Quantum Families of Maps

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Abstract

This dissertation investigates the theory of quantum families of maps, which formulates a non-commutative topological way to study quantum analogs of spaces of continuous mappings, classical objects of interest from general topology. The fundamental element of non-commutative topology is a C^*-algebra. In the theory of C^*-algebras, Gelfand Theorem says that every commutative C^*-algebra is C^*-isomorphic to $C_0(X)$, where X is a locally compact Hausdorff space. Extending this Gelfand duality conceptually to all C^*-algebras (not only those commutative ones), the non-commutative or quantum topology views any C^*-algebra A as the function algebra of a corresponding "virtual" space $\mathcal{QS}(A)$, called a quantum space.

Piotr Sołtan defined a quantum concept of the family of all maps from a quantum space $\mathcal{QS}(M)$ to another quantum space $\mathcal{QS}(B)$, and established some general properties of related objects, extending classical results on such families of mappings. However, a lot of his results carry the assumption that M is finite dimensional (and B is finitely generated), only under which the quantum family of all maps was proved to exist (in a unique way).

In this dissertation, we study the most fundamental and important question about the existence (and uniqueness) of the quantum space of all maps for infinite-dimensional cases, and solve it for the fundamental case of $M = C(\mathbb{N} \cup \{\infty\})$ where $\mathbb{N} \cup \{\infty\}$ is the one-point compactification of $\mathbb{N}$. We find that new structures outside purely C^*-algebraic framework are needed from the von Neumann algebra theory in order to handle such a new situation. This opens up a new direction of research in quantizing spaces of maps betweem more general quantum spaces.

Keywords: C*-algebras, non-commutative topology, quantum space, quantum space of all maps, quantum families of all maps, compact quantum semigroup, compact quantum group.

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Contents

Chapter 1

C*-algebra

1.1 Elementary Theories of C*-algebra

Definition 1.1. A C*-algebra is a Banach algebra A over $\mathbb{C}$ endowed with an anti-linear and anti-multiplicative involutive mapping

$$A \ni a \mapsto a^* \in A$$

such that

$$\|a^*a\| = \|a\|^2 \qquad \text{for all } a \in A. \tag{1.1}$$

If A has a multiplication unit, then we call it *unital*.

A norm satisfying condition (1.1) is called a *C*-norm*. The relation between the norm of a C*-algebra and the algebraic structure expressed by equality (1.1) has very far-reaching consequences. Some textbooks such as [13] provide an enormous amount of knowledge of the C*-algebra theories. I will use [13] as the main reference for most of the results in this chapter.

We first recall a few examples of C*-algebras that will be the main basic ingredients relevant to this thesis.

Example 1.2. The most trivial example of C*-algebras is the field $\mathbb{C}$ itself with the complex conjugation as its involution and the modulus as its norm.

Example 1.3. The vector space $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a Hilbert space $\mathscr{H}$ is a C*-algebra with a multiplicative unit (namely, the identity operator on $\mathscr{H}$) when equipped with:

- the *composition multiplication*: for $S,T \in \mathscr{B}(\mathscr{H})$,

$$TS := T \circ S, \quad \text{i.e.,} \quad (TS)(h) := T(S(h)) \text{ for all } h \in \mathscr{H}$$

- the *operator norm*: for $T \in \mathscr{B}(\mathscr{H})$,

$$\|T\| := \sup_{\|h\| \leq 1, h \in \mathscr{H}} \|T(h)\|$$

- the *adjoint involution*: for $T \in \mathscr{B}(\mathscr{H})$, the adjoint operator $T^* \in \mathscr{B}(\mathscr{H})$ is uniquely determined by

$$< h, T(k) > = < T^*(h), k > \text{ for all } h,k \in \mathscr{H}.$$

This C*-algebra is non-commutative unless $\mathscr{H}$ is one dimensional.

Example 1.4. For any $n \in \mathbb{N}$, we can identify the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with $\mathscr{B}(\mathbb{C}^n)$ in the usual way: i.e., $T \in \mathscr{B}(\mathbb{C}^n)$ corresponds to $A \in M_n(\mathbb{C})$ uniquely determined by $T(v) = Av$. Hence $M_n(\mathbb{C})$ is a C*-algebra equipped with the operator norm and the Hermitian involution

$$A^* := \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}^* = \begin{pmatrix} a^*_{1,1} & \cdots & a^*_{n,1} \\ \vdots & \ddots & \vdots \\ a^*_{1,n} & \cdots & a^*_{n,n} \end{pmatrix}$$

for any $A \in M_n(\mathbb{C})$. Also if B is any C*-algebra, then the algebra $M_n(B) \cong M_n(\mathbb{C}) \otimes B$ of all $n \times n$ matrices with entries in B is also a C*-algebra. (See Chapter 6, [11]).

Before we proceed further, we will always assume that compact and locally compact spaces are Hausdorff.

Example 1.5. Let X be a compact space. Then the space $C(X) \equiv C(X,\mathbb{C})$ of all continuous complex-valued functions on X is a commutative C*-algebra with unit (the constant function **1**) where the operations are defined pointwise as

$$f^*(x) := \overline{f(x)}$$
$$(f+g)(x) := f(x)+g(x)$$
$$(fg)(x) := f(x)g(x)$$
$$(\lambda f)(x) := \lambda f(x)$$

and the norm is the sup-norm

$$\|f\| = \sup_{x\in X}|f(x)|$$

for any $\lambda \in \mathbb{C}$ and $f,g \in C(X)$.

Example 1.6. More generally, let X be a locally compact space. We say that a continuous function f from X to $\mathbb{C}$ *vanishes at infinity*, if for each positive $\varepsilon > 0$, the set $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact. We denote the set of such functions by $C_0(X)$. Equipped with the pointwise operations and sup-norm defined as in the previous example, it becomes a C*-algebra. It is unital if and only if X is compact (and hence $C_0(X) = C(X)$).

Example 1.7. Let $n \in \mathbb{N}$. Then by viewing $\mathbb{C}^n$ as $C(N,\mathbb{C})$, i.e., the algebra of continuous complex-valued functions on the finite set $N := \{1,2,...,n\}$, we see that $\mathbb{C}^n$ is also an example of C*-algebras.

It is not always the case that a C*-algebra has unit. However, we can adjoin a unit to it to make the algebra unital. The technique is as follows:

For any complex *-algebra A, the vector space direct sum $A \oplus \mathbb{C}$ becomes a unital *-algebra A', called the *unitization* of A, when endowed with the multiplication

$$(a,\lambda)(b,\mu) := (ab+\lambda b+\mu a, \lambda\mu)$$

and involution

$$(a,\lambda)^* := (a^*, \bar{\lambda})$$

for $a,b \in A$ and $\lambda,\mu \in \mathbb{C}$. When A is a C*-algebra, then there is a (necessarily unique) norm on its unitization A', extending the norm of A and making A' into a C*-algebra. (See Theorem 2.1.6, [13]).

Definition 1.8. For a unital complex algebra A, we define the *spectrum* of $a \in A$ as

$$\sigma_A(a) := \{z \in \mathbb{C} \mid a - z \text{ is not invertible in } A\}.$$

For a non-unital complex algebra A, we define the *spectrum* of $a \in A$ as

$$\sigma_A(a) := \sigma_{A'}(a).$$

Definition 1.9. Let A be a unital C*-algebra with the unit $\mathbb{1}_A$.

- An element $a \in A$ is *self-adjoint* if $a = a^*$.
- An element $a \in A$ is a *projection* if $a = a^2 = a^*$.
- An element $a \in A$ is a *unitary* if $a^*a = aa^* = \mathbb{1}_A$.
- An element $a \in A$ is *positive* if a is self-adjoint and $\sigma_A(a) \subseteq \mathbb{R}^+ := \{x \in \mathbb{R} \mid x \geq 0\}$.

Remark 1.10. For each $a \in A$, there exist unique self-adjoint elements $b,c \in A$ such that

$$a = b + ic$$

$(b = \frac{1}{2}(a+a^*)$ and $c = \frac{1}{2i}(a-a^*))$.

If a is a self-adjoint element of the closed unit ball of a unital C*-algebra A, then $\mathbb{1}_A - a^2$ is positive and the unique elements

$$u = a + i\sqrt{\mathbb{1}_A - a^2} \qquad \text{and} \qquad v = a - i\sqrt{\mathbb{1}_A - a^2}$$

are unitaries such that $a = \frac{\mathbb{1}_A}{2}(u+v)$. Hence, the unitaries linearly span A.

Remark 1.11. We shall denote the set of all self-adjoint elements in A by A_{sa}.

Definition 1.12. Let A be a C*-algebra and denote the set of all positive elements in A by A^+. If $T : A \to B$ is a linear map between C*-algebras, it is said to be *positive* if $T(A^+) \subseteq B^+$. A *state* on a C*-algebra A is a positive linear functional $\phi : A \to \mathbb{C}$ of norm one. We shall denote the set of states of A by $S(A)$.

Theorem 1.13 (Theorem 3.3.7, [13])**.** *Suppose that τ is a positive linear functional on a C*-algebra A. Then the inequality*

$$\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$$

holds for every $a, b \in A$.

We shall look into finite dimensional and finitely generated C*-algebras later in Chapter 2. So we recall what these two notions mean:

Definition 1.14. Let k be a field. A k-algebra A is called *finite dimensional* if there exist elements $a_1, a_2, ..., a_n \in A$ linearly span A. We call A a *finitely generated* k-algebra if there are elements $a_1, a_2, ...a_n \in A$ such that A is a quotient algebra of $k[a_1, a_2, .., a_n]$, where $k[a_1, a_2, .., a_n]$ is the polynomial ring in n indeterminates.

Theorem 1.15 (Theorem III.1.1 in [7])**.** *Every finite dimensional C*-algebra A is *-isomorphic to the direct sum of matrix algebras*

$$A \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

In particular, every finite dimensional C*-*algebra is unital.*

1.2 Gelfand Duality

As mentioned earlier in the abstract, non-commutative topology aims to extend the study of topological spaces to more general objects by viewing the category of spaces as a subcategory of some larger category of objects endowed with extra structure. In the case that we are studying, we will identify the category of locally compact spaces with a subcategory of the category of C*-algebras. The famous theorem of Gelfand provides us both the motivation and mathematical background for this point of view. Before recalling it, let us introduce the notion of *-homomorphisms between C*-algebras.

Definition 1.16. Let A, B be C*-algebras and let $\Phi : A \to B$. We say that Φ is a *-*homomorphism* if Φ is linear, multiplicative, i.e., $\Phi(ab) = \Phi(a)\Phi(b)$, and involutive, i.e., $\Phi(a^*) = \Phi(a)^*$ for all $a, b \in A$. When A and B are unital, we call Φ *unital* if it maps the unit of A to the unit of B.

We shall just state without proofs a few results about *-homorphisms. Interested readers can refer to ([13] or [2]) for further details.

Proposition 1.17. Any *-homomorphism is necessarily norm decreasing (the result is still true if the domain is just a Banach *-algebra). A *-isomorphism (i.e., a bijective *-homomorphism) is hence isometric.

Proposition 1.18. Every *-homomorphism is positive.

Proposition 1.19. The kernel of a *-homomorphism is a closed two-sided *-ideal.

Theorem 1.20. *Let A be a C*-algebra. If I is a closed two-sided *-ideal of A, then A/I equipped with the quotient norm is a C*-algebra.*

Now we state Gelfand Theorem which is the foundation of non-commutative topology.

Theorem 1.21 (Gelfand Theorem, Theorem 2.1.10, [13])**.** *Let A be a commutative C*-algebra. Then there exists a unique (up to homeomorphism) locally compact space X such that A is *-isomorphic to the C*-algebra $C_0(X)$.*

In fact, X is just $\hat{A}$, the character space of A (the space of algebra *-homomorphisms onto $\mathbb{C}$) endowed with the weakest topology that makes all functions of the form

$$\tau \mapsto \tau(a)$$

where $a \in A$, $\tau \in \hat{A}$, continuous.

By a simple trick of *adjoining the unit*, we can adapt Theorem 1.21 to the case of unital C*-algebra where the corresponding space is a compact space (see Theorem 2.1.20 in [13]).

1.2.1 Multipliers and morphisms

Let A be a non-zero C*-algebra. The *multiplier algebra* of A, is canonically defined as the unital C*-algebra $M(A)$ such that A is identified as an ideal of $M(A)$ which is *essential*, i.e., any other non-zero ideal of $M(A)$ has a non-zero intersection with A. $M(A)$ satisfies a universal property : if C is a unital C*-algebra and we are given an embedding of A into C as an essential ideal, then there exists an injective *-homomorphism of C into $M(A)$ extending the identity map $A \to A$ (considered as a map from a subset of C onto a subset of $M(A)$). Hence $M(A)$ can be regarded as the maximal non-degenerate unitization of a non-unital C*-algebra. Note that we have $M(A) = A$ if and only if A is unital (see Section 2.3 in [8]).

Example 1.22. Let X be locally compact space. Then we have

$$M(C_0(X)) \cong C_b(X) \cong C(\beta X)$$

where $C_b(X) \equiv C_b(X, \mathbb{C})$, the C*-algebra of all bounded continuous complex-valued functions on X and βX is the Stone-Čech compactification of X. (See Proposition 1.10 in [10]).

Now consider C*-algebras A and B. By *morphisms* from A to B, we mean a *-homomorphism $\Phi : A \to M(B)$ such that $\Phi(A)B$ is dense in B. The notation $\Phi(A)B$ means the linear span of all products of the form $\Phi(a)b$ with $a \in A$ and $b \in B$. Such a product will be automatically an element of B since B is an ideal of $M(B)$. The set of all morphisms from A to B will be denoted as $\text{Mor}(A,B)$. Using the fact that A is an essential closed *-ideal of $M(A)$, any $\Phi \in \text{Mor}\ (A,B)$ extends to a unique *-homomorphism $\bar{\Phi}$ of unital algebras $M(A) \to M(B)$. Hence if C is another C*-algebra and $\Psi \in \text{Mor}\ (C,A)$, then we have a well-defined map $\bar{\Phi} \circ \Psi : C \to M(B)$, which is a morphism from C to B. So we can compose morphisms and hence get a category of C*-algebras with morphisms as defined.

1.2.2 Gelfand Duality

We note that the category of all commutative C*-algebras with morphisms as defined above is equivalent to the category of all locally compact spaces with continuous maps as morphisms. In other words, any statement about locally compact spaces and continous maps between them can be formulated as a statement about commutative C*-algebras and their morphisms and vice versa. We will explain more concretely the correspondence as follows:

Let X be a locally compact space. Then the C*-algebra $C_0(X)$ is commutative and $M(C_0(X)) \cong C_b(X)$. A continuous map $\phi : X \to Y$ between locally compact spaces induces a morphism $\Phi \in \text{Mor}(C_0(Y), C_0(X))$ via

$$\Phi(f)(x) = f(\phi(x)), \qquad \text{where } f \in C_0(Y).$$

Conversely, let A be a commutative C*-algebra. By Theorem 1.21, we know that $A \cong C_0(\hat{A})$ with $\hat{A}$ a locally compact space. Any morphism $\Phi : A \to B$ between commutative C*-algebras defines a continuous map between locally compact spaces:

$$\phi : \hat{B} \to \hat{A}$$

$$\tau \mapsto \tau \circ \Phi$$

The above correspondence is called the *Gelfand duality*.

1.2.3 Gelfand Naimark Theorem

Gelfand Theorem tells us that every commutative C*-algebra can be viewed as an algebra of functions on a unique (up to homeomorphism) locally compact space. In 1943, Israel Gelfand and Mark Naimark made a significant development of the theory of general C*-algebras. They showed that any C*-algebra A can be regarded as a C*-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. In order to state the theorem, we shall get sidetracked and learn some vital terms.

Definition 1.23. A *representation* of a *-algebra A is a pair $(\mathscr{H}, \pi)$ where $\mathscr{H}$ is a Hilbert space and $\pi : A \to \mathscr{B}(\mathscr{H})$ is a *-algebra homomorphism. We say that $(\mathscr{H}, \pi)$ is *faithful* if π is injective.

Definition 1.24. If $(\mathscr{H}, \pi)$ is a representation of a *-algebra A, then we say that $v \in \mathscr{H}$ is a π-*cyclic vector* if $\pi(A)(v) := \{\pi(a)(v) \mid a \in A\}$ is dense in $\mathscr{H}$. If $(\mathscr{H}, \pi)$ admits a π-cyclic vector, then we say that it is a *cyclic* representation.

Definition 1.25. Two representations $(\mathscr{H}_1, \pi_1)$ and $(\mathscr{H}_2, \pi_2)$ of a C*-algebra A are *(unitarily) equivalent* if there is a unitary operator $u : \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$\pi_2(a) = u\pi_1(a)u^* \qquad \text{for all } a \in A.$$

Before defining the notion of a direct sum of representations of a *-algebra A, we first recall the notion of a direct sum of Hilbert spaces.

Proposition 1.26. (Proposition 6.2, [5]) If $\mathscr{H}_1, \mathscr{H}_2, \dots$ are Hilbert spaces, the set

$$\mathscr{H} := \left\{ (h_n)_{n=1}^{\infty} \mid h_n \in \mathscr{H}_n \text{ for all } n \text{ and } \sum_{n=1}^{\infty} \|h_n\|^2 < \infty \right\}$$

is a Hilbert space when endowed with an inner product defined by

$$< h, g > := \sum_{n=1}^{\infty} < h_n, g_n >,$$

for all $h = (h_n)_{n=1}^{\infty}$ and $g = (g_n)_{n=1}^{\infty}$ in $\mathscr{H}$. Note that the norm relative to this inner product is $\|h\| := \left(\sum_{n=1}^{\infty} \|h_n\|^2 \right)^{\frac{1}{2}}$.

Definition 1.27. If $\mathscr{H}_1, \mathscr{H}_2, \dots$ are Hilbert spaces, the space $\mathscr{H}$ in the previous proposition is called the *direct sum* of $\mathscr{H}_1, \mathscr{H}_2, \dots$ and is denoted by $\mathscr{H} \equiv \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \cdots$.

Remark 1.28. Note that we can generalize the above definition to arbitrary index set I. That is, we can consider $\{\mathscr{H}_i\}_{i \in I}$ and the direct sum $\bigoplus_{i \in I} \mathscr{H}_i$. The set

$$\bigoplus_{i \in I} \mathscr{H}_i := \left\{ (h_i)_{i \in I} \mid h_i \in \mathscr{H}_i \text{ for every } i \in I \text{ and } \sum_{i \in I} \|h_i\|^2 < \infty \right\}$$

is a Hilbert space when endowed with an inner product defined by

$$< h, g > := \sum_{i \in I} < h_i, g_i >,$$

for all $h, g \in \bigoplus_{i \in I} \mathscr{H}_i$.

Note that the condition $\sum_{i \in I} \|h_i\|^2 < \infty$ implies that $h_i = 0$ for all $i \in I$ outside a countable set $\{i_1, i_2, \dots, i_n, \dots\}_{n \in \mathbb{N}} \subseteq I$ such that $\sum_{i \in I} \|h_i\|^2 = \sum_{j \in \mathbb{N}} \|h_{i_j}\|^2$, and the above inner product is well-defined. Note that the norm induced by the inner product is

$$\|h\| := \left(\sum_{i \in I} \|h_i\|^2 \right)^{\frac{1}{2}} = \left(\sum_{j \in \mathbb{N}} \|h_{i_j}\|^2 \right)^{\frac{1}{2}}.$$

Lemma 1.29. (Exercise 12, Section 1 of [5]) Let $\{\mathscr{H}_i\}_{i\in I}$ be a collection of Hilbert spaces and let the Hilbert space $\mathscr{H} = \oplus_i \mathscr{H}_i$. For each $i \in I$, let $T_i : \mathscr{H}_i \to \mathscr{H}_i$ be a bounded linear operator on $\mathscr{H}_i$ such that the family $\{T_i\}_{i\in I}$ of bounded linear operators is uniformly bounded, i.e., $\sup\{\|T_i\| \mid i \in I\} < \infty$. The *direct sum* of the uniformly bounded family $\{T_i\}_{i\in I}$ is the operator

$$\bigoplus_{i\in I} T_i : \mathscr{H} \to \mathscr{H}$$

defined by

$$\left(\bigoplus_{i\in I} T_i(h)\right)_i := T_i(h_i), \qquad \text{i.e.,} \quad \left(\bigoplus_{i\in I} T_i\right)\Bigg|_{\mathscr{H}_i} = T_i$$

for every $h \in \mathscr{H}$. Then $\bigoplus_{i\in I} T_i$ is a bounded linear operator on $\mathscr{H}$ whose norm is

$$\left\|\bigoplus_{i\in I} T_i\right\| = \sup\{\|T_i\| \mid i \in I\}.$$

Definition 1.30. If π_i with $i \in I$ is a representation of a *-algebra A on $\mathscr{H}_i$, then the *direct sum* $\oplus_{i\in I}\pi_i$ of π_i's is a representation of A on $\oplus_{i\in I}\mathscr{H}_i$. More precisely, we have

$$\left(\left(\bigoplus_{i\in I}\pi_i\right)(a)\right)\left(\bigoplus_{i\in I} h_i\right) = \bigoplus_{i\in I}\pi_i(a)h_i \qquad \text{for all } a \in A,\ h_i \in \mathscr{H}_i \quad i \in I.$$

If each π_i is equivalent to a fixed representation ρ, then $\oplus_{i\in I}\pi_i$ is a *multiple* or *amplification* of ρ by $card(I)$, and we write $\oplus_{i\in I}\pi_i = (card(I))\rho$. By grouping together equivalent π_i's in $\pi := \bigoplus_{i\in I}\pi_i$, we get $\pi = \bigoplus_{j\in J}(m_j)\pi_j$ for some $J \subseteq I$ and $m_j \in \mathbb{N}$ is called the multiplicity of π_j in π, where π_j's are mutually inequivalent.

Proposition 1.31 (page 244, [14])**.** Every representation of a *-algebra on a Hilbert space $\mathscr{H}$ is the direct sum of cyclic representations.

Remark 1.32. For every representation Λ, we shall write

$$\Lambda = \bigoplus_{\rho_\Lambda} (m_{\rho_\Lambda}) \rho_\Lambda$$

where each ρ_Λ is a cyclic representation and m_{ρ_Λ} is its multiplicity.

The GNS construction, discovered independently by Gelfand and Naimark, and Segal is one of the most fundamental ideas in the theory of operator algebras. It provides a method for manufacturing representations of C*-algebras. The construction ultilizes positive linear functionals on a C*-algebra A. With each positive linear functional, there is associated a representation:

Suppose that τ is a positive linear functional on a C*-algebra A. Setting

$$N_\tau = \{a \in A \mid \tau(a^* a) = 0\},$$

and by Theorem 1.13, we see that N_τ is a closed left ideal of A and that the map

$$\begin{aligned} (A/N_\tau) \times (A/N_\tau) &\longrightarrow \mathbb{C} \\ (a + N_\tau, b + N_\tau) &\mapsto \tau(b^* a), \end{aligned}$$

is a well-defined inner product on A/N_τ. We denote by $\mathscr{H}^\tau$ the Hilbert space completion of A/N_τ.

If $a \in A$, define an operator $\psi(a) \in \mathscr{B}(A/N_\tau)$ by setting

$$\psi(a)(b + N_\tau) = ab + N_\tau.$$

The inequality $\|\psi(a)\| \leq \|a\|$ holds since we have

$$\begin{aligned} \|\psi(a)(b + N_\tau)\|^2 &= \tau(b^* a^* ab) \\ &\leq \|a\|^2 \tau(b^* b) \\ &= \|a\|^2 \|b + N_\tau\|^2. \end{aligned}$$

The operator $\psi(a)$ has a unique extension to a bounded linear operator $\psi_\tau(a)$ on $\mathscr{H}^\tau$. The map

$$\psi_\tau : A \to \mathscr{B}(\mathscr{H}^\tau)$$
$$a \mapsto \psi_\tau(a)$$

is a *-homomorphism, i.e., a representation of A.

The representation $(\mathscr{H}^\tau, \psi_\tau)$ of A is the *Gelfand-Naimark-Segal* representation (or *GNS* representation) associated to τ. If A is non-zero, we define its *universal* representation to be the direct sum of all representations $(\mathscr{H}^\tau, \psi_\tau)$, where τ ranges over the state space $S(A)$. Because of the abundance of states (i.e., positive linear functionals of norm 1) on a C*-algebra (see section 3.3, [13]), every C*-algebra has many representations. In fact, we can say even more:

Theorem 1.33 (Gelfand-Naimark, Theorem 3.4.1,[13])**.** *If A is a C*-algebra, then it has a faithful representation. Specifically, its universal representation is faithful.*

So the above theorem says that any C*-algebra can be concretely expressed as a closed *-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. The proof of this theorem is readily available in most C*-algebras textbooks like [2] or [13]. Hence often times, we shall just abuse the notation and regard any C*-algebra as some C*-subalgebra of $\mathscr{B}(\mathscr{H})$.

Before we proceed further, let us note the following:

Let A and B be C*-algebras. Among many norms on the algebraic tensor product $A \otimes_{\mathrm{alg}} B$, there is always the *smallest* norm satisfying the appropriate analog of (1.1) (see Appendix T in [25]). This norm is called the *minimal* C*- norm on $A \otimes_{\mathrm{alg}} B$ and the completion of $A \otimes_{\mathrm{alg}} B$ with respect to this norm, denoted by $A \otimes B$ is called the *minimal tensor product* of A and B. In the case where either A or B is commutative, this is in fact the only C*-norm on $A \otimes_{\mathrm{alg}} B$. We will only consider the minimal tensor product of two C*-algebras in this dissertation.

Then by Lemma T6.16 in [25], we have:

Proposition 1.34. Let X and Y be locally compact spaces. Then the minimal tensor product $C_0(X) \otimes C_0(Y)$ is canonically isomorphic to the C*-algebra $C_0(X \times Y)$. Under this isomorphism the tensor product $f \otimes g \in C_0(X) \otimes C_0(Y)$ is mapped to the function

$$X \times Y \ni (x,y) \mapsto f(x)g(y) \in \mathbb{C}$$

1.3 Universal C*-algebras and Free products

1.3.1 Universal C*-algebras

In this subsection, we shall define a universal C*-algebra on a set $\mathscr{S} = \{x_i\}_{i \in I}$ of generators and a set of relations $\mathscr{R}$. In contrast with universal rings or algebras, where one can consider quotients by free rings to contruct universal objects, universal C*-algebras need to involve algebras of bounded operators on a Hilbert space $\mathscr{H}$ (by Gelfand-Naimark Theorem) and the relations must impose a finite uniform bound on the norm of operators representing each generator. This means that depending on the generators and relations, a universal C*-algebra might not exist. We shall refer to Section II.8.3 in [4] for the definition and examples of universal C*-algebras.

Suppose that a set $\mathscr{S} = \{x_i\}_{i \in I}$ of generators and a set $\mathscr{R}$ of relations are given. We could allow the relations in $\mathscr{R}$ to be fairly general kind of relations but in this thesis, we shall only focus on the following kind of relations

$$p(x_{i_1}, \dots, x_{i_n}, x_{i_1}^*, \dots, x_{i_n}^*)^* p(x_{i_1}, \dots, x_{i_n}, x_{i_1}^*, \dots, x_{i_n}^*) \leq \kappa$$

where p is a polynomial in $2n$ noncommuting variables with complex coefficients and $\kappa \geq 0$. If $\kappa = 0$, then the relation may be rewritten as an algebraic relation among $x_{i_1}, \dots, x_{i_n}, x_{i_1}^*, \dots, x_{i_n}^*$ and the scalars. The only restriction on the relations is that they must be realizable among operators on a Hilbert space and they must impose a finite upper bound on the norm of each generator when realized as an operator. A *representation* of $(\mathscr{S}, \mathscr{R})$ is a set of operators $\{T_i\}_{i \in I}$ on a Hilbert space $\mathscr{H}$ which satisfy

$$\|p(T_{i_1}, \dots, T_{i_n}, T_{i_1}^*, \dots, T_{i_n}^*)\|^2 \leq \kappa$$

whenever

$$(p(x_{i_1},\ldots,x_{i_n},x_{i_1}^*,\ldots,x_{i_n}^*)^* p(x_{i_1},\ldots,x_{i_n},x_{i_1}^*,\ldots,x_{i_n}^*) \le \kappa) \in \mathscr{R}.$$

A representation of $(\mathscr{S},\mathscr{R})$ will then define a *-homomorphism π from the free *-algebra $\mathcal{A}$ generated by $\mathscr{S}$ into $\mathscr{B}(\mathscr{H})$ (i.e., a representation of $\mathcal{A}$ on a Hilbert space $\mathscr{H}$).

Now for any $a \in \mathcal{A}$, we let

$$|||a||| := \sup\{\|\pi(a)\| \mid \pi : A \to \mathscr{B}(\mathscr{H}) \text{ is a representation of } A \text{ on some Hilbert space } \mathscr{H}\}.$$

If this supremum is finite for all $a \in \mathcal{A}$ (it is enough to check this on the generators), it defines a *C*-seminorm* (i.e., a seminorm satisfying (1.1) on $\mathcal{A}$. If ρ is any C*-seminorm on a *-algebra A, then the set $N = \rho^{-1}\{0\}$ is a self adjoint two-sided ideal of A. Hence we get a C*-norm on the quotient *-algebra A/N by setting $\|a+N\| = \rho(a)$. If B denotes the Banach space completion of A/N with this norm, then the multiplication and involution operations of A/N extend uniquely to operations of the same type on B and make B a C*-algebra. We call B the *enveloping* C*-algebra of the pair (A,ρ). The universal C*-algebra of $(\mathscr{S},\mathscr{R})$ is defined to be the enveloping C*-algebra of $(\mathcal{A}, |||\cdot|||)$. Hence, we have

Definition 1.35. The completion of $\mathcal{A}/\{a \mid |||a||| = 0\}$ under the quotient norm $|||\cdot|||$ is called the *universal C*-algebra* A of $(\mathscr{S},\mathscr{R})$. We shall say that $(\mathscr{S},\mathscr{R})$ is a *presentation* of A.

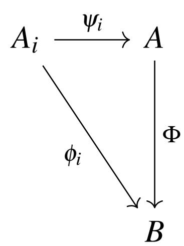
Example 1.36. (Example 1.3(a),[3]) Let A be any C*-algebra, $\mathscr{S} = A$, $\mathscr{R}$ be the set of all *-algebraic relations in A. Then the universal C*-algebra of $(\mathscr{S},\mathscr{R}) \cong A$.

Example 1.37. (Example 1, Section II.8.3.2 in [4]) There is no universal C*-algebra generated by a single self-adjoint element, since there is no bound on the norm of the element. But there is a universal C*-algebra generated by a single self-adjoint element of norm one, with $\mathscr{S} = \{x\}$ and $\mathscr{R} = \{x = x^*, x^*x \le 1\}$. This C*-algebra is isomorphic to $C([-1,1])$.

1.3.2 Free Products

We shall also work with the free products of C*-algebras in this thesis. We first define what is the free product of unital algebras and then our concern, the free product of C*-algebras. (See Chapter 1 of [24]).

Definition 1.38. Let $\{A_i\}_{i\in I}$ be a family of unital *-algebras. Then their *unital *-algebraic free product* $\hat{\star}_{i\in I} A_i$ is the unique unital *-algebra A together with unital *-homomorphisms $\psi_i : A_i \to A$ such that given any unital *-algebra B and unital *-homomorphisms $\phi_i : A_i \to B$ there exists a unique unital *-homomorphism $\Phi : A \to B$ making the diagram



commute.

Remark 1.39. If $A_i = C$ for all $i \in \{1,2,...,n\}$ (i.e., the n copies of A_i are the same), we shall use the notation $C^{\hat{\star} n}$ for $\hat{\star}_{i\in\{1,2,...,n\}} C$.

As a vector space, the free product $\hat{\star}_{i\in I} A_i$ is the quotient of the vector space which has as basis the set

$$B = \{a_1 a_2 \cdots a_n \mid n \in \mathbb{N}, a_j \in A_{\iota_j}, \iota_1 \neq \iota_2 \neq \cdots \neq \iota_n\}$$

by the subspace generated by the relations of the forms

$$\begin{aligned} &a_1 \cdots a_{j-1} \left(\lambda a_j + \mu a_{j'}\right) a_{j+1} \cdots a_n \\ &= \lambda a_1 \cdots a_{j-1} a_j a_{j+1} \cdots a_n + \mu a_1 \cdots a_{j-1} a_{j'} a_{j+1} \cdots a_n \end{aligned}$$

where $\lambda, \mu \in \mathbb{C}$ and

$$a_1 \cdots a_n = a_1 \cdots a_{j-1} a_{j+1} \cdots a_n \qquad \text{where } a_j = 1.$$

With the multiplication and involution defined as

$$(a_1a_2\cdots a_n)(a'_1a'_2\cdots a'_m) := a_1a_2\cdots a_na'_1a'_2\cdots a'_m$$

and

$$(a_1a_2\cdots a_n)^* := a_n^*a_{n-1}^*\cdots a_1^*$$

for any $a_1a_2\cdots a_n, a'_1a'_2\cdots a'_m \in B$ respectively, we say that $\hat{\star}_{i\in I}A_i$ is the *-algebra that is generated by copies of A_i's with no additional relations.

Now consider C*-algebras A_i's with their algebraic free products $\hat{\star}_{i\in I}A_i$. We have canonical unital *-homomorphisms $\pi_i : A_i \to \hat{\star}_{i\in I}A_i$ that embed A_i into $\hat{\star}_{i\in I}A_i$ respectively. For a unital C*-algebra D and unital *-homomorphisms $\phi_i : A_i \to D$, by the universal property of algebraic free products in Definition 1.38, there is a unique *-homomorphism $\Phi : \hat{\star}_{i\in I}A_i \to D$ such that

$$\Phi\circ\pi_i = \phi_i$$

for each $i\in I$. Hence we see that Φ extends ϕ_i for each $i\in I$, when A_i is viewed as a subalgebra of $\hat{\star}_{i\in I}A_i$ by the embedding π_i.

Consider the case where $D = \mathscr{B}(\mathscr{H})$ so that Φ and hence π_i's are Hibert space representations of their respective algebras. From our previous arguments, we see that the representations of $\hat{\star}_{i\in I}A_i$ are in $1-1$ correspondence with collections of representations of A_i's, which act on the same Hilbert space. Then we introduce a C*-seminorm on $\hat{\star}_{i\in I}A_i$ by defining

$$|||a||| := \sup\{\|\Phi(a)\| \mid \Phi \text{ is a representation of } \hat{\star}_{i\in I}A_i\}$$

for each $a\in\hat{\star}_{i\in I}A_i$. Hence we reach the following definition:

Definition 1.40. Let A_i's be unital C*-algebras and $\hat{\star}_{i\in I}A_i$ be the unital *-algebraic free products of A_i's. The enveloping C*-algebra of $\hat{\star}_{i\in I}A_i$ with respect to the C*-seminorm $|||\cdot|||$ is called the unital *C*-free product* of A_i's and will be denoted by $\star_{i\in I}A_i$.

Remark 1.41. Note that by the way we define the C*-free product of A_i's, we can also write the definition alternatively in the following way (similar to the way we defined for the algebraic free product of A_i's) :

If $\{A_i\}_{i\in I}$ is a family of unital C*-algebras, then their unital C*-free product $\star_{i\in I}A_i$ is the unique unital C*-algebra together with unital *-homomorphisms $\psi_i : A_i \to A$ such that given any unital C*-algebra B and unital *-homomorphisms $\phi_i : A_i \to B$, there exists a unique unital *-homomorphism $\Phi = \star_{i\in I}\phi_i : A \to B$ such that the following diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\psi_i} & A \\ & \searrow^{\phi_i} & \downarrow \Phi \\ & & B \end{array}$$

commutes.

Remark 1.42. We shall use the notation $C^{\star n}$ for $\star_{i\in I}A_i$ if all the $A_i = C$. We also see that the free product of A_i's is the universal C*-algebra generated by copies of the A_i's with no additional relations.

1.4 Inductive Limit

As an important construction of operator algebras, we shall also discuss the inductive limit of C*-algebras. The main reference for this section is Section 1, Chapter XIV of [22].

Definition 1.43. An *inductive sequence* of C*-algebras means a sequence $\{A_n \mid n \in \mathbb{N}\}$ of C*-algebras together with a sequence of $\{\pi_n \mid n \in \mathbb{N}\}$ of *-homomorphisms such that π_n maps A_n into A_{n+1} for each $n \in \mathbb{N}$. We write $\{A_n, \pi_n\}$ or

$$A_1 \xrightarrow{\pi_1} A_2 \xrightarrow{\pi_2} A_3 \to \cdots \to A_n \xrightarrow{\pi_n} A_{n+1} \to \cdots.$$

If each A_n is unital and π_n preserves the identity, then the inductive sequence $\{A_n, \pi_n\}$ is called *unital*.

Suppose $\{A_n, \pi_n\}$ is an inductive sequence of C*-algebras. For each $k \in \mathbb{N}$, we set

$$\pi_{n+k,n} = \pi_{n+k-1} \circ \pi_{n+k-2} \circ \cdots \circ \pi_{n+1} \circ \pi_n.$$

Hence we have

$$\pi_{k,j} \circ \pi_{j,i} = \pi_{k,i}, \qquad i < j < k.$$

We shall set $\pi_{n,n} = \mathrm{id}$. Considering $\{A_n\}$ as a sequence of disjoint sets, we take the disjoint union

$$X = \bigsqcup_{n=1}^{\infty} A_n.$$

For $a \in A_n$ and $b \in A_m$, we write $a \sim b$ if

$$\pi_{l,n}(a) = \pi_{l,m}(b)$$

for sufficiently large l. We see that $\sim$ is an equivalence relation in X, giving rise to a quotient set $A_\infty := X/\sim$, consisting of all equivalence classes in X. We shall denote the equivalence class of each $a \in X$ by $[a] \in A_\infty$. We define $\pi_{\infty,n} : A_n \to A_\infty$ by $\pi_{\infty,n}(a) := [a]$ for each $a \in A_n$. Now, we introduce an algebra structure in A_∞ over $\mathbb{C}$ as follows:

For each $\lambda \in \mathbb{C}$, $a \in A_n$ and $b \in A_m$, we choose $l > m, n$ and set

$$\lambda[a] := [\lambda a],$$
$$[a] + [b] := [\pi_{l,n}(a) + \pi_{l,m}(b)],$$
$$[a][b] := [\pi_{l,n}(a)\pi_{l,m}(b)],$$
$$[a]^* := [a^*].$$

It can be shown that the above sum and multiplication do not depend on the choice of representatives and l, i.e., they are determined completely by $[a]$ and $[b]$ (See page 83 in [22]).

Equipped with these algebraic operations, A_∞ becomes an involutive algebra over $\mathbb{C}$. Note also that each $\pi_{\infty,n}$ is a *-homomorphism of A_n into A_∞ and

$$\begin{aligned}
&\pi_{\infty,1}(A_1) \subseteq \pi_{\infty,2}(A_2) \subseteq \cdots \subseteq \pi_{\infty,n}(A_n) \subseteq \cdots, \\
&A_\infty = \bigcup_{n=1}^{\infty} \pi_{\infty,n}(A_n).
\end{aligned}$$

Putting $\overline{A_n} = \pi_{\infty,n}(A_n)$, we obtain a homomorphic image $\overline{A_n}$ of each C*-algebra A_n in A_∞ and $\overline{A_n} \cong A_n/\pi_{\infty,n}^{-1}(0)$. For each $n \in \mathbb{N}$ and $a \in A_n$, $\pi_{\infty,n}(a) = 0$ if and only if $a \sim 0$; if and only if $\pi_{l,n}(a) = 0$ for sufficiently large l. Hence we have

$$\pi_{\infty,n}^{-1}(0) = \bigcap_{k=1}^{\infty} \pi_{n+k,n}^{-1}$$

and

$$\pi_{l,n}^{-1}(0) \subseteq \pi_{m,n}^{-1}(0) \qquad \text{for } l \leq m.$$

Remark 1.44. $\pi_{\infty,n}^{-1}(0)$ need not be closed (refer example 1.2 in [22]).

We say that the inductive sequence $\{A_n, \pi_n\}$ is *proper* if $\pi_{\infty,n}^{-1}(0)$ is closed for sufficiently large n so that $\overline{A_n}$ is a C*-algebra by the natural norm inherited from $A_n/\pi_{\infty,n}^{-1}(0)$. Now we shall return to the discussion of inductive sequence $\{A_n, \pi_n\}$ of C*-algebras with an extra assumption, the properness. Then for large n, $\overline{A_n}$ is a C*-algebra and A_∞ is the union of the increasing sequence of $\{\overline{A_n}\}$ of C^*-algebras. Hence each element of A_∞, say a, has its norm $\|a\|$ as an element of $\overline{A_n}$ for large n. This norm makes A_∞ a pre C*-algebra, i.e., the completion of A of A_∞ under such norm becomes a C*-algebra. Hence we have

Definition 1.45. The C*-algebra A obtained above is called the *inductive limit* of $\{A_n, \pi_n\}$ and written as

$$A = \underrightarrow{\operatorname{Lim}} \{A_n, \pi_n\}.$$

When the connecting homomorphisms π_n's are clearly understood, we write simply

$$A = \varinjlim A_n.$$

Remark 1.46. Replacing A_n by $\overline{A_n}$, we see that each π_n is injective and hence $\pi_{\infty,n}$ is also injective. So $\{A_n\}$ can be viewed as an increasing sequence of C*-subalgebras of A whose union is dense in A.

Chapter 2

Quantum Spaces and Quantum Families of Maps

2.1 Quantum Spaces and their Morphisms

In the last chapter, we see that the theory of locally compact spaces is the same as the theory of commutative C*-algebras. Indeed the two categories are equivalent and any notion pertaining to one class can be expressed using the other. But the class of all C*-algebras (not necessarily commutative ones) with morphisms is also a category (with the commutative ones forming a full subcategory). Now the question is: Can we treat all C*-algebras as algebras of functions? Of course, a non-commutative C*-algebra cannot be isomorphic to an algebra of functions on a space, but it can be viewed as one on a "virtual" space. Non-commutative topology is the study of all C*-algebras from this point of view. Hence we arrive at such definition by S.L. Woronowicz in [26]:

Definition 2.1. A *quantum space* is an object of the category dual to the category of all C*-algebras with morphisms.

We shall use symbols like $\mathbb{X}, \mathbb{Y}, \mathbb{E}, \mathbb{D}$ etc. to denote quantum spaces. Each of them corresponds uniquely to a C*-algebra and the associated C*-algebra will be denoted as $C_0(\mathbb{X})$, $C_0(\mathbb{Y})$, $C_0(\mathbb{E})$ and

$C_0(\mathbb{D})$ respectively. Therefore any C*-algebra $X = C_0(\mathbb{X})$ is thought of as the algebra of functions on a quantum space $\mathbb{X}$. So in this language, the phrase

"Let $\mathbb{X}$ be a quantum space"

means exactly the same thing as

"Let $C_0(\mathbb{X})$ be a C*-algebra"

Notation convention : For any C*-algebra X, we shall use $\mathbb{X}$ (or sometimes $\mathcal{QS}(X)$) to denote its underlying quantum space. In other words, we have $X = C_0(\mathbb{X})$ (or $X = C_0(\mathcal{QS}(X))$).

We shall treat the virtual object $\mathbb{X}$ in a way reminiscent of studying a locally compact space. Hence we have these analogs of definitions that we get from classical spaces (i.e., locally compact spaces):

Definition 2.2.

1. We will say that a quantum space $\mathbb{X}$ is *compact* if $C_0(\mathbb{X})$ is unital. In this case, we will write $C(\mathbb{X})$ for this C*-algebra.

2. We will say that a quantum space $\mathbb{X}$ is *finite* if $C_0(\mathbb{X})$ is finite-dimensional, where in this case, the C*-algebra is automatically unital and $\mathbb{X}$ is compact.

The concept of mappings between quantum spaces is also quite clear. If $\mathbb{X}$ and $\mathbb{Y}$ are quantum spaces, then a *continuous map* from $\mathbb{X}$ to $\mathbb{Y}$ is, by definition, an element of $\mathrm{Mor}(C_0(\mathbb{Y}), C_0(\mathbb{X}))$. In this dissertation, we will only concentrate on compact quantum spaces (i.e., the considered C*-algebras will be unital). Hence we shall write $C(\mathbb{X})$ instead of $C_0(\mathbb{X})$. Note that when a C*-algebra A is unital, $M(A) = A$. Therefore from now on, we will even refrain from writing $\Phi \in \mathrm{Mor}\ (A, B)$ for some unital C*-algebras A, B, favoring the notation $\Phi : A \to B$.

2.2 Quantum Families of Maps

2.2.1 Classical Families of Maps

Consider three sets A, B and C. Clearly, we can see that any family of maps $A \to B$ indexed by elements of C can be equivalently denoted as a mapping from $A \times C \to B$ and vice versa. In fact, this concept can be extended to spaces of continuous mappings of topological spaces. In classical topology the sets of maps between topological spaces are usually themselves given a topology. A particular example is the *compact-open* topology on the space $C(X,Y)$ of continuous maps between topological spaces X and Y in which the collection of the sets

$$U_{\mathbf{K},\mathscr{O}} := \{\psi : X \to Y \text{ continuous } \mid \psi(\mathbf{K}) \subset \mathscr{O}\}$$

with $\mathbf{K}$ compact in X and $\mathscr{O}$ open in Y is a subbasis for the topology. With the space $C(X,Y)$ topologized in this way, we can consider *continuous families of maps* $X \to Y$ which are precisely continuous maps from a topological space E to $C(X,Y)$. We will assume all topological spaces to be Hausdorff. The next theorem depicts the fundamental result about such families (the exponential law for function spaces with the compact-open topology) and provides us the motivation to define quantum families of maps:

Theorem 2.3 (Jackson's Theorem, Theorem 1.1, [9])**.** *Let X, Y and E be topological spaces such that E is locally compact. For $\psi \in C(X \times E, Y)$ define $\sigma(\psi)$ as the mappings from E to $C(X,Y)$ given by*

$$(\sigma(\psi)(e))(x) := \psi(x,e),$$

where $x \in X$ and $e \in E$.

Then σ is a homeomorphism from $C(X \times E, Y)$ onto $C(E, C(X,Y))$ with all spaces of maps topologized by their respective compact-open topologies.

So we see that a continuous family of maps $X \to Y$ labeled by points of E is encoded in a single map $E \to C(X,Y)$ and vice versa.

2.2.2 Definition of Quantum Families of Maps

We wish to extend the classical families of maps into a non-commutative topology setting:

Definition 2.4. Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{E}$ be quantum spaces. A continuous *quantum family of maps* $\mathbb{X} \to \mathbb{Y}$ labeled by $\mathbb{E}$ is a unital *-homomorphism

$$\Phi : C(\mathbb{Y}) \to C(\mathbb{X}) \otimes C(\mathbb{E}).$$

Let us examine the definition in more details. The *-homomorphism Φ describes a continuous map from the quantum space corresponding to $C(\mathbb{X}) \otimes C(\mathbb{E})$ to $\mathbb{Y}$. In view of Jackson's Theorem and Proposition 1.34, we can intepret this as a family of maps $\mathbb{X} \to \mathbb{Y}$ labeled by the quantum space $\mathbb{E}$.

Before we define quantum families of all maps, we shall dwell a little while on the *universal property* of the space $C(X,Y)$ when X and Y are topological spaces. Assume that X is locally compact. and let Γ be the continuous mapping $X \times C(X,Y) \to Y$ defined by

$$\Gamma(x, f) = f(x),$$

where $f \in C(X,Y)$ and $x \in X$.

Now if E is any locally compact space and $\psi : X \times E \to Y$ is continuous, then there exists a unique continuous map $\lambda : E \to C(X,Y)$ such that

$$\Gamma(x, \lambda(e)) = \psi(x, e),$$

where $x \in X$ and $e \in E$. That is, the following diagram:

$$
\begin{array}{ccc}
X \times C(X,Y) & \xrightarrow{\quad\Gamma\quad} & Y \\
{\scriptstyle \mathrm{id}_X \times \lambda}\uparrow & & \Vert \\
X \times E & \xrightarrow{\quad\psi\quad} & Y
\end{array}
$$

commutes.

In fact, the map λ is simply $\sigma(\psi)$ where σ is the homeomorphism in Jackson's Theorem. It is not hard to see that the uniqueness follows from the fact that σ is one-to-one. This universal property of the pair $(C(X,Y),\Gamma)$ characterizes it uniquely. We shall use this property to define the non-commutative analog of $C(X,Y)$ for quantum spaces, namely, the quantum family of all maps.

2.2.3 Existence of Quantum Families of All Maps

In this thesis, we will include some proofs of Sołtan's results for the convenience of the readers to see the useful techniques and arguments involved in the study of this subject.

Definition 2.5. Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{E}$ be quantum spaces and let $\Phi : C(\mathbb{Y}) \to C(\mathbb{X}) \otimes C(\mathbb{E})$ be a quantum family of maps. We say that

- Φ is the *quantum family of all maps* from $\mathbb{X} \to \mathbb{Y}$ and
- $\mathbb{E}$ is the *quantum space of all maps* from $\mathbb{X} \to \mathbb{Y}$

if for any quantum space $\mathbb{D}$ and any quantum family of maps $\Psi : C(\mathbb{Y}) \to C(\mathbb{X}) \otimes C(\mathbb{D})$, there exists a unique $\Lambda : C(\mathbb{E}) \to C(\mathbb{D})$ such that the diagram

$$
\begin{array}{ccc}
C(\mathbb{Y}) & \xrightarrow{\quad\Phi\quad} & C(\mathbb{X}) \otimes C(\mathbb{E}) \\
\Vert & & \downarrow{\scriptstyle \mathrm{id} \otimes \Lambda} \\
C(\mathbb{Y}) & \xrightarrow{\quad\Psi\quad} & C(\mathbb{X}) \otimes C(\mathbb{D})
\end{array}
$$

commutes. We shall call the property of $(C(\mathbb{E}),\Phi)$ described above the *universal property* of $(C(\mathbb{E}),\Phi)$.

Upon specializing to the classical situation (i.e., $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{E}$ are taken to be classical spaces as well as all possible classical spaces $\mathbb{D}$), then Definition 2.5 actually says that $\mathbb{E} = C(X,Y)$, by ultilizing the universal property of $C(X,Y)$, at least when all spaces are assumed finite, so that the compact-open topology is (locally) compact.

The immediate question now is whether given two quantum spaces $\mathbb{X}$ and $\mathbb{Y}$, the quantum space of all maps $\mathbb{X} \to \mathbb{Y}$ (and the corresponding quantum family) exists. The answer is usually negative. The reason for this is that C*-algebras are only suitable to describe locally compact (quantum or classical) spaces and spaces of continuous maps with compact open topology rarely are locally compact. However, in the next theorem, we shall show that under certain conditions, the quantum space of all maps exists. This is a result by Sołtan in his paper [19]. A sketch of the proof is given in that paper. We shall fill in some omitted details here:

Theorem 2.6 (Theorem 3,[19]). *Let $\mathbb{X}$ and $\mathbb{Y}$ be (compact) quantum spaces. Assume that $C(\mathbb{Y})$ is finite dimensional and $C(\mathbb{X})$ is finitely generated and unital. Then*

1. *the quantum space of all maps $\mathbb{Y} \to \mathbb{X}$ exists. We shall denote the corresponding C^*-algebra by $\mathcal{A}$.*

2. *the C^*-algebra $\mathcal{A}$ is unital and generated by $\{(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(x) \mid x \in C(\mathbb{X}), \omega \in C(\mathbb{Y})^*\}$, where $\Phi \in Mor(C(\mathbb{X}), C(\mathbb{Y}) \otimes \mathcal{A})$ is the quantum family of all maps $\mathbb{Y} \to \mathbb{X}$.*

Proof. Let $x_1, x_2, ..., x_N$ be generators of $C(\mathbb{X})$. Since unitaries linearly span $C(\mathbb{X})$, we can assume that $x_1, x_2, ..., x_N$ are unitary. Let $(\mathscr{R}_t)_{t\in\mathscr{T}}$ be the complete list of relations between $x_1, x_2, ..., x_N$, including $x_i^* x_i - 1 = 0$ and $x_i x_i^* - 1 = 0$ for every $1 \le i \le N$ such that

$$(\{x_1, x_2, ..., x_N\}, \mathscr{R}_t(x_1, x_2, ..., x_N), t \in \mathscr{T})$$

is a presentation of $C(\mathbb{X})$.

Since $C(\mathbb{Y})$ is finite-dimensional, we can write $C(\mathbb{Y})$ as the direct sum of matrix algebras:

$$C(\mathbb{Y}) = \bigoplus_{i=1}^{K} M_{n_i}(\mathbb{C}).$$

Now define $\mathscr{A}$ to be the *-algebra generated by

$$\left\{ y^k_{p_{r,s}} \mid p \in \{1,2,...,N\}, k \in \{1,2,...,K\}, r,s \in \{1,2,...,n_k\} \right\}$$

with the relations

$$\mathscr{R}_t\left(\mathcal{Y}_1, \mathcal{Y}_2, ..., \mathcal{Y}_N\right) \qquad\qquad t \in \mathscr{T}$$

where for each $p \in \{1,...,N\}$, $\mathcal{Y}_p$ is the diagonal matrix

$$\mathcal{Y}_p = \begin{pmatrix} Y_p^1 & & \\ & \ddots & \\ & & Y_p^K \end{pmatrix}$$

with

$$Y_p^k = \begin{pmatrix} y^k_{p_{1,1}} & \cdots & y^k_{p_{1,n_k}} \\ \vdots & \ddots & \vdots \\ y^k_{p_{n_k,1}} & \cdots & y^k_{p_{n_k,n_k}} \end{pmatrix}$$

where $1 \leq p \leq N$ and $1 \leq k \leq K$.

In particular, the relations saying that each x_p is unitary give us:

$$\mathcal{Y}_p^* \mathcal{Y}_p = \mathbb{1}_{J \times J}$$

in which $J = \sum_{i=1}^{K} n_i$ and $\mathbb{1}_{J \times J}$ is the $J \times J$ identity matrix.

Note that each entry in the matrix $\mathcal{Y}_p$ is a block matrix. Hence we are talking the multiplication of $\mathcal{Y}_p^* \mathcal{Y}_p$ in terms of block matrices multiplications. We shall let $a^k_{p_{r,s}}$ to be the image of $y^k_{p_{r,s}}$ under the quotient by the relations $\mathscr{R}_t$ where $t \in \mathscr{T}$, for every $p \in \{1,2,...,N\}, k \in \{1,2,...,K\}, r,s \in$

$\{1,2,...,n_k\}$. Note that each $a^k_{p_{r,s}}$ is in $\mathscr{A}$. Therefore for $r,s \in \{1,...,n_k\}$, we have

$$\sum_{q=1}^{n_k} (a^k_{p_{q,r}})^*(a^k_{p_{q,s}}) = \delta_{r,s}\mathbf{1}_{\mathscr{A}}$$

where $\mathbf{1}_{\mathscr{A}}$ is the unit of $\mathscr{A}$ and $\delta_{r,s}$ is the Kronecker delta at (r,s).

In particular, for any Hilbert space *-representation π of $\mathscr{A}$, the norm $\left\|\pi\left(y^k_{p_{r,s}}\right)\right\| \leq 1$ for $p \in \{1,2,...,N\}, k \in \{1,2,...,K\}, r,s \in \{1,2,...,n_k\}$.

Hence for any $a \in \mathscr{A}$, we have

$$\sup_{\pi} \|\pi(a)\|$$

(where the supremum is taken over all cyclic Hilbert space representation of $\mathscr{A}$) is finite.

So we have the following well-defined C*-seminorm,

$$|||a||| := \sup_{\pi} \|\pi(a)\|.$$

Then we define $\mathcal{A}$ to be the enveloping C*-algebra of $(\mathscr{A}, |||\cdot|||)$. Since $\mathcal{A}$ is the completion of $\mathscr{A}$, we shall view $\mathscr{A}$ as a subalgebra of $\mathcal{A}$.

Next let us define a map $\Phi : C(\mathbb{X}) \to C(\mathbb{Y}) \otimes \mathcal{A} = \bigoplus_{i=1}^K M_{n_i}(\mathbb{C}) \otimes \mathcal{A}$ by

$$x_p \mapsto \mathcal{A}_p \in \bigoplus_{i=1}^{K} M_{n_i}(\mathbb{C}) \otimes \mathcal{A}$$

where $\mathcal{A}_p \in \bigoplus_{i=1}^K M_{n_i}(\mathbb{C}) \otimes \mathscr{A} \subseteq \bigoplus_{i=1}^K M_{n_i}(\mathbb{C}) \otimes \mathcal{A}$. We shall explicitly write out what we mean by $\mathcal{A}_p$:

$$\mathcal{A}_p = \begin{pmatrix} A^1_p & & \\ & \ddots & \\ & & A^K_p \end{pmatrix}$$

with

$$A_p^k = \begin{pmatrix} a^k_{p_{1,1}} & \cdots & a^k_{p_{1,n_k}} \\ \vdots & \ddots & \vdots \\ a^k_{p_{n_k,1}} & \cdots & a^k_{p_{n_k,n_k}} \end{pmatrix}$$

for $1 \leq p \leq N$ and $1 \leq k \leq K$.

We will be done if we show that $(\mathcal{A}, \Phi)$ has the universal property as defined in Definition 2.5. So now let $\mathbb{D}$ be any quantum space and $\Psi : C(\mathbb{X}) \to C(\mathbb{Y}) \otimes C(\mathbb{D}) = \bigoplus_{i=1}^K M_{n_i}(\mathbb{C}) \otimes C(\mathbb{D})$ be any quantum family of maps. Since $x_1, ..., x_N$ generate $C(\mathbb{X})$, we just need to consider the image of them under Ψ. Now let the image of each x_p under Ψ be $\mathcal{D}_p$ where

$$\mathcal{D}_p = \begin{pmatrix} D_p^1 & & \\ & \ddots & \\ & & D_p^K \end{pmatrix}$$

with

$$D_p^k = \begin{pmatrix} d^k_{p_{1,1}} & \cdots & d^k_{p_{1,n_k}} \\ \vdots & \ddots & \vdots \\ d^k_{p_{n_k,1}} & \cdots & d^k_{p_{n_k,n_k}} \end{pmatrix}$$

where $d^k_{p_{r,s}} \in C(\mathbb{D})$ for $1 \leq p \leq N$, $1 \leq k \leq K$ and $1 \leq r, s \leq n_k$.

It is not hard to see that the assignment

$$\mathscr{A} \ni a^k_{p_{r,s}} \mapsto d^k_{p_{r,s}} \in C(\mathbb{D})$$

for every $1 \leq p \leq N$, $1 \leq k \leq K$ and $1 \leq r, s \leq n_k$ extends to a well-defined *-homomorphism if and only if $\mathscr{R}_t(\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_N)$, for $t \in \mathscr{T}$. And this is readily available to us since being the image of a *-homomorphism Ψ for $x_1, x_2, ..., x_N$ respectively, $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_N$ fulfill the relations in $\mathscr{R}_t, t \in \mathscr{T}$ because $x_1, x_2, ..., x_N$ do.

Hence we establish the universal property of $(\mathcal{A},\Phi)$. The second part of theorem is obvious by the observation that

$$\{(\omega\otimes \mathrm{id})\Phi(x_p) \mid 1\le p\le N, \omega\in C(\mathbb{Y})^*\}$$

contains the generators $a^k_{p_{r,s}}$ of $\mathcal{A}$. □

Remark 2.7. In the special case where $C(\mathbb{X}) = C(\mathbb{Y})$, we say that a quantum family $\Phi\in \mathrm{Mor}(C(\mathbb{X}), C(\mathbb{X})\otimes C(\mathbb{E}))$ labeled by $\mathbb{E}$ is *trivial* if $\Phi(x) = x\otimes \mathbb{1}_{C(\mathbb{E})}$ for all $x\in C(\mathbb{X})$.

Now if we assume $C(\mathbb{X}) = C(\mathbb{Y})$ finite dimensional (i.e., $\mathbb{X} = \mathbb{Y}$ is a finite compact quantum space), then by the last theorem we are sure that the quantum space of all maps $\mathbb{X}\to\mathbb{X}$ exists. We will analyze the structure of the quantum space of all maps from a finite quantum space to itself. Let $M = C(\mathcal{QS}(M))$ (i.e., $\mathcal{QS}(M)$ is the underlying quantum space of M), be a finite dimensional C*-algebra. So the quantum space of all maps $\mathcal{QS}(M)\to\mathcal{QS}(M)$ exists. Let us denote the corresponding C*-algebra by $\mathcal{A}$ and let $\Phi : M\to M\otimes\mathcal{A}$ be the quantum family of all maps $\mathcal{QS}(M)\to\mathcal{QS}(M)$.

Recall that for any set X, the set of all maps from X to X is endowed canonically with the structure of a semigroup. The semigroup multiplication is given by composition of maps. This phenomemon has its non-commutative counterpart. In order to investigate the non-commutative analog of this phenemenon, we shall introduce the following definitions by S.L. Woronowicz in [28].

Definition 2.8. Let $\mathbb{X}$ be a compact quantum space. We say that

- $\mathbb{X}$ is a *compact quantum semigroup* if there exists a coassociative *-homomorphism

$$\Delta : C(\mathbb{X})\to C(\mathbb{X})\otimes C(\mathbb{X}).$$

  That is, the following diagram

$$
\begin{array}{ccc}
C(\mathbb{X}) & \xrightarrow{\quad\Delta\quad} & C(\mathbb{X}) \otimes C(\mathbb{X}) \\
{\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\
C(\mathbb{X}) \otimes C(\mathbb{X}) & \xrightarrow{\mathrm{id}\otimes\Delta} & C(\mathbb{X}) \otimes C(\mathbb{X}) \otimes C(\mathbb{X})
\end{array}
$$

commutes (i.e., $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$). We shall call this morphism Δ the *commultiplication*.

- $\mathbb{X}$ is a *compact quantum group* if $\mathbb{X}$ is a compact quantum semigroup and the sets

$$\{\Delta(a)(\mathbb{1} \otimes b) \mid a, b \in C(\mathbb{X})\},$$

$$\{(a \otimes \mathbb{1})\Delta(b) \mid a, b \in C(\mathbb{X})\}$$

have dense linear spans in $C(\mathbb{X}) \otimes C(\mathbb{X})$.

- A quantum semigroup $\mathbb{X}$ has a *unit* if $\mathbb{X}$ admits a character ε of $C(\mathbb{X})$ satisfying

$$(\varepsilon \otimes \mathrm{id}_{C(\mathbb{X})}) \circ \Delta = \mathrm{id}_{C(\mathbb{X})} = (\mathrm{id}_{C(\mathbb{X})} \otimes \varepsilon) \circ \Delta.$$

That is, the following diagram

$$
\begin{array}{ccc}
C(\mathbb{X}) & \xrightarrow{\quad\Delta\quad} & C(\mathbb{X}) \otimes C(\mathbb{X}) \\
{\scriptstyle\Delta}\big\downarrow & \searrow^{\mathrm{id}_{C(\mathbb{X})}} & \big\downarrow{\scriptstyle\mathrm{id}_{C(\mathbb{X})}\otimes\varepsilon} \\
C(\mathbb{X}) \otimes C(\mathbb{X}) & \xrightarrow{\varepsilon\otimes\mathrm{id}_{C(\mathbb{X})}} & C(\mathbb{X})
\end{array}
$$

commutes. We shall call ε to the *counit* of $C(\mathbb{X})$.

Definition 2.9. Let $(\mathbb{B}, \Delta_{\mathbb{B}})$ and $(\mathbb{D}, \Delta_{\mathbb{D}})$ be quantum semigroups. An element $\Lambda \in \mathrm{Mor}(C(\mathbb{B}), C(\mathbb{D}))$

is a *quantum semigroup morphism* from $\mathbb{D}$ to $\mathbb{B}$ if it satisfies

$$(\Lambda \otimes \Lambda) \circ \Delta_{\mathbb{B}} = \Delta_{\mathbb{D}} \circ \Lambda.$$

In other words, a quantum semigroup morphism Λ intertwines the comultiplications $\Delta_{\mathbb{B}}$ of B and $\Delta_{\mathbb{D}}$ of D.

We shall show a few results from [19] that the quantum space of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ equipped with a unique comultiplication Δ is a compact quantum semigroup. Using the results that we have stated, we shall prove that unless M is of one-dimensional, then the compact quantum semigroup will not be a compact quantum group.

Before proceeding to the next section, we shall introduce the notion of *composition* of quantum families of maps (see [19]). Let A_1, A_2, B, C and D be C*-algebras and let

$$\Phi_1 : C \to D \otimes A_1, \qquad\qquad \Phi_2 : B \to C \otimes A_2$$

be quantum families of maps $\mathcal{QS}(D) \to \mathcal{QS}(C)$ and $\mathcal{QS}(C) \to \mathcal{QS}(B)$ labeled by $\mathcal{QS}(A_1)$ and $\mathcal{QS}(A_2)$ respectively. We define the quantum family $\Phi_1 \triangle \Phi_2$ of maps $\mathcal{QS}(D) \to \mathcal{QS}(B)$ labeled by $\mathcal{QS}(A_1 \otimes A_2)$ by

$$\Phi_1 \triangle \Phi_2 := (\Phi_1 \otimes \mathrm{id}_{A_2}) \circ \Phi_2.$$

The quantum family of maps will be called the *composition* of the families Φ_1 and Φ_2. We shall also refer to the operation taking Φ_1 and Φ_2 to $\Phi_1 \triangle \Phi_2$ as the operation of composition of quantum families of maps. In case the families are classical (i.e., the C*-algebras A_1 and A_2 are commutative), then the family $\Phi_1 \triangle \Phi_2$ is a classical family consisting of all composition of members of Φ_1 and Φ_2. The crucial property of composition of quantum families of maps is that it is associative.

Proposition 2.10. Let $A_1, A_2, A_3, B_1, B_2, C$ and D be C*-algebras and let

$$\Phi_1 \in \mathrm{Mor}(B_1, C \otimes A_1),$$

$$\Phi_2 \in \mathrm{Mor}(B_2, B_1 \otimes A_2),$$

$$\Phi_3 \in \mathrm{Mor}(D, B_2 \otimes A_3)$$

be quantum families of maps. Then

$$\Phi_1 \triangle (\Phi_2 \triangle \Phi_3) = (\Phi_1 \triangle \Phi_2) \triangle \Phi_3.$$

Proof. Note that

$$\begin{aligned}
\Phi_1 \triangle (\Phi_2 \triangle \Phi_3) &= (\Phi_1 \otimes \mathrm{id}_{A_2 \otimes A_3}) \circ (\Phi_2 \triangle \Phi_3) \\
&= (\Phi_1 \otimes \mathrm{id}_{A_2} \otimes \mathrm{id}_{A_3}) \circ (\Phi_2 \otimes \mathrm{id}_{A_3}) \circ \Phi_3 \\
&= ([(\Phi_1 \otimes \mathrm{id}_{A_2}) \circ \Phi_2] \otimes \mathrm{id}_{A_3}) \circ \Phi_3 \\
&= ([\Phi_1 \triangle \Phi_2] \otimes \mathrm{id}_{A_3}) \circ \Phi_3 \\
&= (\Phi_1 \triangle \Phi_2) \triangle \Phi_3.
\end{aligned}$$

□

Remark 2.11. We shall remark here that the symbol $\triangle$ is for the composition of quantum families whilst the symbol Δ is for the comultiplication of algebra.

2.2.4 Quantum Semigroup Structure

In this section, we shall analyze the structure of the quantum space of all maps from a finite quantum space to itself. Let $M = C(\mathcal{QS}(M))$ (i.e., $\mathcal{QS}(M)$ is the underlying quantum space of M) be a finite dimensional C*-algebra. So the quantum space of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ exists. Let us denote the corresponding C*-algebra by $\mathcal{A}$ and let $\Phi : M \to M \otimes \mathcal{A}$ to be the quantum family of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$.

Theorem 2.12 (Theorem 6, [19])**.**

1. *There exists a unique morphism* $\Delta \in Mor(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ *such that*
$$(\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi = (\mathrm{id}_M \otimes \Delta) \circ \Phi. \tag{2.1}$$

2. *The morphism* Δ *satisfies*
$$(\Delta \otimes \mathrm{id}_{\mathcal{A}}) \circ \Delta = (\mathrm{id}_{\mathcal{A}} \otimes \Delta) \circ \Delta. \tag{2.2}$$

3. *There exists a unique character* ε *of* $\mathcal{A}$ *such that*
$$(\mathrm{id}_M \otimes \varepsilon) \circ \Phi = \mathrm{id}_M. \tag{2.3}$$

4. *The character* ε *is the counit of* $\mathcal{A}$*, i.e., it satisfies*
$$(\mathrm{id}_{\mathcal{A}} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}_{\mathcal{A}}) \circ \Delta = \mathrm{id}_{\mathcal{A}}.$$

Proof. 1. Let us consider the quantum family $\Phi \triangle \Phi$ of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by $\mathcal{QS}(\mathcal{A} \otimes \mathcal{A})$. The universal property of $(\mathcal{A}, \Phi)$ says that there exists a unique $\Delta \in \mathrm{Mor}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad\Phi\quad} & M \otimes \mathcal{A} \\ \Big\| & & \Big\downarrow \mathrm{id}_M \otimes \Delta \\ M & \xrightarrow{\Phi \triangle \Phi = (\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi} & M \otimes \mathcal{A} \otimes \mathcal{A} \end{array}$$

commutes. This is precisely equation (2.1).

2. To prove equation (2.2) we use equation (2.1) to compute $(\Phi \otimes \mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}}) \circ (\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi$ in

two ways:

$$
\begin{aligned}
(\Phi\otimes \mathrm{id}_{\mathcal{A}}\otimes \mathrm{id}_{\mathcal{A}})\circ(\Phi\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi &= (\Phi\otimes \mathrm{id}_{\mathcal{A}}\otimes \mathrm{id}_{\mathcal{A}})\circ(\mathrm{id}_M\otimes\Delta)\circ\Phi\\
&= (\mathrm{id}_M\otimes \mathrm{id}_{\mathcal{A}}\otimes\Delta)\circ(\Phi\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi\\
&= (\mathrm{id}_M\otimes \mathrm{id}_{\mathcal{A}}\otimes\Delta)\circ(\mathrm{id}_M\otimes\Delta)\circ\Phi
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi\otimes \mathrm{id}_{\mathcal{A}}\otimes \mathrm{id}_{\mathcal{A}})\circ(\Phi\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi &= ([(\Phi\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi]\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi\\
&= ([(\mathrm{id}_M\otimes\Delta)\circ\Phi]\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi\\
&= (\mathrm{id}_M\otimes\Delta\otimes \mathrm{id}_{\mathcal{A}})\circ(\Phi\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi\\
&= (\mathrm{id}_M\otimes\Delta\otimes \mathrm{id}_{\mathcal{A}})\circ(\mathrm{id}_M\otimes\Delta)\circ\Phi.
\end{aligned}
$$

Ler ω be a linear functional on M. Applying $(\omega\otimes \mathrm{id}_{\mathcal{A}}\otimes \mathrm{id}_{\mathcal{A}}\otimes \mathrm{id}_{\mathcal{A}})$ to both sides of the equation:

$$
(\mathrm{id}_M\otimes \mathrm{id}_{\mathcal{A}}\otimes\Delta)\circ(\mathrm{id}_M\otimes\Delta)\circ\Phi = (\mathrm{id}_M\otimes\Delta\otimes \mathrm{id}_{\mathcal{A}})\circ(\mathrm{id}_M\otimes\Delta)\circ\Phi
$$

we obtain

$$
[(\mathrm{id}_{\mathcal{A}}\otimes\Delta)\circ\Delta]\,((\omega\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi(m)) = [(\Delta\otimes \mathrm{id}_{\mathcal{A}})\circ\Delta]\,((\omega\otimes \mathrm{id})\Phi(m))
$$

for any $m\in M$.

Note that from Theorem 2.6(2), we see that $\mathcal{A}$ is generated by $\{(\omega\otimes \mathrm{id}_{\mathcal{A}})\circ\Phi(m)\mid m\in M,\omega\in M^*\}$. Since m is arbitrary, we see that equation (2.2) is true.

3. The statement follows from the universal property of $(\mathcal{A},\Phi)$ where we let $C(\mathbb{X})=C(\mathbb{Y})=M$ and $C(\mathbb{D})=\mathbb{C}$ in Definition 2.5. More precisely, we identify $M\otimes C(\mathbb{D})=M\otimes\mathbb{C}$ with M

and take Ψ be the identity morphism from M to $M \otimes C(\mathbb{D}) = M$ (i.e., $\Psi = \mathrm{id}_M$). Hence the universal property of $(\mathcal{A}, \Phi)$ guarantees that there exists a unique $\varepsilon \in \mathrm{Mor}\,(\mathcal{A}, \mathbb{C})$ such that this diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad \Phi \quad} & M \otimes \mathcal{A} \\ \Big\| & & \Big\downarrow {\scriptstyle \mathrm{id}_M \otimes \varepsilon} \\ M & \xrightarrow{\Psi = \mathrm{id}_M} & M \otimes \mathbb{C} \end{array}$$

commutes, which is precisely the equation (2.3).

4. Note that we have

$$\begin{aligned}(\mathrm{id}_M \otimes (\varepsilon \otimes \mathrm{id}_{\mathcal{A}})\Delta) \circ \Phi &= (\mathrm{id}_M \otimes \varepsilon \otimes \mathrm{id}_{\mathcal{A}})(\mathrm{id}_M \otimes \Delta) \circ \Phi \\ &= (\mathrm{id}_M \otimes \varepsilon \otimes \mathrm{id}_{\mathcal{A}})(\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi && \text{by equation (2.1)} \\ &= ((\mathrm{id}_M \otimes \varepsilon)\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi \\ &= (\mathrm{id}_M \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi && \text{by equation (2.3)} \\ &= \Phi\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id}_M \otimes (\mathrm{id}_{\mathcal{A}} \otimes \varepsilon)\Delta) \circ \Phi &= (\mathrm{id}_M \otimes \mathrm{id}_{\mathcal{A}} \otimes \varepsilon)(\mathrm{id}_M \otimes \Delta) \circ \Phi \\ &= (\mathrm{id}_M \otimes \mathrm{id}_{\mathcal{A}} \otimes \varepsilon)(\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi && \text{by equation (2.1)} \\ &= \Phi \circ (\mathrm{id}_M \otimes \varepsilon) \circ \Phi \\ &= \Phi && \text{by equation (2.3).}\end{aligned}$$

Hence we have

$$(\mathrm{id}_M \otimes (\varepsilon \otimes \mathrm{id}_{\mathcal{A}})\Delta) \circ \Phi = \Phi = (\mathrm{id}_M \otimes (\mathrm{id}_{\mathcal{A}} \otimes \varepsilon)\Delta) \circ \Phi.$$

Let ω be a linear functional on M. Applying $\omega \otimes \mathrm{id}_{\mathcal{A}}$ to the above equations, we obtain

$$[(\varepsilon \otimes \mathrm{id}_{\mathcal{A}}) \circ \Delta]((\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m)) = (\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m) = [(\mathrm{id}_{\mathcal{A}} \otimes \varepsilon) \circ \Delta]((\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m))$$

for any $m \in M$.

With $\mathcal{A}$ generated by $\{(\omega \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi(m) \mid m \in M, \omega \in M^*\}$, we get $(\varepsilon \otimes \mathrm{id}_{\mathcal{A}}) \circ \Delta = \mathrm{id}_{\mathcal{A}} = (\mathrm{id}_{\mathcal{A}} \otimes \varepsilon) \circ \Delta$.

□

Remark 2.13. Equation (2.2) simply says that the following diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \otimes \mathcal{A} \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes \Delta} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\Delta \otimes \mathrm{id}} & \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \end{array}$$

is commutative. In other words, $(\mathcal{QS}(\mathcal{A}), \Delta)$ is a compact quantum semigroup.

We define an *action of a quantum semigroup* $(\mathbb{B}, \Delta_{\mathbb{B}})$ on a quantum space $\mathcal{QS}(C)$ to be a morphism $\Psi \in \mathrm{Mor}(C, C \otimes B)$ satisfying:

$$\Psi \triangle \Psi = (\Psi \otimes \mathrm{id}_B) \circ \Psi = (\mathrm{id}_C \otimes \Delta_B) \circ \Psi.$$

Hence from equation (2.2), we see that the quantum family $\Phi \in \mathrm{Mor}(M, M \otimes \mathcal{A})$ of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ is then an action of $\mathcal{QS}(\mathcal{A})$ on the quantum space $\mathcal{QS}(M)$. Since $\mathcal{A}$ is a unital C*-algebra (by Theorem 2.6(2)), the quantum semigroup $\mathcal{QS}(\mathcal{A})$ is compact.

Next we shall show two results from [19] and use them to prove a result in [18], namely:

$(\mathcal{QS}(\mathcal{A}), \Delta)$ is not a compact quantum group unless $\mathcal{QS}(M)$ is a (classical) one point space, i.e., M is of one-dimensional. Before that, we shall digress and define a few notations that we will use later.

Definition 2.14. Let B, N be C*-algebras and N be finite dimensional. Let $\Psi \in \mathrm{Mor}(N, N \otimes B)$ be a quantum family of maps $\mathcal{QS}(N) \to \mathcal{QS}(N)$ labeled by $\mathcal{QS}(B)$ and let ω be a state on N. We say that ω is *invariant* for Ψ if

$$(\omega \otimes \mathrm{id}_B)\Psi(n) = \omega(n)\mathbb{1}_B$$

for all $n \in N$. We also say that the quantum family of maps Ψ preserves the state ω.

Theorem 16 in [19] describes the quantum subsemigroup of $\mathcal{QS}(\mathcal{A})$ preserving a given state ω on M. That is precisely the non-commutative analog of the phenomenon observed in classical case: If X is a classical topological space and μ is a measure on X, then the set of those continuous maps $X \to X$ that preserve μ is a subsemigroup of $C(X, X)$. Since states on commutative C*-algebras correspond to integrations with respect to probability measures, hence it is not surprising that we have such analog. However, we shall not survey more on this concept in this dissertation. Interested readers can refer to the paper [19].

Proposition 2.15 (Proposition 11, [19])**.** The action Φ of $\mathcal{QS}(\mathcal{A})$ on $\mathcal{QS}(M)$ satisfies:

$$\Phi(m) = m \otimes \mathbb{1}_{\mathcal{A}} \Longrightarrow m \in \mathbb{C}\mathbb{1}_M$$

for all $m \in M$.

Proof. Consider the quantum family of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by $\mathcal{QS}(M)$ which is given by $\Psi \in \mathrm{Mor}(M, M \otimes M)$,

$$\Psi(m) = \mathbb{1}_M \otimes m.$$

The universal property of $(\mathcal{A}, \Phi)$ ensures us that there exists a unique element $\Lambda \in \mathrm{Mor}(\mathcal{A}, M)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad\Phi\quad} & M \otimes \mathcal{A} \\ \Big\| & & \Big\downarrow \mathrm{id}_M \otimes \Lambda \\ M & \xrightarrow{\quad\Psi\quad} & M \otimes M \end{array}$$

commutes. Hence we have $\Psi(m) = (\mathrm{id}_M \otimes \Lambda) \circ \Phi(m)$. So we have

$$\begin{aligned} \Psi(m) = \mathbb{1}_M \otimes m &= (\mathrm{id}_M \otimes \Lambda) \circ \Phi(m) \\ &= (\mathrm{id}_M \otimes \Lambda)(m \otimes \mathbb{1}_{\mathcal{A}}) \\ &= m \otimes \mathbb{1}_M. \end{aligned}$$

Hence we have $m \in \mathbb{C}\mathbb{1}_M$. □

Proposition 2.16 (Proposition 15, [19])**.** Let ω be a state on M. Assume that ω is invariant under the action of $\mathcal{QS}(\mathcal{A})$. Then M is one dimensional.

Proof. Let $\Lambda \in \mathrm{Mor}(\mathcal{A}, M)$ be such that for any $m \in M$ we have $(\mathrm{id}_M \otimes \Lambda)\Phi(m) = \Psi(m) = \mathbb{1} \otimes m$, as in the proof of Proposition 2.15. For any $m \in M$, applying Λ to both sides of

$$(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m) = \omega(m)\mathbb{1}_{\mathcal{A}},$$

we get

$$\Lambda((\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m)) = \omega(m)\Lambda(\mathbb{1}_{\mathcal{A}}) = \omega(m)\mathbb{1}_M.$$

Note that

$$\begin{aligned} \Lambda((\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m)) &= ((\omega \otimes \mathrm{id}_M)(\mathrm{id}_M \otimes \Lambda))\Phi(m) \\ &= (\omega \otimes \mathrm{id}_M)\Psi(m) \\ &= (\omega \otimes \mathrm{id}_M)(\mathbb{1}_M \otimes m) = m. \end{aligned}$$

Hence we have $m = \omega(m)\mathbb{1}_M$ for any $m \in M$. Since ω is a state on M, we see that M is of dimension one. $\square$

Theorem 2.17 (Theorem 2.3, [28]). *Let $(\mathcal{QS}(A), \Delta_A)$ be a compact quantum group. Then there exists a unique state h on A such that*

$$(h \otimes \mathrm{id}_A)\Delta_A(a) = (\mathrm{id}_A \otimes h)\Delta_A(a) = h(a)\mathbb{1}_A$$

for all $a \in A$. Such a state h is called the Haar measure.

We shall see that the compact quantum semigroup $(\mathcal{QS}(\mathcal{A}), \Delta)$ of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ does not possess the quantum group structure unless M is of dimension one. The author likes to thank Prof Van Daele for directing her towards this result from Sołtan.

Theorem 2.18 (Proposition 2.1, [18]). *The compact quantum semigroup $(\mathcal{QS}(\mathcal{A}), \Delta)$ of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ is not a compact quantum group unless M is of dimension one.*

Proof. If $(\mathcal{QS}A, \Delta)$ were a compact quantum group, then it would have a Haar measure h. Now consider the map

$$f : m \in M \mapsto (\mathrm{id}_M \otimes h)\Phi(m) \in M.$$

We have

$$\begin{aligned}
\Phi(f(m)) &= \Phi\left((\mathrm{id}_M \otimes h)\Phi(m)\right) \\
&= (\mathrm{id}_M \otimes \mathrm{id}_{\mathcal{A}} \otimes h)\left((\Phi \otimes \mathrm{id}_{\mathcal{A}})\Phi(m)\right) \\
&= (\mathrm{id}_M \otimes \mathrm{id}_{\mathcal{A}} \otimes h)\left((\mathrm{id}_M \otimes \Delta)\Phi(m)\right) && \text{by equation (2.1)} \\
&= [id_M \otimes (\mathrm{id}_{\mathcal{A}} \otimes h)\Delta]\,\Phi(m) \\
&= (\mathrm{id}_M \otimes h)(\Phi(m)) \otimes \mathbb{1}_{\mathcal{A}} && \text{by Haar measure property} \\
&= f(m) \otimes \mathbb{1}_{\mathcal{A}}
\end{aligned}$$

Hence by Proposition 2.15, we have $f(m) \in \mathbb{C}\mathbb{1}_M$.

Define $\omega : M \to \mathbb{C}$ by $\omega(m)\mathbb{1}_m = f(m)$. Since f is defined using the *-homorphism Φ and the state h on M, ω is hence a state on M. In fact, ω will be proven to be invariant for Φ:

$$\begin{aligned}
\mathbb{1}_M \otimes [(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m)] &= (f \otimes id_{\mathcal{A}})\Phi(m) \\
&= [(\mathrm{id}_M \otimes h)\Phi \otimes \mathrm{id}_{\mathcal{A}}]\,\Phi(m) \\
&= (\mathrm{id}_M \otimes h \otimes \mathrm{id}_{\mathcal{A}})(\Phi \otimes \mathrm{id}_{\mathcal{A}})\Phi(m) \\
&= (\mathrm{id}_M \otimes h \otimes \mathrm{id}_{\mathcal{A}})(\mathrm{id}_M \otimes \Delta)\Phi(m) && \text{by equation (2.1)} \\
&= [\mathrm{id}_M \otimes (h \otimes \mathrm{id}_{\mathcal{A}})\Delta]\,\Phi(m) \\
&= (\mathrm{id}_M \otimes h)\Phi(m) \otimes \mathbb{1}_{\mathcal{A}} \\
&= f(m) \otimes \mathbb{1}_{\mathcal{A}} \\
&= \omega(m)\mathbb{1}_M \otimes \mathbb{1}_{\mathcal{A}} \\
&= \mathbb{1}_M \otimes \omega(m)\mathbb{1}_{\mathcal{A}}
\end{aligned}$$

for any $m \in M$. So $(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m) = \omega(m)\mathbb{1}_{\mathcal{A}}$, which implies that $m \in \mathbb{C}\mathbb{1}_{\mathcal{A}}$ by Proposition 2.16. So M has to be of dimension one. □

Remark 2.19. We should understand the above theorem as saying that the compact quantum space $\mathcal{QS}(\mathcal{A})$ is not a compact quantum group when it is equipped with the canonical comultiplication Δ. In fact, in the paper [17], the author explicitly constructed that for $M = \mathbb{C}^2 \cong C(\mathcal{QS}(M))$ where $\mathcal{QS}(M)$ is the classical 2-point space, $\mathcal{QS}(\mathcal{A})$ when equipped with some other comultiplications, is a compact quantum group.

We shall end this section by introducing a result that we will use in our future investigation:

Proposition 2.20 (Proposition 12, [19])**.** Let B be C*-algebra and let $\Psi \in \mathrm{Mor}(M, M \otimes B)$ be a quantum family of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by $\mathcal{QS}(B)$. Assume that there exists a morphism $\Delta_B \in \mathrm{Mor}(B, B \otimes B)$ such that

$$(\mathrm{id}_M \otimes \Delta_B) \circ \Psi = (\Psi \otimes \mathrm{id}_B) \circ \Psi$$

and let $\Lambda \in \mathrm{Mor}(\mathcal{A}, B)$ be the unique morphism such that $(\mathrm{id}_M \otimes \Lambda) \circ \Phi = \Psi$. Then Λ satisfies

$$(\Lambda \otimes \Lambda) \circ \Delta = \Delta_B \otimes \Lambda.$$

Proof. Looking at the commutative diagram

$$\begin{array}{ccccc}
M & \xrightarrow{\Phi} & M \otimes \mathcal{A} & \xrightarrow{\mathrm{id} \otimes \Delta} & M \otimes \mathcal{A} \otimes \mathcal{A} \\
\| & & & & \| \\
M & \xrightarrow{\Phi} & M \otimes \mathcal{A} & \xrightarrow{\Phi \otimes \mathrm{id}} & M \otimes \mathcal{A} \otimes \mathcal{A} \\
\| & & \downarrow{\scriptstyle \mathrm{id} \otimes \Lambda} & & \downarrow{\scriptstyle \mathrm{id} \otimes \Lambda \otimes \Lambda} \\
M & \xrightarrow{\Psi} & M \otimes B & \xrightarrow{\Psi \otimes \mathrm{id}} & M \otimes B \otimes B
\end{array}$$

we have

$$(\mathrm{id}_M \otimes \Lambda \otimes \Lambda) \circ (\mathrm{id}_M \otimes \Delta) \circ \Phi = (\Psi \otimes \mathrm{id}_B) \circ (\mathrm{id}_M \otimes \Lambda) \circ \Phi.$$

But we know that $(\mathrm{id}_M \otimes \Lambda) \circ \Phi = \Psi$ and by using this fact twice we obtain

$$\begin{aligned}
(\mathrm{id}_M \otimes \Lambda \otimes \Lambda) \circ (\mathrm{id}_M \otimes \Delta) \circ \Phi &= (\Psi \otimes \mathrm{id}_B) \circ (\mathrm{id}_M \otimes \Lambda) \circ \Phi \\
&= (\Psi \otimes \mathrm{id}_B) \circ \Psi \\
&= (\mathrm{id}_M \otimes \Delta_B) \circ \Psi \\
&= (\mathrm{id}_M \otimes \Delta_B) \circ (\mathrm{id}_M \otimes \Lambda) \circ \Phi.
\end{aligned}$$

Take $\omega \in M^*$ and let us apply $\omega \otimes \mathrm{id}_B \otimes \mathrm{id}_B$ to both sides of the above equality. We find that for all $m \in M$

$$((\Lambda \otimes \Lambda) \circ \Delta)((\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m)) = (\Delta_B \otimes \Lambda)((\omega \otimes \mathrm{id}_{\mathcal{A}})(\Phi(m))).$$

Since we see from Theorem 2.6(2) that these elements $(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(m)$, for any $m \in M$ and $\omega \in M^*$ generate $\mathcal{A}$, we have

$$(\Lambda \otimes \Lambda) \circ \Delta = \Delta_B \otimes \Lambda.$$

□

2.2.5 Examples

We shall look at two examples in this section. The first one often appears at the very beginning of the theory of quantum families of maps whilst the latter one is about $M_2(\mathbb{C})$.

Example 2.21. Let $\mathcal{QS}(M)$ be the classical two-point space: $C(\mathcal{QS}(M)) = \mathbb{C}^2$. Clearly the classical space of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ has four points. But, as we shall see, the quantum family of all maps from $M \to M$ is infinite dimensional:

We can consider $\mathbb{C}^2$ as the universal C*-algebra generated by an unitary element $m \equiv (1,-1)$ satisfying the relation $m = m^*$. Now using the construction that we had shown in Theorem 2.6, with $N = 1$ and $K = 2$ (i.e., $\mathbb{C}^2 \equiv \mathbb{C} \oplus \mathbb{C}$), we see that $\mathcal{A}$ is the universal C*-algebra generated by two unitary elements a and b satisfying the relations $a = a^*$ and $b = b^*$. By the result in Section 1 in [16], we see that the C*-algebra $\mathcal{A}$ is C$^*(\mathbb{Z}_2 * \mathbb{Z}_2)$, the group algebra of the free product of $\mathbb{Z}_2$ by itself. More concretely, we can describe this C*-algebra $\mathcal{A}$ as the algebra of continuous functions $[0,1] \to M_2(\mathbb{C})$ whose values at the end-points are diagonal, i.e.,

$$\mathcal{A} = \{f \in C([0,1], M_2(\mathbb{C})) \mid f(0), f(1) \text{ are diagonal matrices}\}$$

(see Theorem 1.3 in [16]). We shall let the generators a and b be

$$a = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) \\ \sin(\pi t) & -\cos(\pi t) \end{pmatrix}$$

and

$$b = \begin{pmatrix} -\cos(\pi t) & \sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}$$

where $t \in [0,1]$. Then we have

$$\Phi : M = \mathbb{C}^2 \to M \otimes \mathcal{A} = \mathbb{C}^2 \otimes \mathcal{A} = \mathcal{A} \oplus \mathcal{A}$$

which is defined by

$$\Phi(m) = \Phi(1,-1) = (1,0) \otimes a + (0,1) \otimes b = (a,b).$$

Since $(1,1)$ is the unit in $M = \mathbb{C}^2$, we have $\Phi(1,1) = (1,1) \otimes \mathbb{1}_{\mathcal{A}}$ where $\mathbb{1}_{\mathcal{A}}$ can be considered as a map on $[0,1]$ with constant value

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So we have

$$\begin{aligned} \Phi(1,0) &= \frac{\Phi(1,-1) + \Phi(1,1)}{2} \\ &= \left(\frac{1}{2}, 0\right) \otimes (\mathbb{1}_{\mathcal{A}} + a) + \left(0, \frac{1}{2}\right) \otimes (\mathbb{1}_{\mathcal{A}} + b) \end{aligned}$$

and

$$\begin{aligned} \Phi(0,1) &= \frac{\Phi(1,1) - \Phi(1,-1)}{2} \\ &= \left(\frac{1}{2}, 0\right) \otimes (\mathbb{1}_{\mathcal{A}} - a) + \left(0, \frac{1}{2}\right) \otimes (\mathbb{1}_{\mathcal{A}} - b) \end{aligned}$$

From equation (2.1), we know that $\mathcal{A}$ has a comultiplication $\Delta \in \mathrm{Mor}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ that satisfies

$$(\mathrm{id}_M \otimes \Delta) \circ \Phi = (\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi.$$

Evaluating both sides of the equation at $m = (1, -1)$, we have

$$\begin{aligned}
(\Delta(a), \Delta(b)) &= (\mathrm{id}_M \otimes \Delta)(a, b) \\
&= (\mathrm{id}_M \otimes \Delta)\Phi(1, -1) \\
&= (\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi(1, -1) \\
&= (\Phi \otimes \mathrm{id}_{\mathcal{A}})((1,0) \otimes a + (0,1) \otimes b) \\
&= \Phi(1,0) \otimes a + \Phi(0,1) \otimes b \\
&= \left[\left(\frac{1}{2}, 0\right) \otimes (\mathbb{1}_{\mathcal{A}} + a) + \left(0, \frac{1}{2}\right) \otimes (\mathbb{1}_{\mathcal{A}} + b)\right] \otimes a \\
&\quad + \left[\left(\frac{1}{2}, 0\right) \otimes (\mathbb{1}_{\mathcal{A}} - a) + \left(0, \frac{1}{2}\right) \otimes (\mathbb{1}_{\mathcal{A}} - b)\right] \otimes b.
\end{aligned}$$

Hence we have

$$\Delta(a) = \frac{\mathbb{1}_{\mathcal{A}} + a}{2} \otimes a + \frac{\mathbb{1}_{\mathcal{A}} - a}{2} \otimes b$$

and

$$\Delta(b) = \frac{\mathbb{1}_{\mathcal{A}} + b}{2} \otimes a + \frac{\mathbb{1}_{\mathcal{A}} - b}{2} \otimes b.$$

The infinite dimensionality of $\mathcal{A}$ justifies that the quantum space of all maps from a two-point space to itself is infinite.

Next we shall look at the case of $M = M_2(\mathbb{C})$. This example is shown in Sołtan, [18]. We shall fill in the omitted details here.

Example 2.22. (Proposition 4.1, [18]) Now consider $M = M_2(\mathbb{C})$. We shall use the fact that the C*-algebra $M_2(\mathbb{C})$ is the universal C*-algebra generated by an element n satisfying the relations

$$n^2 = 0 \qquad \text{and} \qquad nn^* + n^*n = \mathbb{1}_M.$$

where one can take

$$n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(see example 11,[3]). We know that $\mathcal{A}$ exists and it is endowed with $\Phi \in \mathrm{Mor}(M, M \otimes \mathcal{A})$ which is the quantum family of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$. We shall examine the quantum semigroup structure of $\mathcal{QS}(\mathcal{A})$. Let

$$\Phi : M \ni n \mapsto \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{g} & \mathfrak{d} \end{pmatrix} \in M_2(\mathcal{A}) = M \otimes \mathcal{A},$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{g}$ and $\mathfrak{d}$ are some elements of $\mathcal{A}$.

Note that we have

$$n^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad nn^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad n^*n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $\{n, n^*, nn^*, n^*n\}$ is a basis of M. Therefore we have

$$\begin{aligned} \Phi(nn^*) = \Phi(n)\Phi(n)^* &= \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{g} & \mathfrak{d} \end{pmatrix} \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{g} & \mathfrak{d} \end{pmatrix}^* \\ &= \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{g} & \mathfrak{d} \end{pmatrix} \begin{pmatrix} \mathfrak{a}^* & \mathfrak{g}^* \\ \mathfrak{b}^* & \mathfrak{d}^* \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{a}\mathfrak{a}^* + \mathfrak{b}\mathfrak{b}^* & \mathfrak{a}\mathfrak{g}^* + \mathfrak{b}\mathfrak{d}^* \\ \mathfrak{g}\mathfrak{a}^* + \mathfrak{d}\mathfrak{b}^* & \mathfrak{g}\mathfrak{g}^* + \mathfrak{d}\mathfrak{d}^* \end{pmatrix}, \end{aligned}$$

$$\Phi(n^*n) = \Phi(n)^*\Phi(n) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$
$$= \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$
$$= \begin{pmatrix} \alpha^*\alpha + \gamma^*\gamma & \alpha^*\beta + \gamma^*\delta \\ \beta^*\alpha + \delta^*\gamma & \beta^*\beta + \delta^*\delta \end{pmatrix}$$

and

$$\Phi(n^2) = \Phi(n)^2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^2$$
$$= \begin{pmatrix} \alpha^2 + \beta\gamma & \alpha\beta + \beta\delta \\ \gamma\alpha + \delta\gamma & \gamma\beta + \delta^2 \end{pmatrix}.$$

By the relations $nn^* + n^*n = \mathbb{1}_M$ and $n^2 = 0_M$, we have the following relations

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma + \alpha\alpha^* + \beta\beta^* &= \mathbb{1}_{\mathcal{A}}, \\ \alpha^*\beta + \gamma^*\delta + \alpha\gamma^* + \beta\delta^* &= 0_{\mathcal{A}}, \\ \beta^*\beta + \delta^*\delta + \gamma\gamma^* + \delta\delta^* &= \mathbb{1}_{\mathcal{A}} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
\mathfrak{a}^2 + \mathfrak{b}\mathfrak{c} &= 0_{\mathcal{A}}, \\
\mathfrak{a}\mathfrak{b} + \mathfrak{b}\mathfrak{d} &= 0_{\mathcal{A}}, \\
\mathfrak{c}\mathfrak{a} + \mathfrak{d}\mathfrak{c} &= 0_{\mathcal{A}}, \\
\mathfrak{c}\mathfrak{b} + \mathfrak{d}^2 &= 0_{\mathcal{A}}.
\end{aligned} \tag{2.5}$$

To construct $\mathcal{A}$ directly and explicitly, we now forget about the existing $\mathcal{A}$ and define directly an algebra $\mathcal{A}$ as the universal C*-algebra generated by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{d}$ with all the relations in equation (2.4) and equation (2.5). Define $\Psi_{\mathcal{A}}$ as the unique *-homomorphism $M \to M \otimes \mathcal{A}$ such that

$$\Psi_{\mathcal{A}}(n) = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix} \in M \otimes \mathcal{A},$$

which exists because M is the universal C*-algebra generated by n subject to the relations

$$n^2 = 0 \qquad \text{and} \qquad nn^* + n^*n = \mathbb{1}_M.$$

Now if we consider any C*-algebra B and any $\Psi_B \in \mathrm{Mor}(M, M \otimes B)$, the four entries of $\Psi_B(n) \in M \otimes B = M_2(B)$ must satisfy the similar relations that we had calculated in equation (2.4) and equation (2.5). Hence, by the universality of $\mathcal{A}$, there is a unique map λ from $\mathcal{A}$ to B such that $\Psi_B = (id_M \otimes \lambda) \circ \Psi_{\mathcal{A}}$. These precisely say that $\mathcal{QS}(\mathcal{A})$ is the quantum space and $\Psi_{\mathcal{A}}$ is the quantum family of all maps from $\mathcal{QS}(M) \to \mathcal{QS}(M)$, where $\mathcal{A}$ is the universal C*-algebra generated by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{d}$ with all the relations in equation (2.4) and equation (2.5).

Next we shall compute the value of each generator $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{d}$ under the comultiplication Δ. By the equation (2.1), we have

$$(\mathrm{id}_M \otimes \Delta) \circ \Phi = (\Phi \otimes \mathrm{id}_{\mathcal{A}}) \circ \Phi.$$

Applying both sides to $n \in M$, we obtain

$$\begin{pmatrix} \Delta(\alpha) & \Delta(\beta) \\ \Delta(\gamma) & \Delta(\delta) \end{pmatrix} = (\mathrm{id}_M \otimes \Delta) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\mathrm{id}_M \otimes \Delta)\Phi(n)$$

$$= (\Phi \otimes \mathrm{id}_{\mathcal{A}})\Phi(n)$$

$$= (\Phi \otimes \mathrm{id}_{\mathcal{A}}) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$= (\Phi \otimes \mathrm{id}_{\mathcal{A}})(nn^* \otimes \alpha + n \otimes \beta + n^* \otimes \gamma + n^* n \otimes \delta)$$

$$= \begin{pmatrix} (\alpha\alpha^* + \beta\beta^*) \otimes \alpha & (\alpha\gamma^* + \beta\delta^*) \otimes \alpha \\ (\gamma\alpha^* + \delta\beta^*) \otimes \alpha & (\gamma\gamma^* + \delta\delta^*) \otimes \alpha \end{pmatrix}$$

$$+ \begin{pmatrix} \alpha \otimes \beta & \beta \otimes \beta \\ \gamma \otimes \beta & \delta \otimes \beta \end{pmatrix} + \begin{pmatrix} \alpha^* \otimes \gamma & \gamma^* \otimes \gamma \\ \beta^* \otimes \gamma & \delta^* \otimes \gamma \end{pmatrix}$$

$$+ \begin{pmatrix} (\alpha^*\alpha + \gamma^*\gamma) \otimes \delta & (\alpha^*\beta + \gamma^*\delta) \otimes \delta \\ (\beta^*\alpha + \delta^*\gamma) \otimes \delta & (\beta^*\beta + \delta^*\delta) \otimes \delta \end{pmatrix}.$$

Hence we have

$$\Delta(\alpha) = \alpha\alpha^* \otimes \alpha + \beta\beta^* \otimes \alpha + \alpha \otimes \beta + \alpha^* \otimes \gamma + \alpha^*\alpha \otimes \delta + \gamma^*\gamma \otimes \delta,$$

$$\Delta(\beta) = \alpha\gamma^* \otimes \alpha + \beta\delta^* \otimes \alpha + \beta \otimes \beta + \gamma^* \otimes \gamma + \alpha^*\beta \otimes \delta + \gamma^*\delta \otimes \delta,$$

$$\Delta(\gamma) = \gamma\alpha^* \otimes \alpha + \delta\beta^* \otimes \alpha + \gamma \otimes \beta + \beta^* \otimes \gamma + \beta^*\alpha \otimes \delta + \delta^*\gamma \otimes \delta,$$

$$\Delta(\delta) = \gamma\gamma^* \otimes \alpha + \delta\delta^* \otimes \alpha + \delta \otimes \beta + \delta^* \otimes \gamma + \beta^*\beta \otimes \delta + \delta^*\delta \otimes \delta.$$

Finally the values of the counit ε is determined by the fact that $(\mathrm{id}_M \otimes \varepsilon) \circ \Phi(m) = m$ for all $m \in$

$M = M_2$. Hence we have

$$\begin{pmatrix} \varepsilon(\mathbb{\alpha}) & \varepsilon(\mathbb{\beta}) \\ \varepsilon(\mathbb{\gamma}) & \varepsilon(\mathbb{\delta}) \end{pmatrix} = (\mathrm{id}_M \otimes \varepsilon)\Phi(n)$$

$$= n$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

which implies that ε maps $\mathbb{\alpha}, \mathbb{\gamma}$ and $\mathbb{\delta}$ to 0 and $\mathbb{\beta}$ to 1.

We shall come back to this example again in the next chapter.

Chapter 3

Quantum Commutants

3.1 Motivation and Definition

Consider this situation: For a given finite set X and a family $\mathscr{E}$ of maps from $X \to X$. Then the set of all maps from $X \to X$ commuting with elements of $\mathscr{E}$ is a semigroup under composition of maps. This phenomenon also has its non-commutative analog. In this chapter, we shall investigate it. We shall retain the notation used in Chapter 2. Hence M is a finite-dimensional C*-algebra and $(\mathfrak{QS}(\mathcal{A}),\Delta)$ is the quantum semigroup of all maps $\mathfrak{QS}(M) \to \mathfrak{QS}(M)$. The action of $(\mathfrak{QS}(\mathcal{A}),\Delta)$ on $\mathfrak{QS}(M)$ will, as before, be denoted by Φ.

In order to describe our main object, we first introduce the notion of commutation of quantum family of maps.

Definition 3.1. Let B and C be two C*-algebras and consider two quantum families

$$\Psi_B \in \mathrm{Mor}(M, M \otimes B) \qquad \text{and} \qquad \Psi_C \in \mathrm{Mor}(M, M \otimes C)$$

of maps $\mathfrak{QS}(M) \to \mathfrak{QS}(M)$ labeled by $\mathfrak{QS}(B)$ and $\mathfrak{QS}(C)$ respectively. We shall say that Ψ_B *commutes* with Ψ_C if

$$(\mathrm{id}_M \otimes \sigma_{B,C}) \circ (\Psi_B \triangle \Psi_C) = \Psi_C \triangle \Psi_B,$$

where $\sigma_{B,C} \in \mathrm{Mor}(B \otimes C, C \otimes B)$ is the flip

$$B \otimes C \ni b \otimes c \mapsto c \otimes b \in C \otimes B.$$

This is a straightforward generalization of the notion of commutation of classical families of maps. Note that Ψ_B commutes with Ψ_C if and only if Ψ_C commutes with Ψ_B. We shall analyze the most basic properties of this notion:

Proposition 3.2 (Proposition 18,[19])**.** Let B be a C*-algebra and let $\Psi_B \in \mathrm{Mor}(M, M \otimes B)$ be a quantum family of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by $\mathcal{QS}(B)$. Then

1. If Ψ_B is trivial, then any quantum family of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ commutes with Ψ_B.
2. If Ψ_B commutes with $\Phi \in \mathrm{Mor}(M, M \otimes \mathcal{A})$, then Ψ_B is trivial.

Proof. (1). Let C be a C*-algebra and let $\Psi_C \in \mathrm{Mor}(M, M \otimes C)$ be a quantum family of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$. Since Ψ_B is assumed to be trivial, we have

$$\begin{aligned}
(\mathrm{id}_M \otimes \sigma_{C,B})(\Psi_C \triangle \Psi_B)(m) &= (\mathrm{id}_M \otimes \sigma_{C,B})(\Psi_C \otimes \mathrm{id}_B)\Psi_B(m) \\
&= (\mathrm{id}_M \otimes \sigma_{C,B})(\Psi_C \otimes \mathrm{id}_B)(m \otimes \mathbb{1}_B) \\
&= (\mathrm{id}_M \otimes \sigma_{C,B})(\Psi_C(m) \otimes \mathbb{1}_B) \\
&= (\Psi_B \otimes \mathrm{id}_C)\Psi_C(m) \qquad \text{since } \Psi_B(\tilde{m}) = \tilde{m} \otimes \mathbb{1}_B \\
&= (\Psi_B \triangle \Psi_C)(m)
\end{aligned}$$

for any $m \in M$.

(2). Let $\Lambda \in \mathrm{Mor}(\mathcal{A}, M)$ be the unique morphism such that $(\mathrm{id} \otimes \Lambda)\Phi(m) = \mathbb{1}_M \otimes m$ for all $m \in M$ (Λ exists from the proof of Proposition 2.15). Let us apply $\mathrm{id}_M \otimes \Lambda \otimes \mathrm{id}_B$ to both sides of the equation:

$$(\mathrm{id}_M \otimes \sigma_{B,\mathcal{A}})(\Psi_B \triangle \Phi)(m) = (\Phi \triangle \Psi_B)(m).$$

We have for all $m \in M$,

$$(\mathrm{id}_M \otimes \Lambda \otimes \mathrm{id}_B)(\Phi \triangle \Psi_B)(m) = (\mathrm{id}_M \otimes \Lambda \otimes \mathrm{id}_B)(\Phi \otimes \mathrm{id}_B)\Psi_B(m)$$
$$= [((\mathrm{id}_M \otimes \Lambda) \circ \Phi) \otimes \mathrm{id}_B]\, \Psi_B(m)$$
$$= \mathbb{1}_M \otimes \Psi_B(m)$$

and

$$(\mathrm{id}_M \otimes \Lambda \otimes \mathrm{id}_B)(\mathrm{id}_M \otimes \sigma_{B,\mathcal{A}})(\Psi_B \triangle \Phi)(m) = (\mathrm{id}_M \otimes \sigma_{B,\mathcal{A}})(\Psi_B \otimes \mathrm{id}_M)(\mathrm{id}_M \otimes \Lambda)\Phi(m)$$
$$= (\mathrm{id}_M \otimes \sigma_{B,\mathcal{A}})(\Psi_B \otimes \mathrm{id}_M)(\mathbb{1}_M \otimes m)$$
$$= (\mathrm{id}_M \otimes \sigma_{B,\mathcal{A}})(\Psi_B(\mathbb{1}_M) \otimes m)$$
$$= (\mathrm{id}_M \otimes \sigma_{B,\mathcal{A}})(\mathbb{1}_M \otimes \mathbb{1}_B \otimes m)$$
$$= \mathbb{1}_M \otimes m \otimes \mathbb{1}_B.$$

Hence, $\Psi_B(m) = m \otimes \mathbb{1}_B$ for all $m \in M$. So we see that Ψ_B is trivial.

□

Next, we shall see that if two quantum families of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ commute with a third one, then so does their composition.

Proposition 3.3 (Proposition 19, [19])**.** Let B, B' and C be C*-algebras and let

$$\Psi_B \in \mathrm{Mor}(M, M \otimes B),$$
$$\Psi_{B'} \in \mathrm{Mor}(M, M \otimes B'),$$
$$\Psi_C \in \mathrm{Mor}(M, M \otimes C)$$

be quantum families of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by $\mathcal{QS}(B)$, $\mathcal{QS}(B')$ and $\mathcal{QS}(C)$ respectively. Assume that Ψ_B commutes with Ψ_C and that $\Psi_{B'}$ commutes with Ψ_C. Then $\Psi_B \triangle \Psi_{B'}$ commutes with Ψ_C.

Proof. We shall use the fact that

$$\sigma_{B\otimes B',C} = (\sigma_{B,C} \otimes \mathrm{id}_{B'}) \circ (\mathrm{id}_B \otimes \sigma_{B',C})$$

Now

$$\begin{aligned}
&(\mathrm{id}_M \otimes \sigma_{B\otimes B',C}) \circ ((\Psi_B \triangle \Psi_{B'}) \triangle \Psi_C) \\
&= (\mathrm{id}_M \otimes \sigma_{B\otimes B',C}) \circ (\Psi_B \triangle \Psi_{B'} \triangle \Psi_C) \\
&= (\mathrm{id}_M \otimes \sigma_{B\otimes B',C}) \circ ((\Psi_B \otimes \mathrm{id}_{B'} \otimes \mathrm{id}_C) \circ (\Psi_{B'} \triangle \Psi_C) \\
&= (\mathrm{id}_M \otimes \sigma_{B,C} \otimes \mathrm{id}_{B'}) \circ (\mathrm{id}_M \otimes \mathrm{id}_B \otimes \sigma_{B',C}) \circ ((\Psi_B \otimes \mathrm{id}_{B'} \otimes \mathrm{id}_C) \circ (\Psi_{B'} \triangle \Psi_C) \\
&= (\mathrm{id}_M \otimes \sigma_{B,C} \otimes \mathrm{id}_{B'}) \circ ((\Psi_B \otimes \mathrm{id}_C \otimes \mathrm{id}_{B'}) \circ (\mathrm{id}_M \otimes \sigma_{B',C}) \circ (\Psi_{B'} \triangle \Psi_C) \\
&= (\mathrm{id}_M \otimes \sigma_{B,C} \otimes \mathrm{id}_{B'}) \circ ((\Psi_B \otimes \mathrm{id}_C \otimes \mathrm{id}_{B'}) \circ (\mathrm{id}_M \otimes \sigma_{B',C}) \circ (\Psi_{B'} \triangle \Psi_C) \\
&= (\mathrm{id}_M \otimes \sigma_{B,C} \otimes \mathrm{id}_{B'}) \circ ((\Psi_B \otimes \mathrm{id}_C \otimes \mathrm{id}_{B'}) \circ (\Psi_C \triangle \Psi_{B'}) \\
&= (\mathrm{id}_M \otimes \sigma_{B,C} \otimes \mathrm{id}_{B'}) \circ ((\Psi_B \otimes \mathrm{id}_C \otimes \mathrm{id}_{B'}) \circ (\Psi_C \otimes \mathrm{id}_{B'}) \circ \Psi_{B'} \\
&= \left(\left[(\mathrm{id}_M \otimes \sigma_{B,C}) \circ (\Psi_B \otimes \mathrm{id}_C) \circ \Psi_C\right] \otimes \mathrm{id}_{B'}\right) \otimes \Psi_{B'} \\
&= \left(\left[(\mathrm{id}_M \otimes \sigma_{B,C}) \circ (\Psi_B \triangle \Psi_C)\right] \otimes \mathrm{id}_{B'}\right) \otimes \Psi_{B'} \\
&= ([\Psi_C \triangle \Psi_B] \otimes \mathrm{id}_{B'}) \otimes \Psi_{B'} \\
&= (\Psi_C \triangle \Psi_B) \triangle \Psi_{B'} \\
&= \Psi_C \triangle (\Psi_B \triangle \Psi_{B'}).
\end{aligned}$$

Hence we have $\Psi_B \triangle \Psi_{B'}$ commuting with Ψ_C. □

Now we are ready to construct the quantum family of all maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ commuting with a given quantum family. In fact, we shall also show that it has a structure of a quantum semigroup, i.e., a comultiplication exists. Sołtan states the following theorem in his paper [19] without proofs. We shall fill in the omitted details here.

Theorem 3.4 (Theorem 21, [19])**.** *Let B be a C^*-algebra and let $\Phi_B \in Mor(M, M \otimes B)$ be a quan-*

tum family of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ *labeled by* $\mathcal{QS}(B)$*. Let* K *be the close ideal of* $\mathcal{A}$ *generated by*

$$\{(\omega \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta)(\Phi \triangle \Psi_B)(m) - (\omega \otimes \eta \otimes \mathrm{id}_{\mathcal{A}})(\Psi_B \triangle \Phi)(m) \mid m \in M, \omega \in M^*, \eta \in B^*\}$$

Let $\bar{A}$ *be the quotient* $\mathcal{A}/K$*,* $\rho : \mathcal{A} \to \bar{A}$ *be the canonical epimorphism and* $\bar{\Phi} = (\mathrm{id}_M \otimes \rho) \circ \Phi$*. Then we have*

1. *The quantum family* $\bar{\Phi} \in Mor(M, M \otimes \bar{A})$ *of maps* $\mathcal{QS}(M) \to \mathcal{QS}(M)$ *labeled by* $\mathcal{QS}(\bar{A})$ *commutes with the quantum family* Ψ_B*.*

2. *For any* C^**-algebra* C *and any quantum family* $\Psi_C \in Mor(M, M \otimes C)$ *of maps* $\mathcal{QS}(M) \to \mathcal{QS}(M)$ *which commutes with* Ψ_B*, there exists a unique* $\Lambda \in Mor(\bar{A}, C)$ *such that* $\Psi_C = (\mathrm{id} \otimes \Lambda) \circ \bar{\Phi}$*. That is, the following diagram*

$$\begin{array}{ccc} M & \xrightarrow{\quad \bar{\Phi} \quad} & M \otimes \bar{A} \\ \| & & \downarrow \mathrm{id} \otimes \Lambda \\ M & \xrightarrow{\quad \Psi_C \quad} & M \otimes C \end{array}$$

   commutes.

3. *There exists a unique* $\bar{\Delta} \in Mor(\bar{A}, \bar{A} \otimes \bar{A})$ *such that*

$$(\bar{\Phi} \otimes \mathrm{id}_{\bar{A}}) \circ \bar{\Phi} = (\mathrm{id}_{\bar{A}} \otimes \bar{\Delta}) \circ \bar{\Phi}.$$

   Hence $\bar{\Phi}$ *is an action of* $(\bar{A}, \bar{\Delta})$ *on* $\mathcal{QS}(M)$*.*

4. *The morphism* $\bar{\Delta}$ *is coassociative. That is,* $(\bar{A}, \bar{\Delta})$ *is a compact quantum semigroup. Moreover, the counit* $\bar{\varepsilon}$ *of* $\bar{A}$ *exists and*

$$\bar{\varepsilon} \circ \rho = \varepsilon$$

   where ε *is the counit of* $\mathcal{A}$*.*

5. *ρ is a quantum semigroup morphism.*

We say that $(\mathcal{QS}(\bar{A}),\bar{\Phi})$ is universal in the sense of (2).

Proof. (1) We wish to show that the two quantum families $\Psi_B \in \mathrm{Mor}(M,M,\otimes B)$ and $\bar{\Phi} \in \mathrm{Mor}(M,M\otimes \bar{A})$ of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by $\mathcal{QS}(B)$ and $\mathcal{QS}(\bar{A})$ respectively fulfills the equation

$$(\mathrm{id}_M \otimes \sigma_{\bar{A},B})(\bar{\Phi}\triangle\Psi_B)(m) = (\Psi_B\triangle\bar{\Phi}(m)).$$

Note that

$$\begin{aligned}
&(\omega\otimes\mathrm{id}_{\mathcal{A}}\otimes\eta)(\Phi\triangle\Psi_B)(m) - (\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})(\Psi_B\triangle\Phi)(m)\\
&= (\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})(\mathrm{id}_M\otimes\sigma_{A,B})((\Phi\triangle\Psi_B)(m)) - (\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})(\Psi_B\triangle\Phi)(m)\\
&= (\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})\left[(\mathrm{id}_M\otimes\sigma_{A,B})(\Phi\triangle\Psi_B)(m) - (\Psi_B\triangle\Phi)(m)\right] \in K
\end{aligned}$$

for all $\omega \in M^*$, $\eta \in B^*$ and $m \in M$.

Since $\bar{A} = \mathcal{A}/K$ and $\rho : \mathcal{A} \to \bar{A}$, then we have

$$\rho\left((\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})\left[(\mathrm{id}_M\otimes\sigma_{A,B})(\Phi\triangle\Psi_B)(m) - (\Psi_B\triangle\Phi)(m)\right]\right) = 0.$$

However we see that

$$\begin{aligned}
&\rho\left((\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})\left[(\mathrm{id}_M\otimes\sigma_{A,B})(\Phi\triangle\Psi_B)(m) - (\Psi_B\triangle\Phi)(m)\right]\right)\\
&= (\omega\otimes\eta\otimes\mathrm{id}_{\bar{A}})\left[(\mathrm{id}_M\otimes\mathrm{id}_B\otimes\rho)\left((\mathrm{id}_M\otimes\sigma_{\mathcal{A},B})(\Phi\triangle\Psi_B)(m) - (\Psi_B\triangle\Phi)(m)\right)\right]\\
&= 0.
\end{aligned}$$

Since this is true for any $\omega \in M^*$, $\eta \in B^*$ and $m \in M$, we have

$$(\mathrm{id}_M\otimes\mathrm{id}_B\otimes\rho)\left((\mathrm{id}_M\otimes\sigma_{\mathcal{A},B})(\Phi\triangle\Psi_B)(m) - (\Psi_B\triangle\Phi)(m)\right) = 0,$$

which is exactly

$$(\mathrm{id}_M \otimes \sigma_{\bar{A},B})(\bar{\Phi} \triangle \Psi_B(m) - (\Psi_B \triangle \bar{\Phi})(m)) = 0.$$

(2) Let C be a C*-algebra and let $\Psi_C \in \mathrm{Mor}(M, M \otimes C)$ of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by $\mathcal{QS}(C)$ which commutes with Ψ_B. By the universal property of $(\mathcal{A}, \Phi)$, there exists a unique map $\Lambda_0 \in \mathrm{Mor}(\mathcal{A}, C)$ such that

$$(\mathrm{id}_M \otimes \Lambda_0) \circ \Phi = \Psi_C.$$

Note that Λ_0 maps all elements of the form

$$(\omega \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta)(\Phi \triangle \Psi_B)(m) - (\omega \otimes \eta \otimes \mathrm{id}_{\mathcal{A}})(\Psi_B \triangle \Phi)(m)$$

to 0 for $m \in M$, $\omega \in M^*$ and $\eta \in B^*$. For we have

$$\begin{aligned}
&\Lambda_0\left((\omega \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta)(\Phi \triangle \Psi_B)(m) - (\omega \otimes \eta \otimes \mathrm{id}_{\mathcal{A}})(\Psi_B \triangle \Phi)(m)\right) \\
&= \Lambda_0\left((\omega \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta)(\Phi \triangle \Psi_B)(m)\right) - \Lambda_0\left((\omega \otimes \eta \otimes \mathrm{id}_{\mathcal{A}})(\Psi_B \triangle \Phi)(m)\right)
\end{aligned}$$

with

$$\begin{aligned}
&\Lambda_0\left((\omega \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta)(\Phi \triangle \Psi_B)(m)\right) \\
&= \Lambda_0\left((\omega \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta)(\Phi \otimes \mathrm{id}_B)\Psi_B(m)\right) \\
&= (\omega \otimes \mathrm{id}_C \otimes \eta)(\mathrm{id}_M \otimes \Lambda_0 \otimes \mathrm{id}_B)(\Phi \otimes \mathrm{id}_B)\Psi_B(m) \\
&= (\omega \otimes \mathrm{id}_C \otimes \eta)\left[(\mathrm{id}_M \otimes \Lambda_0)\Phi \otimes \mathrm{id}_B)\right]\Psi_B(m) \\
&= (\omega \otimes \mathrm{id}_C \otimes \eta)\left[\Psi_C \otimes \mathrm{id}_B)\right]\Psi_B(m) \\
&= (\omega \otimes \mathrm{id}_C \otimes \eta)(\Psi_C \triangle \Psi_B)(m)
\end{aligned}$$

and

$$
\begin{aligned}
&\Lambda_0\left((\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})(\Psi_B\triangle\Phi)(m)\right)\\
&=\Lambda_0\left((\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})(\Psi_B\otimes\mathrm{id}_{\mathcal{A}})\Phi(m)\right)\\
&=(\omega\otimes\eta\otimes\mathrm{id}_C)(\mathrm{id}_M\otimes\mathrm{id}_B\otimes\Lambda_0)(\Psi_B\otimes\mathrm{id}_{\mathcal{A}})\Phi(m)\\
&=(\omega\otimes\eta\otimes\mathrm{id}_C)(\Psi_B\otimes\mathrm{id}_C)(\mathrm{id}_M\otimes\Lambda_0)\Phi(m)\\
&=(\omega\otimes\eta\otimes\mathrm{id}_C)(\Psi_B\otimes\mathrm{id}_C)\Psi_C(m)\\
&=(\omega\otimes\eta\otimes\mathrm{id}_C)(\Psi_B\triangle\Psi_C)(m).
\end{aligned}
$$

Since Ψ_C commutes with Ψ_B, we have $(\mathrm{id}_M\otimes\sigma_{B,C})(\Psi_B\triangle\Psi_C)=\Psi_C\triangle\Psi_B$. So

$$
\begin{aligned}
&\Lambda_0\left((\omega\otimes\mathrm{id}_{\mathcal{A}}\otimes\eta)(\Phi\triangle\Psi_B)(m)-(\omega\otimes\eta\otimes\mathrm{id}_{\mathcal{A}})(\Psi_B\triangle\Phi)(m)\right)\\
&=(\omega\otimes\mathrm{id}_C\otimes\eta)(\Psi_C\triangle\Psi_B)(m)-(\omega\otimes\eta\otimes\mathrm{id}_C)(\Psi_B\triangle\Psi_C)(m)\\
&=(\omega\otimes\mathrm{id}_C\otimes\eta)(\mathrm{id}_M\otimes\sigma_{B,C})(\Psi_B\triangle\Psi_C)(m)-(\omega\otimes\eta\otimes\mathrm{id}_C)(\Psi_B\triangle\Psi_C)(m)\\
&=0.
\end{aligned}
$$

Hence $K\subseteq\ker\Lambda_0$. Since $\rho:\mathcal{A}\to\bar{A}$ is the canonical epimorphism, for $\Lambda_0\in\mathrm{Mor}(\mathcal{A},C)$, there is a unique $\Lambda\in\mathrm{Mor}(\bar{A},C)$ such that this diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\ \rho\ } & \bar{A}\\
 & {\scriptstyle\Lambda_0}\searrow & \downarrow{\scriptstyle\Lambda}\\
 & & C
\end{array}
$$

commutes. That is, $\Lambda \circ \rho = \Lambda_0$. Therefore this implies that

$$\begin{aligned}(\mathrm{id}_M \otimes \Lambda)\bar{\Phi} &= (\mathrm{id}_M \otimes \Lambda)(\mathrm{id}_M \otimes \rho)\Phi \\ &= [\mathrm{id}_M \otimes (\Lambda \circ \rho)]\,\Phi \\ &= (\mathrm{id}_M \otimes \Lambda_0)\Phi \\ &= \Psi_C.\end{aligned}$$

So $(\mathcal{QS}(\bar{A}), \bar{\Phi})$ poses the universal property that for any C*-algebra C and any quantum family $\Psi_C \in \mathrm{Mor}(M, M \otimes C)$ of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ which commutes with Ψ_B, then there exists a unique $\Lambda \in$ Mor $(\bar{A}, C)$ such that this diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad \bar{\Phi} \quad} & M \otimes \bar{A} \\ \Big\| & & \Big\downarrow \mathrm{id} \otimes \Lambda \\ M & \xrightarrow{\quad \Psi_C \quad} & M \otimes C \end{array}$$

commutes.

(3) Note that by Proposition 3.3, $\bar{\Phi} \triangle \bar{\Phi}$ is a quantum family of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by $\mathcal{QS}(\bar{A} \otimes \bar{A})$ that commutes with Ψ_B. Then by the universal property in (2), there exists a unique $\bar{\Delta} \in \mathrm{Mor}(\bar{A}, \bar{A} \otimes \bar{A})$ such that

$$(\mathrm{id}_{\bar{A}} \otimes \bar{\Delta}) \circ \bar{\Phi} = \bar{\Phi} \triangle \bar{\Phi} = (\bar{\Phi} \otimes \mathrm{id}_{\bar{A}}) \circ \bar{\Phi}.$$

(4) Since ρ is an epimorphism from $\mathcal{A}$ to $\bar{A}$ and $\bar{A}$ is generated by

$$\{(\omega \otimes \mathrm{id}_{\bar{A}})\bar{\Phi}(m) \mid m \in M, \omega \in M^*\},$$

the proofs that $\bar{\Delta}$ is a comultiplication and $\bar{\varepsilon}$ is the counit can be copied verbatim from that of Theorem 2.12 by substituting Φ's and Δ's in equation (2.2) by $\bar{\Phi}$'s and $\bar{\Delta}$'s.

(5) Note that we have $\bar{\Delta} \in \mathrm{Mor}(\bar{A}, \bar{A} \otimes \bar{A})$ such that

$$(\mathrm{id}_{\bar{A}} \otimes \bar{\Delta}) \circ \bar{\Phi} = (\bar{\Phi} \otimes \mathrm{id}_{\bar{A}}) \circ \bar{\Phi}$$

and

$$(\mathrm{id}_M \otimes \rho) \circ \Phi = \bar{\Phi}.$$

Hence by Proposition 2.20, we have

$$(\rho \otimes \rho) \circ \Delta = \bar{\Delta} \circ \rho.$$

Therefore ρ is a quantum semigroup morphism. □

3.2 Example

In this section, we shall continue examining the case when $M = M_2(\mathbb{C})$ (see the last example in Chapter 2). We shall study some simple example of quantum commutants of families of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$. Of course any classical family of morphisms $M \to M$ can be interpreted as a quantum family of maps $\mathcal{QS}(M) \to \mathcal{QS}(M)$ labeled by a classical space. The simplest possible such family to be considered will be the classical family consisting of a single automorphism of $M_2(\mathbb{C})$. We will use the notation $\mathrm{Aut}(M_2(\mathbb{C}))$ to indicate the group of *-algebra automorphisms from $M_2(\mathbb{C}) \to M_2(\mathbb{C})$.

Let $\phi \in \mathrm{Aut}(M_2(\mathbb{C}))$. The singleton family $\{\phi\}$ can be described in our non-commutative language by taking $B = \mathbb{C}$ and $\Psi_B : M_2(\mathbb{C}) \ni m \mapsto \phi(m) \otimes \mathbb{1}_{\mathbb{C}} \in M_2(\mathbb{C}) \otimes B$. Now the quantum commutant of Ψ_B (or in other words of $\{\phi\}$) is $(\mathcal{QS}(\bar{A}), \bar{\Delta})$, where $\bar{A}$ is the quotient of $\mathcal{A}$ (in Example 2.22) by the closed ideal K (in Theorem 3.4) :

$$\{(\omega \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta)(\Phi \triangle \Psi_B)(m) - (\omega \otimes \eta \otimes \mathrm{id}_{\mathcal{A}})(\Psi_B \triangle \Phi)(m) \mid m \in M, \omega \in M^*, \eta \in B^*\}.$$

Since $\Psi_B(m) = \phi(m) \otimes \mathbb{1}_{\mathbb{C}}$ for all $m \in M$ and $\eta \in B^* = \mathbb{C}^* = \mathbb{C}$, we have

$$\begin{aligned}&(\omega \otimes \mathrm{id}_{\mathcal{A}} \otimes \eta)(\Phi \triangle \Psi_B)(m) - (\omega \otimes \eta \otimes \mathrm{id}_{\mathcal{A}})(\Psi_B \triangle \Phi)(m)\\ &= \eta(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(\phi(m)) - \eta(\omega \otimes \mathrm{id}_{\mathcal{A}})(\phi \otimes \mathrm{id}_{\mathcal{A}})\Phi(m).\end{aligned}$$

Since η is just a complex number, we say that K, in this special case, is generated by:

$$\{(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(\phi(m)) - ((\omega \otimes \mathrm{id}_{\mathcal{A}})(\phi \otimes \mathrm{id}_{\mathcal{A}})\Phi(m) \mid m \in M, \omega \in M^*\}.$$

Example 3.5. Let $M = M_2(\mathbb{C})$ and $\phi \in \mathrm{Aut}(M)$. Consider the singleton family $\{\phi\}$ where ϕ defined by

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

We shall use the notations in Theorem 3.4. So $\bar{A}$ is the quotient $\mathcal{A}/K$, $\rho : \mathcal{A} \to \bar{A}$ be the canonical epimorphism and $\bar{\Phi} = (\mathrm{id}_M \otimes \rho) \circ \Phi$. Here, K is the closed ideal generated by

$$\{(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(\phi(m)) - ([\omega \circ \phi] \otimes \mathrm{id}_{\mathcal{A}})\Phi(m) \mid m \in M, \omega \in M^*\}.$$

Recall that in Example 2.22, $\mathbb{a}, \mathbb{b}, \mathbb{g}$ and $\mathbb{d}$ are generators of $\mathcal{A}$. Let α, β, γ and δ be the images of these generators respectively in $\bar{A}$ under ρ. We also know that M is the universal C*-algebra generated by

$$n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

satisfying the relations

$$n^*n + nn^* = \mathbb{1}_M \qquad \text{and} \qquad n^2 = 0$$

and

$$\Phi(n) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Now consider the followings:

$$\begin{aligned}
\Phi(\phi(n)) &= \Phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \\
&= \Phi(n^*) = \Phi(n)^* \\
&= \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
(\phi \otimes \mathrm{id}_{\mathcal{A}})\Phi(n) &= (\phi \otimes \mathrm{id}_{\mathcal{A}}) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\
&= \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}.
\end{aligned}$$

Since we have

$$\begin{aligned}
&(\omega \otimes \mathrm{id}_{\mathcal{A}})\Phi(\phi(n)) - ((\omega \otimes \mathrm{id}_{\mathcal{A}})(\phi \otimes \mathrm{id}_{\mathcal{A}})\Phi(n) \\
&= (\omega \otimes \mathrm{id}_{\mathcal{A}})\left[\Phi(\phi(n)) - (\phi \otimes \mathrm{id}_{\mathcal{A}})\Phi(n)\right] \\
&= (\omega \otimes \mathrm{id}_{\mathcal{A}})\left[\begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} - \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}\right] \\
&= (\omega \otimes \mathrm{id}_{\mathcal{A}})\begin{pmatrix} \alpha^* - \delta & \gamma^* - \gamma \\ \beta^* - \beta & \delta^* - \alpha \end{pmatrix} \in K
\end{aligned}$$

we see that

$$\rho(\omega \otimes \mathrm{id}_{\mathcal{A}}) \begin{pmatrix} \mathbb{\alpha}^* - \mathbb{\delta} & \mathbb{\gamma}^* - \mathbb{\gamma} \\ \mathbb{\beta}^* - \mathbb{\beta} & \mathbb{\delta}^* - \mathbb{\alpha} \end{pmatrix} = 0$$

which then implies that

$$(\omega \otimes \mathrm{id}_{\bar{A}}) \begin{pmatrix} \alpha^* - \delta & \gamma^* - \gamma \\ \beta^* - \beta & \delta^* - \alpha \end{pmatrix} = 0.$$

Since this is true for all $\omega \in M^*$, we have $\alpha^* = \delta$, $\gamma = \gamma^*$ and $\beta = \beta^*$. Note that the following relations hold automatically in the quotient algebra $\bar{A}$:

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma + \alpha\alpha^* + \beta\beta^* &= \mathbb{1}_{\bar{A}}, \\ \alpha^*\beta + \gamma^*\delta + \alpha\gamma^* + \beta\delta^* &= 0_{\bar{A}}, \\ \beta^*\beta + \delta^*\delta + \gamma\gamma^* + \delta\delta^* &= \mathbb{1}_{\bar{A}} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \alpha^2 + \beta\gamma &= 0_{\bar{A}}, \\ \alpha\beta + \beta\delta &= 0_{\bar{A}}, \\ \gamma\alpha + \delta\gamma &= 0_{\bar{A}}, \\ \gamma\beta + \delta^2 &= 0_{\bar{A}}. \end{aligned} \tag{3.2}$$

Similarly, we have

$$\begin{aligned} \Delta(\alpha) &= \alpha\alpha^* \otimes \alpha + \beta\beta^* \otimes \alpha + \alpha \otimes \beta + \alpha^* \otimes \gamma + \alpha^*\alpha \otimes \delta + \gamma^*\gamma \otimes \delta, \\ \Delta(\beta) &= \alpha\gamma^* \otimes \alpha + \beta\delta^* \otimes \alpha + \beta \otimes \beta + \gamma^* \otimes \gamma + \alpha^*\beta \otimes \delta + \gamma^*\delta \otimes \delta, \\ \Delta(\gamma) &= \gamma\alpha^* \otimes \alpha + \delta\beta^* \otimes \alpha + \gamma \otimes \beta + \beta^* \otimes \gamma + \beta^*\alpha \otimes \delta + \delta^*\gamma \otimes \delta, \end{aligned} \tag{3.3}$$

and $\bar{\varepsilon}(\alpha) = \bar{\varepsilon}(\gamma) = 0$, $\bar{\varepsilon}(\beta) = 1$.

To contruct $\bar{A}$ directly and explicitly, we now forget about the existing $\bar{A}$ and define directly an algebra $\bar{A}$ as the universal C*-algebra generated by α, β, γ and δ with all the relations in equations (3.1) and (3.2) (with $\delta = \alpha^*$, $\beta = \beta^*$ and $\gamma = \gamma^*$). Define $\Phi_{\bar{A}}$ as the unique *-homomorphism $M \to M \otimes \bar{A}$ which commutes with $\{\phi\}$ by

$$\Phi_{\bar{A}}(n) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M \otimes \bar{A},$$

which exists because M is the universal C*-algebra generated by n subject to

$$n^2 = 0 \qquad \text{and} \qquad n^* n + n n^* = \mathbb{1}_M.$$

Now if we consider any C*-algebra B and any $\Phi_B \in$Mor$(M, M \otimes B)$ which commutes with $\{\phi\}$, the four entries of $\Psi_B(n) \in M \otimes B$ must satisfy the similar relations as in equations (3.1) and (3.2) (with $\delta = \alpha^*$, $\beta = \beta^*$ and $\gamma = \gamma^*$). Hence, by the universality of $\bar{A}$, there is a unique map λ from $\bar{A}$ to B such that $\Phi_B = (\mathrm{id}_M \otimes \lambda) \circ \Phi_{\bar{A}}$. These precisely say that $\mathcal{QS}(\bar{A})$ is the quantum space and $\Phi_{\bar{A}}$ is the quantum commutant of the singleton family $\{\phi\}$, where $\bar{A}$ is the universal C*-algebra generated by α, β, γ and δ with all the relations in equations (3.1) and (3.2) (with $\delta = \alpha^*$, $\beta = \beta^*$ and $\gamma = \gamma^*$).

Remark 3.6. If we let $\beta = 0_{M_2(\mathbb{C})} = \gamma$ and

$$\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C}),$$

then we see that α, β and γ satisfy the above relations in 3.1 and 3.2 (with $\delta = \alpha^*$). Moreover, α, β and γ generate $M_2(\mathbb{C})$. Since we see that $M_2(\mathbb{C})$ is non-commutative and $\bar{A}$ is the universal C*-algebra generated by 3 elements satisfying the relations (3.1) and (3.2) (with $\delta = \alpha^*$, $\beta = \beta^*$ and $\gamma = \gamma^*$) together with the requirements that two of the generators are self-adjoint, then $\bar{A}$ is non-commutative.

We shall later see that for the case when $M = M_2(\mathbb{C})$ and $\Psi_B = \{\phi\}$, $(\mathcal{QS}(\bar{A}), \bar{\Delta})$ does not admit a compact quantum group structure. Before that, let us digress a little bit and examine some properties of compact quantum groups. We shall start off by defining what is a Hopf *-algebra:

Definition 3.7. Let $\mathscr{A}$ be a unital *-algebra and $\Delta : \mathscr{A} \to \mathscr{A} \otimes_{\mathrm{alg}} \mathscr{A}$ be a unital *-algebra homomorphism such that such that $(\Delta \otimes \mathrm{id}_{\mathscr{A}}) \circ \Delta = (\mathrm{id}_{\mathscr{A}} \otimes \Delta) \circ \Delta)$ (coassociativity). We say that $(\mathscr{A}, \Delta)$ is a *Hopf *-algebra* if there exist linear mappings $e : \mathscr{A} \to \mathbb{C}$ and $\kappa : \mathscr{A} \to \mathscr{A}$ such that

$$(e \otimes \mathrm{id}_{\mathscr{A}})\Delta(a) = a$$

$$(\mathrm{id}_{\mathscr{A}} \otimes e)\Delta(a) = a$$

$$m(\kappa \otimes \mathrm{id}_{\mathscr{A}})\Delta(a) = e(a)\mathbb{1}_{\mathscr{A}}$$

$$m(\mathrm{id}_{\mathscr{A}} \otimes \kappa)\Delta(a) = e(a)\mathbb{1}_{\mathscr{A}}$$

for any $a \in \mathscr{A}$. In the above formulae, m denotes the multiplication map $m : \mathscr{A} \otimes_{\mathrm{alg}} \mathscr{A} \to \mathscr{A}$, i.e., the linear map such that $m(a \otimes b) = ab$ for any $a, b \in \mathscr{A}$. Note that e is just the counit. Also we will call κ the *antipode* of $\mathscr{A}$.

It is known that the counit and antipode of $\mathscr{A}$ are uniquely determined. e is a unital *-algebra homomorphism and κ is antimultiplicative, anticomultiplicative and

$$\kappa(\kappa(a^*)^*) = a$$

for any $a \in \mathscr{A}$ (see [23]).

We shall see that the corresponding C*-algebra $\mathcal{A}$ of any compact quantum group $(\mathcal{QS}(\mathcal{A}), \Delta)$ has a dense *-subalgebra $\mathscr{A}$ such that $(\mathscr{A}, \Delta|_{\mathscr{A}})$ is a Hopf *-algebra. We will need the notion of finite dimensional representation (see [27]).

Definition 3.8. Let $(\mathcal{A}, \Delta)$ be a compact quantum group and $v = (v_{kl})_{k,l=1,2,\ldots,N}$ be an $N \times N$ matrix with entries in $\mathcal{A}$. Then we say that v is an *N-dimensional unitary representation* of $\mathcal{A}$ if v is a unitary element of $M_N(\mathcal{A}) \equiv M_N(\mathbb{C}) \otimes \mathcal{A}$ and

$$\Delta(v_{kl}) = \sum_r v_{kr} \otimes v_{rl}$$

for all $k, l = 1, 2, ..., N$.

Theorem 3.9 (Theorem 1.2, [28]). *Let* $(\mathcal{QS}(\mathcal{A}), \Delta)$ *be a compact quantum group and* $\mathscr{A}$ *be the set of linear combinations of matrix elements of all finite-dimensional unitary representations of* $\mathcal{A}$. *Then* $\mathscr{A}$ *is a dense* **-subalgebra of* $\mathcal{A}$ *and* $\Delta(\mathscr{A}) \subset \mathscr{A} \otimes_{alg} \mathscr{A}$. *Moreover* $(\mathscr{A}, \Delta|_{\mathscr{A}})$ *is a Hopf* **-algebra.*

We shall call $\mathscr{A}$ the *associated* Hopf *-algebra of $(\mathcal{QS}(\mathcal{A}), \Delta)$. In fact it has the following uniqueness property.

Theorem 3.10 (Theorem A1, [1]). *The associated Hopf* **-algebra* $\mathscr{A}$ *of a compact quantum group* $(\mathcal{QS}(\mathcal{A}), \Delta)$ *is the unique dense Hopf* **-subalgebra of* $(\mathcal{QS}(\mathcal{A}), \Delta)$.

By a Hopf *-subalgebra $\mathscr{A}$ of a compact quantum group $(\mathcal{QS}(\mathcal{A}), \Delta)$, we mean a Hopf *-algebra such that $\mathscr{A}$ is a *-subalgebra of $\mathcal{A}$ with co-multiplication given by restricting the co-multiplication Δ from $\mathcal{A}$ to $\mathscr{A}$. For any C*-algebra B, we shall say that a state $h \in B^*$ is *faithful* if $h(b^*b) = 0 \Rightarrow b = 0$ for any $b \in B$.

Theorem 3.11 (Theorem 4.3, [27] and Theorem 1.6, [28]). *Let* $(\mathcal{QS}(\mathcal{A}), \Delta)$ *be a compact quantum group and* $\mathscr{A}$ *be the associated Hopf* **-algebra. Then the Haar measure h is faithful on* $\mathscr{A}$. *Moreover, if h is faithful on* $\mathcal{A}$, *then* $\mathscr{A} = \{a \in \mathcal{A} \mid \Delta(a) \in \mathcal{A} \otimes_{alg} \mathcal{A}\}$.

Theorem 3.11 implies that the intersection of $\mathscr{A}$ with the space

$$J = \{a \in \mathcal{A} \mid h(a^*a) = 0\}$$

is trivial. Moreover, by Proposition 7.9 in [12], we see that for any $a \in \mathcal{A}$, $h(a^*a) = 0$ if and only if $h(aa^*) = 0$. Since h is a state, by Theorem 1.13, we have

$$h(b^*a^*ab) \leq \|a^*a\| h(b^*b)$$

for any $a, b \in \mathcal{A}$. Hence J is a two-sided ideal of $\mathcal{A}$. Let $\mathcal{A}_r = \mathcal{A}/J$ and denote by λ the canonical quotient map $\mathcal{A} \to \mathcal{A}_r$. The map is injective on the dense subalgebra $\mathscr{A}$, so $\mathcal{A}_r$ can also be viewed as a different completion of $\mathscr{A}$. Also, there is a comultiplication $\Delta_r : \mathcal{A}_r \to \mathcal{A}_r \otimes \mathcal{A}_r$ extending that of $\mathscr{A}$ such that $(\mathcal{QS}(\mathcal{A}_r), \Delta_r)$ is a compact quantum group which is called the *reduced quantum group* of $(\mathcal{QS}(\mathcal{A}), \Delta)$. Moreover, $(\lambda \otimes \lambda) \circ \Delta = \Delta_r \otimes \lambda$. The Haar measure h_r of $\mathcal{A}_r$ is the unique state h_r of $\mathcal{A}_r$ such that $h = h_r \circ \lambda$. In fact, h_r is faithful on $\mathcal{A}_r$. It can also be shown that if $\mathcal{A}_r$ is commutative, then $\mathcal{A}_r = \mathcal{A}$ (see section 3, [21]).

Now we are ready to prove that for the case when $M = M_2(C)$ and $\Psi_B = \{\phi\}$ where $\phi : M \to M$ is defined by

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix},$$

the quantum commutant of Ψ_B does not admit a compact quantum group structure. When I started to work on proving the result, I was not aware that Piotr Sołtan has shown in his paper [18] about this result. However, his proof was incorrect, in which equation (4.13) was wrong which leads to the later implications in the proof being wrong too. Furthermore, in his proof, his claim about $C(Z)$ (page 10) can be seen to be incorrect by drawing the mentioned relations on some mathematics software like Matlab or Mathematica. Nonetheless, the result is still true and I will give a modified proof here by using the similar techniques in his paper.

Proposition 3.12. Let $M = M_2(C)$ and $\Psi_B = \{\phi\}$ where $\phi : M \to M$ is defined by

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Then the quantum commutant of Ψ_B does not admit a compact quantum group structure.

Proof. We shall use the notations in Example 3.5. Hence α, β and γ are generators of $\bar{A}$ where $\beta = \beta^*$ and $\gamma = \gamma^*$. We shall let $X = \alpha + \alpha^*$ and $Y = \beta + \gamma$. Hence X and Y are self-adjoint. Consider

$$XY = (\alpha + \alpha^*)(\beta + \gamma) = \alpha\beta + \alpha^*\beta + \alpha\gamma + \alpha^*\gamma$$

and

$$YX = (\beta + \gamma)(\alpha + \alpha^*) = \beta\alpha + \beta\alpha^* + \gamma\alpha + \gamma\alpha^*.$$

Note that from equations (3.1) and (3.2), we have

$$\begin{aligned} \alpha^*\beta + \gamma\alpha^* + \alpha\gamma + \beta\alpha &= 0_{\bar{A}}, \\ \alpha\beta + \beta\alpha^* &= 0_{\bar{A}}, \\ \alpha^*\gamma + \gamma\alpha &= 0_{\bar{A}}. \end{aligned}$$

Hence we see that X and Y are anticommutative, i.e.,

$$XY + YX = 0_{\bar{A}}.$$

Moreover, we also have

$$\begin{aligned} \alpha^*\alpha + \gamma^2 + \alpha\alpha^* + \beta^2 &= \mathbb{1}_{\bar{A}}, \\ \alpha^2 + \beta\gamma &= 0_{\bar{A}}, \\ \alpha^{*2} + \alpha\beta &= 0_{\bar{A}}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
X^2 + Y^2 &= (\alpha + \alpha^*)^2 + (\beta + \gamma)^2 \\
&= \alpha^2 + \alpha\alpha^* + \alpha^*\alpha + \alpha^{*2} + \beta^2 + \beta\gamma + \gamma\beta + \gamma^2 \\
&= \mathbb{1}_{\bar{A}}.
\end{aligned}
$$

We will prove this result by contradiction. So we shall assume that $(\mathcal{QS}(\bar{A}), \bar{\Delta})$ is a compact quantum group. Hence by Theorem 2.17, there exists the Haar measure $\bar{h}$ of $\bar{A}$ such that

$$
(\bar{h} \otimes \mathrm{id}_{\bar{A}})\bar{\Delta}(a) = (\mathrm{id}_{\bar{A}} \otimes \bar{h})\bar{\Delta}(a) = \bar{h}(a)\mathbb{1}_{\bar{A}}
$$

for any $a \in \bar{A}$. From equations in (3.3), we know the value of each generator α, β and γ under the comultiplication $\bar{\Delta}$. Therefore we have

$$
\begin{aligned}
\bar{\Delta}(X) &= \bar{\Delta}(\alpha + \alpha^*) \\
&= \bar{\Delta}(\alpha) + \bar{\Delta}(\alpha^*) \\
&= \mathbb{1}_{\bar{A}} + \alpha + (\alpha^*\alpha + \gamma^2) \otimes (\alpha^* - \alpha) + \alpha \otimes \beta + \alpha^* \otimes \gamma \\
&\qquad + 1 \otimes \alpha^* + (\alpha^*\alpha + \gamma^2) \otimes (\alpha - \alpha^*) + \alpha^* \otimes \beta + \alpha \otimes \gamma \\
&= \mathbb{1}_{\bar{A}} \otimes (\alpha + \alpha^*) + (\alpha + \alpha^*) \otimes (\beta + \gamma) \\
&= \mathbb{1}_{\bar{A}} \otimes X + X \otimes Y
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\Delta}Y &= \bar{\Delta}(\beta+\gamma) \\
&= \bar{\Delta}(\beta)+\bar{\Delta}(\gamma) \\
&= (\alpha\gamma+\beta\alpha)\otimes(\alpha-\alpha^*)+\beta\otimes\beta+\gamma\otimes\gamma \\
&\qquad +(\beta\alpha+\alpha\gamma)\otimes(\alpha^*-\alpha)+\gamma\otimes\beta+\beta\otimes\gamma \\
&= (\beta+\gamma)\otimes\beta+(\beta+\gamma)\otimes\gamma \\
&= (\beta+\gamma)\otimes(\beta+\gamma) \\
&= Y\otimes Y.
\end{aligned}
$$

We shall next consider the Haar measure property on X. So

$$
\begin{aligned}
\bar{h}(X)\mathbb{1}_{\bar{A}} &= (\bar{h}\otimes \mathrm{id}_{\bar{A}})\bar{\Delta}(X) \\
&= (\bar{h}\otimes \mathrm{id}_{\bar{A}})(\mathbb{1}_{\bar{A}}\otimes X+X\otimes Y) \\
&= X+\bar{h}(X)Y.
\end{aligned}
$$

Hence we have

$$X+\bar{h}(X)Y=\bar{h}(X)\mathbb{1}_{\bar{A}}.$$

Multiplying Y to the above equation from left and right hand sides, we get

$$
\begin{aligned}
XY+\bar{h}(X)Y^2 &= \bar{h}(X)Y \\
YX+\bar{h}(X)Y^2 &= \bar{h}(X)Y.
\end{aligned}
$$

So we have $XY=YX$. But we also know that X and Y are anticommutative. Hence $XY=YX=0_{\bar{A}}$.

Next we multiply Y to the equation $X^2+Y^2=\mathbb{1}_{\bar{A}}$ from the right, we have

$$\begin{aligned} X^2Y+Y^3 &= Y \\ X(XY)+Y^3 &= Y \\ \Rightarrow Y^3 &= Y. \end{aligned}$$

So Y^2 is a projection. We shall exclude the possibilities where Y^2 is a trivial projections. That is, cases where $Y^2=0_{\bar{A}}$ and $Y^2=1_{\bar{A}}$ are impossible. If $Y^2=0_{\bar{A}}$, then we have $\beta=-\gamma$ (because $Y=0_{\bar{A}}$). Recall that in Example 3.5, the counit $\bar{\varepsilon}$ is such that $\bar{\varepsilon}(\alpha)=\bar{\varepsilon}(\gamma)=0$, $\bar{\varepsilon}(\beta)=1$. Hence contradiction occurs.

Now if $Y^2=\mathbb{1}_{\bar{A}}$, then by the equation $X^2+Y^2=\mathbb{1}_{\bar{A}}$, we have $X^2=0_{\bar{A}}$. Hence $X=\alpha+\alpha^*=0_{\bar{A}}$. By substituting $\alpha=-\alpha^*$ into equations in (3.1) and (3.2), we have

$$\begin{aligned} -\alpha^2+\gamma^2-\alpha^2+\beta^2 &= \mathbb{1}_{\bar{A}} \\ -\alpha\beta-\gamma\alpha+\alpha\gamma+\beta\alpha &= 0_{\bar{A}} \\ \alpha^2+\beta\gamma &= 0_{\bar{A}} \\ \alpha\beta-\beta\alpha &= 0_{\bar{A}} \\ \gamma\alpha-\alpha\gamma &= 0_{\bar{A}}. \end{aligned}$$

Since β and γ are self-adjoint and $\alpha=-\alpha^*$,

$$\alpha^2+\beta\gamma=0_{\bar{A}} \Rightarrow \alpha^2+\gamma\beta=0_{\bar{A}}.$$

So $\beta\gamma=\gamma\beta$. Therefore, we see that all the generators α, β and γ of $\bar{A}$ are commutative with each other which then implies that $\bar{A}$ is commutative. Again, this is a contradiction by Remark 3.6. So Y^2 is a proper projection.

Consider the Haar measure property on Y^2

$$\begin{aligned}\bar{h}(Y^2)\mathbb{1}_{\bar{A}} &= (\bar{h}\otimes \mathrm{id}_{\bar{A}})\bar{\Delta}(Y^2)\\ &= (\bar{h}\otimes \mathrm{id}_{\bar{A}})(Y^2\otimes Y^2)\\ &= \bar{h}(Y^2)Y^2.\end{aligned}$$

Note that $\bar{h}(Y^2) = 0$ (otherwise $Y^2 = \mathbb{1}_{\bar{A}}$ which is a contradiction). If we consider the canonical reducing map $\lambda : \bar{A} \to \bar{A}_r$, then Y^2 will be mapped to $0_{\bar{A}_r}$. We want to show that this is again, impossible. If $Y^2 = 0_{\bar{A}_r}$, then the equation $X^2 + Y^2 = \mathbb{1}_{\bar{A}}$, under the canonical reducing map λ, will imply that $X^2 = \mathbb{1}_{\bar{A}_r}$. Hence we see that $X = \mathbb{1}_{\bar{A}_r}$. Applying λ to equations in (3.1) and (3.2), together with $\beta = -\gamma\,(Y = 0_{\bar{A}_r})$ and $\alpha = \mathbb{1}_{\bar{A}_r} - \alpha^*$ $(X = \mathbb{1}_{\bar{A}_r})$, we have that $\bar{A}_r$ is generated by two elements α and β satisfying

$$\begin{aligned}\beta &= \beta^*,\\ \alpha+\alpha^* &= \mathbb{1}_{\bar{A}_r},\\ \alpha^*\alpha+2\beta^2+\alpha\alpha^* &= \mathbb{1}_{\bar{A}_r},\\ \alpha^2 &= \beta^2,\\ \alpha\beta+\beta\alpha^* &= 0_{\bar{A}_r},\\ \beta\alpha+\alpha^*\beta &= 0_{\bar{A}_r}.\end{aligned}$$

Note that from the above equations, we have

$$\begin{aligned}\beta\alpha^* &= -\alpha\beta\\ \alpha^*\beta &= -\beta\alpha.\end{aligned}$$

Multiplying β from left and right hand side of $\alpha + \alpha^* = \mathbb{1}_{\bar{A}_r}$, we have

$$\alpha\beta + \alpha^*\beta = \beta\alpha + \beta\alpha^*$$

Combining the last two results, we have

$$\alpha\beta - \beta\alpha = \beta\alpha - \alpha\beta$$
$$2\alpha\beta = 2\beta\alpha$$
$$\Rightarrow \alpha\beta = \beta\alpha.$$

So the generators of $\bar{A}_r$ commute with each other which implies that $\bar{A}_r$ is commutative. Hence $\bar{A}_r = \bar{A}$ which is a contradiction.

Therefore, $(\mathcal{QS}(\bar{A}), \bar{\Delta})$ does not admit compact quantum group structure. □

Chapter 4

Quantum families of maps for an infinite dimensional case

In this chapter, we wish to extend the most fundamental result of the study of the quantum families of maps, namely Theorem 2.6 about existence and uniqueness. We shall investigate the unexplored case of a quantum space $\mathcal{QS}(A)$ of all maps from $\mathcal{QS}(C) \to \mathcal{QS}(B)$ when C is infinite dimensional. In particular, we will work on the fundamental case of $C = C(\mathbb{N} \cup \{\infty\})$, where $\mathbb{N} \cup \{\infty\}$ is the one-point compactification of $\mathbb{N}$. In our investigation, we shall let B to remain as a finitely generated C*-algebra and use a result in [20] to give us some sort of motivations to tackle our considerations of the case of $C = C(\mathbb{N} \cup \{\infty\})$. This chapter presents findings of a joint venture with my advisor Albert Sheu to explore this unknown area. It turns out that new structures outside purely C*-algebraic framework are needed from the von Neumann algebra theory in order to handle such a new situation. This opens up a new direction of research in quantizing spaces of maps between more general quantum spaces. Before proceeding, we shall first look at a few fundamental theorems that will serve as main tools in our later work.

4.1 Topologies on $\mathscr{B}(\mathscr{H})$

Recall that in Chapter 1, we did mention that each C*-algebra can be concretely realized as a C*-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. We shall ultilize such characterization in this chapter. Before proceeding to talk about the constructions of our results, we shall briefly study the topologies on $\mathscr{B}(\mathscr{H})$. There are eight important topologies on $\mathscr{B}(\mathscr{H})$ (See [15]). We shall content ourselves here with only three of these:

- The *norm operator topology* which is determined by the *operator norm* $T \in \mathscr{B}(\mathscr{H}) \mapsto \|T\|$,
- The *strong operator topology* which is determined by the seminorms $T \in \mathscr{B}(\mathscr{H}) \mapsto \|Tv\|$ with $v \in \mathscr{H}$,
- The *weak operator topology* which is determined by the seminorms $T \in \mathscr{B}(\mathscr{H}) \mapsto | < w, Tv > |$ with $v, w \in \mathscr{H}$.

Note that we have the norm operator topology to be stronger (or finer) than the strong operator topology and the strong operator topology is stronger than the weak operator topology (see Chapter 4, [13]). Moreover, we have

Definition 4.1. If A is a strongly closed *-subalgebra of $\mathscr{B}(\mathscr{H})$, then we call A a *von Neumann* algebra on $\mathscr{H}$.

Remark 4.2. Since the strong operator topology is weaker than the norm operator topology, a strongly closed set is also norm-closed. Hence a von Neumann algebra is a C*-algebra.

Remark 4.3. It can be proven that if A is a C*-subalgebra of $\mathscr{B}(\mathscr{H})$, then A is weakly closed if and only if it is strongly closed. Hence, in some literature, we do see that some authors claimed that a weakly closed *-subalgebra of $\mathscr{B}(\mathscr{H})$ is a von Neumann algebra on $\mathscr{H}$.

We shall also list a few definitions and results that will be useful later.

Definition 4.4. Let M and N be von Neumann algebra in $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}(\mathscr{K})$, respectively. A positive linear map p of M to N is said to be *normal* if for each bounded, monotone increasing net $\{x_i\}_{i\in I}$ in M_{sa} with limit x, the net $\{p(x_i)\}_{i\in I}$ increases to $p(x)$ in N_{sa}.

Definition 4.5. Let A be a C^*-algebra, π_u the universal representation of A, and B the strong closure of $\pi_u(A)$. Then B is said to be the *enveloping von Neumann algebra* of A and π_u is the *canonical morphism* of A into B. We shall write $B = \mathcal{VN}(A)$.

The pair (B, π_u) is the solution of a universal problem. In fact:

Proposition 4.6 (Proposition 12.1.5, [6])**.** Let A be a C^*-algebra, B the enveloping von Neumann algebra of A, and $\pi : A \to B$ the canonical morphism. Let ρ be a representation of A on $\mathscr{H}_\rho$. There exists exactly one normal representation $\bar{\rho}$ of B in $\mathscr{H}_\rho$ such that $\bar{\rho}(\pi(x)) = \rho(x)$ for every $x \in A$. The image $\bar{\rho}(B)$ is the weak closure of $\rho(A)$.

4.2 Motivations and Constructions

Now, we are ready to start working on the quantum family of maps when B is a finitely generated C^*-algebra and $C = C(\mathbb{N} \cup \{\infty\})$. Recall that in Chapter 2, we see that if B is finitely generated and C is finite dimensional, then the quantum family and quantum space of all maps from $\mathcal{QS}(C) \to \mathcal{QS}(B)$ exist. In particular, when we talk about a special case where $C = \mathbb{C}^n$ for some $n \in \mathbb{N}$, we have the following result:

Theorem 4.7 (Theorem 2.1, [20])**.** *Let B be a unital finitely generated C^*-algebra and $C = \mathbb{C}^n$ a commutative finite dimensional C^*-algebra. Let A be the C^*-algebra corresponding to the quantum space of all maps $\mathcal{QS}(C) \to \mathcal{QS}(B)$ and let*

$$\Phi : B \to C \otimes A$$

be the quantum family of all maps $\mathcal{QS}(C) \to \mathcal{QS}(B)$. Then A is isomorphic to the free product $B^{\star n}$ and with this isomorphism, $\Phi : B \to C \otimes B^{\star n}$ defined by

$$\Phi(b) = \sum_{i=1}^{n} e_i \otimes \iota_i(b), \tag{4.1}$$

where e_i's are the standard basis of C and $\iota_1,\dots,\iota_n$ are the natural inclusions $B \hookrightarrow B^{\star n}$.

Proof. The conclusion of the theorem may be reached by analyzing the construction of A given in Theorem 2.6. However, in Chapter 1, we see the construction of the free product of n-copies of B. We shall prove this theorem by simply checking that $(B^{\star n},\Phi)$, with Φ given by (4.1), has the universal property of the quantum family of all maps $\mathcal{QS}(C) \to \mathcal{QS}(B)$. First note that for any C*-algebra D and any $\Psi \in \mathrm{Mor}(B, C \otimes D)$, Ψ is of the form

$$\Psi(b) = \sum_{i=1}^{n} e_i \otimes \Psi_i(b)$$

where $\Psi_1,\dots,\Psi_n$ are unital *-homorphisms $B \to D$. The universal property of $(B^{\star n},\iota_1,\dots,\iota_n)$ is precisely that for any collection $\Psi_1,\dots,\Psi_n$ of *-homomorphisms $B \to D$, there exists a unique $\Lambda : B^{\star n} \to D$ such that $\Lambda \circ \iota_i = \Psi_i$ for each $i \in \{1,\dots,n\}$. But that also fulfills our requirement for the universal property of the quantum family of all maps $\mathcal{QS}(C) \to \mathcal{QS}(B)$. Hence we get the desired result. □

Looking at the result, intuitively, in order to get to our case of $C(\mathbb{N} \cup \{\boldsymbol{\infty}\})$, we shall let $n \to \infty$ and probably we shall consider $B^{\star\infty} := \cup_{n\in\mathbb{N}} B^{\star n}$. Note that we have canonical embedding $B^{\star n} \hookrightarrow B^{\star n+1}$ for each $n \in \mathbb{N}$. Recall that from Chapter 1 (Inductive limit), we see that $B^{\star\infty}$ has a pre C*-algebra structure. We wish to stick to our convention of the universality of quantum space and quantum family of maps. So we should consider the norm closure of $B^{\star\infty}$ which is a C*-algebra. We shall denote this C*-algebra as $\tilde{B}$. Following the notation in Theorem 4.7, we let $\iota_n : B \to \tilde{B}$ be inclusion maps and "wish" to have a *-homomorphism $\iota_{\boldsymbol{\infty}} : B \to \tilde{B}$ such that

$$\lim_{n\to\infty} \iota_n(b) = \iota_{\boldsymbol{\infty}}(b) \qquad \text{for every } b \in B.$$

Note that we are talking about norm convergence in $\tilde{B}$. Then $\tilde{B}$ and $\Phi \in \mathrm{Mor}\,(B, C(\mathbb{N} \cup \{\boldsymbol{\infty}\}) \otimes \tilde{B})$, where Φ is defined to be

$$\Phi(b) = \sum_{n \in \mathbb{N}} \delta_n \otimes \iota_n(b) + \delta_\infty \otimes \iota_\infty(b) \qquad \text{for every } b \in B$$

with δ_i's and δ_∞ being the evaluation maps at i and ∞ respectively for every $i \in \mathbb{N}$, would satisfy the universal property of quantum space and quantum family of maps from $\mathfrak{QS}(C(\mathbb{N} \cup \{\infty\}))$ to $\mathfrak{QS}(B)$ respectively. However, unless $b \in \mathbb{C}$, the sequence $\{\iota_i(b)\}_{i \in \mathbb{N}}$ does not converge in norm and $\iota_\infty(b)$ is not well-defined in $\tilde{B}$, because free products of algebras tends to be pathological due to the fact that it is being "freely" constructed. Hence such an approach fails to work for our consideration of the case $C(\mathbb{N} \cup \{\infty\})$.

Then we wish to see what other topology that we can use to get our job done and the next immediate one is the strong operator topology. But we cannot use strong operator topology unless we are talking about bounded linear operators on some Hilbert space $\mathscr{H}$. Since every C*-algebra can be concretely represented as a C*-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, from now on all C*-algebras are understood to be a C*-subalgebra of some underlying $\mathscr{B}(\mathscr{H})$ with an unambiguous strong closure. So by abuse of notation, we can talk about strong operator topology on any C*-algebra D discussed below. We remark here that for any compact space X and C*-algebra D, $C(X) \otimes D \cong C(X, D)$.

Now for any C*-subalgebra $D \subseteq \mathscr{B}(\mathscr{H})$ for a Hilbert space $\mathscr{H}$, let us consider the *-algebra

$$C(\mathbb{N} \cup \{\infty\}) \tilde{\otimes} D := \{f : \mathbb{N} \to D \text{ bounded} \mid \text{s-}\lim_{n \to \infty} f(n) =: f(\infty) \text{ exists in } \overline{D}^{\text{strong}} \subseteq \mathscr{B}(\mathscr{H})\}$$

where $\text{s-}\lim_{n \to \infty} f(n) =: f(\infty)$ denote the strong operator limit (unique if exists) of the sequence $\{f(n)\}_{n \in \mathbb{N}}$ and $\overline{D}^{\text{strong}}$ is the strong closure of D.

In general, for any h in some Hilbert space $\mathscr{H}$, if $\{T_n\}_{n \in \mathbb{N}} \subset \mathscr{B}(\mathscr{H})$ converges strongly to $T \in \mathscr{B}(\mathscr{H})$, we have

$$\|T(h)\| = \lim_{n \to \infty} \|T_n(h)\| \leq \liminf_{n \to \infty} \|T_n\| \|h\|$$

for every $h \in \mathscr{H}$. So we have $\|T\| \leq \liminf_{n\to\infty} \|T_n\| \leq \sup_{n\in\mathbb{N}} \|T_n\|$. Moreover, since we know that norm convergence implies strong convergence, we have that for any C*-algebra D:

$$C(\mathbb{N} \cup \{\boldsymbol{\infty}\}) \otimes D \subseteq C(\mathbb{N} \cup \{\boldsymbol{\infty}\}) \tilde{\otimes} D \subseteq C_b(\mathbb{N}) \otimes D.$$

Note that since norm operator topology is stronger than the strong operator topology, $C(\mathbb{N} \cup \{\boldsymbol{\infty}\}) \tilde{\otimes} D$ is a C*-subalgebra of $C_b(\mathbb{N}) \otimes D$ equipped with a sup norm. Hence for C*-algebras B and D, any $\Psi \in \mathrm{Mor}(B, C(\mathbb{N} \cup \{\boldsymbol{\infty}\}) \tilde{\otimes} D)$ can be expressed as:

$$\Psi(b) = \sum_{n\in\mathbb{N}} \delta_n \otimes \Psi_n(b) + \delta_{\boldsymbol{\infty}} \otimes \Psi_{\text{s-}\boldsymbol{\infty}}(b)$$

where $\{\delta_n\}_{n\in\mathbb{N}}$ and $\delta_{\boldsymbol{\infty}}$ are evaluation maps at the point $n \in \mathbb{N}$ and $\boldsymbol{\infty}$ respectively and $\Psi_n : B \to D$ is representation of B on Hilbert space $\mathscr{H}_\Psi$ for each $n \in \mathbb{N}$ converges strongly to $\Psi_{\text{s-}\boldsymbol{\infty}}(b) \in \overline{D}^{\text{strong}}$.

Since we will investigate our case by considering strong operator topology, we shall digress a bit here to see that the direct sum of bounded linear operators preserves strong convergence.

Proposition 4.8. Suppose we have sequences $\{T_{i,n}\}_{n\in\mathbb{N}} \subseteq \mathscr{B}(\mathscr{H}_i)$ converging strongly to $T_i \in \mathscr{B}(\mathscr{H}_i)$ respectively for $i \in I$. Furthermore, we assume that $\{T_{i,n}\}_{n\in\mathbb{N}}$ are uniformly bounded over all $i \in I$ and $n \in \mathbb{N}$. Then $\left\{\bigoplus_{i\in I} T_{i,n}\right\}_{n\in\mathbb{N}}$ converges strongly to $\bigoplus_{i\in I} T_i$.

Proof. Note that $\bigoplus_{i\in I} T_{i,n}$ acts on $\bigoplus_{i\in I} H_i$ for each $n \in \mathbb{N}$ and in order to prove that $\left\{\bigoplus_{i\in I} T_{i,n}\right\}_{n\in\mathbb{N}}$ converges strongly to $\bigoplus_{i\in I} T_i$, we need to show that as $n \to \infty$,

$$\left\| \bigoplus_{i\in I} T_{i,n}(h) - \bigoplus_{i\in I} T_i(h) \right\| \to 0 \qquad \text{for every } h \in \bigoplus_{i\in I} \mathscr{H}_i.$$

Also, we know that the families $\{T_{i,n}\}_{n\in\mathbb{N}}$ are uniformly bounded over all $i \in I$ and $n \in \mathbb{N}$. Hence there is a $M > 0$ such that

$$\|T_{i,n}\|, \|T_i\| \leq M \qquad \text{for all } i \in I \text{ and } n \in \mathbb{N}.$$

So we have

$$\left\| \bigoplus_{i\in I} T_{i,n}(h) - \bigoplus_{i\in I} T_i(h) \right\|^2 = \sum_{i\in I} \|T_{i,n}(h_i) - T_i(h_i)\|^2$$

$$= \sum_{j\in\mathbb{N}} \left\|(T_{i_j,n} - T_{i_j})(h_{i_j})\right\|^2 \qquad \text{where } i_j\text{'s are as in Remark 1.28}$$

For $\varepsilon > 0$, since $h \in \bigoplus_{i\in I} \mathscr{H}_i$, there exists an $N \in \mathbb{N}$ such that

$$\sum_{j>N} \|h_{i_j}\|^2 < \frac{\varepsilon}{8M^2}$$

Also, since $\{T_{i,n}\}_{n\in\mathbb{N}}$ converges strongly to T_i for every $i \in I$, we have

$$\sum_{j\leq N} \|(T_{i_j,n} - T_{i_j})(h_{i_j})\|^2 < \frac{\varepsilon}{2}.$$

for n sufficiently large. So,

$$\begin{aligned}
\left\| \bigoplus_{i\in I} T_{i,n}(h) - \bigoplus_{i\in I} T_i(h) \right\|^2 &= \sum_{j\in\mathbb{N}} \left\|(T_{i_j,n} - T_{i_j})(h_{i_j})\right\|^2 \\
&\leq \sum_{j\leq N} \|(T_{i_j,n} - T_{i_j})(h_{i_j})\|^2 + \sum_{j>N} \left(\|T_{i_j,n}\| + \|T_{i_j}\|\right)^2 \|h_{i_j}\|^2 \\
&< \frac{\varepsilon}{2} + 4M^2 \sum_{j>N} \|h_{i_j}\|^2 \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence we see that strong convergence is preserved under direct sum of uniformly bounded linear operators. □

We would also like to remark here that norm operator convergence is not preserved under direct sum.

Remark 4.9. Suppose that we have a sequence $\{S_j\}_{j\in\mathbb{N}} \subset \mathscr{B}(\mathscr{H})$ converging in norm to $S \in$

$\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, i.e., we have

$$\|S_j - S\| \longrightarrow 0 \qquad \text{as } j \to \infty$$

We shall also assume that $S_1 \neq S$. Now let us define sequences of bounded linear operators on Hilbert space $\mathscr{H}$ in the following way.

For each i, define the sequence $\{T_{i,n}\}_{n\in\mathbb{N}} \subseteq \mathscr{B}(\mathscr{H})$:

$$T_{i,n} := \begin{cases} S_{n-i} & \text{if } n > i \\ S_1 & \text{if } n \leq i \end{cases}$$

So we see that when

$$\begin{array}{llllllll} i = 1, & \text{we have} & T_{1,1} = S_1, & T_{1,2} = S_1, & T_{1,3} = S_2, & T_{1,4} = S_3, & T_{1,5} = S_4 & \cdots \\ i = 2, & \text{we have} & T_{2,1} = S_1, & T_{2,2} = S_1, & T_{2,3} = S_1, & T_{2,4} = S_2, & T_{2,5} = S_3 & \cdots \\ i = 3, & \text{we have} & T_{3,1} = S_1, & T_{3,2} = S_1, & T_{3,3} = S_1, & T_{3,4} = S_1, & T_{3,5} = S_2 & \cdots \end{array}$$

$$\vdots$$

Note that by our construction, for every $i \in \mathbb{N}$, $\|T_{i,n} - S\| \longrightarrow 0$ when $n \to \infty$. So for fixed N,

$$\begin{aligned} \left\| \bigoplus_{i\in\mathbb{N}} T_{i,N} - \bigoplus_{i\in\mathbb{N}} S \right\| &= \left\| \bigoplus_{i\in\mathbb{N}} (T_{i,N} - S) \right\| \\ &= \sup_i \|T_{i,N} - S\| \\ &\geq \|S_1 - S\| \end{aligned}$$

Hence we see that $\{\bigoplus_{i\in\mathbb{N}} T_{i,n}\}_{n\in\mathbb{N}}$ does not converge in norm to $\bigoplus_{i\in\mathbb{N}} S$ even though $T_{i,n} \to S$ in norm as $n \to \infty$ for every $i \in \mathbb{N}$.

Let us go back to the result from Theorem 4.7 again. The theorem says that for any $N \in \mathbb{N}$, we have that for any C*-algebra D and any $\Psi \in \mathrm{Mor}(B, C^N \otimes D)$, we have $\mathcal{QS}(B^{\star N})$ and Φ acting as the quantum space and quantum family of all maps $\mathcal{QS}(C^N) \to \mathcal{QS}(B)$ respectively. That is, the following diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad\Phi\quad} & C^N \otimes B^{\star N} \\ \Big\| & & \Big\downarrow \mathrm{id}_{C^N} \otimes \Lambda_N \\ B & \xrightarrow{\quad\Psi\quad} & C^N \otimes D \end{array}$$

commutes for a unique $\Lambda_N : B^{\star N} \to D$. Recall that $\Psi : B \to C^N \otimes D$ is of the form

$$\Psi(b) = \sum_{i=1}^{N} e_i \otimes \Psi_i(b)$$

where $\Psi_i : B \to D$ is a representation of B (since we regard D as a C*-subalgebra of $\mathscr{B}(\mathscr{H}_\Psi)$) on some Hilbert space $\mathscr{H}_\Psi$ whilst $\Phi : B \to C^N \otimes B^{\star N}$ is precisely

$$\Phi(b) = \sum_{i=1}^{N} e_i \otimes \iota_i(b)$$

where $\iota_i : B \hookrightarrow B^{\star N}$ is the i^{th} natural inclusion for each $i = 1, \ldots N$.

Note that

$$\Psi_i(b) = \Lambda_N(\iota_i(b)) \qquad \text{for every } i = 1, \ldots N.$$

Moreover, the above equation is truely independent of $N \geq i$, i.e.,

$$\Psi_i(b) = \Lambda_M(\iota_i(b))$$

is still true for $M \geq N$.

Now let us look at our case of $C(\mathbb{N} \cup \{\infty\})$. For any C*-algebra D and $\Psi \in$Mor($B, C(\mathbb{N} \cup \{\infty\}) \tilde{\otimes} D$), we have that Ψ can be expressed as

$$\Psi(b) = \sum_{n \in \mathbb{N}} \delta_n \otimes \Psi_n(b) + \delta_\infty \otimes \Psi_{\text{s-}\infty}(b)$$

where $\{\delta_n\}_{n\in\mathbb{N}}$ and δ_∞ are evaluation maps at the point $n \in \mathbb{N}$ and ∞ respectively and $\Psi_n : B \to D$ is representation of B on Hilbert space $\mathscr{H}_\Psi$ for each $n \in \mathbb{N}$ with $\{\Psi_n(b)\}_{n\in\mathbb{N}}$ converging strongly to $\Psi_{\text{s-}\infty}(b) \in \overline{D}^{\text{strong}}$ for all $b \in B$. The previous argument told us that for any $N \in \mathbb{N}$, we have a unique $\Lambda_N : B^{\star N} \subseteq B^{\star\infty} \to D$ such that

$$\Psi_i(b) = \Lambda_N(\iota_i(b))$$

and Λ_N's are compatible with the embeddings $B^{\star N} \hookrightarrow B^{\star N+1}$. So we have a unique well-defined $\Lambda : B^{\star\infty} = \cup_{n\in\mathbb{N}} B^{\star n} \to D$ such that

$$\Psi_i(b) = \Lambda(\iota_i(b)) \qquad \text{for every } i \in \mathbb{N}$$

and $\Lambda|_{B^{\star N}} = \Lambda_N$ for every $N \in \mathbb{N}$. Since $\Psi_i(b) = \Lambda(\iota_i(b))$ for every $i \in \mathbb{N}$ and $\{\Psi_i(b)\}_{i\in\mathbb{N}}$ converges strongly to $\Psi_{\text{s-}\infty}(b)$, hence we have $\{\Lambda(\iota_i(b))\}_{i\in\mathbb{N}}$ converging strongly to $\Psi_{s-\infty}(b) \in \overline{D}^{\text{strong}}$. Note that Λ is in a unique one-to-one correspondence with $\Psi \in \text{Mor}(B, C(\mathbb{N}\cup\{\infty\})\tilde{\otimes}D)$.

We shall now construct the "C*-algebra" corresponding to the quantum space of all maps from $\mathcal{QS}(C(\mathbb{N}\cup\{\infty\}))$ to $\mathcal{QS}(B)$. Note that we remarked earlier that norm topology is too strong a condition for our consideration of the case $C(\mathbb{N}\cup\{\infty\})$ and so we shift our attention towards using strong operator topology by viewing C*-algebra D as a C*-subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ so that every C*-algebra has an underlying Hilbert space. So we are not really following the definition of quantum space of all maps in Chapter 2. Hence putting the double quotation marks on the term "C*-algebra" simply reminds that our construction is different from the one in Chapter 2.

With $B^{\star\infty} = \cup_{n\in\mathbb{N}} B^{\star n}$ and $\iota_i : B \hookrightarrow B^{\star\infty}$ the natural inclusion maps for each $i \in \mathbb{N}$, we consider a cyclic representation $\rho : B^{\star\infty} \to \mathscr{B}(\mathscr{H}_\rho)$ such that

$$\text{s-}\lim_{i\to\infty} \rho(\iota_i(b)) \text{ exists in } \mathscr{B}(\mathscr{H}_\rho) \text{ for every } b \in B. \qquad (❦)$$

Define C*-algebra A as the norm closure of $B^{\star\infty}$ under the direct sum of all cyclic representations ρ of $B^{\star\infty}$ satisfying (❦), i.e.,

$$A := \overline{\left(\bigoplus_{\substack{\rho \text{ cyclic,}\\ \text{satisfying (❦)}}} \rho\right)(B^{\star\infty})} \subseteq \mathscr{B}\left(\bigoplus_{\substack{\rho \text{ cyclic,}\\ \text{satisfying (❦)}}} \mathscr{H}_\rho\right)$$

We wish to claim that $\mathcal{QS}(A)$ and $\Phi \in \mathrm{Mor}(B, C(\mathbb{N}\cup\{\boldsymbol{\infty}\})\tilde{\otimes}A)$ defined by:

$$\Phi(b) := \sum_{i\in\mathbb{N}} \delta_i \otimes \bigoplus_{\substack{\rho \text{ cyclic,}\\ \text{satisfying (❦)}}} (\rho(\iota_i(b)) \quad + \quad \delta_{\boldsymbol{\infty}} \otimes \bigoplus_{\substack{\rho \text{ cyclic,}\\ \text{satisfying (❦)}}} (\rho_{\boldsymbol{\infty}}(b))$$

where δ_i's and $\delta_{\boldsymbol{\infty}}$ are evaluation maps at $i \in \mathbb{N}$ and $\boldsymbol{\infty}$ respectively and

$$\rho_{\boldsymbol{\infty}}(b) := \text{s-}\lim_{i\to\infty} \rho(\iota_i(b))$$

are, in the precise sense as described in Theorem 4.10 below, the quantum space and quantum family of all maps from $\mathcal{QS}(C(\mathbb{N}\cup\{\boldsymbol{\infty}\}))$ to $\mathcal{QS}(B)$ respectively. Now in order to prove our claim about the universality of quantum space and quantum family of maps from $\mathcal{QS}(C(\mathbb{N}\cup\{\boldsymbol{\infty}\}))$ to $\mathcal{QS}(B)$, we need that given any $\Psi \in \mathrm{Mor}(B, C(\mathbb{N}\cup\{\boldsymbol{\infty}\})\tilde{\otimes}D)$ for a C*-algebra $D \subseteq \mathscr{B}(\mathscr{H}_\Psi)$, there is a unique $\Gamma : A \to D$ which extends to a unique normal homomorphism $\overline{\Gamma} : \overline{A}^{\text{strong}} \to \overline{D}^{\text{strong}}$ such that the following diagram

$$\begin{array}{ccc}
B & \xrightarrow{\quad\Phi\quad} & C(\mathbb{N}\cup\{\boldsymbol{\infty}\})\tilde{\otimes}A \\
\Big\| & & \Big\downarrow \mathrm{id}\tilde{\otimes}\Gamma \\
B & \xrightarrow{\quad\Psi\quad} & C(\mathbb{N}\cup\{\boldsymbol{\infty}\})\tilde{\otimes}D
\end{array}$$

commutes, where $\mathrm{id}\tilde{\otimes}\Gamma$ can be formally expressed as $\sum\limits_{i\in\mathbb{N}} \delta_i \otimes \Gamma + \delta_{\boldsymbol{\infty}} \otimes \overline{\Gamma}$.

Recall that we have a unique representation $\Lambda : B^{\star\infty} \to D$ such that $\Psi_i(b) = \Lambda(\iota_i(b))$ for every $i \in \mathbb{N}$. Since every representation can be written as the direct sum of cyclic representations, we can write

$$\Lambda = \bigoplus_{\rho_\Lambda \text{ cyclic}} (m_{\rho_\Lambda})\rho_\Lambda, \qquad \text{where } m_{\rho_\Lambda} \text{ is the multiplicity of } \rho_\Lambda \text{ in } \Lambda.$$

Furthermore, since $\{\Lambda(\iota_i(b)) = \Psi_i(b)\}_{i\in\mathbb{N}}$ converges strongly to $\Psi_{\text{s-}\infty}(b)$, $\{\rho_\Lambda(\iota_i(b))\}_{i\in\mathbb{N}}$ converges strongly too in $\mathscr{B}(\mathscr{H}_{\rho_\Lambda})$ for each ρ_Λ. Hence, ρ_Λ's satisfy (❦).

Summing up all these arguments so far, we see that for every $\Psi \in \mathrm{Mor}(B, C(\mathbb{N}\cup\{\infty\})\tilde{\otimes}D)$, there is a unique representation Λ and we can write Λ as the direct sum of cyclic representations ρ_Λ with $\{\rho_\Lambda(\iota_i(b))\}_{i\in\mathbb{N}}$ converging strongly in $\mathscr{B}(\mathscr{H}_{\rho_\Lambda})$. Note that by the way we define A as

$$A = \overline{\left(\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying (❦)}}} \rho\right)(B^{\star\infty})} \subseteq \mathscr{B}\left(\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying (❦)}}} \mathscr{H}_\rho\right),$$

we are letting A to run over all the cyclic representations ρ where $\{\rho(\iota_i(b))\}_{i\in\mathbb{N}}$ converges strongly in $\mathscr{B}(\mathscr{H}_\rho)$. We actually have our candidate of Γ readily available. We shall just let Γ be the combination of canonical projection map and amplification map from A to the the C*-subalgebra generated by all $\Psi_i(B)$'s in D in term of sequences $\{\Psi_i(b) = \Lambda(\iota_i(b))\}_{i\in\mathbb{N}}$ which converge strongly to $\Psi_{\text{s-}\infty}(b)$ for every $b \in B$. Hence, technically speaking, for the unique Λ that is corresponding to Ψ, if $\Lambda = \bigoplus_{\rho_\Lambda \text{ cyclic}} (m_{\rho_\Lambda})\rho_\Lambda$, then we shall first "pick" those ρ_Λ's from all the cyclic representations of $B^{\star\infty}$ that satisfy (❦) (which are the basic ingredients on how we define A) and then amplify each ρ_Λ to its multiplicity (m_{ρ_Λ}). Hence we define $\Gamma : A \longrightarrow D$ to be the map by:

$$\Gamma:\ A = \overline{\left(\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying (❦)}}} \rho\right)(B^{\star\infty})} \longrightarrow D$$

$$\bigoplus_{\rho} a_\rho \longmapsto \bigoplus_{\rho_\Lambda} (m_{\rho_\Lambda}) a_{\rho_\Lambda}, \qquad \text{where } a_\rho \in \overline{\rho(B^{\star\infty})}.$$

By Section 2.5.1 of [15], the following extension $\overline{\Gamma} : \overline{A}^{\text{strong}} \longrightarrow \overline{D}^{\text{strong}}$ of $\Gamma : A \to D$ is a normal representation of A, we define $\overline{\Gamma} : \overline{A}^{\text{strong}} \longrightarrow \overline{D}^{\text{strong}}$ by

$$\begin{aligned} \overline{\Gamma} :\ & \overline{A}^{\text{strong}} \longrightarrow \overline{D}^{\text{strong}} \\ & \bigoplus_{\rho} a_\rho \longmapsto \bigoplus_{\rho_\Lambda} (m_{\rho_\Lambda}) a_{\rho_\Lambda}, \qquad \text{where } a_\rho \in \overline{\rho(B^{\star\infty})}^{\text{strong}}. \end{aligned}$$

In particular, we have

$$\left(\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying (❦)}}} \rho \right) (\iota_i(b)) \longmapsto \Psi_i(b) = \Lambda(\iota_i(b)) = \bigoplus_{\rho_\Lambda \text{ cyclic}} (m_{\rho_\Lambda}) \rho_\Lambda(\iota_i(b)) \qquad \text{for every } i \in \mathbb{N}$$

and

$$\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying (❦)}}} \rho_{\boldsymbol{\infty}}(b) \longmapsto \bigoplus_{\rho_\Lambda \text{ cyclic}} (m_{\rho_\Lambda}) \rho_{\Lambda \boldsymbol{\infty}}(b)$$

where

$$\rho_{\Lambda \boldsymbol{\infty}}(b) := \text{s-}\lim_{i \to \infty} \rho_\Lambda(\iota_i(b)).$$

Note that we have

$$\overline{\left(\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying (❦)}}} \rho \right) (B^{\star\infty})}^{\text{strong}} = \overline{A}^{\text{strong}}$$

since norm operator topology is stronger than strong operator topology.

Note that $\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying (❦)}}} \rho$ is a faithful representation of $B^{\star\infty}$ (it is faithful on each $B^{\star n} \subseteq B^{\star\infty}$, $n \in \mathbb{N}$), hence it identifies $B^{\star\infty}$ with a norm dense *-subalgebra of A, and hence $\tilde{B} \cong A$ (recall that $\tilde{B}$ is the norm closure of $B^{\star\infty}$).

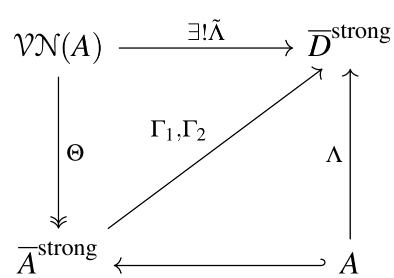
Note that $\Lambda : \cup_{n \in \mathbb{N}} B^{\star n} \to D$ is norm decreasing on each $B^{\star n}$ so it is norm decreasing on $B^{\star\infty}$.

Since $\cup_{n\in\mathbb{N}} B^{\star n}$ is dense in $\tilde{B} = A$, hence Λ can be extended uniquely to a representation, which we shall still call it as Λ, of A. So we have $\Lambda : A \to D$ which coincides with Γ by the way that we defined $\Gamma : A \to D$

To get our job done about the universality of (A, Φ), we just need to show that $\overline{\Gamma}$ is in fact the only normal extension of $\Gamma = \Lambda$ (since Λ is uniquely defined from $A = B^{\star\infty} \to D$). We will use Proposition 4.6 in proving this. Note that Γ in general does not have a strong continuous extension to $\overline{A}^{\text{strong}}$, and we will have to adapt our convention of requiring Γ to be a morphism (from Definition 2.5) into a normal representation which is a commonly used concept in the von Neumann algebra theory.

Suppose that we have two distinct normal representations $\Gamma_1, \Gamma_2 : \overline{A}^{\text{strong}} \to \overline{D}^{\text{strong}}$ that extend $\Lambda : A \to D$. By Proposition 4.6, we know that there exists a unique normal representation extension $\tilde{\Lambda}$ of Λ to the enveloping von Neumann algebra of A, which we denote as $\mathcal{VN}(A)$. That is, $\tilde{\Lambda} : \mathcal{VN}(A) \to \overline{D}^{\text{strong}}$ is the unique normal representation that extends $\Lambda : A \to D \subseteq \overline{D}^{\text{strong}}$. Note that $\mathcal{VN}(A)$ is the strong closure of $\pi_u(A)$ where π_u is the universal representation of A. Recall that the universal representation of a C*-algebra C is the direct sum $\bigoplus_{\tau \in S(C)} \pi_\tau$ of all GNS-representations associated with states of C. Every cyclic representation is a GNS-representation associated to some state. Hence the universal representation $\bigoplus_{\tau \in S(A)} \pi_\tau$ clearly contains $\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying } (❦)}} \rho$. Hence there is a canonical projection map from $\mathcal{VN}(A)$ to $\overline{A}^{\text{strong}}$ which we shall denote as $\Theta : \mathcal{VN}(A) \twoheadrightarrow \overline{A}^{\text{strong}}$.

We shall gather all the information so far into the following commutative diagram which then clearly implies that $\Gamma_1 = \Gamma_2$.



Note that $\Theta : \mathcal{VN}(A) \twoheadrightarrow \overline{A}^{\text{strong}}$ is a projection map. So it is normal. Furthermore, the composition of two normal maps is still normal. Hence if we have $\Gamma_1, \Gamma_2 : \overline{A}^{\text{strong}} \to \overline{D}^{\text{strong}}$ to be two distinct normal representations of $\overline{A}^{\text{strong}}$, then $\Gamma_1 \circ \Theta$ and $\Gamma_2 \circ \Theta$ are two distinct normal representations of $\mathcal{VN}(A)$ which extend $\Lambda : A \to D$. This is a contradiction by Proposition 4.6. Hence there will be only one normal representation extension of $\Lambda : A \to D$ to $\overline{A}^{\text{strong}}$. Therefore, $\overline{\Gamma} : \overline{A}^{\text{strong}} \to \overline{D}^{\text{strong}}$ is the unique normal representation that extends Λ.

In summary, we have the following theorem:

Theorem 4.10. *Let B be a unital finitely generated C^*-algebra, $B^{\star\infty} = \cup_{n\in\mathbb{N}} B^{\star n}$ and A be the C^*-algebra defined as the following:*

$$A := \overline{\left(\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying } (❦)}} \rho \right) (B^{\star\infty})} \subseteq \mathscr{B} \left(\bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying } (❦)}} \mathscr{H}_\rho \right)$$

where ρ's are cyclic representations of $B^{\star\infty}$ on a Hilbert space $\mathscr{H}_\rho$ such that

$$s\text{-}\lim_{i\to\infty} \rho(\iota_i(b)) \text{ exists in } \mathscr{B}(\mathscr{H}_\rho) \text{ for every } b \in B \tag{❦}$$

and

$$\iota_i : B \hookrightarrow B^{\star\infty}$$

is the natural inclusion map for each $i \in \mathbb{N}$.

Consider the C^*-algebra

$$C(\mathbb{N} \cup \{\infty\}) \tilde{\otimes} A := \{ f : \mathbb{N} \to A \text{ bounded} \mid s\text{-}\lim_{n\to\infty} f(n) =: f(\infty) \text{ exists in } \overline{A}^{strong} \}$$

and $\Phi \in Mor(B, C(\mathbb{N} \cup \{\infty\}) \tilde{\otimes} A)$ where

$$\Phi(b) := \sum_{i\in\mathbb{N}} \delta_i \otimes \bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying } (❦)}} (\rho(\iota_i(b))) \quad + \quad \delta_\infty \otimes \bigoplus_{\substack{\rho \text{ cyclic,} \\ \text{satisfying } (❦)}} \rho_\infty(b)$$

where δ_i's and δ_∞ are evaluation maps at $i \in \mathbb{N}$ and ∞ respectively and

$$\rho_\infty(b) := s\text{-}\lim_{i\to\infty} \rho(\iota_i(b))$$

Then for any C^*-algebra $D \subseteq \mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ and $\Psi \in Mor(B, C(\mathbb{N}\cup\{\infty\})\tilde{\otimes}D)$, there exists a unique representation $\Gamma : A \to D \subseteq \mathscr{B}(\mathscr{H})$ which extends uniquely to a normal representation $\overline{\Gamma} : \overline{A}^{strong} \to \overline{D}^{strong}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad\Phi\quad} & C(\mathbb{N}\cup\{\infty\})\tilde{\otimes}A \\ \Big\| & & \Big\downarrow \mathrm{id}\tilde{\otimes}\Gamma \\ B & \xrightarrow{\quad\Psi\quad} & C(\mathbb{N}\cup\{\infty\})\tilde{\otimes}D \end{array}$$

commutes in the canonical way.

Remark 4.11. The above theorem shows that (A, Φ) possesses an analog of the universal property mentioned in Definition 2.5. From Remark 4.9, since norm operator convergence is not preserved under direct sum, the above result will not work if we only consider norm closed A without including its strong closure.

4.3 Future Projects

An analog of Theorem 4.7 is shown in the above theorem. Some natural questions about extending other results of Sołtan to such an infinite dimensional case arise :

1. Suppose that we let $B = C = C(\mathbb{N}\cup\{\infty\})$ in Theorem 4.7. Will the quantum space $\mathcal{QS}(A)$ and quantum family of all maps Φ from $\mathcal{QS}(C)$ to $\mathcal{QS}(B)$ exist? Since $B = C(\mathbb{N}\cup\{\infty\})$ is not finitely generated, do similar constructions in Theorem 4.10 (which assumes B finitely generated as Sołtan does) still work for the "universality" of (A, Φ)? Can the requirement of "finitely generated" be removed or relaxed?

2. If $\mathcal{QS}(A)$ and Φ exist from Question 1, how will we define a "comultiplication" $\mathbb{\Delta}$ on A such

that $(A, \mathbb{A})$ is a "compact quantum semigroup"? Recall that in Chapter 2, when we let M be a finite dimensional C*-algebra, the quantum space $\mathcal{QS}(\mathcal{A})$ and quantum family of all maps Φ from $\mathcal{QS}(M)$ to $\mathcal{QS}(M)$ exist and there is a canonical comultiplication Δ such that $(\mathcal{A}, \Delta)$ is a compact quantum semigroup where Δ is a *-homomorphism. However, our contructions in the previous section require the representation $\overline{\Gamma} : \overline{A}^{\text{strong}} \to \overline{D}^{\text{strong}}$ to be normal. How will we adapt our constructions in the previous section to obtain a "comultiplication" that is normal?

3. How should we adapt the notion of quantum commutants (Chapter 3) in such an infinite dimensional case?

References

[1] E. Bédos, G. J. Murphy, and L. Tuset, *Co-amenability of compact quantum groups*, J. Geom. Phys. **40** (2001), no. 2, 130–153. MR1862084

[2] B.V.Rajarama Bhat, Elliot A. George, and Peter A. Fillmore, *Lectures on operator theory*, American Mathematical Society, 2000.

[3] Bruce Blackadar, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), no. 2, 249–275. MR813640

[4] ______, *Operator algebras: Theory of C^*-algebras and von neumann algebras*, Encyclopaedia of Mathematical Sciences, Springer, 2006.

[5] John B. Conway, *A course in functional analysis*, Graduate texts in mathematics, Springer, New York, 1985.

[6] J. Dixmier, *C*-algebras*, North-Holland mathematical library, North-Holland, 1982.

[7] Kenneth R. Dvidson, *C^*-algebras by examples*, American Mathematical Society, 1991.

[8] Raeburn Iain and Williams Dana P., *Morita equivalence and continuous-trace C^*-algebras*, American Mathematical Society, 1991.

[9] James R. Jackson, *Spaces of mappings on topological products with applications to homotopy theory*, Proc. Amer. Math. Soc. **3** (1952), 327–333. MR0047322

[10] Gracia-Bondia José M., Várilly Joseph C., and Figueroa Héctor, *Elements of noncommutative geometry*, Springer Science+Business media, 1990.

[11] Terry A. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, American Mathematical Society, 1997.

[12] Ann Maes and Alfons Van Daele, *Notes on compact quantum groups*, Nieuw Arch. Wisk. (4) **16** (1998), no. 1-2, 73–112. MR1645264

[13] Gerard J. Murphy, *C^*-algebras and operator theory*, 1st ed., Academic Press, 1990.

[14] M.A. Naimark, *Normed algebras*, SpringerLink : Bücher, Springer Netherlands, 2012.

[15] G.K. Pedersen, *C*-algebras and their automorphism groups*, L.M.S. monographs, Academic Press, 1979.

[16] Iain Raeburn and Allan M. Sinclair, *The C^*-algebra generated by two projections*, Math. Scand. **65** (1989), no. 2, 278–290. MR1050869

[17] Maysam Maysami Sadr, *A kind of compact quantum semigroups*, Int. J. Math. Math. Sci. (2012), Art. ID 725270, 10. MR3009563

[18] Piotr M. Sołtan, *Examples of quantum commutants*, Arab. J. Sci. Eng. Sect. C Theme Issues **33** (2008), no. 2, 447–457. MR2500052

[19] ______, *Quantum families of maps and quantum semigroups on finite quantum spaces*, J. Geom. Phys. **59** (2009), no. 3, 354–368. MR2501746

[20] Piotr M. Soltan, *On quantum maps into quantum semigroups*, Houston J. Math. **40** (2014), no. 3, 779–790. MR3275623

[21] Piotr Mikołaj Sołtan, *When is a quantum space not a group?*, Banach algebras 2009, 2010, pp. 353–364. MR2777489

[22] M. Takesaki, *Theory of Operator Algebras iii*, Encyclopaedia of Mathematical Sciences, Springer Berlin Heidelberg, 2013.

[23] A. Van Daele, *Dual pairs of Hopf $*$-algebras*, Bull. London Math. Soc. **25** (1993), no. 3, 209–230. MR1209245

[24] D.V. Voiculescu, K.J. Dykema, and A. Nica, *Free random variables*, CRM Monograph Series, American Mathematical Soc.

[25] Niels Erik Wegge-Olsen, *K-theory and C^*-algebras: A friendly approach*, Oxford University Press, 1993.

[26] S. L. Woronowicz, *Pseudospaces, pseudogroups and Pontriagin duality*, Mathematical problems in theoretical physics (Proc. Internat. Conf. Math. Phys., Lausanne, 1979), 1980, pp. 407–412. MR582650

[27] ______, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), no. 4, 613–665. MR901157

[28] ______, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), 1998, pp. 845–884. MR1616348