CERTAIN PROPOSITIONS of the TRIANGLE.

by

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Submitted to the Department of Mathematics and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Master of Arts.

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June, 1932.
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CERTAIN PROPOSITIONS of the TRIANGLE

**Introduction**

The pages of the Histories of Mathematics are filled with propositions which have been handed down from age to age. Some of these propositions are true, others false. Without doubt each merits some attention, since, in every case, there are underlying principles which lead to a better understanding of mathematics.

In his "Recreations in Mathematics", Lick suggests this interesting proposition:

"Let ABC be any triangle. On the sides as bases construct the isosceles triangles AcB, BaC, CbA, so that the equal sides of each make angles of 30° with its base. Draw ab, ac, bc. Then each of the angles of the triangles abc is 60°." Is this proposition true or false?

The first part of this paper gives several methods of proof of the above proposition. It also

# Lick's, Recreations in Mathematics, p. 58.
proves that the joins of the vertices of the set of isosceles triangles whose equal sides make 30° angles internally with the sides of triangle ABC form a second equilateral triangle. The second part of the paper sets forth some interesting properties discovered in the proof of the original proposition.

The proposition is proved by four different methods, namely: plane geometry, analytical geometry, vector analysis, and trigonometry.
Fig. 2
Section I
Solution of the Problem

Theorem I. If isosceles triangles whose equal angles are 30° angles are constructed on the sides of any triangle as bases, (a) the lines joining the vertices of the isosceles triangles constructed externally form an equilateral triangle, and (b) the lines joining the vertices of the isosceles triangles constructed internally form an equilateral triangle.

A. Proof by Plane Geometry.

Given a triangle ABC.

Construct the external isosceles triangles AR, B, AR, C, BR, C, and the internal isosceles triangles AR, B, AR, C, BR, C, so that the angles \( P_1 AB, P_2 BA, P_3 AB, P_4 BA, P_5 CB, P_6 BC, P_7 BC, P_8 CB, P_9 CA, P_9 AC, P_9 CA, P_9 AC \) are each equal to 30°.

Draw the lines \( P_1 P_1', P_2 P_2', P_3 P_3', P_4 P_4', P_5 P_5', P_6 P_6', P_7 P_7', P_8 P_8', P_9 P_9' \).

To prove the triangles \( P, P_1 P_2 \) and \( P' P_1 P_2' \) are equilateral.
Proof of (b):

\[ \angle BAP_1 = \angle A + 30^\circ. \]
\[ \angle BAP_2 = 30^\circ. \]
\[ \angle P_2'AP_1 = \angle BAP_1 - BAP_2' = \angle A + 30^\circ - 30^\circ = \angle A. \]

By symmetry,
\[ \angle P_1'AP_3 = \angle A. \]
\[ \angle BBP_2' = \angle B. \]
\[ \angle P_2'BP_3' = \angle B. \]
\[ \angle P_2'CP_1 = \angle C. \]
\[ \angle P_2'CP_1' = \angle C. \]

By construction, where \(a, b, c\) are opposite \(\triangle A, B, C\) respectively.

\[ CE = EB = \frac{a}{2}. \]
\[ AD = DC = \frac{b}{2}. \]
\[ AF = FB = \frac{c}{2}. \]

\[ BP_2 = CP_2 = BP_2' = CP_2' = \frac{a\sqrt{3}}{3}. \]
\[ AP_1 = CP_1 = AP_1' = CP_1' = \frac{b\sqrt{3}}{3}. \]
\[ AP_3 = BP_3 = AP_3' = BP_3' = \frac{c\sqrt{3}}{3}. \]
Then

\[ \frac{P_1'C}{AC} = \frac{\sqrt{3}S}{5} = \frac{\sqrt{3}}{5} \]

\[ \frac{P_2'C}{BC} = \frac{\sqrt{3}S}{5} = \frac{\sqrt{3}}{5} \]

Therefore

\[ \triangle P_1'P_2'C \cong \triangle ABC. \]

By the same method of proof, triangles \( P_1P_2'C \), \( P_3'BP_2 \), \( P_4'BP_2 \), \( AP_3P_1' \), \( AP_3'P \), are also similar to triangle \( ABC \).

Furthermore, it can be shown that the triangles \( AP_3'P \), \( AP_3P_1' \), \( P_1P_2'C \), \( P_3'P_2'C \), \( P_3'BP_2 \), and \( P_4'BP_2 \) are congruent.

Hence

\[ P_1P_2 = P_3P_4' \]
\[ P_1'P_2 = P_2'P_4' \]
\[ P_2'P_3 = P_3'P_2' \]

Also

\[ P_3P_4' \perp AP_3 \]
\[ P_2P_4' \perp BC \]
\[ P_1P_4' \perp AC \]

and

\[ P_3P_4' = AP_3' = \frac{\sqrt{3}}{5} \] 
\[ \angle P_1'AP_3' = 60' - \angle A \]
\[ \angle P_3'P_2P_4' = 60' - \angle A. \]
But from triangles \( P_3 P_2' \) and \( AP_3' P_1 \)
\[
P_3 P_2' = AP_3' = AP_1'.
\]
Thus the triangles \( P_3' P_2' \) and \( P_3' AP_1' \) are congruent, and
\[
P_3' P_2' = P_3' P_1'.
\]
In a similar way it can be shown that \( P_2' P_3' = P_2' P_2' \).
Therefore
\[
\triangle P_2' P_3' \triangle P_3' P_2' \quad \text{is equilateral.}
\]

Proof of (a):

\[
\angle P_1 P_3' P_5 = \angle P_1 P_5' A + \angle AP_3' P_5 = \angle B + 60°.
\]
\[
\angle P_1 P_2' P_2 = \angle P_1 P_2' C + \angle OP_2' P_2 = \angle B + 60°.
\]
By congruent triangles
\[
P_1 P_2' = P_2' U.
\]
But
\[
P_2' C = P_2' P_1.
\]
Hence
\[
P_1 P_3' = P_2' P_2'.
\]
Similarly
\[
P_2' P_3' = P_2' P_2', \quad \text{and the triangles}
\]
\( P_1 P_2' P_3' \) and \( P_2' P_2' P_2' \) are congruent.
Likewise
\[
\triangle P_2' P_3' P_2' \quad \text{and} \quad P_2' P_2' P_2' \quad \text{are congruent,}
\]
and
\[
P_1 P_2 = P_2 P_3 = P_2 P_3'.
\]
This proves that the triangle $P_1P_2P_3$ is equilateral.

**Corollary.** If triangle ABC is equilateral, $AP_1' = AP_2'$, $BE_1' = BE_2'$, $CP_1' = CP_2'$, and the triangle $P_1'P_2'P_3'$ disappears.

**B. Proof by Plane Geometry.**

The following is another interesting proof by plane geometry that triangle $P_1P_2P_3$ is equilateral.

Given a triangle ABC.

Construct the triangles $AP_1A'$, $AP_2A'$, $BP_2B'$ isosceles with equal angles equal to $30^\circ$; circles $D, E, F$ with centers at $P_1, P_2, P_3$ and radii $P_1C = AP_1$, $P_2C = P_2B'$, $P_3A = P_3B$ respectively.

The circles $D$ and $E$ intersect at $C$ and $C'$, circles $D$ and $F$ at $A$ and $A'$, and circles $E$ and $F$ at $B$ and $B'$.

Hence, the following angles are equal:

$z' = z$, $y' = y$, $x' = x$,

$w' = w$, $v' = v$, $u' = u$,

and

$P_1CP_2 = P_3C'P_2$. 
\[ \begin{align*}
P_2 B'P &= P_2 B'P_1, \\
P_1 A'P &= P_1 A'P_2.
\end{align*} \]

**Lemma 1.** Either the points \( A', B', C' \) are all distinct or they are identical.

**Proof:**

Let \( A' \) and \( B' \) coincide.

Since \( A' \) is on the circles \( D \) and \( F \), and \( B' \) is on the circles \( E \) and \( F \), the three circles intersect in the point \( A' = B' \).

But \( C' \) is on the circles \( D \) and \( E \).

Hence

\[ C' = A' = B'. \]

Therefore, the points \( A', B', C' \) are either all distinct or identical.

**Lemma 2.** If the points \( A', B', C' \) are distinct, then the sectors \( C'B_1A', A'B_2B', B'C_2C' \) of the three circles are all common respectively to the pairs of triangles \( C'B_1C \) and \( A'B_2A \), \( A'B_1B \) and \( E'B_2B \), \( B'C_2C \), or no one of these sectors is included in any of the triangles.

**Proof:**

Let the three points be distinct and
designate the angles $C', B, B', A', P, B'$ by $\beta$, $\gamma$, and $\delta$ respectively.

Since the points are distinct

$P_1, A' = P_1, B'$ is $\geq$ or $\leq P_1, C'$.

Consider first $P_1, A' = P_1, B' \geq P_1, C'$.

The points $P_1$ and $A'$ lie outside of circle $E$ since the two circles $D$ and $E$ can intersect in only two actual points, and those points are $C$ and $C'$.

This must be true, otherwise $P_1, A' > P_1, C'$.

In a similar way it can be shown that $B'$ is outside of circle $D$.

Hence, no one of the sectors is a part of any of the triangles in question.

Now consider $P_1, A' = P_1, B' \geq P_1, C'$.

Then $B'$ is on the arc $CC'$ of circle $E$, and $A'$ is on the arc $CC'$ of circle $D$ by the same method of reasoning as in the previous case. If this is true, then the sectors are all common respectively to the pairs of triangles. Thus the lemma is proved.

But

$$z + y + c + 60' = 180'$$

$$u + v + a + 60' = 180'$$

$$x + w + b + 60' = 180'$$
Hence
\[ z' + y' = 120^\circ - C, \]
\[ u' + v' = 120^\circ - A, \]
\[ x' + u' = 120^\circ - B, \]
and
\[ x' + y' + u' + v' + w' + z' = 180^\circ. \]

But
\[ x' + y' + u' + v' + w' + z' + (\beta + \gamma + \delta) = 180^\circ. \]

That is, by the lemma, the three angles must either all be added or subtracted.

Then by Lemma 2,
\[ \beta = y = \delta = 0^\circ, \]
and by Lemma 1,
\[ A' \equiv B' \equiv C'. \]

Then
\[ \angle P_1 P_2 P_3 = z' + u', \]
\[ \angle P_2 P_3 P_1 = u' + w', \]
\[ \angle P_3 P_1 P_2 = x' + y'. \]

But
\[ \angle C P_3 A = 2z' + 2u' = 120^\circ, \]
from which
\[ z' + u' = 60^\circ. \]

Similarly
\[ u' + w' = 60^\circ, \]
and

\[ x' + y' = 60'. \]

Therefore, the triangle \( P_1 P_2 P_3 \) is equiangular.

C. Proof by Analytical Geometry.

The proof by analytical geometry is of interest chiefly in that it leads most readily to other interesting properties. It is for this reason that the various elements of the proposition are set forth in some detail.

Let the given triangle be \( OAB \) with vertices whose coordinates are:

\[ O(0,0), \ A(a,c), \ B(b,c). \]

The equations of the sides of the triangle \( OAB \) are:

- \( OA \): \( y = c \).
- \( AB \): \( cx + (a-b)y = ac \).
- \( OB \): \( cx - by = c \).

The equations of the sides of the isosceles triangles are:

- \( P_1 A \): \( \sqrt{3}x - 3y = a\sqrt{3} \).
- \( P_1 O \): \( \sqrt{3}x + 3y = 0 \).
\[ P_2 A: (a\sqrt{3} - b\sqrt{3} + 3c)x + (3a - 3b - cv\sqrt{3})y = 3ac + a^2\sqrt{3} - ab\sqrt{3}. \]

\[ P_2 B: (-a\sqrt{3} + b\sqrt{3} + 3c)x + (3a - 3b + cv\sqrt{3})y = 3ac - ab\sqrt{3} + b^2\sqrt{3} + c^2\sqrt{3}. \]

\[ P_2 P: (-b\sqrt{3} + 3c)x - (3b + cv\sqrt{3})y = -b^2\sqrt{3} - c^2\sqrt{3}. \]

\[ P_2 O: (b\sqrt{3} + 3c)x - (3b - cv\sqrt{3})y = 0. \]

\[ P_2' A: \sqrt{3}x + 3y = a\sqrt{3}. \]

\[ P_2' O: \sqrt{3}x - 3y = 0. \]

\[ P_2' A: (-a\sqrt{3} + b\sqrt{3} + 3c)x + (3a - 3b + cv\sqrt{3})y = 3ac - a^2\sqrt{3} + ab\sqrt{3}. \]

\[ P_2' B: (a\sqrt{3} - b\sqrt{3} + 3c)x + (3a - 3b - cv\sqrt{3})y = 3ac + ab\sqrt{3} - b^2\sqrt{3} - c^2\sqrt{3}. \]

\[ P_3' B: (b\sqrt{3} + 3c)x + (-3b + cv\sqrt{3})y = b^2\sqrt{3} + c^2\sqrt{3}. \]

\[ P_2' O: (-b\sqrt{3} + 3c)x - (3b + cv\sqrt{3})y = 0. \]

The coordinates of the points \( P_1, P_2, P_3, \) and \( P_1', P_2', P_3' \) are:

\[ P_1 \left( \frac{3a}{6}, \frac{-a\sqrt{3}}{6} \right), \]

\[ P_2 \left( \frac{3a + 3b + cv\sqrt{3}}{6}, \frac{a\sqrt{3} - b\sqrt{3} + 3c}{6} \right), \]

\[ P_3 \left( \frac{3b}{6}, \frac{b\sqrt{3} + 3c}{6} \right). \]
The equations of the sides of the triangle \( P, P_2, P_3 \) and \( P', P_2', P_3' \) are:

\[
P, P_2 : (2a\sqrt{3} - b\sqrt{3} + 3c)x - (3b + cv3)y = a^2\sqrt{3} + 2ac.
\]

\[
P, P_3 : (a\sqrt{3} + b\sqrt{3} + 3c)x + (3a - 3b + cv3)y = ab\sqrt{3} + ac.
\]

\[
P_2, P_3 : (-a\sqrt{3} + 2b\sqrt{3})x + (3a + 2cv3)y = 2ac + b^2\sqrt{3} + c^2\sqrt{3}.
\]

\[
P, P_2' : (2a\sqrt{3} - b\sqrt{3} - 3c)x + (3b - cv3)y = a^2\sqrt{3} - 2ac.
\]

\[
P, P_3' : (a\sqrt{3} + b\sqrt{3} - 3c)x + (-3a + 3b + cv3)y = ab\sqrt{3} - ac.
\]

\[
P_2, P_3' : (-a\sqrt{3} + 2b\sqrt{3})x + (-3a + 2cv3)y = -2ac + b^2\sqrt{3} + c^2\sqrt{3}.
\]

The length of the sides of the triangles \( P, P_2, P_3 \) and \( P', P_2', P_3' \) respectively are:

\[
\frac{1}{3} \sqrt{3(a^2 + b^2 + c^2 - ab + ac\sqrt{3})},
\]

and

\[
\frac{1}{3} \sqrt{3(a^2 + b^2 + c^2 - ab - ac\sqrt{3})}.
\]

Therefore, the triangles are equilateral.
D. Proof by Vector Analysis.

The external and internal isosceles triangles are discussed separately as in the "A" proof by plane geometry.

\textbf{Proof of (a):}

Let the vectors on the sides $AB$, $AC$, and $BC$ be $a\mathbf{\alpha}$, $b\mathbf{\beta}$, and $c\mathbf{\gamma}$ respectively, where $\mathbf{\alpha}$, $\mathbf{\beta}$, and $\mathbf{\gamma}$ are unit vectors and $a$, $b$, $c$ are the tensors.

Then

$$c\mathbf{\gamma} = -a\mathbf{\alpha} + b\mathbf{\beta}.$$ 

Let the mid points of $AB$, $AC$, and $BC$ be $D$, $F$, and $E$ respectively.

The vectors representing $AD$, $AF$, and $BE$ are

$$\frac{a\mathbf{\alpha}}{2}, \frac{b\mathbf{\beta}}{2}, -\frac{a\mathbf{\alpha} + b\mathbf{\beta}}{2}.$$

The tensors of $D_1^\mathbf{\alpha}$, $F_2^\mathbf{\beta}$, and $E_2^\mathbf{\gamma}$ are

$$\frac{a\sqrt{3}}{3}, \frac{b\sqrt{3}}{3}, \text{ and } \frac{c\sqrt{3}}{3}$$

respectively.

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Since the angle $\angle BD &= 90^\circ$, the vector of $DP$ is
\[ \frac{-a\sqrt{3} \times i}{6} \]

Similarly, the vectors of $EP_2$ and $FP_3$ are
\[ \frac{(a - b \beta) \sqrt{3} i}{6}, \text{ and } \frac{b\sqrt{3} \beta i}{6} \]

Then, by vector addition
\[ AP_1 = \frac{3a \alpha - a\sqrt{3} \times i}{6} \]
\[ AP_2 = \frac{3a \alpha + 3b \beta + (a - b \beta) \sqrt{3} i}{6} \]
\[ AP_3 = \frac{3b \beta + b\sqrt{3} \beta i}{6} \]
\[ P_1 P_2 = \frac{2a\sqrt{3} \times i + 3b \beta + b\sqrt{3} \beta i}{6} \]
\[ P_1 P_3 = \frac{-3a \alpha - a\sqrt{3} \times i + 3b \beta + b\sqrt{3} \beta i}{6} \]
\[ P_2 P_3 = \frac{-3a \alpha - a\sqrt{3} \times i + 2b\sqrt{3} \beta i}{6} \]

Drop a perpendicular from $P_1$ to $P_2 P_3$. Let $R$ be the point at which this perpendicular intersects $P_2 P_3$.

Then
\[ P_1 R = -x(P_2 P_3) i, \text{ where } x \text{ is the tensor.} \]
\[ P_1 R = -x\left( \frac{-3a \alpha - a\sqrt{3} \times i + 2b\sqrt{3} \beta i}{6} \right) i \]
\[ P_2 R = y(P_2 P_3), \text{ where } y \text{ is the tensor.} \]
\[ P_2 \mathbf{R} = \mathbf{y} \left( \frac{-3a \mathbf{x} + ay\sqrt{3} \mathbf{i} + 2b\sqrt{3} \beta \mathbf{i}}{6} \right) \]

But

\[ P_2 \mathbf{R} + \mathbf{R} \mathbf{P}_3 = P_2 \mathbf{P}_3 \]

\[ \mathbf{R} \mathbf{P}_3 = P_2 \mathbf{P}_3 - P_2 \mathbf{R} \]

\[ \mathbf{R} \mathbf{P}_3 = \frac{-3a \mathbf{x} + sy\sqrt{-a\mathbf{v}3} \mathbf{i} + ay\mathbf{v}3 \mathbf{x} \mathbf{i} + 2b\mathbf{v}3 \beta \mathbf{i}}{6} \quad \frac{-2by\mathbf{v}3 \beta \mathbf{i}}{6} \]

\[ \mathbf{R} \mathbf{P}_3 = (1-y) \left( \frac{-3a \mathbf{x} - ay\sqrt{3} \mathbf{i} + 2b\mathbf{v}3 \beta \mathbf{i}}{6} \right) \]

Similarly:

\[ P_1 \mathbf{P}_2 = \frac{3ax \mathbf{x} + ax\sqrt{3} \mathbf{i} - 2b\mathbf{v}3 \beta \mathbf{i} + 3ay \mathbf{x}}{6} + \frac{ay\sqrt{3} \mathbf{i} - 2by\sqrt{3} \beta \mathbf{i}}{6} \]

or

\[ \frac{2a\sqrt{3} \mathbf{i} + 3b \beta - b\mathbf{v}3 \beta \mathbf{i}}{6} = \frac{-ax\sqrt{3} \mathbf{x} + 3ax \mathbf{x} \mathbf{i} + 2b\mathbf{v}3 \beta + 3ay \mathbf{x} + ay\sqrt{3} \mathbf{x} \mathbf{i}}{6} \quad \frac{-2by\sqrt{3} \beta \mathbf{i}}{6} \]

Equating tensors of like vectors:

\[ 2a\sqrt{3} = 3ax + ay\sqrt{3} \]

\[ 3b = 2b\mathbf{v}3 \]

\[ b\mathbf{v}3 = 2by\sqrt{3} \]

\[ ax\sqrt{3} = 3ay \]
Hence, \( R \) is the midpoint of \( P_2 P_3 \), and
\[ \angle RP_2 P = 60^\circ. \]
Thus, the triangle \( P_1 P_2 P_3 \) is equilateral.

**Proof of (b):**

As in proof (a), it can be shown that
\[
\begin{align*}
P_1'P_2' &= \frac{-3a\alpha - a\sqrt{3} \alpha i + 3b\beta + b\sqrt{3} \beta i}{6}, \\
P_2'P_3' &= \frac{-3a\alpha + a\sqrt{3} \alpha i - 2b\sqrt{3} \beta i}{6}, \\
P_3'P_1' &= \frac{-3a\alpha + a\sqrt{3} \alpha i - 2b\sqrt{3} \beta i}{6}.
\end{align*}
\]

Drop a perpendicular from \( P_1' \) to \( P_2' P_3' \), and let it meet \( P_2'P_3' \) at \( R' \).

Then
\[
\begin{align*}
P_1'R' &= \frac{-3ax\alpha i - ax\sqrt{3} \alpha + 2bx\sqrt{3} \beta}{6}, \\
R'P_3' &= \frac{-3ay\alpha + ay\sqrt{3} \alpha i - 2by\sqrt{3} \beta i}{6},
\end{align*}
\]
where \( x \) and \( y \) are tensors.

Equating vectors
\[
\begin{align*}
\frac{-3a\alpha - a\sqrt{3} \alpha i + 3b\beta + b\sqrt{3} \beta i}{6} &= \frac{-3ax\alpha i}{6} \\
\frac{-ax\sqrt{3} \alpha + 2bx\sqrt{3} \beta - 3ay\alpha + ay\sqrt{3} \alpha i - 2by\sqrt{3} \beta i}{6} &= \frac{-3ay\alpha + ay\sqrt{3} \alpha i - 2by\sqrt{3} \beta i}{6}
\end{align*}
\]
EQUATING TENSORS OF LIKE VECTORS

\[ x = \frac{\sqrt{3}}{2}, \quad y = \frac{1}{2}, \quad \tan \angle P'_1P'_2R' = \sqrt{3}. \]

Hence, \( R \) is the mid point of \( P'_1P'_2 \), and

\[ \angle P'_1P'_2R' = 60^\circ. \]

Therefore, the triangle \( P'_1P'_2P'_3 \) is equilateral.

E. PROOF BY TRIGONOMETRY.

Construction. Drop a perpendicular from \( C \) to \( AB \). Call the intersection \( M \).

Proof:

Let \( CM = x \), \( BM = y \).

Then

\[ \overline{P'_1P'_2}^2 = \overline{P'_1B}^2 + \overline{P'_2B}^2 - 2 \overline{P'_1B} \cdot \overline{P'_2B} \cos (B + 60^\circ). \]

\[ = \frac{a^2}{3} + \frac{c^2}{3} - \frac{2ac \cos (B + 60^\circ)}{3}. \]

\[ \overline{P'_1P'_3}^2 = \frac{a^2}{3} + \frac{b^2}{3} - 2ab \cos (C + 60^\circ). \]

\[ \overline{P'_2P'_3}^2 = \frac{b^2}{3} + \frac{c^2}{3} - 2bc \cos (A + 60^\circ). \]

\[ \overline{P'_1P'_2} - \overline{P'_2P'_3} = \frac{b^2}{3} - \frac{c^2}{3} - \frac{2ab \cos (C + 60^\circ)}{3} + \frac{2ac \cos (B + 60^\circ)}{3}. \]

#Refer to results of "A" proof by plane geometry.
\[
\overline{P_1 P_2} - \overline{P_2 P_3} = \frac{b^2 - c^2 - 2ab \cos \left[ \frac{240^\circ - (A + B)}{3} \right]}{3} + \frac{2ac \cos (B + 60^\circ)}{3}
\]

\[
= \frac{b^2 - c^2 + ab \cos A \cos B}{3}
\]

\[-\frac{ab \sin A \sin B + ac \cos B + ab\sqrt{3} \sin A \cos B}{3}
\]

\[+ \frac{ab\sqrt{3} \cos A \sin B - ac\sqrt{3} \sin B}{3} \]

But

\[
\sin A = \frac{x}{b}, \quad \cos A = \frac{\sqrt{b^2 - x^2}}{b}
\]

\[
\sin B = \frac{x}{a}, \quad \cos B = \frac{\sqrt{a^2 - x^2}}{a}
\]

Hence

\[
\overline{P_1 P_2} - \overline{P_2 P_3} = \frac{b^2 - c^2 - \sqrt{b^2 - x^2} \sqrt{a^2 - x^2}}{3}
\]

\[-\frac{x^2 - c\sqrt{a^2 - x^2} - \sqrt{3}x \sqrt{a^2 - x^2} + \sqrt{3}x \sqrt{b^2 - x^2} - c\sqrt{3}x}{3} \]

But

\[
b^2 - x^2 = (c + y)^2,
\]

\[
a^2 - x^2 = y^2,
\]

\[
b^2 - c^2 = a^2 + 2cy,
\]

and

\[
\overline{P_1 P_2} - \overline{P_2 P_3} = a^2 + 2cy - cy - y^2 - cy - \sqrt{3}xy + \sqrt{3}xc + \sqrt{3}xy - \sqrt{3}xc = 0.
\]
In a similar way it can be shown that

\[ \overline{P_1 P_2}^2 - \overline{P_2 P_3}^2 = 0 , \]

\[ \overline{P_1' P_2'}^2 - \overline{P_2' P_3'}^2 = 0 , \]

\[ \overline{P_1' P_2'}^2 - \overline{P_2' P_3'}^2 = 0 . \]

Therefore, the triangles \( P_1 P_2 P_3 \) and \( P_1' P_2' P_3' \) are equilateral.
Section II
Related Problems

Most of the theorems in Section II were discovered in working out the solution of the original proposition. The proofs of these theorems are given by Analytical Geometry with the one exception that Theorem VI is proved by Projective Geometry.

Theorem II. The four circles, two inscribed and two circumscribed, of the two derived triangles have a common center.

Proof:

Let the mid points of \( P, P_2, P_3, P_2 P_3, P' P'_2, P' P'_3, \) and \( P'_2 P'_3 \) be

\[ R_2, R_3, R_4, R'_2, R'_3, \]

respectively.

Their coordinates are *

\[ R_2 \left( \frac{6a + 3b + c\sqrt{3}}{12}, \frac{-b\sqrt{3} + 3c}{12} \right), \]

\[ R_3 \left( \frac{3a + 3b - c\sqrt{3}}{12}, \frac{-a\sqrt{3} + b\sqrt{3} + 3c}{12} \right), \]

\[ R'_3 \left( \frac{3a + 6b}{12}, \frac{a\sqrt{3} + 3c}{12} \right), \]

* These values may be obtained from those on p. 12.
Since the triangles $P_1, P_2, P_3$ and $P'_1, P'_2, P'_3$ are equilateral, the coordinates of the point which divides the line joining any vertex of either triangle to the midpoint of the opposite side in the ratio $2:1$ will be the coordinates of the centers of both the inscribed and circumscribed circles for that triangle.

The coordinates of the centers for all four circles are

$$\left( \frac{a+b}{3}, \frac{c}{3} \right).$$

Therefore, the centers of the circles coincide.
Theorem III. The sum of the areas of the two derived triangles is always equal to the area of the original triangle.

Proof: * 

The area of triangle OAB = \( \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a & 0 & 1 \\ b & c & 1 \end{vmatrix} \).

The area of triangle \( P_1P_2P_3 \) + the area of triangle \( P'_1P'_2P'_3 \) =

\[
\frac{1}{2} \begin{vmatrix} \frac{3a}{6} & \frac{-a\sqrt{3}}{6} & 1 \\
\frac{3a + 3b + cv3}{6} & \frac{a\sqrt{3} - b\sqrt{3} + 3c}{6} & 1 \\
\frac{3b - cv3}{6} & \frac{b\sqrt{3} + 3c}{6} & 1 \\
\frac{3a}{6} & \frac{a\sqrt{3}}{6} & 1 \\
\frac{3a + 3b - cv3}{6} & \frac{-a\sqrt{3} + b\sqrt{3} + 3c}{6} & 1 \\
\frac{3b + cv3}{6} & \frac{-b\sqrt{3} + 3c}{6} & 1 \end{vmatrix} +
\]

Therefore, the area of triangle OAB = the area of triangle \( P_1P_2P_3 \) + the area of triangle \( P'_1P'_2P'_3 \).

Corollary. If the original triangle is equilateral, the internally derived triangle disappears. * Figure 8. Values, p. 12.
and the externally derived triangle is congruent to the original triangle.

**Theorem IV.** The lines joining the mid points of the corresponding sides of the two derived triangles are perpendicular to the sides of the original triangle upon which they do not depend.

**Definition of terms.** By corresponding sides of the two derived triangles is meant the two sides, one from each triangle, which are determined by the same elements of the original triangle. For example, \( P_1P_2 \) and \( P_1'P_2' \) are corresponding sides since each side depends upon \( OA \) and \( AB \) of the original triangle.

By the side upon which they do not depend is meant that side of the original triangle which in no way determines their position. For example, \( P_1P_2 \) does not depend upon \( OB \).

**Proof:**

The slopes of the lines joining \( R_1R_{1'} \), \( R_2R_{2'} \), \( R_3R_{3'} \), are

\[
\frac{-b}{c}, \quad \frac{a-b}{c}, \quad \infty
\]

respectively, while the slopes of the lines of the original
triangle upon which they are not dependent are
\[
\frac{c}{b}, \frac{c}{a-b}, 0
\]
respectively.

It is evident that these lines are perpendicular and the theorem is proved.

**Theorem V.** If the sides of the original triangle are extended so that each intersects that side of the derived triangle which is not dependent upon it, the three meet due to each of the derived triangles are on a line.

**Proof:**

The coordinates of the intersections of:

- **OB and P_1P_2 (N)**
  \[
  \left( \frac{2abc + a^2b\sqrt{3}}{2ab\sqrt{3} - b^2\sqrt{3} - c^2\sqrt{3}}, \frac{2ac^2 + a^2c\sqrt{3}}{2ab\sqrt{3} - b^2\sqrt{3} - c^2\sqrt{3}} \right)
  \]

- **AB and P_1P_3 (L)**
  \[
  \left( \frac{2a^2c + ac^2\sqrt{3} - 2abc - a^2b\sqrt{3} + abc\sqrt{3}}{-a^2\sqrt{3} + b^2\sqrt{3} + c^2\sqrt{3}}, \frac{-2ac^2 - a^2c\sqrt{3}}{-a^2\sqrt{3} + b^2\sqrt{3} + c^2\sqrt{3}} \right)
  \]

- **OA and P_2P_3 (M)**
  \[
  \left( \frac{2ac + b^2\sqrt{3} + c^2\sqrt{3}}{-a\sqrt{3} + 2b\sqrt{3}}, 0 \right)
  \]

* From the equations on pages 11 and 12.
OB and \( P_1P_2 \) \( (N') \)

\[
\left( \frac{a^2 b \sqrt{3} - 2abc}{2ab \sqrt{3} - b^2 \sqrt{3} - c^2 \sqrt{3}} , \frac{a^2 c \sqrt{3} - 2ac^2}{2ab \sqrt{3} - b^2 \sqrt{3} - c^2 \sqrt{3}} \right).
\]

AB and \( P_1P_2 \) \( (L') \)

\[
\left( \frac{a^2 b \sqrt{3} - ab \sqrt{3} + 2a^2 c - 2abc - ac^2 \sqrt{3}}{a^2 \sqrt{3} - b^2 \sqrt{3} - c^2 \sqrt{3}} , \frac{a^2 c \sqrt{3} - 2ac^2}{a^2 \sqrt{3} - b^2 \sqrt{3} - c^2 \sqrt{3}} \right).
\]

OA and \( P_2P_2 \) \( (M') \)

\[
\left( \frac{2ac - b^2 \sqrt{3} - c^2 \sqrt{3}}{a\sqrt{3} - 2b\sqrt{3}} , 0 \right).
\]

If the two sets of points \( L, M, N \) and \( L', M', N' \) are collinear, the determinants

\[
\begin{vmatrix} x_L & y_L & 1 \\ x_M & y_M & 1 \\ x_N & y_N & 1 \end{vmatrix}
\]

and

\[
\begin{vmatrix} x_{L'} & y_{L'} & 1 \\ x_{M'} & y_{M'} & 1 \\ x_{N'} & y_{N'} & 1 \end{vmatrix}
\]

are each equal to zero (where \( x_L, x_M, x_N \) are the x-coordinates, and \( y_L, y_M, y_N \) are the y-coordinates of the points \( L, M, N \) respectively).

Since this is true, \( LMN \) and \( L'M'N' \) are two straight lines.
Theorem VI. The simple hexagons, whose opposite sides are the pairs of lines of the derived triangles, and the original triangle, whose intersections are on a line, are inscribable in a conic.

Proof by Projective Geometry.*

Consider the conic determined by the projectivity:

\[ F \left[ \text{EIG} \right] \neq K \left[ \text{EIG} \right]. \]

\[ F \left[ \text{EIG} \right] \neq \text{MIP}_2 \neq \frac{M}{K} \text{LBG} \neq K \left[ \text{EIG} \right]. \]

But, \( H \) is the meet of \( MI \) and \( LG \).

Then

\[ \text{MIP}_2 H \neq \frac{M}{K} \text{LBG}, \]

and

\[ F \left[ \text{MIP}_2 H \right] \neq \text{MIP}_2 H \neq \text{LBG} \neq K \left[ \text{LBG} \right]. \]

But

\[ F \left[ \text{MIP}_2 H \right] = F \left[ \text{EIGH} \right], \]

and

\[ K \left[ \text{LBGH} \right] = K \left[ \text{EIGH} \right]. \]

Hence

\[ F \left[ \text{EIGH} \right] \neq K \left[ \text{EIGH} \right]. \]

Therefore, \( H \) is a point of the conic, and the

* Figures 10 and 11.
simple hexagon $EFGHI$ is inscribable in a conic.

This proof holds if the letters are replaced by their primes. Thus, the theorem is proved.

Loci Theorems.

So far the relationships between the various elements of the original and derived triangles have been given in terms of a fixed triangle of reference. In what follows, the relationships will be studied for triangles of reference in which one vertex moves in the plane according to some fixed law. The vertices $O$ and $A$ on the $x$-axis are the fixed vertices. This makes the problem general, since it is possible to obtain every type of triangle by the movement of one vertex.

Case I. Let $B$ move on a line perpendicular to the $x$-axis.

If $B$ moves on a line perpendicular to the $x$-axis, the coordinates of $B$, where $i=1,2,...$, are $(Ka, \lambda_i c)$, where $\lambda$ is a variable and $K = \frac{b}{e}$. It is evident that $P_i$ and $P_i'$ are fixed points, but that $P_1$, $P_1'$, $P_3$, and $P_3'$ are variable points.
The equation of a line joining any two variable points \( P_i \) and \( P_j \), where \( i, j = 1, 2, \ldots \), is:

\[
\begin{vmatrix}
  x & y & 1 \\
  \frac{3a + 3Ka + \lambda \cdot cv3}{6} & \frac{av3 - Kav3 + 3\lambda \cdot c}{6} & 1 \\
  \frac{3a + 3Ka + \lambda \cdot cv3}{6} & \frac{av3 - Kav3 + 3\lambda \cdot c}{6} & 1 \\
\end{vmatrix} = 0.
\]

The slope of this line is \( \sqrt{3} \) and its \( x \)-intercept is \( \pm a(1 + 2K) \) or \( a + 2b \).

In a similar manner the slope and \( x \)-intercept of the lines joining other varying points are:

<table>
<thead>
<tr>
<th>Line</th>
<th>Slope</th>
<th>( x )-intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_i ) ( P_j )</td>
<td>( -\sqrt{3} )</td>
<td>( \frac{2Ka}{3} ) or ( \frac{2b}{3} )</td>
</tr>
<tr>
<td>( P_i' ) ( P_j' )</td>
<td>( -\sqrt{3} )</td>
<td>( \frac{a + 2Ka}{3} )</td>
</tr>
<tr>
<td>( P_i' ) ( P_j' )</td>
<td>( +\sqrt{3} )</td>
<td>( \frac{2Ka}{3} )</td>
</tr>
</tbody>
</table>

It is evident that none of these lines depend upon the variable ordinate of \( B \). Hence, we have
Fig. 13.
Theorem VII. If two of the vertices of a triangle are fixed, and the third vertex moves on a line perpendicular to the side determined by the fixed vertices, (a) one vertex of each of the two derived triangles is fixed, (b) corresponding sides meet on the fixed side of the original triangle, and (c) all of the variable vertices determined by one variable side of the original triangle are on two lines which make positive angles of 60° and 120° with the fixed side of the original triangle.

Case II. Let the vertex B move on a line parallel to the x-axis.

The coordinates of B are \((x, y)\), where \(x\) is a variable. It is evident that \(P_1\) and \(P'_1\) are fixed, while the other four vertices of the two derived triangles are variable.

The slope and x-intercept of the lines joining the variable points are:
<table>
<thead>
<tr>
<th>Line</th>
<th>Slope</th>
<th>x-intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_i\ P_j$</td>
<td>$\frac{-\sqrt{3}}{3}$</td>
<td>$\frac{2a - 2cv3}{3}$</td>
</tr>
<tr>
<td>$P_i\ P_j$</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>$\frac{-2cv3}{3}$</td>
</tr>
<tr>
<td>$P_i\ P_j'$</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>$\frac{2a - 2cv3}{3}$</td>
</tr>
<tr>
<td>$P_i\ P_j'$</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>$\frac{2cv3}{3}$</td>
</tr>
</tbody>
</table>

From the above, we have

**Theorem VIII.** If two vertices of a triangle are fixed and the third vertex moves on a line parallel to the side determined by the two fixed vertices, (a) one vertex of each derived triangle is fixed, (b) all of the variable vertices determined by one variable side of the original triangle are on two lines which make positive angles of $30°$ and $150°$ with the fixed side of the original triangle.
Case III. Let B move on any line through the origin.

The coordinates of B are \((u, b, \mu m b)\), where \(m\) is the slope of the line \(y = mx\) on which B moves, and \(\mu\) is a variable. It is evident that \(P_1\) and \(P_1'\) are fixed, while \(P_3\) and \(P_3'\) move on two lines which make angles of 30° and 30° with \(y = mx\) at the origin.

The slopes of the two lines are

\[
\frac{\sqrt{3} + 3m}{3 - \sqrt{3} m} \quad \text{and} \quad \frac{3m - \sqrt{3}}{m\sqrt{3} + 3}
\]

respectively. The points \(P_{1i}\) fall on a line whose slope is

\[
\frac{3m - \sqrt{3}}{3 + m\sqrt{3}}
\]

and whose x-intercept is

\[
\frac{a(m - \sqrt{3})}{3m - \sqrt{3}}
\]

The points \(P_{1i}'\) fall on a line whose slope is

\[
\frac{\sqrt{3} + 3m}{3 - m\sqrt{3}}
\]

and whose x-intercept is

\[
\frac{a(m + \sqrt{3})}{3m + \sqrt{3}}
\]

From the above, we have
Theorem IX. If two vertices of a triangle are fixed and the third vertex moves on a line through one of the fixed vertices, (a) the two vertices of the derived triangles which depend upon the fixed side are fixed, (b) the two vertices of the derived triangles which depend upon a segment of the line on which B moves make angles of 30° and -30° with this line, and (c) the vertices of the derived triangles which depend upon the remaining side move on two straight lines whose constant angles with the fixed side are determined by the angle between that constant side and the line on which the variable vertex moves.

Case IV. Let B move on a circle about the origin.

It is evident that P₁ and P₁' are fixed, and that P₃ and P₃' move on a circle about the origin whose radius is to the radius on which B moves as \( \sqrt{3} \) is to 3.

The points P₂ and P₂' move so that the distances P₂P₂', P₃P₃' and P₂P₂' are always equal, and the distances P₃P₃', P₃'P₃', and P₂P₂' are also equal.
Consider for the present some fixed triangle OAB whose vertices have the coordinates \((0,0)\), \((a,0)\), and \((b,c)\) respectively. This triangle determines uniquely the triangles \(P_1P_2P_3\) and \(P'_1P'_2P'_3\). The equation of a circle with center at \(P'_1\) and passing through the point \(P_2\) is:

\[
\left( x - \frac{a}{2} \right)^2 + \left( y - \frac{av3}{2} \right)^2 = \frac{b^2 + c^2}{3}.
\]

But \(b^2 + c^2\) is exactly the radius of the circle on which \(B\) moves. Hence, this circle about \(P'_1\) has the same radius as that on which \(P_3\) and \(P'_3\) move.

Similarly, the equation of a circle with center at \(P'_1\) and passing through the point \(P_2'\) is:

\[
\left( x - \frac{a}{2} \right)^2 + \left( y + \frac{av3}{2} \right)^2 = \frac{b^2 + c^2}{3}.
\]

Hence, this circle has a radius equal to that of the circle about \(P'_1\).

Since

\[
P'_1P'_2 = 0P'_1^2 + 0P'_3^2 - 20P'_1 \cdot 0P'_3 \cos (\text{BOA} + 60^\circ),
\]

and

\[
P'_1P'_2 = P'_1P'_2 + P'_3P'_2 - 2P'_1P'_2 \cos (\text{OP}'_3A + 60^\circ),
\]

for all positions of \(P_3\) and \(P_2\).
But

\[ \overline{P_1P'} = \overline{OP_1}, \quad \overline{OP_3} = \overline{P_1P_2}, \]

\[ \overline{AP_2} = \overline{AB} \sqrt{3}, \quad \overline{AP'} = \overline{OA} \sqrt{3}. \]

Then

\[ \triangle P_1P_A = \triangle OAB, \]

and

\[ \angle P_2P'A = \angle BOA. \]

Since \( \overline{OP_1}, \overline{OP_3}, \overline{P_1P_2} \) are constant distances, \( \overline{P_1P_2} \) must also be constant.

Therefore, \( P_2 \) moves on the circle with \( P_1 \) as center.

In a similar way it can be shown that \( P_1' \) moves on the circle with \( P_1 \) as center.

Consider the line \( OB \) coincident with the \( x \)-axis so that \( \angle AOB = 0^\circ \). When \( B \) moves to generate a positive angle at \( O \), the vertices of the derived triangles will move on their respective circle in such way that they will also generate positive angles. It is also evident that the angle \( AOB = 180^\circ \) is a special case. As the angle \( AOB \) passes through \( 180^\circ \), the isosceles triangles which were external become the internal triangles, and vice versa. Thus, the pairs of points \( P, P', P_2P_2', \) and \( P_3P_3' \) change.
From the foregoing discussion, we have

**Theorem X.** If B moves on a circle about the origin as center, starting so that the angle which OB makes with the x-axis is $0^\circ$, and the other two vertices are fixed at the origin and a point on the x-axis, (a) the points $P$, and $P'$ are fixed, except as the point B crosses the negative portion of the x-axis; the points $P$, and $P'$ exchange places, (b) the points $P$, and $P'$ move on a circle about the origin whose radius equals $\frac{OB}{3}$, and these points also exchange positions as B crosses the negative portion of the x-axis, and (c) the points $P_2$ and $P_2'$ move on circles about $P_1'$ and $P_1$, respectively whose radii are equal to the radius of the circle on which $P_1$ and $P_1'$ move; these points also exchange places as B crosses the negative portion of the x-axis.
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