\, i = 0,1,\dots,n_{\varepsilon},\ {}_a\boldsymbol{r}_{[j]};\, j = 0,1,\dots,n_{{}_a r},\ {}^{t\Theta}_{\ s}\boldsymbol{J}_{[k]};\, k = 0,1,\dots,n_{{}^{t\Theta}_{s}J}, \\
&\qquad {}_s\boldsymbol{\sigma}^{[i]};\, i = 0,1,\dots,(m_{{}_s\sigma}-1),\ {}_a\boldsymbol{\sigma}^{[j]};\, j = 0,1,\dots,(m_{{}_a\sigma}-1), \\
&\qquad \boldsymbol{m}^{[k]};\, k = 0,1,\dots,(m_m-1),\ \theta\big) \\
&\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta)
\end{aligned}
\tag{3.209}
$$

Note that the argument tensors of Φ and η are the totality of all of the argument tensors of all of the constitutive variables. At this stage (3.209) is the most general choice. During the derivation of constitutive theories, some arguments of some constitutive variables may be ruled out due to some other considerations.

3.5.2 Entropy Inequality: Further Considerations

Using Φ in (3.209) one can obtain the material derivative of Φ needed in (3.82).

$$
\begin{aligned}
\frac{D\Phi}{Dt} = \dot{\Phi} = & \sum_{j=0}^{n_{\varepsilon}} \frac{\partial \Phi}{\partial (\varepsilon_{[j]})_{ik}} (\dot{\varepsilon}_{[j]})_{ik} + \sum_{j=0}^{n_{ar}} \frac{\partial \Phi}{\partial ({}_{a}r_{[j]})_{ik}} ({}_{a}\dot{r}_{[j]})_{ik} + \sum_{j=0}^{n_{{}^{t\Theta}_{s}J}} \frac{\partial \Phi}{\partial ({}^{t\Theta}_{s}J_{[j]})_{ik}} ({}^{t\Theta}_{s}\dot{J}_{[j]})_{ik} \\
& + \sum_{j=0}^{(m_{s\sigma}-1)} \frac{\partial \Phi}{\partial ({}_{s}\sigma^{[j]})_{ik}} ({}_{s}\dot{\sigma}^{[j]})_{ik} + \sum_{j=0}^{(m_{a\sigma}-1)} \frac{\partial \Phi}{\partial ({}_{a}\sigma^{[j]})_{ik}} ({}_{a}\dot{\sigma}^{[j]})_{ik} \\
& + \sum_{j=0}^{(m_{m}-1)} \frac{\partial \Phi}{\partial (m^{[j]})_{ik}} (\dot{m}^{[j]})_{ik} + \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{\partial \Phi}{\partial \theta} \dot{\theta}
\end{aligned}
\tag{3.210}
$$

Substituting (3.210) into the entropy inequality (3.82) and collecting terms

$$
\begin{aligned}
& \left(\rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ik}} - ({}_{s}\sigma^{[0]})_{ik} \right) (\dot{\varepsilon}_{[0]})_{ik} + \sum_{j=1}^{n_{\varepsilon}} \rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[j]})_{ik}} (\dot{\varepsilon}_{[j]})_{ik} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}_{a}r_{[j]})_{ik}} - ({}_{a}\sigma^{[0]})_{ik} \right) ({}_{a}\dot{r}_{[0]})_{ik} \\
& + \sum_{j=1}^{n_{ar}} \rho_0 \frac{\partial \Phi}{\partial ({}_{a}r_{[j]})_{ik}} ({}_{a}\dot{r}_{[j]})_{ik} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}^{t\Theta}_{s}J_{[j]})_{ik}} - (m^{[0]})_{ik} \right) ({}^{t\Theta}_{s}\dot{J}_{[0]})_{ik} + \sum_{j=1}^{n_{{}^{t\Theta}_{s}J}} \rho_0 \frac{\partial \Phi}{\partial ({}^{t\Theta}_{s}J_{[j]})_{ik}} ({}^{t\Theta}_{s}\dot{J}_{[j]})_{ik} \\
& + \sum_{j=0}^{(m_{s\sigma}-1)} \rho_0 \frac{\partial \Phi}{\partial ({}_{s}\sigma^{[j]})_{ik}} ({}_{s}\dot{\sigma}^{[j]})_{ik} + \sum_{j=0}^{(m_{a\sigma}-1)} \rho_0 \frac{\partial \Phi}{\partial ({}_{a}\sigma^{[j]})_{ik}} ({}_{a}\dot{\sigma}^{[j]})_{ik} + \sum_{j=0}^{(m_{m}-1)} \rho_0 \frac{\partial \Phi}{\partial (m^{[j]})_{ik}} (\dot{m}^{[j]})_{ik} \\
& + \rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) \dot{\theta} + \rho_0 \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_{i}\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{[0]}) \leq 0
\end{aligned}
\tag{3.211}
$$

The entropy inequality (3.211) is satisfied for arbitrary but admissible values of

$$
\begin{gathered}
\dot{\boldsymbol{\varepsilon}}_{[j]};\ j = 1, 2, \ldots, n_{\varepsilon}\ , \qquad {}_{a}\dot{\boldsymbol{r}}_{[j]};\ j = 1, 2, \ldots, n_{ar} \\
{}^{t\Theta}_{s}\dot{\boldsymbol{J}}_{[j]};\ j = 1, 2, \ldots, n_{{}^{t\Theta}_{s}J}\ , \qquad {}_{s}\dot{\boldsymbol{\sigma}}^{[j]};\ j = 0, 1, \ldots, (m_{s\sigma} - 1) \\
{}_{a}\dot{\boldsymbol{\sigma}}^{[j]};\ j = 0, 1, \ldots, (m_{a\sigma} - 1)\ , \qquad \dot{\boldsymbol{m}}^{[j]};\ j = 0, 1, \ldots, (m_{m} - 1) \\
\dot{\theta}\ , \qquad \dot{g}_i
\end{gathered}
\tag{3.212}
$$

if their coefficients are zero. Hence, the following hold.

$$
\rho_0 \left(\eta + \frac{\partial \Phi}{\partial \theta} \right) = 0 \implies \eta = -\frac{\partial \Phi}{\partial \theta}
\tag{3.213}
$$

$$
\begin{aligned}
\frac{\partial \Phi}{\partial g_i} = 0 \implies \Phi \neq \Phi(\boldsymbol{g}) \\
\rho_0 \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}_{[j]}} = 0;\ j = 1,2,\dots,n_\varepsilon \implies \Phi \neq \Phi\big(\boldsymbol{\varepsilon}_{[j]};\ j = 1,2,\dots,n_\varepsilon\big) \\
\rho_0 \frac{\partial \Phi}{\partial {}_a\boldsymbol{r}_{[j]}} = 0;\ j = 1,2,\dots,n_{{}_a r} \implies \Phi \neq \Phi\big({}_a\boldsymbol{r}_{[j]};\ j = 1,2,\dots,n_{{}_a r}\big) \\
\rho_0 \frac{\partial \Phi}{\partial {}^{t\Theta}_{s}\boldsymbol{J}_{[j]}} = 0;\ j = 1,2,\dots,n_{{}^{t\Theta}_{s}J} \implies \Phi \neq \Phi\big({}^{t\Theta}_{s}\boldsymbol{J}_{[j]};\ j = 1,2,\dots,n_{{}^{t\Theta}_{s}J}\big) \\
\rho_0 \frac{\partial \Phi}{\partial {}_s\boldsymbol{\sigma}^{[j]}} = 0;\ j = 0,1,\dots,(m_{{}_s\sigma} - 1) \implies \Phi \neq \Phi\big({}_s\boldsymbol{\sigma}^{[j]};\ j = 0,1,\dots,(m_{{}_s\sigma} - 1)\big) \\
\rho_0 \frac{\partial \Phi}{\partial {}_a\boldsymbol{\sigma}^{[j]}} = 0;\ j = 0,1,\dots,(m_{{}_a\sigma} - 1) \implies \Phi \neq \Phi\big({}_a\boldsymbol{\sigma}^{[j]};\ j = 0,1,\dots,(m_{{}_a\sigma} - 1)\big) \\
\rho_0 \frac{\partial \Phi}{\partial \boldsymbol{m}^{[j]}} = 0;\ j = 0,1,\dots,(m_m - 1) \implies \Phi \neq \Phi\big(\boldsymbol{m}^{[j]};\ j = 0,1,\dots,(m_m - 1)\big)
\end{aligned}
\tag{3.214}
$$

Condition (3.213) implies that η is not a constitutive variable. Using (3.214), the entropy inequality reduces to

$$
\left(\rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ik}} - ({}_s\sigma^{[0]})_{ik}\right)(\dot{\varepsilon}_{[0]})_{ik} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}_a r_{[j]})_{ik}} - ({}_a\sigma^{[0]})_{ik}\right)({}_a\dot{r}_{[0]})_{ik} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}^{t\Theta}_{s}J_{[j]})_{ik}} - (m^{[0]})_{ik}\right)({}^{t\Theta}_{s}\dot{J}_{[0]})_{ik} + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{[0]}) \leq 0 \tag{3.215}
$$

and the argument tensors of Φ are modified as well.

$$
\Phi = \Phi(\boldsymbol{\varepsilon}_{[0]}, {}_a\boldsymbol{r}_{[0]}, {}^{t\Theta}_{s}\boldsymbol{J}_{[0]}, \theta) \tag{3.216}
$$

The argument tensors of the remaining constitutive variables remain the same as in (3.209).

The entropy inequality in the form given by (3.215) is essential. For example, if for arbitrary but admissible choices of $\dot{\boldsymbol{\varepsilon}}_{[0]}$, ${}_a\dot{\boldsymbol{r}}_{[0]}$, and ${}^{t\Theta}_{s}\dot{\boldsymbol{J}}_{[0]}$ it is assumed that their coefficients in (3.215) are zero, then

$$
\begin{aligned}
{}_s\boldsymbol{\sigma}^{[0]} = \rho_0 \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}_{[0]}} \implies {}_s\boldsymbol{\sigma}^{[0]} = {}_s\boldsymbol{\sigma}^{[0]}(\boldsymbol{\varepsilon}_{[0]}, \theta) \\
{}_a\boldsymbol{\sigma}^{[0]} = \rho_0 \frac{\partial \Phi}{\partial {}_a\boldsymbol{r}_{[0]}} \implies {}_a\boldsymbol{\sigma}^{[0]} = {}_a\boldsymbol{\sigma}^{[0]}({}_a\boldsymbol{r}_{[0]}, \theta) \\
\boldsymbol{m}^{[0]} = \rho_0 \frac{\partial \Phi}{\partial {}^{t}_{s}\!\Theta\boldsymbol{J}_{[0]}} \implies \boldsymbol{m}^{[0]} = \boldsymbol{m}^{[0]}({}^{t}_{s}\!\Theta\boldsymbol{J}_{[0]}, \theta)
\end{aligned}
\tag{3.217}
$$

Note that (3.217) are invalid based on (3.209), hence in the entropy inequality the following must hold (leaving the first term as is).

$$
\frac{\partial \Phi}{\partial {}_a\boldsymbol{r}_{[0]}} = 0 \; ; \quad \frac{\partial \Phi}{\partial {}^{t}_{s}\!\Theta\boldsymbol{J}_{[0]}} = 0 \tag{3.218}
$$

Using (3.218) in (3.215), the entropy inequality reduces to

$$
\begin{aligned}
\left(\rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ik}} - ({}_s\sigma^{[0]})_{ik} \right) (\dot{\varepsilon}_{[0]})_{ik} - ({}_a\sigma^{[0]})_{ik} ({}_a\dot{r}_{[0]})_{ik} - (m^{[0]})_{ik} ({}^{t}_{s}\!\Theta\dot{J}_{[0]})_{ik} \\
+ \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{[0]}) \leq 0
\end{aligned}
\tag{3.219}
$$

and the argument tensors of Φ are modified.

$$
\Phi = \Phi(\boldsymbol{\varepsilon}_{[0]}, \theta) \tag{3.220}
$$

In order to proceed further, consider the decomposition of the symmetric Cauchy stress tensor ${}_s\boldsymbol{\sigma}^{[0]}$ into equilibrium ${}_e({}_s\boldsymbol{\sigma})$ and deviatoric ${}_d({}_s\boldsymbol{\sigma})$ stress tensors.

$$
{}_s\boldsymbol{\sigma}^{[0]} = {}_e({}_s\boldsymbol{\sigma}^{[0]}) + {}_d({}_s\boldsymbol{\sigma}^{[0]}) \tag{3.221}
$$

Substituting (3.221) in (3.219)

$$\left(\rho_0 \frac{\partial \Phi}{\partial(\varepsilon_{[0]})_{ik}} - {}_e({}_s\sigma^{[0]})_{ik}\right)(\dot{\varepsilon}_{[0]})_{ik} - \text{tr}\big([{}_d({}_s\sigma^{[0]})][\dot{\varepsilon}_{[0]}]\big) - \text{tr}\big([{}_a\sigma^{[0]}][{}_a\dot{r}_{[0]}]\big)$$
$$- \text{tr}\big([m^{[0]}][{}^{t}{}_{s}^{\Theta}\dot{J}_{[0]}]\big) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{[0]}) \leq 0 \quad (3.222)$$

For small deformation, small strain the matter is incompressible ($|J| = 1$), hence

$$\frac{\partial \Phi}{\partial(\varepsilon_{[0]})_{ki}} = \frac{\partial \Phi}{\partial |J|}\frac{\partial |J|}{\partial(\varepsilon_{[0]})_{ki}} = 0 \qquad \text{as} \qquad \frac{\partial \Phi}{\partial |J|} = 0 \quad (3.223)$$

Thus, the first term in (3.222) cannot be used to derive the constitutive theory for ${}_e({}_s\boldsymbol{\sigma}^{[0]})$. Note that ${}_e({}_s\boldsymbol{\sigma}^{[0]})$ in (3.222) is in fact only valid for compressible matter if the coefficient of $(\dot{\varepsilon}_{[0]})_{kl}$ is set to zero. The incompressibility condition must be introduced in (3.222).

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = \text{tr}[\bar{D}] = \text{tr}[\bar{L}] = \text{tr}(\dot{\boldsymbol{J}}\boldsymbol{J}^{-1}) = \dot{J}_{kl}(J^{-1})_{lk} = \dot{J}_{kl}\delta_{lk} = 0 \quad (3.224)$$

Also

$$\text{tr}([\bar{L}]^T) = \text{tr}\big((\boldsymbol{J}^{-1})^T \dot{\boldsymbol{J}}^T\big) = (J^{-1})_{lk}\dot{J}_{lk} = \dot{J}_{lk}\delta_{lk} = 0 \quad (3.225)$$

Since

$$\text{tr}[\bar{L}] = \text{tr}([\bar{L}]^T) \quad (3.226)$$

it follows that

$$\frac{1}{2}\big(\text{tr}[\bar{L}] + \text{tr}([\bar{L}]^T)\big) = \frac{1}{2}(\dot{J}_{kl}\delta_{lk} + \dot{J}_{lk}\delta_{lk}) = (\dot{\varepsilon}_{[0]})_{kl}\delta_{kl} = 0 \quad (3.227)$$

Let $p(\theta)$ be an arbitrary Lagrange multiplier. Then the incompressibility condition based on (3.227) becomes

$$p(\theta)(\dot{\varepsilon}_{[0]})_{ki}\delta_{ki} = 0 \quad (3.228)$$

Adding (3.228) to the left side of (3.222) and using $\frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}_{[0]}} = 0$,

$$\big(p(\theta)\delta_{ik} - {}_e({}_s\sigma^{[0]})_{ik}\big)\,(\dot{\varepsilon}_{[0]})_{ik} - \text{tr}\big([{}_d({}_s\sigma^{[0]})][\dot{\varepsilon}_{[0]}]\big) - \text{tr}\big([{}_a\sigma^{[0]}][{}_a\dot{r}_{[0]}]\big) - \text{tr}\big([m^{[0]}][{}^{t}_{s}\dot{\overset{\Theta}{J}}_{[0]}]\big) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{[0]}) \leq 0 \quad (3.229)$$

For arbitrary but admissible $\dot{\boldsymbol{\varepsilon}}_{[0]}$, (3.229) holds if

$$p(\theta)\delta_{ki} - {}_e({}_s\sigma^{[0]})_{ki} = 0 \quad (3.230)$$

or

$${}_e({}_s\boldsymbol{\sigma}^{[0]}) = p(\theta)\boldsymbol{I} \quad (3.231)$$

This is the constitutive theory for the equilibrium stress for an incompressible solid. $p(\theta)$ is called the mechanical pressure. If compressive pressure is assumed to be positive, then $p(\theta)$ in (3.231) can be replaced by $-p(\theta)$. The entropy inequality now reduces to

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \text{tr}\big([{}_d({}_s\sigma^{[0]})][\dot{\varepsilon}_{[0]}]\big) - \text{tr}\big([{}_a\sigma^{[0]}][{}_a\dot{r}_{[0]}]\big) - \text{tr}\big([m^{[0]}][{}^{t}_{s}\dot{\overset{\Theta}{J}}_{[0]}]\big) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{[0]}) \leq 0 \quad (3.232)$$

The corresponding energy equation becomes

$$\rho_0\frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \text{tr}\big([{}_d({}_s\sigma^{[0]})][\dot{\varepsilon}_{[0]}]\big) - \text{tr}\big([{}_a\sigma^{[0]}][{}_a\dot{r}_{[0]}]\big) - \text{tr}\big([m^{[0]}][{}^{t}_{s}\dot{\overset{\Theta}{J}}_{[0]}]\big) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{[0]}) = 0 \quad (3.233)$$

The entropy inequality (3.182) is satisfied if

$$\begin{aligned} {}^{s\sigma}\Psi &= \text{tr}\big([{}_d({}_s\sigma^{[0]})][\dot{\varepsilon}_{[0]}]\big) \geq 0 \\ {}^{a\sigma}\Psi &= \text{tr}\big([{}_a\sigma^{[0]}][{}_a\dot{r}_{[0]}]\big) \geq 0 \\ {}^{sm}\Psi &= \text{tr}\big([m^{[0]}][{}^{t}_{s}\dot{\overset{\Theta}{J}}_{[0]}]\big) \geq 0 \end{aligned} \quad (3.234)$$

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \leq 0 \quad (3.235)$$

and

$$ {}_i\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{[0]}) = 0 \tag{3.236} $$

The conditions (3.234) imply that the rate of work due to ${}_d({}_s\boldsymbol{\sigma}^{[0]})$, ${}_a\boldsymbol{\sigma}^{[0]}$, and $\boldsymbol{m}^{[0]}$ must be positive. Inequality (3.235) can be used to derive the constitutive theory for $\boldsymbol{q}$. Equation (3.236) serves as a constraint (compatibility condition) on ${}_i\boldsymbol{\Theta}$ and the antisymmetric components of the Cauchy stress tensor $\boldsymbol{\sigma}^{[0]}$. The rate of work or the work conjugate pairs in (3.232) are in conformity with (3.209). The argument tensors of the constitutive variables in (3.209) can now be revised.

$$
\begin{aligned}
&{}_s\boldsymbol{\sigma}^{[m_{s\sigma}]} = {}_s\boldsymbol{\sigma}^{[m_{s\sigma}]}\big(\boldsymbol{\varepsilon}_{[i]};\ i = 0,1,\dots,n_\varepsilon,\ {}_s\boldsymbol{\sigma}^{[j]};\ j = 0,1,\dots,(m_{s\sigma}-1),\ \theta\big)\\
&{}_a\boldsymbol{\sigma}^{[m_{a\sigma}]} = {}_a\boldsymbol{\sigma}^{[m_{a\sigma}]}\big({}_a\boldsymbol{r}_{[i]};\ i = 0,1,\dots,n_{ar},\ {}_a\boldsymbol{\sigma}^{[j]};\ j = 0,1,\dots,(m_{a\sigma}-1),\ \theta\big)\\
&\boldsymbol{m}^{[m_m]} = \boldsymbol{m}^{[m_m]}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[i]};\ i = 0,1,\dots,n_{{}^{t\Theta}_{s}J},\ \boldsymbol{m}^{[j]};\ j = 0,1,\dots,(m_m-1),\ \theta\big)\\
&\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g},\theta)\\
&{}_e({}_s\boldsymbol{\sigma}^{[0]}) = p(\theta)\boldsymbol{I}\\
&\Phi = \Phi(\theta)
\end{aligned}
\tag{3.237}
$$

and

$$ {}_s\boldsymbol{\sigma}^{[0]} = {}_e({}_s\boldsymbol{\sigma}^{[0]}) + {}_d({}_s\boldsymbol{\sigma}^{[0]}) \tag{3.238} $$

3.5.3 Constitutive Theory for ${}_d({}_s\boldsymbol{\sigma}^{[m_{s\sigma}]})$

Consider the argument tensors of ${}_d({}_s\boldsymbol{\sigma}^{[m_{s\sigma}]})$ in (3.237). Let ${}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i;\ i = 1,2,\dots,N_{s\sigma}$ be the combined generators of the argument tensors of ${}_d({}_s\boldsymbol{\sigma}^{[m_{s\sigma}]})$ that are symmetric tensors of rank two and let ${}^{s\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,M_{s\sigma}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration based on the representation theorem [80–99].

$$ {}_d({}_s\boldsymbol{\sigma}^{[m_{s\sigma}]}) = {}^{s\sigma}\underset{\sim}{\alpha}^0\boldsymbol{I} + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underset{\sim}{\alpha}^i({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i) \tag{3.239} $$

in which

$$ {}^{s\sigma}\underset{\sim}{\alpha}^i = {}^{s\sigma}\underset{\sim}{\alpha}^i({}^{s\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_{s\sigma},\ \theta); \quad i = 1, 2, \dots, N_{s\sigma} \tag{3.240} $$

3.5.3.1 Material Coefficients in the Constitutive Theory for ${}_d({}_s\boldsymbol{\sigma}^{[m_{s\sigma}]})$

To determine the material coefficients in (3.239), expand each ${}^{s\sigma}\underset{\sim}{\alpha}^i$ in a Taylor series in ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{s\sigma}$ and θ about a known configuration $\underline{\Omega}$, retain only up to linear terms in ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{s\sigma}$ and θ, and then substitute these ${}^{s\sigma}\underset{\sim}{\alpha}^i$ in (3.239). Collect coefficients of those terms that are defined in the current configuration to obtain

$$ \begin{aligned} {}_d({}_s\boldsymbol{\sigma}^{[m_{s\sigma}]}) = {}^0_s\sigma\big|_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M_{s\sigma}} {}^{s\sigma}\underline{a}_j\big({}^{s\sigma}\underset{\sim}{I}^j\big)\boldsymbol{I} - {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}\,(\theta - \theta_{\underline{\Omega}})\,\boldsymbol{I} \\ + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underline{b}_i\big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) + \sum_{i=1}^{N_{s\sigma}}\sum_{j=1}^{M_{s\sigma}} {}^{s\sigma}\underset{\sim}{c}_{ij}\big({}^{s\sigma}\underset{\sim}{I}^j\big)\big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underset{\sim}{d}_i\,(\theta - \theta_{\underline{\Omega}})\,\big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) \end{aligned} \tag{3.241} $$

in which

$$ \begin{aligned} {}^0_s\sigma\big|_{\underline{\Omega}} &= {}^{s\sigma}\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} - \sum_{j=1}^{M_{s\sigma}} \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} \big({}^{s\sigma}\underset{\sim}{I}^j\big)_{\underline{\Omega}} \\ {}^{s\sigma}\underline{a}_j &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}}; \quad j = 1, 2, \dots, M_{s\sigma} \\ {}^{s\sigma}\underline{b}_i &= {}^{s\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} - \sum_{j=1}^{M_{s\sigma}} \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} \big({}^{s\sigma}\underset{\sim}{I}^j\big)_{\underline{\Omega}}; \quad i = 1, 2, \dots, N_{s\sigma} \\ {}^{s\sigma}\underset{\sim}{c}_{ij} &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}}; \quad \begin{aligned} i &= 1, 2, \dots, N_{s\sigma} \\ j &= 1, 2, \dots, M_{s\sigma} \end{aligned} \\ {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}} &= -\left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial\theta}\right|_{\underline{\Omega}} \\ {}^{s\sigma}\underset{\sim}{d}_i &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial\theta}\right|_{\underline{\Omega}}; \quad i = 1, 2, \dots, N_{s\sigma} \end{aligned} \tag{3.242} $$

${}^{s\sigma}\underline{a}_j$, ${}^{s\sigma}\underline{b}_i$, ${}^{s\sigma}\underset{\sim}{c}_{ij}$, ${}^{s\sigma}\underset{\sim}{d}_i$, and ${}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}$ are the material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_{s\sigma} + N_{s\sigma} + M_{s\sigma}N_{s\sigma} + N_{s\sigma} + 1)$ material coefficients. The material coefficients defined in (3.241) are functions of $({}^{s\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\theta\big|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{s\sigma}\underset{\sim}{\alpha}^i;\ i = 0, 1, \ldots, N_{s\sigma}$. The complete theory contains too many material coefficients to be determined, experimentally or otherwise. Simplified forms are considered in a later section.

3.5.4 Constitutive Theory for $\boldsymbol{m}^{[m_m]}$

Consider the argument tensors of $\boldsymbol{m}^{[m_m]}$ in (3.237). Let ${}^m\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \ldots, N_m$ be the combined generators of the argument tensors of $\boldsymbol{m}^{[m_m]}$ that are symmetric tensors of rank two and let ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M_m$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration.

$$\boldsymbol{m}^{[m_m]} = {}^m\underset{\sim}{\alpha}^0 \boldsymbol{I} + \sum_{i=1}^{N_m} {}^m\underset{\sim}{\alpha}^i ({}^m\underset{\sim}{\boldsymbol{G}}^i) \tag{3.243}$$

in which

$${}^m\underset{\sim}{\alpha}^i = {}^m\underset{\sim}{\alpha}^i({}^m\underset{\sim}{I}^j;\ j = 1, 2, \ldots, M_m,\ \theta); \quad i = 1, 2, \ldots, N_m \tag{3.244}$$

3.5.4.1 Material Coefficients in the Constitutive Theory for $\boldsymbol{m}^{[m_m]}$

To determine the material coefficients in (3.243), expand each ${}^m\underset{\sim}{\alpha}^i$ in a Taylor series in ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M_m$ and θ about a known configuration $\underline{\Omega}$, retain only up to linear terms in ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M_m$ and θ, and then substitute these ${}^m\underset{\sim}{\alpha}^i$ in (3.243). After the collecting coefficients of those terms that are defined in the current configuration, the following is obtained.

$$\begin{aligned} \boldsymbol{m}^{[m_m]} = {}^0m\big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{j=1}^{M_m} {}^m\underline{a}_j \left({}^m\underset{\sim}{I}^j\right) \boldsymbol{I} - {}^m\underset{\sim}{\alpha}_{\text{tm}} \left(\theta - \theta_{\underline{\Omega}}\right) \boldsymbol{I} \\ + \sum_{i=1}^{N_m} {}^m\underset{\sim}{b}_i \left({}^m\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_m}\sum_{j=1}^{M_m} {}^m\underset{\sim}{c}_{ij} \left({}^m\underset{\sim}{I}^j\right)\left({}^m\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_m} {}^m\underset{\sim}{d}_i \left(\theta - \theta_{\underline{\Omega}}\right)\left({}^m\underset{\sim}{\boldsymbol{G}}^i\right) \end{aligned} \tag{3.245}$$

${}^m\underline{a}_j$, ${}^m\underset{\sim}{b}_i$, ${}^m\underset{\sim}{c}_{ij}$, ${}^m\underset{\sim}{d}_i$, and ${}^m\underset{\sim}{\alpha}_{\text{tm}}$ are the material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_m + N_m + M_m N_m + N_m + 1)$ material coefficients. The material coefficients are functions of $({}^m\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{m}\underset{\sim}{\alpha}^{i};\ i = 0, 1, \dots, N_m$. Explicit forms of the material coefficients can be obtained from (3.242) by simply replacing the back superscript ${}_{s}\sigma$ with m and ${}^{0}_{s}\sigma|_{\underline{\Omega}}$ by ${}^{0}m|_{\underline{\Omega}}$. This complete theory contains too many material coefficients to be determined, experimentally or otherwise. Simplified forms are considered in a later section.

3.5.5 Constitutive Theory for ${}_{a}\boldsymbol{\sigma}^{[m_{a\sigma}]}$

Consider the argument tensors of ${}_{a}\boldsymbol{\sigma}^{[m_{a\sigma}]}$ in (3.237). Let ${}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^{i};\ i = 1, 2, \dots, N_{a\sigma}$ be the combined generators of the argument tensors of ${}_{a}\boldsymbol{\sigma}^{[m_{a\sigma}]}$ that are antisymmetric tensors of rank two and let ${}^{a\sigma}\underset{\sim}{I}^{j};\ j = 1, 2, \dots, M_{a\sigma}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration based on the representation theorem [80–99] (note that the identity tensor is not an antisymmetric tensor of rank two, hence is not a generator of ${}_{a}\boldsymbol{\sigma}^{[m_{a\sigma}]}$).

$$ {}_{a}\boldsymbol{\sigma}^{[m_{a\sigma}]} = \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underset{\sim}{\alpha}^{i} \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \right) \tag{3.246} $$

in which

$$ {}^{a\sigma}\underset{\sim}{\alpha}^{i} = {}^{a\sigma}\underset{\sim}{\alpha}^{i} \left({}^{a\sigma}\underset{\sim}{I}^{j};\ j = 1, 2, \dots, M_{a\sigma},\ \theta \right); \quad i = 1, 2, \dots, N_{a\sigma} \tag{3.247} $$

3.5.5.1 Material Coefficients in the Constitutive Theory for ${}_{a}\boldsymbol{\sigma}^{[m_{a\sigma}]}$

To determine the material coefficients in (3.246), expand each ${}^{a\sigma}\underset{\sim}{\alpha}^{i}$ in a Taylor series in ${}^{a\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, M_{a\sigma}$ and θ about a known configuration $\underline{\Omega}$, retain only up to linear terms in ${}^{a\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, M_{a\sigma}$ and θ, and then substitute these ${}^{a\sigma}\underset{\sim}{\alpha}^{i}$ in (3.246). After collecting the coefficients of those terms that are defined in the current configuration, the following is obtained.

$$ {}_{a}\boldsymbol{\sigma}^{[m_{a\sigma}]} = \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underline{b}_{i} \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \right) + \sum_{i=1}^{N_{a\sigma}} \sum_{j=1}^{M_{a\sigma}} {}^{a\sigma}\underset{\sim}{c}_{ij} \left({}^{a\sigma}\underset{\sim}{I}^{j} \right) \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \right) + \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underset{\sim}{d}_{i} \left(\theta - \theta_{\underline{\Omega}} \right) \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \right) \tag{3.248} $$

${}^{a\sigma}\underline{b}_{i}$, ${}^{a\sigma}\underset{\sim}{c}_{ij}$, and ${}^{a\sigma}\underset{\sim}{d}_{i}$ are the material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(N_{a\sigma} + M_{a\sigma}N_{a\sigma} + N_{a\sigma})$ material coefficients. The material coefficients are functions of $({}^{a\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{a\sigma}\underset{\sim}{Q}^i;\ i = 0, 1, \dots, N_{a\sigma}$. Explicit forms of the material coefficients can be obtained from (3.242) by simply replacing the back superscript s with a. This complete theory contains too many material coefficients to be determined, experimentally or otherwise. Simplified forms are considered in the following section.

3.5.6 Simplified Constitutive Theories for ${}_d({}_s\boldsymbol{\sigma}^{[m_{s\sigma}]})$, $\boldsymbol{m}^{m_m}$, and ${}_a\boldsymbol{\sigma}^{m_{a\sigma}}$

The constitutive theories for ${}_d({}_s\boldsymbol{\sigma}^{[m_{s\sigma}]})$, $\boldsymbol{m}^{m_m}$, and ${}_a\boldsymbol{\sigma}^{m_{a\sigma}}$ derived in Sections 3.5.3, 3.5.4, and 3.5.5 are complete as they are based on integrity, but require determination of too many material coefficients. In this section simplified forms of the theories are considered that require the minimum number of material coefficients, but at the expense of only modeling simplified and compromised physics.

Polymeric fluids are generally classified as dilute polymeric fluids or dense polymeric fluids. Dilute polymers behave much like thermoviscous fluids with some elasticity due to relatively low concentration of polymer molecules in the solvent. Dense polymeric fluids on the other hand have significantly higher concentrations of polymer molecules in the solvent, hence exhibit pronounced elastic behavior. Surana, et al. [105–107] presented derivations of Maxwell, Oldroyd-B, and Giesekus constitutive models for classical continuum theories strictly using the entropy inequality and the representation theorem. The work presented in References [105–107] highlights limitations of the currently used Maxwell, Oldroyd-B, and Giesekus constitutive models for polymeric fluids and provides derivations of the ordered rate theories based on integrity that permit much more comprehensive constitutive models.

Surana, et al. [108–111] presented constitutive theories for classical thermoviscoelastic solids based on integrity. The constitutive models in References [108–111] were derived by making assumptions similar to those used for fluids [105–107] and were compared with phenomenological models such as Kelvin-Voigt model. The simplified constitutive theories presented in this section for non-classical solids parallel those for classical thermoviscoelastic fluids such as Maxwell, Oldroyd-B, and Giesekus models.

Remarks.

(1) The constitutive theories for classical thermoviscoelastic solids are always a subset of the constitutive theories for the non-classical polymeric solids presented in this dissertation. These are obtained by simply removing the internal and Cosserat rotation physics, which leads to $\boldsymbol{m}^{[0]} = 0$ and ${}_a\boldsymbol{\sigma}^{[0]} = 0$, and the Cauchy stress tensor becomes symmetric due to the balance of angular momenta. The resulting constitutive theories are the same as those by Surana, et al. [108–111] for classical thermoviscoelastic solids.

(2) Specific forms of the constitutive theories are all subsets of those derived based on integrity, as done here. Hence, it is possible to derive a single simplified constitutive model using the general theory that should contain various possible models related to specific physics such as those that resemble Maxwell, Oldroyd-B, and Giesekus fluids, etc.

(3) The simplest way to proceed is to assume that constitutive models parallel to Maxwell, Oldroyd-B, and Giesekus models for polymeric fluids are desired for non-classical thermoviscoelastic solids. Furthermore, in order for the non-classical constitutive models to contain the classical models of interest as a subset, the following ordered rates must be chosen:

$$\begin{gathered} n_\varepsilon = 2; \quad n_{{}_a r} = 2; \quad n_{{}^{t}_{s}\!\Theta J} = 2 \\ m_{{}_s\sigma} = 1; \quad m_{{}_a\sigma} = 1; \quad m_m = 1 \end{gathered} \tag{3.249}$$

For this choice, the constitutive variables and the argument tensors are

$$\begin{aligned} {}_d\big({}_s\boldsymbol{\sigma}^{[1]}\big) &= {}_d\big({}_s\boldsymbol{\sigma}^{[1]}\big)\Big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[1]}, \boldsymbol{\varepsilon}_{[2]}, {}_d\big({}_s\boldsymbol{\sigma}^{[0]}\big), \theta\Big) \\ \boldsymbol{m}^{[1]} &= \boldsymbol{m}^{[1]}\Big({}^{t}_{s}\!\Theta\boldsymbol{J}_{[0]}, {}^{t}_{s}\!\Theta\boldsymbol{J}_{[1]}, {}^{t}_{s}\!\Theta\boldsymbol{J}_{[2]}, \boldsymbol{m}^{[0]}, \theta\Big) \\ {}_a\boldsymbol{\sigma}^{[1]} &= {}_a\boldsymbol{\sigma}^{[1]}\Big({}_a\boldsymbol{r}_{[0]}, {}_a\boldsymbol{r}_{[1]}, {}_a\boldsymbol{r}_{[2]}, {}_a\boldsymbol{\sigma}^{[0]}, \theta\Big) \end{aligned} \tag{3.250}$$

The constitutive theories using (3.250) when based on integrity still require too many material coefficients. Consider the following simplifications, the motivation being that when these are further simplified for classical case they will yield standard, familiar constitutive theories.

(i) Consider the constitutive theories to be linear in $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, $\boldsymbol{\varepsilon}_{[2]}$, ${}^{t}_{s}\!\overset{\Theta}{}\boldsymbol{J}_{[0]}$, ${}^{t}_{s}\!\overset{\Theta}{}\boldsymbol{J}_{[1]}$, ${}^{t}_{s}\!\overset{\Theta}{}\boldsymbol{J}_{[2]}$, ${}_{a}\boldsymbol{r}_{[0]}$, ${}_{a}\boldsymbol{r}_{[1]}$, ${}_{a}\boldsymbol{r}_{[2]}$.

(ii) Neglect product terms of the tensors in (i). Also neglect products of the invariants.

(iii) Neglect $(\theta - \theta_{\underline{\Omega}})$ terms.

(iv) Neglect the first term in each constitutive theory (if present) containing the influence of the initial stress or moment tensor.

(v) Consider ${}_{d}({}_{s}\boldsymbol{\sigma}^{[0]})^2$, $(\boldsymbol{m}^{[0]})^2$, and $({}_{a}\boldsymbol{\sigma}^{[0]})^2$ as generators, but neglect quadratic and cubic trace terms in the invariants as well as their products.

3.5.6.1 Simplified Constitutive Theory for ${}_{d}({}_{s}\boldsymbol{\sigma}^{[m_{s\sigma}]})$

Consider

$$
\begin{gathered}
{}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^1 = \boldsymbol{\varepsilon}_{[0]}; \quad {}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^2 = \boldsymbol{\varepsilon}_{[1]}; \quad {}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^3 = \boldsymbol{\varepsilon}_{[2]}; \quad {}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^4 = {}_{d}({}_{s}\boldsymbol{\sigma}^{[0]}); \quad {}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^5 = \big({}_{d}({}_{s}\boldsymbol{\sigma}^{[0]})\big)^2 \\
{}^{s\sigma}\underset{\sim}{I}^1 = \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[0]}\big); \quad {}^{s\sigma}\underset{\sim}{I}^2 = \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[1]}\big); \quad {}^{s\sigma}\underset{\sim}{I}^3 = \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[2]}\big); \quad {}^{s\sigma}\underset{\sim}{I}^4 = \mathrm{tr}\big({}_{d}({}_{s}\boldsymbol{\sigma}^{[0]})\big)
\end{gathered}
\tag{3.251}
$$

then (3.241) reduces to

$$
\begin{aligned}
{}_{d}({}_{s}\boldsymbol{\sigma}^{[1]}) = {}& {}^{s\sigma}\underline{a}_1 \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + {}^{s\sigma}\underline{a}_2 \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{I} + {}^{s\sigma}\underline{a}_3 \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[2]}\big)\boldsymbol{I} + {}^{s\sigma}\underline{a}_4 \mathrm{tr}\big({}_{d}({}_{s}\boldsymbol{\sigma}^{[0]})\big)\boldsymbol{I} \\
& + {}^{s\sigma}\underline{b}_1 (\boldsymbol{\varepsilon}_{[0]}) + {}^{s\sigma}\underline{b}_2 (\boldsymbol{\varepsilon}_{[1]}) + {}^{s\sigma}\underline{b}_3 (\boldsymbol{\varepsilon}_{[2]}) + {}^{s\sigma}\underline{b}_4 \big({}_{d}({}_{s}\boldsymbol{\sigma}^{[0]})\big) + {}^{s\sigma}\underline{b}_5 \big({}_{d}({}_{s}\boldsymbol{\sigma}^{[0]})\big)^2
\end{aligned}
\tag{3.252}
$$

In order to obtain standard forms from (3.252), transfer the ${}^{s\sigma}\underline{b}_4\big({}_d({}_s\boldsymbol{\sigma}^{[0]})\big)$ term to the left hand side and divide the whole equation by $-{}^{s\sigma}\underline{b}_4$. Define

$$
\begin{gathered}
{}^{s\sigma}\underset{\sim}{\lambda} = -\frac{1}{{}^{s\sigma}\underline{b}_4}; \quad 2\mu = -\frac{{}^{s\sigma}\underline{b}_1}{{}^{s\sigma}\underline{b}_4}; \quad \lambda = -\frac{{}^{s\sigma}\underline{a}_1}{{}^{s\sigma}\underline{b}_4}; \quad 2\mu_1 = -\frac{{}^{s\sigma}\underline{b}_2}{{}^{s\sigma}\underline{b}_4} \\
\lambda_1 = -\frac{{}^{s\sigma}\underline{a}_2}{{}^{s\sigma}\underline{b}_4}; \quad 2\mu_2 = -\frac{{}^{s\sigma}\underline{b}_3}{{}^{s\sigma}\underline{b}_4}; \quad \lambda_2 = -\frac{{}^{s\sigma}\underline{a}_3}{{}^{s\sigma}\underline{b}_4}; \quad {}^{s\sigma}k_3 = -\frac{{}^{s\sigma}\underline{a}_4}{{}^{s\sigma}\underline{b}_4}; \quad {}^{s\sigma}\eta_4 = -\frac{{}^{s\sigma}\underline{b}_5}{{}^{s\sigma}\underline{b}_4}
\end{gathered}
\tag{3.253}
$$

Thus, (3.252) can be written as

$$
\begin{aligned}
{}_d({}_s\boldsymbol{\sigma}^{[0]}) + {}^{s\sigma}\underset{\sim}{\lambda}\big({}_d({}_s\boldsymbol{\sigma}^{[1]})\big) = {} & 2\mu(\boldsymbol{\varepsilon}_{[0]}) + \lambda \operatorname{tr}\big(\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + 2\mu_1(\boldsymbol{\varepsilon}_{[1]}) + \lambda_1 \operatorname{tr}\big(\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{I} \\
& + 2\mu_2(\boldsymbol{\varepsilon}_{[2]}) + \lambda_2 \operatorname{tr}\big(\boldsymbol{\varepsilon}_{[2]}\big)\boldsymbol{I} + {}^{s\sigma}k_3 \operatorname{tr}\big({}_d({}_s\boldsymbol{\sigma}^{[0]})\big)\boldsymbol{I} + {}^{s\sigma}\eta_4\big({}_d({}_s\boldsymbol{\sigma}^{[0]})\big)^2
\end{aligned}
\tag{3.254}
$$

in which ${}^{s\sigma}\underset{\sim}{\lambda}$ is the relaxation time and μ and λ are Lamé's constants associated with strains. μ_1, λ_1, μ_2, and λ_2 are the material coefficients associated with strain rates (dissipation or damping mechanism). For incompressible matter (as the case is here), appropriate coefficient(s) can be set to zero. The constitutive theory (3.254) remains the same regardless of the consideration of classical or non-classical continuum theories. This constitutive theory contains models similar to Maxwell, Oldroyd-B, and Giesekus polymeric fluids as a subset. Specific forms of these are presented in the following.

Constitutive Theory Similar to Maxwell Polymeric Fluid

In order to obtain a constitutive theory similar to Maxwell model for fluids (classical theory), use (3.254) with

$$
2\mu_2 = 0, \quad \lambda_2 = 0, \quad {}^{s\sigma}k_3 = 0, \quad {}^{s\sigma}\eta_4 = 0 \tag{3.255}
$$

which results in

$$
{}_d({}_s\boldsymbol{\sigma}^{[0]}) + {}^{s\sigma}\underset{\sim}{\lambda}\big({}_d({}_s\boldsymbol{\sigma}^{[1]})\big) = 2\mu(\boldsymbol{\varepsilon}_{[0]}) + \lambda \operatorname{tr}\big(\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + 2\mu_1(\boldsymbol{\varepsilon}_{[1]}) + \lambda_1 \operatorname{tr}\big(\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{I} \tag{3.256}
$$

When (3.256) is compared with the Maxwell model for fluids (classical theory), the strain terms containing material coefficients 2μ and λ are additional terms due to elasticity. To parallel the Maxwell model for incompressible fluids, the $\text{tr}(\boldsymbol{\varepsilon}_{[1]})$ term should be zero. Therefore, $2\mu \neq 0$, $\lambda \neq 0$, $2\mu_1 \neq 0$, but $\lambda_1 = 0$.

Constitutive Theory Similar to Oldroyd-B Polymeric Fluid

For this constitutive model, set ${}^{s\sigma}k_3 = 0$ and ${}^{s\sigma}\eta_4 = 0$ in (3.254) to obtain

$$
{}_d({}_s\boldsymbol{\sigma}^{[0]}) + {}^{s\sigma}\underset{\sim}{\lambda}\big({}_d({}_s\boldsymbol{\sigma}^{[1]})\big) = 2\mu(\boldsymbol{\varepsilon}_{[0]}) + \lambda \text{tr}\big(\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + 2\mu_1(\boldsymbol{\varepsilon}_{[1]}) + \lambda_1 \text{tr}\big(\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{I} + 2\mu_2(\boldsymbol{\varepsilon}_{[2]}) + \lambda_2 \text{tr}\big(\boldsymbol{\varepsilon}_{[2]}\big)\boldsymbol{I} \quad (3.257)
$$

To match models for incompressible fluids, the $\text{tr}(\boldsymbol{\varepsilon}_{[1]})$ and $\text{tr}(\boldsymbol{\varepsilon}_{[2]})$ terms should be zero, so $\lambda_1 = 0$ and $\lambda_2 = 0$.

Constitutive Theory Similar to Giesekus Polymeric Fluid

In this case set $2\mu_2 = 0$, $\lambda_2 = 0$, and ${}^{s\sigma}k_3 = 0$ in (3.254).

$$
{}_d({}_s\boldsymbol{\sigma}^{[0]}) + {}^{s\sigma}\underset{\sim}{\lambda}\big({}_d({}_s\boldsymbol{\sigma}^{[1]})\big) = 2\mu(\boldsymbol{\varepsilon}_{[0]}) + \lambda \text{tr}\big(\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + 2\mu_1(\boldsymbol{\varepsilon}_{[1]}) + \lambda_1 \text{tr}\big(\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{I} + {}^{s\sigma}\eta_4\big({}_d({}_s\boldsymbol{\sigma}^{[0]})\big)^2 \quad (3.258)
$$

To match models for incompressible fluids, the $\text{tr}(\boldsymbol{\varepsilon}_{[1]})$ term is set to zero, which implies $\lambda_1 = 0$.

3.5.6.2 Simplified Constitutive Theory for $\boldsymbol{m}^{[m_m]}$

Consider

$$
\begin{gathered}
{}^m\underset{\sim}{\boldsymbol{G}}^1 = {}^t_s\Theta\boldsymbol{J}_{[0]}; \quad {}^m\underset{\sim}{\boldsymbol{G}}^2 = {}^t_s\Theta\boldsymbol{J}_{[1]}; \quad {}^m\underset{\sim}{\boldsymbol{G}}^3 = {}^t_s\Theta\boldsymbol{J}_{[2]}; \quad {}^m\underset{\sim}{\boldsymbol{G}}^4 = \boldsymbol{m}^{[0]}; \quad {}^m\underset{\sim}{\boldsymbol{G}}^5 = \big(\boldsymbol{m}^{[0]}\big)^2 \\
{}^m\underset{\sim}{I}^1 = \text{tr}\big({}^t_s\Theta\boldsymbol{J}_{[0]}\big); \quad {}^m\underset{\sim}{I}^2 = \text{tr}\big({}^t_s\Theta\boldsymbol{J}_{[1]}\big); \quad {}^m\underset{\sim}{I}^3 = \text{tr}\big({}^t_s\Theta\boldsymbol{J}_{[2]}\big); \quad {}^m\underset{\sim}{I}^4 = \text{tr}\big(\boldsymbol{m}^{[0]}\big)
\end{gathered} \quad (3.259)
$$

Thus, (3.245) leads to the following.

$$
\begin{gathered}
\boldsymbol{m}^{[1]} = {}^m\underline{a}_1 \text{tr}\big({}^t_s\Theta\boldsymbol{J}_{[0]}\big)\boldsymbol{I} + {}^m\underline{a}_2 \text{tr}\big({}^t_s\Theta\boldsymbol{J}_{[1]}\big)\boldsymbol{I} + {}^m\underline{a}_3 \text{tr}\big({}^t_s\Theta\boldsymbol{J}_{[2]}\big)\boldsymbol{I} + {}^m\underline{a}_4 \text{tr}\big(\boldsymbol{m}^{[0]}\big)\boldsymbol{I} \\
+ {}^m\underline{b}_1({}^t_s\Theta\boldsymbol{J}) + {}^m\underline{b}_2({}^t_s\Theta\boldsymbol{J}_{[1]}) + {}^m\underline{b}_3({}^t_s\Theta\boldsymbol{J}_{[2]}) + {}^m\underline{b}_4\big(\boldsymbol{m}^{[0]}\big) + {}^m\underline{b}_5\big(\boldsymbol{m}^{[0]}\big)^2
\end{gathered} \quad (3.260)
$$

Transfer the ${}^{m}\underset{\sim}{b}_4(\boldsymbol{m}^{[0]})$ term to the left side of (3.260) and divide the whole equation by $-{}^{m}\underset{\sim}{b}_4$. Define

$$\begin{gathered}{}^{m}\underset{\sim}{\lambda} = -\frac{1}{{}^{m}\underset{\sim}{b}_4}; \quad \alpha = -\frac{{}^{m}\underset{\sim}{b}_1}{{}^{m}\underset{\sim}{b}_4}; \quad B = -\frac{{}^{m}\underset{\sim}{a}_1}{{}^{m}\underset{\sim}{b}_4}; \quad \alpha_1 = -\frac{{}^{m}\underset{\sim}{b}_2}{{}^{m}\underset{\sim}{b}_4} \\ B_1 = -\frac{{}^{m}\underset{\sim}{a}_2}{{}^{m}\underset{\sim}{b}_4}; \quad \alpha_2 = -\frac{{}^{m}\underset{\sim}{b}_3}{{}^{m}\underset{\sim}{b}_4}; \quad B_2 = -\frac{{}^{m}\underset{\sim}{a}_3}{{}^{m}\underset{\sim}{b}_4}; \quad {}^{sm}\underset{\sim}{k}_3 = -\frac{{}^{m}\underset{\sim}{a}_4}{{}^{m}\underset{\sim}{b}_4}; \quad {}^{sm}\eta_4 = -\frac{{}^{m}\underset{\sim}{b}_5}{{}^{m}\underset{\sim}{b}_4}\end{gathered} \tag{3.261}$$

Then, (3.260) can be written as

$$\begin{aligned}\boldsymbol{m}^{[0]} + {}^{m}\underset{\sim}{\lambda}\big(\boldsymbol{m}^{[1]}\big) = {} & \alpha\big({}^{t\Theta}_{s}\boldsymbol{J}_{[0]}\big) + B\,\mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[0]}\big)\boldsymbol{I} + \alpha_1\big({}^{t\Theta}_{s}\boldsymbol{J}_{[1]}\big) + B_1\mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[1]}\big)\boldsymbol{I} \\ & + \alpha_2\big({}^{t\Theta}_{s}\boldsymbol{J}_{[2]}\big) + B_2\mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[2]}\big)\boldsymbol{I} + {}^{m}\underset{\sim}{k}_3\mathrm{tr}\big(\boldsymbol{m}^{[0]}\big)\boldsymbol{I} + {}^{m}\eta_4\big(\boldsymbol{m}^{[0]}\big)^2\end{aligned} \tag{3.262}$$

In which ${}^{m}\underset{\sim}{\lambda}$ is the relaxation time associated with $\boldsymbol{m}$. The other material coefficients have similar meanings as those defined in Section 3.5.6.1. Constitutive theories for the Cauchy moment tensor in non-classical thermoviscoelastic solids similar to Maxwell, Oldroyd-B, and Giesekus polymeric fluids can be obtained by setting $\alpha_2 = 0$, ${}^{m}\underset{\sim}{k}_3 = 0$, and ${}^{m}\eta_4 = 0$; ${}^{m}\underset{\sim}{k}_3 = 0$ and ${}^{m}\eta_4 = 0$; and $\alpha_2 = 0$ and ${}^{m}\underset{\sim}{k}_3 = 0$, respectively. The details are straightforward.

3.5.6.3 Simplified Constitutive Theory for ${}_{a}\boldsymbol{\sigma}^{[m_{a\sigma}]}$

Based on the simplifications in (3.249) for the constitutive variable ${}_{a}\boldsymbol{\sigma}$ with argument tensors (3.250), the generators are given by

$${}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^1 = {}_{a}\boldsymbol{r}_{[0]}; \quad {}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^2 = {}_{a}\boldsymbol{r}_{[1]}; \quad {}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^3 = {}_{a}\boldsymbol{r}_{[2]}; \quad {}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^4 = {}_{a}\boldsymbol{\sigma}^{[0]} \tag{3.263}$$

There are no invariants of the argument tensors of ${}_{a}\boldsymbol{\sigma}^{[1]}$ that are linear in their components and do not contain the product terms of the components of argument tensors. Invariants containing quadratic terms in the argument tensors are not considered either. That is, the invariants

$$\begin{gathered}\mathrm{tr}\big([{}_{a}r_{[0]}][{}_{a}r_{[1]}]\big), \quad \mathrm{tr}\big([{}_{a}r_{[0]}][{}_{a}r_{[2]}]\big), \quad \mathrm{tr}\big([{}_{a}r_{[0]}][{}_{a}\sigma^{[0]}]\big), \quad \text{etc}\ldots \\ \mathrm{tr}\big([{}_{a}r_{[0]}]^2\big), \quad \mathrm{tr}\big([{}_{a}r_{[1]}]^2\big), \quad \text{etc}\ldots\end{gathered} \tag{3.264}$$

are not considered. Likewise, the generators containing the product and quadratic and higher degree terms in the argument tensors are not considered either. If there is a need for any of these invariants and generators to be considered, they can easily be incorporated in the derivation of the constitutive theory. Neglecting $\theta - \underline{\theta}_\Omega$ term, the following holds

$$ {}_a\boldsymbol{\sigma}^{[1]} = {}^{a\sigma}\underline{b}_1({}_a\boldsymbol{r}_{[0]}) + {}^{a\sigma}\underline{b}_2({}_a\boldsymbol{r}_{[1]}) + {}^{a\sigma}\underline{b}_3({}_a\boldsymbol{r}_{[2]}) + {}^{a\sigma}\underline{b}_4({}_a\boldsymbol{\sigma}^{[0]}) \tag{3.265} $$

Transferring the ${}^{a\sigma}\underline{b}_4$ term to the left side of (3.265), dividing throughout by $-{}^{a\sigma}\underline{b}_4$, and defining

$$ {}^{a\sigma}\underset{\sim}{\lambda} = -\frac{1}{{}^{a\sigma}\underline{b}_4}; \quad \kappa = -\frac{{}^{a\sigma}\underline{b}_1}{{}^{a\sigma}\underline{b}_4}; \quad \kappa_1 = -\frac{{}^{a\sigma}\underline{b}_2}{{}^{a\sigma}\underline{b}_4}; \quad \kappa_2 = -\frac{{}^{a\sigma}\underline{b}_3}{{}^{a\sigma}\underline{b}_4} \tag{3.266} $$

leads to

$$ {}_a\boldsymbol{\sigma}^{[0]} + {}^{a\sigma}\underset{\sim}{\lambda}\left({}_a\boldsymbol{\sigma}^{[1]}\right) = \kappa({}_a\boldsymbol{r}_{[0]}) + \kappa_1({}_a\boldsymbol{r}_{[1]}) + \kappa_2({}_a\boldsymbol{r}_{[2]}) \tag{3.267} $$

This constitutive theory is obviously linear. As previously noted, by including generators and invariants that are nonlinear functions of the argument tensors, more elaborate forms of (3.267) can be obtained, depending on the need. Due to the antisymmetric nature of ${}_a\boldsymbol{\sigma}^{[1]}$, there are no simplifications of this constitutive theory corresponding to the Maxwell, Oldroyd-B, and Giesekus models for classical polymeric fluids.

Remarks.

(1) The most general simplified constitutive theories require 22 material coefficients. μ and λ are Lamé's constants for the strain terms, μ_1 and λ_1 are corresponding material coefficients related to strain rates, and μ_2 and λ_2 are associated with the second time derivative of strain, all present in classical theories. α, α_1, and α_2 are required in the presence of rotations and rotation rates (internal or Cosserat). κ, κ_1, κ_2, B, B_1, and B_2 are only needed when Cosserat rotations and rotation rates are present. $\underset{\sim}{\lambda}$, k_3, and η_4 (with appropriate back superscripts) are the coefficients for the stress and moment terms in the constitutive theories.

(2) In the case of purely internal polar non-classical continuum theories, the constitutive theory for ${}_a\boldsymbol{\sigma}^{[m_{a\sigma}]}$ is not needed as these are balanced by gradients of the moment tensor in the balance of angular momenta, hence in this case κ, κ_1, and κ_2 are zero. ${}^{t\Theta}_{s}\boldsymbol{J}$ and its rates become the symmetric part of the internal rotation gradient tensor and its rates, and the coefficients B, B_1, and B_2 are no longer required as the trace of the gradient of internal rotations is zero.

(3) If the balance of moments of moments balance law is neglected, then the moment tensor is not symmetric. Following the procedure presented, the constitutive theories in such a case would be identical to when the balance of moments of moments is considered, with the addition of the constitutive theory for the antisymmetric part of the moment tensor.

$$
\begin{aligned}
{}_d\big({}_s\boldsymbol{\sigma}^{[0]}\big) + {}^{s\sigma}\underset{\sim}{\lambda}\Big({}_d\big({}_s\boldsymbol{\sigma}^{[1]}\big)\Big) &= 2\mu\big(\boldsymbol{\varepsilon}_{[0]}\big) + \lambda \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + 2\mu_1\big(\boldsymbol{\varepsilon}_{[1]}\big) + \lambda_1 \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{I} \\
&\quad + 2\mu_2\big(\boldsymbol{\varepsilon}_{[2]}\big) + \lambda_2 \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[2]}\big)\boldsymbol{I} + {}^{s\sigma}k_3 \mathrm{tr}\Big({}_d\big({}_s\boldsymbol{\sigma}^{[0]}\big)\Big)\boldsymbol{I} + {}^{s\sigma}\eta_4\Big({}_d\big({}_s\boldsymbol{\sigma}^{[0]}\big)\Big)^2
\end{aligned} \tag{3.268}
$$

$$
\begin{aligned}
{}_s\boldsymbol{m}^{[0]} + {}^{sm}\underset{\sim}{\lambda}\big({}_s\boldsymbol{m}^{[1]}\big) &= \alpha\big({}^{t\Theta}_{s}\boldsymbol{J}_{[0]}\big) + B \mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[0]}\big)\boldsymbol{I} + \alpha_1\big({}^{t\Theta}_{s}\boldsymbol{J}_{[1]}\big) + B_1 \mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[1]}\big)\boldsymbol{I} \\
&\quad + \alpha_2\big({}^{t\Theta}_{s}\boldsymbol{J}_{[2]}\big) + B_2 \mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[2]}\big)\boldsymbol{I} + {}^{sm}k_3 \mathrm{tr}\big({}_s\boldsymbol{m}^{[0]}\big)\boldsymbol{I} + {}^{sm}\eta_4\big({}_s\boldsymbol{m}^{[0]}\big)^2
\end{aligned} \tag{3.269}
$$

$$
{}_a\boldsymbol{\sigma}^{[0]} + {}^{a\sigma}\underset{\sim}{\lambda}\big({}_a\boldsymbol{\sigma}^{[1]}\big) = \kappa\big({}_a\boldsymbol{r}_{[0]}\big) + \kappa_1\big({}_a\boldsymbol{r}_{[1]}\big) + \kappa_2\big({}_a\boldsymbol{r}_{[2]}\big) \tag{3.270}
$$

$$
{}_a\boldsymbol{m}^{[0]} + {}^{am}\underset{\sim}{\lambda}\big({}_a\boldsymbol{m}^{[1]}\big) = \beta\big({}^{t\Theta}_{a}\boldsymbol{J}_{[0]}\big) + \beta_1\big({}^{t\Theta}_{a}\boldsymbol{J}_{[1]}\big) + \beta_2\big({}^{t\Theta}_{a}\boldsymbol{J}_{[2]}\big) \tag{3.271}
$$

(4) These theories can be simplified as needed to match commonly recognized models for polymeric fluids. All such models are a subset of the general ordered rate constitutive theories presented here.

3.5.6.4 Retardation and Memory Moduli

Define

$$
{}^{\sigma}_{s}\boldsymbol{Q} = 2\mu\big(\boldsymbol{\varepsilon}_{[0]}\big) + \lambda \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + 2\mu_1\big(\boldsymbol{\varepsilon}_{[1]}\big) + \lambda_1 \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{I} + 2\mu_2\big(\boldsymbol{\varepsilon}_{[2]}\big) + \lambda_2 \mathrm{tr}\big(\boldsymbol{\varepsilon}_{[2]}\big)\boldsymbol{I} \tag{3.272}
$$

$$
{}^{m}\boldsymbol{Q} = \alpha({}^{t\Theta}_{s}\boldsymbol{J}_{[0]}) + B\,\mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[0]}\big)\boldsymbol{I} + \alpha_1({}^{t\Theta}_{s}\boldsymbol{J}_{[1]}) + B_1\mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[1]}\big)\boldsymbol{I} + \alpha_2({}^{t\Theta}_{s}\boldsymbol{J}_{[2]}) + B_2\mathrm{tr}\big({}^{t\Theta}_{s}\boldsymbol{J}_{[2]}\big)\boldsymbol{I} \tag{3.273}
$$

$$
{}^{\sigma}_{a}\boldsymbol{Q} = \kappa({}_{a}\boldsymbol{r}_{[0]}) + \kappa_1({}_{a}\boldsymbol{r}_{[1]}) + \kappa_2({}_{a}\boldsymbol{r}_{[2]}) \tag{3.274}
$$

Consider the constitutive theories (3.254), (3.262), and (3.267) and discard the stress and moment terms on the right sides of the equations, then

$$
{}_{d}({}_{s}\boldsymbol{\sigma}^{[0]}) + {}^{s\sigma}\underset{\sim}{\lambda}\big({}_{d}({}_{s}\boldsymbol{\sigma}^{[1]})\big) = {}^{\sigma}_{s}\boldsymbol{Q} \tag{3.275}
$$

$$
\boldsymbol{m}^{[0]} + {}^{m}\underset{\sim}{\lambda}\big(\boldsymbol{m}^{[1]}\big) = {}^{m}\boldsymbol{Q} \tag{3.276}
$$

$$
{}_{a}\boldsymbol{\sigma}^{[0]} + {}^{a\sigma}\underset{\sim}{\lambda}\big({}_{a}\boldsymbol{\sigma}^{[1]}\big) = {}^{\sigma}_{a}\boldsymbol{Q} \tag{3.277}
$$

Equations (3.275) – (3.277) are first order differential equations in time in ${}_{d}({}_{s}\boldsymbol{\sigma}^{[0]})$, $\boldsymbol{m}^{[0]}$, and ${}_{a}\boldsymbol{\sigma}^{[0]}$, hence can be integrated using the following method.

The differential equation

$$
\frac{d\phi}{dx} + P(x)\phi = Q(x) \tag{3.278}
$$

has the solution

$$
\phi = \mathrm{e}^{-\int P(x)dx}\left[\int Q(x)\mathrm{e}^{\int P(x)dx}dx + C\right] \tag{3.279}
$$

where C is a constant of integration. Consider (3.275) and rewrite

$$
{}_{d}({}_{s}\boldsymbol{\sigma}^{[1]}) + \frac{1}{{}^{s\sigma}\underset{\sim}{\lambda}}\big({}_{d}({}_{s}\boldsymbol{\sigma}^{[0]})\big) = \frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}} \tag{3.280}
$$

Hence, using (3.278) and (3.279),

$$
\begin{aligned}
{}_{d}({}_{s}\boldsymbol{\sigma}^{[0]}) &= \mathrm{e}^{-\int 1/{}^{s\sigma}\underset{\sim}{\lambda}dt}\left[\int \frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{\int 1/{}^{s\sigma}\underset{\sim}{\lambda}dt}dt + \boldsymbol{C}\right] \\
&= \mathrm{e}^{-t/{}^{s\sigma}\underset{\sim}{\lambda}}\left[\int \frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{t'/{}^{s\sigma}\underset{\sim}{\lambda}}dt' + \boldsymbol{C}\right] \\
&= \frac{\int\limits_{-\infty}^{t} \frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{t'/{}^{s\sigma}\underset{\sim}{\lambda}}dt'}{\mathrm{e}^{t/{}^{s\sigma}\underset{\sim}{\lambda}}} + \mathrm{e}^{-t/{}^{s\sigma}\underset{\sim}{\lambda}}\boldsymbol{C}
\end{aligned} \tag{3.281}
$$

Based on Reference [112], the choice of $-\infty$ is arbitrary. Some other value could result in a different value of $\boldsymbol{C}$. If it is prescribed that the stress at $t=-\infty$ is finite, then $\boldsymbol{C}$ must be zero. The first term in (3.281) also requires attention, since both the numerator and the denominator go to zero as t goes to $-\infty$. Using L'Hôpital's rule,

$$\lim_{t\to-\infty} {}_d({}_s\boldsymbol{\sigma}^{[0]}) = \lim_{t\to-\infty} \frac{\frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{t/{}^{s\sigma}\underset{\sim}{\lambda}}}{\frac{1}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{t/{}^{s\sigma}\underset{\sim}{\lambda}}} = {}^{\sigma}_{s}\boldsymbol{Q}(-\infty) \tag{3.282}$$

Thus, if ${}^{\sigma}_{s}\boldsymbol{Q}(-\infty)$ is finite, the strain is finite at $t=-\infty$. Hence, (3.281) reduces to

$$ {}_d({}_s\boldsymbol{\sigma}^{[0]}) = \int\limits_{-\infty}^{t} \left(\frac{1}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{-(t-t')/{}^{s\sigma}\underset{\sim}{\lambda}}\right) {}^{\sigma}_{s}\boldsymbol{Q}(t')dt' \tag{3.283}$$

The quantity in parentheses in the integrand in (3.283) is called the *retardation modulus*. When ${}^{\sigma}_{s}\boldsymbol{Q}$ only contains $\dot{\boldsymbol{\varepsilon}}$, as in the case of fluids, the relaxation modulus is deterministic from (3.283). This is omitted here as it requires approximating ${}^{\sigma}_{s}\boldsymbol{Q}$. The retardation modulus is as valid a measure of rheology as the relaxation modulus.

Using similar derivations, the following can be determined from (3.276) and (3.277), first by rewriting them by dividing by ${}^{m}\underset{\sim}{\lambda}$ and ${}^{a\sigma}\underset{\sim}{\lambda}$, respectively

$$\boldsymbol{m}^{[1]} + \frac{1}{{}^{m}\underset{\sim}{\lambda}}(\boldsymbol{m}^{[0]}) = \frac{{}^{m}\boldsymbol{Q}}{{}^{m}\underset{\sim}{\lambda}} \tag{3.284}$$

$${}_a\boldsymbol{\sigma}^{[1]} + \frac{1}{{}^{a\sigma}\underset{\sim}{\lambda}}({}_a\boldsymbol{\sigma}^{[0]}) = \frac{{}^{\sigma}_{a}\boldsymbol{Q}}{{}^{a\sigma}\underset{\sim}{\lambda}} \tag{3.285}$$

and then by following the derivation for ${}_d({}_s\boldsymbol{\sigma}^{[0]})$.

$$\boldsymbol{m}^{[0]} = \int\limits_{-\infty}^{t} \left(\frac{1}{{}^{m}\underset{\sim}{\lambda}}\mathrm{e}^{-(t-t')/{}^{m}\underset{\sim}{\lambda}}\right) {}^{m}\boldsymbol{Q}(t')dt' \tag{3.286}$$

$$ {}_{a}\boldsymbol{\sigma}^{[0]} = \int_{-\infty}^{t} \left(\frac{1}{{}^{\sigma}_{a}\lambda\!\!\!\sim} \mathrm{e}^{-(t-t')/{}^{\sigma}_{a}\underset{\sim}{\lambda}} \right) {}^{\sigma}_{a}\boldsymbol{Q}(t')dt' \tag{3.287} $$

The terms in parentheses in (3.286) and (3.287) are the retardation moduli for $\boldsymbol{m}$ and ${}_{a}\boldsymbol{\sigma}$.

3.6 Constitutive Theories for Heat Vector $\boldsymbol{q}$

Recall the inequality (3.92) resulting from the entropy inequality.

$$ \boldsymbol{q} \cdot \boldsymbol{g} \leq 0 \quad (\text{as } \theta > 0) \tag{3.288} $$

In (3.288), $\boldsymbol{q}$ and $\boldsymbol{g}$ are conjugate. The simplest possible constitutive theory for $\boldsymbol{q}$ can be derived by assuming the $\boldsymbol{q}$ is proportional to $\boldsymbol{g}$ which leads to the following for $\boldsymbol{q}$ [78, 79].

$$ \boldsymbol{q} = -k(\theta)\boldsymbol{g} \tag{3.289} $$

Alternatively if

$$ \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{3.290} $$

then using the representation theorem, $\boldsymbol{g}$ is the only combined generator of $\boldsymbol{g}$ and θ that is a tensor of rank one and

$$ \boldsymbol{q} = -{}^{q}\alpha\boldsymbol{g} \tag{3.291} $$

in which

$$ {}^{q}\alpha = {}^{q}\alpha({}^{q}\underset{\sim}{I}, \theta) \ ; \quad {}^{q}\underset{\sim}{I} = \boldsymbol{g} \cdot \boldsymbol{g} \tag{3.292} $$

${}^{q}\underset{\sim}{I}$ is the only invariant of the argument tensors $\boldsymbol{g}$ and θ. Expanding ${}^{q}\alpha$ in a Taylor series in ${}^{q}\underset{\sim}{I}$ and θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^{q}\underset{\sim}{I}$ and θ, the following holds [78].

$$ \boldsymbol{q} = -k\big|_{\underline{\Omega}}\boldsymbol{g} - k_1\big|_{\underline{\Omega}}\{g\}^T\{g\}\boldsymbol{g} - k_2\big|_{\underline{\Omega}}(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \tag{3.293} $$

where

$$
\begin{aligned}
k\big|_{\underline{\Omega}} &= {}^q\alpha\big|_{\underline{\Omega}} + \left.\frac{\partial\, {}^q\alpha}{\partial\, {}^q\underset{\sim}{I}}\right|_{\underline{\Omega}} (\{g\}^T\{g\})_{\underline{\Omega}} \\
k_1\big|_{\underline{\Omega}} &= \left.\frac{\partial\, {}^q\alpha}{\partial\, {}^q\underset{\sim}{I}}\right|_{\underline{\Omega}} \\
k_2\big|_{\underline{\Omega}} &= \left.\frac{\partial\, {}^q\alpha}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned}
\tag{3.294}
$$

The constitutive theory (3.293) is the simplest possible constitutive theory based on the representation theorem (using (3.290)). The only assumption in this constitutive theory is the truncation of the Taylor series beyond linear terms in ${}^q\underset{\sim}{I}$ and θ. It is based on integrity, hence is complete. Obviously (3.289), the Fourier heat conduction law, is a subset of (3.293) when k is the only material coefficient and it only depends on temperature θ.

Remarks.

(1) If the principle of equipresence holds, then one could possibly consider

$$
\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \boldsymbol{\varepsilon}, {}^{t\Theta}\boldsymbol{J}, {}_a\boldsymbol{r}, \theta) \tag{3.295}
$$

even though (3.92) (or (3.288)) only supports $\boldsymbol{q}$ and $\boldsymbol{g}$ as a conjugate pair. Using (3.295) and the representation theorem, a more comprehensive constitutive theory for $\boldsymbol{q}$ is possible as long as the final constitutive equation for $\boldsymbol{q}$ is not in violation of (3.288).

(2) It is straightforward to verify that the constitutive theory for $\boldsymbol{q}$ in (3.293) satisfies (3.288) when the material coefficients k, k_1, and k_2 are positive.

Chapter 4

Mathematical Models for Fluid Continua

The conservation and balance laws are derived for homogeneous, isotropic fluids. From the energy equation and entropy inequality, energy conjugate pairs are determined. Using the conjugate pairs, constitutive theories are derived for thermoviscous fluids and compared with those presented by Eringen [9, 10, 19, 20] for micropolar fluids. Additionally, constitutive theories for thermoviscoelastic fluids with memory are also derived.

4.1 Conservation and Balance Laws

The non-classical continuum theory presented in this dissertation for fluids incorporates new physics due to internal rotation rates defined by $\bar{\boldsymbol{L}}$ (therefore known in terms of velocity gradients), as well as the Cosserat rotation rates as additional unknown three degrees of freedom at each material point. This new physics is absent in the currently used thermodynamic framework for fluent continua. The influence of this physics on the conservation and balance laws can only be determined by initiating their derivations from the most fundamental stage, as done in classical continuum theories [78,79]. In this process of deriving conservation and balance laws with the new rotation rate physics, it is possible that some conservation and balance laws are not affected, however such conclusions without rigorous derivations are not possible. In the non-classical continuum theory for fluids with velocities, velocity gradients, strain rate tensor, internal and Cosserat rotation rates, and their gradients describing the kinematics of deformation, the following conservation and balance laws describe the physics based on the assumption of thermodynamic equilibrium during the evolution of the deforming matter: (1) conservation of mass, (2) balance of linear momenta, (3) balance of angular momenta, (4) balance of moments of moments, (5) first law of thermodynamics (i.e., balance of energy), and (6) second law of thermodynamics (i.e., entropy inequality). The conservation and balance laws are derived using the basis independent Cauchy stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$ and the convected time derivatives of the basis independent strain tensor [78], as well as the corresponding rotation rates, as these are more general and may easily be substituted for measures in agreement with the physics of the deformed tetrahedron in the current configuration.

Note that in the present work, the Cosserat rotation rate physics is not intrinsically associated with any physical configuration or its deformation at the microscale. With this new rotation rate physics (both internal and Cosserat), the resulting thermodynamic framework must undoubtedly be able to describe deformation physics over and beyond the scope of the classical continuum theories used presently in fluid dynamics. The microfluid theories of Eringen [9, 10, 19, 20] consider physics associated with microfibers. The rotation rates of these microfibers are assumed to be Cosserat rotation rates. It is advocated that the deformation physics associated with these microstructures require *conservation of inertia moments* and *balance of first stress moments*. In the work presented in this dissertation, since there is no physics of microfibers, these additional balance laws cannot be used. However, the presence of internal and Cosserat rotation rates and the conjugate moments in the deforming volume of fluid and the equilibrium of the entire volume of matter do require an additional balance law. This of course is the balance of moments of moments balance law, first presented by Yang, et al. [77]. This balance law is necessitated purely due to rotation rate physics and associated conjugate moments and is not restricted in its form or derivation details by the micromechanical behavior. This balance law ensures that in the presence of rotation rates and associated moments, the entire volume of a body of fluid will remain in equilibrium [103, 104].

4.1.1 Conservation of Mass

The conservation of mass in a deforming volume of fluid leads to the continuity equation that remains the same in the present work as for classical continuum theory [78, 79], and is given in the following using the Eulerian description for compressible fluent continua.

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\boldsymbol{\nabla}} \cdot (\bar{\rho}\, \bar{\boldsymbol{v}}) = 0 \tag{4.1}$$

or

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\, \mathrm{div}(\bar{\boldsymbol{v}}) = 0 \tag{4.2}$$

in which $\bar{\rho}(\bar{\boldsymbol{x}}, t)$ is the density at a material point at $\bar{\boldsymbol{x}}$ in the current configuration.

4.1.2 Balance of Linear Momenta

For a deforming volume of matter, the rate of change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. The derivation is identical to that for the classical continuum theory. Following Reference [78] and using the basis independent Cauchy stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$, the following holds.

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b - \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}} = 0 \tag{4.3}$$

or

$$\bar{\rho}\frac{\partial \bar{v}_i}{\partial t} + \bar{\rho}\bar{v}_j\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{\rho}\bar{F}_i^b - \frac{\partial {}^{(0)}\bar{\sigma}_{ji}}{\partial \bar{x}_j} = 0 \tag{4.4}$$

in which $\bar{\boldsymbol{F}}^b$ are body forces per unit mass and ${}^{(0)}\bar{\boldsymbol{\sigma}}$ is the Cauchy stress tensor. Equations (4.3) or (4.4) are the momentum equations in the x_1, x_2, and x_3 directions.

4.1.3 Balance of Angular Momenta

The principle of balance of angular momenta for a non-classical continuum can be stated as: *The material derivative (time rate of change) of the moments of momenta must be equal to the vector sum of the moments of forces and the moments*. Thus, due to the surface stress $\bar{\boldsymbol{P}}$, total surface moment $\bar{\boldsymbol{M}}$ (per unit area), body force $\bar{\boldsymbol{F}}^b$ (per unit mass), and the momentum $\bar{\rho}\bar{\boldsymbol{v}}d\bar{V}$ for an elemental mass $\bar{\rho}d\bar{V}$ in the current configuration, the following holds in the Eulerian description.

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}}\, d\bar{V} = \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b\, d\bar{V} \tag{4.5}$$

The negative sign for $\bar{\boldsymbol{M}}$ is due to clockwise rotation rates being positive. Consider each term in (4.5) individually. First consider

$$\begin{aligned}
\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}} d\bar{V} &= \frac{D}{Dt}\int\limits_{\bar{V}(t)} \epsilon_{ijk}\bar{x}_i\bar{v}_j\boldsymbol{e}_k\bar{\rho}\, d\bar{V} \\
&= \frac{D}{Dt}\int\limits_{V} \epsilon_{ijk}x_i v_j\boldsymbol{e}_k\rho_0\, dV \\
&= \int\limits_{V} \epsilon_{ijk}\boldsymbol{e}_k\frac{D}{Dt}\left(x_i v_j\right)\rho_0\, dV \\
&= \int\limits_{\bar{V}(t)} \epsilon_{ijk}\boldsymbol{e}_k\frac{D}{Dt}\left(\bar{x}_i\bar{v}_j\right)\bar{\rho}\, d\bar{V} \\
&= \int\limits_{\bar{V}(t)} \epsilon_{ijk}\boldsymbol{e}_k\left(\frac{D\bar{x}_i}{Dt}\bar{v}_j + x_i\frac{D\bar{v}_j}{Dt}\right)\bar{\rho}\, d\bar{V} \\
&= \int\limits_{\bar{V}(t)} \epsilon_{ijk}\boldsymbol{e}_k\left(\bar{v}_i\bar{v}_j + x_i\frac{D\bar{v}_j}{Dt}\right)\bar{\rho}\, d\bar{V} \\
&= \int\limits_{\bar{V}(t)} \epsilon_{ijk}\boldsymbol{e}_k\left(x_i\frac{D\bar{v}_j}{Dt}\right)\bar{\rho}\, d\bar{V}\ ; \quad \epsilon_{ij}\bar{v}_i\bar{v}_j = 0
\end{aligned} \tag{4.6}$$

Consider the first term on the right side of (4.5). Using the basis independent Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$

$$
\begin{aligned}
\int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} &= \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \left({}^{(0)}\bar{\boldsymbol{\sigma}}\right)^T \cdot \bar{\boldsymbol{n}} - \left({}^{(0)}\bar{\boldsymbol{m}}\right)^T \cdot \bar{\boldsymbol{n}}\right) d\bar{A} \\
&= \int\limits_{\partial\bar{V}(t)} \boldsymbol{e}_k \left(\epsilon_{ijk}\bar{x}_i{}^{(0)}\bar{\sigma}_{mj}\bar{n}_m - {}^{(0)}\bar{m}_{mk}\bar{n}_m\right) d\bar{A} \\
&= \int\limits_{\bar{V}(t)} \boldsymbol{e}_k \left(\epsilon_{ijk}\left(\bar{x}_i{}^{(0)}\bar{\sigma}_{mj}\right)_{,m} - {}^{(0)}\bar{m}_{mk,m}\right) d\bar{V} \\
&= \int\limits_{\bar{V}(t)} \boldsymbol{e}_k \left(\epsilon_{ijk}\left(\bar{x}_{i,m}{}^{(0)}\bar{\sigma}_{mj} + \bar{x}_i{}^{(0)}\bar{\sigma}_{mj,m}\right) - {}^{(0)}\bar{m}_{mk,m}\right) d\bar{V} \\
&= \int\limits_{\bar{V}(t)} \boldsymbol{e}_k \left(\epsilon_{ijk}\left({}^{(0)}\bar{\sigma}_{ij} + \bar{x}_i{}^{(0)}\bar{\sigma}_{mj,m}\right) - {}^{(0)}\bar{m}_{mk,m}\right) d\bar{V}
\end{aligned}
\tag{4.7}
$$

Next consider the second term on the right side (4.5).

$$
\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b d\bar{V} = \int\limits_{\bar{V}(t)} \boldsymbol{e}_k\epsilon_{ijk}\bar{x}_i\bar{F}_j^b\bar{\rho}\, d\bar{V} \tag{4.8}
$$

Substituting from (4.6), (4.7), and (4.8) into (4.5)

$$
\begin{aligned}
&\int\limits_{\bar{V}(t)} \boldsymbol{e}_k\epsilon_{ijk}\left(\bar{x}_i\frac{D\bar{v}_j}{Dt}\right)\bar{\rho}\, d\bar{V} \\
&\qquad = \int\limits_{\bar{V}(t)} \boldsymbol{e}_k \left(\epsilon_{ijk}\left({}^{(0)}\bar{\sigma}_{ij} + x_i{}^{(0)}\bar{\sigma}_{mj,m}\right) - {}^{(0)}\bar{m}_{mk,m}\right) d\bar{V} + \int\limits_{\bar{V}(t)} \boldsymbol{e}_k\epsilon_{ijk}\bar{x}_i\bar{F}_j^b\bar{\rho}\, dV
\end{aligned}
\tag{4.9}
$$

or

$$
\int\limits_{\bar{V}(t)} \boldsymbol{e}_k\epsilon_{ijk}\bar{x}_i\left(\bar{\rho}\frac{D\bar{v}_j}{Dt} - \bar{\rho}\bar{F}_j^b - {}^{(0)}\bar{\sigma}_{mk,m}\right) d\bar{V} + \int\limits_{\bar{V}(t)} \boldsymbol{e}_k\left({}^{(0)}\bar{m}_{mk,m} - \epsilon_{ijk}{}^{(0)}\bar{\sigma}_{ij}\right) d\bar{V} = 0 \tag{4.10}
$$

Using the balance of linear momenta (4.4), (4.10) reduces to

$$\int_{\bar{V}(t)} \boldsymbol{e}_k \left({}^{(0)}\bar{m}_{mk,m} - \epsilon_{ijk} {}^{(0)}\bar{\sigma}_{ij} \right) d\bar{V} = 0 \tag{4.11}$$

Since the volume $\bar{V}$ is arbitrary,

$$ {}^{(0)}\bar{m}_{mk,m} - \epsilon_{ijk} {}^{(0)}\bar{\sigma}_{ij} = 0 \tag{4.12}$$

or

$$\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}} - \boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} = 0 \tag{4.13}$$

Equation (4.13) represents the balance of angular momenta. The Cauchy stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$ is non-symmetric and so is the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$.

Remarks.

(1) From the balance of angular momenta (4.13), note that the antisymmetric components of the Cauchy stress tensor are balanced by the gradients of the Cauchy moment tensor (non-symmetric at this stage).

(2) In the case of classical continuum theories for fluids, the balance of angular momenta is a statement of self-equilibrating moments due to symmetry of the Cauchy stress tensor.

$$\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} = 0 \tag{4.14}$$

(3) In the case of non-classical fluent continua, the existence of the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ due to material constitution resisting the rotation rates, both internal and Cosserat, results in shear stress couples from the antisymmetric part of the Cauchy stress tensor that are balanced by internal moments. Thus, for a non-classical continuum theory, the balance of angular momenta yields (4.13) instead of (4.14), which only holds in the case of classical continuum

theories.

(4) It is important to point out that the theory presented here does not assume the existence of the Cauchy moment ${}^{(0)}\bar{\boldsymbol{m}}$. The non-classical continuum theory presented in this dissertation demonstrates that varying rotation rates, both internal and Cosserat, at neighboring material points, when resisted by the deforming fluent continua necessitate the existence of the Cauchy moment tensor. The balance of angular momenta establishes a relationship between ${}^{(0)}\bar{\boldsymbol{m}}$ and ${}^{(0)}\bar{\boldsymbol{\sigma}}$.

4.1.4 Balance of Moments of Moments

As discussed in detail in Chapters 1 and 3, non-classical continuum theories incorporating polar physics require this additional balance law. Consider the current configuration at time t in the Eulerian description. For the deforming volume of fluid to be in equilibrium, the moments of moments must vanish. In the moments of moments balance law, consider $\bar{\boldsymbol{M}}$ and the antisymmetric components of the stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$, i.e., $\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}$. Thus, the following holds (neglecting inertial terms) in the Eulerian description.

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times (\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}})\, d\bar{V} - \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} = 0 \tag{4.15}$$

Expand the second term in (4.15) and then convert the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using the divergence theorem.

$$\begin{aligned}\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} &= \int_{\partial\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} x_i \bar{M}_j\, d\bar{A} \\ &= \int_{\partial\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{x}_i {}^{(0)}\bar{m}_{mj} \bar{n}_m\, d\bar{A} \\ &= \int_{\bar{V}} \boldsymbol{e}_k (\epsilon_{ijk} \bar{x}_i {}^{(0)}\bar{m}_{mj})_{,m}\, d\bar{V}\end{aligned} \tag{4.16}$$

$$
\begin{aligned}
&= \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} ({}^{(0)}\bar{m}_{ij} + \bar{x}_i {}^{(0)}\bar{m}_{mj,m}) \, d\bar{V} \\
&= \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} {}^{(0)}\bar{m}_{ij} \, d\bar{V} + \int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}} \right) \, d\bar{V}
\end{aligned}
\tag{4.16}
$$

Using equation (4.16) in (4.15) and collecting terms

$$
\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(-\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}} + \boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) d\bar{V} - \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} {}^{(0)}\bar{m}_{ij} \, d\bar{V} = 0 \tag{4.17}
$$

The first term in (4.17) vanishes due to the balance of angular momenta (4.13), giving

$$
\int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} {}^{(0)}\bar{m}_{ij} \, d\bar{V} = 0 \tag{4.18}
$$

and since $\bar{V}$ is arbitrary, the final form is given by

$$
\epsilon_{ijk} {}^{(0)}\bar{m}_{ij} = 0 \tag{4.19}
$$

Equation (4.73) implies that the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ is symmetric. Thus, in the non-classical continuum theory presented here for fluids, the Cauchy moment tensor is symmetric but the Cauchy stress tensor is non-symmetric. In the classical continuum theory, the Cauchy stress tensor is symmetric and the Cauchy moment tensor does not exist as the rotation rates are not considered in the theory. Note here also, as in Chapter 3, that in most reported works on non-classical theories (specifically for solids), except Yang, et al. [77], this balance law is not considered. As a consequence the Cauchy moment tensor remains non-symmetric, requiring additional constitutive theories for the non-symmetric part of the moment tensor. However, the constitutive theory for the symmetric part of the Cauchy moment tensor remains the same regardless of whether one uses the balance of moments of moments as a balance law. This dissertation considers symmetric ${}^{(0)}\bar{\boldsymbol{m}}$, although a constitutive theory for non-symmetric ${}^{(0)}\bar{\boldsymbol{m}}$ (in the absence of the balance of moments of moments) is presented to compare with the work of Eringen [9, 10, 19, 20].

4.1.5 First Law of Thermodynamics

The sum of work and heat added to a deforming volume of matter results in a change of the total energy of the system. This is expressed as a rate equation in the Eulerian description in the following.

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{4.20}$$

$\bar{E}_t$, $\bar{Q}$, and $\bar{W}$ are the total energy, heat added, and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} \tag{4.21}$$

$$\frac{D\bar{Q}}{Dt} = -\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}\,d\bar{A} \tag{4.22}$$

$$\frac{D\bar{W}}{Dt} = \int\limits_{\partial\bar{V}(t)} (\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot{}_t^t\bar{\boldsymbol{\Theta}})\,d\bar{A} \tag{4.23}$$

where $\bar{e}$ is the specific internal energy, $\bar{\boldsymbol{F}}^b$ is the body force vector per unit mass, and $\bar{\boldsymbol{q}}$ is the rate of heat. Note that the additional term $\bar{\boldsymbol{M}}\cdot{}_t^t\bar{\boldsymbol{\Theta}}$ in $\frac{D\bar{W}}{Dt}$ contributes additional rate of work due to rates of total rotations ${}_t^t\bar{\boldsymbol{\Theta}}$. Expanding the integrals and following Reference [78], the following can be derived.

$$\begin{aligned}\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} &= \int\limits_{V}\left(\rho_0\frac{De}{Dt} + \rho_0\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt} - \rho_0\boldsymbol{F}^b\cdot\boldsymbol{v}\right)dV \\ &= \int\limits_{\bar{V}(t)}\left(\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\rho}\bar{\boldsymbol{v}}\cdot\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{v}}\right)d\bar{V}\end{aligned} \tag{4.24}$$

$$-\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}\,d\bar{A} = -\int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}\,d\bar{V} \tag{4.25}$$

$$\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}}\,d\bar{A} = \bar{\boldsymbol{v}}\cdot({}^{(0)}\bar{\boldsymbol{\sigma}})^T\cdot\bar{\boldsymbol{n}}\,d\bar{A} = (\bar{\boldsymbol{v}}\cdot({}^{(0)}\bar{\boldsymbol{\sigma}})^T)\cdot d\bar{\boldsymbol{A}} \tag{4.26}$$

$$\bar{\boldsymbol{M}}\cdot{}_t^t\bar{\boldsymbol{\Theta}}\,d\bar{A} = {}_t^t\bar{\boldsymbol{\Theta}}\cdot({}^{(0)}\bar{\boldsymbol{m}})^T\cdot\bar{\boldsymbol{n}}\,d\bar{A} = ({}_t^t\bar{\boldsymbol{\Theta}}\cdot({}^{(0)}\bar{\boldsymbol{m}})^T)\cdot d\bar{\boldsymbol{A}} \tag{4.27}$$

Thus, the following holds for (4.20).

$$\int\limits_{\bar{V}(t)} \left(\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\rho}\bar{\boldsymbol{v}} \cdot \frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{v}} \right) dV$$

$$= - \int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}}\, d\bar{V} + \int\limits_{\partial\bar{V}(t)} (\bar{\boldsymbol{v}} \cdot ({}^{(0)}\bar{\boldsymbol{\sigma}})^T) \cdot d\bar{\boldsymbol{A}} + \int\limits_{\partial\bar{V}(t)} ({}_t^t\bar{\boldsymbol{\Theta}} \cdot ({}^{(0)}\bar{\boldsymbol{m}})^T) \cdot d\bar{\boldsymbol{A}} \tag{4.28}$$

$$= - \int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}}\, d\bar{V} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}} \cdot (\bar{\boldsymbol{v}} \cdot ({}^{(0)}\bar{\boldsymbol{\sigma}})^T) d\bar{V} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}} \cdot ({}_t^t\bar{\boldsymbol{\Theta}} \cdot ({}^{(0)}\bar{\boldsymbol{m}})^T) d\bar{V}$$

Following Reference [78],

$$\begin{aligned} \bar{\boldsymbol{\nabla}} \cdot \left(\bar{\boldsymbol{v}} \cdot ({}^{(0)}\bar{\boldsymbol{\sigma}})^T\right) &= \bar{\boldsymbol{v}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}}\right) + {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} \\ \bar{\boldsymbol{\nabla}} \cdot \left({}_t^t\bar{\boldsymbol{\Theta}} \cdot ({}^{(0)}\bar{\boldsymbol{m}})^T\right) &= {}_t^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) + {}^{(0)}\bar{m}_{ji}\frac{{}_t^t\bar{\Theta}_i}{\partial \bar{x}_j} \end{aligned} \tag{4.29}$$

Substituting from (4.29) into (4.28)

$$\int\limits_{\bar{V}(t)} \left(\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\rho}\bar{\boldsymbol{v}} \cdot \frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{v}} \right) dV$$

$$= - \int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}}\, d\bar{V} + \int\limits_{\bar{V}(t)} \left(\bar{\boldsymbol{v}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}}\right) + {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} + {}_t^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) + {}^{(0)}\bar{m}_{ji}\frac{\partial {}_t^t\bar{\Theta}_i}{\partial \bar{x}_j} \right) d\bar{V} \tag{4.30}$$

Using (4.4) (the balance of linear momenta), (4.30) reduces to

$$\int\limits_{\bar{V}(t)} \left(\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - {}^{(0)}\bar{m}_{ji}\frac{\partial {}_t^t\bar{\Theta}_i}{\partial \bar{x}_j} - {}_t^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) \right) dV = 0 \tag{4.31}$$

Since volume $\bar{V}$ is arbitrary, the following holds.

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \left({}^{(0)}\bar{m}_{ji}\frac{\partial {}_t^t\bar{\Theta}_i}{\partial \bar{x}_j} + {}_t^t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) \right) = 0 \tag{4.32}$$

Note that in the term ${}^{t}_{t}\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right)$, $\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}$ can be substituted from the balance of angular momenta (4.13), thereby eliminating gradients of ${}^{(0)}\bar{\boldsymbol{m}}$ but instead introducing the Cauchy stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$.

4.1.6 Second Law of Thermodynamics

Let $\bar{\eta}$ be the entropy density in the deformed volume $\bar{V}(t)$, $\bar{h}$ be the entropy flux between $\bar{V}(t)$ and the volume of matter surrounding it (i.e., contacting sources), and $\bar{s}$ be the source of entropy in $\bar{V}(t)$ due to non-contacting bodies. Then the rate of increase of entropy in the volume $\bar{V}(t)$ is at least equal to that supplied to $\bar{V}(t)$ from all contacting and non-contacting sources [78]. Thus

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho}\, d\bar{V} \geq \int\limits_{\partial\bar{V}(t)} \bar{h}\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\, d\bar{V} \tag{4.33}$$

Cauchy's postulate for $\bar{h}$ can be stated as

$$\bar{h} = -\bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \tag{4.34}$$

Using (4.34) in (4.33)

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\eta}\bar{\rho}\, d\bar{V} \geq -\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\, d\bar{V} \tag{4.35}$$

Using $\bar{\rho}\, d\bar{V} = \rho_0\, dV$ in the left hand side of (4.35)

$$\frac{D}{Dt}\int\limits_{V} \eta\rho_0\, dV \geq -\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\, d\bar{V} \tag{4.36}$$

$$\int\limits_{V} \frac{D\eta}{Dt}\rho_0\, dV \geq -\int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\, d\bar{V} \tag{4.37}$$

$$\int\limits_{\bar{V}(t)} \bar{\rho}\frac{D\bar{\eta}}{Dt}\,d\bar{V} \geq -\int\limits_{\bar{V}(t)} \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\psi}}\,d\bar{V} + \int\limits_{\bar{V}(t)} \bar{s}\bar{\rho}\,d\bar{V} \tag{4.38}$$

$$\int\limits_{\bar{V}(t)} \left(\bar{\rho}\frac{D\bar{\eta}}{Dt} + \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\psi}} - \bar{s}\bar{\rho}\right) d\bar{V} \geq 0 \tag{4.39}$$

Since the volume $\bar{V}$ is arbitrary,

$$\bar{\rho}\frac{D\bar{\eta}}{Dt} + \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\psi}} - \bar{s}\bar{\rho} \geq 0 \tag{4.40}$$

Inequality (4.40), referred to as the entropy inequality, is the most fundamental form resulting from the second law of thermodynamics.

An alternate form of the entropy inequality is possible using a relationship between $\bar{\boldsymbol{\psi}}$ and $\bar{\boldsymbol{q}}$ and the energy equation. Since the energy equation has all possible mechanisms that result in energy storage and dissipation, this form of the entropy inequality derived using the energy equation is expected to be helpful in the derivations of constitutive theories. Using

$$\bar{\boldsymbol{\psi}} = \frac{\bar{\boldsymbol{q}}}{\bar{\theta}}\,, \qquad \bar{s} = \frac{\bar{r}}{\bar{\theta}} \tag{4.41}$$

where $\bar{\theta}$ is absolute temperature, $\bar{\boldsymbol{q}}$ is the heat vector, and $\bar{r}$ is a suitable potential, then

$$\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\psi}} = \bar{\psi}_{i,i} = \frac{\bar{q}_{i,i}}{\bar{\theta}} - \frac{\bar{q}_i\bar{\theta}_{,i}}{\bar{\theta}^2} = \frac{\bar{q}_{i,i}}{\bar{\theta}} - \frac{\bar{q}_i\bar{g}_i}{\bar{\theta}^2} = \frac{\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}}{\bar{\theta}} - \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}^2} \tag{4.42}$$

Substituting from (4.41) and (4.42) into (4.40) and multiplying throughout by $\bar{\theta}$ yields

$$\bar{\rho}\bar{\theta}\frac{D\bar{\eta}}{Dt} + \left(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}} - \bar{\rho}\bar{r}\right) - \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} \geq 0 \tag{4.43}$$

From the energy equation (4.32) (after inserting the $\bar{\rho}\bar{r}$ term)

$$\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}} - \bar{\rho}\bar{r} = -\bar{\rho}\frac{D\bar{e}}{Dt} + {}^{(0)}\bar{\sigma}_{ji}\frac{\partial\bar{v}_i}{\partial\bar{x}_j} + {}^{(0)}\bar{m}_{ji}\frac{{}_t^t\bar{\Theta}_i}{\partial\bar{x}_j} + {}_t^t\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\boldsymbol{\nabla}}\cdot{}^{(0)}\bar{\boldsymbol{m}}\right) \tag{4.44}$$

Substituting (4.44) in (4.43)

$$\bar{\rho}\bar{\theta}\frac{D\bar{\eta}}{Dt} - \bar{\rho}\frac{D\bar{e}}{Dt} + {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} + {}^{(0)}\bar{m}_{ji}\frac{\partial\, {}^t_t\bar{\Theta}_i}{\partial \bar{x}_j} + {}^t_t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) - \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \geq 0 \tag{4.45}$$

or

$$\bar{\rho}\left(\frac{D\bar{e}}{Dt} - \bar{\theta}\frac{D\bar{\eta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - {}^{(0)}\bar{m}_{ji}\frac{\partial\, {}^t_t\bar{\Theta}_i}{\partial \bar{x}_j} - {}^t_t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) \leq 0 \tag{4.46}$$

Let $\bar{\Phi}$ be the Helmholtz free energy density defined by

$$\bar{\Phi} = \bar{e} - \bar{\eta}\bar{\theta} \tag{4.47}$$

Then

$$\frac{D\bar{e}}{Dt} - \bar{\theta}\frac{D\bar{\eta}}{Dt} = \frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt} \tag{4.48}$$

Substituting (4.48) into (4.46) gives

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - {}^{(0)}\bar{m}_{ji}\frac{\partial\, {}^t_t\bar{\Theta}_i}{\partial \bar{x}_j} - {}^t_t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) \leq 0 \tag{4.49}$$

or

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - \operatorname{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left[\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[\frac{\partial\{{}^t_t\bar{\Theta}\}}{\partial\{\bar{x}\}}\right]\right) - {}^t_t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) \leq 0 \tag{4.50}$$

or

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - \operatorname{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left[\bar{L}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^t_t\bar{\Theta}\bar{J}\right]\right) - {}^t_t\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) \leq 0 \tag{4.51}$$

in which $\left[{}^t_t\bar{\Theta}\bar{J}\right]$ is the gradient of total rotation rates. Inequality (4.51) resulting from the second law of thermodynamics is the most fundamental form of the entropy inequality in the Helmholtz free energy density $\bar{\Phi}$. A slightly more expanded form of (4.51) that is more useful in the derivations of the constitutive theories can be derived using the decomposition of various tensors in (4.51) into symmetric and antisymmetric tensors, presented in a later section.

Noting that

$$ {}_t^t\bar{\boldsymbol{\Theta}} = {}_i^t\bar{\boldsymbol{\Theta}} - {}_e^t\bar{\boldsymbol{\Theta}} \tag{4.52} $$

$$ \left[{}_t^t\bar{\Theta}\bar{J}\right] = \left[{}_i^t\bar{\Theta}\bar{J}\right] - \left[{}_e^t\bar{\Theta}\bar{J}\right] = \left[\frac{\partial\{{}_i^t\bar{\Theta}\}}{\partial\{\bar{x}\}}\right] - \left[\frac{\partial\{{}_e^t\bar{\Theta}\}}{\partial\{\bar{x}\}}\right] \tag{4.53} $$

and from the balance of angular momenta (4.13)

$$ \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}} = \boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \tag{4.54} $$

Recall (2.40) and (2.41)

$$ [\bar{L}] = [\bar{\mathbb{L}}] + [{}_e\bar{W}] \tag{4.55} $$

Using (4.52) – (4.55) in (4.51),

$$ \bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - \operatorname{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left([\bar{\mathbb{L}}] + [{}_e\bar{W}]\right)\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}_t^t\bar{\Theta}\bar{J}\right]\right) \\ - \left({}_i^t\bar{\boldsymbol{\Theta}} - {}_e^t\bar{\boldsymbol{\Theta}}\right) \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \tag{4.56} $$

or

$$ \bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - \operatorname{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left[\bar{\mathbb{L}}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left[{}_e\bar{W}\right]\right) \\ - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}_t^t\bar{\Theta}\bar{J}\right]\right) - {}_i^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) + {}_e^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \tag{4.57} $$

A simple calculation shows that

$$ \operatorname{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left[{}_e\bar{W}\right]\right) = {}_e^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \tag{4.58} $$

Using (4.58) in (4.57), (4.57) reduces to

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}-\operatorname{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left[\bar{\mathbb{L}}\right]\right)-\operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{t}\bar{J}\right]\right)-{}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}\right)\leq 0 \tag{4.59}$$

Inequality (4.59) is the desired form of the entropy inequality that is useful in deriving constitutive theories. By making similar substitutions and simplifications, the energy equation (4.32) can be written as

$$\bar{\rho}\frac{D\bar{e}}{Dt}+\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}-\operatorname{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left[\bar{\mathbb{L}}\right]\right)-\operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{t}\bar{J}\right]\right)-{}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}\right)=0 \tag{4.60}$$

4.2 Rate of Work Conjugate Pairs in the Entropy Inequality

As well known, determination of conjugate pairs using either the energy equation or the entropy inequality is essential in deriving constitutive theories. From the entropy inequality it appears that $\left([{}^{(0)}\bar{\sigma}],[\bar{\mathbb{L}}]\right)$ and $\left([{}^{(0)}\bar{m}],[{}^{t\bar{\Theta}}_{t}\bar{J}]\right)$ are rate of work conjugate pairs. The additional term ${}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}\right)$ also needs to be accounted for. Additionally, $(\bar{\boldsymbol{q}},\bar{\boldsymbol{g}})$ appears to be a conjugate pair. Once the conjugate pairs are established from the energy equation or the entropy inequality, the constitutive theories can be derived using the representation theorem in conjunction with the entropy inequality.

Note that $[{}^{(0)}\bar{\sigma}]$ and $[\bar{\mathbb{L}}]$ are both non-symmetric tensors whereas $[{}^{(0)}\bar{m}]$ is a symmetric tensor but $[{}^{t\bar{\Theta}}_{t}\bar{J}]$ is a non-symmetric tensor. Whether $\left([{}^{(0)}\bar{\sigma}],[\bar{\mathbb{L}}]\right)$ and $\left([{}^{(0)}\bar{m}],[{}^{t\bar{\Theta}}_{t}\bar{J}]\right)$ are actually rate of work conjugate must be established.

Consider the entropy inequality (4.59). Decompose ${}^{(0)}\bar{\boldsymbol{\sigma}}$ into symmetric (${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$) and antisymmetric (${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$) tensors and use $\bar{\mathbb{L}}$ from (2.41). Also decompose ${}^{t\bar{\Theta}}_{t}\bar{\boldsymbol{J}}$ into symmetric (${}^{t\bar{\Theta}}_{s}\bar{\boldsymbol{J}}$) and antisymmetric (${}^{t\bar{\Theta}}_{a}\bar{\boldsymbol{J}}$) tensors and substitute these into the entropy inequality (4.59).

$${}^{(0)}\bar{\boldsymbol{\sigma}}={}^{(0)}_{s}\bar{\boldsymbol{\sigma}}+{}^{(0)}_{a}\bar{\boldsymbol{\sigma}} \tag{4.61}$$

$${}^{t\bar{\Theta}}_{t}\bar{\boldsymbol{J}}={}^{t\bar{\Theta}}_{s}\bar{\boldsymbol{J}}+{}^{t\bar{\Theta}}_{a}\bar{\boldsymbol{J}} \tag{4.62}$$

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}-\text{tr}\left(\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]+\left[{}^{(0)}_{a}\bar{\sigma}\right]\right)\left(\left[\bar{D}\right]+\left[{}_{t}\bar{W}\right]\right)\right)$$

$$-\text{tr}\left(\left[{}^{(0)}\bar{m}\right]\left(\left[{}^{t\bar{\Theta}}_{t\,s}\bar{J}\right]+\left[{}^{t\bar{\Theta}}_{t\,a}\bar{J}\right]\right)\right)-{}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}\right)\leq 0 \quad (4.63)$$

Since

$$\text{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[{}_{t}\bar{W}\right]\right)=0 \quad (4.64)$$

$$\text{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[\bar{D}\right]\right)=0 \quad (4.65)$$

$$\text{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{t\,a}\bar{J}\right]\right)=0 \quad (4.66)$$

the entropy inequality (4.63) can be written as

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}-\text{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right)-\text{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[{}_{t}\bar{W}\right]\right)$$

$$-\text{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{t\,s}\bar{J}\right]\right)-{}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}\right)\leq 0 \quad (4.67)$$

In (4.67), $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{g}}$, $[{}^{(0)}_{s}\bar{\sigma}]$ and $[\bar{D}]$, $[{}^{(0)}_{a}\bar{\sigma}]$ and $[{}_{t}\bar{W}]$, and $[{}^{(0)}\bar{m}]$ and $[{}^{t\bar{\Theta}}_{t\,s}\bar{J}]$ are the rate of work conjugate pairs that are in conformity with the works of Spencer, Wang, Zheng, etc. [80–99], i.e., symmetric tensors are conjugate with symmetric tensors and antisymmetric tensors are conjugate with antisymmetric tensors. Using (4.61), (4.62), and (4.64) – (4.66), the energy equation can also be written as

$$\bar{\rho}\frac{D\bar{e}}{Dt}+\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}-\text{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right)-\text{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[{}_{t}\bar{W}\right]\right)-\text{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{t\,s}\bar{J}\right]\right)-{}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}\right)=0 \quad (4.68)$$

From the conjugate pairs it is straightforward to conclude that for the simplest possible case

$$\begin{aligned}
{}^{(0)}_{s}\bar{\boldsymbol{\sigma}} &= {}^{(0)}_{s}\bar{\boldsymbol{\sigma}}(\bar{\boldsymbol{D}},\bar{\theta})\\
{}^{(0)}_{a}\bar{\boldsymbol{\sigma}} &= {}^{(0)}_{a}\bar{\boldsymbol{\sigma}}({}_{t}\bar{\boldsymbol{W}},\bar{\theta})\\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}({}^{t\bar{\Theta}}_{t\,s}\bar{\boldsymbol{J}},\bar{\theta})
\end{aligned} \quad (4.69)$$

Furthermore, this indicates that ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ is not a function of ${}_{t}\bar{\boldsymbol{W}}$ and ${}^{t\bar{\Theta}}_{s}\bar{\boldsymbol{J}}$, ${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$ is not a function of $\bar{\boldsymbol{D}}$ and ${}^{t\bar{\Theta}}_{s}\bar{\boldsymbol{J}}$, and ${}^{(0)}\bar{\boldsymbol{m}}$ is not a function of $\bar{\boldsymbol{D}}$ and ${}_{t}\bar{\boldsymbol{W}}$.

4.2.1 Summary of the Conservation and Balance Laws

$$\frac{\partial\bar{\rho}}{\partial t} + \bar{\boldsymbol{\nabla}}\cdot(\bar{\rho}\bar{\boldsymbol{v}}) = 0 \tag{4.70}$$

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^{b} - \bar{\boldsymbol{\nabla}}\cdot{}^{(0)}\bar{\boldsymbol{\sigma}} = 0 \tag{4.71}$$

$$\bar{\boldsymbol{\nabla}}\cdot{}^{(0)}\bar{\boldsymbol{m}} - \boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} = 0 \tag{4.72}$$

$$\epsilon_{ijk}{}^{(0)}\bar{m}_{ij} = 0 \tag{4.73}$$

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}} - \operatorname{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[{}_{t}\bar{W}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) = 0 \tag{4.74}$$

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} - \operatorname{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[{}_{t}\bar{W}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \tag{4.75}$$

In the mathematical model (4.70) – (4.75), the dependent variables are: $\bar{\boldsymbol{v}}$ (3), ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ (6), ${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$ (3), ${}^{(0)}\bar{\boldsymbol{m}}$ (6), $\bar{\boldsymbol{q}}$ (3), $\bar{\theta}$ (1), and ${}^{t}_{e}\bar{\boldsymbol{\Theta}}$ (3), a total of 25. $\bar{\Phi}$, $\bar{e}$, and $\bar{\eta}$ are not dependent variables as it can be shown that these are deterministic from others. The numbers in brackets refer to the number of variables. The equations in the model are: linear momentum (3), angular momentum (3), energy (1), constitutive theories for ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ (6), ${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$ (3), ${}^{(0)}\bar{\boldsymbol{m}}$ (6), and $\bar{\boldsymbol{q}}$ (3), totaling 25, hence the mathematical model has closure.

4.3 Constitutive Theories for Thermoviscous Fluids

In the case of thermoviscous fluids there is energy storage as well as dissipation. Dissipation results in entropy production. While stored energy by definition is recoverable, the rate of mechanical work resulting into rate of entropy production is not recoverable. Thus, the approach for deriving constitutive theories for thermoviscous fluids is quite different than for thermoelastic solids.

4.3.1 Dependent Variables in the Constitutive Theories

It is straightforward to conclude from the conservation and balance laws that $\bar{\Phi}$, $\bar{\eta}$, ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}_{a}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and $\bar{\boldsymbol{q}}$ are possible dependent variables in the constitutive theories. For compressible fluid physics, density must be incorporated as an argument of all dependent variables in the constitutive theories. Note that compressibility is due to the determinant of the Jacobian of deformation $|J| = \left|\frac{\partial\{\bar{\boldsymbol{x}}\}}{\partial\{\boldsymbol{x}\}}\right|$. Recall that in the Lagrangian description (from continuity) $\rho_0 = |J|\rho(\boldsymbol{x},t)$, hence $|J| = \frac{\rho_0}{\rho(\boldsymbol{x},t)}$ in which ρ_0 is density in the reference configuration (constant), i.e., instead of $|J|$, $\frac{1}{\rho(\boldsymbol{x},t)}$ in the Lagrangian description or $\frac{1}{\bar{\rho}(\bar{\boldsymbol{x}},t)}$ in the Eulerian description can be used as an argument of all dependent variables in the constitutive theories. At later stages $\frac{1}{\bar{\rho}(\bar{\boldsymbol{x}},t)}$ can be replaced by $\bar{\rho}(\bar{\boldsymbol{x}},t)$ using simple calculus. Temperature $\bar{\theta}$ is certainly a valid choice for thermoviscous behavior. From (4.75), note that $\bar{\boldsymbol{D}}$, ${}_{t}\bar{\boldsymbol{W}}$, ${}_{s}^{t}{}_{t}^{\Theta}\bar{\boldsymbol{J}}$, and $\bar{\boldsymbol{g}}$ are natural choices of argument tensors for ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}_{a}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and $\bar{\boldsymbol{q}}$ dependent variables in the constitutive theories, respectively.

Additionally $\bar{\boldsymbol{D}}$, ${}_{t}\bar{\boldsymbol{W}}$, ${}_{s}^{t}{}_{t}^{\Theta}\bar{\boldsymbol{J}}$, $\bar{\boldsymbol{g}}$, and $\bar{\theta}$ all must be considered as argument tensors of $\bar{\Phi}$ and $\bar{\eta}$ due to the axiom of equipresence. Thus, at this stage the dependent variables in the constitutive theories and their argument tensors consist of the following.

$$
\begin{aligned}
\bar{\Phi} &= \bar{\Phi}(1/\bar{\rho}, \bar{\boldsymbol{D}}, {}_{t}\bar{\boldsymbol{W}}, {}_{s}^{t}{}_{t}^{\Theta}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta}) \\
\bar{\eta} &= \bar{\eta}(1/\bar{\rho}, \bar{\boldsymbol{D}}, {}_{t}\bar{\boldsymbol{W}}, {}_{s}^{t}{}_{t}^{\Theta}\bar{\boldsymbol{J}}, \bar{\boldsymbol{g}}, \bar{\theta})
\end{aligned}
\tag{4.76}
$$

$$
\begin{aligned}
{}^{(0)}_{s}\bar{\boldsymbol{\sigma}} &= {}^{(0)}_{s}\bar{\boldsymbol{\sigma}}({}^{1}\!/\!{}_{\bar{\rho}}, \bar{\boldsymbol{D}}, \bar{\theta}) \\
{}^{(0)}_{a}\bar{\boldsymbol{\sigma}} &= {}^{(0)}_{a}\bar{\boldsymbol{\sigma}}({}^{1}\!/\!{}_{\bar{\rho}}, {}_{t}\bar{\boldsymbol{W}}, \bar{\theta}) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}({}^{1}\!/\!{}_{\bar{\rho}}, {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}, \bar{\theta}) \\
\bar{\boldsymbol{q}} &= \bar{\boldsymbol{q}}({}^{1}\!/\!{}_{\bar{\rho}}, \bar{\boldsymbol{g}}, \bar{\theta})
\end{aligned}
\tag{4.76}
$$

4.3.2 Entropy Inequality: Further Considerations

Using $\bar{\Phi}$ in (4.76), the material derivative of $\bar{\Phi}$ needed in (4.75) is defined.

$$
\frac{D\bar{\Phi}}{Dt} = \dot{\bar{\Phi}} = \frac{\partial \bar{\Phi}}{\partial ({}^{1}\!/\!{}_{\bar{\rho}})}\left(-\frac{1}{\bar{\rho}^2}\right)\dot{\bar{\rho}} + \frac{\partial \bar{\Phi}}{\partial \bar{D}_{ik}}\dot{\bar{D}}_{ik} + \frac{\partial \bar{\Phi}}{\partial_t \bar{W}_{ik}}{}_{t}\dot{\bar{W}}_{ik} + \frac{\partial \bar{\Phi}}{\partial {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{J}_{ik}}{}^{t}_{t}{}^{\bar{\Theta}}_{s}\dot{\bar{J}}_{ik} + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i}\dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}}\dot{\bar{\theta}} \tag{4.77}
$$

From the continuity equation (4.1)

$$
\frac{D\bar{\rho}}{Dt} = \dot{\bar{\rho}} = -\bar{\rho}\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}} = -\bar{\rho}\bar{D}_{kk} = -\bar{\rho}\bar{D}_{ki}\delta_{ik} \tag{4.78}
$$

and

$$
-\frac{\partial \bar{\Phi}}{\partial ({}^{1}\!/\!{}_{\bar{\rho}})} = \bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \tag{4.79}
$$

Using (4.78) and (4.79) in (4.77)

$$
\frac{D\bar{\Phi}}{Dt} = \dot{\bar{\Phi}} = \bar{\rho}\frac{\partial \bar{\Phi}}{\partial \bar{\rho}}\bar{D}_{ik}\delta_{ki} + \frac{\partial \bar{\Phi}}{\partial \bar{D}_{ik}}\dot{\bar{D}}_{ik} + \frac{\partial \bar{\Phi}}{\partial_t \bar{W}_{ik}}{}_{t}\dot{\bar{W}}_{ik} + \frac{\partial \bar{\Phi}}{\partial {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{J}_{ik}}{}^{t}_{t}{}^{\bar{\Theta}}_{s}\dot{\bar{J}}_{ik} + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i}\dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}}\dot{\bar{\theta}} \tag{4.80}
$$

Substituting (4.80) into (4.75) and regrouping terms

$$
\begin{aligned}
&\left(\bar{\rho}^2\frac{\partial \bar{\Phi}}{\partial \bar{\rho}}\delta_{ki} - {}^{(0)}_{s}\bar{\sigma}_{ik}\right)\bar{D}_{ik} + \bar{\rho}\frac{\partial \bar{\Phi}}{\partial \bar{D}_{ik}}\dot{\bar{D}}_{ik} + \bar{\rho}\frac{\partial \bar{\Phi}}{\partial_t \bar{W}_{ik}}{}_{t}\dot{\bar{W}}_{ik} + \bar{\rho}\frac{\partial \bar{\Phi}}{\partial {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{J}_{ik}}{}^{t}_{t}{}^{\bar{\Theta}}_{s}\dot{\bar{J}}_{ik} + \bar{\rho}\frac{\partial \bar{\Phi}}{\partial \bar{g}_i}\dot{\bar{g}}_i \\
&+ \bar{\rho}\left(\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta}\right)\dot{\bar{\theta}} + \frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}} - \operatorname{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[{}_{t}\bar{W}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0
\end{aligned}
\tag{4.81}
$$

For inequality (4.81) to hold for arbitrary but admissible $\dot{\bar{\boldsymbol{D}}}$, ${}_t\dot{\bar{\boldsymbol{W}}}$, ${}^{t\bar{\Theta}}_{t\,s}\dot{\bar{\boldsymbol{J}}}$, $\dot{\bar{\boldsymbol{g}}}$, and $\dot{\bar{\theta}}$, the following must hold.

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{D}_{ik}} = 0 \implies \frac{\partial\bar{\Phi}}{\partial\bar{D}_{ik}} = 0 \tag{4.82}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial{}^{t\bar{\Theta}}_{t\,s}\bar{J}_{ik}} = 0 \implies \frac{\partial\bar{\Phi}}{\partial{}^{t\bar{\Theta}}_{t\,s}\bar{J}_{ik}} = 0 \tag{4.83}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial_t\bar{W}_{ik}} = 0 \implies \frac{\partial\bar{\Phi}}{\partial_t\bar{W}_{ik}} = 0 \tag{4.84}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{g}_i} = 0 \implies \frac{\partial\bar{\Phi}}{\partial\bar{g}_i} = 0 \tag{4.85}$$

$$\bar{\rho}\left(\frac{\partial\bar{\Phi}}{\partial\bar{\theta}} + \bar{\eta}\right) = 0 \implies \frac{\partial\bar{\Phi}}{\partial\bar{\theta}} + \bar{\eta} = 0 \tag{4.86}$$

$$\left(\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ki} - {}^{(0)}_{\;s}\bar{\sigma}_{ik}\right)\bar{D}_{ik} + \frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} - \operatorname{tr}\left(\left[{}^{(0)}_{\;a}\bar{\sigma}\right]\left[{}_t\bar{W}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{t\,s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \tag{4.87}$$

Equations (4.82) – (4.87) are fundamental relations resulting from the entropy inequality.

Remarks.

(1) Equations (4.82) – (4.85) imply that $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{D}}$, ${}_t\bar{\boldsymbol{W}}$, ${}^{t\bar{\Theta}}_{t\,s}\bar{\boldsymbol{J}}$, and $\bar{\boldsymbol{g}}$.

(2) Based on (4.86), $\bar{\eta}$ is not a dependent variable in the constitutive theory as $\bar{\eta} = -\frac{\partial\bar{\Phi}}{\partial\bar{\theta}}$, hence $\bar{\eta}$ is deterministic from $\bar{\Phi}$.

(3) The inequality (4.87) in this form is essential. For example, if

$$\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ki} - {}^{(0)}_{\;s}\bar{\sigma}_{ik} = 0 \tag{4.88}$$

and

$$\frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} - \operatorname{tr}\left(\left[{}^{(0)}_{\;a}\bar{\sigma}\right]\left[{}_t\bar{W}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{t\,s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \tag{4.89}$$

then from (4.88) ${}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}$ is not a function of $\bar{\boldsymbol{D}}$ as $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{D}}$, which is a contradiction as $[{}^{(0)}_{\;s}\bar{\sigma}]$ and $[\bar{D}]$ are conjugate.

In view of the remarks, the arguments of the dependent variables in the constitutive theories in (4.76) can be modified. Use $\bar{\rho}(\bar{\boldsymbol{x}}, t)$ instead of $\frac{1}{\bar{\rho}(\bar{\boldsymbol{x}}, t)}$.

$$\begin{aligned}
\bar{\Phi} &= \bar{\Phi}(\bar{\rho}, \bar{\theta}) \\
{}_{s}^{(0)}\bar{\boldsymbol{\sigma}} &= {}_{s}^{(0)}\bar{\boldsymbol{\sigma}}(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\theta}) \\
{}_{a}^{(0)}\bar{\boldsymbol{\sigma}} &= {}_{a}^{(0)}\bar{\boldsymbol{\sigma}}(\bar{\rho}, {}_{t}\bar{\boldsymbol{W}}, \bar{\theta}) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}(\bar{\rho}, {}_{t}^{t}\bar{\Theta}_{s}\bar{\boldsymbol{J}}, \bar{\theta}) \\
\bar{\boldsymbol{q}} &= \bar{\boldsymbol{q}}(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta})
\end{aligned} \tag{4.90}$$

Note that even though in (4.90) argument tensors of the dependent variables in the constitutive theory are defined, resolution of some terms in the entropy inequality (4.87) is essential before proceeding further.

4.3.2.1 Decomposition of the Symmetric Cauchy Stress Tensor ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$

Consider decomposition of ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$ into equilibrium $\left({}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})\right)$ and deviatoric $\left({}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})\right)$ stress tensors. The motivation for doing so is to separate the stress tensor ${}_{s}^{(0)}\bar{\boldsymbol{\sigma}}$ into one that is purely responsible for change in volume and another one that only causes change in shape, i.e., distortion.

$${}_{s}^{(0)}\bar{\boldsymbol{\sigma}} = {}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) + {}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) \tag{4.91}$$

$$\begin{aligned}
{}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) &= {}_{e}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})(\bar{\rho}, \bar{\theta}) \\
{}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) &= {}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\theta}) \\
{}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}}) &= {}_{d}({}_{s}^{(0)}\bar{\boldsymbol{\sigma}})(\bar{\rho}, 0, \bar{\theta}) = 0
\end{aligned} \tag{4.92}$$

That is, ${}_e({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})$ is not a function of $\bar{\boldsymbol{D}}$ and ${}_d({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})$ vanishes when $\bar{\boldsymbol{D}}$ is zero. Substituting (4.91) into the entropy inequality (4.87) and rearranging terms

$$\left(\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ki} - {}_e({}^{(0)}_{\ s}\bar{\sigma})_{ik}\right)\bar{D}_{ik} + \frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}_d({}^{(0)}_{\ s}\bar{\sigma})_{ik}\bar{D}_{ik}$$
$$- \operatorname{tr}\left(\left[{}^{(0)}_{\ a}\bar{\sigma}\right]\left[{}_t\bar{W}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \quad (4.93)$$

4.3.2.2 Constitutive Theory for Equilibrium Stress ${}_e({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})$: Compressible Thermoviscous Fluids

Since $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{D}}$ and neither is ${}_e({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})$ (due to (4.92)), the constitutive theory for ${}_e({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})$ must be derivable from

$$\begin{aligned} {}_e({}^{(0)}_{\ s}\bar{\sigma})_{ik} &= \bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ki} = \bar{p}(\bar{\rho},\bar{\theta})\delta_{ki} \\ \left[{}_e({}^{(0)}_{\ s}\bar{\sigma})\right] &= \bar{p}(\bar{\rho},\bar{\theta})[I] \end{aligned} \quad (4.94)$$

in which

$$\bar{p}(\bar{\rho},\bar{\theta}) = \bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \quad (4.95)$$

$\bar{p}(\bar{\rho},\bar{\theta})$ is called the thermodynamic pressure and is generally referred to as the equation of state [78, 79] in which $\bar{p}$ is expressed as a function of $\bar{\rho}$ and $\bar{\theta}$ or $\bar{v} = \frac{1}{\bar{\rho}}$ and $\bar{\theta}$, where $\bar{v}$ is specific volume. If compressive pressure is assumed to be positive, then $\bar{p}(\bar{\rho},\bar{\theta})$ in (4.94) can be replaced by $-\bar{p}(\bar{\rho},\bar{\theta})$. Using (4.94), inequality (4.93) reduces to

$$\frac{\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}_d({}^{(0)}_{\ s}\bar{\sigma})_{ik}\bar{D}_{ik} - \operatorname{tr}\left(\left[{}^{(0)}_{\ a}\bar{\sigma}\right]\left[{}_t\bar{W}\right]\right) - \operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \quad (4.96)$$

4.3.2.3 Constitutive Theory for Equilibrium Stress ${}_e({}_s\bar{\boldsymbol{\sigma}}^{(0)})$: Incompressible Thermoviscous Fluids

For incompressible matter, density is constant, hence $\bar{\rho} = \rho_0$. For this case $\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} = 0$, hence the constitutive theory for this case cannot be derived using (4.94). Instead consider $|J| = 1$.

The incompressibility condition must be incorporated in the entropy inequality. Recall that the incompressibility condition in the Eulerian description is given by

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = \text{tr}[\bar{D}] = \bar{D}_{ik}\delta_{ki} = 0 \tag{4.97}$$

Based on (4.97),

$$\bar{p}(\bar{\theta})\bar{D}_{ik}\delta_{ki} = 0 \tag{4.98}$$

in which $\bar{p}(\bar{\theta})$ is an arbitrary Lagrange multiplier. Adding (4.98) to (4.93) and realizing that for incompressible matter $\frac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0$,

$$\begin{aligned}\left(\bar{p}(\bar{\theta})\delta_{ki} - {}_{e}({}^{(0)}_{s}\bar{\sigma})_{ik}\right)\bar{D}_{ik} + \frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - {}_{d}({}^{(0)}_{s}\bar{\sigma})_{ik}\bar{D}_{ik} \\ - \text{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[{}_{t}\bar{W}\right]\right) - \text{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{t\,s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0\end{aligned} \tag{4.99}$$

In the case of incompressible fluids, ${}_{e}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ is a function of $\bar{\theta}$ only, hence

$${}_{e}({}^{(0)}_{s}\bar{\sigma})_{ik} = \bar{p}(\bar{\theta})\delta_{ik} \quad \text{or} \quad \left[{}_{e}({}^{(0)}_{s}\bar{\sigma})\right] = \bar{p}(\bar{\theta})[I] \tag{4.100}$$

$\bar{p}(\bar{\theta})$ is called the mechanical pressure. Since $\bar{p}(\bar{\theta})$ is an arbitrary Lagrange multiplier, it is not deterministic from the deformation field. In view of (4.100), inequality (4.99) also reduces to (4.96), i.e., (4.96) holds for both compressible and incompressible matter.

4.3.3 Final Choice of the Dependent Variables and Their Argument Tensors in the Constitutive Theories

In view of the stress decomposition, constitutive theories for ${}_{e}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$, and the conjugate pairs in (4.96), the final list of dependent variables and their argument tensors is given in the following.

Compressible Matter

$$
\begin{gathered}
\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta}) \\
{}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}_{e}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) + {}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) \\
{}_{e}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = \bar{p}(\bar{\rho}, \bar{\theta})\boldsymbol{I} \ ; \quad \bar{p}(\bar{\rho}, \bar{\theta}) = \bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \\
{}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = {}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\theta}) \\
{}^{(0)}_{a}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{a}\bar{\boldsymbol{\sigma}}(\bar{\rho}, {}_{t}\bar{\boldsymbol{W}}, \bar{\theta}) \\
{}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}\bar{\boldsymbol{m}}(\bar{\rho}, {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}, \bar{\theta}) \\
\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta})
\end{gathered}
\tag{4.101}
$$

Incompressible Matter

In this case $\bar{\rho} = \rho_0$, constant, hence

$$
\begin{gathered}
\bar{\Phi} = \bar{\Phi}(\bar{\theta}) \\
{}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}_{e}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) + {}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) \\
{}_{e}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = \bar{p}(\bar{\theta})\boldsymbol{I} \ ; \quad \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0 \\
{}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = {}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})(\bar{\boldsymbol{D}}, \bar{\theta}) \\
{}^{(0)}_{a}\bar{\boldsymbol{\sigma}} = {}^{(0)}_{a}\bar{\boldsymbol{\sigma}}({}_{t}\bar{\boldsymbol{W}}, \bar{\theta}) \\
{}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}\bar{\boldsymbol{m}}({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}, \bar{\theta}) \\
\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\bar{\boldsymbol{g}}, \bar{\theta})
\end{gathered}
\tag{4.102}
$$

If compressive pressure is considered positive, then $\bar{p}$ can be replaced by $-\bar{p}$ in (4.101) and (4.102).

The choice of argument tensors for ${}_{d}({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ can be modified and made more general by recognizing that

$$
\bar{\boldsymbol{D}} = \boldsymbol{\gamma}_{(1)} = \boldsymbol{\gamma}^{(1)} = {}^{(1)}\boldsymbol{\gamma} \tag{4.103}
$$

$\boldsymbol{\gamma}_{(1)}$ and $\boldsymbol{\gamma}^{(1)}$ being the first convected time derivatives of the Green and Almansi strain tensors in covariant and contravariant bases. Let

$$ {}^{(k)}\boldsymbol{\gamma}\ ; \quad k = 1, 2, \ldots, n \tag{4.104} $$

be the convected time derivatives of the basis independent strain tensor up to order n. Thus with ${}_d({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})$ as deviatoric stress measure its argument $\bar{\boldsymbol{D}} = {}^{(0)}\boldsymbol{\gamma}$ can be replaced by ${}^{(k)}\boldsymbol{\gamma}\ ;\ k = 1, 2, \ldots, n$ in (4.101) and (4.102). That is, consider

$$ {}_d({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}) = {}_d({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})(\bar{\rho}, {}^{(k)}\boldsymbol{\gamma};\ k = 1, 2, \ldots, n\ , \bar{\theta}) \tag{4.105} $$

in (4.101) for the compressible case (all other arguments remaining the same) and

$$ {}_d({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}) = {}_d({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})({}^{(k)}\boldsymbol{\gamma};\ k = 1, 2, \ldots, n\ , \bar{\theta}) \tag{4.106} $$

in (4.102) for the incompressible case (all other arguments remaining the same). In addition to (4.101), (4.102), (4.105), and (4.106), the condition (4.96) resulting from the entropy inequality also holds. Inequality (4.96) is satisfied if

$$ \begin{aligned} {}^{s\sigma}\Psi &= \left({}_d({}^{(0)}_{\ s}\bar{\sigma})_{ik}\right)\left(\bar{D}_{ik}\right) \geq 0 \\ {}^{a\sigma}\Psi &= \left({}^{(0)}_{\ a}\bar{\sigma}_{ik}\right)\left({}_t\bar{W}_{ik}\right) \geq 0 \\ {}^{m}\Psi &= \left({}^{(0)}\bar{m}_{ik}\right)\left({}^{t\bar{\Theta}}_{\ s}\bar{J}_{ik}\right) \geq 0 \end{aligned} \tag{4.107} $$

$$ \frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - {}^{t}_{i}\bar{\boldsymbol{\Theta}} \boldsymbol{\cdot} \left(\boldsymbol{\epsilon} \boldsymbol{:} {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \tag{4.108} $$

To ensure inequality (4.108) is satisfied, the following must hold.

$$ \frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} \leq 0 \tag{4.109} $$

$$ {}^{t}_{i}\bar{\boldsymbol{\Theta}} \boldsymbol{\cdot} \left(\boldsymbol{\epsilon} \boldsymbol{:} {}^{(0)}\bar{\boldsymbol{\sigma}}\right) = 0 \tag{4.110} $$

The inequalities in (4.107) imply that the rate of work due to ${}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$, ${}^{(0)}_{\,a}\bar{\boldsymbol{\sigma}}$, and ${}^{(0)}\bar{\boldsymbol{m}}$ (i.e., ${}^{s\sigma}\Psi$, ${}^{a\sigma}\Psi$, and ${}^{m}\Psi$) must be positive.

4.3.4 Constitutive Theory for ${}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$

Consider compressible matter and

$$ {}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big) = {}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)(\bar{\rho}, {}^{(k)}\boldsymbol{\gamma};\ k = 1, 2, \dots, n\ , \bar{\theta}) \tag{4.111} $$

${}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$ is a symmetric tensor of rank two whose arguments are $\bar{\rho}$, a tensor of rank zero, ${}^{(k)}\boldsymbol{\gamma};\ k = 1, 2, \dots, n$, all symmetric tensors of rank two, and $\bar{\theta}$, a tensor of rank zero. Based on the representation theorem [80–99], ${}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$ can be expressed as a linear combination of the combined generators of its argument tensors that are symmetric tensors of rank two.

Let ${}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \dots, N$ be the combined generators of the argument tensors of ${}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$ that are symmetric tensors of rank two and ${}^{s\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M$ be the combined invariants of the same argument tensors of ${}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$, then

$$ {}_d\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big) = {}^{s\sigma}\underset{\sim}{\alpha}^0 \boldsymbol{I} + \sum_{i=1}^{N} {}^{s\sigma}\underset{\sim}{\alpha}^i \big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) \tag{4.112} $$

in which

$$ {}^{s\sigma}\underset{\sim}{\alpha}^i = {}^{s\sigma}\underset{\sim}{\alpha}^i(\bar{\rho}, {}^{s\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M,\ \bar{\theta}) \tag{4.113} $$

4.3.4.1 Material Coefficients

Note that (4.112) and (4.113) hold in the current configuration in which the deformation is not yet known, hence ${}^{s\sigma}\underset{\sim}{\alpha}^i$ are not material coefficients. To determine or establish material coefficients from (4.113), consider the Taylor series expansion of each ${}^{s\sigma}\underset{\sim}{\alpha}^i;\ i = 0, 1, \dots, N$ in ${}^{s\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M$ and $\bar{\theta}$ about a known configuration $\underline{\Omega}$ of the deforming volume of matter and retain only up to linear terms in the invariants and $\bar{\theta}$ (for simplicity).

$$
{}^{s\sigma}\underset{\sim}{\alpha}^i = {}^{s\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j - \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}}\right) + \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right); \quad i = 0, 1, \dots, N \tag{4.114}
$$

Note that

$$
{}^{s\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}}, \quad \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}};\ j = 1, 2, \dots, M, \quad \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \tag{4.115}
$$

are functions of $\bar{\rho}|_{\underline{\Omega}}$, $({}^{s\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}$, and $\bar{\theta}|_{\underline{\Omega}}$ whereas ${}^{\sigma}\underset{\sim}{\alpha}^i;\ i = 0, 1, \dots, N$ are functions of the same quantities but in the current configuration. When (4.114) is substituted in (4.112), the final expression is the most general form of the constitutive theory for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ up to orders n. The final expression defines the material coefficients in the known configuration $\underline{\Omega}$. Substituting (4.114) in (4.112),

$$
\begin{aligned}
{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = &\left({}^{s\sigma}\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{M} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j - \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}}\right) + \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\right)\boldsymbol{I} \\
&+ \sum_{i=1}^{N}\left({}^{s\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j - \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}}\right) + \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\right)\left({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\right)
\end{aligned} \tag{4.116}
$$

Collect coefficients (quantities defined in $\underline{\Omega}$) of the terms in (4.116) that are defined in the current configuration and group those terms that are completely defined in the known configuration $\underline{\Omega}$. Define

$$
\begin{aligned}
{}^0\bar{\sigma}|_{\underline{\Omega}} &= {}^{s\sigma}\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} - \sum_{j=1}^{M} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}} \\
{}^{s\sigma}\underline{a}_j &= \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}}; \quad j = 1, 2, \dots, M \\
{}^{s\sigma}\underline{b}_i &= {}^{s\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} - \sum_{j=1}^{M} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}}; \quad i = 1, 2, \dots, N \\
{}^{s\sigma}\underset{\sim}{c}_{ij} &= \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\bigg|_{\underline{\Omega}}; \quad \begin{array}{l} i = 1, 2, \dots, N \\ j = 1, 2, \dots, M \end{array}
\end{aligned} \tag{4.117}
$$

$$
\begin{aligned}
{}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}} &= - \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \\
{}^{s\sigma}\underset{\sim}{d}_i &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} ; \quad i = 1, 2, \dots, N
\end{aligned} \tag{4.117}
$$

Using (4.117) in (4.116)

$$
\begin{aligned}
{}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) = {}^{0}\bar{\sigma}\big|_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M} {}^{s\sigma}\underset{\sim}{a}_j \big({}^{s\sigma}\underset{\sim}{I}^j\big)\boldsymbol{I} - {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}} \big(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\big)\, \boldsymbol{I} \\
+ \sum_{i=1}^{N} {}^{s\sigma}\underset{\sim}{b}_i \big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) + \sum_{i=1}^{N}\sum_{j=1}^{M} {}^{s\sigma}\underset{\sim}{c}_{ij} \big({}^{s\sigma}\underset{\sim}{I}^j\big)\big({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) + \sum_{i=1}^{N} {}^{s\sigma}\underset{\sim}{d}_i \big(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\big)\, \big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big)
\end{aligned} \tag{4.118}
$$

${}^{s\sigma}\underset{\sim}{a}_j$, ${}^{s\sigma}\underset{\sim}{b}_i$, ${}^{s\sigma}\underset{\sim}{c}_{ij}$, ${}^{s\sigma}\underset{\sim}{d}_i$, and ${}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}$ are the material coefficients defined in known configuration $\underline{\Omega}$. This constitutive theory requires $(M+N+MN+N+1)$ material coefficients. The material coefficients defined in (4.117) are functions of $\bar{\rho}\big|_{\underline{\Omega}}$, $({}^{s\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}$, and $\bar{\theta}\big|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{s\sigma}\underset{\sim}{\alpha}^i;\ i = 0, 1, \dots, N$.

4.3.4.2 Rate Constitutive Theory of Order One $(n = 1)$ for ${}_d({}_s\bar{\boldsymbol{\sigma}}^{(0)})$

In this case the number of argument tensors of ${}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big)$ to $\bar{\rho}$, ${}^{(1)}\boldsymbol{\gamma}$ (or $\bar{\boldsymbol{D}}$), and $\bar{\theta}$ are limited by choosing $n = 1$. That is,

$$
{}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) = {}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big)(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\theta}) \tag{4.119}
$$

Based on (4.119)

$$
{}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^1 = \bar{\boldsymbol{D}}\ ; \quad {}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^2 = \bar{\boldsymbol{D}}^2\ ; \quad N = 2 \tag{4.120}
$$

and

$$
{}^{s\sigma}\underset{\sim}{I}^1 = \text{tr}(\bar{\boldsymbol{D}})\ ; \quad {}^{s\sigma}\underset{\sim}{I}^2 = \text{tr}(\bar{\boldsymbol{D}}^2)\ ; \quad {}^{s\sigma}\underset{\sim}{I}^3 = \text{tr}(\bar{\boldsymbol{D}}^3)\ ; \quad M = 3 \tag{4.121}
$$

In (4.121) the principal invariants of $\bar{\boldsymbol{D}}$ could also be considered. Since the two sets of invariants are related, the resulting constitutive theory is unaffected. Thus

$$
{}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) = {}^{s\sigma}\underset{\sim}{\alpha}^0\boldsymbol{I} + \sum_{i=1}^{2} {}^{s\sigma}\underset{\sim}{\alpha}^i \big({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\big) \tag{4.122}
$$

Using (4.120) and (4.121) for $N = 2$ and $M = 3$ in the general expression (4.118), the following explicit expression for the first order ($n = 1$) constitutive theory for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ holds.

$$
\begin{aligned}
{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = {}^0\bar{\sigma}|_{\underline{\Omega}}\boldsymbol{I} &+ {}^{s\sigma}\underline{a}_1\text{tr}(\bar{\boldsymbol{D}})\boldsymbol{I} + {}^{s\sigma}\underline{a}_2\text{tr}(\bar{\boldsymbol{D}}^2)\boldsymbol{I} + {}^{s\sigma}\underline{a}_3\text{tr}(\bar{\boldsymbol{D}}^3)\boldsymbol{I} - {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\boldsymbol{I} \\
&+ {}^{s\sigma}\underline{b}_1(\bar{\boldsymbol{D}}) + {}^{s\sigma}\underline{b}_2(\bar{\boldsymbol{D}}^2) + {}^{s\sigma}\underset{\sim}{c}_{11}\big(\text{tr}(\bar{\boldsymbol{D}})\big)\bar{\boldsymbol{D}} + {}^{s\sigma}\underset{\sim}{c}_{12}\big(\text{tr}(\bar{\boldsymbol{D}}^2)\big)\bar{\boldsymbol{D}} + {}^{s\sigma}\underset{\sim}{c}_{13}\big(\text{tr}(\bar{\boldsymbol{D}}^3)\big)\bar{\boldsymbol{D}} \\
&+ {}^{s\sigma}\underset{\sim}{c}_{21}\big(\text{tr}(\bar{\boldsymbol{D}})\big)\bar{\boldsymbol{D}}^2 + {}^{s\sigma}\underset{\sim}{c}_{22}\big(\text{tr}(\bar{\boldsymbol{D}}^2)\big)\bar{\boldsymbol{D}}^2 + {}^{s\sigma}\underset{\sim}{c}_{23}\big(\text{tr}(\bar{\boldsymbol{D}}^3)\big)\bar{\boldsymbol{D}}^2 + {}^{s\sigma}\underset{\sim}{d}_1\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\bar{\boldsymbol{D}} \\
&+ {}^{s\sigma}\underset{\sim}{d}_2\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\bar{\boldsymbol{D}}^2
\end{aligned}
\tag{4.123}
$$

This constitutive theory requires 14 material coefficients and contains up to fifth degree terms in the components of $\bar{\boldsymbol{D}}$.

4.3.4.3 Linear Rate Constitutive Theory of Order One $(n = 1)$ for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$

To obtain a constitutive theory for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ that is linear in the components of $\bar{\boldsymbol{D}}$, neglect the terms that are of degree two or higher in the components of $\bar{\boldsymbol{D}}$ in (4.123).

$$
{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = {}^0\bar{\sigma}|_{\underline{\Omega}}\boldsymbol{I} + {}^{s\sigma}\underline{a}_1\text{tr}(\bar{\boldsymbol{D}})\boldsymbol{I} + {}^{s\sigma}\underline{b}_1(\bar{\boldsymbol{D}}) - {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\boldsymbol{I} \tag{4.124}
$$

Define $\lambda = {}^{s\sigma}\underline{a}_1$, $2\eta = {}^{s\sigma}\underline{b}_1$, then

$$
{}_d({}^{(d)}_{s}\bar{\boldsymbol{\sigma}}) = {}^0\bar{\sigma}|_{\underline{\Omega}}\boldsymbol{I} + \lambda\text{tr}(\bar{\boldsymbol{D}})\boldsymbol{I} + 2\eta(\bar{\boldsymbol{D}}) - {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\boldsymbol{I} \tag{4.125}
$$

The material coefficients η and λ can be functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$, and the invariants of $\bar{\boldsymbol{D}}$ in the known configuration $\underline{\Omega}$. This constitutive theory is the simplest constitutive theory for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$.

Note that the constitutive theory for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ for non-classical thermoviscous fluids is identical to the constitutive theory for ${}^{(0)}_{d}\bar{\boldsymbol{\sigma}}$ for classical thermoviscous fluids. The fundamental difference is that the same constitutive theory is for two different stress measures. In the case of non-classical thermoviscous fluids the stress measure is deviatoric part of the symmetric part of Cauchy stress tensor, whereas in the case of classical thermoviscous fluids the stress measure is the deviatoric part of the total Cauchy stress tensor. In the absence of the first and last terms in (4.125), the right side of (4.125) is the standard Newton's law of viscosity for compressible fluids.

The material coefficients η, λ are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$, and the invariants of $\bar{\boldsymbol{D}}$ in the known configuration $\underline{\Omega}$, and they can be defined using power law, Carreau-Yasuda model, Sutherland model, etc., similar to classical generalized Newtonian fluids (see Reference [113]).

4.3.5 Constitutive Theory for ${}^{(0)}\bar{\boldsymbol{m}}$

Using (4.101) defining the argument tensors of ${}^{(0)}\bar{\boldsymbol{m}}$,

$$
{}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}\bar{\boldsymbol{m}}(\bar{\rho}, {}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}, \bar{\theta}) \tag{4.126}
$$

${}^{(0)}\bar{\boldsymbol{m}}$ and ${}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}$ are both symmetric tensors of rank two and $\bar{\rho}$ and $\bar{\theta}$ are tensors of rank of zero. Based on the representation theorem [80–99], ${}^{(0)}\bar{\boldsymbol{m}}$ can be expressed as a linear combination of the combined generators of its argument tensors that are symmetric tensors of rank two. $\boldsymbol{I}$, ${}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}$, and $\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2$ are the combined generators of $\bar{\rho}$, ${}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}$, and $\bar{\theta}$ that are symmetric tensors of rank two.

$$
{}^{(0)}\bar{\boldsymbol{m}} = {}^{m}\underset{\sim}{\alpha}^{0}\boldsymbol{I} + {}^{m}\underset{\sim}{\alpha}^{1}\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) + {}^{m}\underset{\sim}{\alpha}^{2}\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2 \tag{4.127}
$$

in which

$$
{}^{m}\underset{\sim}{\alpha}^{i} = {}^{m}\underset{\sim}{\alpha}^{i}(\bar{\rho}, {}^{m}\underset{\sim}{I}^{j};\ j = 1,2,3,\ \bar{\theta}) \tag{4.128}
$$

${}^{m}\underset{\sim}{I}^{j};\ j = 1,2,3$ are the combined invariants of the argument tensors of ${}^{(0)}\bar{\boldsymbol{m}}$ in (4.126). Choose either

$$
{}^{m}\underset{\sim}{I}^{1} = \text{tr}\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) \ ; \quad {}^{m}\underset{\sim}{I}^{2} = \text{tr}\left(\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2\right) \ ; \quad {}^{m}\underset{\sim}{I}^{3} = \text{tr}\left(\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^3\right) \tag{4.129}
$$

or the principal invariants of ${}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}$, i.e., $I_{({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{J})}$, $II_{({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{J})}$, and $III_{({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{J})}$. In the following derivation, (4.129) is considered.

4.3.5.1 Material Coefficients

To derive the material coefficients using (4.128), expand each ${}^{m}\underset{\sim}{\alpha}^{i}$; $i = 0, 1, 2$ in a Taylor series in ${}^{m}\underset{\sim}{I}^{j}$; $j = 1, 2, 3$ and $\bar{\theta}$ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in the invariants ${}^{m}\underset{\sim}{I}^{j}$ and $\bar{\theta}$ (for simplicity).

$$
{}^{m}\underset{\sim}{\alpha}^{i} = {}^{m}\underset{\sim}{\alpha}^{i}\Big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial({}^{m}\underset{\sim}{\alpha}^{i})}{\partial({}^{m}\underset{\sim}{I}^{j})}\bigg|_{\underline{\Omega}} \left({}^{m}\underset{\sim}{I}^{j} - \left({}^{m}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}\right) + \frac{\partial({}^{m}\underset{\sim}{\alpha}^{i})}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right); \quad i = 0, 1, 2 \tag{4.130}
$$

Substituting (4.130) into (4.127) and collecting coefficients (those defined in $\underline{\Omega}$) of $\boldsymbol{I}$, $({}^{m}\underset{\sim}{I}^{j})\boldsymbol{I}$, $({}^{m}\underset{\sim}{I}^{j})({}^{t\bar{\Theta}}_{i\ s}\bar{\boldsymbol{J}})$, $({}^{m}\underset{\sim}{I}^{j})({}^{t\bar{\Theta}}_{i\ s}\bar{\boldsymbol{J}})^{2}$, $(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})\boldsymbol{I}$, $(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})({}^{t\bar{\Theta}}_{i\ s}\bar{\boldsymbol{J}})$, and $(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})({}^{t\bar{\Theta}}_{i\ s}\bar{\boldsymbol{J}})^{2}$ and defining

$$
\begin{aligned}
{}^{0}\bar{m}\big|_{\underline{\Omega}} &= {}^{m}\underset{\sim}{\alpha}^{0}\Big|_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial({}^{m}\underset{\sim}{\alpha}^{0})}{\partial({}^{m}\underset{\sim}{I}^{j})}\bigg|_{\underline{\Omega}} \left({}^{m}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \\
{}^{m}\underline{a}_{j} &= \frac{\partial({}^{m}\underset{\sim}{\alpha}^{0})}{\partial({}^{m}\underset{\sim}{I}^{j})}\bigg|_{\underline{\Omega}}; \quad j = 1, 2, 3 \\
{}^{m}\underline{b}_{i} &= {}^{m}\underset{\sim}{\alpha}^{i}\Big|_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial({}^{m}\underset{\sim}{\alpha}^{i})}{\partial({}^{m}\underset{\sim}{I}^{j})}\bigg|_{\underline{\Omega}} \left({}^{m}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}; \quad i = 1, 2 \\
{}^{m}\underset{\sim}{c}_{ij} &= \frac{\partial({}^{m}\underset{\sim}{\alpha}^{i})}{\partial({}^{m}\underset{\sim}{I}^{j})}\bigg|_{\underline{\Omega}}; \quad \begin{matrix} i = 1, 2 \\ j = 1, 2, 3 \end{matrix} \\
{}^{m}\underset{\sim}{\alpha}_{\text{tm}} &= -\frac{\partial({}^{m}\underset{\sim}{\alpha}^{0})}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \\
{}^{m}\underset{\sim}{d}_{i} &= \frac{\partial({}^{m}\underset{\sim}{\alpha}^{i})}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}}; \quad i = 1, 2
\end{aligned} \tag{4.131}
$$

the following holds for ${}^{(0)}\bar{\boldsymbol{m}}$.

$$
\begin{aligned}
{}^{(0)}\bar{\boldsymbol{m}} = {}& {}^{0}\bar{m}\big|_{\underline{\Omega}}\boldsymbol{I} + {}^{m}\underline{a}_1 \mathrm{tr}\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)\boldsymbol{I} + {}^{m}\underline{a}_2 \mathrm{tr}\left(\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2\right)\boldsymbol{I} + {}^{m}\underline{a}_3 \mathrm{tr}\left(\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^3\right)\boldsymbol{I} - {}^{m}\underset{\sim}{\alpha}_{\mathrm{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\boldsymbol{I} \\
& + {}^{m}\underline{b}_1\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) + {}^{m}\underline{b}_2\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2 + {}^{m}\underset{\sim}{c}_{11}\mathrm{tr}\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) + {}^{m}\underset{\sim}{c}_{12}\mathrm{tr}\left(\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2\right)\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) \\
& + {}^{m}\underset{\sim}{c}_{13}\mathrm{tr}\left(\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^3\right)\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) + {}^{m}\underset{\sim}{c}_{21}\mathrm{tr}\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2 + {}^{m}\underset{\sim}{c}_{22}\mathrm{tr}\left(\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2\right)\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2 \\
& + {}^{m}\underset{\sim}{c}_{23}\mathrm{tr}\left(\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^3\right)\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2 + {}^{m}\underset{\sim}{d}_1\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) + {}^{m}\underset{\sim}{d}_2\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)^2
\end{aligned}
\tag{4.132}
$$

4.3.5.2 Linear Constitutive Theory for ${}^{(0)}\bar{\boldsymbol{m}}$

The simplest constitutive theory for ${}^{(0)}\bar{\boldsymbol{m}}$ would be a constitutive theory that is linear in the components of ${}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}$. Using (4.132) and neglecting quadratic and higher degree terms in ${}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}$, (also neglecting ${}^{0}\bar{m}\big|_{\underline{\Omega}}$ and $\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)$ terms)

$$
{}^{(0)}\bar{\boldsymbol{m}} = {}^{m}\underline{b}_1\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) + {}^{m}\underline{a}_1 \mathrm{tr}\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)\boldsymbol{I}
\tag{4.133}
$$

Defining $\alpha = {}^{m}\underline{b}_1$ and $B = {}^{m}\underline{a}_1$

$$
{}^{(0)}\bar{\boldsymbol{m}} = \alpha\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right) + B \mathrm{tr}\left({}_{t}^{t}{}_{s}^{\bar{\Theta}}\bar{\boldsymbol{J}}\right)\boldsymbol{I}
\tag{4.134}
$$

4.3.6 Constitutive Theory for ${}_{a}^{(0)}\bar{\boldsymbol{\sigma}}$

Consider (using (4.101))

$$
{}_{a}^{(0)}\bar{\boldsymbol{\sigma}} = {}_{a}^{(0)}\bar{\boldsymbol{\sigma}}(\bar{\rho}, {}_{t}\bar{\boldsymbol{W}}, \bar{\theta})
\tag{4.135}
$$

${}_{a}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}_{t}\bar{\boldsymbol{W}}$ are antisymmetric tensors of rank two and $\bar{\rho}$ and $\bar{\theta}$ are tensors of rank zero. Since ${}_{t}\bar{\boldsymbol{W}}$ is an antisymmetric tensor, the following holds.

$$
\begin{gathered}
I_{({}_{t}\bar{W})} = 0\,; \quad II_{({}_{t}\bar{W})} = -\frac{1}{2}\mathrm{tr}\left(\left({}_{t}\bar{\boldsymbol{W}}\right)^2\right) \neq 0\,; \quad III_{({}_{t}\bar{W})} = 0 \\
i_{({}_{t}\bar{W})} = 0\,; \quad \ddot{u}_{({}_{t}\bar{W})} = \mathrm{tr}\left(\left({}_{t}\bar{\boldsymbol{W}}\right)^2\right) \neq 0\,; \quad \dddot{m}_{({}_{t}\bar{W})} = 0
\end{gathered}
\tag{4.136}
$$

Thus the only non-zero invariants in this case are $II_{({}_t\bar{W})}$ and $\ddot{u}_{({}_t\bar{W})}$. These are related.

$$II_{({}_t\bar{W})} = -\frac{1}{2}\ddot{u}_{({}_t\bar{W})} \tag{4.137}$$

Let ${}^{a\sigma}\underset{\sim}{I}^1 = \ddot{u}_{({}_t\bar{W})}$, the only combined invariant of the argument tensors of ${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}$ in (4.135). The combined generators of the argument tensors of ${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}$ in (4.135) include only ${}_t\bar{\boldsymbol{W}}$, forming the basis of the space containing ${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}$. Thus,

$${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}} = {}^{a\sigma}\underset{\sim}{\alpha}\big({}_t\bar{\boldsymbol{W}}\big) \tag{4.138}$$

in which

$${}^{a\sigma}\underset{\sim}{\alpha} = {}^{a\sigma}\underset{\sim}{\alpha}(\bar{\rho}, {}^{a\sigma}\underset{\sim}{I}^1, \bar{\theta}) \tag{4.139}$$

4.3.6.1 Material Coefficients

To determine the material coefficients in the constitutive theory for ${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}$ in (4.138), expand ${}^{a\sigma}\underset{\sim}{\alpha}$ in (4.139) in a Taylor series in ${}^{a\sigma}\underset{\sim}{I}^1$ and $\bar{\theta}$ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in ${}^{a\sigma}\underset{\sim}{I}^1$ and $\bar{\theta}$.

$${}^{a\sigma}\underset{\sim}{\alpha} = {}^{a\sigma}\underset{\sim}{\alpha}\Big|_{\underline{\Omega}} + \frac{\partial({}^{a\sigma}\underset{\sim}{\alpha})}{\partial({}^{a\sigma}\underset{\sim}{I}^1)}\bigg|_{\underline{\Omega}} \Big({}^{a\sigma}\underset{\sim}{I}^1 - \big({}^{a\sigma}\underset{\sim}{I}^1\big)_{\underline{\Omega}}\Big) + \frac{\partial({}^{a\sigma}\underset{\sim}{\alpha})}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \big(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\big) \tag{4.140}$$

Substituting (4.140) in (4.138) and collecting coefficients of ${}_t\bar{\boldsymbol{W}}$, $\big({}^{a\sigma}\underset{\sim}{I}^1\big){}_t\bar{\boldsymbol{W}}$, and $\big(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\big)\,{}_t\bar{\boldsymbol{W}}$,

$${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}} = \left({}^{a\sigma}\underset{\sim}{\alpha}\Big|_{\underline{\Omega}} - \frac{\partial({}^{a\sigma}\underset{\sim}{\alpha})}{\partial({}^{a\sigma}\underset{\sim}{I}^1)}\bigg|_{\underline{\Omega}} \big({}^{a\sigma}\underset{\sim}{I}^1\big)_{\underline{\Omega}}\right){}_t\bar{\boldsymbol{W}} + \frac{\partial({}^{a\sigma}\underset{\sim}{\alpha})}{\partial({}^{a\sigma}\underset{\sim}{I}^1)}\bigg|_{\underline{\Omega}} \big({}^{a\sigma}\underset{\sim}{I}^1\big){}_t\bar{\boldsymbol{W}} + \frac{\partial({}^{a\sigma}\underset{\sim}{\alpha})}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \big(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\big)\,{}_t\bar{\boldsymbol{W}} \tag{4.141}$$

Let

$${}^{a\sigma}\underline{b}_1 = {}^{a\sigma}\underset{\sim}{\alpha}\Big|_{\underline{\Omega}} - \frac{\partial({}^{a\sigma}\underset{\sim}{\alpha})}{\partial({}^{a\sigma}\underset{\sim}{I}^1)}\bigg|_{\underline{\Omega}} \big({}^{a\sigma}\underset{\sim}{I}^1\big)_{\underline{\Omega}}\,; \quad {}^{a\sigma}\underset{\sim}{c}_{11} = \frac{\partial({}^{a\sigma}\underset{\sim}{\alpha})}{\partial({}^{a\sigma}\underset{\sim}{I}^1)}\bigg|_{\underline{\Omega}}\,; \quad {}^{a\sigma}\underset{\sim}{d}_2 = \frac{\partial({}^{a\sigma}\underset{\sim}{\alpha})}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \tag{4.142}$$

Substituting (4.142) into (4.141),

$${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}} = \big({}^{a\sigma}\underline{b}_1\big){}_t\bar{\boldsymbol{W}} + \big({}^{a\sigma}\underset{\sim}{c}_{11}\big)\big({}^{a\sigma}\underset{\sim}{I}^1\big){}_t\bar{\boldsymbol{W}} + \big({}^{a\sigma}\underset{\sim}{d}_2\big)\big(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\big)\,{}_t\bar{\boldsymbol{W}} \tag{4.143}$$

This constitutive theory requires three material coefficients, ${}^{a\sigma}\underline{b}_1$, ${}^{a\sigma}\underset{\sim}{c}_{11}$, and ${}^{a\sigma}\underset{\sim}{d}_2$. However, if the $\big(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\big)$ term is neglected then the constitutive theory (4.143) only requires two material coefficients, ${}^{a\sigma}\underline{b}_1$ and ${}^{a\sigma}\underset{\sim}{c}_{11}$. This constitutive theory contains up to cubic terms in the components of the antisymmetric part of the total rotation rate tensor ${}_t\bar{\boldsymbol{W}}$.

4.3.6.2 Linear Constitutive Theory for ${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}$

If only up to linear terms in the components of ${}_t\bar{\boldsymbol{W}}$ in (4.143) are retained (neglecting the $\big(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\big)$ term), then

$$
{}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}} = \kappa\big({}_t\bar{\boldsymbol{W}}\big) \ ; \quad \kappa = {}^{a\sigma}\underline{b}_1 \tag{4.144}
$$

This constitutive theory obviously requires only one material coefficient. The material coefficient κ can be a function of $\bar{\rho}|_{\underline{\Omega}}$, $\bar{\theta}|_{\underline{\Omega}}$, and ${}^{a\sigma}\underset{\sim}{I}^1|_{\underline{\Omega}}$, or could be a constant.

Remarks.

(1) These constitutive theories require only five material coefficients. η and λ are the usual viscosity material coefficients. α, B, and κ are three new coefficients. α is required in the presence of rotation rates (${}^{t}_{i}\bar{\boldsymbol{\Theta}}$ or ${}^{t}_{e}\bar{\boldsymbol{\Theta}}$ or both) and κ and B are only needed when Cosserat rotation rates are present.

(2) In the case of a purely internal polar non-classical continuum theory in which the Cosserat rotation rates are absent, the constitutive theory for ${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}$ is not needed as these are balanced by the gradients of the Cauchy moment tensor in the balance of angular momenta. In this case $\kappa = 0$ and ${}^{t\bar{\Theta}}_{t\ s}\bar{\boldsymbol{J}}$ becomes ${}^{t\bar{\Theta}}_{i\ s}\bar{\boldsymbol{J}}$, the symmetric part of the internal rotation rate gradient tensor. The coefficient B is no longer required as the trace of the gradient of internal rotation rates is zero.

(3) The material coefficient κ is only necessitated due to the presence of Cosserat rotation rates.

(4) If the balance of moments of moments balance law is neglected, then the moment tensor is not symmetric. Following the procedure presented, the constitutive theories in such a

case would be identical to when the balance of moments of moments is considered with the addition of the constitutive theory for the antisymmetric part of the moment tensor.

$$
{}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) = 2\eta(\bar{\boldsymbol{D}}) + \lambda \mathrm{tr}(\bar{\boldsymbol{D}})\boldsymbol{I} \tag{4.145}
$$

$$
{}^{(0)}_{s}\bar{\boldsymbol{m}} = \alpha\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big) + B\mathrm{tr}\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big)\boldsymbol{I} \tag{4.146}
$$

$$
{}^{(0)}_{a}\bar{\boldsymbol{\sigma}} = \kappa\big({}_t\bar{\boldsymbol{W}}\big) \tag{4.147}
$$

$$
{}^{(0)}_{a}\bar{\boldsymbol{m}} = \beta\big({}^{t}_{t}{}^{\bar{\Theta}}_{a}\bar{\boldsymbol{J}}\big) \tag{4.148}
$$

4.3.7 Mathematical Model of Eringen

In this section the conservation and balance laws by Eringen [9, 10, 19, 20] and the associated constitutive theories for thermoviscous fluids are presented. The continuity and balance of momenta equations resulting from the conservation of mass, balance of linear momenta, and balance of angular momenta are the same as those presented in this dissertation. The balance of moments of moments balance law is not used in the work presented by Eringen. The consequence of this is that the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ is not symmetric.

Recall equations (2.40) – (2.41):

$$
\bar{\mathbb{L}} = \bar{\boldsymbol{L}} - {}_e\bar{\boldsymbol{W}} = \bar{\boldsymbol{D}} + \bar{\boldsymbol{W}} - {}_e\bar{\boldsymbol{W}} = \bar{\boldsymbol{D}} + {}_t\bar{\boldsymbol{W}} \tag{4.149}
$$

$$
{}_e\bar{\boldsymbol{W}} = \bar{\boldsymbol{L}} - \bar{\mathbb{L}} \tag{4.150}
$$

in which $\bar{\boldsymbol{W}}$ contains internal rotation rates due to $\bar{\boldsymbol{L}}$ and ${}_e\bar{\boldsymbol{W}}$ are Cosserat rotation rates. When these are resisted by the deforming matter, the conjugate Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ is created which together with ${}_t\bar{\boldsymbol{W}}$ results in rate of work as presented in this dissertation. When comparing the work presented in this dissertation with Eringen's work, there are several major differences:

(1) In References [9, 10, 19, 20], only the rate of work due to ${}^{t}_{e}\bar{\boldsymbol{\Theta}}$ is considered, i.e., the rate of

work due to ${}^{t}_{i}\bar{\boldsymbol{\Theta}}$ (internal rotation rates due to $\bar{\boldsymbol{L}}$) is neglected. As a consequence the energy equation and the entropy inequality only contain gradients of the rotation rates ${}^{t}_{e}\bar{\boldsymbol{\Theta}}$ (gradients of ${}^{t}_{i}\bar{\boldsymbol{\Theta}}$ are neglected).

(2) Another consequence of remark (1) is that the ${}^{t}_{i}\bar{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}})$ term that appears in the energy equation and the entropy inequality derived in this dissertation is absent in the works in References [9, 10, 19, 20].

(3) Due to remark (1), the conjugate pairs in the entropy inequality are affected.

4.3.7.1 Conservation and Balance Laws

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\text{div}(\bar{\boldsymbol{v}}) = 0 \tag{4.151}$$

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^{b} - \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\sigma}} = 0 \tag{4.152}$$

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} - \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}} = 0 \tag{4.153}$$

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \text{tr}\left(\left[\bar{\sigma}\right]\left[\bar{\mathbb{L}}\right]\right) - \text{tr}\left(\left[\bar{m}\right]\left[{}^{t}_{e}\bar{\Theta}\bar{J}\right]\right) = 0 \tag{4.154}$$

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - \text{tr}\left(\left[\bar{\sigma}\right]\left[\bar{\mathbb{L}}\right]\right) - \text{tr}\left(\left[\bar{m}\right]\left[{}^{t}_{e}\bar{\Theta}\bar{J}\right]\right) \leq 0 \tag{4.155}$$

Remarks.

(1) $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{m}}$ are the non-symmetric Cauchy stress and moment tensors in which no distinction is made between covariant and contravariant measures, which is essential in the Eulerian description [78].

(2) The rate of work due to the internal rotation rates ${}^{t}_{i}\bar{\boldsymbol{\Theta}}$ is neglected. Due to neglecting the ${}^{t}_{i}\bar{\boldsymbol{\Theta}}$ term, the ${}^{t}_{i}\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}\right)$ term is absent in (4.154) and (4.155).

(3) From (4.154) and (4.155), one could conclude that $\bar{\boldsymbol{\sigma}}, \bar{\mathbb{L}}$ and $\bar{\boldsymbol{m}}, {}^{t}_{e}\bar{\Theta}\bar{\boldsymbol{J}}$ are conjugate pairs. This conclusion requires further considerations.

4.3.7.2 Constitutive Theories

The Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$ is decomposed into the equilibrium Cauchy stress tensor ${}_e\bar{\boldsymbol{\sigma}}$ and the deviatoric Cauchy stress tensor ${}_d\bar{\boldsymbol{\sigma}}$.

$$\bar{\boldsymbol{\sigma}} = {}_e\bar{\boldsymbol{\sigma}} + {}_d\bar{\boldsymbol{\sigma}} \tag{4.156}$$

The constitutive theory for ${}_e\bar{\boldsymbol{\sigma}}$ is established for both compressible and incompressible fluids using the Helmholtz free energy density and the same approach as presented in Sections 4.3.2.2 and 4.3.2.3.

$$\begin{aligned} {}_e\bar{\boldsymbol{\sigma}} &= \bar{p}(\bar{\rho},\bar{\theta})\boldsymbol{I} \ ; \quad \text{Compressible} \\ {}_e\bar{\boldsymbol{\sigma}} &= \bar{p}(\bar{\theta})\boldsymbol{I} \ ; \quad \text{Incompressible} \end{aligned} \tag{4.157}$$

$\bar{p}(\bar{\rho},\bar{\theta})$ and $\bar{p}(\bar{\theta})$ are the thermodynamic and mechanical pressures. Note that $\bar{\boldsymbol{\sigma}}$ is non-symmetric and ${}_e\bar{\boldsymbol{\sigma}}$ is diagonal, hence ${}_d\bar{\boldsymbol{\sigma}}$ is a non-symmetric tensor. In Eringen's work [9, 10, 19, 20], ${}_d\bar{\boldsymbol{\sigma}}$ and $\bar{\mathbb{L}}$ are used as a conjugate pair and $\bar{\boldsymbol{m}}$ and ${}^{\,t}_{e}\bar{\Theta}\bar{\boldsymbol{J}}$ are also used as a conjugate pair. It is assumed that there exist dissipation potentials $\bar{\phi}_1(\bar{\rho},\bar{\mathbb{L}},\bar{\theta})$ and $\bar{\phi}_2(\bar{\rho},{}^{\,t}_{e}\bar{\Theta}\bar{\boldsymbol{J}},\bar{\theta})$ such that

$${}_d\bar{\sigma}_{ij} = \bar{\rho}\frac{\partial\bar{\phi}_1}{\partial\bar{\mathbb{L}}_{ij}} \ ; \qquad \bar{m}_{ij} = \bar{\rho}\frac{\partial\bar{\phi}_2}{\partial{}^{\,t}_{e}\bar{\Theta}\bar{J}_{ij}} \tag{4.158}$$

$\bar{\phi}_1$ is considered as a function of tr$(\bar{\mathbb{L}})$, tr$(\bar{\mathbb{L}}^2)$, tr$(\bar{\mathbb{L}}\bar{\mathbb{L}}^T)$, etc. Likewise $\bar{\phi}_2$ contains similar tr$(\cdot)$ terms in ${}^{\,t}_{e}\bar{\Theta}\bar{\boldsymbol{J}}$. No explanation or rationale is given for these choices. Using (4.158) and the definition of the potentials $\bar{\phi}_1$ and $\bar{\phi}_2$, the following constitutive theories are given by Eringen [9, 10, 19, 20].

$${}_d\bar{\sigma}_{kl} = \lambda_v \text{tr}\big(\bar{\mathbb{L}}\big)\delta_{kl} + (\mu_v + \kappa_v)\bar{\mathbb{L}}_{kl} + \mu_v\bar{\mathbb{L}}_{lk} \tag{4.159}$$

$$\bar{m}_{kl} = \alpha_v \text{tr}\big({}^{\,t}_{e}\bar{\Theta}\bar{\boldsymbol{J}}\big)\delta_{kl} + \beta_v {}^{\,t}_{e}\bar{\Theta}\bar{J}_{kl} + \gamma_v {}^{\,t}_{e}\bar{\Theta}\bar{J}_{lk} \tag{4.160}$$

The coefficients $(\lambda_v, \mu_v, \kappa_v)$ are called viscosity coefficients for the stress tensor. $(\alpha_v, \beta_v, \gamma_v)$ are the new viscosity coefficients due to rotation rate physics. These constitutive theories by Eringen contain six material coefficients.

Remarks.

(1) By decomposing the Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$ into symmetric $({}_s\bar{\boldsymbol{\sigma}})$ and antisymmetric $({}_a\bar{\boldsymbol{\sigma}})$ tensors and likewise decomposing $\bar{\mathbb{L}}$ into symmetric $({}_s\bar{\mathbb{L}})$ and antisymmetric $({}_a\bar{\mathbb{L}})$ tensors and using them in $\text{tr}\big([\bar{\sigma}][\bar{\mathbb{L}}]\big)$, it is straightforward to show that $\text{tr}\big([{}_s\bar{\sigma}][{}_a\bar{\mathbb{L}}]\big)$ and $\text{tr}\big([{}_a\bar{\sigma}][{}_s\bar{\mathbb{L}}]\big)$ do not result in rate of work and that only $\text{tr}\big([{}_s\bar{\sigma}][{}_s\bar{\mathbb{L}}]\big)$ and $\text{tr}\big([{}_a\bar{\sigma}][{}_a\bar{\mathbb{L}}]\big)$ produce rate of work. Thus, the derivation of the constitutive theory (4.159) contains terms that do not result in work. Similar arguments hold for (4.160).

(2) If the balance of moments of moments as a balance law is considered, the Cauchy moment tensor becomes symmetric, which requires that β_v and γ_v must be replaced by a single material coefficient as there will not be a constitutive theory for the antisymmetric part of the moment tensor. The remaining material coefficients in the constitutive theory for the moment tensor will therefore be reduced by one. This leaves five material coefficients, three in (4.159) and two in (4.160).

(3) Note that the constitutive theories for ${}_s\bar{\boldsymbol{\sigma}}$ and ${}_a\bar{\boldsymbol{\sigma}}$ are completely unrelated. Hence, the material coefficients in the constitutive theory for ${}_a\bar{\boldsymbol{\sigma}}$ should not appear in the constitutive theory for ${}_s\bar{\boldsymbol{\sigma}}$. Regardless, the linear constitutive theory for ${}_s\bar{\boldsymbol{\sigma}}$ requires two material coefficients and the linear constitutive theory for ${}_a\bar{\boldsymbol{\sigma}}$ requires only one coefficient, i.e., the constitutive theory for stress tensor $\bar{\boldsymbol{\sigma}}$ requires only three material coefficients. This is in conformity with (4.159), even though (4.159) is completely different compared to what has been presented. The linear constitutive theory for $\bar{\boldsymbol{m}}$ requires two material coefficient when the balance of moments of moments is used as a balance law as opposed to three material coefficients in its absence. This is also in general agreement with the number of material coefficients required

in the linear constitutive theory for the moment tensor.

(4) Some more disturbing aspects of the derivations of the constitutive theories based on dissipation potentials $\bar{\phi}_1$ and $\bar{\phi}_2$ are that (i) there is no real basis given for the construction of $\bar{\phi}_1$ and $\bar{\phi}_2$ compared to the well-founded methodology based on the representation theorem, (ii) material coefficients are limited to constant values, and (iii) the constitutive theories are limited to linear theories.

(5) Lastly, it can be shown that the constitutive theories (4.159) and (4.160) cannot be supported by the representation theorem when $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{m}}$, $\bar{\mathbb{L}}$, and ${}^{t}_{e}\bar{\Theta}\bar{\boldsymbol{J}}$ are all non-symmetric tensors. Consider $\text{tr}\big([{}_{d}\bar{\sigma}][\bar{\mathbb{L}}]\big)$. ${}_{d}\bar{\boldsymbol{\sigma}}$ and $\bar{\mathbb{L}}$ are non-symmetric tensors of rank two and appear as rate of work conjugate. Thus, based on Eringen's work, if there is a dissipation potential (density function) $\bar{\phi}_1$ such that (4.158) holds, then $\bar{\phi}_1$ must be a function of the invariants of $\bar{\mathbb{L}}$, $\bar{\rho}$, and $\bar{\theta}$

$$\bar{\phi}_1 = \bar{\phi}_1(\bar{\rho}, I_{\bar{\mathbb{L}}}, II_{\bar{\mathbb{L}}}, III_{\bar{\mathbb{L}}}, \bar{\theta}) \tag{4.161}$$

in which $I_{\bar{\mathbb{L}}}$, $II_{\bar{\mathbb{L}}}$, and $III_{\bar{\mathbb{L}}}$ are principal invariants of $\bar{\mathbb{L}}$. Using (4.158) and (4.161) it is straightforward to derive the following [78].

$${}_{d}\bar{\boldsymbol{\sigma}} = {}^{\sigma}\underline{c}_0 \boldsymbol{I} + {}^{\sigma}\underline{c}_1 \bar{\mathbb{L}} + {}^{\sigma}\underline{c}_2 \bar{\mathbb{L}}^{-1} \tag{4.162}$$

Using Cayley-Hamilton theorem $\bar{\mathbb{L}}^{-1}$ can be expressed in terms of $\boldsymbol{I}$, $\bar{\mathbb{L}}$, and $\bar{\mathbb{L}}^2$ and then substituted into (4.162) to obtain

$${}_{d}\bar{\boldsymbol{\sigma}} = {}^{\sigma}c_0 \boldsymbol{I} + {}^{\sigma}c_1 \bar{\mathbb{L}} + {}^{\sigma}c_2 \bar{\mathbb{L}}^2 \tag{4.163}$$

Expression (4.163), a linear combination of $\bar{\boldsymbol{I}}$, $\bar{\mathbb{L}}$, and $\bar{\mathbb{L}}^2$, suggests that when ${}_{d}\bar{\boldsymbol{\sigma}} = {}_{d}\bar{\boldsymbol{\sigma}}(\bar{\rho}, \bar{\mathbb{L}}, \bar{\theta})$, then in (4.163) $\bar{\boldsymbol{I}}$, $\bar{\mathbb{L}}$, and $\bar{\mathbb{L}}^2$ must be the combined generators of the argument tensors $\bar{\mathbb{L}}$, $\bar{\rho}$, and $\bar{\theta}$ that constitute the basis of the space of ${}_{d}\bar{\boldsymbol{\sigma}}$. Based on the works of Spencer, Wang, Zheng, etc. [80–99], when ${}_{d}\bar{\boldsymbol{\sigma}}$ and $\bar{\mathbb{L}}$ are both non-symmetric tensors and when

${}_d\bar{\boldsymbol{\sigma}} = {}_d\bar{\boldsymbol{\sigma}}(\bar{\rho}, \bar{\mathbb{L}}, \bar{\theta})$, then $\bar{\boldsymbol{I}}$, $\bar{\mathbb{L}}$, and $\bar{\mathbb{L}}^2$ are not the combined generators of the arguments of ${}_d\bar{\boldsymbol{\sigma}}$, i.e., based on the representation theorem (4.163) is not valid. On the other hand if ${}_d\bar{\boldsymbol{\sigma}}$ and $\bar{\mathbb{L}}$ are both symmetric tensors of rank two then (4.158), (4.162), and (4.163) are valid and (4.163) is supported by the representation theorem, hence in this case $\bar{\boldsymbol{I}}$, $\bar{\mathbb{L}}$, and $\bar{\mathbb{L}}^2$ form a basis of the space containing ${}_d\bar{\boldsymbol{\sigma}}$.

(6) All constitutive theories derived in this dissertation are strictly in accordance with the representation theorem.

4.3.8 Incompressible Matter

The conservation and balance laws and the constitutive theories presented for compressible fluent continua can be easily modified for the incompressible case.

(a) Since for incompressible matter there is no change in density, $\bar{\rho}(\bar{\boldsymbol{x}}, t) = \rho_0(\boldsymbol{x}, 0)$ holds, i.e., the density of the matter in all configurations remains the same as it is in the reference configuration.

(b) Due to (a), the continuity equation modifies and becomes (divergence free velocity field)

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = 0 \tag{4.164}$$

(c) Due to (4.164)

$$\text{tr}\left(\bar{\boldsymbol{D}}\right) = \text{tr}\left(\bar{\mathbb{L}}\right) = 0 \tag{4.165}$$

Thus, the constitutive theories containing (4.165) need modifications. For example, the linear constitutive theory for ${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})$ for incompressible matter becomes

$${}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = 2\eta\bar{\boldsymbol{D}} \tag{4.166}$$

reducing the material coefficients required by one.

(d) Since the density is constant, the dependence of the material coefficients on $\bar{\rho}$ can be eliminated.

(e) The constitutive theory for the equilibrium Cauchy stress $_e({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})$ is related to the mechanical pressure only, i.e., $_e({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}) = \bar{p}(\bar{\theta})\boldsymbol{I}$. If compressive pressure is assumed positive then $\bar{p}(\bar{\theta})$ can be replaced by $-\bar{p}(\bar{\theta})$.

With these modifications, the conservation and balance laws and the constitutive theories for incompressible fluent continua can be easily obtained from those for compressible case.

4.4 Constitutive Theories for Thermoviscoelastic Fluids with Memory

Thermoviscoelastic (i.e., polymeric) fluids have both elastic and dissipation mechanisms. Broadly speaking, such liquids may be classified as dense or dilute solutions. Historically, different models have been used for either classification [112]. Using the rate of work conjugate pairs, a general constitutive theory that is valid for both classes of polymeric fluids is derived using the representation theorem in conjunction with the conditions set by the entropy inequality. Note that the dependent variables and their argument tensors are pairs of symmetric or antisymmetric tensors as required by the representation theorem [80–99].

4.4.1 Dependent Variables in the Constitutive Theories

It is straightforward to conclude from the conservation and balance laws that $\bar{\Phi}$, $\bar{\eta}$, ${}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and $\bar{\boldsymbol{q}}$ are possible choices of dependent variables in the constitutive theories. For compressible fluent continua physics, density must be incorporated as an argument tensor of all dependent variables in the constitutive theories.

Compressibility is due to the determinant of the Jacobian of deformation $|J| = \left|\frac{\partial\{\bar{\boldsymbol{x}}\}}{\partial\{\boldsymbol{x}\}}\right|$. Recall that in the Lagrangian description (from continuity) $\rho_0 = |J|\rho(\boldsymbol{x},t)$, hence $|J| = \frac{\rho_0}{\rho(\boldsymbol{x},t)}$ in which ρ_0

is the density in the reference configuration (constant), i.e., instead of $|J|$, $\frac{1}{\rho(\boldsymbol{x},t)}$ in the Lagrangian description or $\frac{1}{\bar{\rho}(\bar{\boldsymbol{x}},t)}$ in the Eulerian description may be an argument of all dependent variables in the constitutive theories. At later stages $\frac{1}{\bar{\rho}(\bar{\boldsymbol{x}},t)}$ can be replaced by $\bar{\rho}(\bar{\boldsymbol{x}},t)$ using simple calculus. From (4.67), note that $\bar{\boldsymbol{D}}$, ${}_t\bar{\boldsymbol{W}}$, ${}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}$, and $\bar{\boldsymbol{g}}$ are natural choices for the argument tensors of ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and $\bar{\boldsymbol{q}}$ dependent variables in the constitutive theories, respectively. If it is assumed that higher order convected time derivatives of the strain tensor also contribute to dissipation, then ${}^{(k)}\boldsymbol{\gamma}$; $k = 1, 2, \ldots, n$ need to be argument tensors of ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$. To introduce viscoelasticity and memory, the stress must be a rate equation, which implies that ${}^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}}$ is a constitutive variable with ${}^{(i)}_{s}\bar{\boldsymbol{\sigma}}$; $i = 0, 1, \ldots, (m_\sigma - 1)$ as arguments. Further assume that the moment and antisymmetric stress also contribute to memory and viscoelasticity. Temperature $\bar{\theta}$ is certainly a valid choice in all constitutive variables for thermoviscoelastic behavior.

$$
\begin{aligned}
\bar{\Phi} &= \bar{\Phi}\big(1/\bar{\rho}, {}^{(i)}\boldsymbol{\gamma};\ i=1,2,\ldots,n\ , {}_t\bar{\boldsymbol{W}}, {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}, {}^{(j)}_{s}\bar{\boldsymbol{\sigma}};\ j=0,1,\ldots,(m_{s\sigma}-1)\ , \\
&\qquad {}^{(j)}\bar{\boldsymbol{m}};\ j=0,1,\ldots,(m_{m}-1)\ , {}^{(j)}_{a}\bar{\boldsymbol{\sigma}};\ j=0,1,\ldots,(m_{a\sigma}-1)\ , \bar{\boldsymbol{g}}, \bar{\theta}\big) \\
\bar{\eta} &= \bar{\eta}\big(1/\bar{\rho}, {}^{(i)}\boldsymbol{\gamma};\ i=1,2,\ldots,n\ , {}_t\bar{\boldsymbol{W}}, {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}, {}^{(j)}_{s}\bar{\boldsymbol{\sigma}};\ j=0,1,\ldots,(m_{s\sigma}-1)\ , \\
&\qquad {}^{(j)}\bar{\boldsymbol{m}};\ j=0,1,\ldots,(m_{m}-1)\ , {}^{(j)}_{a}\bar{\boldsymbol{\sigma}};\ j=0,1,\ldots,(m_{a\sigma}-1)\ , \bar{\boldsymbol{g}}, \bar{\theta}\big) \\
{}^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}} &= {}^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}}\big(1/\bar{\rho}, {}^{(i)}\boldsymbol{\gamma};\ i=1,2,\ldots,n\ , {}^{(j)}_{s}\bar{\boldsymbol{\sigma}};\ j=0,1,\ldots,(m_{s\sigma}-1)\ , \bar{\theta}\big) \\
{}^{(m_m)}\bar{\boldsymbol{m}} &= {}^{(m_m)}\bar{\boldsymbol{m}}\big(1/\bar{\rho}, {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}, {}^{(j)}\bar{\boldsymbol{m}};\ j=0,1,\ldots,(m_m-1)\ , \bar{\theta}\big) \\
{}^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}} &= {}^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}}\big(1/\bar{\rho}, {}_t\bar{\boldsymbol{W}}, {}^{(j)}_{a}\bar{\boldsymbol{\sigma}};\ j=0,1,\ldots,(m_{a\sigma}-1)\ , \bar{\theta}\big) \\
\bar{\boldsymbol{q}} &= \bar{\boldsymbol{q}}\big(1/\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta}\big)
\end{aligned}
\tag{4.167}
$$

Note that the argument tensors of $\bar{\Phi}$ and $\bar{\eta}$ are the totality of all of the argument tensors of all of the constitutive variables. At this stage (4.167) is the most general choice. During the derivation of constitutive theories, some arguments of some constitutive variables may be ruled out due to some other considerations.

4.4.2 Entropy Inequality: Further Considerations

Using $\bar{\Phi}$ in (4.167), the material derivative of $\bar{\Phi}$ needed in (4.75) is defined.

$$\frac{D\bar{\Phi}}{Dt} = \dot{\bar{\Phi}} = \frac{\partial \bar{\Phi}}{\partial(1/\bar{\rho})}\left(-\frac{1}{\bar{\rho}^2}\right)\dot{\bar{\rho}} + \sum_{j=1}^{n}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\gamma)_{ik}}\left({}^{(j)}\dot{\gamma}\right)_{ik} + \frac{\partial\bar{\Phi}}{\partial({}^{t\Theta}_{s}\bar{J})_{ik}}\left({}^{t\Theta}_{s}\dot{\bar{J}}\right)_{ik}$$
$$+ \frac{\partial\bar{\Phi}}{\partial({}_t\bar{W})_{ik}}\left({}_t\dot{\bar{W}}\right)_{ik} + \sum_{j=0}^{m_{s\sigma}-1}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}_{s}\bar{\sigma})_{ik}}\left({}^{(j)}_{s}\dot{\bar{\sigma}}\right)_{ik} + \sum_{j=0}^{m_m-1}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\bar{m})_{ik}}\left({}^{(j)}\dot{\bar{m}}\right)_{ik} \tag{4.168}$$
$$+ \sum_{j=0}^{m_{a\sigma}-1}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}_{a}\bar{\sigma})_{ik}}\left({}^{(j)}_{a}\dot{\bar{\sigma}}\right)_{ik} + \frac{\partial\bar{\Phi}}{\partial\bar{g}_i}\dot{\bar{g}}_i + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}}$$

From the continuity equation (4.2)

$$\dot{\bar{\rho}} = -\bar{\rho}\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}} = -\bar{\rho}\bar{D}_{kk} = -\bar{\rho}\bar{D}_{ik}\delta_{ik} \tag{4.169}$$

and

$$\frac{\partial\bar{\Phi}}{\partial(1/\bar{\rho})} = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \tag{4.170}$$

Using (4.169) and (4.170) in (4.168),

$$\dot{\bar{\Phi}} = -\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\bar{D}_{ik}\delta_{ik} + \sum_{j=1}^{n}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\gamma)_{ik}}\left({}^{(j)}\dot{\gamma}\right)_{ik} + \frac{\partial\bar{\Phi}}{\partial({}^{t\Theta}_{s}\bar{J})_{ik}}\left({}^{t\Theta}_{s}\dot{\bar{J}}\right)_{ik}$$
$$+ \frac{\partial\bar{\Phi}}{\partial({}_t\bar{W})_{ik}}\left({}_t\dot{\bar{W}}\right)_{ik} + \sum_{j=0}^{m_{s\sigma}-1}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}_{s}\bar{\sigma})_{ik}}\left({}^{(j)}_{s}\dot{\bar{\sigma}}\right)_{ik} + \sum_{j=0}^{m_m-1}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\bar{m})_{ik}}\left({}^{(j)}\dot{\bar{m}}\right)_{ik} \tag{4.171}$$
$$+ \sum_{j=0}^{m_{a\sigma}-1}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}_{a}\bar{\sigma})_{ik}}\left({}^{(j)}_{a}\dot{\bar{\sigma}}\right)_{ik} + \frac{\partial\bar{\Phi}}{\partial\bar{g}_i}\dot{\bar{g}}_i + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}}$$

Substituting (4.171) into the entropy inequality (4.75) and collecting terms

$$
\begin{aligned}
&\left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}^{(0)}_{s}\bar{\sigma}_{ik}\right)\bar{D}_{ik} + \sum_{j=1}^{n}\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\gamma)_{ik}}\big({}^{(j)}\dot{\gamma}\big)_{ik} + \bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{t}_{s}{}^{\Theta}_{}\bar{J})_{ik}}\big({}^{t\Theta}_{s}\dot{\bar{J}}\big)_{ik} \\
&+ \bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}_{t}\bar{W})_{ik}}\big({}_{t}\dot{\bar{W}}\big)_{ik} + \sum_{j=0}^{m_{s\sigma}-1}\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}_{s}\bar{\sigma})_{ik}}\big({}^{(j)}_{s}\dot{\bar{\sigma}}\big)_{ik} + \sum_{j=0}^{m_m-1}\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\bar{m})_{ik}}\big({}^{(j)}\dot{\bar{m}}\big)_{ik} \\
&+ \sum_{j=0}^{m_{a\sigma}-1}\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}_{a}\bar{\sigma})_{ik}}\big({}^{(j)}_{a}\dot{\bar{\sigma}}\big)_{ik} + \bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{g}_i}\dot{\bar{g}}_i + \bar{\rho}\left(\frac{\partial\bar{\Phi}}{\partial\bar{\theta}} + \bar{\eta}\right)\dot{\bar{\theta}} \\
&+ \frac{\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{q}}}{\bar{\theta}} - {}^{(0)}\bar{m}_{jk}\big({}^{t\Theta}_{s}\bar{J}_{jk}\big) - {}^{(0)}_{a}\bar{\sigma}_{jk}\big({}_{t}\bar{W}_{jk}\big) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\big(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\big) \le 0
\end{aligned}
\tag{4.172}
$$

Inequality (4.172) will be satisfied for arbitrary but admissible values of ${}^{(j)}\dot{\boldsymbol{\gamma}}$; $j = 1, 2, \ldots, n$, ${}^{t\Theta}_{s}\dot{\bar{\boldsymbol{J}}}$, ${}_{t}\dot{\bar{\boldsymbol{W}}}$, ${}^{(j)}_{s}\dot{\bar{\boldsymbol{\sigma}}}$; $j = 0, 1, \ldots, (m_{s\sigma} - 1)$, ${}^{(j)}_{s}\dot{\bar{\boldsymbol{m}}}$; $j = 0, 1, \ldots, (m_m - 1)$, ${}^{(j)}_{a}\dot{\bar{\boldsymbol{\sigma}}}$; $j = 0, 1, \ldots, (m_{a\sigma} - 1)$, $\dot{\bar{\theta}}$, and $\dot{\bar{\boldsymbol{g}}}$ if the following hold.

$$
\bar{\rho}\left(\frac{\partial\bar{\Phi}}{\partial\bar{\theta}} + \bar{\eta}\right) = 0 \implies \frac{\partial\bar{\Phi}}{\partial\bar{\theta}} + \bar{\eta} = 0 \tag{4.173}
$$

$$
\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{g}_i} = 0 \implies \bar{\Phi} \ne \bar{\Phi}(\bar{\boldsymbol{g}}) \tag{4.174}
$$

$$
\begin{aligned}
\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\gamma)_{ik}} = 0;\ j = 1, 2, \ldots, n &\implies \bar{\Phi} \ne \bar{\Phi}({}^{(j)}\boldsymbol{\gamma};\ j = 1, 2, \ldots, n) \\
\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{t\Theta}_{s}\bar{J})_{ik}} = 0 &\implies \bar{\Phi} \ne \bar{\Phi}({}^{t\Theta}_{s}\bar{\boldsymbol{J}}) \\
\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}_{t}\bar{W})_{ik}} = 0 &\implies \bar{\Phi} \ne \bar{\Phi}({}_{t}\bar{\boldsymbol{W}}) \\
\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}_{s}\bar{\sigma})_{ik}} = 0;\ j = 0, 1, \ldots, (m_{s\sigma} - 1) &\implies \bar{\Phi} \ne \bar{\Phi}({}^{(j)}_{s}\bar{\boldsymbol{\sigma}};\ j = 0, 1, \ldots, (m_{s\sigma} - 1)) \\
\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}\bar{m})_{ik}} = 0;\ j = 0, 1, \ldots, (m_m - 1) &\implies \bar{\Phi} \ne \bar{\Phi}({}^{(j)}\bar{\boldsymbol{m}};\ j = 0, 1, \ldots, (m_m - 1)) \\
\bar{\rho}\frac{\partial\bar{\Phi}}{\partial({}^{(j)}_{a}\bar{\sigma})_{ik}} = 0;\ j = 0, 1, \ldots, (m_{a\sigma} - 1) &\implies \bar{\Phi} \ne \bar{\Phi}({}^{(j)}_{a}\bar{\boldsymbol{\sigma}};\ j = 0, 1, \ldots, (m_{a\sigma} - 1))
\end{aligned}
\tag{4.175}
$$

Using (4.175), the entropy inequality can be written as

$$
\left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}^{(0)}_{s}\bar{\sigma}_{ik}\right)\bar{D}_{ik} + \frac{\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{q}}}{\bar{\theta}} - {}^{(0)}\bar{m}_{jk}\big({}^{t\Theta}_{s}\bar{J}_{jk}\big) - {}^{(0)}_{a}\bar{\sigma}_{jk}\big({}_{t}\bar{W}_{jk}\big) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\big(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\big) \le 0 \tag{4.176}
$$

Remarks.

(1) Equations (4.175) and (4.174) imply that $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{D}}$, ${}_t\bar{\boldsymbol{W}}$, ${}^{t\bar{\Theta}}_{s}\bar{\boldsymbol{J}}$, and $\bar{\boldsymbol{g}}$.

(2) Based on (4.173), $\bar{\eta}$ is not a dependent variable in the constitutive theory as $\bar{\eta} = -\frac{\partial\bar{\Phi}}{\partial\bar{\theta}}$, hence $\bar{\eta}$ is deterministic from $\bar{\Phi}$.

(3) The inequality (4.172) in this form is essential. For example, if

$$\bar{\rho}^2 \frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \delta_{ki} - {}^{(0)}_{s}\bar{\sigma}_{ik} = 0 \tag{4.177}$$

and

$$\frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - \text{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[{}_t\bar{W}\right]\right) - \text{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{t\bar{\Theta}}_{s}\bar{J}\right]\right) - {}^{t}_{i}\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \leq 0 \tag{4.178}$$

then from (4.177) ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ is not a function of $\bar{\boldsymbol{D}}$ as $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{D}}$, which is a contradiction as $\left[{}^{(0)}_{s}\bar{\sigma}\right]$ and $[\bar{D}]$ are conjugate according to (4.167).

In view of the remarks, the argument tensors of $\bar{\Phi}$ can be modified. The arguments of the other constitutive variables are unchanged.

$$\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta}) \tag{4.179}$$

4.4.2.1 Decomposition of the Symmetric Cauchy Stress Tensor ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$

Consider the decomposition of ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ into equilibrium $\left({}_e({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})\right)$ and deviatoric $\left({}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})\right)$ stress tensors. The motivation for doing so is to separate the stress tensor ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ into one that is purely responsible for change in volume and another one that only causes change in shape, i.e., distortion.

$${}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}_e({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) + {}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) \tag{4.180}$$

$$\begin{aligned}
&{}_e({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = {}_e({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})(\bar{\rho}, \bar{\theta}) \\
&{}_d({}^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}}) = {}_d({}^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}})(\bar{\rho}, {}^{(j)}\boldsymbol{\gamma};\ j = 1, 2, \dots, n,\ {}_d({}^{(j)}_{s}\bar{\boldsymbol{\sigma}});\ j = 0, 1, \dots, (m_{s\sigma} - 1),\ , \bar{\theta}) \\
&{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = {}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}})(\bar{\rho}, 0, \bar{\theta}) = 0
\end{aligned} \tag{4.181}$$

That is, ${}_{e}\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$ is not a function of $\bar{\boldsymbol{D}}$ and ${}_{d}\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$ vanishes when $\bar{\boldsymbol{D}}$ is zero. Substituting (4.180) into the entropy inequality (4.176) and rearranging terms

$$\left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}_{e}\big({}^{(0)}_{\,s}\bar{\sigma}\big)_{ik}\right)\bar{D}_{ik} - {}_{d}\big({}^{(0)}_{\,s}\bar{\sigma}\big)_{ik}\bar{D}_{ik} + \frac{\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{q}}}{\bar{\theta}} - {}^{(0)}\bar{m}_{jk}\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{J}_{jk}\big)$$
$$- {}^{(0)}_{\,a}\bar{\sigma}_{jk}\big({}_{t}\bar{W}_{jk}\big) - {}^{t}_{i}\bar{\boldsymbol{\Theta}}\cdot\big(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\big) \leq 0 \quad (4.182)$$

4.4.2.2 Constitutive Theory for Equilibrium Stress ${}_{e}\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$: Compressible Thermoviscoelastic Fluids

Based on (4.181), the coefficient of $\bar{D}_{ik}$ in the first term in (4.182) can be set to zero, giving

$${}_{e}\big({}^{(0)}_{\,s}\bar{\sigma}\big)_{ik} = \left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\right)\delta_{ik} = \bar{p}(\bar{\rho},\bar{\theta})\delta_{ik} \quad (4.183)$$

$\bar{p}(\bar{\rho},\bar{\theta})$ is called the thermodynamic pressure, which can be derived using $\bar{\Phi}(\bar{\rho},\bar{\theta})$. If compressive pressure is assumed to be positive, then $\bar{p}(\bar{\rho},\bar{\theta})$ in (4.183) can be replaced by $-\bar{p}(\bar{\rho},\bar{\theta})$. Equation (4.183) is the constitutive theory for ${}_{e}\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$ when the matter is compressible.

4.4.2.3 Constitutive Theory for Equilibrium Stress ${}_{e}\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$: Incompressible Thermoviscoelastic Fluids

For incompressible fluids, $\bar{\rho} = \rho_0 =$ constant and the constitutive variables are no longer a function of density, hence $\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} = 0$. Thus, for this case (4.183) does not yield the constitutive theory for ${}_{e}\big({}^{(0)}_{\,s}\bar{\boldsymbol{\sigma}}\big)$. For incompressible matter, $|J| = 1$ and

$$\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}} = \text{tr}[\bar{D}] = \bar{D}_{ik}\delta_{ik} = 0 \quad (4.184)$$

This incompressibility condition must be enforced. Based on (4.184),

$$\bar{p}(\bar{\theta})\bar{D}_{ik}\delta_{ik} = 0 \quad (4.185)$$

In (4.185), $\bar{p}(\bar{\theta})$ is an arbitrary Lagrange multiplier. Adding (4.185) to the entropy inequality (4.182) and setting $\frac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0$,

$$\left(\bar{p}(\bar{\theta})\delta_{ik} - {}_{e}\big({}^{(0)}_{\ s}\bar{\sigma}\big)_{ik}\right) \bar{D}_{ik} - {}_{d}\big({}^{(0)}_{\ s}\bar{\sigma}\big)_{ik}\bar{D}_{ik} + \frac{\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{q}}}{\bar{\theta}} - {}^{(0)}\bar{m}_{jk}\big({}^{t\bar{\Theta}}_{\ t\ s}\bar{J}_{jk}\big) \\ - {}^{(0)}_{\ a}\bar{\sigma}_{jk}\big({}_{t}\bar{W}_{jk}\big) - {}^{t}_{i}\bar{\boldsymbol{\Theta}} \cdot \big(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\big) \leq 0 \quad (4.186)$$

Setting the coefficient of $\bar{D}_{ik}$ in the first term to zero, the following constitutive theory for ${}_{e}\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big)$ is obtained for the incompressible case.

$$ {}_{e}\big({}^{(0)}_{\ s}\bar{\sigma}\big)_{ik} = \bar{p}(\bar{\theta})\delta_{ik} \quad (4.187)$$

$\bar{p}(\bar{\theta})$ is called the mechanical pressure. If compressive pressure is assumed to be positive, then $\bar{p}(\bar{\theta})$ in (4.187) can be replaced by $-\bar{p}(\bar{\theta})$.

The entropy inequality now reduces to

$$\frac{\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{q}}}{\bar{\theta}} - {}_{d}\big({}^{(0)}_{\ s}\bar{\sigma}\big)_{ik}\bar{D}_{ik} - {}^{(0)}\bar{m}_{jk}\big({}^{t\bar{\Theta}}_{\ t\ s}\bar{J}_{jk}\big) - {}^{(0)}_{\ a}\bar{\sigma}_{jk}\big({}_{t}\bar{W}_{jk}\big) - {}^{t}_{i}\bar{\boldsymbol{\Theta}} \cdot \big(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}}\big) \leq 0 \quad (4.188)$$

4.4.3 Final Choice of the Dependent Variables and Their Argument Tensors in the Constitutive Theories

In view of the stress decomposition, the constitutive theories for ${}_{e}\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big)$, and the conjugate pairs in (4.188), the final list of dependent variables and their argument tensors is given in the following.

Compressible Matter

$$\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta})$$
$$\,^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}_{e}\big(\,^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) + {}_{d}\big(\,^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big)$$
$${}_{e}\big(\,^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) = \bar{p}(\bar{\rho}, \bar{\theta})\boldsymbol{I} \;; \quad \bar{p}(\bar{\rho}, \bar{\theta}) = \bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}}$$
$${}_{d}\big(\,^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}}\big) = {}_{d}\big(\,^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}}\big)\Big(\bar{\rho}, \,^{(i)}\boldsymbol{\gamma};\; i = 1, 2, \dots, n \;, {}_{d}\big(\,^{(j)}_{s}\bar{\boldsymbol{\sigma}}\big);\; j = 0, 1, \dots, (m_{s\sigma} - 1) \;, \bar{\theta}\Big) \tag{4.189}$$
$$\,^{(m_m)}\bar{\boldsymbol{m}} = \,^{(m_m)}\bar{\boldsymbol{m}}\Big(\bar{\rho}, \,^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}, \,^{(j)}\bar{\boldsymbol{m}};\; j = 0, 1, \dots, (m_m - 1) \;, \bar{\theta}\Big)$$
$$\,^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}} = \,^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}}\Big(\bar{\rho}, {}_{t}\bar{\boldsymbol{W}}, \,^{(j)}_{a}\bar{\boldsymbol{\sigma}};\; j = 0, 1, \dots, (m_{a\sigma} - 1) \;, \bar{\theta}\Big)$$
$$\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta})$$

Incompressible Matter

In this case $\bar{\rho} = \rho_0$, constant, hence

$$\bar{\Phi} = \bar{\Phi}(\bar{\theta})$$
$$\,^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}_{e}\big(\,^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) + {}_{d}\big(\,^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big)$$
$${}_{e}\big(\,^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) = \bar{p}(\bar{\theta})\boldsymbol{I} \;; \quad \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0$$
$${}_{d}\big(\,^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}}\big) = {}_{d}\big(\,^{(m_{s\sigma})}_{s}\bar{\boldsymbol{\sigma}}\big)\Big(\,^{(i)}\boldsymbol{\gamma};\; i = 1, 2, \dots, n \;, {}_{d}\big(\,^{(j)}_{s}\bar{\boldsymbol{\sigma}}\big);\; j = 0, 1, \dots, (m_{s\sigma} - 1) \;, \bar{\theta}\Big) \tag{4.190}$$
$$\,^{(m_m)}\bar{\boldsymbol{m}} = \,^{(m_m)}\bar{\boldsymbol{m}}\Big(\,^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}, \,^{(j)}\bar{\boldsymbol{m}};\; j = 0, 1, \dots, (m_m - 1) \;, \bar{\theta}\Big)$$
$$\,^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}} = \,^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}}\Big({}_{t}\bar{\boldsymbol{W}}, \,^{(j)}_{a}\bar{\boldsymbol{\sigma}};\; j = 0, 1, \dots, (m_{a\sigma} - 1) \;, \bar{\theta}\Big)$$
$$\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\bar{\boldsymbol{g}}, \bar{\theta})$$

If compressive pressure is considered positive, then $\bar{p}$ can be replaced by $-\bar{p}$ in (4.189) and (4.190).

The entropy inequality (4.188) is satisfied if

$$\,^{s\sigma}\Psi = \Big({}_{d}\big(\,^{(0)}_{s}\bar{\sigma}\big)_{ik}\Big)\Big(\bar{D}_{ik}\Big) \geq 0$$
$$\,^{a\sigma}\Psi = \Big(\,^{(0)}_{a}\bar{\sigma}_{ik}\Big)\Big({}_{t}\bar{W}_{ik}\Big) \geq 0 \tag{4.191}$$
$$\,^{m}\Psi = \Big(\,^{(0)}\bar{m}_{ik}\Big)\Big(\,^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{J}_{ik}\Big) \geq 0$$

$$\frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - {}_i^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) \leq 0 \tag{4.192}$$

To ensure inequality (4.192) is satisfied, the following must hold.

$$\frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} \leq 0 \tag{4.193}$$

$${}_i^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) = 0 \tag{4.194}$$

The inequalities in (4.191) imply that the rate of work due to ${}_d({}_s^{(0)}\bar{\boldsymbol{\sigma}})$, ${}_a^{(0)}\bar{\boldsymbol{\sigma}}$, and ${}^{(0)}\bar{\boldsymbol{m}}$ (i.e., ${}^{s\sigma}\Psi$, ${}^{a\sigma}\Psi$, and ${}^{m}\Psi$) must be positive. Equation (4.194) serves as a constraint on ${}_i^t\bar{\boldsymbol{\Theta}}$ and the antisymmetric components of the Cauchy stress tensor.

4.4.4 Constitutive Theory for ${}_d({}_s^{(m_{s\sigma})}\bar{\boldsymbol{\sigma}})$

Consider the argument tensors of ${}_d({}_s^{(m_{s\sigma})}\bar{\boldsymbol{\sigma}})$ in (4.189). Let ${}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i$; $i = 1, 2, \dots, N_{s\sigma}$ be the combined generators of the argument tensors of ${}_d({}_s^{(m_{s\sigma})}\bar{\boldsymbol{\sigma}})$ that are symmetric tensors of rank two and let ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{s\sigma}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration based on the representation theorem [80–99].

$${}_d({}_s^{(m_{s\sigma})}\bar{\boldsymbol{\sigma}}) = {}^{s\sigma}\underset{\sim}{\alpha}^0 \boldsymbol{I} + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underset{\sim}{\alpha}^i ({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i) \tag{4.195}$$

in which

$${}^{s\sigma}\underset{\sim}{\alpha}^i = {}^{s\sigma}\underset{\sim}{\alpha}^i(\bar{\rho}, {}^{s\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_{s\sigma}, \bar{\theta}); \quad i = 0, 1, \dots, N_{s\sigma} \tag{4.196}$$

4.4.4.1 Material Coefficients

To determine the material coefficients in (4.195), expand each ${}^{s\sigma}\underset{\sim}{\alpha}^i$ in a Taylor series in ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{s\sigma}$ and $\bar{\theta}$ about a known configuration $\underline{\Omega}$, retaining only up to linear terms in ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{s\sigma}$ and $\bar{\theta}$ (for simplicity). Substitute these ${}^{s\sigma}\underset{\sim}{\alpha}^i$ in (4.195) and collect coefficients of those terms that are defined in the current configuration to obtain the following.

$$
\begin{aligned}
{}_d\left({}^{(m_{s\sigma})}_{\ s}\bar{\boldsymbol{\sigma}}\right) = {}^0_s\sigma\big|_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M_{s\sigma}} {}^{s\sigma}\underline{a}_j\left({}^{s\sigma}\underset{\sim}{I}^j\right)\boldsymbol{I} - {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\boldsymbol{I} \\
+ \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underline{b}_i\left({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_{s\sigma}}\sum_{j=1}^{M_{s\sigma}} {}^{s\sigma}\underset{\sim}{c}_{ij}\left({}^{s\sigma}\underset{\sim}{I}^j\right)\left({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underset{\sim}{d}_i\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\left({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\right)
\end{aligned}
\tag{4.197}
$$

in which

$$
\begin{aligned}
{}^0_s\sigma\big|_{\underline{\Omega}} &= {}^{s\sigma}\underset{\sim}{\alpha}^0\big|_{\underline{\Omega}} - \sum_{j=1}^{M_{s\sigma}} \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}} \\
{}^{s\sigma}\underline{a}_j &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} ; \quad j = 1, 2, \dots, M_{s\sigma} \\
{}^{s\sigma}\underline{b}_i &= {}^{s\sigma}\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} - \sum_{j=1}^{M_{s\sigma}} \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}} ; \quad i = 1, 2, \dots, N_{s\sigma} \\
{}^{s\sigma}\underset{\sim}{c}_{ij} &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\right|_{\underline{\Omega}} ; \quad \begin{array}{l} i = 1, 2, \dots, N_{s\sigma} \\ j = 1, 2, \dots, M_{s\sigma} \end{array} \\
{}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}} &= -\left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \\
{}^{s\sigma}\underset{\sim}{d}_i &= \left.\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} ; \quad i = 1, 2, \dots, N_{s\sigma}
\end{aligned}
\tag{4.198}
$$

${}^{s\sigma}\underline{a}_j$, ${}^{s\sigma}\underline{b}_i$, ${}^{s\sigma}\underset{\sim}{c}_{ij}$, ${}^{s\sigma}\underset{\sim}{d}_i$, and ${}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}$ are the material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_{s\sigma} + N_{s\sigma} + M_{s\sigma}N_{s\sigma} + N_{s\sigma} + 1)$ material coefficients. The material coefficients defined in (4.198) are functions of $({}^{s\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{s\sigma}\underset{\sim}{\alpha}^i;\ i = 0, 1, \dots, N_{s\sigma}$. Simplified forms of this constitutive theory are considered in a later section.

4.4.5 Constitutive Theory for ${}^{(m_m)}\bar{\boldsymbol{m}}$

Consider the argument tensors of ${}^{(m_m)}\bar{\boldsymbol{m}}$ in (4.189). Let ${}^{m}\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \dots, N_m$ be the combined generators of the argument tensors of ${}^{(m_m)}\bar{\boldsymbol{m}}$ that are symmetric tensors of rank two and let ${}^m\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_m$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration based on the representation theorem [80–99].

$$ {}^{(m_m)}\bar{\boldsymbol{m}} = {}^m\alpha^0 \boldsymbol{I} + \sum_{i=1}^{N_m} {}^m\alpha^i \left({}^m\underset{\sim}{\boldsymbol{G}}^i \right) \tag{4.199} $$

in which

$$ {}^m\alpha^i = {}^m\alpha^i(\bar{\rho}, {}^m\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_m,\ \bar{\theta}) \tag{4.200} $$

4.4.5.1 Material Coefficients

To determine the material coefficients in (4.199), expand each ${}^m\alpha^i$ in a Taylor series in ${}^m\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_{sm}$ and $\bar{\theta}$ about a known configuration $\underline{\Omega}$, retaining only up to linear terms in ${}^m\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_m$ and $\bar{\theta}$. Substitute these ${}^m\alpha^i$ in (4.199) and collect coefficients of those terms that are defined in the current configuration to obtain the following.

$$ \begin{aligned} {}^{(m_m)}\bar{\boldsymbol{m}} = {}^0m\big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{j=1}^{M_m} {}^m\underline{a}_j \left({}^m\underset{\sim}{I}^j \right) \boldsymbol{I} - {}^m\underset{\sim}{\alpha}_{\text{tm}} \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}} \right) \boldsymbol{I} \\ + \sum_{i=1}^{N_m} {}^m\underset{\sim}{b}_i \left({}^m\underset{\sim}{\boldsymbol{G}}^i \right) + \sum_{i=1}^{N_m}\sum_{j=1}^{M_m} {}^m\underset{\sim}{c}_{ij} \left({}^m\underset{\sim}{I}^j \right) \left({}^m\underset{\sim}{\boldsymbol{G}}^i \right) + \sum_{i=1}^{N_m} {}^m\underset{\sim}{d}_i \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}} \right) \left({}^m\underset{\sim}{\boldsymbol{G}}^i \right) \end{aligned} \tag{4.201} $$

${}^m\underline{a}_j$, ${}^m\underset{\sim}{b}_i$, ${}^m\underset{\sim}{c}_{ij}$, ${}^m\underset{\sim}{d}_i$, and ${}^m\underset{\sim}{\alpha}_{\text{tm}}$ are the material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_m + N_m + M_mN_m + N_m + 1)$ material coefficients. The material coefficients are functions of $({}^m\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\bar{\theta}\big|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^m\alpha^i;\ i = 0, 1, \dots, N_m$. Explicit forms of the material coefficients can be obtained from (4.198) by simply replacing the back superscript ${}_s\sigma$ with m and ${}^0_s\sigma\big|_{\underline{\Omega}}$ by ${}^0m\big|_{\underline{\Omega}}$. Simplified forms of the constitutive theory are considered in later sections.

4.4.6 Constitutive Theory for ${}^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}}$

Consider the argument tensors of ${}^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}}$ in (4.189). Let ${}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i$; $i = 1, 2, \dots, N_{a\sigma}$ be the combined generators of the argument tensors of ${}^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}}$ that are symmetric tensors of rank two and let ${}^{a\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{a\sigma}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration based on the representation theorem [80–99].

$$
{}^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}} = \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underset{\sim}{\alpha}^i \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \tag{4.202}
$$

in which

$$
{}^{a\sigma}\underset{\sim}{\alpha}^i = {}^{a\sigma}\underset{\sim}{\alpha}^i(\bar{\rho}, {}^{a\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_{a\sigma},\ \bar{\theta}) \tag{4.203}
$$

4.4.6.1 Material Coefficients

To determine the material coefficients in (4.202), expand each ${}^{a\sigma}\underset{\sim}{\alpha}^i$ in a Taylor series in ${}^{a\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M_{a\sigma}$ and $\bar{\theta}$ about a known configuration $\underline{\Omega}$, retaining only up to linear terms in ${}^{a\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M_{a\sigma}$ and $\bar{\theta}$. Substitute these ${}^{a\sigma}\underset{\sim}{\alpha}^i$ in (4.202) and collect coefficients of those terms that are defined in the current configuration to obtain the following.

$$
{}^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}} = \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underline{b}_i \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_{a\sigma}} \sum_{j=1}^{M_{a\sigma}} {}^{a\sigma}\underset{\sim}{\underline{c}}_{ij} \left({}^{a\sigma}\underset{\sim}{I}^j\right) \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underset{\sim}{\underline{d}}_i \left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right) \left({}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \tag{4.204}
$$

${}^{a\sigma}\underline{b}_i$, ${}^{a\sigma}\underset{\sim}{\underline{c}}_{ij}$, and ${}^{a\sigma}\underset{\sim}{\underline{d}}_i$ are the material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(N_{a\sigma} + M_{a\sigma}N_{a\sigma} + N_{a\sigma})$ material coefficients. The material coefficients are functions of $({}^{a\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}$ and $\bar{\theta}|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{a\sigma}\underset{\sim}{\alpha}^i;\ i = 0, 1, \dots, N_{a\sigma}$. Explicit forms of the material coefficients can be obtained from (4.198) by simply replacing the back superscript s with a. Simplified forms of the constitutive theory are considered in later sections.

4.4.7 Simplified Constitutive Theories for ${}_d\big({}^{(m_{s\sigma})}_{\;s}\bar{\boldsymbol{\sigma}}\big)$, ${}^{(m_m)}\bar{\boldsymbol{m}}$, and ${}^{(m_{a\sigma})}_{\;a}\bar{\boldsymbol{\sigma}}$

Among the most commonly used constitutive models for polymeric liquids are Maxwell, Oldroyd-B, and Giesekus models. The origin and historic developments of these constitutive models can be found in Reference [112]. Different treatments of dilute and dense polymeric liquids, phenomenological methodologies, and brownian motion have been the basis of these derivations. Surana, et al. [105–107] derived ordered rate constitutive theories for polymeric fluids using the convected time derivatives of the strain tensor up to order n and the convected time derivatives of the Cauchy stress tensor of up to order m based on classical continuum mechanics. They showed that (i) the Maxwell model is a simplified linear constitutive model corresponding to $n = 1$ and $m = 1$, (ii) the Oldroyd-B model is a simplified quasilinear constitutive model corresponding to $n = 2$ and $m = 1$ that only contains Cauchy stress, its first convected time derivative, and the first and second convected time derivatives of the strain tensor, and (iii) the Giesekus model is the same as the Maxwell model but additionally contains a quadratic term of the Cauchy stress tensor, thus this constitutive model is nonlinear. Equivalent models are derived from the constitutive theories presented in Sections 4.4.4 – 4.4.6 for non-classical compressible polymeric fluids.

Remarks.

(1) The ordered rate constitutive theories presented here for non-classical polymeric fluids incorporating internal and Cosserat rotation rates naturally contain the ordered rate constitutive theories for classical polymeric fluids as a subset. These are easily obtained by removing the internal rotation rate physics, which leads to ${}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}} = 0$, and the Cauchy stress tensor becomes symmetric due to the balance of angular momenta. This constitutive theory is the same as in References [105–107] for classical polymeric fluids.

(2) Since the non-classical constitutive theories presented here are based on integrity, specific forms of the constitutive models for non-classical as well as classical cases are all subsets of these. Hence, it should be possible to present a single simplified non-classical constitutive

model for dilute as well as dense polymeric fluids which would also contain commonly used current constitutive models (based on classical theory).

(3) In order for the non-classical constitutive theories to contain at least the classical constitutive theories of interest as a subset, the following are chosen for the ordered rates.

$$n = 2, \quad m_{s\sigma} = 1, \quad m_m = 1, \quad m_{a\sigma} = 1$$

For this choice the constitutive variables and their argument tensors are

$$\begin{aligned} {}_d\big({}^{(1)}_{\ s}\bar{\boldsymbol{\sigma}}\big) &= {}_d\big({}^{(1)}_{\ s}\bar{\boldsymbol{\sigma}}\big)\big(\bar{\rho}, {}^{(1)}\boldsymbol{\gamma}, {}^{(2)}\boldsymbol{\gamma}, {}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big), \bar{\theta}\big) \\ {}^{(1)}\bar{\boldsymbol{m}} &= {}^{(1)}\bar{\boldsymbol{m}}\big(\bar{\rho}, {}^{t\bar{\Theta}}_{t\ s}\bar{\boldsymbol{J}}, {}^{(0)}\bar{\boldsymbol{m}}, \bar{\theta}\big) \\ {}^{(1)}_{\ a}\bar{\boldsymbol{\sigma}} &= {}^{(1)}_{\ a}\bar{\boldsymbol{\sigma}}\big(\bar{\rho}, {}_t\bar{\boldsymbol{W}}, {}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}, \bar{\theta}\big) \end{aligned} \tag{4.205}$$

The constitutive theories derived using (4.205) require too many material coefficients that are difficult to determine experimentally. Consider the following simplifications, which are motivated by the fact that these non-classical constitutive theories are desired to be extensions of classical constitutive theories. Based on this, the influence of internal and Cosserat rotation rates on the classical theories should be clear.

(i) Consider the constitutive theories to be linear in ${}^{(1)}\boldsymbol{\gamma}$, ${}^{(2)}\boldsymbol{\gamma}$, ${}^{t\bar{\Theta}}_{t\ s}\bar{\boldsymbol{J}}$, and ${}_t\bar{\boldsymbol{W}}$.

(ii) Neglect the products of ${}^{(1)}\boldsymbol{\gamma}$, ${}^{(2)}\boldsymbol{\gamma}$, ${}^{t\bar{\Theta}}_{t\ s}\bar{\boldsymbol{J}}$, ${}_t\bar{\boldsymbol{W}}$, ${}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big)$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}_{\ a}\bar{\boldsymbol{\sigma}}$.

(iii) Neglect the $(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})$ terms to conform to the currently used models based on classical theories.

(iv) Also neglect the first term containing the influence of the initial stress or moment.

(v) Consider the generators $\big({}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big)\big)^2$, $\big({}^{(0)}\bar{\boldsymbol{m}}\big)^2$, but neglect the invariants $\ddot{u}_{{}_d({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})}$, $\dddot{w}_{{}_d({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}})}$, $\ddot{u}_{{}^{(0)}\bar{\boldsymbol{m}}}$, and $\dddot{w}_{{}^{(0)}\bar{\boldsymbol{m}}}$.

4.4.7.1 Simplified Constitutive Theory for ${}_d\big({}^{(m_{s\sigma})}_{\;s}\bar{\boldsymbol{\sigma}}\big)$

Consider

$$
\begin{gathered}
{}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^1 = {}^{(1)}\boldsymbol{\gamma}\;;\quad {}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^2 = {}^{(2)}\boldsymbol{\gamma}\;;\quad {}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^3 = {}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\;;\quad {}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^4 = \big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)^2 \\
{}^{s\sigma}\underset{\sim}{I}^1 = \operatorname{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\;;\quad {}^{s\sigma}\underset{\sim}{I}^2 = \operatorname{tr}\big({}^{(2)}\boldsymbol{\gamma}\big)\;;\quad {}^{s\sigma}\underset{\sim}{I}^1 = \operatorname{tr}\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)
\end{gathered}
\tag{4.206}
$$

Then from (4.197) based on the restrictions (i) – (v):

$$
\begin{aligned}
{}_d\big({}^{(1)}_{\;s}\bar{\boldsymbol{\sigma}}\big) = {}&{}^{s\sigma}\underline{a}_1\operatorname{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} + {}^{s\sigma}\underline{a}_2\operatorname{tr}\big({}^{(2)}\boldsymbol{\gamma}\big)\boldsymbol{I} + {}^{s\sigma}\underline{a}_3\operatorname{tr}\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)\boldsymbol{I} \\
&+ {}^{s\sigma}\underline{b}_1\big({}^{(1)}\boldsymbol{\gamma}\big) + {}^{s\sigma}\underline{b}_2\big({}^{(2)}\boldsymbol{\gamma}\big) + {}^{s\sigma}\underline{b}_3\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big) + {}^{s\sigma}\underline{b}_4\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)^2
\end{aligned}
\tag{4.207}
$$

To express (4.207) in the standard and easily recognizable form, transfer the ${}^{s\sigma}\underline{b}_3\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)$ term to the left side and then divide the entire equation by $-{}^{s\sigma}\underline{b}_3$ to obtain

$$
\begin{aligned}
{}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big) + \left(-\frac{1}{{}^{s\sigma}\underline{b}_3}\right){}_d\big({}^{(1)}_{\;s}\bar{\boldsymbol{\sigma}}\big) = {}&\left(-\frac{{}^{s\sigma}\underline{a}_1}{{}^{s\sigma}\underline{b}_3}\right)\operatorname{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} + \left(-\frac{{}^{s\sigma}\underline{a}_2}{{}^{s\sigma}\underline{b}_3}\right)\operatorname{tr}\big({}^{(2)}\boldsymbol{\gamma}\big)\boldsymbol{I} + \left(-\frac{{}^{s\sigma}\underline{a}_3}{{}^{s\sigma}\underline{b}_3}\right)\operatorname{tr}\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)\boldsymbol{I} \\
&+ \left(-\frac{{}^{s\sigma}\underline{b}_1}{{}^{s\sigma}\underline{b}_3}\right)\big({}^{(1)}\boldsymbol{\gamma}\big) + \left(-\frac{{}^{s\sigma}\underline{b}_2}{{}^{s\sigma}\underline{b}_3}\right)\big({}^{(2)}\boldsymbol{\gamma}\big) + \left(-\frac{{}^{s\sigma}\underline{b}_4}{{}^{s\sigma}\underline{b}_3}\right)\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)^2
\end{aligned}
\tag{4.208}
$$

Defining [78, 105–107, 112]

$$
\begin{gathered}
{}^{s\sigma}\underset{\sim}{\lambda} = \left(-\frac{1}{{}^{s\sigma}\underline{b}_3}\right);\qquad 2\eta = \left(-\frac{{}^{s\sigma}\underline{b}_1}{{}^{s\sigma}\underline{b}_3}\right);\qquad \lambda = \left(-\frac{{}^{s\sigma}\underline{a}_1}{{}^{s\sigma}\underline{b}_3}\right) \\
{}^{s\sigma}k_3 = \left(-\frac{{}^{s\sigma}\underline{a}_3}{{}^{s\sigma}\underline{b}_3}\right);\qquad \lambda_2 = \left(-\frac{{}^{s\sigma}\underline{a}_2}{{}^{s\sigma}\underline{b}_3}\right);\qquad 2\eta_2 = \left(-\frac{{}^{s\sigma}\underline{b}_2}{{}^{s\sigma}\underline{b}_3}\right);\qquad {}^{s\sigma}\eta_4 = \left(-\frac{{}^{s\sigma}\underline{b}_4}{{}^{s\sigma}\underline{b}_3}\right)
\end{gathered}
\tag{4.209}
$$

equation (4.208) can be written as

$$
\begin{aligned}
{}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big) + {}^{s\sigma}\underset{\sim}{\lambda}\,{}_d\big({}^{(1)}_{\;s}\bar{\boldsymbol{\sigma}}\big) = {}&2\eta\big({}^{(1)}\boldsymbol{\gamma}\big) + \lambda\operatorname{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} + 2\eta_2\big({}^{(2)}\boldsymbol{\gamma}\big) + \lambda_2\operatorname{tr}\big({}^{(2)}\boldsymbol{\gamma}\big)\boldsymbol{I} \\
&+ {}^{s\sigma}k_3\operatorname{tr}\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)\boldsymbol{I} + {}^{s\sigma}\eta_4\big({}_d\big({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}}\big)\big)^2
\end{aligned}
\tag{4.210}
$$

The notations and coefficients in (4.210) are consistent with those used in Sections 4.3.4.3 and 3.5.6.1. It should be noted that Bird [112] defines the coefficients somewhat differently, making some dependent on others and assigning physical meaning to them.

$$\begin{gathered}\lambda = \left(-\frac{1}{{}^{s\sigma}\underline{b}_3}\right); \qquad 2\eta = \left(-\frac{{}^{s\sigma}\underline{b}_1}{{}^{s\sigma}\underline{b}_3}\right); \qquad k = \left(-\frac{{}^{s\sigma}\underline{a}_1}{{}^{s\sigma}\underline{b}_3}\right) \\ {}^1k = \left(-\frac{{}^{s\sigma}\underline{a}_3}{{}^{s\sigma}\underline{b}_3}\right); \qquad {}^2k = \left(-\frac{{}^{s\sigma}\underline{a}_2}{{}^{s\sigma}\underline{b}_3}\right); \qquad 2\eta\lambda_2 = \left(-\frac{{}^{s\sigma}\underline{b}_2}{{}^{s\sigma}\underline{b}_3}\right); \qquad \frac{\lambda}{\eta}\alpha = \left(-\frac{{}^{s\sigma}\underline{b}_4}{{}^{s\sigma}\underline{b}_3}\right)\end{gathered} \tag{4.211}$$

$$\begin{aligned} {}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big) + \lambda_d\big({}^{(1)}_{\ s}\bar{\boldsymbol{\sigma}}\big) &= 2\eta\big({}^{(1)}\boldsymbol{\gamma}\big) + 2\eta\lambda_2\big({}^{(2)}\boldsymbol{\gamma}\big) + \frac{\lambda}{\eta}\alpha\Big({}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big)\Big)^2 \\ &\quad + k\,\mathrm{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} + {}^1k\,\mathrm{tr}\Big({}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big)\Big)\boldsymbol{I} + {}^2k\,\mathrm{tr}\big({}^{(2)}\boldsymbol{\gamma}\big)\boldsymbol{I}\end{aligned} \tag{4.212}$$

in which η is viscosity, λ is relaxation time, λ_2 is retardation time, ${}^1k$ and ${}^2k$ are the second viscosity and the viscosity associated with ${}^{(2)}\boldsymbol{\gamma}$, and α is the mobility factor. While (4.210) and (4.212) are equivalent, (4.210) will be used in further derivations to maintain consistency with the rest of this dissertation. This constitutive model holds for compressible polymeric fluids. In case of incompressible fluids, $\mathrm{tr}({}^{(1)}\boldsymbol{\gamma}) = \mathrm{tr}(\bar{\boldsymbol{D}}) = 0$ can be used to modify (4.210).

Maxwell Model

Using (4.210) with

$$\eta_2 = 0, \quad \lambda_2 = 0, \quad {}^{s\sigma}k_3 = 0, \quad {}^{s\sigma}\eta_4 = 0$$

the resulting constitutive theory is given by

$${}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big) + {}^{s\sigma}\underset{\sim}{\lambda}_d\big({}^{(1)}_{\ s}\bar{\boldsymbol{\sigma}}\big) = 2\eta\big({}^{(1)}\boldsymbol{\gamma}\big) + \lambda\,\mathrm{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} \tag{4.213}$$

This is obviously a linear viscoelastic model.

Oldroyd-B Model

Using (4.210) with

$$\lambda_2 = 0, \quad {}^{s\sigma}k_3 = 0, \quad {}^{s\sigma}\eta_4 = 0$$

the resulting constitutive theory is given by

$$ {}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big) + {}^{s\sigma}\underset{\sim}{\lambda}_d\big({}^{(1)}_{\ s}\bar{\boldsymbol{\sigma}}\big) = 2\eta\big({}^{(1)}\boldsymbol{\gamma}\big) + \lambda \mathrm{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} + 2\eta_2\big({}^{(2)}\boldsymbol{\gamma}\big) \tag{4.214}$$

This is a quasilinear (nonlinearity due to ${}^{(2)}\boldsymbol{\gamma}$) viscoelastic model.

Giesekus Model

Using (4.210) with

$$\eta_2 = 0, \quad \lambda_2 = 0, \quad {}^{s\sigma}k_3 = 0$$

the resulting constitutive theory is given by

$$ {}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big) + {}^{s\sigma}\underset{\sim}{\lambda}_d\big({}^{(1)}_{\ s}\bar{\boldsymbol{\sigma}}\big) = 2\eta\big({}^{(1)}\boldsymbol{\gamma}\big) + \lambda \mathrm{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} + {}^{s\sigma}\eta_4\Big({}_d\big({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\big)\Big)^2 \tag{4.215}$$

This is a nonlinear viscoelastic model.

4.4.7.2 Simplified Constitutive Theory for ${}^{(m_m)}\bar{\boldsymbol{m}}$

This constitutive theory is absent in classical continuum mechanics framework. Consider the dependent variable ${}^{(1)}\bar{\boldsymbol{m}}$ and is argument tensors in (4.205). The constitutive theory based on integrity using argument tensors in (4.205) still requires too many material coefficients. The following considers a simplified constitutive theory based on the restrictions stated earlier in (i) – (v). Let

$$\begin{gathered} {}^{m}\underset{\sim}{\boldsymbol{G}}^1 = {}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}} \ ; \quad {}^{m}\underset{\sim}{\boldsymbol{G}}^2 = {}^{(0)}\bar{\boldsymbol{m}} \ ; \quad {}^{m}\underset{\sim}{\boldsymbol{G}}^3 = \big({}^{(0)}\bar{\boldsymbol{m}}\big)^2 \\ {}^{m}\underset{\sim}{I}^1 = \mathrm{tr}\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big) \ ; \quad {}^{m}\underset{\sim}{I}^2 = \mathrm{tr}\big({}^{(0)}\bar{\boldsymbol{m}}\big) \end{gathered} \tag{4.216}$$

Then,

$$
{}^{(1)}\bar{\boldsymbol{m}} = {}^{m}\underline{a}_1 \mathrm{tr}\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big)\boldsymbol{I} + {}^{m}\underline{a}_2 \mathrm{tr}\big({}^{(0)}\bar{\boldsymbol{m}}\big)\boldsymbol{I} + {}^{m}\underline{b}_1\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big) + {}^{m}\underline{b}_2\big({}^{(0)}\bar{\boldsymbol{m}}\big) + {}^{m}\underline{b}_3\big({}^{(0)}\bar{\boldsymbol{m}}\big)^2 \tag{4.217}
$$

Transfer ${}^{m}\underline{b}_2\big({}^{(0)}\bar{\boldsymbol{m}}\big)$ to the left side and divide the whole equation by $-{}^{m}\underline{b}_2$.

$$
\begin{aligned}
{}^{(0)}\bar{\boldsymbol{m}} + \left(-\frac{1}{{}^{m}\underline{b}_2}\right){}^{(1)}\bar{\boldsymbol{m}} = \left(-\frac{{}^{m}\underline{a}_1}{{}^{m}\underline{b}_2}\right)\mathrm{tr}\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big)\boldsymbol{I} + \left(-\frac{{}^{m}\underline{a}_2}{{}^{m}\underline{b}_2}\right)\mathrm{tr}\big({}^{(0)}\bar{\boldsymbol{m}}\big)\boldsymbol{I} \\
+ \left(-\frac{{}^{m}\underline{b}_1}{{}^{m}\underline{b}_2}\right)\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big) + \left(-\frac{{}^{m}\underline{b}_3}{{}^{m}\underline{b}_2}\right)\big({}^{(0)}\bar{\boldsymbol{m}}\big)^2
\end{aligned}
\tag{4.218}
$$

Define

$$
\begin{aligned}
{}^{m}\underset{\sim}{\lambda} = \left(-\frac{1}{{}^{m}\underline{b}_2}\right); \qquad \alpha = \left(-\frac{{}^{m}\underline{b}_1}{{}^{m}\underline{b}_2}\right); \qquad B = \left(-\frac{{}^{m}\underline{a}_1}{{}^{m}\underline{b}_2}\right) \\
{}^{m}\eta_4 = \left(-\frac{{}^{m}\underline{b}_3}{{}^{m}\underline{b}_2}\right); \qquad {}^{m}k_3 = \left(-\frac{{}^{m}\underline{a}_2}{{}^{m}\underline{b}_2}\right)
\end{aligned}
\tag{4.219}
$$

then (4.218) can be written as

$$
{}^{(0)}\bar{\boldsymbol{m}} + {}^{m}\underset{\sim}{\lambda}{}^{(1)}\bar{\boldsymbol{m}} = \alpha\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big) + B\mathrm{tr}\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big)\boldsymbol{I} + {}^{m}k_3\mathrm{tr}\big({}^{(0)}\bar{\boldsymbol{m}}\big)\boldsymbol{I} + {}^{m}\eta_4\big({}^{(0)}\bar{\boldsymbol{m}}\big)^2 \tag{4.220}
$$

Maxwell and Oldroyd-B Models

Following derivations similar to the constitutive thoery for the deviatoric part of the symmetric Cauchy stress tensor, if

$$
{}^{m}k_3 = 0\,; \qquad {}^{m}\eta_4 = 0
$$

in equation (4.220), then the model reduces to

$$
{}^{(0)}\bar{\boldsymbol{m}} + {}^{m}\underset{\sim}{\lambda}{}^{(1)}\bar{\boldsymbol{m}} = \alpha\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big) + B\mathrm{tr}\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big)\boldsymbol{I} \tag{4.221}
$$

This is a linear viscoelastic model.

Giesekus Model

Using (4.220) with

$$^{m}k_3 = 0\ ; \qquad ^{m}\eta_4 = 0$$

results in

$$^{(0)}\bar{\boldsymbol{m}} + {}^{m}\underset{\sim}{\lambda}{}^{(1)}\bar{\boldsymbol{m}} = \alpha\big({}^{t}_{t}\bar{\Theta}^{}_{s}\bar{\boldsymbol{J}}\big) + B\mathrm{tr}\big({}^{t}_{t}\bar{\Theta}^{}_{s}\bar{\boldsymbol{J}}\big)\boldsymbol{I} + {}^{m}\eta_4\big({}^{(0)}\bar{\boldsymbol{m}}\big)^2 \tag{4.222}$$

This is a nonlinear viscoelastic model.

4.4.7.3 Simplified Constitutive Theory for ${}^{(m_{a\sigma})}_{a}\bar{\boldsymbol{\sigma}}$

Consider the dependent variable ${}^{(1)}_{a}\bar{\boldsymbol{\sigma}}$ and is argument tensors in (4.205). ${}^{(1)}_{a}\bar{\boldsymbol{\sigma}}$ and its argument tensors ${}_{t}\bar{\boldsymbol{W}}$ and ${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$ are antisymmetric tensors of rank two, hence the combined generators of the argument tensors that are antisymmetric tensors of rank two and the invariants of the argument tensors are quite limited. For this reason, consider the constitutive theory based on integrity before undertaking and simplifications. Based on (4.204), the following are the combined generators and invariants.

$$\begin{gathered} {}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^1 = {}_{t}\bar{\boldsymbol{W}}\ ; \qquad {}^{a\sigma}\underset{\sim}{\boldsymbol{G}}^2 = {}^{(0)}_{a}\bar{\boldsymbol{\sigma}} \\ [{}^{a\sigma}\underset{\sim}{G}^3] = [{}_{t}\bar{W}][{}^{(0)}_{a}\bar{\sigma}] - [{}^{(0)}_{a}\bar{\sigma}][{}_{t}\bar{W}] \\ {}^{a\sigma}\underset{\sim}{I}^1 = \mathrm{tr}\big(({}_{t}\bar{\boldsymbol{W}})^2\big)\ ; \quad {}^{a\sigma}\underset{\sim}{I}^2 = \mathrm{tr}\big(({}^{(0)}_{a}\bar{\boldsymbol{\sigma}})^2\big)\ ; \quad {}^{a\sigma}\underset{\sim}{I}^3 = \mathrm{tr}\big([{}_{t}\bar{W}][{}^{(0)}_{a}\bar{\sigma}]\big) \end{gathered} \tag{4.223}$$

This constitutive theory will require fifteen material coefficients ($N_{a\sigma} = 3$, $M_{a\sigma} = 3$). The choice of which generators and invariants to retain is not so obvious as it clearly depends upon the physics of interest. For illustration purposes, consider a constitutive theory that is linear in ${}_{t}\bar{\boldsymbol{W}}$ and ${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$ but includes their product terms, i.e., consider all three generators but only the invariant ${}^{a\sigma}\underset{\sim}{I}^3$. In this constitutive theory the products of the generators and the invariant ${}^{a\sigma}\underset{\sim}{I}^3$ are not considered either. Thus, the following is derived from (4.204) (after neglecting $(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})$ terms).

$${}^{(1)}_{a}\bar{\boldsymbol{\sigma}} = {}^{a\sigma}\underline{b}_1\big({}_{t}\bar{\boldsymbol{W}}\big) + {}^{a\sigma}\underline{b}_2\big({}^{(0)}_{a}\bar{\boldsymbol{\sigma}}\big) + {}^{a\sigma}\underline{b}_3\big(({}_{t}\bar{\boldsymbol{W}})({}^{(0)}_{a}\bar{\boldsymbol{\sigma}}) - ({}^{(0)}_{a}\bar{\boldsymbol{\sigma}})({}_{t}\bar{\boldsymbol{W}})\big) \tag{4.224}$$

Transfer ${}^{a\sigma}\underline{b}_2\big({}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}}\big)$ to the left side and divide by $-{}^{a\sigma}\underline{b}_2$.

$$
{}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}} + \left(-\frac{1}{{}^{a\sigma}\underline{b}_2}\right)\left({}^{(1)}_{\;a}\bar{\boldsymbol{\sigma}}\right) = \left(-\frac{{}^{a\sigma}\underline{b}_1}{{}^{a\sigma}\underline{b}_2}\right)\left({}_t\bar{\boldsymbol{W}}\right) + \left(-\frac{{}^{a\sigma}\underline{b}_3}{{}^{a\sigma}\underline{b}_2}\right)\left(({}_t\bar{\boldsymbol{W}})({}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}}) - ({}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}})({}_t\bar{\boldsymbol{W}})\right) \quad (4.225)
$$

Let

$$
{}^{a\sigma}\underset{\sim}{\lambda} = \left(-\frac{1}{{}^{a\sigma}\underline{b}_2}\right); \qquad \kappa = \left(-\frac{{}^{a\sigma}\underline{b}_1}{{}^{a\sigma}\underline{b}_2}\right); \qquad {}^{a\sigma}\eta_4 = \left(-\frac{{}^{a\sigma}\underline{b}_3}{{}^{a\sigma}\underline{b}_2}\right) \quad (4.226)
$$

then

$$
{}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}} + {}^{a\sigma}\underset{\sim}{\lambda}\big({}^{(1)}_{\;a}\bar{\boldsymbol{\sigma}}\big) = \kappa\big({}_t\bar{\boldsymbol{W}}\big) + {}^{a\sigma}\eta_4\big(({}_t\bar{\boldsymbol{W}})({}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}}) - ({}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}})({}_t\bar{\boldsymbol{W}})\big) \quad (4.227)
$$

This constitutive theory is linear in ${}_t\bar{\boldsymbol{W}}$ and ${}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}}$ but contains their product terms.

Maxwell Model

The Maxwell model is a linear constitutive model that can easily be extracted from (4.227) by setting ${}^{a\sigma}\eta_4 = 0$, which gives

$$
{}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}} + {}^{a\sigma}\underset{\sim}{\lambda}\big({}^{(1)}_{\;a}\bar{\boldsymbol{\sigma}}\big) = \kappa\big({}_t\bar{\boldsymbol{W}}\big) \quad (4.228)
$$

Oldroyd-B and Giesekus Models

Derivation of these models for the antisymmetric moment tensor that are parallel to the corresponding constitutive models in classical continuum mechanics is not possible as this constitutive model does not contain the second convected time derivative of the rotation rate tensor (needed for Oldroyd-B model) and the $({}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}})^2$ term (needed for Giesekus constitutive model). Even the constitutive model based on integrity does not contain these terms. Thus, at this stage (4.227) is the only choice until calibration of the model suggests some modifications.

Remarks.

(1) The most general simplified constitutive theories require 15 material coefficients. η and λ are the standard first and second viscosities for the strain rate terms, η_2 and λ_2 are corresponding material coefficients related to the second convected time derivative of the strains, all present in classical theories. α is required in the presence of rotations and rotation rates (internal or Cosserat). κ and B are only needed when Cosserat rotations and rotation rates are present. $\underset{\sim}{\lambda}$, k_3, and η_4 (with appropriate back superscripts) are the coefficients for the stress and moment terms in the constitutive theories.

(2) In the case of purely internal polar non-classical continuum theories, the constitutive theory for ${}_a\boldsymbol{\sigma}^{[m_{a\sigma}]}$ is not needed as these are balanced by gradients of the moment tensor in the balance of angular momenta, hence in this case κ is zero. ${}^{t}_{s}{}^{\bar{\Theta}}_{t}\bar{\boldsymbol{J}}$ become the symmetric part of the internal rotation rate gradient tensor, and the coefficient B is no longer required as the trace of the gradient of internal rotation rates is zero.

(3) If the balance of moments of moments balance law is neglected, then the moment tensor is not symmetric. Following the procedure presented, the constitutive theories in such a case would be identical to when the balance of moments of moments is considered, with the addition of the constitutive theory for the antisymmetric part of the moment tensor.

$$
{}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) + {}^{s\sigma}\underset{\sim}{\lambda}_d\big({}^{(1)}_{s}\bar{\boldsymbol{\sigma}}\big) = 2\eta\big({}^{(1)}\boldsymbol{\gamma}\big) + \lambda\operatorname{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} + 2\eta_2\big({}^{(2)}\boldsymbol{\gamma}\big) + \lambda_2\operatorname{tr}\big({}^{(2)}\boldsymbol{\gamma}\big)\boldsymbol{I} + {}^{s\sigma}k_3\operatorname{tr}\Big({}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big)\Big)\boldsymbol{I} + {}^{s\sigma}\eta_4\Big({}_d\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big)\Big)^2 \quad (4.229)
$$

$$
{}^{(0)}_{s}\bar{\boldsymbol{m}} + {}^{sm}\underset{\sim}{\lambda}\big({}^{(1)}_{s}\bar{\boldsymbol{m}}\big) = \alpha\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big) + B\operatorname{tr}\big({}^{t}_{t}{}^{\bar{\Theta}}_{s}\bar{\boldsymbol{J}}\big)\boldsymbol{I} + {}^{sm}k_3\operatorname{tr}\big({}^{(0)}_{s}\bar{\boldsymbol{m}}\big)\boldsymbol{I} + {}^{sm}\eta_4\big({}^{(0)}_{s}\bar{\boldsymbol{m}}\big)^2 \quad (4.230)
$$

$$
{}^{(0)}_{a}\bar{\boldsymbol{\sigma}} + {}^{a\sigma}\underset{\sim}{\lambda}\big({}^{(1)}_{a}\bar{\boldsymbol{\sigma}}\big) = \kappa\big({}_t\bar{\boldsymbol{W}}\big) + {}^{a\sigma}\eta_4\Big(\big({}_t\bar{\boldsymbol{W}}\big)\big({}^{(0)}_{a}\bar{\boldsymbol{\sigma}}\big) - \big({}^{(0)}_{a}\bar{\boldsymbol{\sigma}}\big)\big({}_t\bar{\boldsymbol{W}}\big)\Big) \quad (4.231)
$$

$$
{}^{(0)}_{a}\bar{\boldsymbol{m}} + {}^{am}\underset{\sim}{\lambda}\big({}^{(1)}_{a}\bar{\boldsymbol{m}}\big) = \beta\big({}^{t}_{t}{}^{\bar{\Theta}}_{a}\bar{\boldsymbol{J}}\big) + {}^{am}\eta_4\Big(\big({}^{t}_{t}{}^{\bar{\Theta}}_{a}\bar{\boldsymbol{J}}\big)\big({}^{(0)}_{a}\bar{\boldsymbol{m}}\big) - \big({}^{(0)}_{a}\bar{\boldsymbol{m}}\big)\big({}^{t}_{t}{}^{\bar{\Theta}}_{a}\bar{\boldsymbol{J}}\big)\Big) \quad (4.232)
$$

(4) These theories can be simplified as needed to match commonly recognized models for poly-

meric fluids. All such models are a subset of the general ordered rate constitutive theories presented here. Furthermore, all classical theories are also a subset of these general ordered rate non-classical theories.

4.4.7.4 Retardation and Memory Moduli

Define

$$ {}^{\sigma}_{s}\boldsymbol{Q} = 2\eta\big({}^{(1)}\boldsymbol{\gamma}\big) + \lambda \mathrm{tr}\big({}^{(1)}\boldsymbol{\gamma}\big)\boldsymbol{I} + 2\eta_2\big({}^{(2)}\boldsymbol{\gamma}\big) + \lambda_2 \mathrm{tr}\big({}^{(2)}\boldsymbol{\gamma}\big)\boldsymbol{I} \tag{4.233} $$

$$ {}^{m}\boldsymbol{Q} = \alpha\big({}^{t}_{s}{}^{\bar{\Theta}}_{t}\bar{\boldsymbol{J}}\big) + B\mathrm{tr}\big({}^{t}_{s}{}^{\bar{\Theta}}_{t}\bar{\boldsymbol{J}}\big)\boldsymbol{I} \tag{4.234} $$

$$ {}^{\sigma}_{a}\boldsymbol{Q} = \kappa\big({}_{t}\bar{\boldsymbol{W}}\big) \tag{4.235} $$

Consider constitutive theories (4.210), (4.220), and (4.227) and discard the stress and moment terms on the right sides of the equations, then

$$ {}_{d}\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big) + {}^{s\sigma}\underset{\sim}{\lambda}_{d}\big({}^{(1)}_{s}\bar{\boldsymbol{\sigma}}\big) = {}^{\sigma}_{s}\boldsymbol{Q} \tag{4.236} $$

$$ {}^{(0)}\bar{\boldsymbol{m}} + {}^{m}\underset{\sim}{\lambda}{}^{(1)}\bar{\boldsymbol{m}} = {}^{m}\boldsymbol{Q} \tag{4.237} $$

$$ {}^{(0)}_{a}\bar{\boldsymbol{\sigma}} + {}^{a\sigma}\underset{\sim}{\lambda}\big({}^{(1)}_{a}\bar{\boldsymbol{\sigma}}\big) = {}^{\sigma}_{a}\boldsymbol{Q} \tag{4.238} $$

Equations (4.236) – (4.238) are first order differential equations in time in ${}_{d}\big({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\big)$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$, hence can be integrated using the following method.

The differential equation

$$ \frac{d\phi}{dx} + P(x)\phi = Q(x) \tag{4.239} $$

has the solution

$$ \phi = \mathrm{e}^{-\int P(x)dx}\left[\int Q(x)\mathrm{e}^{\int P(x)dx}dx + C\right] \tag{4.240} $$

where C is a constant of integration. Consider (4.236) and rewrite

$$
{}_d({}^{(1)}_{s}\bar{\boldsymbol{\sigma}}) + \frac{1}{{}^{s\sigma}\underset{\sim}{\lambda}}\,{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = \frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}} \tag{4.241}
$$

Hence, using (4.239) and (4.240)

$$
\begin{aligned}
{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) &= \mathrm{e}^{-\int 1/{}^{s\sigma}\underset{\sim}{\lambda} dt}\left[\int \frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{\int 1/{}^{s\sigma}\underset{\sim}{\lambda} dt}dt + \boldsymbol{C}\right] \\
&= \mathrm{e}^{-t/{}^{s\sigma}\underset{\sim}{\lambda}}\left[\int \frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{t'/{}^{s\sigma}\underset{\sim}{\lambda}}dt' + \boldsymbol{C}\right] \\
&= \frac{\int\limits_{-\infty}^{t} \frac{{}^{\sigma}_{s}\boldsymbol{Q}}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{t'/{}^{s\sigma}\underset{\sim}{\lambda}}dt'}{\mathrm{e}^{t/{}^{s\sigma}\underset{\sim}{\lambda}}} + \boldsymbol{C}\mathrm{e}^{-t/{}^{s\sigma}\underset{\sim}{\lambda}}
\end{aligned} \tag{4.242}
$$

Based on Reference [112], the choice of $-\infty$ is arbitrary. Some other value could result in a different value of $\boldsymbol{C}$. If it is prescribed that the stress at $t = -\infty$ is finite, then $\boldsymbol{C}$ must be zero (null). The first term in (4.242) also requires attention, since both the numerator and the denominator go to zero as t goes to $-\infty$. Using L'Hôpital's rule,

$$
\lim_{t\to-\infty}{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = \lim_{t\to-\infty}\frac{\frac{{}^{\sigma}_{s}\boldsymbol{Q}}{\lambda}\mathrm{e}^{t/\lambda}}{\frac{1}{\lambda}\mathrm{e}^{t/\lambda}} = {}^{\sigma}_{s}\boldsymbol{Q}(-\infty) \tag{4.243}
$$

Thus, if ${}^{\sigma}_{s}\boldsymbol{Q}(-\infty)$ is finite, the strain is finite at $t = -\infty$. Hence, (3.281) reduces to

$$
{}_d({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}) = \int\limits_{-\infty}^{t}\left(\frac{1}{{}^{s\sigma}\underset{\sim}{\lambda}}\mathrm{e}^{-(t-t')/{}^{s\sigma}\underset{\sim}{\lambda}}\right){}^{\sigma}_{s}\boldsymbol{Q}(t')dt' \tag{4.244}
$$

The quantity in parentheses in the integrand in (4.244) is called the *retardation modulus*. When ${}^{\sigma}_{s}\boldsymbol{Q}$ only contains the $2\eta({}^{(1)}\boldsymbol{\gamma})$ term, the relaxation modulus is deterministic from (4.244). This is straightforward, but is omitted here as it requires approximating ${}^{\sigma}_{s}\boldsymbol{Q}$. The retardation modulus is as valid a measure of rheology as the relaxation modulus.

Using similar derivations, the following can be determined from (4.237) and (4.238), first by rewriting them by dividing by ${}^{m}\underset{\sim}{\lambda}$ and ${}^{a\sigma}\underset{\sim}{\lambda}$, respectively

$$ {}^{(1)}\bar{\boldsymbol{m}} + \frac{1}{{}^{m}\underset{\sim}{\lambda}}\, {}^{(0)}\bar{\boldsymbol{m}} = \frac{{}^{m}\boldsymbol{Q}}{{}^{m}\underset{\sim}{\lambda}} \tag{4.245} $$

$$ {}^{(1)}_{\;a}\bar{\boldsymbol{\sigma}} + \frac{1}{{}^{a\sigma}\underset{\sim}{\lambda}}\, {}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}} = \frac{{}^{\sigma}_{a}\boldsymbol{Q}}{{}^{a\sigma}\underset{\sim}{\lambda}} \tag{4.246} $$

and then following the derivation for ${}_{d}({}^{(0)}_{\;s}\bar{\boldsymbol{\sigma}})$.

$$ {}^{(0)}\bar{\boldsymbol{m}} = \int_{-\infty}^{t} \left(\frac{1}{{}^{m}\underset{\sim}{\lambda}} \mathrm{e}^{-(t-t')/{}^{m}\underset{\sim}{\lambda}} \right) {}^{m}\boldsymbol{Q}(t')\,dt' \tag{4.247} $$

$$ {}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}} = \int_{-\infty}^{t} \left(\frac{1}{{}^{a\sigma}\underset{\sim}{\lambda}} \mathrm{e}^{-(t-t')/{}^{a\sigma}\underset{\sim}{\lambda}} \right) {}^{\sigma}_{a}\boldsymbol{Q}(t')\,dt' \tag{4.248} $$

The terms in the parentheses in (4.247) and (4.248) in the integrands are the retardation moduli for ${}^{(0)}\bar{\boldsymbol{m}}$ and ${}^{(0)}_{\;a}\bar{\boldsymbol{\sigma}}$, respectively.

4.5 Constitutive Theories for Heat Vector $\bar{\boldsymbol{q}}$

Recall the inequality (4.109) resulting from the second law of thermodynamics.

$$ \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}} \leq 0 \quad (\text{as } \bar{\theta} > 0) \tag{4.249} $$

In (4.249), $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{g}}$ are conjugate. The simplest possible constitutive theory for $\bar{\boldsymbol{q}}$ can be derived by assuming that $\bar{\boldsymbol{q}}$ is proportional to $-\bar{\boldsymbol{g}}$ which leads to the following constitutive theory for $\bar{\boldsymbol{q}}$ [78].

$$ \bar{\boldsymbol{q}} = -k(\bar{\theta})\bar{\boldsymbol{g}} \tag{4.250} $$

Alternatively, if

$$ \bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta}) \tag{4.251} $$

then using the representation theorem (as $\bar{\boldsymbol{g}}$ is the only combined generator of $\bar{\rho}$, $\bar{\boldsymbol{g}}$, and $\bar{\theta}$ that is a tensor of rank 1),

$$\bar{\boldsymbol{q}} = {}^q\underset{\sim}{\alpha}\bar{\boldsymbol{g}} \tag{4.252}$$

in which

$${}^q\underset{\sim}{\alpha} = {}^q\underset{\sim}{\alpha}(\bar{\rho}, {}^q\underset{\sim}{I}, \bar{\theta}) \; ; \quad {}^q\underset{\sim}{I} = \bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}} \tag{4.253}$$

${}^q\underset{\sim}{I}$ is the only invariant of the argument tensors $\bar{\rho}$, $\bar{\boldsymbol{g}}$, and $\bar{\theta}$. Expanding ${}^q\underset{\sim}{\alpha}$ in a Taylor series in ${}^q\underset{\sim}{I}$ and $\bar{\theta}$ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^q\underset{\sim}{I}$ and $\bar{\theta}$,

$$\bar{\boldsymbol{q}} = -k\big|_{\underline{\Omega}}\bar{\boldsymbol{g}} - k_1\big|_{\underline{\Omega}}(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})\bar{\boldsymbol{g}} - k_2\big|_{\underline{\Omega}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)\bar{\boldsymbol{g}} \tag{4.254}$$

in which

$$k\big|_{\underline{\Omega}} = {}^q\underset{\sim}{\alpha}\Big|_{\underline{\Omega}} - \left.\frac{\partial({}^q\underset{\sim}{\alpha})}{\partial({}^q\underset{\sim}{I})}\right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}\right)_{\underline{\Omega}} \; ; \quad k_1\big|_{\underline{\Omega}} = \left.\frac{\partial({}^q\underset{\sim}{\alpha})}{\partial({}^q\underset{\sim}{I})}\right|_{\underline{\Omega}} \; ; \quad k_2\big|_{\underline{\Omega}} = -\left.\frac{\partial({}^q\underset{\sim}{\alpha})}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \tag{4.255}$$

This constitutive theory is complete based on the representation theorem. The only assumption in the constitutive theory is the truncation of the Taylor series beyond linear terms in ${}^q\underset{\sim}{I}$ and $\bar{\theta}$. Obviously the standard Fourier heat conduction law (4.250) is a subset of (4.255) when k is the only material coefficient and only depends on temperature $\bar{\theta}$. It is straightforward to verify that the constitutive theory (4.255) for $\bar{\boldsymbol{q}}$ satisfies (4.249) when all of the material coefficients are positive.

Chapter 5

Summary and Conclusions

The conservation and balance laws and the constitutive theories have been derived and presented for non-classical continuum theories for solid and fluent continua. For solid continua, the motivation for the non-classical theory presented here is two-fold:

(i) Since the displacements and the Jacobian of deformation are complete measures of deformation, these in their entirety must form the basis for the conservation and the balance laws. Polar decomposition of the Jacobian of deformation at a material point into the stretch tensor and the rotation tensor shows that this can be accomplished by incorporating the rotation tensor in the current classical continuum theories. The rotation tensor physics is in fact in the antisymmetric part of the Jacobian of deformation that contains rotation angles at a material point. The resulting theory is a non-classical continuum theory with internal rotation physics due to the deformation gradient tensor. This theory provides conservation and balance laws that are consistent with the complete deformation physics. Such theories are referred to as internal polar theories or non-classical continuum theories with internal rotations. The internal rotations can be viewed as rotations about the axes of a triad located at each material point with its axes parallel to the x-frame.

(ii) The second motivation is to enhance the non-classical theories in (i) so that more complex physics of deformation can possibly be described by the resulting theory. This is done by incorporating additional three unknown rotational degrees of freedom about the axes of the same triad as in (i). These are called Cosserat rotations. Thus, now each material point has six degrees of freedom, three displacements and three Cosserat rotations.

The continuum theories for solid conitnua resulting due to (i) and (ii) are non-classical continuum theories incorporating internal and Cosserat rotations.

In the case of fluent continua, velocities are observable quantities, hence the non-classical continuum theories for fluent continua, both compressible and incompressible, are motivated by:

(i) Incorporating velocities and the velocity gradient tensor in their entirety in the conservation and balance laws. This requires that internal rotation rates (the antisymmetric part of the velocity gradient tensor) be incorporated in the existing classical continuum theories for fluent continua. The resulting theories are internal polar non-classical continuum theories for fluent continua.

(ii) The second motivation is to enhance the non-classical continuum theory in (i) so that more complex physics of deformation can possibly be described by the resulting theory. This is done by incorporating an additional three rotation rates at each material point as unknown degrees of freedom, giving rise to six degrees of freedom at each material point, three velocities and three rotation rates. These rotation rates are called Cosserat rotation rates. Both internal and Cosserat rotation rates are about the axes of the same triad at each material point with its axes parallel to the x-frame.

The continuum theories for fluids resulting due to (i) and (ii) are non-classical continuum theories incorporating internal and Cosserat rotation rates. Based on the conservation and balance laws presented in this dissertation, consider the following specific remarks.

(1) It is shown that the conservation and balance laws used for classical continuum theories require modifications to be valid for non-classical continuum theories.

(2) It is shown by Yang, et al. [77] and Surana, et al. [103, 104] (and reported here) that non-classical continuum theories require an additional balance law, *the balance of moments of*

moments, to ensure equilibrium of the deforming matter in the presence of rotation and rotation rate physics.

(3) In non-classical theories, the Cauchy stress tensor is always non-symmetric. The Cauchy moment tensor resulting due to moments conjugate to the rotations or the rotation rates is: (i) symmetric if the balance of moments of moments is used as a balance law and (ii) non-symmetric if the balance of moments of moments is not used as a balance law. Yang, et al. [77] and Surana, et al. [103, 104] have shown the necessity of the balance law in non-classical continuum theories.

(4) For both solids and fluids, the constitutive theories are derived using conjugate pairs in the entropy inequality, as well as consideration of additional physics that may not be obvious from the conditions resulting from the entropy inequality, in conjunction with the representation theorem. All constitutive theories are first presented using integrity (hence are complete), followed by their simplified forms that eventually result in linear constitutive theories.

(5) For solid matter, constitutive theories are derived for thermoelastic behavior and thermoviscoelastic behavior with and without memory. Energy storage, dissipation, and memory mechanisms are identified. Simplified constitutive models resembling Maxwell, Oldroyd-B, and Giesekus models for classical polymeric liquids are also derived. Retardation moduli are derived for polymeric behavior. The work presented here only considers small deformation and small strain physics.

(6) For fluent continua, both compressible and incompressible, constitutive theories are derived for thermoviscous as well as thermoviscoelastic fluids (polymeric liquids). Simplified forms of the constitutive theories including those that resemble Maxwell, Oldroyd-B, and Giesekus constitutive models for the classical theory are derived.

(7) The constitutive theories in (5) and (6) for solid and fluent continua are considered in the absence of the balance of moments of moments balance law. This case is more general, as for

this case the Cauchy moment tensor is non-symmetric, hence requires constitutive theories for symmetric as well as antisymmetric moment tensors. When the balance of moments of moments is used as a balance law, the Cauchy moment tensor is symmetric. The constitutive theory for the symmetric part of the moment tensor remains unaffected regardless of whether the balance of moments of moments is used as a balance law.

(8) Constitutive theories for the heat vector based on the representation theorem as well as based on the conditions resulting from the entropy inequality are also presented.

(9) In all cases the conservation and balance laws in conjunction with the constitutive theories provide mathematical descriptions in which the mathematical models have closure.

The work presented here is compared with published work (including Eringen [14, 17–20]) to demonstrate its thermodynamic consistency, generality, and versatility.

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