THE RIEMANN SURFACE FOR THE FUNCTION

\[ w = z^4 + 2a^2 z + b \]

by

Iva Oman

A. B., Baker University, Baldwin, Kansas, 1912

Submitted to the Department of Mathematics and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Master of Arts.

Approved by:

\hspace{1cm}

Instructor in Charge

Head or Chairman of the Department of Mathematics

\hspace{1cm}

Date: May 1932
<table>
<thead>
<tr>
<th>Table of Contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
</tr>
<tr>
<td>The Function ( w^2 = z^2 + 2a^2 z + b^2 ),</td>
</tr>
<tr>
<td>Case I, ( a = 0, b \neq 0 ),</td>
</tr>
<tr>
<td>Case I, ( a = 0, b = 0 ),</td>
</tr>
<tr>
<td>Case I, ( a = 0, b ) is imaginary,</td>
</tr>
<tr>
<td>Case II, ( a = 1, b &gt; a' ),</td>
</tr>
<tr>
<td>Case II, ( a = 1, b = a' ),</td>
</tr>
<tr>
<td>Case II, ( a = 1, b &lt; a' ),</td>
</tr>
<tr>
<td>Case II, ( a = 1, b = 0 ),</td>
</tr>
<tr>
<td>Case II, ( a = 1, b ) is imaginary,</td>
</tr>
<tr>
<td>Case III, ( a ) is imaginary, ( b = 1 ),</td>
</tr>
<tr>
<td>Case III, ( a' &gt; b^2 ), ( b = 1 ),</td>
</tr>
<tr>
<td>Case III, ( a' &lt; b^2 ), ( b = 1 ),</td>
</tr>
<tr>
<td>Case IV, ( a ) is complex, ( b = 0 ),</td>
</tr>
<tr>
<td>Case IV, ( a ) is complex, ( b = 1 ),</td>
</tr>
<tr>
<td>Case IV, ( a ) is complex, ( b ) is imaginary,</td>
</tr>
<tr>
<td>Case V, ( a = 0 ), ( b ) is complex,</td>
</tr>
<tr>
<td>The General Equation ( w = (1 - z^2)(1 - k^2 z^2) ),</td>
</tr>
</tbody>
</table>
A multiple-valued function differs from a single-valued function in this respect: In a single-valued function the paths described by \( w \) between \( w_1 \) and \( w_2 \), which correspond to those described by \( z \) between \( z_1 \) and \( z_2 \), always end in the same point \( w \). In a multiple-valued function there are several values of \( w \) for the same value of \( z \) and therefore different paths may lead to different points.

At certain points, two or more values of the function \( w \), in general different, become equal to one another. Riemann, a German mathematician (1826-1866), called such points branch-points.

In order to treat a multiform function as though it were a uniform function Riemann assumed that a plane consists of \( n \) sheets or leaves extending over the entire plane. To each point in each sheet corresponds only a single value of the function and to the \( n \) points lying immediately one below the other in all the \( n \)-sheets correspond the \( n \) different values of the function which belong to the same value of \( w \). At the branch-points two or more of the sheets are connected for at these points values, elsewhere different, become equal.

In the following pages the multiform function \( w^2 = z^2 + 2az + b \) will be considered.
The function to be considered is

\[ w^2 = z^2 + 2a^2 z + b^2 \]  

(1).

For every value of \( z \) there are two values of \( w \) and for every value of \( w \) there are four values of \( z \). Only the case where there are four values of \( z \) for each value of \( w \) will be studied.

In order to have a 1-1 correspondence between \( z \) and \( w \) four sheets must be made in the \( w \)-plane and each of the sheets will correspond to a certain region of the \( z \)-plane. To find where the sheets are connected it is necessary to find the points where one value of \( z \) changes into another, or where at least two values of \( z \) are the same. The condition that makes two values equal is that the roots must satisfy the equation and also the derivative of the equation.

Taking the derivative with respect to \( z \) in (1),

\[ 4z^3 + 4azz = 4z(z^2 + a) = 0. \]

\[ z = 0 \text{ and } z = \pm a, \]

Substituting these values of \( z \) in (1),

\[ w = \pm b \text{ and } w = \pm \sqrt{b^2 - a^2}. \]

Substituting these values of \( w \) in (1) the corresponding values of \( z \) are obtained; that is, when \( w = b \),

\[ b^2 = z^2 + 2a^2 z + b^2 / \]

\[ z^2 (z^2 + 2a^2) = 0 \]

\[ z = 0, 0, \text{ and } z = \pm a \sqrt{2}. \]

Therefore \( \pm b \) and \( -b \) are branch-points since one of the values of \( z \) corresponding to them is the double point \((0,0)\).
When \( w = \pm b^2 - a^2 \), then

\[
\begin{align*}
    b^2 - a^2 &= z + 2az^2 + b^2 \\
    z + 2az^2 + a^2 &= 0 \\
    z^2 &= -a^2, \text{ twice.} \\
    z &= ia, -ia, \text{ and } -ia, -ia.
\end{align*}
\]

Therefore \( +\sqrt{b^2 - a^2} \) and \( -\sqrt{b^2 - a^2} \) are branch-points since there are double points in the \( z \)-plane corresponding to them. At least two sheets are connected at each branch-point.

If \( w = 0 \) the equation \( w = z + 2az^2 + b \) reduces to

\[
    z + 2az^2 + b = 0;
\]

whence \( z^2 = -a^2 + b^2 \)

\[
    z = \pm i/\sqrt{a^2 - b^2} \quad \text{and} \quad \pm \sqrt{a^2 + b^2}
\]

If \( w = \infty \) then \( z = \infty \). All the sheets are connected at \( \infty \).

Tabulating these corresponding values of \( w \) and \( z \):

\[
\begin{align*}
    w &= \pm b, \text{ branch-points,} \quad z &= 0, 0, \text{ double point;} \\
    &\quad z = \pm ia/2, \text{ ordinary points.} \\
    w &= \pm \sqrt{b^2 - a^2}, \text{ branch-points,} \quad z &= ia, ia, \text{ double point;} \\
    &\quad z = -ia, -ia, \text{ double point.} \\
    w &= 0, \text{ ordinary point,} \quad z &= \pm i/\sqrt{a^2 - b^2} \quad z = \pm i/\sqrt{a^2 + b^2}, \text{ ordinary points.} \\
    w &= \infty, \text{ branch point,} \quad z &= \infty.
\end{align*}
\]
Since all the sheets of the \( w \)-surface are connected at \( \infty \) any of the branch-points may be joined to infinity by means of cuts. Assuming that the branch-point \( w = +b \) lies in the second and third sheets a cut may be drawn from there along the \( u \)-axis to \( \infty \); but such a cut would lie only in the second and third sheets.

In order to find what lines of the \( z \)-plane correspond to the lines of the \( w \)-surface set \( w = u + iv \) and \( z = x + iy \). Equation

\[
w^2 = z^2 + 2a_z^2 + b^2
\]

becomes

\[
u^2 + 2uv - v^2 = x^2 + 4xyi - 6y^2 = 4xyi + y^4 + 2a_x^2 + 4axyi - 2ay^2 + b^2.
\]

Equating real parts and imaginary parts:

\[
u^2 - v^2 = x^2 - 6y^2 + y^4 + 2a_x^2 - 2ay^2 + b^2
\]

and \( uv = 2xy(x^2 - y^2 + a^2) \).

If either \( u \) or \( v \) equals zero then

\[
2xy(x^2 - y^2 + a^2) = 0,
\]

whence \( 2xy = 0 \); and \( y = x = a \), an equilateral hyperbola.

From the above results it would seem that both the \( u \)- and the \( v \)-axes of the \( w \)-surface correspond to the \( x \)- and the \( y \)-axes and an hyperbola of the \( z \)-plane. It will be shown that varying portions of the \( u \)- and the \( v \)-axes map into the \( x \)-and the \( y \)-axes and the hyperbola—the portions depending on the values of \( a \) and \( b \).

In the following cases values have been assigned to \( a \) and \( b \) and figures have been drawn to show the behavior of the function with respect to the \( w \)-surface and the \( z \)-plane.
Case I

Let \( a \) be a constant, 0, and \( b \) be a variable taking on all positive and negative values and also zero. Since \( b \) is squared only the positive values need be considered.

Case I,

\[ b^2 > a^2 \]

Let \( a = 0 \) and \( b = 2 \) and the equation \( w^2 = z^2 + 2a^2 z^2 + b^2 \) reduces to the form:

\[ w^2 = z^2 + 4. \]

For convenience in following the maps denote the branch-point \( w = \pm b \) by \( M' \) and the corresponding critical values of \( z \) by \( M_{1,2,3,4} \); the point \( w = 0 \) by \( R' \) and the corresponding values of \( z \) by \( R_{1,2,3,4} \).

Substituting the values of \( a \) and \( b \) in the table on page 2:

\[ (M'), \ w = 2, \ \text{branch-point,--}\ z = 0, 0, 0, 0. \]

\[ (R'), \ w = 0, \ \text{ordinary point,--}\ z = \pm \sqrt{2} i, \ \pm \sqrt{2} i, \ \text{twice, which in complex form are:} \]

\( (1 + i), (1 - i), (-1 + i), (-1 - i). \)

These are obtained by setting \( (x + iy) = \sqrt{2} i \)

\( w = \infty, \ \text{branch-point,--}\ z = \infty. \)

The equation of the hyperbola becomes \( y^2 - x^2 = 0 \); this equation represents a pair of straight lines through the origin.

In order to determine the behavior of \( w \) as it describes a small circle about the branch-point \( w = 2 \) and of \( z \) as it turns about the corresponding critical point \( z = 0 \), in the equation \( w^2 - 4 = z^2 \) or
Case I,
$w^2 = z^4 + 4$

Plate I
let \( w - 2 = \rho e^{\phi i} \). Then \( w = \rho e^{\phi i} + 2 \) and \( w + 2 = \rho e^{\phi i} + 4 \); let \( z = \text{re}^{\phi i} \). Substituting these values, \( \rho e^{\phi i}(\rho e^{\phi i} + 4) = \text{r'e}^{\phi i} \).

Let \( \phi \) increase by \( \alpha \). Multiply both members of the equation by \( e^{\alpha i} \). Then \( \rho e^{\phi i}(\rho e^{\phi i} + 4) = \text{r'e}^{\phi i(\phi + \alpha i)} \).

Therefore for small values of \( \rho \) as \( \phi \) increases by \( \alpha \), \( \Theta \) increases by \( \frac{\alpha}{\sqrt{\rho}} \). That is, at the branch-point \( M' \), \( w \) describes a small circle four times as fast as \( z \) turns about the corresponding point \( M_{1,2,3,4} \).

To determine the behavior of \( w \) and \( z \) at \( R' \) and \( R_{1,2,3,4} \), in the equation \( \frac{w}{z} = 4 = (z + 2i)(z - 2i) = (z + \sqrt{2i})(z - \sqrt{2i})(z + 2i) \), let \( w = \rho e^{\phi i} \) and \( (z - 2i) = \text{re}^{\phi i} \). Then
\[
\rho e^{\phi i} = \text{re}^{i(\phi + 2\sqrt{2i})} (\text{r'e}^{-2\sqrt{2i}} + 4i)
\]
Let \( \phi \) increase by \( \alpha \), then \( \rho e^{\phi i} = \text{re}^{i(\phi + \alpha i) + \phi i} \).

Therefore as \( \Theta \) increases by \( \alpha \), \( \phi \) increases only half as fast. That is, at the origin \( R' \) on the \( w \)-surface, \( w \) turns only half as fast as \( z \) turns at \( R' \).

Let \( z \) describe the \( x \)-axis; then in \( z = x + iy \), \( y = 0 \) and therefore
\[
x'' = w'' - 4
\]
which shows that \( x \) is real if \( w > 2 \) or \( < -2 \). Therefore the \( x \)-axis maps into the \( u \)-axis beyond \( \pm 2 \). In the same manner it can be shown that the \( y \)-axis also maps into the \( u \)-axis beyond \( \pm 2 \). These parts are colored red. (See Plate I). Let \( z \) describe the hyperbola; then \( z \) is complex of the form \( k(1+i) \). If \( k = 1 \), \( w = 0 \), the point \( R' \). If \( k > 1 \), as \( 2 \), then \( w = 2^{\frac{1}{4}}(1 + i)^4 + 4 \), from which
which shows that \( w \) is imaginary for values of \( k > 1 \). Therefore the \( v \)-axis corresponds to the part of the hyperbola beyond the points \( R_{123y} \). These parts are colored green. If \( k < 1 \) as \( \frac{1}{2} \), then \( w = \frac{1}{16}(-4) + 4 = 3.75 \) from which \( w = \pm 2.75 \), real values between +2 and -2. Therefore the parts of the hyperbola between the origin and the points \( R_{123y} \) map into the \( u \)-axis between +2 and -2. These parts are colored black.

Make four cuts from \( M' \) to \( \infty \) along the \( u \)-axis, such that the sheets are connected thus: first and second; second and third; third and fourth; fourth and first.

We are now ready to trace out the regions of the \( z \)-plane which correspond to the sheets of the \( w \)-surface. (See Plate I).

If \( z \) is large, real or imaginary, positive or negative, in \( w^2 = z^2 + 4 \), \( w \) is large, real, and positive or negative. Consider only the positive value of \( w \). Following the yellow lines, as \( w \) comes in from large positive values and reaches \( M' \) and turns through 180° in the positive direction, \( z \) comes in from large positive values and reaches \( M_{123y} \) where it turns through 45°. When \( w \) reaches \( R' \) and turns through 90° \( z \) reaches \( R \) and goes straight ahead. If \( w \) travels in the negative direction about \( M' \) \( z \) comes in from large, positive, imaginary values along the \( y \)-axis and on reaching \( M_{123y} \) turns through negative 45° and goes out along the hyperbola. Region I⁺, I⁻ is thus determined. Regions II, III, IV are traced in a similar manner in blue, brown, and purple.
Case I,

\[ b^2 = a^4 \]

Let \( a = 0 \) and \( b = 0 \) and the equation reduces to the form:

\[ w = z^2 \]

which factors into \( w = z^2 \) and \( -w = z^2 \).

Considering the positive value of \( w \) only, the Riemann surface in this case has only two sheets with branch-points at 0 and \( \infty \).

\((M')\), \( w = 0 \), a branch-point, \( \infty \) = 0, 0, a double point.

\( w = \infty \), a branch-point, \( \infty \) = \( \infty \).

Let \( w = u + iv \) and \( z = x + iy \); then \( w = z \) becomes

\[ u + iv = x - y + 2xy \cdot \]

Equating real parts and imaginary parts:

\[ u = x - y^2 \]

\[ v = 2xy \]

Set \( w = \rho e^{\phi i} \) and \( z = r e^{\theta i} \). Then \( \rho e^{\phi i} = r e^{\theta i} \). Let \( \phi \) increase by \( \lambda \), then \( \rho e^{\phi i} = r e^{\theta i} \) and therefore for small values of \( \rho \) as \( \phi \) increases by \( \lambda \), \( \theta \) increases by \( \frac{\lambda \rho}{r} \) which means that \( w \) describes a small circle twice as fast as \( z \) does at the points \( M' \) and \( M/\lambda \).

Let \( z \) describe the \( x \)-axis. Then \( y = 0 \) and \( x^2 = w \). Therefore the \( x \)-axis corresponds to the positive end of the \( u \)-axis. These parts are colored red. (See Plate II).

Let \( z \) describe the \( y \)-axis. Then \( x = 0 \) and \( -y^2 = w \). Therefore \( y \) is real if \( w \leq 0 \) and so the \( y \)-axis corresponds to the negative end of the \( u \)-axis. These parts are colored green.
Case I,
\[ w = z^2 \]

z-plane

w-surface

Plate II
Let \( w \) describe the \( v \)-axis. Then \( u = 0 \) and \( x^2 - y^2 = 0 \), therefore the hyperbola \( x^2 - y^2 = 0 \) corresponds to the \( v \)-axis. These parts are colored black. (See Plate II).

Make a cut from 0 to \( \infty \) along the \( u \)-axis.

If \( z \) is large and real, \( w \) is large, real, and positive. Following the yellow lines, as \( w \) comes in from large, real, positive values in the first sheet and reaches \( M' \), \( z \) comes in from large, positive values and reaches \( M_\alpha \). If \( w \) turns through a positive 90° at \( M' \) and goes along the \( v \)-axis, \( z \) turns through positive 45° at \( M \), and goes along the line \( x = y \). If \( z \) comes in from large, negative values and turns through a negative angle at \( M_\alpha \), \( w \) turns through a negative angle twice as large at \( M' \). Region I+, I−, or the upper half of the \( z \)-plane thus corresponds to the first sheet of the \( w \)-surface.

Following the blue lines, as \( w \) comes in from large, real, positive values in the second sheet and goes in the direction of increasing angles about \( M' \), \( z \) comes in from large, negative values and turns through a positive angle at \( M_\alpha \). If \( w \) turns through a negative angle about \( M' \), \( z \) comes in from large, positive values and makes a negative angle about \( M_\alpha \). Region II+, II−, or the lower half of the \( z \)-plane is thus determined. It corresponds to the second sheet of the \( w \)-surface.
Case \( I_3 \)

\[ w^2 = z^2 - 4 \]

**z-plane**

I+  
II+  
III+  
IV+  

I-  
II-  
III-  
IV-  

**w-surface**

Plate III
Case I₃

Let $a = 0$ and $b$ be imaginary as $± 2i$; the equation becomes:

$$w^2 = z^2 - 4$$

Substituting these values of $a$ and $b$ in the table on page 2:

$(M')$, $w = 2i$, branch-point, $z = 0, 0, 0, 0$.

$(R')$, $w = 0$, ordinary point, $z = ±\sqrt{2}$, and $±i\sqrt{2}$.

$w = \infty$, branch-point, $z = \infty$.

As in Case I, $w$ turns about the branch-point $M'$ four times as fast as $z$ turns about the corresponding critical points, $M, \ldots$ and at $R'$, the origin, the usual relation between the angles is not preserved.

The $x$-axis beyond $±\sqrt{2}$ and the $y$-axis beyond $±i\sqrt{2}$ map into the $u$-axis. These parts are colored red. (See Plate III).

The $x$-axis between $±\sqrt{2}$ and $±i\sqrt{2}$ and the $y$-axis between $±\sqrt{2}$ and $±i\sqrt{2}$ map into the $v$-axis between $±2$ and $±2$. These parts are colored black.

The hyperbola $x^2 - y^2 = 0$ maps into the $v$-axis beyond $±2$.

These parts are colored green.

Make a cut from $M'$ along the $v$-axis to $\infty$. If $z$ is large and complex, as $(k + i)$, $w$ is large and imaginary. Following the yellow lines as $w$ comes in from large, imaginary values and reaches $M'$, in the first sheet, $z$ comes in from some point on $x = y$ and reaches $M$. If $w$ turns through positive $180°$, $z$ turns through positive $45°$. When $w$ is at $R'$, $z$ is at $R$, and as $w$ goes along the $u$-axis $z$ continues along the $x$-axis. Region $I+$ is thus traced and Region $I-$ is traced if $w$ and $z$ turn through negative angles.
Let $a$ be a constant, 1, and $b$ be a variable taking on all positive and negative values and also zero.

Case II.

$b^2 > a^2$

Let $a = 1$ and $b = 2$ and the equation $w^2 = z^2 + 2a^2 z^2 + b^2$ becomes

$$w^2 = z^2 + 2z^2 + 4.$$  

Substituting the values of $a$ and $b$ in the table on page 2:

(M'), $w = 2$, branch-point, $----- z = 0, 0$, and $\pm i\sqrt{2}$

(P'), $w = \sqrt{3}$, branch-point, $----- z = i, i$, and $-i, -i$.

(R'), $w = 0$, ordinary point, $----- z = \sqrt{2}(1 \pm i\sqrt{3}), \sqrt{2}(-1 \pm i\sqrt{3})$

$w = \infty$, branch-point, $----- z = \infty$.

The equation of the hyperbola becomes $y^2 - x^2 = 1$. The points $R_{1,2,4}$ satisfy this equation. In order to determine the behavior of $w$ as it describes a small circle about the branch-points set

$w - 2 = re^{i\phi}$ and $z = re^{i\phi}$ in the equation $w^2 - 4 = z^2 + 2z^2$. Factoring this equation and substituting the above values we have

$$r e^{i\phi}(r e^{i\phi} + 4) = r e^{i\phi}(r e^{i\phi} + 2).$$

Let $\phi$ increase by $\alpha$, then

$$r e^{i(\phi + \alpha)}(r e^{i(\phi + \alpha)} + 4) = r e^{i(\phi + \alpha)}(r e^{i(\phi + \alpha)} + 2).$$

Therefore as $\phi$ increases by $\alpha$, $\theta$ increases by $\frac{\alpha}{2}$, or $w$ describes a small circle at $M'$ twice as fast as $z$ at $M_{1,2,4}$. At the branch-point $\sqrt{3}$, or $P'$, $w$ again turns twice as fast as $z$ at $P_{1,2,4}$. To determine this put the equation in the form $z^2 + 2z^2 + 1 + 3 = w^2$, whence $(z^2 + 1)^2 = w^2 - 5 = 0$, if $w^2 = 3$. 


Let \( z \) describe the \( x \)-axis. Set \( z = x + iy \) and \( y = 0 \) and the equation becomes \( w^2 = x^2 + 2x + 4 \). Therefore \( w \) is real for all values of \( x \) which means that the \( x \)-axis corresponds to the \( u \)-axis or a part of it. Let \( w \) describe the \( u \)-axis; then \( z^2 + 2z + 4 - u^2 = 0 \) and this equation shows that \( z \) is real when \( u > 2 \) or \( u < -2 \). Therefore the \( x \)-axis maps into the \( u \)-axis beyond \( \pm 2 \). These parts are colored red. (See Plate IV).

If \( u \) is between \( \pm \sqrt{3} \) and \( \pm 2 \), \( z \) is imaginary and corresponds to the \( y \)-axis between \( M \), and \( M' \). These parts are colored black. The part of the \( y \)-axis beyond \( M \), and \( M' \) corresponds to the \( u \)-axis beyond \( \pm 2 \) and is colored red.

If \( u \) is between \( +\sqrt{3} \) and \( -\sqrt{3} \), \( z \) is complex and by testing points on the hyperbola as far as the points \( R, x, y \), it is found that they map into the axis of reals between \( +\sqrt{3} \) and \( -\sqrt{3} \). These parts are colored orange.

If \( w \) describes the \( v \)-axis then \( u = 0 \) and the equation is

\[
z^2 + 2z + 4 + v^2 = 0
\]

\[
z = -1 \pm \sqrt{-3 - v^2}
\]

which shows that \( z \) is complex. To see if the points of the hyperbola beyond \( R_{12} \), map into the \( v \)-axis, take some point on the hyperbola as \((1 \pm i\sqrt{2})\) and substitute in the given equation. Then

\[
w^2 = 1 \pm 4i\sqrt{2} - 12 \mp 8i\sqrt{2} + 1 + 2 \pm 4i\sqrt{2} - 4 + 4
\]

\[
w^2 = -8
\]

\[
w = \pm i\sqrt{8}, \quad \text{a pure imaginary.}
\]

These parts are colored green.
Case II,
\[ w^2 = z^4 + 2z^2 + 4 \]

Plate IV
Make a cut in the second and third sheets along the $u$-axis from $+2$ to $\infty$. Make a cut in the first and second sheets and another in the third and fourth sheets from $+\sqrt{3}$ to the origin and then along the $v$-axis to $\infty$. (See Plate IV).

If $z$ is large, positive or negative, real or imaginary, $z''$ is large and real and therefore $w$ is large and real.

We are now ready to trace out the regions of the $z$-plane which correspond to the four sheets of the $w$-surface.

Following the yellow lines in the first sheet: $w$ comes in from large, positive values and reaches $M'$, which is not a branch-point in the first sheet, $z$ is at $M$, in Region I; when $w$ reaches $P'$, a branch-point, $z$ is at $P, P_2$, a double point; if $w$ turns through $180^\circ$ about $P'$, $z$ turns through $90^\circ$ at $P, P_2$; when $w$ is at $R'$, $z$ is at $R_1$, and goes on along the hyperbola. Region I+ is thus determined if $w$ and $z$ travel in the direction of increasing angles and Region I− is mapped when $z$ and $w$ travel in the direction of decreasing angles. Region IV, in purple, is determined in the same manner.

Following the blue lines in the second sheet: $w$ comes in from large, positive values and reaches $M'$, a branch-point, $z$ is at $M_2 M_3$, a double point; if $w$ turns through $180^\circ$ about $M'$, $z$ turns through $90^\circ$ at $M_2 M_3$, $w$ reaches $P'$, a branch-point, $z$ is at $P, P_2$, a double point; if $w$ turns through $180^\circ$ about $P'$, $z$ turns through $90^\circ$ at $P, P_2$; $w$ reaches $R'$, $z$ is at $R_1$. Region II+ is thus determined when $z$ and $w$ travel through positive angles and Region II− is mapped when they go in the negative direction. Region III, in brown, is determined in the same manner.
Any continuous curve on the \( w \)-surface maps into a continuous curve on the \( z \)-plane. Below is a model of the \( w \)-surface showing \( w \) on a continuous curve. As \( w \) passes from one sheet to another \( z \) passes from one region to another.
Case $\text{II}_2$

\[ w = z^2 + 1 \]
Case II.
\[ a^2 = b^2 \]

Let \( a = 1 \) and \( b = 1 \) and the equation becomes
\[ w^2 = z^2 + 2z^2 + 1 = (z^2 + 1) \]
\[ w = z^2 + 1 \text{ and } -w = z^2 + 1. \]

Considering only the positive value of \( w \), the Riemann surface in this case consists of two sheets connected at \( 1 \) and \( \infty \).

Let \( w = u + iv \) and \( z = x + iy \) and \( w = z^2 + 1 \) becomes
\[ u + iv = x^2 - y^2 + 2xyi + 1. \]

Equating reals and imaginaries,
\[ u = x^2 - y^2 + 1 \text{ and } v = 2xy. \]

Let \( z \) describe the \( x \)-axis. Then \( y = 0 \) and \( w = x^2 + 1 \) and
\[ x^2 = w - 1. \] From this equation it is seen that \( x \) is real if \( w > 1 \).
Therefore the \( x \)-axis corresponds to the \( u \)-axis to the right of \( 1 \);
these parts are colored red. (See Plate V).

Let \( z \) describe the \( y \)-axis. Then \( x = 0 \) and \( w = -y^2 + 1 \) and
\[ y^2 = 1 - w. \] From this equation it is seen that \( y \) is real if \( w < 1 \).
Therefore the \( y \)-axis corresponds to the \( u \)-axis to the left of \( 1 \);
these parts are colored green.

Let \( w \) describe the \( v \)-axis. Then \( u = 0 \) and \( x^2 - y^2 = -1 \) or
\[ y^2 - x^2 = 1. \] Therefore the hyperbola corresponds to the \( v \)-axis;
these parts are colored black.

Make a cut in the \( w \)-surface from \( 1 \) to \( \infty \) along the \( u \)-axis.

As in similar cases, \( w \) describes a circle about a branch-
point twice as fast as \( z \) turns about the corresponding critical point.
Case $II_3$

$$w^2 = z^2 + 2z^2 + \frac{1}{4}$$

$z$-plane

$w$-surface

Plate VI
Case II_3

\[ b^2 < a \]

Let \( a = 1 \) and \( b = \frac{1}{2} \) and the equation becomes

\[ w^2 = z^4 + 2z^2 + \frac{1}{4} \]

Substituting the values of \( a \) and \( b \) in the table on page 2:

- \( (M') \), \( w = \frac{1}{2} \), branch-point
- \( (P') \), \( w = \frac{1}{3} \sqrt{3} \), branch-point
- \( (R') \), \( w = 0 \), ordinary point

\[ \pm \frac{1}{2} \sqrt{3} \quad \pm \frac{1}{2} \sqrt{3} \quad \frac{1}{2} \sqrt{3} \quad \frac{1}{2} \sqrt{3} \]

\[ w = \infty \], branch-point

\( w \) describes a small circle about the branch-points \( M' \) and \( P' \) twice as fast as \( z \) describes a small circle about the corresponding critical points.

The hyperbola maps into the \( v \)-axis beyond \( \pm \frac{1}{2} \sqrt{3} \). These parts are colored green. (See Plate VI).

The \( x \)-axis and the part of the \( y \)-axis beyond \( \pm \sqrt{2} \) map into the \( u \)-axis to the right of \( \frac{1}{2} \) and to the left of \( -\frac{1}{2} \); these parts are colored red; the part of the \( y \)-axis between \( R \) and \( R' \), and \( R \) and \( R'' \), map into the \( v \)-axis between \( P' \) and \( P'' \) and these parts are colored orange; the rest of the \( y \)-axis maps into the \( u \)-axis between \( \frac{1}{2} \) and \( -\frac{1}{2} \) and is colored black.

Make a cut in sheets two and three along the \( u \)-axis from \( M' \) to \( \infty \). Make a cut in sheets one and two along the \( v \)-axis from \( P' \) to \( \infty \) and a similar cut in sheets three and four.

If \( z \) is large, real or imaginary, positive or negative, \( w \) is large and real. The regions corresponding to the four sheets of the Riemann surface are traced in colors on Plate VI.
Case II.

Let $a = 1$ and $b = 0$ and the equation $w^2 = z^2 + 2az + b$ becomes

$$w^2 = z^2 + 2a.$$  

Substituting the values of $a$ and $b$ in the table on page 2:

$(M'), w = 0$, branch-point---------$z = 0, 0$, and $z/2$

$(P'), w = i$, branch-point---------$z = i, i$, and $-i, -i$.

$w = \infty$, branch-point---------$z = \infty$.

To determine the behavior of $w$ at the branch-points $M'$ and $P'$ where $w = 0$ and $w = i$ respectively, put the equation in the form

$$w^2 = z^2 (z + 2).$$

If $w = 0$, then $z = 0$ and $z + 2 = 0$.

Let $w = 0^1 \phi i$ and $z = r e^i$. Then $\phi = \frac{\pi}{3} e^i (r e^i + 2)$ and if $\phi$ increases by $\pi$, $\theta$ increases by the same amount, therefore at $M'$ and $M$, the angles increase at the same rate.

When $z + 2 = 0$, this part may be written $z + 1 + 1 = 0$ or

$$z^2 + 1 = -1 = \bar{w} \text{ if } w = \pm i.$$  

So at the branch-point $w = i$,

$$w^2 = z^2 + 1$$

$$(w - 1)(w + 1) = z^2.$$  

Let $w - 1 = \rho e^{\phi i}$ and let $z = r e^{\theta i}$. Then $\rho e^{\phi i} (\rho e^{\phi i} + 2) = r e^{\theta i}$ and if $\phi$ increases by $\pi$ then $\theta$ increases by $\frac{\pi}{2}$. Therefore at the branch-point $P'$, $w$ describes a small circle twice as fast as $z$ does at the points $P_{1, 3, 3, \gamma}$.

Make a cut in sheets two and three from $0$ to $\infty$ and a cut in sheets one and two along the $y$-axis from $i$ to $\infty$; make another cut from $i$ to $\infty$ in sheets three and four. (See Plate VII).
Case II
\[ w^2 = z^4 + 2z^2 \]

z-plane

III-

III+

II-

II+

M1

M2

M3

F1

F2

F3

III-

III+

F4

IV-

IV+

w-surface

Plate VII
The v-axis beyond \( P' \) and \( P'' \) corresponds to the hyperbola, \( y^2 - x^2 = 1 \); the y-axis above \( i/2 \) and below \(-i/2 \) maps into the u-axis. The y-axis between \( i/2 \) and \(-i/2 \) corresponds to the v-axis between \( P' \) and \( P'' \). The x-axis corresponds to the u-axis. The corresponding parts are colored the same.

If \( z \) is large, positive or negative, real or imaginary, \( w \) is large and real. Following the yellow lines in the first sheet: if \( z \) comes in from large positive, imaginary values, \( w \) comes in from large, positive values and reaches \( M' \) which is not a branch-point in the first sheet, then \( z \) is at \( M' \). When \( w \) reaches \( P' \) and turns through 180° in a positive direction, \( z \) turns through positive 90° at \( P' \), and goes along the hyperbola. Region I+ is thus determined and Region I− is traced when \( w \) and \( z \) travel in the negative direction. Region IV, in purple is determined in a similar manner.

Following the blue lines in the second sheet: when \( w \) is at \( M'' \), a branch-point, and turns through 90° in a positive direction, \( z \) is at \( M'' \), and turns through 90° also. But at \( P'' \) \( w \) turns twice as fast as \( z \) does at \( P' \). Region II+ is thus mapped out and Region II− is determined when \( w \) and \( z \) go in the negative direction. Region III, in brown, is traced in a similar manner. (See Plate VII).
Case II: $w^2 = z^2 + 2z^2 - 4$

Plate VIII
Let \( a = 1 \) and \( b \) be an imaginary quantity, as \( 2i \), and the equation \( w^2 = z^2 + 2az^2 + b^2 \) becomes \( w^2 = z^2 + 2z^2 = 4 \).

Substituting the values of \( a \) and \( b \) in the table on page 2:

\[
(M'), \ w = 2i, \ branch-point, \ z = 0, 0, \pm \sqrt{2}i
\]

\[
(P'), \ w = \sqrt{5}, \ branch-point, \ z = i, i, \ and -i, -i.
\]

\[
(R'), \ w = 0, \ ordinary \ point, \ z = \pm \sqrt{1+5}, \pm \sqrt{-1+5}
\]

\[
w = \infty, \ branch-point, \ z = \infty.
\]

The hyperbola, \( y^2 - x^2 = 1 \), corresponds to the part of the \( v \)-axis beyond \( R' \) and \( P'' \) and these parts are colored green. The parts of the \( x \)- and \( y \)-axes beyond the points \( R \) correspond to the \( u \)-axis and are colored red. The rest of the \( x \)- and \( y \)-axes correspond to the remaining part of the \( v \)-axis and are colored black and orange. (See Plate VIII).

At the branch-points \( M' \) and \( P' \), \( w \) describes a small circle twice as fast as \( z \) does at the corresponding critical points.

Make a cut in sheets two and three from \( M' \) to the origin and then along the \( u \)-axis to \( \infty \). Make a cut in sheets one and two and another in sheets three and four from \( P' \) along the \( v \)-axis to \( \infty \).

The regions corresponding to the different sheets are traced as in previous cases. It should be noted that \( R' \), which is on the cut in sheets two and three, corresponds to the points \( R \) which are on the border between Regions II and III.
Case III,
\[ w^2 = (z^2 - 1)^2 \]
Case III,

If a is imaginary as ± i and b = 1, the equation reduces to 
\[ w = z - 2z^2 + 1 = (z^2 - 1). \]

Solving the above equation, \(-w = z^2 - 1\) and \(w = z^2 - 1\).

Considering the positive value of \(w\) only, the Riemann surface in this case has only two sheets with branch-points at \(-1\) and \(\infty\).

Let \(w = u + iv\) and \(z = x + iy\). The equation then has the form 
\[ u + iv = x^2 - y^2 + 2xyi - 1. \]

Equating real parts and imaginary parts:
\[ u = x^2 - y^2 - 1, \text{ and } v = 2xy. \]

Let \(z\) describe the \(x\)-axis. Then \(y = 0\) and \(w = x^2 - 1\). Transposing \(1, w + 1 = x^2\). From this equation it is seen that \(x\) is real if \(w > -1\). Therefore the \(x\)-axis corresponds to the \(u\)-axis at the right of \(-1\); these parts are colored red.

Let \(z\) describe the \(y\)-axis. Then \(x = 0\) and \(w = -y^2 - 1\) and \(y^2 = -1 - w\). From this equation it is seen that \(y\) is real if \(w < -1\). These parts are colored green.

Let \(w\) describe the \(v\)-axis. Then \(u = 0\) and \(x^2 - y^2 = 1\). Therefore the hyperbola corresponds to the \(v\)-axis; these parts are colored black.

Make a cut in the \(w\)-surface from \(-1\) to \(\infty\) along the \(u\)-axis.

As in preceding cases \(w\) is found to describe a small circle about a branch-point twice as fast as \(z\) describes a small circle about the corresponding critical point. The upper half of the \(z\)-plane corresponds to the first sheet of the \(w\)-surface and the lower half of the \(z\)-plane to the second sheet of the \(w\)-surface. The mapping is shown on Plate IX.
Case $\text{III}_2$

$$w = z^3 - 3z^2 + 1$$

Plate X
Let $a = 2$ and $b = 1$ and the equation $w = z^2 + 2az + b$ becomes $w = z^2 - 8z + 1$.

Substitute the values of $a$ and $b$ in the table on page 2:

- $(M^{'})$, $w = 1$, branch-point, $z = 0, 0, \pm 2\sqrt{2}$
- $(P^{'})$, $w = i/\sqrt{15}$, branch-point, $z = 2, 2, \text{and} -2, -2$
- $(R^{'})$, $w = 0$, ordinary point, $z = \pm 2.3, \text{and} \pm 3.6$
- $w = \infty$, branch-point, $z = \infty$

The hyperbola, $x^2 - y^2 = 4$, corresponds to the $v$-axis beyond $P^{'1}$ and $P^{''1}$. (See Plate X). Make a cut along the $u$-axis in the second and third sheets from $M^{'1}$ to $\infty$, and a double cut along the $v$-axis from $P^{'1}$ to $\infty$. The regions corresponding to the different sheets of the Riemann surface may be traced as in previous cases.
Case III
\[ w = z^2 - \frac{1}{2}z^4 + 1 \]
Case III

Let $a = \frac{1}{2}$ and $b = 1$ and the equation $w^2 = z^2 + 2az + b$ becomes $w^2 = z^2 - \frac{1}{2}z^2 + 1$.

Substituting the values of $a$ and $b$ in the table on page 2:

$(M'), w = 1$, branch-point, $z = 0, 0, \pm \frac{\sqrt{2}}{2}$

$(P'), w = \frac{1}{\sqrt{15}}$, branch-point, $z = \frac{1}{3}, \frac{1}{6}, \text{ and } -\frac{1}{6}, -\frac{1}{3}$

$(R'), w = 0$, ordinary point, $z = (0.78 + 0.6i), (0.78 - 0.6i), (-0.78 + 0.6i), (-0.78 - 0.6i)$

$w = \infty$, branch-point, $z = \infty$.

The hyperbola, $x^2 - y^2 = \frac{1}{r}$, beyond the points $R$ corresponds to the entire $v$-axis and is colored green on Plate XI. The regions shown on Plate XI may be traced as in previous cases.
Case IV, \( w = z^2 + 4iz^2 \)

z-plane

w-surface

Plate XII
Case IV,

Let \( a \) be complex, as \((1 + i)\), and let \( b = 0 \) and the equation

\[ w^2 = z^4 + 2a z^2 + b^2 \]

reduces to \( w^2 = z^4 + 4iz^2 \).

Let \( w = u + iv \) and \( z = x + iy \), then:

\[ u^2 - v^2 + 2uv = x^4 + 4ix^3 - 6x^2y^2 - 4ixy^3 + y^4 + 4x^3 - 4y^3z - 8xy \]

and equating imaginary parts:

\[ 2uv = 4xy - 4x^3 + 4x^2 - 4y^2 = 4\left[ xy(x^2 - y^2) + (x^2 - y) \right] \]

\[ = 4\left[ (x^2 - y^2)(xy + 1) \right] \]

\[ uv = 2\left[ (x^2 - y^2)(xy + 1) \right] \]

If \( u \) or \( v \) equals zero then \( x^2 - y^2 = 0 \) and \( xy = -1 \).

Therefore the \( u \)-axis or the \( v \)-axis corresponds to the curves

\[ x^2 - y^2 = 0 \] and \( xy = -1. \)

Substituting the values of \( a \) and \( b \) in the table on page 21:

\[ (M'), w = 0, \text{ branch-point, } z = 0,0, \sqrt{2}(1 \pm i) \]

\[ (P'), w = 2, \text{ branch-point, } z = \pm 1(1 \pm i) \]

\[ w = \infty, \text{ branch-point, } z = \infty . \]

By testing points on the curves, the hyperbola is found to correspond to the \( u \)-axis beyond \( \pm 2 \) and these parts are colored red. The line \( x = y \) and the part of the line \( x = -y \) beyond \( M \), and \( M' \), correspond to the \( v \)-axis and are colored green. The part of the line \( x = -y \) between \( M \) and \( M' \), corresponds to the \( u \)-axis between \( \pm 2 \) and these parts are colored black. (See Plate XII).

Make a cut along the \( u \)-axis from \( P' \) to \( \infty \) in sheets one and two and another in sheets three and four. Make a cut along the \( v \)-axis from \( M' \) to \( \infty \) in sheets two and three. The regions are traced on Plate XII.
Case IV
\[ w^2 = z^4 + 4iz^2 + 1 \]

Plate XIII
Case IVa

Let \( a \) be complex, as \((1 + i)\), and let \( b = 1 \) and the equation
\[ w^2 = z^2 + 2a^2 z^2 + b^2 \]
reduces to \( w^2 = z^2 + 4iz^2 + 1. \)

As in Case IV set \( w = u + iv \) and \( z = x + iy \) and by equating
the imaginary parts we again have \( uv = 2 \left[ (x - y)(xy + 1) \right] \), and if
\( u \) or \( v \) equals zero then \( x^2 - y^2 = 0 \) and \( xy = -1. \)

Substituting the values of \( a \) and \( b \) in the table on page 21:

\[
\begin{align*}
(M'), w &= 1, \text{branch-point}, \quad z = 0,0, \sqrt{2}(-1 + i), \sqrt{2}(-1 - i) \\
(P'), w &= \sqrt{5}, \text{branch-point}, \quad z = (1 - i)(1 + i), (-1 + i)(-1 - i) \\
(R'), w &= 0, \text{ordinary point}, \quad z = \pm 1.45(-1 + i) \text{ etc. and} \quad \\
& \quad \pm 34(1 + i) \text{ etc. } \\
\end{align*}
\]

\[ w = \infty, \text{branch-point}, \quad z = \infty \]

The mapping is shown on Plate XIII. By testing points on the
curves the hyperbola \( xy = -1 \) is found to correspond to the \( u \)-axis
beyond \( \pm \sqrt{5} \); these parts are colored red. Points on the line
\( x = -y \) beyond \( R \), and \( R \), correspond to the \( v \)-axis and are colored
green. Points on the line \( x = y \) beyond the points \( R \) also corre-
spond to the \( v \)-axis. Points between \( M \) and \( R \) on the lines \( x = -y \)
and \( x = y \) correspond to the part of the \( u \)-axis between \( M' \) and \( M'' \)
and are colored orange. Points on \( x = -y \) between \( M' \) and \( M'' \), and \( M'' \) and \( P'' \),
and are colored black.

Make a cut along the \( u \)-axis from \( P' \) to \( \infty \) in sheets one and
two and another cut in sheets three and four. Make a cut from \( M' \)
along the \( u \)-axis to \( P' \) and then along the \( v \)-axis to \( \infty \) in sheets
two and three. The regions are traced in colors on Plate XIII.
Case IV₃

\[ w = z^2 + 2i \]
Case IV

Let \( a \) be complex, as \((1 + i)\), and \( b \) be imaginary, as \(2i\). The equation \( w^2 = z^2 + 2a^2z + b^2 \) becomes

\[
w^2 = z^2 + 4iz^2 - 4 = (z^2 + 2i)^2
\]

\( w = z^2 + 2i \) and \(-w = z^2 + 2i\).

Considering only the positive value of \( w \), the Riemann surface in this case has only two sheets; these are connected at \( \pm 2i \) and \( \infty \).

Let \( w = u + iv \) and \( z = x + iy \) and then

\[
u + iv = x^2 - y^2 + 2xyi + 2i.
\]

Equating real parts and imaginary parts:

\[
u = x^2 - y^2 \text{ and } v = 2(xy + 1).
\]

If \( u = 0 \), \( x^2 - y^2 = 0 \); and if \( v = 0 \), \( xy = -1 \).

Therefore \( x^2 - y^2 = 0 \) corresponds to the \( v \)-axis. By testing points, the line \( x = y \) is found to correspond to the \( v \)-axis above \( M' \) and the line \( x = -y \) to the \( v \)-axis below \( M' \); these parts are colored black. (See Plate XIV).

\[xy = -1 \text{ corresponds to the } u \text{-axis and is colored red.}
\]

The intercepts of \( xy = -1 \) with \( x = -y \) are at \((1 - i)\) and \((-1 + i)\).

\((M')\), \( w = 2 \), branch-point, \(-\infty(z = 0,0)\).

\((R')\), \( w = 0 \), ordinary point, \(-\infty(z = (1 - i) \text{ and } (-1 + i)\).

\( w = \infty \), branch-point, \(-\infty(z = \infty\).

Make a cut from \( M' \) along the \( v \)-axis to \( \infty \). If \( z \) comes in on the line \( x = y \), \( w \) is large and imaginary. (See Plate XIV).
Case V
\[ w^2 = z^2 + 2i \]
Case V.

Let \( a = 0 \) and \( b = (1 + i) \). Then the equation \( w^2 = z' + 2az + b^2 \) reduces to \( w^2 = z' + 2i \).

Substituting the values of \( a \) and \( b \) in the table on page 2:

\((M'), \ w = (1 + i), \ \text{branch-point}, \ -z = 0,0,0,0.\)

\((R'), \ w = 0, \ \text{ordinary point}, \ -z = \pm 1,45 \left[ (1 - \sqrt{2}) + i \right] \)

\(w = \infty, \ \text{branch-point}, \ -z = \infty\).

Let \( w = u + iv \) and \( z = x + iy \); then

\[ u^2 - v^2 + 2uv i = x' + 4xy i - 6x'y - 4xy^2 + y' + 2i. \]

Equating real parts:

\[ u^2 - v^2 = x' - 6x'y + y' \]

If \( u = iv, \ x' - 6x'y + y' = 0; \) or \((x^2 - y^2 - 2xy)(x^2 - y^2 + 2xy) = 0.\)

\[ \begin{cases} x + y \pm y/2 = 0 \\ x - y \pm y/2 = 0 \end{cases} \]

Pairs of straight lines through the origin.

By the method for rotating axes, from the second factor:

\[ \tan 2\phi = \frac{B}{A - 0} = \frac{2}{-1 + 1} = 1 \]

\(2\phi = 45^\circ\) and \( \phi = 22.5^\circ. \) If the axes are rotated through positive \( 22.5^\circ \) the above equation reduces to \( x_1^2 - y_1^2 = 0 \), the degenerate form of an equilateral hyperbola. If the axes are rotated through a negative \( 22.5^\circ \) the other factor reduces to \( x_1^2 - y_1^2 = 0.\)

The line \( u = v \) corresponds to the pair of lines \( x - y \pm y/2 = 0 \)

The line \( u = -v \) corresponds to the pair of lines \( x + y \pm y/2 = 0 \).

The mapping is shown on Plate XV.
The General Equation

Consider the general equation \( w = (1 - z^2)(1 - k^2 z^2) \)
which may be written in the form \( w = z^2 - \frac{(1 + k^2)z^2}{k^2} + \frac{1}{k^2} \)
and the equation \( w = z^2 + 2a^2 z + b^2 \). Comparing the two equations:
\[
\frac{1}{1 + k^2} = \frac{1}{-2k^2}, \quad \text{and} \quad b = \frac{1}{k^2}.
\]

Substituting the values of \( a \) and \( b \) in the table on page 21:

\[ (M') \), \( w = \frac{1}{k} \), branch-point, \( z = 0, 0, \pm \frac{1 + k^2}{k^2} \).
\[ (P') \), \( w = i \frac{(1 - k^2)}{2k^2} \), branch-point, \( z = \pm \frac{1 + k^2}{2k^2} \), twice.
\[ (R') \), \( w = 0 \), ordinary point, \( z = \pm 1, \pm \frac{1}{k} \).
\]

\( w = \infty \), branch-point, \( z = \infty \).

The equation of the hyperbola is \( x^2 - y^2 = \frac{1 + k^2}{2k^2} \), and its intercepts are \( \pm \frac{1 + k^2}{2k^2} \).

Case where \( k \) is real.

(a). Let \( k = 0 \). Then \( w \) and \( z \) become infinite. The only branch-point is at infinity.
\[ w = z^2 - \frac{5z^2}{4} + \frac{1}{4} \]

**Plate XVI**
Case when $k$ is real.

(b). Let $k = \pm 2$. Then the equation becomes $w = z^2 - \frac{5z^2}{4} + \frac{1}{4}$

Substituting the values of $a$ and $b$ in the table on page 2:

$(M')$, $w = \pm \frac{1}{2}$, branch-points, $z = 0, 0, \pm \frac{\sqrt{5}}{4}$

$(P')$, $w = \pm \frac{2}{3}i$, branch-points, $z = \pm \frac{\sqrt{5}}{3}$ twice.

$(R')$, $w = 0$, ordinary point, $z = \pm \frac{1}{2}, \pm 1$

$w = \infty$, branch-point, $z = \infty$.

The equation of the hyperbola is $x^2 - y^2 = \frac{5}{6}$; its intercepts are on the x-axis.

The mapping is shown on Plate XVI.
$w^2 = z^4 - 2z^2 + 1$

Plate XVII
Case where $k$ is real.

(c). Let $k = \pm 1$. Then the equation becomes $w^2 = z^2 - 2z + 1$ or

$$w^2 = (z^2 - 1)^2$$

$w = z^2 - 1$, and $-w = z^2 - 1$.

Considering only the positive value of $w$, the Riemann surface in this case has only two sheets. The sheets are connected at $-1$ and $\infty$.

$(M')$, $w = -1$, branch-point, $z = 0, 0$.

$(R')$, $w = 0$, ordinary point, $z = \pm 1$.

$w = \infty$, branch-point, $z = \infty$.

The mapping is shown on Plate XVII.
Case where $k$ is imaginary.

Let $k = \pm \frac{1}{2} i$. Then the equation becomes $w^2 = z^2 + 3z^2 - 4$.

Substituting the values of $a$ and $b$ in the table on page 2:

$(M'), w = 2i$, branch-point, $z = 0, 0, \pm i/3$.

$(P'), w = 5i$, branch-point, $z = \pm i/2$ twice.

$(R'), w = 0$, ordinary point, $z = \pm 1$ or $\pm 2i$.

$w = \infty$, branch-point, $z = \infty$.

The mapping is shown on Plate XVIII.
Case where $k$ is imaginary.

Let $k = \pm i$. Then the equation becomes $w^2 = z^2 - 1$.

Substituting the values of $a$ and $b$ in the table on page 2:

$(M'), w = i$, branch-point, $z = 0, 0, 0, 0$.

$(R'), w = 0$, ordinary point, $z = \pm 1$, and $\pm i$.

$w = \infty$, branch-point, $z = \infty$.

In this case the hyperbola becomes a pair of straight lines through the origin.

The mapping is shown on Plate XIX.
BIBLIOGRAPHY

Ashton, Dr. C.H.—Class Lectures in Function Theory, 1932.

   (Translation by Fisher & Schwatt.)