Canonical Expansions for the Equations of Curves

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It is well known that the equation of any plane curve may be expressed in the form of a development in a power series in the neighborhood of a point, by the use of Taylor's expansion. The values of the coefficients in the series depend upon the relation of the coordinate system to the curve. It is the purpose of this paper to establish a coordinate system that will simplify the equation of the curve, by making as many of the coefficients as possible equal to zero or simple constants. The corresponding problem for space curves will also be studied.

This problem has been investigated from the standpoint of Projective Differential Geometry for both plane and space curves by several men. Wilczynski(1) has studied the problem from this standpoint, and Stouffer(2) has obtained similar canonical developments, and at the same time simplified the methods of Wilczynski. Sannia(3) and Halpen(4) have also solved the problem for both plane and space curves. This is essentially a problem of Differential Geometry, but it is the purpose of this paper to study it without the use of differential equations.

I. Plane Curves

Let $y=f(x)$ be any plane curve, and (x_o,y_o) be any general point on the curve. Let x, y represent projective non-homogeneous coordinates, and x_1, x_2, x_3 the corresponding projective homogeneous coordinates with $x = x_2/x_1$, $y = x_3/x_1$.

The function $f(x)$ may be developed in a power series in $(x-x_o)$ by the use of Taylor's expansion. Thus

$$(1) \qquad y = a_o + a_1(x-x_o) + a_2(x-x_o)^2 + \dots ,$$

where

$$a_o = f(x_o),$$
$$a_1 = f'(x_o),$$
$$a_2 = f''(x_o)/2!,$$
$$, \quad , \quad , \quad , \quad .$$

In this paper we shall use the same letters $a_1, a_2, a_3, \dots$ for the coefficients of $x, x^2, x^3, \dots$ respectively, but the coefficients will take on new values whenever a transformation of the coordinate system is made.

In the projective homogeneous coordinate system we have a triangle of reference with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, which may be fixed in any position with reference to the curve. Suppose the vertex $(1, 0, 0)$ is placed at the point (x_o,y_o). Then since the function is being developed about the point $x_o=0$, $y_o=0$, we have $f(x_o)=0$. Therefore $a_o=0$ and the series becomes

$$(2) \qquad y = a_1 x + a_2 x^2 + a_3 x^3 + \dots ,$$

where $a_1, a_2, a_3, \dots$ have new values determined by the condition placed upon the coordinate system.

The vertex (1, 0, 0) has been placed at the point (x_0, y_0), but this does not determine in any way the position of either of the other vertices of the triangle of reference. Let us make the line $x_3 = 0$ tangent to the curve at the point (1, 0, 0). The curve and the line will have first order of contact, and x^2 must be a factor of the equation

$$0 = a_1 x + a_2 x^2 + a_3 x^3 + \dots .$$

Therefore $a_1 = 0$, and the equation of the plane curve is reduced to the form

$$y = a_2 x^2 + a_3 x^3 + a_4 x^4 \dots ,$$

where again $a_2, a_3, a_4, \dots$ have new values. It is evident that if $a_2 = 0$, the plane curve and the line $x_3 = 0$ would have second order of contact. We should then have a point of inflection at (1, 0, 0), and our point would not be a general point on the curve. Therefore we assume in all future equations that $a_2 \neq 0$.

In order to locate further the triangle of reference let us obtain the equation of the conic osculating the curve at the point (1, 0, 0). The general equation of any conic has the form

$$(4) \qquad Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 .$$

Since a conic is determined by five points, the conic and the plane curve will have order of contact four. Therefore we let the value of y in (3) be substituted for y in equation (4), and make x^5 a factor of the equation. We then find the following values for the coefficients of equation (4):

$$A = -E a_2$$
$$B = E\left(a_3^2/a_2^3 - a_4/a_2^2\right)$$
$$C = -E a_3/a_2$$
$$D = F = 0$$

The equation of the osculating conic therefore becomes

$$-a_2^4 x_2^2 + (a_3^2 - a_4 a_2) y^2 - a_3 a_2^2 xy + a_2^3 y = 0$$

or expressed in homogeneous coordinates

(5) $$-a_2^4 x_2^2 + (a_3^2 - a_4 a_2) x_3^2 - a_3 a_2^2 x_2 x_3 + a_2^3 x_1 x_3 = 0$$

The polar of the point (0, 1, 0) with respect to the conic (5) is

$$-2 a_2^2 x_2 - a_3 x_3 = 0$$

But let us make this the line $x_2=0$. We then must have $a_3=0$, and the equation of the conic now becomes

$$-a_2^3 x_2^2 - a_4 x_3^2 + a_2^2 x_1 x_3 = 0$$

Let us now choose the vertex (0, 0, 1) on the conic, and it is evident that

$$-a_4 = 0$$

Therefore we have $a_4=0$ and the equation of the plane curve is now reduced to

$$y = a_2 x^2 + a_5 x^5 + a_6 x^6 + a_7 x^7 \ldots$$

where as before a_2, a_5, a_6,... have new values.

The line $x_2=0$ is the polar of the point (0, 1, 0), but their positions have not yet been fixed. In order to do this we must obtain another curve. Since we have used the tangent line and the osculating conic, the next simplest curve which we have is the cubic. Let us find the cubic which has a double point at (1, 0, 0) and highest possible order of contact with the plane curve. The general equation for such a cubic is

$$(7)\qquad Ax^3+By^3+Cx^2y+Dy^2x+Ex^2+Fy^2+Gxy+H=0\ .$$

Since we have seven coefficients to determine we may put seven conditions on our equation. Let us substitute the value of y in equation (6) for y in (7), and make x^7 a factor of the equation. We then find the following values for the coefficients in equation (7).

$$A=-Ga_2\ ,$$
$$B=-Ga_5/a_2^3\ ,$$
$$C=Ga_6/a_5\ ,$$
$$D=E=H=0\ ,$$
$$F=-Ga_6/a_2a_5\ .$$

and the equation of the osculating nodal cubic becomes

$$-a_2^4a_5x^3-a_5^2y^3+a_2^3a_6x^2y-a_2^2a_6y^2+a_2^3a_5xy=0,$$

or in homogeneous coordinates

$$-a_2^4a_5X_2^3-a_5^2X_3^3+a_2^3a_6X_2^2X_3-a_2^2a_6X_3^2X_1+a_2^3a_5X_1X_2X_3=0\ .$$

Now let us make the line $x_2=0$ the second tangent to the cubic at the point (1, 0, 0). Substituting this in equation (8) we have

$$-a_5^2X_3^3-a_2^2a_6X_3^2X_1=0\ .$$

or in non-homogeneous coordinates

$$-a_5^2y^3-a_2^2a_6y^2=0$$

Since the nodal cubic and the line have orderof contact two, x^3 must be a factor of the above equation. We have therefore $-a_2^2 a_6=0$, and since $a_2\neq 0$, $a_6=0$. The equation of the plane curve now has been reduced to the form

$$y = a_2 x^2 + a_5 x^5 + a_7 x^7 + a_8 x^8 + \cdots,$$

where again $a_2, a_5, a_7, \ldots$ have new values.

All the vertices of the triangle of reference are now fixed. The vertex $(1, 0, 0)$ is any general point (x_0, y_0) of the plane curve. The line $x_3=0$ is a common tangent to the plane curve and the nodal cubic, and the line $x_1=0$ is the second tangent to the nodal cubic at the point $(1, 0, 0)$. The vertex $(0, 0, 1)$ is the point where the line $x_2=0$ cuts the osculating conic, and the vertex $(0, 1, 0)$ is the pole of this same line. Since the line $x_2=0$ is fixed, the vertices $(0, 0, 1)$ and $(0, 1, 0)$ are also fixed.

Let us now determine a unit point and the coordinate system will be established. The most general transformation which does not disturb the triangle of reference is of the form

$$y = k_1 \bar{y}, \qquad x = k_2 \bar{x},$$

where k_1 and k_2 are constants. Substituting these values for y *and* x in equation (6) we find

$$(10) \qquad a_2' = a_2 k_2^2/k_1, \qquad a_5' = a_5 k_2^5/k_1,$$

where a_2' and a_5' are the new coefficients of x^2 and x^5 respectively in the equation of the plane curve. Solving for k_1 and k_2 we have

$$k_2 = \epsilon \sqrt[3]{a_2 a_5'/a_2' a_5}, \qquad k_1 = (\epsilon^2 a_2/a_2') \sqrt{(a_2 a_5'/a_2' a_5)^2}$$

where ϵ represents one of the cube roots of unity. We may give a_2' and a_5' any values we please except zero, and it is evident that we have three values for k_1 and k_2 for every value of a_2' and a_5'. We have already shown that $a_2 \neq 0$, and a_5' must be different from zero or the nodal cubic degenerates. If we substitute any of the three values for k_1 and k_2 in equations (10) we obtain the values we want for a_2' and a_5'. Let us first choose k_1 and k_2 such that $a_2' = 1$, and the equation of our conic becomes

$$-X_2^2 + X_1 X_3 = 0$$

It is now evident that the point (1, 1, 1) is on the conic.

In order to determine what is the best value to choose for a_5', let us find the equation of the second line through (0, 1, 0) which is tangent to the nodal cubic. Any line through (0, 1, 0) is $X_3 + K_1 X_1 = 0$. Solving this with the simplified equation of the nodal cubic

$$-X_2^3 - a_5' X_3^3 + X_1 X_2 X_3 = 0 ,$$

we have

$$-X_2^3 + a_5' K^3 X_1 - K X_1^2 X_2 = 0 ,$$

or in homogeneous coordinates

$$X^3 + K X - a_5' K^3 = 0$$

This is of the form $x^3 + px + q = 0$, and since we want two values of x to be equal we must have

$$q^2/4 + p^3/27 = 0$$

Therefore

$$a_5'^2 K^6/4 + K^3/27 = 0$$

$$K^3(a_5'^2 K^3/4 + 1/27) = 0$$

$$K = -\epsilon \sqrt[3]{4} / 3 \sqrt[3]{a_5'^2}$$

where ϵ is again one of the cube roots of unity, and the equation of the tangent line is

$$(11) \qquad x_3 - \epsilon \sqrt[3]{4}\, x_1 / 3\sqrt[3]{a_5'^2} = 0.$$

Let us choose a_5' such that the point (1, 1, 1) will be on this tangent line. The solutions for k_1 and k_2 showed us that we have three choices for both k_1 and k_2 for every value of a_2' and a_5'. This means that there will be three lines tangent to the cubic, from this point, one of the lines being real and the other two imaginary. These three values for k_1 and k_2, however, always give us the same values for a_2' and a_5', so our canonical expansion is not effected. Letting $x_3 = x_1 = 1$ in equation (11) we see that $a_5' = \epsilon' 2\sqrt{3}/9$ where $\epsilon' = \pm 1$. Therefore if we choose k_1 and k_2 such ~~that~~ that $a_5' = \epsilon' 2\sqrt{3}/9$, our unit point is at the point where the second tangent line to the nodal cubic through the vertex (0, 1, 0) cuts the conic. We now have two choices for our unit point depending upon the sign of ϵ', and the equation of plane curve is of the form

$$y = x^2 + \epsilon' 2\sqrt{3}\, x^5/9 + a_7 x^7 + a_8 x^8$$

where $a_7, a_8, \ldots$ again have new values and contain ϵ'.

Another way to determine the unit point is to choose k_1 and k_2 in (10) such that $a_2' = 1$ and $a_5' = 1$. It is evident that the unit point is again on the conic, but not at the same point as in the above work. If $a_5' = 1$ the equation of the second line tangent to the nodal cubic through (0, 1, 0) is seen from (11) to be

$$3x_3 = \epsilon x_1 \sqrt[3]{4}$$

where again ϵ equals one of the cube roots of unity. We have now determined three lines through $(0, 1, 0)$, and the unit point $(1, 1, 1)$ determines a fourth line, $x_1 - x_3 = 0$, through $(0, 1, 0)$. The four lines are

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_3 &= 0 \\ x_1 &= 0 \\ 3x_3 &= \epsilon x_1 \sqrt[3]{4} \end{aligned}$$

These four lines cut the line $x_2 = 0$ in four points. Expressed in non-homogeneous coordinates these points are $(0, 1)$, $(0, 0)$, $(0, \infty)$, $(0, \epsilon\sqrt[3]{4}/3)$ respectively. Now let us find the double ratio of $(A\ B\ C\ D)$ where $A = \epsilon\sqrt[3]{4}/3$, $B = 1$, $C = 0$, $D = \infty$. The numerical value of $(A\ B\ C\ D)$ is $\epsilon\sqrt[3]{4}/3$. This determines our unit point, for in the double ratio $(A\ B\ C\ D) = \epsilon\sqrt[3]{4}/3$, three of the points, A, C, and D, are fixed, therefore the fourth is also determined. We have three choices for our unit point depending upon the value of ϵ. Two of these are imaginary and one real. The equation of our general plane curve now becomes

$$y = x^2 + x^5 + a_7 x^7 + a_8 x^8 + \ldots,$$

where a_7, a_8,... have new values and contain ϵ.

II. Space Curves

Let $y = f_1(x)$
$z = f_2(x)$

be any space curve, and (x_0, y_0, z_0) any general point on the curve. As in the previous work let x, y, z represent projective non-homogeneous coordinates, and x_1, x_2, x_3, x_4 the corresponding projective homogeneous coordinates with $x = x_2/x_1$, $y = x_3/x_1$, $z = x_4/x_1$.

The functions $f_1(x)$ and $f_2(x)$ may be developed in a power series in $(x - x_0)$ by Taylor's expansion, and we have

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots,$$
$$z = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \dots,$$

where

$$\begin{aligned} a_0 &= f_1(x_0), & b_0 &= f_2(x_0), \\ a_1 &= f_1'(x_0), & b_1 &= f_2'(x_0), \\ a_2 &= f_1''(x_0)/2!, & b_2 &= f_2''(x_0)/2!, \\ &\dots, & &\dots \end{aligned}$$

As before we shall use $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ as the coefficients of $x, x^2, x^3 \dots$ respectively, throughout the paper, but the coefficients will take on new values whenever a transformation is made.

In Projective Geometry the coordinate system in space is established by means of a tetrahedron of reference with vertices (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), which may be placed in any position with relation to the curve. Let us place the vertex (1, 0, 0, 0) at the point (x_0, y_0, z_0). Then since the funstions are being developed about the point $x_0 = 0$, $y_0 = 0$, $z_0 = 0$, we have $f_1(x_0) = 0$, $f_2(x_0) = 0$, and the equations of the space curve become

$$(2)\qquad \begin{aligned} y &= a_1 x + a_2 x^2 + a_3 x^3 + \dots, \\ z &= b_1 x + b_2 x^2 + b_3 x^3 + \dots \end{aligned}$$

where $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ take on new values.

Although we have the vertex (1, 0, 0, 0) on any general point of the space curve, this in no way determines the position of the other three vertices. Suppose we make the line $x_3 = 0$, $x_4 = 0$ tangent to the space curve. In non-homogeneous coordinates the equations $x_3 = 0$, $x_4 = 0$ become $y = 0$, $z = 0$, and x^2 must be a factor of the equations

$$(3)\qquad \begin{aligned} 0 &= a_1 x + a_2 x^2 + a_3 x^3 + \dots, \\ 0 &= b_1 x + b_2 x^2 + b_3 x^3 + \dots. \end{aligned}$$

Therefore $a_1 = 0$ and $b_1 = 0$. Moreover let us make the plane $z = 0$ osculate the curve at the point (1, 0, 0, 0). Then x^3 must be a factor of the second equation of (3), and therefore $b_2 = 0$. The equations of the space curve are now reduced to the form

$$(4)\qquad \begin{aligned} y &= a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots, \\ z &= b_3 x^3 + b_4 x^4 + b_5 x^5 + \dots. \end{aligned}$$

Following somewhat the methods used in the study of the plane curve, let us find the space cubic which osculates the space curve at the point (1, 0, 0, 0). The parametric representation of the general space cubic is

$$(5)\qquad \begin{cases} x_1 = A_0 + A_1 t + B_1 t^2 + C_1 t^3, \\ x_2 = A_0' + A_2 t + B_2 t^2 + C_2 t^3, \\ x_3 = A_0'' + A_3 t + B_3 t^2 + C_3 t^3, \\ x_4 = A_0''' + A_4 t + B_4 t^2 + C_4 t^3. \end{cases}$$

The plane $x_4=0$ osculates the cubic at $(1, 0, 0, 0)$, therefore $A_0''' = A_4 = B_4 = 0$. Also the line $x_3=0$, $x_4=0$ is tangent to the cubic at the point $(1, 0, 0, 0)$, therefore $A_0'' = A_3 = 0$. Let us choose the parameter t such that for $t=0$ we have the point $(1, 0, 0, 0)$. We then have $A_0' = 0$, and $A_0 = k \neq 0$. Let us divide all the equations by k and then our new A_0 equals unity. If we replace $A_2 t$ by t' our new A_2 also equals unity. The osculating cubic meets the plane $x_3=0$ in three points, two of them are at the vertex $(1, 0, 0, 0)$, but we have not found the third point yet. When $x_3=0$ we have

$$B_3 t^2 + C_3 t^3 = 0,$$

$$t^2(B_3 + C_3 t) = 0,$$

$$t = -B_3/C_3.$$

The point $t = -B_3/C_3$ is therefore the other point of intersection of the cubic with the plane $x_3=0$. There is some point on the cubic where $t=\infty$. Let the parameter t be such that the points $t=\infty$ and $t=-B_3/C_3$ coincide. We then must have $C_3=0$.

The parametric representation of the cubic has now been reduced to

$$\begin{cases} x_1 = 1 + A_1 t + B_1 t^2 + C_1 t^3, \\ x_2 = t + B_2 t^2 + C_2 t^3, \\ x_3 = B_3 t^2, \\ x_4 = C_4 t^3. \end{cases}$$

To simplify this cubic still further let us put the vertex $(0,0,0,1)$ on the point of the cubic where $t=\infty$, which is also the third point of intersection of the cubic with the plane $x_3=0$. Replacing t by

$1/t'$ and letting t' approach zero, we see that $C_1 = C_2 = 0$.

At any point (x_1, x_2, x_3, x_4) of the cubic the equation of the osculating plane is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 + A_1 t + B_1 t^2 & t + B_2 t^2 & B_3 t^2 & C_4 t^3 \\ A_1 + 2B_1 t & 1 + 2B_2 t & 2B_3 t & 3C_4 t^2 \\ 2B_1 & 2B_2 & 2B_3 & 6C_4 t \end{vmatrix} = 0$$

which becomes by expansion

$$(6) \qquad x_1(2B_3C_4t^3) - x_2(6B_3C_4t^2 + 2A_1B_3C_4t^3) + x_3[6C_4t + 6B_2C_4t^2 + 2(A_1B_2C_4 - B_1C_4)t^3] - x_4(2B_3 + 2B_1B_3t^3) = 0.$$

Let us now find the osculating plane at the point $t = \infty$. Making the substitution $t = 1/t'$, and letting t' approach zero, we see that the osculating plane at this point is

$$2B_3C_4x_1 - 2A_1B_3C_4x_2 + 2A_1B_2C_4x_3 - 2B_1C_4x_3 = 0.$$

Let us make this the plane $x_1 = 0$, and we see that $B_1 = A_1 = 0$, for if either $B_3 = 0$, or $C_4 = 0$ the cubic degenerates.

Let us also choose the line $x_1 = 0$, $x_2 = 0$ tangent to the cubic at $(0, 0, 0, 1)$. Then since at this point $t = \infty$, it is easily seen that $B_2 = 0$. The parametric representation of our cubic is now reduced to

$$\begin{cases} x_1 = 1, \\ x_2 = t, \\ x_3 = B_3t^2, \\ x_4 = C_4t^3. \end{cases}$$

We have not yet determined the value of the coefficients of the osculating space cubic with relation to the space curve. Since a space cubic is determined by six points, the canonical expansion for the equation of the cubic osculating the space curve at the point (1, 0, 0, 0) must agree with the equation of the space curve up to and including the fifth powers of x. From the parametric equations (7), it is easy to find the equations for x, y, and z in non-homogeneous coordinates:

$$x = x_2/x_1 = t,$$
$$y = x_3/x_1 = B_3 t^2,$$
$$z = x_4/x_1 = C_4 t^3.$$

Eliminating the parameter t we have the equations of the osculating cubic in the form

$$y = B_3 x^2,$$
$$z = C_4 x^3.$$

Since these equations must agree with the equations of the space curve up to and including the fifth powers of x, we see by comparing them with equations (4) that

$$B_3 = a_2, \quad C_4 = b_3, \quad a_3 = a_4 = a_5 = b_4 = b_5 = 0.$$

Our cubic, therefore, becomes

$$(8) \qquad \begin{cases} x_1 = 1, \\ x_2 = t, \\ x_3 = a_2 t^2, \\ x_4 = b_3 t^3. \end{cases}$$

or

$$(9) \qquad \begin{aligned} y &= a_2 x^2, \\ z &= b_3 x^3. \end{aligned}$$

and the equations of the space curve are

$$(10)\qquad \begin{aligned} y &= a_2x^2 + a_6x^6 + a_7x^7 + \dots, \\ z &= b_3x^3 + b_6x^6 + b_7x^7 + \dots . \end{aligned}$$

A theorem due to Halpen[5] states that if two space curves have contact of order n at a point P, there exists a unique plane, called the principal plane which passes through the common tangent, and which has the property that the cones projecting the curves from any point O of this plane have contact of at least order n ~~+1~~ along the line PO. Let us find the principal plane of our general space curve and the osculating space cubic. Using the method given in a paper, "On the Contact of Space Curves"[6] by Stouffer, let us make a transformation of coordinates which shall involve changing the fourth vertex (0, 0,0,1) of the homogeneous coordinate system to a point whose coordinates are (a′, b′, c′, 1), where a′, b′, c′ are constants whose values may be assigned arbitrarily. This transformation is expressed in terms of the non-homogeneous coordinates by the equations

$$(11)\qquad \begin{aligned} \bar{x} &= (x - b'z)/(1 - a'z), \\ \bar{y} &= (y - c'z)/(1 - a'z), \\ \bar{z} &= z/(1 - a'z). \end{aligned}$$

Let us substitute equation (10) into the second equation of (11). We have equated the coefficients of the cubic and the space curve up to and including the fifth powers of x, therefore we are interested in the coefficients of x^6. By the above substitution we see that the coefficient of x^6 in our new equation is $(a_6 - c'b_6 - a'c'b_3^2)$. Substituting equations (9) into the second equation of (11) we see that the coefficient of x^6 in this new equation is $-a'c'b_3^2$. Therefore, since the new equations have order of contact six, our equations must agree up to and including the sixth powers of x. Therefore,

$$a_6 - c'b_6 - a'c'b_3^2 = -a'c'b_3^2, \qquad \text{or } a_6 = c'b_6$$

Let us make the plane $x_3 = 0$ the principal tangent plane, then $c' = 0$, and we must have $a_6 = 0$. The equations of the space curve now become

$$y = a_2 x^2 + a_7 x^7 + a_8 x^8 + \cdots,$$
$$z = b_3 x^3 + b_6 x^6 + b_7 x^7 + \cdots,$$

where again $a_2, a_7, a_8, \ldots$ and $b_3, b_6, b_7, \ldots$ have new values.

All the vertices of the tetrahedron of reference are now fixed. The vertex (1, 0, 0, 0) is at (x_0, y_0, z_0), any general point of the space curve, and the plane $x_4 = 0$ osculates the curve at this point. The line $x_3 = 0$, $x_4 = 0$ is tangent to the curve at the vertex (1, 0, 0, 0). The plane $x_3 = 0$ is the principal plane of the space curve and the osculating cubic. The vertex (0, 0, 0, 1) is the third point of intersection of the plane $x_3 = 0$ and the osculating cubic. $x_1 = 0$ is the plane that osculates the cubic at the point (0, 0, 0, 1), and $x_1 = 0$, $x_2 = 0$ is the line tangent to the cubic at the same point. The vertex (0, 0, 1, 0) is the point where the line $x_1 = 0$, $x_2 = 0$ cuts the plane $x_4 = 0$, and the vertex (0, 1, 0, 0) is the point where the line $x_3 = 0$, $x_4 = 0$ intersects the plane $x_1 = 0$.

In order to establish completely the coordinate system let us now determine a unit point. First let us find the equation of the quadric through the points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and having highest order of contact possible with the space curve at

(1, 0, 0, 0). The general equation of any quadric is

$$(13) \qquad Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Kz + L = 0$$

Expressed in homogeneous coordinates (13) becomes

$$(14) \qquad Ax_2^2 + Bx_3^2 + Cx_4^2 + Dx_2x_3 + Ex_2x_4 + Fx_3x_4 + Gx_1x_2 + Hx_1x_3 + Kx_1x_4 + Lx_1^2 = 0$$

We have ten constants to determine; therefore we may put ten conditions on the equation. The quadric goes through the points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), so it is easily seen that $L = A = B = 0$. Let us also substitute for y and z in (13) their values given in equations (12), and make x^7 a factor of the equation. We then have

$$E = F = G = H = 0\,,$$
$$D = -Kb_3/a_2\,,$$
$$C = -Kb_6/b_3^2\,,$$

and the equation of the quadric becomes

$$(15) \qquad -a_2b_6x_4^2 - b_3^2x_2x_3 + b_3^2a_2x_1x_4 = 0$$

From (15) we see that the polar plane of the point (0, 0, 0, 1) is

$$(16) \qquad -2b_6x_4 + b_3^2$$

Again let us assume the most general transformation which does not disturb the tetrahedron of reference by choosing

$$x = k_1\bar{x}\,, \qquad y = k_2\bar{y}\,, \qquad z = k_3\bar{z}$$

where k_1, k_2, and k_3 are constants. Substituting these values for x, y, and z in (12) we see that

$$a_2' = a_2k_1^2/k_2\,, \quad b_3' = b_3k_1^3/k_3\,, \quad b_6' = b_6k_1^6/k_3\,,$$

where a_2', b_3', and b_6' are new coefficients in the equations of the space curve. As in the study of the plane curve we may choose our constants, k_1, k_2, and k_3, so that a_2', b_3' and b_6' will have any values we desire. Work which follows shows that it is to our advantage to choose k_1, k_2 and k_3 such that $a_2'=1$, $b_3'=1$, and $b_6'=1/2$. We have then

$$b_3' k_1^3/k_3 = 2\, b_6 k_1^6/k_3\,,$$
$$k_1^3 = b_3/2b_6\,,$$
$$k_1 = \epsilon\sqrt[3]{b_3/2b_6}\,.$$

where ϵ is one of the cube roots of unity. It is evident therefore that we have three choices for our unit point. By the last transformation our polar plane becomes

$$-x_4 + x_1 = 0$$

and it is evident that the unit point is on that plane. A cubic intersects a plane in three points, so let us find the points of intersection of our osculating space cubic and the polar plane. Since $a_2'=1$ and $b_3'=1$ the parametric equations of the osculating cubic are

$$(17)\qquad \begin{cases} x_1 = 1 \\ x_2 = t \\ x_3 = t^2 \\ x_4 = t^3 \end{cases}$$

and it is evident that this cubic cuts the polar plane at the points $t=\sqrt[3]{1}$. Let us choose the point where the cubic cuts the polar plane at $t=1$ as the unit point of our coordinate system, and the equations of the space curve are now

$$y = x^2 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \ldots\,,$$
$$z = x^3 + x^6/2 + b_7 x^7 + b_8 x^8 + \ldots\,,$$

where a_7, a_8, ..., b_7, b_8, ... again have new values, *and depend on ϵ.*

Let us also find the equation of the conic which is enveloped by the totality of the intersections of the osculating planes to the bubic with the plane $x_4 = 0$. This is called the osculating conic. Equation (6) is the osculating plane at any point t, but we have found that $B_1 = A_1 = B_2 = 0$, and $B_3 = a_2 = 1$, $C_4 = b_3 = 1$. The equation of the intersections of the two planes therefore takes the form

(18) $$2x_1 t^3 - 6x_2 t^2 + 6x_3 t = 0, \qquad x_4 = 0.$$

Taking the first derivative we have

(19) $$6x_1 t^2 - 12x_2 t + 6x_3 = 0.$$

Solving equations (18) and (19) simultaneously for t we find $t = 2x_3/x_2$. Substituting this value for t in (18) we have

$$4x_1 x_3 - 3x_2^2 = 0, \qquad x_4 = 0,$$

which is the osculating conic.

From equation (17) it is evident that the projection of the osculating cubic on the plane $x_4 = 0$ is

$$x_3 x_1 = x_2^2, \qquad x_4 = 0.$$

which is a conic of the same form as the osculating conic, but the two conics will never intersect except at the points (1, 0, 0, 0) and (0, 0, 1, 0).

From equations (13) the equation of the projection of the general space curve from the vertex (0, 0, 0, 1) on the plane $x_4 = 0$ is seen to be

$$y = x^2 + a_7 x^7 + a_8 x^8 + \dots, \qquad z = 0.$$

It is evident that the osculating conic of this plane curve is

$$y = x^2, \qquad z = 0$$

which is also the projection of the osculating cubic from the vertex $(0, 0, 0, 1)$ on the plane $x_4 = 0$. Therefore, the projection of the osculating cubic from the vertex $(0, 0, 0, 1)$ on the plane $x_4 = 0$ lies on the osculating conic of the plane curve, which is the projection of our general space curve from the same vertex on the same plane.

References:

(1) Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces", Chapters 2, 3 and 13.

(2) Stouffer, "Some Canonical Forms and Associated Canonical Expansions in Projective Differential Geometry", Bulletin of the American Mathematical Society, May- June, 1928.

(3) Sannia, "Rendiconti dei Lincei", (5), vol. 31, 1° sem., p.p. 450-454 and pp. 503-506; (5) vol. 31, 2° sem., pp. 17-19 and pp. 432-434; "Annali di Matematica Pura ed Applicata", (4) vol. 1, pp. 1-18; (4) vol. 3, pp. 1-25.

(4) Halpen, "Journal de L'Ecole Polytechnique", t. 47, 1880, pp. 23-29.

(5) Halpen, "Journal de L'Ecole Polytechnique, t. 47, 1880, pp. 25-27.

(6) "On the Contact of Two Space Curves ", Stouffer, Bulletin of American Mathematical Society, vol. 38.