CONCERNING SPECIAL CURVES OF CYCLOIDAL TYPE.

by


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# CONCERNING SPECIAL CURVES OF CYCLOIDAL TYPE.

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I THE GENERAL CONSIDERATION.

If a circle rolls along a straight line, the curve generated by a point on the circumference is called a cycloid. If the generating point is taken anywhere on the radius of the rolling circle or on the radius produced, the curve is called the prolate cycloid or curvate cycloid according to whether the generating point is without or within the circle. In each case the distance from the generating point to the center of the rolling circle is constant.

The generation of the curves to be considered in this paper differs from that of the cycloid and the prolate and curvate cycloids in this respect: the distance of the generating point from the center of the rolling circle, instead of being constant, is to be a variable, \( a f(\theta) \), where \( a \) is the radius of the rolling circle, and \( \theta \) the angle thru which the rolling circle has turned. This distance we shall call the generating arm. We shall study first the general case, later examining particular curves generated when different functions of \( \theta \) are used as the length of the generating arm.

The parametric equations of the general curve (figure 1) are obtained from the geometry of the figure. \( x \) and \( y \), the coordinates of \( P \), any point on the curve are:
\[ x = CA - BA = \widehat{EA} - DG = a\theta - GP \sin (\pi - \theta) \]
\[ = a(\theta - f(\theta) \sin \theta) \]
\[ y = BD + DP = a + GP \cos (\pi - \theta) \]
\[ = a(1 - f(\theta) \cos \theta) \]

An investigation of some of the general properties of the curve may be made and the results applied when desirable to the particular curves obtained by fixing \( f(\theta) \) as a particular function.

**INTERCEPTS.**

When the curve crosses the \( x \) axis, \( y = 0 \).

\[ f(\theta) = \sec \theta. \]

When the curve crosses the \( y \) axis, \( x = 0 \).

\[ f(\theta) = \theta \csc \theta. \]

**CRITICAL POINTS.**

When a point on the curve is a maximum or minimum point,

\[ \frac{dy}{dx} = 0 \quad \text{ie} \]

\[ \frac{f(\theta) \sin \theta - f'(\theta) \cos \theta}{1 - f(\theta) \cos \theta - f'(\theta) \sin \theta} = 0, \]

and the sign of \( \frac{dy}{dx} \) changes at this point. Unless the denominator of \( \frac{dy}{dx} \) vanishes, \( \frac{dy}{dx} \) equals zero when the numerator is zero, ie

\[ f(\theta) \sin \theta - f'(\theta) \cos \theta = 0 \quad \text{or} \]

\[ \frac{f'(\theta)}{f(\theta)} = \tan \theta \]
Integrating both sides,
\[ \log f(\theta) = \log \sec \theta + c, \]
\[ = \log \sec \theta + \log c \]
\[ = \log (c \sec \theta) \]
\[ \therefore f(\theta) = c\sec \theta, \]
which is a condition for a maximum or minimum point;
also for a horizontal tangent.

AT INFLECTION POINTS, \( \frac{d^2y}{dx^2} \) equals zero and changes sign.

\[ \frac{d^2y}{dx^2} = \frac{d}{d\theta} \frac{dy}{dx} \frac{d\theta}{dx} \]
\[ = \frac{-2 f''(\theta) + f(\theta) f'(\theta) - f'(\theta) + [f(\theta) - f'(\theta)] \cos \theta + 2 f'(\theta) \sin \theta}{a [1 - f(\theta) \cos \theta - f'(\theta) \sin \theta]^3} \]

The tangent to the curve will be perpendicular to the x axis when \( \frac{dy}{dx} = \infty \) ie
\[ \frac{f(\theta) \sin \theta - f'(\theta) \cos \theta}{1 - f(\theta) \cos \theta - f'(\theta) \sin \theta} = \infty \]
or, providing the numerator is not equal to zero, when the denominator vanishes, ie
\[ f(\theta) \cos \theta + f'(\theta) \sin \theta = 1 \]

SINGULAR POINTS. If \( \frac{dy}{dx} \) takes the indeterminate form \( \frac{0}{0} \) for some value or values of \( \theta \), that is if
\[ f(\theta) \sin \theta = f'(\theta) \cos \theta \quad \text{and} \]
\[ f(\theta) \cos \theta + f'(\theta) \sin \theta = 1 \]
the curve has a singular point for such a value of \( \theta \).
PERIOD. The necessary and sufficient conditions that the period of the curve be \( \pi \), where \( n \) has any real value are: the value of \( x \) when \( \theta \) equals \( n + \theta \), minus the value of \( x \) when \( \theta \) equals \( n \), must equal the value of \( x \) when \( \theta \) equals \( \theta \), minus the value of \( x \) where \( \theta \) equals zero; and the value of \( y \) when \( \theta \) equals \( n + \theta \) must equal the value of \( y \) when \( \theta \) equals \( \theta \). That is, in Figure 2,

\[
\begin{align*}
DE &= AC \quad \text{ie} \\
OE - OD &= OC - OA \quad \text{or} \\
x_q(\theta_q = \theta + n\pi) - x_3(\theta_q = n\pi) &= x_2(\theta_q = \theta) - x_1(\theta_q = 0)
\end{align*}
\]

and \( EQ = CP \) or

\[
\begin{align*}
y_q(\theta_q = \theta + n\pi) &= y_2(\theta_q = \theta)
\end{align*}
\]

![Figure 2](image_url)

Let \( x_i \) and \( y_i \) be the rectangular coordinates of \( P \), any point on the curve for which the angle thru which the rolling circle has turned is \( \theta_i \). Substituting in the above conditions the values of \( x_i \) and \( y_i \) in terms of functions of \( \theta \) the conditions become:

\[
\begin{align*}
f(n\pi) \sin n\pi - f(n\pi + \theta) \sin(n\pi + \theta) &= f(0) \sin 0 - f(\theta) \sin \theta \\
\text{and } f(n\pi + \theta) \cos (n\pi + \theta) &= f(\theta) \cos \theta
\end{align*}
\]
SYMMETRY. In order that the curve be symmetrical with respect to the line \( x = x_o (\theta_o = n\pi) \), where \( n \) is real and rational, the following conditions must be satisfied:

\[ x_2 (\theta_2 = n\pi + \theta) = -x_1 = x_o = x, (\theta_1 = n\pi - \theta) \]

and \( y_2 = y_1 \).

Substituting for \( x_1 \) and \( y_1 \) in terms of functions of \( \theta \), the conditions of symmetry with respect to the line \( x = x_o \) are:

\[
\begin{align*}
 f(n\pi + \theta) \sin(n\pi + \theta) + f(n\pi - \theta) \sin(n\pi - \theta) &= 2f(n\pi) \sin n\pi \\
\text{and} &
 f(n\pi + \theta) \cos(n\pi + \theta) = f(n\pi - \theta) \cos(n\pi - \theta)
\end{align*}
\]

It is seen by the above conditions that the two curves whose generating arms are the negative of each other, will have the same period and lines of symmetry. They are however not identical curves.

AREA UNDER ONE ARCH.

\[
A = \int_{\phi = \phi_1}^{\phi = \phi_2} \frac{1}{2} r^2 \, d\phi
\]

Change to rectangular coordinates. This method will be used in finding area under one arch of curves.

\[
\begin{align*}
 r^2 &= x^2 + y^2 \\
\frac{y}{x} &= \tan \phi \\
x \, dy - y \, dx &= \sec^2 \phi \, d\phi = \frac{x^2 + y^2}{x} \, d\phi \\
\phi &= \frac{x \, dy - y \, dx}{x^2 + y^2} \\
\therefore A &= \frac{1}{2} \int_{\phi = \phi_1}^{\phi = \phi_2} (x \, dy - y \, dx)
\end{align*}
\]
A comparative study of several of the particular curves obtained by variations of $f(\theta)$ may now be of interest and illustrate the above observations. The details of graphing these curves will not be included.
Fig 3

\[ a + (t) = a \cos \theta \]
\[ x = a (\theta - \sin \theta \cos \theta) \]
\[ y = a \sin^2 \theta \]

Cycloid; rolling circle radius a.
II PARTICULAR CURVES.

# 1. Figure 3. Let \( f(\theta) = a \cos \theta \)

The equations of the curve are:

\[
\begin{align*}
    x &= a(\theta - \sin \theta \cos \theta) \\
    y &= a \sin^2 \theta
\end{align*}
\]

Using the substitution \( b = \frac{a}{2} \) and \( \phi = 2\theta \),
the equations of the curve become

\[
\begin{align*}
    x &= b(\phi - \sin \phi) \\
    y &= b(1 - \cos \phi)
\end{align*}
\]

which are the parametric equations of the cycloid whose
rolling circle has a radius \( \frac{a}{2} \). Hence the length of arch,
area under arch, volume formed by revolving one arch about
\( x \) axis, and surface of revolution formed by revolving one
arch about the \( x \) axis, may be written at once.

\[
\begin{align*}
    S &= 3b = 4a \\
    A &= 3\pi b^3 = \frac{3}{4} \pi a^2 \\
    V &= 5\pi^2 b^3 = \frac{5}{8} \pi a^2 \\
    S &= \frac{64}{3} \pi b^2 = \frac{16}{3} \pi a^2
\end{align*}
\]

The period of the curve is \( \pi \), is one half the period of
the cycloid since \( Q = \frac{\phi}{2} \).
Fig 4

\[ a \sin \theta = 0 \]

\[ x = a \cos \theta \]

\[ y = a (1 - \sin \theta \cos \theta) \]
Figure 4. Let \( f(\theta) = \sin \theta \).

Then \( x = a(\theta - \sin^2 \theta) \)

\[ y = a(1 - \sin \theta \cos \theta) \]

By the conditions of chapter I the period is found to be \( \pi \).

Applying the test of symmetry, \( x = x_0 \) is found to be a line of symmetry of the curve when \( \theta = \pi \) equals \( \frac{\pi}{4} \), \( \frac{3\pi}{4} \), and \( \frac{2m - 1}{4} \pi \), where \( m \) is any integer.

The slope of the tangent to the curve at \( \theta = \frac{\pi}{4} \) is \( \infty \), therefore the tangent is perpendicular to the \( x \) axis.

Shifting the origin to the point on the curve at which \( \theta = \frac{\pi}{4} \) i.e \( x = \frac{a}{2}(\pi - 2) \), \( y = \frac{a}{2} \) by the transformation:

\[ x' = x + \frac{a}{4}(\pi - 2), \quad y' = y + \frac{a}{2}, \quad \theta = \theta' + \frac{\pi}{4}, \]

\[ x' + \frac{a}{4} - \frac{a}{2} = a\left[ \theta' + \frac{\pi}{4} - \sin^2 (\theta' + \frac{\pi}{4}) \right] \]

\[ = a\left[ \theta' + \frac{\pi}{4} - \left( \sin \theta' + \cos \theta' \right)^2 \right] \]

\[ = a(\theta' + \frac{\pi}{4} - \frac{1}{2} - \sin \theta' \cos \theta' ) \]

\[ \therefore x' = a(\theta' - \sin \theta' \cos \theta' ) \]

\[ y' + \frac{a}{2} = a\left[ 1 - \sin (\theta' + \frac{\pi}{4}) \cos (\theta' + \frac{\pi}{4}) \right] \]

\[ = a\left[ 1 - \frac{1}{2}(\sin \theta' + \cos \theta') \cos (\theta' - \sin \theta' ) \right] \]

\[ = a\left[ 1 - \frac{1}{2}(\cos^2 \theta' - \sin^2 \theta') \right] = a\left[ 1 - \frac{1}{2}(1 - 2 \sin^2 \theta') \right] \]

\[ \therefore y' = \sin^2 \theta' \]

These resulting equations are those of curve \( \# 1 \) for which \( f(\theta) = \cos \theta \). Therefore we obtain the same curve, though translated, when \( f(\theta) = \sin \theta \) as when \( f(\theta) = \cos \theta \).
Fig. 5

\[ a \cos(\theta) = 2a - a \cos \theta \]

\[ x = a(\theta - 2 \sin \theta \cos \theta + \sin \theta \cos \theta) \]

\[ y = a(1 - 2 \cos \theta + \cos^2 \theta) \]
Figure 5. Let \( f(\theta) = 2a - a \cos \theta. \)

Then \( x = a(\theta - 2 \sin \theta + \sin \theta \cos \theta), \)
\[ y = a(1 - 2 \cos \theta + \cos^2 \theta) \]

The period is \( 2\pi \) by the conditions of Chapter I.

Lines of symmetry are \( x = x_1, \) where \( \theta \) equals 0, \( \pi, \) \( 2\pi \) or \( n\pi, \) where \( n \) is any integer.

\[
\frac{dy}{dx} = \frac{2 \sin \theta (1 - \cos \theta)}{(1 - \cos \theta)^2 - (1 - \cos^2 \theta)} = \frac{\sin \theta}{(1 - \cos \theta) - (1 + \cos \theta)} = \frac{\sin \theta}{-\cos \theta} = -\tan \theta
\]

Then \( \frac{dy}{dx} = 0 \) when \( \theta \) equals 0, \( \pi, \) \( 2\pi \) or \( n\pi, \) where \( n \) is any integer. Since \( \frac{dy}{dx} \) changes from + to - as \( x \) decreases, when \( \theta \) equals 0, \( 2\pi, \) or \( 2n\pi, \) \( n \) any integer, these are minima points.

\[
\frac{dy}{dx} = \infty \text{ when } \theta = \frac{\pi}{2}, \frac{3\pi}{2} \text{ or } (2n - 1) \frac{\pi}{2}, \text{ } n \text{ any integer;}
\]

therefore, the tangent to the curve at each of these points is perpendicular to the x axis.

**LENGTH OF ARCH.**

\[
s = 2 \int_{\theta = 0}^{\theta = \pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta
\]
\[
= 4a \int_{\theta = 0}^{\pi} (1 - \cos \theta) \, d\theta
\]
\[
= 4\pi a
\]

which is exactly twice the length of the base of the arch.

**AREA UNDER ONE ARCH.**

\[
A = \frac{1}{2} \int_{\theta = 0}^{\theta = 2\pi} (x \, dy - y \, dx)
\]
\[ \frac{\pi a^2}{4} \int_0^\frac{\pi}{2} (4\theta \sin \theta - 2\sin 2\theta - \sin^2 2\theta - 10 + 16 \cos \theta - 2 \cos 2\theta - 4 \cos^4 \theta) \, d\theta = \frac{\pi}{2} \pi a^2 \]

VOLUME OF REVOLUTION OF ONE ARCH ABOUT THE X AXIS.

Using the cylindrical shell method,

\[ \frac{1}{2} V = \int_0^{\pi} 2\pi y (\pi a - x) \, dy \]

\[ = 2\pi a \int_0^{\pi} y \, dy - 2\pi \int_0^{\pi} xy \, dy \]

\[ = 16\pi a^3 - 4\pi a^3 \int_0^{\pi} (\theta \sin \theta - \frac{3}{4} 2\theta \sin 2\theta + 3\theta \sin \theta \cos^2 \theta - \theta \sin \theta \cos^3 \theta + 7 \sin^2 \theta \cos \theta + 5 \cos^3 \theta - 5 \cos^5 \theta - 12 \sin^2 \theta + 11 \sin^4 \theta - \sin^6 \theta) \, d\theta \]

\[ = 16\pi a^3 - \frac{23}{8} \pi^2 a^3 \]

\[ = \frac{105}{8} \pi^2 a^3 \]

\[ V = \frac{105}{2} \pi^2 a^3 \]

SURFACE OF REVOLUTION OF ONE ARCH ABOUT THE X AXIS.

Since \( x \) decreases from \( x_2 \) (\( \theta_2 = 0 \)) to \( x_1 \) (\( \theta_1 = \frac{\pi}{2} \)),

\[ \int_{x_3}^{x_1} ds \] is negative and likewise the portion of the surface of revolution \( \int_{x_3}^{x_1} 2\pi y \, ds \) would be negative.

Hence we can not take \( \frac{1}{2} S = \int_{x_3}^{x_1} 2\pi y \, ds \) in one integration.

But \( \frac{1}{2} S \) may be broken into two positive areas and their sum be taken as follows:

\[ \frac{1}{2} S = \int_{x_1}^{x_2} 2\pi y \, ds + \int_{x_1}^{x_3} 2\pi y \, ds \]

\[ = 4\pi a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (-\cos^3 \theta + 3 \cos^2 \theta - 3 \cos \theta + 1) \, d\theta + 4\pi a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (-\cos^3 \theta + 3 \cos^2 \theta - 3 \cos \theta + 1) \, d\theta \]

\[ S = \frac{176\pi a^2}{2} \]
Fig. 6

\[ f(c) = a \cos \theta - 2a \]

\[ x = a \left( \theta - \sin \theta \cos \theta + 2 \sin \theta \right) \]

\[ y = a \left( \sin^2 \theta + 2 \cos \theta \right) \]
# 4. Figure 6. Let \( f(\theta) = a \cos \theta - 2a \) Then \( x = a (\theta - \sin \theta \cos \theta + 2 \sin \theta) \)
\( y = a (\sin^2 \theta + 2 \cos \theta) \)

The generating arms of curves # 3 and # 4 are the negative of each other; therefore the curves have the same period, \( 2\pi \), and lines of symmetry, \( x = x \); where \( \theta \) equals \( 0, \pi, \) and \( 2\pi \). However upon shifting the axes of curve # 4 to the same relative position as those of curve # 3, that is, so the origin of curve # 4 will be at the point \((\pi a; -2a)\), we get by the transformation.

\[
x = x' + a\pi, \quad y = y' - 2a, \quad \theta = \theta' + \pi
\]

the new equations of # 4 to be

\[
x' = a(\theta' - \sin \theta' \cos \theta' - 2 \sin \theta')
\]
\[
y' = a(2 - 2 \cos \theta' + \sin^2 \theta')
\]

which are not the equations of curve # 3. The two curves are therefore not identical.

AREA included between one half arch, the \( y \) axis, and the line \( y = -2a \):

Move the origin down \( 2a \) by the transformation

\[
x = x', \quad y = y' - 2a, \quad \theta = \theta'.
\]

The new equations are

\[
x' = a(\theta' - \sin \theta' \cos \theta' + 2 \sin \theta')
\]
\[
y' = a(2 + \sin^2 \theta' + 2 \cos \theta')
\]

Integrate over the area under one half the arch.
\[ \frac{1}{2} A = \frac{1}{2} \int_0^\pi \left( x' \, dy' - y' \, dx' \right) \]

\[ = \frac{\pi a^2}{2} \int_0^\pi \left( \theta' \sin 2 \theta' - 4 \cos \theta' \sin \theta' - 6 \sin^2 \theta' \right) \, d\theta' \]

\[ = \frac{\pi a^2}{2} \left[ \frac{1}{2}(-2 \theta' \cos 2 \theta' - \sin 2 \theta') - 2(-\theta' \cos \theta' + \sin \theta') - 4 \theta' - 4 \sin \theta' - 6 \left( \frac{\theta'}{2} - \frac{1}{4} \sin 2 \theta' \right) \right]_0^\pi \]

\[ = 4\pi a^2 \]

VOLUME of revolution of one half arch about the line 

\[ y' = 0, \] using the above translation of axes:

\[ \frac{1}{2} V = \int_0^\pi y'^2 \, dx' \]

\[ = \int_0^\pi \pi a^2 \left( \sin^2 \theta' + 2 \cos \theta' + 2 \right)^2 \left( 2 \cos \theta' + \sin \theta' \right) \, d\theta' \]

\[ = 2\pi a^3 \int_0^\pi \left( \sin^5 \theta' + 5 \cos \theta' \sin^4 \theta' + 8 \cos \theta' \sin^2 \theta' + 8 \cos \theta' \right) \, d\theta' \]

\[ = 2\pi a^3 \left[ -\frac{\sin^5 \theta' \cos \theta'}{6} - \frac{5}{3} \left( -\sin^3 \theta' \cos \theta' + 2 \theta' - 3 \sin 2 \theta' \right) \right. \]

\[ + \sin^5 \theta' + 8 \sin^3 \theta' \left( -\sin \theta' + 8 \theta' + 8 \theta' + 4 \left( \frac{\cos \theta' \sin^3 \theta'}{4} \right) \right) \]

\[ + \frac{\theta'}{8} - \frac{\sin 2 \theta'}{16} \right]_0^\pi \]

\[ = \frac{141 \pi^2 a^3}{8} \]
Fig. 7

\[ a f(x) = a (\cos \theta + 2) \]

\[ x = a (\theta - \sin \theta \cos \theta - 2 \sin \theta) \]

\[ y = a (\sin^2 \theta - 2 \cos \theta) \]
# 5. Figure 7. Let \( f(\theta) = a \cos \theta + 2a \)

Then \( x = a(\theta - \sin \theta \cos \theta - 2 \sin \theta) \)
\[
\begin{align*}
y &= a(\sin^2 \theta - 2 \cos \theta)
\end{align*}
\]

If the axes be translated by the substitution
\[
\begin{align*}
x &= x' + wa, \\
y &= y', \\
\theta &= \theta' + w,
\end{align*}
\]
the transformed equations are
\[
\begin{align*}
x' &= a(\theta' - \sin \theta' \cos \theta' + 2 \sin \theta') \\
y' &= a(\sin^2 \theta' + 2 \cos \theta')
\end{align*}
\]

which are the equations of curve \# 4. Curves \# 4 and \# 5 are therefore identical.
Fig. 8

\[ a \cdot f(\theta) = a \cdot (1 - \cos \theta) \]

\[ x = a \cdot (\theta + \sin \theta \cdot \cos \theta - \sin \theta) \]

\[ y = a \cdot (1 + \cos^2 \theta - \cos \theta) \]
# 6. Figure 8. Let \( f(\theta) = a(1 - \cos \theta) \).

Then \( x = a(\theta + \sin \theta \cos \theta - \sin \theta) \),
\[ y = a(1 + \cos^2 \theta - \cos \theta) \]

The period is \( 2\pi \).

Lines of symmetry are \( x = 0 \), \( x = \pi a \), \( x = 2\pi a \).

Maxima points are at \( \theta = 0 \), \( \theta = \pi \), and \( \theta = 2\pi \).

At \( \theta = \frac{\pi}{2} \) and \( \theta = \frac{3\pi}{2} \) the tangent to the curve is perpendicular to the \( x \) axis.

\[
\frac{dy}{dx} = \frac{-a \sin \theta (2 \cos \theta - 1) d\theta}{a \cos \theta (2 \cos \theta - 1) d\theta} = \frac{\tan \theta}{\tan \frac{3\pi}{2}} \]

which takes the form \( \theta = \frac{\pi}{2} \) and \( \frac{3\pi}{2} \). These values of \( \theta \) therefore give singular points. Since the above fraction reduces to \( -\tan \theta \), the values of \( \frac{dy}{dx} \) at these singular points are \( -\sqrt{3} \) and \( +\sqrt{3} \) respectively.

**Area under one arch.**

Move the origin along the \( x \) axis to the line of symmetry \( x = \pi a \), by the transformation
\[
x = x' + \pi a, \quad y = y', \quad \theta = \theta' + \pi.
\]

\[
\frac{1}{2} A = \int_0^{2\pi} \frac{a^2}{r} d\phi + \frac{\pi a^2}{2} \]

which is the right triangle whose base is one half the base of the arch, and whose vertex is the point on the curve at which \( \theta' \) equals \( \pi \).

\[
\frac{1}{2} A = \frac{a^2}{2} \int_0^{\pi} \left( -4 \cos^2 \theta' - 4 \cos \theta' - 1 - 2 \theta' \sin \theta' \cos \theta' - \theta' \sin \theta' \right) d\theta' + \frac{\pi a^2}{2}
\]

\[
A = 4\frac{1}{2} \pi a^2
\]
It makes no difference if \( \varphi \) moves part of the time, or all of the time or not at all in the negative sense between the points where \( \varphi' = \frac{2\pi}{3} \) and the singular point \( \varphi' = \frac{2\pi}{3} \); for if any area under the curve is duplicated, or a negative area or an area not under the curve introduced during integration, it will vanish as the integral is taken between its two limits or with the addition of the triangle. The following exaggerated diagram will illustrate the integration between the limits \( \varphi' = \frac{2\pi}{3} \) and \( \varphi' = 0 \) for the area under one half arch.

\[
\begin{align*}
\int_{\varphi' = \frac{2\pi}{3}}^{\varphi' = \frac{2\pi}{3}} (x' \, dy' - y' \, dx') &= \text{area } \text{DPBD} - \text{area } \text{OCBO}' \\
\int_{\varphi' = 0}^{\varphi' = \frac{2\pi}{3}} (x' \, dy' - y' \, dx') &= \text{area } \text{OCBO}' \\
\int_{\varphi' = 0}^{\varphi' = \frac{2\pi}{3}} (x' \, dy' - y' \, dx') &= \text{area } \text{DPBD} - \text{area } \text{OCBO}' - \text{area } \text{CDFC} \\
\int_{\varphi' = \frac{2\pi}{3}}^{\varphi' = \frac{2\pi}{3}} (x' \, dy' - y' \, dx') + \Delta \text{OAP} &= \text{area under one half arch}
\end{align*}
\]

The area under one arch may also be obtained by dividing half the area into three parts:
having shifted the origin by the transformation \( x = x' + w a, \ y = y', \ \theta = \theta' + \pi \) for the second area; the third area being the triangle whose vertex is the singular point and whose base is one half the base of the arch.

**VOLUME OF REVOLUTION OF ONE ARCH ABOUT THE X AXIS.**

\[
\frac{1}{2} V = \pi \int_{0}^{\pi} y^2 \, dx - \pi \int_{0}^{\pi} y^3 \, dx + \pi \int_{0}^{\pi} y^3 \, dx
\]

\[
= \pi \int_{0}^{\pi} y^2 \, dx
\]

\[
y^2 \, dx = a^3 \left[ -\cos \theta + 4 \cos^2 \theta - 7 \cos^3 \theta + 3 \cos^4 \theta - 5 \cos^5 \theta + 2 \cos^6 \theta \right] \, d\theta
\]

\[
V = \frac{45 \pi^2 a^3}{4}
\]

**LENGTH OF ARCH.**

\[
s = \int_{0}^{2\pi} \sqrt{\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2} \, d\theta
\]

\[
in = a \int_{0}^{2\pi} (1 - 2 \cos \theta) \, d\theta
\]

\[
= 2\pi a
\]

**SURFACE OF REVOLUTION OF ONE ARCH ABOUT THE X AXIS.**

\[
\frac{1}{2} S = \pi \int_{0}^{\pi} y \, ds
\]

\[
= \pi \int_{0}^{\pi} \left[ 2 \pi y \, ds + \pi y \, ds + 2 \pi y \, ds \right]
\]

\[
y \, ds = a^2 \left( 1 - 3 \cos \theta + 3 \cos^2 \theta - 2 \cos^3 \theta \right) \, d\theta
\]

\[
S = \frac{\pi a^2}{3} \left( 104 - 45 \sqrt{3} + 20 \pi \right)
\]
Fig. 9
\[
a(t) = a(t + \theta) - a
\]
\[
x = a(\theta - \sin \theta \cos \theta + \sin \theta)
\]
\[
y = a(\sin \theta + \cos \theta)
\]

Fig. 10
\[
a(t) = a(t + \theta + 1)
\]
\[
x = a(\theta - \sin \theta \cos \theta - \sin \theta)
\]
\[
y = a(\sin \theta - \cos \theta)
\]
Figure 9. Let \( f(\theta) = a(\cos \theta - 1) \)

Then \( x = a(\theta - \sin \theta \cos \theta + \sin \theta) \)

\[ y = a(\sin^2 \theta + \cos \theta) \]

The period of the curve is \( 2\pi \).

Lines of symmetry are \( x = 0 \), \( x = \pi a \), \( x = 2\pi a \).

Minima points occur when \( \theta \) equals \( 0 \), \( \pi \), and \( 2\pi \);

maxima points when \( \theta \) equals \( \frac{\pi}{3} \) and \( \frac{5\pi}{3} \).

The area between one half arch and the lines \( y = -a \),

and \( x = 0 \).

\[ A = \frac{1}{2} \int_{\theta = \pi}^{e = 0} (x \, dy - y \, dx) + \left( \frac{m^2}{2} \right), \text{ which is the area of the triangle whose base is one half the base of the arch and whose vertex is at the origin} \]

\[ \frac{1}{2} \int_{\theta = \pi}^{e = 0} (x \, dy - y \, dx) = \frac{a^2}{2} \left( 2\theta \sin \theta \cos \theta - \theta \sin \theta - \frac{3}{4} + 2 \cos^2 \theta \right) \, d\theta \]

\[ = \frac{\pi}{4} a^2 \]

\[ \therefore A = 2\frac{1}{4} \pi a^2 \]

**Volume of Revolution** of one half arch about the line \( y = -a \) as an axis.

Translate the origin to the point \( (\pi a, -a) \) by the transformation \( x = x' + \pi a \), \( y = y' - a \), \( \theta = \theta' + \pi \). The new equations are

\[ x' = a(\theta' - \sin \theta' - \sin \theta' \cos \theta') \]

\[ y' = a(1 + \sin^2 \theta' - \cos \theta') \]

\[ \frac{1}{2} V = \int_{\theta = 0}^{\theta = \pi} \pi y'^2 \, dx' \]
\[\pi a^3 \int_0^\pi \left(-6 \sin^2 \theta' \cos \theta' - 5 \sin^4 \theta' \cos \theta' + 2 \sin^6 \theta' - \cos \theta' - \cos^3 \theta' + 2 + 4 \sin^2 \theta' + 2 \sin^2 \theta' \cos^2 \theta' \right) d\theta' \]
\[= \frac{39\pi a^3}{8}\]

\# 8. Figure 10. Let \( f(\theta) = a(\cos \theta + 1) \)
Then \( x = a(\theta - \sin \theta \cos \theta - \sin \theta) \)
\( y = a(\sin^2 \theta - \cos \theta) \)
Translating the origin to the point \((\pi a, 0)\) by the substitution \( x = x' + \pi a, y = y', \theta = \theta' + \pi \), the curve is seen to be identical to curve \# 7, where \( f(\theta) = a(\cos \theta - 1) \)
Fig. 11
\[ a \tan^2 \theta = \frac{a - \tan \theta}{\cos \theta} \]
\[ x = a(\theta - \tan \theta) \]
\[ y = 0 \]

Fig. 12
\[ a \tan^2 \theta = \frac{a - \tan \theta}{\sin \theta} \]
\[ x = 0 \]
\[ y = a(1 - \cot \theta) \]

Fig. 13
\[ a \tan^2 \theta = \frac{a - \tan \theta}{\sin \theta} \]
\[ x = a(\theta - 1) \]
\[ y = a(1 - \cot \theta) \]
9. Figure 11. Let \( f(\theta) = \frac{a}{\cos \theta} \)

Then \( x = a(\theta - \tan \theta) \)
\( y = 0 \)

The curve is a straight line \( y = 0 \), of period \( \pi \); the locus of a point starting at \( x = 0 \) and moving always in a negative direction along the \( x \) axis, covering during each cycle the entire \( x \) axis with the exception of a length \( \pi a \), which length is a different portion of the \( x \) axis for each cycle. When \( \theta = n\pi \), \( x = n\pi a \), where \( n \) is any integer.

10. Figure 12. Let \( f(\theta) = \frac{a\theta}{\sin \theta} \)

Then \( x = 0 \)
\( y = a(1 - \theta \cot \theta) \)

The curve has no period. It is the locus of a point which moves in a positive direction along the \( y \) axis, starting at \( y = 0 \) when \( \theta = 0 \), and reaching \( y = a \) when \( \theta = (2n - 1)\frac{\pi}{2} \), and passing from \( y = +\infty \) to \( y = -\infty \) when \( \theta = n\pi \) (\( n \) any integer.)

11. Figure 13. Let \( f(\theta) = \frac{a}{\sin \theta} \)

Then \( x = a(\theta - 1) \)
\( y = a(1 - \cot \theta) \)

The period is \( \pi \). Each cycle of the curve passes successively through all values of \( y \) from \(-\infty\) to \( +\infty \), over a base equal to \( \pi a \).

An inflection point occurs at \( \theta = \frac{\pi}{2} \)
\[ \frac{dy}{dx} = \cos^2 \theta, \] which is always greater than zero, hence there is no horizontal tangent at any point on the curve.
\[ x = a \left( \theta - 2 \sin \theta \right) \]
\[ y = a \left( 1 - 2 \cos \theta + \cos^2 \theta \right) \]
12. Figure 14. Let \( f(\theta) = a(1 - \cos 2 \theta) \)

Then \( x = a(\theta - 2 \sin^3 \theta) \)
\[ y = a(1 - 2 \cos \theta + 2 \cos^2 \theta) \]

The period is \( 2\pi \).

Lines of symmetry are \( x = x_0 (\theta \) equaling \( \pi \) and 0) Minima points occur at \( \theta = \cos \frac{\sqrt{3}}{2} \) and \( \theta = \pi \); maxima, when \( \theta = \cos \left(-\frac{\sqrt{3}}{2}\right) \) and \( \theta = 2\pi \) and \( \theta = 0 \).

A, the area included between the curve and the line \( y = a \), is obtained by shifting the origin to the starting point of the curve, \((0,a)\), by the substitution
\[ x = x', y = y' + a, \theta = \theta'. \]

The new equations are then
\[ x' = a(\theta' - 2 \sin \theta' + 2 \sin \theta' \cos \theta') \]
\[ y' = a(-2 \sin^2 \theta' \cos \theta') \]

\[
\frac{1}{2} A = \int_{\theta' = 0}^{\theta' = \pi} \left( x' \, dy' - y' \, dx' \right) \\
= a^2 \int_{\theta' = 0}^{\theta' = \pi} \left( \theta' \sin \theta' - 2 \sin^4 \theta' - 3 \theta' \sin \theta' \cos^2 \theta' + \sin^2 \theta' \cos \theta' \right) \, d\theta' \\
A = 1\frac{\pi}{2} a^2 \]
III CONCLUSION.

The number of curves which can be obtained by changing the function of $\theta$ which is the variable generating arm is of course infinitely large; for, since $f(\theta)$ is arbitrary, any function of $\theta$ leads to the parametric equations of some curve.

The variables taken as generating arms in the twelve curves here considered were chosen because they give curves which are comparatively easily plotted, and some of whose properties are obtainable. They are all variables involving trigonometric functions of $\theta$. The choice of variables of this type alone is without limit.

Curves whose generating arms are algebraic functions of $\theta$ give rise to many integrals and expressions of a nature almost prohibitive to further investigation.