

Projective Normality for some families of surfaces of general type

By

Biswajit Rajaguru

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Chair: Prof. Bangere Purnaprajna

Prof. Satyagopal Mandal

Prof. Marc Greenberg,
School of Languages

Prof. Jeffrey Lang

Prof. Yunfeng Jiang

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The dissertation committee for Biswajit Rajaguru
certifies that this is the approved version of the following dissertation:

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Chair: Prof. Bangere Purnaprajna

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Abstract

In this thesis, we present the author's joint research with Lei Song, published in [28]. We show this: Suppose X is a minimal surface, which is a ramified double covering $\pi : X \rightarrow S$, of a rational surface S , with $\dim | -K_S| \geq 1$. And suppose L is a divisor on S , such that $L^2 \geq 7$ and $L \cdot C \geq 3$ for any curve C on S . Then $K_X + \pi^*L$ is base-point free and the natural map $Sym^r(H^0(K_X + \pi^*L)) \rightarrow H^0(r(K_X + \pi^*L))$, is surjective for all $r \geq 1$. In particular this implies, when S is also smooth and L is an ample line bundle on S , that $K_X + n\pi^*L$ embeds X as a projectively normal variety for all $n \geq 3$.

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Contents

1	Introduction	2
2	Preliminaries	9
2.1	Rational maps induced by the sections of a line bundle	9
2.2	The property N_0	11
2.3	The property N_p	14
2.4	Cyclic covers of surfaces	16
3	On the projective normality of double coverings over a rational surface	19
3.1	Fixed curves, adjoint divisors on anticanonical rational surfaces . . .	21
3.2	Double coverings over anticanonical rational surfaces	32
3.3	Proof of the main theorem	34

Chapter 1

Introduction

Let L be a very ample line bundle on a projective variety. Let $\phi_L : X \rightarrow \mathbb{P}^n$ be the mapping induced by the complete linear series of L . We would like to know how simple is this embedding. In particular we are interested in knowing various characteristics of the homogeneous coordinate ring of the image variety. A first question is: is the coordinate ring R of the image of X normal. For R to be normal X has to be normal to begin with, so a prerequisite is that X is normal. After ensuring this, we only have to check if $R_{(0)}$ is normal. Also in this situation the section ring of L , $R(L) = \bigoplus_{k \geq 0} H^0(L^{\otimes k})$ is the integral closure of R . Hence another way of checking normality is checking the surjectivity of the maps $R_k = \text{Sym}^k(H^0(L))|_{\phi_L(X)} \rightarrow H^0(L^{\otimes k})$ for $k \geq 1$. If it is so, the next question we are interested in knowing is: if the homogeneous ideal of the image of X is generated as simply as possible. Since by the definition of ϕ_L , $R_1 \rightarrow H^0(L)$ is bijective, the homogeneous ideal doesnot contain deg 1 forms. Hence the homogeneous ideal I , is most simply generated when it is generated by quadratic forms. Furthermore we are interested in the minimal free resolution of R and if it is as simple as possible. The property of N_p -ness is a measure of this simplicity. Let

$$0 \rightarrow F_n \xrightarrow{\psi_n} \dots \xrightarrow{\psi_3} F_2 \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 \rightarrow R \rightarrow 0$$

be the minimal free resolution of R . Then we say:

- L satisfies N_0 if R is normal.
- L satisfies N_1 if L satisfies N_0 and the homogeneous ideal I of the image of X is generated by degree 2 forms.
- L satisfies N_p if L satisfies N_{p-1} and the matrix corresponding to the map ψ_p has linear entries, but the matrix corresponding to ψ_{p+1} is either 0 or does not have linear entries.

When X is a smooth complex algebraic curve of genus g , and L is a line bundle on X , conditions on the degree of L , imply various properties of L . By the Riemann-Roch theorem if the degree of L is greater than or equal to $2g$ then L is globally generated, and if $\deg L \geq 2g + 1$, then L is very ample. Castelnuovo proved in the 19th century that if $\deg L \geq 2g + 1$, then L satisfies N_0 , and if $\deg L \geq 2g + 2$ then L satisfies N_1 . In the same century Noether and Enriques-Petri proved results describing precisely when the canonical line bundle is N_0 , and when it N_1 . Generalizing the result of Castelnuovo, Mark Green proved that if L is a line bundle on a smooth curve X and $\deg L \geq 2g + 1 + p$ then L satisfies the property N_p . This result is sharp. It is a result that if L is a line bundle of degree $3 + p$ on an elliptic curve, then L satisfies N_p but not N_{p+1} .

Let A be an ample line bundle on a curve. Then the line bundle $K_X + nA$ satisfies N_p if $n \geq 3 + p$, by Green's theorem mentioned above. This has suggested a possible extension of Green's result for line bundles of the type $K_X + nA$ to higher dimensions.

Two conjectures in this direction, which have attracted attention are:

Conjecture 1 (Fujita [8]). *Let X be a smooth variety of dimension d , and A an ample line bundle on X . If $n \geq d + 1$, then $K_X + nA$ is globally generated, and if $n \geq d + 2$, then $K_X + nA$ is very ample.*

Conjecture 2 (Mukai). *Let X be a smooth algebraic surface and A an ample line bundle on X . If $n \geq 4 + p$ then $L = K_X + nA$ satisfies N_p .*

Fujita's conjecture holds for $d = 1$ by the Riemann-Roch theorem. Reider in [29] proved the case for $d = 2$. The part referring to global generation was proved in dimension 3 by Ein and Lazarsfeld in [6], and in dimension 4 by Kawamata in [23], and it is open in dimension greater than 4. The very ample-ness part of the Fujita's Conjecture is open in dimension greater than 2.

Ein-Lazarsfeld in [5] proved that if A is a very ample line bundle on a smooth projective variety of dimension d , then the line bundle $K_X + (d + 1 + p)A + B$ satisfies N_p , where B is a nef line bundle. But it seems that the methods used to prove this result, where A is very ample do not help when A is just ample to begin with, as is the case in the statement of Mukai's conjecture. Much less is known about Mukai's conjecture. Even the simplest case $p = 0$, is not known in general. But results are known in certain classes of surfaces. For geometrically ruled surfaces Butler in [4] has proved that if $n \geq 5$, then $K_X + nA$ satisfies N_0 , and if $n \geq 4p + 4$ and $p \geq 1$, then L satisfies the property N_p . When X is geometrically ruled over an elliptic curve Homma in [20, 21], proved that if $n \geq 4$ then $K_X + nA$ satisfies N_0 . This is the lower

bound claimed by Mukai's conjecture for N_0 . She in fact characterized all line bundles on elliptic ruled surfaces which satisfy N_0 . Again for a smooth elliptic ruled surface Gallego-Purnaprajna in [9] proved that if $n \geq 2p + 3$, and $p \geq 1$, then $K_X + nA$ satisfies N_p . In particular this implies the N_1 case of Mukai's conjecture for these surfaces. In this paper Gallego-Purnaprajna in fact characterized all line bundles on an smooth elliptic ruled surface which satisfy N_1 . In [15], Gallego-Purnaprajna proved results on higher syzygies for rational surfaces X . If X is in particular anti canonical, then they characterized all linebundles L which satisfy the property N_p . As a corollary they proved higher syzygy analogues of Reider's result on base point freeness and very ampleness for these surfaces. This in particular proves Mukai's conjecture for an anti canonical rational surface X .

For surfaces X of Kodaira dimension 0, in [12] Gallego-Purnaprajna proved generalized Mukai's conjecture for $p = 0, 1$. They also showed that the line bundle $K_X + nA$ satisfies N_p for all $n \geq 2p + 2$ if A is an ample line bundle. For a minimal surface X of general type, in [12] and [13] they have proved various results about projective normality and higher syzygies for line bundles on X . In particular, they showed that if X is a minimal surface of general type, then the property N_p holds for $K + nB$, when B is a base point free and ample line bundle in a certain open set in $Pic(X)$. But the conjecture is open in it's full generality even for the so called Horikawa surfaces in particular and genus two fibrations in general. We now concentrate our attention on these surfaces.

A minimal surface of general type on which Noether's inequality is an equality $K_X^2 = 2p_g - 4$ is called a Horikawa surface. Horikawa in [22] has shown that these

surfaces are canonical double covers of surfaces of minimal degree in particular of degree $p_g - 2$ in \mathbb{P}^{p_g-1} . It is well known that the surfaces of minimal degree are rational. So double covers of rational surfaces are interesting cases in the study of Algebraic surfaces. So one can ask a question if Mukai's conjecture holds good for these surfaces. We address this question in our paper [28] where we got the following result.

Theorem 1 (R-Song [28]). *Let S be an anti canonical rational surface. $\pi : X \rightarrow S$ be a ramified double covering of S by a minimal surface. Let L be a line bundle on S such that $K_S + L$ is nef and $L.C \geq 3$ for each irrd. curve C in S . Then the line bundle $K_X + \pi^*L$ is basepoint free and the natural map*

$$S^r(H^0(K_X + \pi^*L)) \rightarrow H^0(r(K_X + \pi^*L))$$

surjects for all $r \geq 1$.

The condition $K_S + L$ is nef and $L.C \geq 3$ is equivalent to $L^2 \geq 7$ and $L.C \geq 3$. We proved this in [28]. Hence the following corollary follows.

Corollary. *Let S be an anti canonical rational surface, and let $\pi : X \rightarrow S$ be a ramified double covering of S by a minimal smooth surface X . Let L be an ample divisor on S . Then for every $r \geq 3$, $K_X + r\pi^*L$ is very ample and $|K_X + r\pi^*L|$ embeds X as a projectively normal variety.*

Remark: Please note the very ampleness and the projective normality stated in the corollary follow from the surjectivity in our theorem by the following theorem of

Mumford.

Theorem 2 (Mumford [26]). *Let L be a line bundle on a smooth algebraic surface X . If L is basepoint free and $S^r(H^0(L)) \rightarrow H^0(rL)$ is surjective for all $r \geq 0$. Then L is very ample and satisfies the property N_0 .*

To prove the surjectivity in our theorem, we push the line bundles on X down to S , where the verification of the surjectivity is reduced to proving surjectivity of two multiplication maps on S . One of these surjectivities follows directly from the theorem of Gallego-Purnaprajna proving Mukai's conjecture for anti canonical rational surfaces [15]. The other surjectivity is to show

$$H^0(K_S + B + L) \otimes H^0(K_S + L) \rightarrow H^0(2K_S + B + 2L)$$

is surjective where B is a divisor such that the branch locus of π is a member of $|2B|$. The surjectivity of this map can be reduced to the surjectivity of certain multiplication maps over 2 curves on S . One curve is a member of the linear system $|K_S + L|$, and the other is either the fixed part of $|-K_S|$ or a member of $|-K_S|$, when the fixed part of it is empty.

I also have projects exploring projective normality and higher syzygies of line bundles for surfaces of general type that are finite canonical covers of rational surfaces. An example of these are the quadruple canonical covers of surfaces of minimal degree. These covers behave generically from many perspectives and a successful handling of these cases I believe will give a clue for the more general case. The methods we have

developed so far also yield some interesting results for elliptic surfaces, which I am also exploring.

Chapter 2

Preliminaries

2.1 Rational maps induced by the sections of a line bundle

Let $D = \{(U_i, f_i)\}$ be a Cartier divisor on a complex variety X . Let $L = \mathcal{O}_X(D)$ be the associated invertible sheaf. Then a section $s \in \Gamma(U_i, L)$ of L over U_i is a rational function of the form $s = \frac{s'}{f_i}$ for some regular function $s' \in \Gamma(U_i, \mathcal{O}_X)$. If $\{s_0, \dots, s_n\}$ is a basis for $H^0(X, L)$. Then $s_j|_{U_i} = \frac{s'_j}{f_i}$ for some $s'_j \in \Gamma(U_i, \mathcal{O}_X)$. Then $(s_0, \dots, s_n) = \frac{1}{f_i}(s'_0, \dots, s'_n)$. If x is not a base-point of L we can define $[s_0(x) : \dots : s_n(x)] := [s'_0(x) : \dots : s'_n(x)]$, as a homogeneous coordinate in $\mathbb{P}(H^0(L))$. Also if $s_j = \frac{s''_j}{f_l}$ over U_l , then for $x \in U_i \cap U_l$, $s_j = \frac{s''_j}{f_l(x)} = \frac{s'_j(x)}{f_i(x)}$. So $s''_j(x) = s'_j(x)(f_l(x)/f_i(x))$. Hence $[s''_1(x) : s''_2(x) : \dots : s''_n(x)] = [s'_0(x) : s'_1(x) : \dots : s'_n(x)]$. So $[s_0(x) : s_1(x) : \dots : s_n(x)]$ is well defined for all $x \in X$, which are not base-points of L . Hence we can define a map $\phi_L^s : U \rightarrow \mathbb{P}(H^0(L))$ by setting $x \mapsto [s_0(x) : s_1(x) : \dots : s_n(x)]$, where $U = X \setminus \{\text{base-points of } L\}$. Please note that, the last argument also implies that the definition of ϕ_L^s does not depend upon the particular Cartier divisor representation of L , we choose to define ϕ_L^s . Also if we choose a different basis for $H^0(L)$, then the two basis are related as $(t_0, \dots, t_n)^t = A(s_0, \dots, s_n)^t$, for some element of $GL(H^0(L))$. Hence $\phi_L^t = \alpha_A \circ \phi_L^s$, where α_A , is the linear automorphism of $\mathbb{P}(H^0(L))$, induced by A . So the geometric properties of ϕ_L^s and ϕ_L^t are same up to an automorphism of

the image. In future we will omit the superscript s , and simply use the notation ϕ_L , when referring to the mapping induced by L .

For a section $s \in \Gamma(X, L), s \neq 0$, we define an effective divisor $div_L(s)$ by defining $div_L(s)|_{U_i} := (s')$, where s' is as it was defined earlier. $s' = s|_{U_i} f_i$, and $(s') = ((s) + D)|_{U_i}$. Hence $div_L(s) = (s) + D$. Please note $s \in \Gamma(X, L)$ if and only if $(s) + D \geq 0$. If (U_i, f_i) is a representation of L over U_i , and $s = \frac{s''}{f_i}$, then over $(U_i \cap U_l)$, $f_i = h f_l$ for some $h \in (X, \mathcal{O}_{U_i \cap U_l}^*)$, and $s = \frac{s'}{f_i} = \frac{s''}{f_l}$. Hence $s' = s'' h$ on $U_i \cap U_l$. So $(s')|_{(U_i \cap U_l)} = (s'')|_{(U_i \cap U_l)}$ since $(h) = 0$ over $U_i \cap U_l$. So $div_L(s)$ is well defined over X . Since s' is regular over U_i , for each i , $div_L(s)$ is effective. By the argument we gave to show that $div_L(s)$, is well defined over X , also shows that $div_L(s)$, is independent of the particular cartier divisor representation we choose to calculate it. So we have $div_L(s) = (s) + D$. In particular $div_L(s)$ is linearly equivalent to D . If $div_L(s_1) = div_L(s_2)$ for two distinct sections of L , then $(s_1) + D = (s_2) + D$. So $(s_1/s_2) = 0$. So $s_1/s_2 \in \Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$. Conversely if D_1 is an effective divisor linearly equivalent to D , then $D_1 = (s) + D \geq 0$. So $s \in \Gamma(X, L)$. and $D_1 = div_L(s)$. So $div_L(\cdot)$ gives a 1 – 1 correspondence

$$(H^0(X, L) \setminus \{0\})/\mathbb{C}^* = \mathbb{P}(H^0(X, L)) \rightarrow \{\text{effective divisors linearly equivalent to } D\}$$

We are just recalling here some properties of line bundles. Let us recall that the linear system corresponding to $W \subseteq H^0(L)$ denoted by $|W|$, is called globally generated or base-point free, if given any point $x \in X$, $|W|$ has a member which does not pass through the point x . Please note when ever we speak of a point in

geometrical context, we mean a geometrical point, i.e a closed point. When W is base-point free, a basis for W , induces a mapping $\phi_W : X \rightarrow \mathbb{P}(W)$, similar to ϕ_L . and the members of the linear system $|W|$ are inverse images of the members of $|\mathcal{O}_{\mathbb{P}(W)}(1)|$ i.e the hyper planes in $\mathbb{P}(W)$. W is said to separate two points $x, y \in X$, if $|W|$ contains a member which passes through x but not through y , or vice versa. W is said to separate points on X , if $|W|$ has this property with respect to any given pair of points $x, y \in X$. If $|W|$ separates points on X , then ϕ_W is injective. To see this: let x, y be a pair of points in X , let $\delta \in |W|$, separate x, y . Let H be a hyperplane in $\mathbb{P}(W)$ such that $\phi_W^{-1}(H) = \delta$, then H separates $\phi_W(x), \phi_W(y)$. W is said to separate tangent vectors at a point $x \in X$, if the image of $\{s \in W \mid \text{div}_W(s) \text{ passes through } x\}$ in the cotangent space at x (i.e $\{ds' \in \mathfrak{m}_x/\mathfrak{m}_x^2 \mid s'(x) = 0, s \in W\}$) under a local isomorphism $L|_{U_x} \rightarrow \mathcal{O}_X|_{U_x}$ is surjective. Here U_x is some neighbourhood about x , in which L has a trivialization. Hence the mapping ϕ_W is a closed immersion if and only if $|W|$ is base-point free, separates points in X , and separates tangent vectors at any given point in X . If for a line bundle L , ϕ_L induces a closed immersion, then L is said to be very ample. In this case $L = \phi_L^*(\mathcal{O}(1))$. and the linear system $|L| = \phi_L^{-1}(\{\text{the set of hyperplanes in } \mathbb{P}(H^0(L))\})$.

2.2 The property N_0

Let L be a very ample line bundle on X . Let $V = H^0(L)$, and $\phi_L : X \rightarrow \mathbb{P}(V)$ be the mapping induced by L . Let us choose $S = \mathbb{C}[V]$, to be the homogenous coordinate ring of $\mathbb{P}(V)$. By abuse of notation let us denote the image of X , in $\mathbb{P}(V)$ under ϕ_L

by X . Let I_X , be the homogeneous ideal of $\mathbb{C}[V]$ defining X as projective subvariety of $\mathbb{P}(V)$. Let $S(X) = S/I_X$, be the homogeneous coordinate ring of X . Let $\mathcal{O}_X(1) = \mathcal{O}(1)|_X$. Then by the properties of ϕ_L , $L = \mathcal{O}_X(1)$. Hence $L^{\otimes n} = \mathcal{O}(n)|_X$. Again by the properties of ϕ_L , $H^0(X, L) = H^0(\mathbb{P}(V), \mathcal{O}(1))|_X = S(X)_1$. But for $n > 1$, the inclusion $S(X)_n = H^0(\mathbb{P}(V), \mathcal{O}(n))|_X \subseteq H^0(X, L^{\otimes n})$ maynot be an equality. We define the section ring of the line bundle L , by $R(L) := \bigoplus_{k \geq 0} H^0(L^{\otimes k})$. Similarly define $R(\mathcal{O}_1) = \bigoplus_{n \geq 0} H^0(\mathcal{O}(n))$. Then $S = R(\mathcal{O}(1))$, and $S(X) = R(\mathcal{O}(1))|_X$. We know $S = \text{Sym}^\bullet(V)$. So we have a map of graded rings $\text{Sym}^\bullet(V) \xrightarrow{|_X} S(X) \hookrightarrow R(L)$. On individual components this map is $\text{Sym}^n(H^0(L)) = H^0(\mathcal{O}(n)) \xrightarrow{|_X} H^0(L^{\otimes n})$. this restriction of sections corresponds geometrically to the restriction of the linear system of degree n hypersurfaces on $\mathbb{P}(V)$ to X . i.e $(|\mathcal{O}(n)|)|_X \subseteq |L^{\otimes n}|$. Since X is a normal variety, by [19] (Ex. II.5.14) $R(L)$ is the integral closure of $S(X)$, in it's quotient field. Hence the homogeneous coordinate ring of X , $S(X)$ is normal if and only if the map $\text{Sym}^n(H^0(L)) \rightarrow H^0(L^{\otimes n})$ is surjective for all $n \geq 0$, or geometrically we have $|\mathcal{O}(n)|_X = |L^{\otimes n}|$. In words, every member of $|L^{\otimes n}|$ is the intersection of a degree n hypersurface with X .

We say a line bundle L on X , embeds X as projectively normal variety if L is very ample, and the homogeneous coordinate ring of the image of X under ϕ_L is normal. We also refer to this by saying that L has the property N_0 . So L satisfies the property N_0 , if the restriction maps $\text{Sym}^n(H^0(L)) \rightarrow H^0(L^{\otimes n})$ is surjective for all $n > 0$. One way to prove this family of maps is surjective is by induction. Please observe that we have the natural sequence of maps $H^0(L)^{\otimes n} \rightarrow \text{Sym}^n(H^0(L)) \rightarrow H^0(L^{\otimes n})$. Hence it is enough to prove $H^0(L)^{\otimes n} \rightarrow H^0(L^{\otimes n})$ is surjective. Given a $N > 0$, suppose

the maps for $n < N$ are surjective. Then for $n = N$, the above map factorizes as $H^0(L) \otimes H^0(L)^{\otimes(N-1)} \rightarrow H^0(L) \otimes H^0(L^{\otimes(N-1)}) \rightarrow H^0(L^{\otimes N})$. Hence it is enough to prove $H^0(L) \otimes H^0(L^{\otimes(N-1)}) \otimes H^0(L^{\otimes N})$ is surjective. So to use induction we have to prove that the family of maps $H^0(L) \otimes H^0(L^{\otimes n}) \otimes H^0(L^{\otimes(n+1)})$ are surjective for $n > 0$. So to test a line bundle L for N_0 property, we have to check two things:

1. L is very ample.
2. The map $Sym^n(H^0(L)) \rightarrow H^0(L^{\otimes n})$ is surjective for all $n > 0$ or the map $H^0(L) \otimes H^0(L^{\otimes n}) \rightarrow H^0(L^{\otimes(n+1)})$ is surjective for all $n > 0$.

In this context the following theorem of David Mumford is useful.

Theorem 1 (Mumford). *Let X be a smooth projective variety. If L is ample and base-point free and*

$$Sym^k(H^0(L)) \rightarrow H^0(L^k)$$

is surjective, then L is normally generated.

Hence by the above theorem we only have to check for ampleness and base-point freeness, and the surjectivities, to prove L satisfies N_0 , instead of checking for very-ampleness of L . This is useful in the following scenario: Let $\pi : X \rightarrow Y$ be a finite map of projective varieties. and L is ample and base-point free on Y , then $\pi^*(L)$ is ample and base-point free on X . But the property of very-ampleness maynot be retained. Suppose we want to verify $H^0(\pi^*L) \otimes H^0(\pi^*L^{\otimes n}) \rightarrow H^0(\pi^*L^{\otimes(n+1)})$ is surjective. Since π is finite, $H^0(\pi_*\pi^*L) \cong H^0(\pi^*L)$. So it is enough to verify $H^0(\pi_*\pi^*L) \otimes H^0(\pi_*\pi^*L^{\otimes n}) \rightarrow H^0(\pi_*\pi^*L^{\otimes(n+1)})$ is surjective. Also this map is the natural map induced by the multiplication of the associated line bundles. So testing

π^*L for N_0 on X , can be done by testing various things about $\pi_*\pi^*L$ on Y . This is useful, if we are more familiar with the geometry in Y .

The following lemma is useful in checking the surjectivities, needed for verifying N_0 , but generally it doesnot solve the problem completely. In particular we have found that for the case $n = 1$, we have to use other techniques.

Theorem 2 (Castelnuovo-Mumford Regularity). *Let F be a coherent sheaf on a complex projective variety X , and L is a base-point free line bundle on X . Then*

$$H^0(F) \otimes H^0(L) \rightarrow H^0(F \otimes L)$$

is surjective if and only if $H^i(F \otimes L^{-i}) = 0$ for all $i \geq 1$.

In the above lemma, if we set $F = L^{\otimes N}$, for $N \geq 2$, then we would have the surjectivity we need for the case $n = N$, if $H^1(L^{\otimes(N-1)}) = 0$, and $H^2(L^{\otimes(N-2)}) = 0$. since $\dim X = 2$, we donot have to worry for $i > 2$. So if L is ample and base-point free and $H^1(L^{\otimes(n+1)}) = 0$ and $H^2(L^{\otimes n}) = 0$ for all $n \geq 0$, and $H^0(L) \otimes H^0(L) \rightarrow H^0(L^{\otimes 2})$ is surjective, then L satisfies N_0 .

2.3 The property N_p

Suppose a linebundle L satisfies N_0 . So the coordinate ring R of X in $\mathbb{P}(H^0(L))$ is a normal S module. A measure of "nice"-ness of the embedding ϕ_L is, how "simple" is the minimal graded free resolution of R as a S -module. By Hilbert's Syzygy theorem

R has a minimal graded free resolution of length atmost $h^0(L)$. Let

$$\dots \xrightarrow{\phi_3} \oplus_j S(-j)^{\beta_{2,j}} \xrightarrow{\phi_2} \oplus_j S(-j)^{\beta_{1,j}} \xrightarrow{\phi_1} S \xrightarrow{\phi_0} R \rightarrow 0$$

be a minimal graded free resolution of R . Since the betti numbers $\beta_{i,j}$ of a minimal graded free resolution are an invariant of the module, we can call this resolution, the minimal graded free resolution for our purposes. $\ker \phi_0 = I_X$. Clearly $(\phi_0)_0$, the degree 0 component of ϕ_0 , is an injection. By the definition of ϕ_L , there is a 1 – 1 correspondence between the linear systems $|\mathcal{O}(1)|$ and $|L|$ given by restriction to X , and $H^0(\mathcal{O}(1)) \rightarrow H^0(L)$ is bijective. So $(\phi_0)_1$ is also injective. So I_X doesnot have any component in degree 1. A pleasant consequence of this is that X is not contained in a linear subspace of $\mathbb{P}(H^0(L))$ of lower dimension. So I_X is generated by forms of degree 2 or higher. So $\beta_{1,j} = 0$ for $j < 2$. Since the maps ϕ_i , are maps between free modules, the maps ϕ_i 's, can be represented by matrices whose entries are homogeneous elements in S , whose degree must be atleast 1, since the resolution is minimal. Hence $\beta_{2,j} = 0$ for $j < 3$, $\beta_{3,j} = 0$ for $j < 4$ and so on. So $I_X = \ker \phi_0$ is most simply generated when it is generated by degree 2 forms. In this case $\beta_{1,j} = 0$ for $j \neq 2$. If this is the case we say that L satisfies N_1 . If L satisfies N_1 and if each entry of ϕ_2 is linear(or 0), then $\beta_{2,j} = 0$ if $j \neq 3$, in this case we say L satisfies N_2 . We define the property N_p for $p > 0$, inductively as follows: L satisfies N_p , if L satisfies N_{p-1} , and the matrix of ϕ_p has linear entries. In this case $\beta_{p,j} = 0$ if $j \neq p + 1$. To sum up L satisfies N_p for $p > 1$, when L is very ample, the homogeneous coordinate ring of $R = \phi_L(X)$, is normal, the homogeneous ideal I_X defining $\phi_L(X)$ is generated by quadratic elements of S , and the minimal graded free resolution of R is linear from

the second to the p^{th} stage. If L satisfies N_p , then the minimal graded free resolution of R upto the p^{th} stage will have the shape

$$\dots \xrightarrow{\phi_{p+1}} S(-p-1)^{\beta_{p,p+1}} \xrightarrow{\phi_p} \dots \xrightarrow{\phi_3} S(-3)^{\beta_{2,3}} \xrightarrow{\phi_2} S(-2)^{\beta_{1,2}} \xrightarrow{\phi_1} S \xrightarrow{\phi_0} R \rightarrow 0$$

2.4 Cyclic covers of surfaces

Lazarsfeld's Positivity in Algebraic Geometry I[24] section 4.1.B Pages 242-244, has a very nice section describing the construction of cyclic covers. In this section we present this construction and a few other facts we need for the computations in the next chapter.

Let X be an affine variety, and $s \in \mathbf{C}[X]$, is a regular function on X . We want to construct a finite cover $\pi : Y \rightarrow X$, such that, there is a regular function $s' \in \mathbf{C}[Y]$ on Y , which satisfies the equation $s'^n = \pi^*s$. Let $p_1 : X \times \mathbb{A}^1 \rightarrow X$, be the projection to the first coordinate. Then $p_1^\# : \mathbf{C}[X] \rightarrow \mathbf{C}[X][t]$, is the corresponding map of the coordinate rings. Let $Y \subset X \times \mathbb{A}^1$, be the closed subset defined by the equation $t^n - s = 0$. Let $s' := t|_Y$, and let $\pi := p_1|_Y : Y \rightarrow X$. Then $\pi^\# : \mathbf{C}[X] \rightarrow \mathbf{C}[X][t]/(t^n - s)$, is the corresponding map of the coordinate rings. $\mathbf{C}[X][t]/(t^n - s) \cong \mathbf{C}[X] \oplus \mathbf{C}[X]t \oplus \mathbf{C}[X]t^2 \oplus \dots \oplus \mathbf{C}[X]t^{(n-1)}$. So $\mathbf{C}[Y]$ is a finite $\mathbf{C}[X]$ module of rank n , so π is a finite map of degree n . Also $s' \in \mathbf{C}[Y]$ is a regular function on Y , and $s'^n = \pi^*s$. This gives a construction of a local cyclic cover.

We now construct a global cyclic cover, when X is not necessarily affine. Let L

be a line bundle on X . Let s be a section of $L^{\otimes n}$, defining a divisor $D := \text{div}_{L^{\otimes n}}(s)$ on X . We want to construct a finite covering $\pi : Y \rightarrow X$ such that π^*L , has a global section s' on Y , which satisfies the equation $s'^n = \pi^*s$, in the section ring of π^*L , that is in $R(\pi^*L) := \bigoplus_{m \geq 0} H^0(\pi^*L^{\otimes m})$. Let $p : \mathbf{L} \rightarrow X$, be the total space of the line bundle L . Then p^*L has the tautological section" $t \in H^0(\mathbf{L}, p^*L)$, which is defined by $t(l) = l$, where $l \in \mathbf{L}$. t serves as the global fibre coordinate, similar to what t was for $X \times \mathbf{A}^1$, in our previous discussion. We have $\text{div}_{p^*L}(t)$ is the zero section of $p : \mathbf{L} \rightarrow X$. Let

$$Y := \text{div}_{p^*L^{\otimes n}}(t^n - p^*s) \subset \mathbf{L}$$

Let $s' = t|_Y$, and $\pi = p|_Y : Y \rightarrow X$. Then $s' \in H^0(Y, \pi^*L)$ and $s'^n - \pi^*s = 0$ in $H^0(Y, \pi^*L^{\otimes n})$. The following lemma collects various properties of cyclic covers, we would need in the next chapter.

Lemma 2.1 ([24] Sec 4.1.B, Proposition 4.1.6, Remark 4.1.7. and [3] Sec I.17, Lemma 17.1 and 17.2). *Let $\pi : Y \rightarrow X$ be as above. Then Y is a n -cyclic covering of X , branched over D , determined by L .*

1. *Let D' be the reduced divisor $\pi^{-1}(D)$ on Y . Then $\mathcal{O}_Y(D') = \pi^*L$.*
2. *$\pi^*D = nD'$. In particular n is the branching order along D' .*
3. *$K_Y = \pi^*(K_X \otimes L^{n-1})$.*

4. $\pi_* \mathcal{O}_Y = \mathcal{O}_X \oplus L^{-1} \oplus \dots \oplus L^{-(n-1)}$.

5. *If X and D are nonsingular, then Y and D' are nonsingular.*

Proof. Please refer to [24] Sec 4.1.B, Proposition 4.1.6, Remark 4.1.7. and [3] Sec I.17, Lemma 17.1 and 17.2 .

Chapter 3

On the projective normality of double coverings over a rational surface

In this chapter we study the projective normality of a minimal surface X which is a ramified double covering over a rational surface S with $\dim | -K_S| \geq 1$. This chapter is based on [28]. We show that divisors of the form $K_X + r\pi^*A$ are normally generated, when the integer $r \geq 3$. Here A is an ample divisor on S , and π is the covering map.

In [28] our main result is,

Theorem 3.1. *Let S be a rational surface with $\dim | -K_S| \geq 1$. Let $\pi : X \rightarrow S$ be a ramified double covering of S by a minimal surface X (possibly singular). Let L be a divisor on S with the property that $K_S + L$ is nef and $L.C \geq 3$ for any curve C . Then $K_X + \pi^*L$ is base point free and the natural map*

$$S^r H^0(K_X + \pi^*L) \rightarrow H^0(r(K_X + \pi^*L)) \quad (3.1)$$

surjects for every $r \geq 1$.

Remark. The condition that $K_S + L$ is nef and $L.C \geq 3$ for any curve C on S is

equivalent to $L^2 \geq 7$ and $L.C \geq 3$ see Proposition 3.10. Hence we have the following corollary.

Corollary 3.1.1. *Let S be rational surface with $\dim | -K_S| \geq 1$. Let $\pi : X \rightarrow S$ be a ramified double covering of S by a minimal smooth surface X . Then for every $r \geq 3$, and an ample divisor L , $K_X + r\pi^*L$ is very ample and $|K_X + r\pi^*L|$ embeds X as a projectively normal variety.*

The idea of the proof of theorem 3.1 is as follows. By the projection formula, the surjectivity in the theorem can be reduced to the surjectivity of two multiplication maps, of which the difficult one is to show

$$H^0(K_S + B + L) \otimes H^0(K_S + L) \rightarrow H^0(2K_S + B + 2L)$$

surjects, where B is the divisor class such that the branch locus of π is a member of $|2B|$. Via an appropriate commutative diagram, the surjectivity can be reduced to surjectivity of multiplication maps over two curves in S . One curve is a member of the linear system $|K_S + L|$, and the other is either the fixed part of $| -K_S|$ (in case the fixed part is not empty), or a member of $| -K_S|$. The fixed part of $| -K_S|$ is in general nonreduced; however its special structure enables us to proceed by induction on the summation of coefficients of its components.

3.1 Fixed curves, adjoint divisors on anticanonical rational surfaces

The next two lemmas help to reduce the verification of projective normality of line bundles on surfaces to the verification of their restriction to certain curves on surfaces.

Lemma 3.2. *(Gallego and Purnaprajna [12],p.154) Let X a smooth variety, with $H^1(\mathcal{O}_X) = 0$. Let E be a vector bundle, and $L = \mathcal{O}_X(C)$ be a base point free line bundle with the property that $H^1(E \otimes L^{-1}) = 0$. if the natural map $H^0(E|_C) \otimes H^0(L|_C) \rightarrow H^0(E \otimes L|_C)$ is surjective then so is the natural map $H^0(E) \otimes H^0(L) \rightarrow H^0(E \otimes L)$.*

Proof:

Consider the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(\mathcal{O}_X) \otimes H^0(E) & \longrightarrow & H^0(L) \otimes H^0(E) & \longrightarrow & H^0(L|_C) \otimes H^0(E) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H^0(E) & \longrightarrow & H^0(E \otimes L) & \longrightarrow & H^0((E \otimes L)|_C) \longrightarrow \dots
 \end{array}$$

got from the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow L \rightarrow L|_C \rightarrow 0$$

by tensoring with $H^0(E)$ for the top row, and with E for the bottom row. the left hand column map is commutative, because $H^0(\mathcal{O}_X)$ contains 1. the right hand column map can be split, as follows

$$H^0(E) \otimes H^0(L|_C) \rightarrow H^0(E|_C) \otimes H^0(L|_C) \rightarrow H^0((E \otimes L)|_C)$$

The first map is surjective, when we consider the short exact sequence $0 \rightarrow E \otimes L^{-1} \rightarrow E \rightarrow (E|_C) \rightarrow 0$, and the portion of the consequent long exact sequence

$$H^0(E) \rightarrow H^0(E|_C) \rightarrow H^1(E \otimes L^{-1}) \rightarrow \dots$$

and noting $H^1(E \otimes L^{-1}) = 0$, while the second map is the condition in the lemma. Hence under the conditions of the lemma, the right hand map is surjective. Now by snake lemma, the middle map will be surjective.

The second is the so called Green's H^0 -lemma.

Lemma 3.3. *(Green [16]) Let C be a smooth projective curve. Let L, M be line bundles on C , and $W \subseteq H^0(L)$ be a basepoint free subspace, such that $h^1(M \otimes L^{-1}) \leq \dim W - 2$, then the natural map $W \otimes H^0(M) \rightarrow H^0(L \otimes M)$ is surjective.*

Proof: cf [16].

Let S be a surface, and K_S be it's canonical divisor. S is a rational surface if each plurigenus of S is 0, that is $P_i = 0$ for all $i > 0$, and $h^1(S) = 0$. This implies $h^2(S) = 0$. Hence on a rational surface the Riemann-Roch theorem has the form $\chi(D) = 1 + \frac{D \cdot (D - K_S)}{2}$. A rational surface is anticanonical if $h^0(-K_S) \geq 1$. In this paper we will be concerned mostly with anticanonical surfaces with $h^0(-K_S) \geq 2$. The rational ruled surfaces \mathbb{F}_e for $e > 0$, and their blowups at less than 8 points are of this type. In the beautiful papers [17] and [18], Brian Harbourne has given many conditions relating the intersection numbers of line bundles with $-K_S$, when S is an

anticanonical rational surface, to their geometrical properties like nefness and base-point freeness. In those papers a principal idea, is that *the geometrical properties of a nef line bundle F on an anticanonical rational surface depend upon the intersection number, $-K_X \cdot F$.* In the following we have collected a few of his results which we have used, in our study. So the results are not ours, but I have tried to present them in our context.

Lemma 3.4. ([18], Lemma II.2, Corollary II.3, Lemma II.5) *Let X be a smooth projective rational surface. and L be a divisor on X .*

1. $h^0(L) - h^1(L) + h^2(L) = 1 + (L^2 - K_X \cdot L)/2$
2. *If L is effective then $h^2(L) = 0$.*
3. *If L is nef, then $h^2(L) = 0$ and $L^2 \geq 0$.*
4. *If L is nef, and $L \cdot -K_X \geq 0$, then L is effective. In particular if $-K_X$ is effective, then every nef divisor on X is effective.*
5. *If X is an anticanonical surface, and C be an effective divisor on X and let L be any divisor on X . Then $h^0(\mathcal{O}_C) = h^1(\mathcal{O}_C) = 1$, and $h^0(C, L) - h^1(C, L) = -K_X \cdot L$.*

Proof. See [18], Lemma II.2, Corollary II.3, Lemma II.5 .

Lemma 3.5. ([18], Theorem III.1) *Let X be a smooth projective rational anticanonical surface with a numerically effective class L and let C be a nonzero section of $-K_X$. Let $|L| = F + |M|$*

1. If $-K_X \cdot L \geq 2$, then $h^1(L) = 0$ and L is base-point and hence fixed component free.
2. If $-K_X \cdot L = 1$, then $h^1(L) = 0$. If L is fixed component free, then the sections of L have a unique base-point, which is on C .
3. If $-K_X \cdot L = 0$, then either $F = 0$ (in which case L is base-point free) or F is a smooth rational curve of self-intersection number -2 , or $F + K_X$ is effective.

Proof. See [18] Theorem III.1 .

The next proposition is about fixed curves on an anticanonical rational surface, and is probably well known to experts.

Proposition 3.6. *Let S be a rational surface with $\dim | -K_S| \geq 1$. Suppose that C is a curve on S with $h^0(\mathcal{O}_S(C)) = 1$. Then $h^1(C, \mathcal{O}_C) = 0$.*

Proof. Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{O}_C(C|_C) \rightarrow 0$$

Taking the long exact sequence and noting that S is a rational surface, we have the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{O}_S) \rightarrow H^0(\mathcal{O}_S(C)) \rightarrow H^0(\mathcal{O}_C(C|_C)) \rightarrow \\ \rightarrow 0 \rightarrow H^1(\mathcal{O}_S(C)) \rightarrow H^1(\mathcal{O}_C(C|_C)) \rightarrow 0 \rightarrow H^2(\mathcal{O}_S(C)) \rightarrow 0 \end{aligned}$$

Hence $h^2(\mathcal{O}_S(C)) = 0$. Using Riemann-Roch theorem for $\mathcal{O}_S(C)$, we get $h^0(C) - h^1(C) + 0 = 1 + \frac{D^2 + D \cdot (-K_S)}{2}$. Hence $-2h^1(C) = C^2 + (-K_S) \cdot C$. hence $2g_a - 2 = C^2 + (-K_S) \cdot C + 2K_S \cdot C \leq 0$ since $(-K_S \cdot C \geq 0)$ as $C \neq_{lin} -K$, as $h^0(C) = 1$, while $h^0(-K_S) \geq 2$. . Why is this strictly negative.

Remark (Artin, [1] Theorem 1.7). Any curve C on a smooth surface with $h^1(\mathcal{O}_C) = 0$ has the property that for each sub-curve $C' \subset C$ we also have $h^1(C', \mathcal{O}_{C'}) = 0$. which implies that C is a chain of \mathbb{P}^1 's, and the intersections are transversal.

Let $|-K_S| = F + |M|$, where F is the fixed part of the linear system $|-K_S|$, and M is the moving part. In this case M is nef. hence effective by 3.4.

Lemma 3.7. *Let S be an anticanonical rational surface with $\dim |-K_S| \geq 1$. Suppose that $|-K_S|$ has the fixed part F , and write $|-K_S| = F + |M|$, where $|M|$ is the moving part . Then $M.F \geq 2$.*

Proof. Since $\dim |-K_S| \geq 1$, $|M|$ is nonempty. From the exact sequence $0 \rightarrow \mathcal{O}_S(-F) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_F \rightarrow 0$, we see that $h^0(\mathcal{O}_F) = 1 + h^1(\mathcal{O}_S(-F))$. as $h^1(\mathcal{O}_S) = 0$, and $h^0(\mathcal{O}_S(-F)) = 0$ as F is effective.

$$= 1 + h^1(\mathcal{O}_S(-M)) \text{ (using Serre duality)}$$

$$= 1 - \chi(\mathcal{O}_S(-M)), \text{ as } h^2(\mathcal{O}_S(-M)) = h^0(\mathcal{O}_S(-F)) = 0, \text{ similarly } h^0(\mathcal{O}_S(-M)) = 0$$

$$\begin{aligned} &= -\frac{(-M)(-K_S - M)}{2} \\ &= \frac{M.F}{2} \geq 1 \end{aligned}$$

This implies that $M.F \geq 2$.

Corollary 3.7.1. *Let S be an anticanonical rational surface, with nontrivial fixed part of $|-K_S|$. With notations as above, Suppose B is a divisor on S such that $K_S + B$ is nef. Then $H^1(B - F) = 0$.*

Proof We have $B - F = K_S + B + M$, is nef, since M is nef. Also by Lemma 3.7, $(B - F).(-K_S) \geq M.(-K_S) \geq M^2 + M.F \geq M.F \geq 2$, since $M^2 \geq 0$. Then it follows that $H^1(B - F) = 0$ by ([18], Theorem III.1)

The following is a proposition in [15], which we use in our next proposition:

Proposition 3.8. *(Gallego, Purnaprajna [15], Proposition 1.10) Let X be an anti-canonical rational surface and let L be an ample line bundle such that $1 \leq K_X^2 \leq 7$. Then $-K_X \cdot L \geq K_X^2 + 3$ unless one of the following happens:*

1. $L = -K_X$, in which case $-K_X \cdot L = K_X^2$
2. $K_X^2 = 1$ and $L = -2K_X$, in which case $-K_X \cdot L = K_X^2 + 1$
3. $K_X^2 = 1$ and $L = -3K_X$, in which case $-K_X \cdot L = K_X^2 + 2$
4. $K_X^2 = 2$ and $L = -2K_X$, in which case $-K_X \cdot L = K_X^2 + 2$
5. $K_X + L$ is a base-point free line bundle, L is very ample and (X, L) is a conic fibration under $|K_X + L|$, in which case $-K_X \cdot L \geq K_X^2 + 2$.

Proposition 3.9. *Let S be an anti-canonical rational surface, and L a line bundle on S with the property that $L.C \geq 3$ for every curve C on the surface. Then $(K_S + L).(-K_S) \geq 3$ unless we have one of the following cases.*

1. $S = \mathbb{P}^2, L = \mathcal{O}_{\mathbb{P}^2}(3)$

2. $K_S^2 = 1, L = -3K_S$. In this case $-K_S$ is ample and has a unique base point.

Proof Under the conditions in the proposition, L is nef, hence effective by 3.4. Hence $L^2 \geq 3$. Now by Nakai-Moishezon criterion for ampleness L is ample. Since $|-K_S|$ is effective $L \cdot (-K_S) \geq 3$. If $K_S^2 \leq 0$, then we have $(K_S + L) \cdot (-K_S) \geq 3$. Also every rational surface is got by finite number of monoidal transformations of the minimal rational surfaces (\mathbb{P}^2 whose $K_{\mathbb{P}^2}^2 = 9$) or (\mathbb{F}_e for $e = 0$ or $e \geq 2$, whose $K_{\mathbb{F}_e}^2 = 8$), and each monoidal transformation decreases K^2 by 1. So for a rational surface S , $K_S^2 \leq 9$. Since we have already treated the case $K_S^2 \leq 0$, for the rest of the proof we focus on the case $0 < K_S^2 \leq 9$.

1. $K_S^2 = 9$, Then $S = \mathbb{P}^2$, and $L = \mathcal{O}_{\mathbb{P}^2}(m)$. for some $m \geq 3$. In this case $K_S = \mathcal{O}_{\mathbb{P}^2}(-3)$. Hence $(K_S + L) \cdot (-K_S) \geq 3$, unless $L = -K_S$.
2. $K_S^2 = 8$. In this case S is a rational ruled surface \mathbb{F}_e for some $e \geq 0$, and since L is an ample line bundle on a rational ruled surface, $L = aC_0 + bf$. Since $L \cdot C \geq 3 \Rightarrow -ae + b \geq 3$, and $L \cdot f \geq 3 \Rightarrow a \geq 3$. This also ensures $L \cdot C \geq 3$ for every curve C .

$$\begin{aligned}
(K_S + L) \cdot (-K_S) &= -8 + L \cdot (-K_S) \\
&= -8 + (aC_0 + bf)(2C_0 + (2 + e)f) \\
&= -8 - 2ae + 2b + a(2 + e) = -8 + 2(b - ae) + a(2 + e) \\
&\geq -8 + 2(3) + 2(3) = 4 \text{ since } b - ae \geq 3, e \geq 0, a \geq 3
\end{aligned}$$

3. $0 < K_S^2 \leq 7$. In this case by Gallego-Purnaprajna[15] Proposition 1.10, for an

ample divisor L , $(K_S + L) \cdot (-K_S) \geq 3$ unless one of the following cases occurs.

- (a) $L = -K_S$
- (b) $K_S^2 = 1$ and $L = -2K_S$
- (c) $K_S^2 = 1$ and $L = -3K_S$
- (d) $K_S^2 = 2$ and $L = -2K_S$
- (e) $K_S + L$ is base point free, L is very ample and (S, L) is a conic fibration over \mathbb{P}^1 under $|K_S + L|$.

Note (b) is impossible since $3 \leq L \cdot (-K_S) = (-2K_S) \cdot (-K_S) = 2K_S^2 = 2$. For parts (a) and (d) note that S is a del Pezzo surface of degree ≤ 7 , so there exists a (-1) rational curve $C \subset S$. For this curve, adjunction formula $2(0) - 2 = C^2 + C \cdot K_S$, give $C \cdot K_S = -1$. Hence $L \cdot C \leq (-2K_S) \cdot C = 2$, which contradicts the condition on L . For the case (e), $(K_S + L)^2 = 0$, since $|K_S + L|$ gives a conic fibration. (This is so because in this case $K_S + L$ is a linear combination of the fibers of the fibration. whose mutual and self intersections are 0). Now $(K_S + L) \cdot (-K_S) = 2$, so $(K_S + L) \cdot L = 2$. Which implies $K_S + L$ can't be effective. Also $h^i(K_S + L) = 0$, since L is ample. Hence $\chi(K_S + L) = 0$. But using the Riemann-Roch formula we get $\chi(K_S + L) = 1 + \frac{L \cdot (L - K_S)}{2} = 2$, which is a contradiction. So case (e) does not occur. So case (c) is the only exceptional case possible. In this case since L is ample, and $L = -3K_S$, $-K_S$ is also ample. The assertion about the base locus of $|-K_S|$ follows from [[18], Theorem III.1 (b)].

Proposition 3.10. *Let S be an anticanonical rational surface and L be a divisor*

on S with the property that $L.C \geq 3$, for any curve C on S . Then the following are equivalent,

- (a) $K_S + L$ is nef,
- (b) $K_S + L$ is base-point free,
- (c) $L^2 \geq 7$

Moreover if any of the equivalent conditions holds, then $K_S + L$ is ample, unless $K_S + L = 0$.

Proof (b) \implies (a): Since $K_S + L$ is base-point free, $|K_S + L| = F + M \neq \emptyset$, where F, M are the fixed component and the moving part of $K_S + L$ respectively. Since $K_S + L$ is base-point free, $F = 0$ so $(K_S + L)^2 = M^2 \geq 0$. So $(K_S + L).C \geq 0$ for every curve C on S .

(c) \implies (b) by [[29], Theorem 1].

(a) \implies (c). Since $K_S + L$ is nef, it is effective by 3.4. So $(K_S + L).L \geq 4$, since it has to be an even number by Riemann-Roch theorem. Now $(-K_S) \cdot L \geq 3$, hence $K_S \cdot L \leq -3$. So $4 \leq (K_S + L) \cdot L \leq L^2 - 3$. Finally $L^2 \geq 7$.

To complete the proof we have to show that $K_S + L$ is ample. We know $(K_S + L)^2 \geq 0$, since $(K_S + L)$ is nef. Just to recall this so because the cone of nef line bundles, in the vector space of line bundles in S , is the closure of the cone of ample line bundles. From now on we assume $K_S + L \neq 0$. We will first show that $(K_S + L)^2 > 0$. By Bertini's theorem for base point free line bundles, a general member of $K_S + L$ is

smooth, but may not be connected. Let $C = \sum C_i$, be a smooth member of $|K_S + L|$, where C_i are the connected components of C . Suppose $(K_S + L)^2 = 0$. Then $C^2 = 0$. But since $C_i \cdot C_j = 0$ as C is smooth, so $C^2 = \sum C_i^2$. But also $C_i^2 = C_i \cdot (K_S + L) \geq 0$, since $K_S + L$ is nef. So $C_i^2 = (K_S + L) \cdot C_i = 0$. Hence $K_S \cdot C_i = -L \cdot C_i \leq -3$. By adjunction $2g(C_i) - 2 = C_i^2 + K_S \cdot C_i \leq -3$. Hence $g(C_i) < 0$. This is impossible since each C_i is an smooth irreducible curve. Thus $(K_S + L)^2 > 0$, when $K_S + L \neq 0$. Next if C is an integral curve such that $(K_S + L) \cdot C = 0$. Then $K_S \cdot C = -L \cdot C \leq -3$. Since $(K_S + L)^2 > 0$, we have by the hodge index theorem $C^2(K_S + L)^2 \leq (C \cdot (K_S + L))^2 = 0$. Since C is not a multiple of $K_S + L$, as $C \cdot (K_S + L) = 0$, we must have $C^2 < 0$. It follows that $2g(C) - 2 = C^2 + C \cdot K_S \leq -4$, since this number has to be even. which implies $g(C) < 0$. This is again impossible since C is an integral curve. Hence $(K_S + L) \cdot C > 0$, for any integral curve in S . Hence $K_S + L$ must be ample by Nakai-Moishezon criterion.

Lemma 3.11. (*Gallego-Purnaprajna[15], Theorem 1.3*) *Let X be a rational surface and L be an ample line bundle on X . If L is base-point free and $-K_X \cdot L \geq 3$, then L satisfies property N_0 .*

Proof. To show L satisfies the property N_0 , we have to show that $Sym^r(H^0(L)) \rightarrow H^0(L^{\otimes r})$ is surjective for $r \geq 1$. To show this, it is enough to show that

$$H^0(rL) \otimes H^0(L) \rightarrow H^0((r+1)L) \text{ for all } r \geq 1$$

is surjective. Since L is ample and base-point free, we can choose a smooth and irreducible curve $C \in |L|$. We want to apply 3.2 with $E = rL$. We need to show $H^1(rL) = 0$ for $r \geq 0$. for $r = 0$, this holds since X is a rational surface. for $r \geq 1$ we

show it by induction. So let us assume $H^1((r-1)L) = 0$, for some r . Consider the short exact sequence $0 \rightarrow (r-1)L \rightarrow rL \rightarrow \mathcal{O}_C(rL|_C) \rightarrow 0$. then we have the long exact sequence

$$\cdots \rightarrow H^1((r-1)L) \rightarrow H^1(rL) \rightarrow H^1(rL|_C) \rightarrow \cdots$$

Now $(rL) \cdot L > L^2 + L \cdot K_X = 2g(C) - 2$, since $L \cdot -K_X > 0$. Hence $H^1(rL|_C) = 0$. Hence $H^1(rL) = 0$. hence by induction for $r > 0$. Hence we can apply lemma 3.2. To complete the proof of the forward implication, we have to show

$$H^0(rL|_C) \otimes H^0(L_C) \rightarrow H^0((r+1)L_C)$$

is surjective for all $r > 0$. Since $-K_X \cdot L \geq 3$, we have

$$\deg L_C = L^2 \geq 3 + K_X \cdot L + L^2 = 2g(C) + 1$$

So by Castelnuovo's criterion for projective normality of line bundles on curves, $L|_C$ is projectively normal. Hence

$$H^0(rL|_C) \otimes H^0(L_C) \rightarrow H^0((r+1)L_C)$$

is surjective for all $r > 0$.

3.2 Double coverings over anticanonical rational surfaces

Let $\pi : X \rightarrow S$ be a ramified cyclic double covering of S , an anticanonical rational surface by a minimal surface X , which may be singular. Let B be a divisor such that the branch locus of π , $\Gamma \in |2B|$. Let $R \subset X$ be the ramification locus of π . Then the ramification divisor of π , $\mathcal{O}_X(R) \in |(2-1)\pi^*(\mathcal{O}_S(B)) = \pi^*\mathcal{O}_S(B)|$ and the induced morphism $\pi : R \rightarrow \Gamma$ is an isomorphism, since π is a cyclic double cover. Also $\pi_*\mathcal{O}_X \cong \mathcal{O}_S \oplus \mathcal{O}_S(-B)$, cf [[24], p.243].

Proposition 3.12. *With assumptions and notations as above we have*

1. $K_X = \pi^*K_S + R \cong \pi^*(K_S + B)$
2. $(K_S + B) \cdot C \geq 0$, for any curve $C \subset S$.
3. $K_S^2 + B \cdot K_S \leq 0$; and if $K_S^2 > 0$ then $K_S^2 + B \cdot K_S < 0$
4. B is effective.

Proof. (1) is simply Hurwitz's formula applied to double covers.

$$(2) \quad 2(K_S + B) \cdot C = \pi^*(K_S + B) \cdot \pi^*C = K_X \cdot \pi^*C \geq 0, \text{ since } X \text{ is minimal.}$$

(3) The first inequality is a special case of (2) since $-K_S$ is effective. If $K_S^2 > 0$, and $(K_S + B) \cdot K_S = 0$, then by Hodge index theorem $(K_S + B)^2 \leq 0$. But $(K_S + B)^2 \neq 0$. Hence $(K_S + B)^2 < 0$. But this contradicts the nefness of $K_S + B$. Hence $(K_S + B) \cdot K_S < 0$.

(4) Since $B = (K_S + B) + (-K_S)$, and $-K_S$ is effective and $K_S + B$ is effective by 3.4, B is effective.

Remark. Let L, M be effective line bundles on a variety X . Then there exists an injection from $H^0(L) \rightarrow H^0(L + M)$. To see this consider the short exact sequence $0 \rightarrow \mathcal{O}_X(-L) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_L \rightarrow 0$, tensor this sequence with $\mathcal{O}_X(L + M)$ to get the exact sequence $0 \rightarrow \mathcal{O}_X(M) \rightarrow \mathcal{O}_X(L + M) \rightarrow \mathcal{O}_L(L + M) \rightarrow 0$. The long exact sequence for this short exact sequence starts with $0 \rightarrow H^0(M) \rightarrow H^0(L + M) \rightarrow \dots$.

In the setting of the previous lemma,

Lemma 3.13. *Let L be a divisor on S such that $L \cdot C \geq 2$ for any curve C on S .*

Then

1. $K_S + B + L$ is ample and base-point free and $H^1(r(K_S + B + L)) = 0$ for any $r \geq 1$.
2. If in addition $L \cdot (-K_S) \geq 3$, then $K_S + B + L$ is very ample and normally generated.

Proof. (1) Since $K_S + B$ is nef, and L is ample, $K_S + B + L$ is ample, since the sum of an ample line bundle and a nef line bundle is an ample line bundle. Since $(-K_S)$ is effective, we have

$$(-K_S) \cdot r(K_S + B + L) = r(-K_S) \cdot (K_S + B) + r(-K_S) \cdot (L) \geq 0 + 2$$

The assertion about the base locus and H^1 s now follow from ([18], Theorem III.1)

(2) If $L \cdot (-K_S) \geq 3$, then $(-K_S) \cdot (r(K_S + B + L)) \geq 3$. The second assertion in the theorem now follows from the criterion for N_p property on rational surfaces in ([15], Theorem 1.3)

Remark. The sum of an ample line bundle and a nef line bundle is an ample line bundle. clearly $(N + A) \cdot C > 0$, for any curve C on S . Also $N \cdot kA \geq 0$, for a k such that kA is very ample. Hence $N \cdot A \geq 0$. Then $(N + A)^2 = N^2 + A^2 + 2N \cdot A \geq A^2 > 0$. So by Nakai-Moishzon criterion $N + A$ is ample. Another way to say this is, $N + A$ lies in the interior of the nef cone, hence is ample.

3.3 Proof of the main theorem

Lemma 3.14. *Let M be an ample and base-point free divisor, on a regular anticanonical rational surface S and C a curve on S , such that $h^0(\mathcal{O}_C) = 1$, and $-K_S - C$ is effective. Then*

$$H^i(M - C) = 0 \text{ for } i > 0$$

Proof $h^2(M - C) = h^0(K_S + C - M)$. But $-(K_S + C), M$ are effective so $-(K_S + C - M)$ is effective. hence $h^0(-(-K_S - C + M)) = 0$.

The short exact sequence

$$0 \rightarrow \mathcal{O}_S(-C) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_C \rightarrow 0$$

yields the long exact sequence

$$0 \rightarrow H^0(\mathcal{O}_S(-C)) \rightarrow H^0(\mathcal{O}_S) \rightarrow H^0(\mathcal{O}_C) \rightarrow H^1(\mathcal{O}_S(-C)) \rightarrow 0 \rightarrow \dots$$

Also $h^0(\mathcal{O}_S) = h^0(\mathcal{O}_C) = 1$ and $h^0(\mathcal{O}_S(-C)) = 0$ since $\mathcal{O}_S(C)$ is effective, so $h^1(\mathcal{O}_S(-C)) = 0$.

Now take a smooth irreducible member $\Delta \in |M|$. Such a member exists by Bertini's theorem, for base-point free line bundles. Tensoring the short exact sequence

$$0 \rightarrow \mathcal{O}_S(-\Delta) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

with $\Delta - C$ yields the short exact sequence

$$0 \rightarrow \mathcal{O}_S(-C) \rightarrow \mathcal{O}_S(\Delta - C) \rightarrow \mathcal{O}_\Delta(\Delta - C) \rightarrow 0$$

Taking the long exact sequence we get

$$\dots \rightarrow H^1(\mathcal{O}_S(-C)) \rightarrow H^1(\mathcal{O}_S(M-C)) \rightarrow H^1(\mathcal{O}_\Delta(\Delta-C)) \rightarrow H^2(\mathcal{O}_S(-C)) \rightarrow H^2(\mathcal{O}_S(\Delta-C)) \rightarrow \dots$$

Also $\deg(\Delta - C)|_\Delta = (\Delta - C) \cdot \Delta = (\Delta + K_S) \cdot \Delta + (-K_S - C) \cdot \Delta > \deg K_\Delta$ if $(-K_S - C) \neq 0$, since Δ is ample. So when $-K_S - C \neq 0$, this implies $h^1(\mathcal{O}_\Delta(\Delta - C)) = 0$. Since $h^1(\mathcal{O}_S(-C)) = 0$, this will imply $h^1(\mathcal{O}_S(M - C)) = 0$, when $-K_S - C \neq 0$.

If $-K_S - C = \mathcal{O}_S$, then $M - C = M + K_S$, and $\mathcal{O}_\Delta(M + K_S) = K_\Delta$ by adjunction. hence $h^1(\mathcal{O}_\Delta(M - C)) = h^1(K_\Delta) = 1$ by Serre duality, since Δ is an irreducible curve. We have $h^2(\mathcal{O}_S(-C)) = h^2(\mathcal{O}_S(K_S)) = 1$, $h^2(\mathcal{O}_S(M - C)) = 0$, as we proved earlier, and $h^1(\mathcal{O}_S(-C)) = 0$. Hence when $K_S + C = \mathcal{O}_S$, the long exact sequence

becomes

$$0 \rightarrow H^1(\mathcal{O}_S(M - C)) \rightarrow H^1(\mathcal{O}_\Delta(\Delta - C)) \rightarrow H^2(\mathcal{O}_S(-C)) \rightarrow 0$$

now looking at the the dimensions of the last 2 terms, we can conclude that $H^1(\mathcal{O}_S(M - C)) = 0$, when $K_S + C = 0$.

Lemma 3.15. *Let S be a smooth anticanonical rational surface and C a curve with $h^1(\mathcal{O}_C) = 0$. Put $C = \sum a_i \Gamma_i$, where Γ_i are the irreducible components of C . Let L_1, L_2 be two divisors on S such that $L_1 \cdot \Gamma_i > 0$ for every i and L_2 is ample and base-point free. Suppose that $-K_S - \Gamma_i$ is effective for every i . Then the natural map*

$$H^0(L_1|_C) \otimes H^0(L_2) \rightarrow H^0((L_1 + L_2)|_C)$$

is surjective.

Proof. We proceed by induction on $\sum a_i$. the numbers a_i are positive integers. So when $\sum a_i = 1$. C is isomorphic to \mathbb{P}^1 . (why is C smooth.) Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_S(-C) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_C \rightarrow 0$$

Tensor this short exact sequence with L_2 to get the short exact sequence

$$0 \rightarrow \mathcal{O}_S(L_2 - C) \rightarrow \mathcal{O}_S(L_2) \rightarrow \mathcal{O}_C(L_2|_C) \rightarrow 0$$

Note that by 3.14 $H^1(\mathcal{O}_S(L_2 - C)) = 0$, which when applied to the long exact sequence induced by the last short exact sequence implies that the map $H^0(L_2) \rightarrow H^0(L_2|_C)$

is surjective. The map in the lemma can be split as the composition of the maps

$$H^0(L_1|C) \otimes H^0(L_2) \rightarrow H^0(L_1|C) \otimes H^0(L_2|_C) \rightarrow H^0((L_1 + L_2)|_C)$$

By what we have just said the first map in the sequence is surjective. Also noting that on \mathbb{P}^1 , if M, N are line bundles of nonnegative degrees then the map $H^0(L) \otimes H^0(N) \rightarrow H^0(M + N)$ is surjective. The degree of the map $\deg L_1|_C = L_1 \cdot C > 0$ by assumption and $\deg L_2|_C = L_2 \cdot C > 0$, since L_2 is ample and C is a curve. so the second map in the above sequence is also surjective.

For the general case since $h^1(\mathcal{O}_C) = 0$, there exists a component Γ_0 of C such that $\Gamma_0 \cdot C \leq 1 + \Gamma_0^2$. Set $C' = C - \Gamma_0$. we have $C' \cdot \Gamma_0 \leq 1$.

Consider the decomposition sequence for $C = C' + \Gamma_0$.

$$0 \rightarrow \mathcal{O}_{\Gamma_0}(-C') \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{C'} \rightarrow 0$$

This gives the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0((L_1 - C')|_{\Gamma_0}) \otimes H^0(L_2) & \longrightarrow & H^0(L_1|_C) \otimes H^0(L_2) & \longrightarrow & H^0(L_1|_{C'}) \otimes H^0(L_2) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0((L_1 + L_2 - C')|_{\Gamma_0}) & \longrightarrow & H^0((L_1 + L_2)|_C) & \longrightarrow & H^0((L_1 + L_2)|_{C'}) \longrightarrow 0 \end{array}$$

$\deg((L_1 - C')|_{\Gamma_0}) = (L_1 - C') \cdot \Gamma_0 \geq 0 - 1 = -1$. Also by ([1], Theorem 1.7) $p_a(\Gamma_0) = h^1(\Gamma_0) = 0$. So $\deg(L_1 - C')|_{\Gamma_0} > 2p_a(\Gamma_0) - 2$. Hence $h^1((L_1 - C')|_{\Gamma_0}) = 0$. It follows that the first row is exact. Similarly so is the second one, because $(L_1 + L_2 - C') \cdot \Gamma_0 = (L_1 - C') \cdot \Gamma_0 + L_2 \cdot \Gamma_0 > (L_1 - C') \cdot \Gamma_0$, since L_2 is ample. The surjectivity of the right column map is by induction hypothesis. The left column map

factors as

$$H^0((L_1 - C')|_{\Gamma_0}) \otimes H^0(L_2) \xrightarrow{m_1} H^0((L_1 - C')|_{\Gamma_0}) \otimes H^0(L_2|_{\Gamma_0}) \xrightarrow{m_2} H^0((L_1 + L_2 - C')|_{\Gamma_0})$$

By lemma 3.14 $H^1(L_2 - \Gamma_0) = 0$, so the 1st map would be surjective when we consider the long exact sequence induced by the short exact sequence $0 \rightarrow \mathcal{O}_S(L_2 - \Gamma_0) \rightarrow \mathcal{O}_S(L_2) \rightarrow \mathcal{O}_{\Gamma_0}(L_2) \rightarrow 0$. For some reason the map m_2 is also surjective. Hence the left column map of the commutative diagram is surjective. Now by the snake lemma, the middle map of the commutative diagram is also surjective.

Lemma 3.16. *Let S be a surface and $C = m\Gamma$, where Γ is a smooth irreducible curve with $\Gamma^2 = 0$ and $m \geq 1$. Let L_1, L_2 be two divisors on S such that $L_1 \cdot \Gamma > \max \{2g(\Gamma), 4g(\Gamma) - L_2 \cdot \Gamma - 2h^1(L_2|_{\Gamma})\}$, and L_2 is ample. Then the natural map*

$$H^0(L_1|_C) \otimes H^0(L_2|_C) \rightarrow H^0((L_1 + L_2)|_C)$$

is surjective.

Proof. For $m = 1$, the surjection holds good by ([4], Proposition 2.2). Since the proposition is only true for irreducible curves, for $m > 1$, we proceed by induction as in the previous lemma. Using the notations used in the previous lemma. Let us choose $\Gamma_0 = \Gamma$. Then $\Gamma_0 \cdot C = 0 \leq 1 + \Gamma_0^2$. $C' = (m - 1)\Gamma$. We have $\Gamma_0 \cdot C' = 0 \leq 1$. Consider the commutative diagram in the proof of the previous lemma. $H^1((L_1 - (m - 1)\Gamma)|_{\Gamma}) = 0$ since $(L_1 - (m - 1)\Gamma) \cdot \Gamma = L_1 \cdot \Gamma \geq 2g(\Gamma) > 2g(\Gamma) - 2$. So the

top row of the commutative diagram is exact. Since L_2 is ample, $(L_1 + L_2 - C') \cdot \Gamma_0 > (L_1 - C') \cdot \Gamma_0 > 2g(\Gamma_0) - 2$. Hence the bottom row is exact. The right column is exact by induction hypothesis. the left column map factors into two maps as in the previous proof, into $m_2 \circ m_1$. And m_2 is surjective by ([4], Proposition 2.2), since $(L_1 - C') \cdot \Gamma_0 = L_1 \cdot \Gamma$, and $\Gamma_0 = \Gamma$, is an irreducible smooth curve.

Lemma 3.17. *Let S be a smooth anticanonical rational surface, and L be a divisor on S such that $K_S + L$ is big and base-point free. Let B' be a divisor on S , such that $h^1(B') = 0$. Also suppose $(K_S + L) \cdot (B' - 2K_S) \geq 5$. Then the natural map*

$$H^0(K_S + L + B') \otimes H^0(K_S + L) \rightarrow H^0(2K_S + 2L + B')$$

is surjective.

Proof. We want to use 3.15. We have S is smooth and $H^1(\mathcal{O}_S) = 0$. Let $E = K_S + L + B'$. Let $\Delta \in |K_S + L|$ be a smooth, irreducible member. Since $(K_S + L)$ is big, we can choose a smooth Δ , which is also irreducible. Then $h^1((K_S + L + B') - (K_S + L)) = 0$. Hence by 3.2, if

$$H^0((K_S + L + B')|_{\Delta}) \otimes H^0((K_S + L)|_{\Delta}) \rightarrow H^0((2K_S + 2L + B')|_{\Delta})$$

is surjective, then the map in the lemma would be surjective. Also let us choose a Δ which is smooth (as $K_S + L$ is base-point free we can make such a choice) and irreducible (we can make this choice as $K_S + L$ is big). Now we want to apply 3.3 to prove the last surjectivity. Δ is a smooth irreducible curve. $(K_S + L)|_{\Delta}$ is base-point free as it is the restriction of a base-point free line bundle. To prove the surjectivity

we have to prove

$$h^1(B'|_{\Delta}) \leq h^0((K_S + L)|_{\Delta}) - 2$$

Using Riemann-Roch Theorem for $(K_S + L)|_{\Delta}$,

$$\begin{aligned} h^0((K_S + L)|_{\Delta}) &\geq 1 + (K_S + L)^2 - \frac{(K_S + (K_S + L)) \cdot (K_S + L) + 2}{2} \\ &\geq \frac{L \cdot (K_S + L)}{2} \end{aligned}$$

While using Serre duality,

$$h^1(B'|_{\Delta}) = h^0(K_{\Delta} - B'|_{\Delta}) = h^0((2K_S + L - B')|_{\Delta})$$

But since B' is effective, $B' \cdot (K_S + L) \geq 0$. So

$$\deg(2K_S + L - B')|_{\Delta} = (2K_S + L - B') \cdot (K_S + L) \leq (2K_S + L) \cdot (K_S + L) = \deg K_{\Delta}$$

Hence we have that $(2K_S + L - B')$ is special and effective. Now Clifford's theorem for effective special divisors on curves tells us that

$$h^0(2K_S + L - B') \leq \frac{(2K_S + L - B') \cdot (K_S + L)}{2} + 1$$

Putting these analysis of the two sides of the inequality together

$$h^0(K_S + L) - h^1(B) - 2$$

$$\begin{aligned}
&\geq \frac{L \cdot (K_S + L)}{2} - \frac{(2K_S + L - B') \cdot (K_S + L)}{2} - 3 \\
&= \frac{(K_S + L) \cdot (B' - 2K_S) - 6}{2} \geq \frac{5 - 6}{2} = \frac{-1}{2} \text{ using the condition given in the lemma}
\end{aligned}$$

But since the betti numbers are integers. we must have $h^0(K_S+L)-h^1(B')-2 \geq 0$. This completes the proof.

We are now ready to prove the main theorem.

Theorem 3.18. *Let S be a rational surface with $\dim | -K_S| \geq 1$. Let $\pi : X \rightarrow S$ be a ramified double covering of S by a minimal surface X (possibly singular). Let L be a divisor on S with the property that $K_S + L$ is nef and $L.C \geq 3$ for any curve C . Then $K_X + \pi^*L$ is base point free and the natural map*

$$S^r H^0(K_X + \pi^*L) \rightarrow H^0(r(K_X + \pi^*L)) \tag{3.2}$$

surjects for every $r \geq 1$.

Proof. $K_S + B + L$ is ample and base-point free by 3.13(1). Then $K_X + \pi^*L = \pi^*(K_S + B + L)$, is base-point free, since pullback of base-point free line bundles is base-point free. It is also ample since the pullback of ample line bundles under affine morphisms is ample, and a finite morphism is affine.

It is clear that the surjectivity in the theorem will follow if we can prove

$$H^0(r(K_X + \pi^*L)) \otimes H^0(K_X + \pi^*L) \rightarrow H^0((r+1)(K_X + \pi^*L))$$

is surjective for all $r \geq 1$.

Since π is finite, it is affine, hence for any quasicoherent sheaf F on X , $R^i \pi_* F = 0$, for $i > 0$. Hence $H^i(F) \cong H^i(\pi_* F)$ for $i \geq 0$. So in particular $H^i(r(K_X + \pi^* L)) \cong H^i((r)\pi_*(K_X + \pi^* L)) \cong H^i((r)\pi_* \pi^*(K_S + B + L))$. Hence using the projection formula for double covers

$$H^i(r(K_X + \pi^* L)) \cong H^i((r)((K_S + B + L) \oplus (K_S + L)))$$

We want to use Castelnuovo-Mumford regularity Lemma, to prove surjectivity, in the case $r \geq 2$. Toward's that end, let us verify the vanishing of the various cohomologies, that is needed to use the theorem. Note that L is ample, $K_S + L$ is nef, and $(K_S + B + L)$ is ample.

hence for $r = 2, i = 1$, and $r = 3, i = 2$, $(r - i) = 1$, $H^i((r - i)(K_X + \pi^* L)) = 0$, by Kodaira Vanishing theorem.

for $r = 2, i = 2$, $r - i = 0$, and the two cohomology groups in the direct sum are 0, since $H^2(\mathcal{O}_S) = 0$, as S is a rational surface.

For $r > 3, i \geq 1$ and $r = 3, i = 1$, we can write $(r - i)(K_X + \pi^* L) = K_X + \pi^* L + (r - i - 1)(K_X + \pi^* L)$, and $\pi^* L + (r - i - 1)(K_X + \pi^* L)$ is sum of two

ample line bundles hence ample. In these cases Kodaira vanishing theorem will imply $H^i((r-i)(K_X + \pi^*L)) = 0$.

Hence for $r \geq 2$, the map

$$H^0((r)(K_X + \pi^*L)) \otimes H^0(K_X + \pi^*L) \rightarrow H^0((r+1)(K_X + \pi^*L))$$

is surjective by Castelnuovo-Mumford Regularity lemma. It remains to treat the case

$$H^0(K_X + \pi^*L) \otimes H^0(K_X + \pi^*L) \rightarrow H^0(2K_X + 2\pi^*L)$$

Pushing down the line bundles to S , and using the canonical isomorphism between cohomology groups, verifying this surjectivity is equivalent to verifying the following surjectivity

$$\begin{aligned} & [H^0(K_S + B + L) \oplus H^0(K_S + L)] \otimes [H^0(K_S + B + L) \oplus H^0(K_S + L)] \\ & \rightarrow H^0(2K_S + 2B + 2L) \oplus H^0(2K_S + 2L + B) \end{aligned}$$

looking at degrees of line bundles the above map splits into two maps

$$H^0(K_S + B + L) \otimes H^0(K_S + B + L) \oplus H^0(K_S + L) \otimes H^0(K_S + L) \rightarrow H^0(2K_S + 2B + 2L)$$

and

$$H^0(K_S + B + L) \otimes H^0(K_S + L) \oplus H^0(K_S + L) \otimes H^0(K_S + B + L) \rightarrow H^0(2K_S + 2L + B)$$

But it is enough to prove the following two maps are surjective

$$H^0(K_S + B + L) \otimes H^0(K_S + B + L) \xrightarrow{f_1} H^0(2K_S + 2B + 2L)$$

$$H^0(K_S + B + L) \otimes H^0(K_S + L) \xrightarrow{f_2} H^0(2K_S + 2L + B)$$

By 3.13(2) f_1 is surjective.

To prove surjectivity of f_2 , we will choose a curve C_0 on S , such that the rows of the following commutative diagram are exact, and the two outside columns are exact. Then the surjectivity of f_2 will follow from Snake lemma.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \begin{array}{c} H^0(K_S+L+B-C_0) \\ \otimes \\ H^0(K_S+L) \end{array} & \longrightarrow & \begin{array}{c} H^0(K_S+L+B) \\ \otimes \\ H^0(K_S+L) \end{array} & \longrightarrow & \begin{array}{c} H^0(K_S+L+B)|_{C_0} \\ \otimes \\ H^0(K_S+L) \end{array} \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow f_2 & & \downarrow \beta \\ 0 & \longrightarrow & H^0(2K_S + 2L + B - C_0) & \longrightarrow & H^0(2K_S + 2L + B) & \longrightarrow & H^0((2K_S + 2L + B)|_{C_0}) \longrightarrow 0 \end{array}$$

If $K_S + L = 0$, then f_2 is obviously surjective, so for the rest of this proof we assume $K_S + L \neq 0$. Thus $K_S + L$ is ample and base-point free by 3.10. we analyze three different cases depending upon $-K_S$.

Case I: $-K_S$ has a fixed part.

Let $-K_S = F + M$, where F is the fixed part of $-K_S$. Let us take $C_0 = F$, and set $B' = B - C_0$. Then $B' = B + K_S + M$ is nef as $B + K_S$ is nef and M being the

moving part of a line bundle is always nef, as it's self-intersection is nonnegative. Also since S is a anticanonical rational surface, B' is effective. Hence by Kodaira vanishing $H^1(K_S + L + B') = 0$, as the sum of an ample line bundle and a nef line bundle is an ample line bundle. Hence the top row of the commutative diagram is exact in this case. Similarly $2K_S + 2L + B' = K_S + (L + B') + (K_S + L)$, and $K_S + L$ is ample. So Kodaira vanishing theorem implies $H^1(2K_S + 2L + B') = 0$. So the bottom row is exact. Because $|-K_S|$ has nonzero fixed part, Corollary 3.9 implies $H^1(B') = 0$. Now Proposition 3.10 implies that $(K_S + L) \cdot (-K_S) \geq 3$, as the two exceptional cases in the corollary donot occur in our case, as $K_S + L \neq 0$, and (why case 2 doesnot occur). Now we can apply lemma 3.17(why ≥ 5) to conclude that α is surjective.

For the surjectivity of β we will apply lemma 3.15, with $L_1 = K_S + L + B$ and $L_2 = K_S + L$. We need to verify two conditions to be able to apply this lemma. First $h^1(C_0) = 0$ since $h^0(C_0) = 1$ (as $C_0 = F$ is the fixed part of $-K_S$) by Proposition 3.6. Secondly for every component Γ_i of C_0 , $-K_S - \Gamma_i = (C_0 - \Gamma_i) + M$ is effective since $C_0 - \Gamma_i$ and M are effective. Now Lemma 3.15 implies β is surjective.

Case II: $-K_S$ has no fixed part and $K_S^2 > 0$.

In this case we can directly prove that f_2 is surjective, using Lemma 3.17. In this case $-K_S$ is effective and has no fixed component. so $-K_S$ is nef. Now $K_S^2 > 0$, implies it is big. Since $B = (K_S + B) + (-K_S)$, and $(K_S + B)$ is nef, B is big and nef. Then by ([17], Theorem 8.) we have $H^1(B) = 0$. By Proposition 3.9, $(K_S + L) \cdot (-K_S) \geq 2$ (what about the second exceptional case). Since $0 \neq B$ is

effective, we must have $(K_S + L) \cdot (B - 2K_S) \geq 5$. Hence we can apply lemma 3.17 to get that f_2 is surjective in this case.

Case III: $-K_S$ has no fixed part and $K_S^2 = 0$.

In this case $-K_S$ is nef, since it has no fixed component. and hence base-point free, as S is an anticanonical rational surface by ([18], Theorem III.1(c)). Similarly $B = (K_S + B) + (-K_S)$ is nef as $(K_S + B)$ is nef, hence it is base-point free. As in case II, if $H^1(B) = 0$, we are done by lemma 3.17. Hence let us consider the case $H^1(B) \neq 0$. Then $B \cdot (-K_S) = 0$ by ([18], Theorem III.1). But the Hodge Index theorem will imply $B = m(-K_S)$ for some $m > 0$.

Set $B_k := k(-K_S)$. We will use the commutative diagram above to prove the surjectivity of f_2 in this case. Let $C_0 \in |-K_S|$ be a smooth member. Note we just now proved that $-K_S$ is base-point free. We will do induction on k to show that for all $k \geq 0$, the map

$$\alpha_k : H^0(K_S + L + B_k) \otimes H^0(K_S + L) \rightarrow H^0(2K_S + 2L + B_k)$$

is surjective.

When $k = 0$, this is true because ([15], Theorem 1.3) and the fact that $(K_S + L) \cdot (-K_S) \geq 3$. Suppose α_k is surjective for some $k \geq 0$. Since $H^1(K_S + L + B_k) = 0$ by Kodaira vanishing as $B_k = k(-K_S)$ is nef, the top row of the diagram is surjective.

For reasons similar to that in case I, the bottom row of the diagram is exact. then the middle column = α_{k+1} would be surjective as soon as

$$\beta_{k+1} : H^0((K_S + L + B_{K+1})|_{C_0}) \otimes H^0(K_S + L) \rightarrow H^0((2K_S + 2L + B)|_{C_0})$$

is surjective.

First $H^1(K_S + L - C_0) = H^1(K_S + (K_S + L)) = 0$ by Kodaira vanishing. Let $C_0 = \sum \Gamma_i$, where Γ_i are the smooth components. Also $C_0^2 = \sum \Gamma_i^2$ as $\Gamma_i \cdot \Gamma_j = 0$, since C_0 is smooth. Suppose $\Gamma_i^2 < 0$ for some i . Then $\deg(\Gamma_i|_{\Gamma_i}) < 0$. Hence $H^0(\mathcal{O}_{\Gamma_i}(\Gamma_i|_{\Gamma_i})) = 0$. Now consider the short exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(\Gamma_i) \rightarrow \mathcal{O}_{\Gamma_i}(\Gamma_i|_{\Gamma_i}) \rightarrow 0$$

got by tensoring the structure sequence for Γ_i , with $\mathcal{O}_S(\Gamma_i)$. Taking the long exact sequence of this, we get $H^0(\mathcal{O}_S) \cong H^0(\mathcal{O}_S(\Gamma_i))$. Hence Γ_i , must be a fixed curve. But then it will be a component of the fixed component of $|-K_S|$. which in our case is empty. So Γ_i^2 cannot be negative, and in particular for any i . Hence $0 = C_0^2 = \sum \Gamma_i^2$, then forces each of the $\Gamma_i^2 = 0$ for each i . Hence $\Gamma_i \cdot (-K_S) = 0$. Now the adjunction formula $2g(\Gamma_i) - 2 = \Gamma_i^2 + \Gamma \cdot K_S = 0$, gives Γ_i is an elliptic curve for each i . Hence to prove B_{k+1} it suffices to show

$$H^0((K_S + L + B_{k+1})|_{\Gamma_i}) \otimes H^0((K_S + L)|_{\Gamma_i}) \rightarrow H^0((2K_S + 2L + B_{k+1})|_{\Gamma_i})$$

is surjective for each i . This is elementary, because

$$\deg((K_S + L + B_{k+1})|_{\Gamma_i}) = \deg((K_S + L)|_{\Gamma_i}) = L \cdot \Gamma_i \geq 3 = 2g(\Gamma_i) + 1$$

This completes the proof for this case, and hence of the theorem.

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