A LOGICAL PARADOX
AND THE FINITENESS OF NATURAL LANGUAGE

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Two powerful and contradictory intuitions govern a priori reflections on whether natural languages contain finitely or infinitely many sentences. The first of these is connected with the criterion that a sentence of a natural language be recognizable as such by a native speaker. Simple considerations connected with the finiteness of human capacities and ultimately of human life insistently suggest that the number of sentences of a natural language, though undoubtedly very large, must be finite.

This point is made rather pungently in Olmstead 1967 (304), "The assumption that English sentences are not always less than a million words in length smells of the computer center rather than of the field situation." Even if the native speaker of our natural language were possessed of a monumental memory and were heroically loquacious, it hardly makes sense to speak of him as uttering or recognizing sentences of such stupendous length. This unless he were idealized to such an extent as to equip him for conversation only with God.

On the other hand, the existence of recursive grammatical constructions in natural languages leads one to the opposite conclusion. For if there were only finitely many sentences of a natural language, then there must be one of maximal length. Applying such a construction to this sentence would then produce a still longer sentence. Hence the number of sentences in a natural language is infinite.

This intuition is rather clearly expressed in
Grinder and Elgin 1973 (24)

.. while any particular sentence of English is finite in length, the set of all well-formed sentences in English is infinite. There can be no "longest possible sentence," to any given sequence constituting a grammatical English sentence, you can add another relative clause, and no natural limit can be set at which such structures become incomprehensible.

The aim of what follows is to show that considerations of the sort just mentioned are specious. My method of attack will consist in showing the close similarities between the recursiveness argument and an important logical paradox.

There are many different ways in which the paradox I am concerned to expose may be formulated. It can in particular be formulated for virtually any concrete empirical predicate which admits of a class of clear instances and also of a class of clear counterinstances. I have selected the predicate 'is poor' primarily for reasons of technical convenience.

Consider the following argument:

(Al) (Pl) An American family with a yearly income of only 10 dollars is poor.
(Px) For any x, if an American family with a yearly income of only x dollars is poor, then an American family with a yearly income of only x+5 dollars is poor. Therefore,
(Cl) An American family with a yearly income of only 15 dollars is poor.

I am presuming it to be uncontroversial that (Pl) is in fact true. To be sure, one could imagine bizarre circumstances in which (Pl) might fail to hold completely invariably. But I am supposing that such unusual circumstances do not in fact obtain.
There is even less difficulty in accepting \((Px)\) as true, since it amounts merely to the claim that enriching a poor family by five dollars will not remove them from their poverty. Five dollars is too small a sum to make enough of a difference to remove a family from the class of poor families and firmly establish them in the class of the non-poor.

Next, according to the principles of modern quantifier logic (and the additional tacit premiss '10+5=15'), argument \((Al)\) is valid if its premises are true then its conclusion cannot be false. Since its premises are in fact obviously true, we are committed by modern logic to the truth of \((Cl)\). And we see in this case that \((Cl)\) is in fact also true.

Suppose we now make conclusion \((Cl)\) the first premiss of a new argument \((A2)\):

\begin{align*}
(A2) & \quad (P2) \text{ An American family with a yearly income of only 15 dollars is poor} \\
& \quad (Px) \text{ For any } x, \text{ if an American family with a yearly income of only } x \text{ dollars is poor, then an American family with a yearly income of only } x+5 \text{ dollars is poor. Therefore, } (C2) \\
& \quad \text{An American family with a yearly income of only 20 dollars is poor}
\end{align*}

Exactly the same reflections appropriate to argument \((Al)\) continue to apply to argument \((A2)\). Additionally, however, we note that the truth of \((P2)\) is guaranteed by virtue of its having been the conclusion of argument \((Al)\), and so we have the stronger result which is expressed by argument \((A2')\):

\begin{align*}
(A2') & \quad (P1) \text{ An American family with a yearly income of only 10 dollars is poor} \\
& \quad (Px) \text{ For any } x, \text{ if an American family with a yearly income of only } x \text{ dollars is poor, then an American family with a yearly income of only } x+5 \text{ dollars is poor. Therefore, } (C2) \\
& \quad \text{An American family with a yearly income of only 20 dollars is poor}
\end{align*}
But now the paradox arises. For there is nothing to prevent us from continuing to construct a long list of arguments, using a computer or a bevy of graduate students if need be, which includes such arguments as (A998*) and (A3998*).

(A998*) (Pl) An American family with a yearly income of only 10 dollars is poor.
(Px) For any x, if an American family with a yearly income of only x dollars is poor, then an American family with a yearly income of only x+5 dollars is poor. Therefore, (C998) An American family with a yearly income of only 5000 dollars is poor.

(A3998*) (Pl). (Px) Therefore, (C3998) An American family with a yearly income of only 20000 dollars is poor.

But here we are obviously in trouble. For (C3998) is surely false. And yet the validity of each argument in the series guarantees the truth of each conclusion, and hence the truth of (C3998).

Notice incidentally that none of the arguments is excessively long. That is, we are not faced here with the problem of requiring that the arguments in the series contain premisses of indefinite length. If we were to continue the series of arguments indefinitely, though, this problem might well arise, since the numerals in the premisses and conclusions would become excessively long.

This paradox may not be lightly dismissed. For surely the principles of logic must not become suspect merely because applied too often. And if argument (A1) is valid, then argument (A3998*) is also valid, since each of the previous arguments in the series is valid. If we reject the reasoning which has led us to (C3998), then we must be able to explain where the reasoning breaks down, and why. The force of the paradox is that it seems impossible to point the finger of blame at any one of the arguments in the series.
Solving the paradox by rejecting either premiss \((P_1)\) or \((P_x)\) is of course something which is open to us. But these premisses are obviously true, and I doubt whether it would occur to anyone to doubt them were it not for the fact that they lead us to the false conclusion \((C_3998)\).

I wish now to set out what seems to me to be the most coherent solution to the paradox. It would be wrong to dismiss out of hand other proposed solutions, the great thing about a good paradox is that it opens up various lines of inquiry for tracing precisely what are the hidden premisses of a piece of reasoning. A really good paradox, such as I think ours is, fascinates us because the hidden premisses are so well hidden and so few.

Although modern quantifier logic requires very little of the arguments to which it is to be applied, it does make several stipulations or requirements. The two most important of these may be expressed as the requirements of bivalence and non-contradiction.

The requirement of non-contradiction is that the sentences to which logical theory is applied may not be both true and false. This requirement does not play a role in the paradox.

The requirement of bivalence is that the sentences to which logical theory is applied are to be either true or false. It is not that modern logic assumes that every declarative sentence is either true or false. It is just that modern quantifier logic has little to say about arguments constructed from sentences which fail to satisfy the bivalence requirement.

The bivalence requirement is normally taken for granted, due in part to the desire of logicians to deal with the form of sentences as abstracted from the concrete circumstances of their use. Normally, that is, all that a logician will require is that the arguments he examines be composed of sentences having a transparent logical structure.
The arguments in the paradox have a sufficiently transparent logical structure to make the validity of their common form quite evident. And under normal circumstances the logical form is all that matters. But the paradox reveals that more than the form of the arguments is at issue.

Modern logic does guarantee that none of the separate arguments (A1)-(A3998) permits us to pass from true premises to a false conclusion. But it does not rule out the possibility that we may pass from true premises to an indeterminate conclusion, that is, a conclusion which is neither true nor false, or from premises one of which is indeterminate to a false conclusion. And this is, I would suggest, precisely what has happened during the series of arguments. Somewhere in the region of (A998) the conclusions have become indeterminate.

As a result, the argument (A3998*) may well be expected to have true premises and a false conclusion, since by transitivity of entailment the indeterminate conclusions have all dropped out. Only an underlying presumption that all of the sentences under consideration are either true or false has led to the expectation that all of the conclusions must be true.

The most common source of failure of a sentence to satisfy the bivalence requirement is the application of an empirical predicate to a borderline case. It has become a philosophical truism to observe that empirical predicates do not have sharp boundaries. Normally there are some cases to which it clearly applies, some cases to which it clearly does not apply, and a region of borderline cases to which it neither clearly applies nor clearly does not apply. And the most coherent interpretation of sentences which result from applying an empirical predicate, such as 'is poor', to a borderline case is I would suggest to treat them as neither true nor false, and thus as failing to satisfy the bivalence requirement.

It should be noted that one seldom is able to say precisely where the borderline cases begin. It seems...
clear enough to me, for example, that (C998) is neither true nor false. Yet I am not prepared to say just where the cases become borderline. An attempt to be more precise will often result in a rather arbitrary stipulation, such as a guideline laid down by the Department of Health, Education, and Welfare.

If the empirical predicate in question is made sharp, then paradox is avoided in another way, namely, by observing that for such predicates the second premiss (Px) is false. Indeed, I think it is clear that it is precisely the existence of indeterminate cases which guarantees the plausible truth of (Px). A strict definition of poverty, such as 'having a yearly income of less than 5000 dollars', does make it possible sometimes for five dollars to remove a family from poverty, no doubt to their dismay.

Being unable to say precisely where the indeterminate cases begin and end must not blind us to their existence. For an inability to take account of borderline cases may lead us into a very common fallacy, which might be called the Where do you draw the line fallacy. Noticing correctly that there is no sharp distinction between cases of poverty and cases of non-poverty, but rather a continuous blending of the one into the other through an indeterminate region, we may if we are not careful fall into the trap of concluding that there is no difference between the two conditions.

Similarly, our inability in general to say precisely which straw made the camel's burden too heavy to bear should not lead us to conclude that it could bear any burden. A poignant image for us to respect in this regard may be derived from Truffaut's great film, The 400 Blows. There is no single point in the film where the young boy has been injured too badly to recover, but he has clearly suffered irreversible damage at the end.

Although the point to be made against the recursiveness argument is probably clear enough by now, I shall nevertheless spell it out in order to make sure.
The recursiveness argument can be rephrased in such a way that its being an instance of the paradox under discussion is made evident. For consider the following argument.

\[(Arl) \quad (Pr1) \text{ There is an English sentence containing } 10 \text{ words.} \]
\[(Prx) \quad \text{For any } x, \text{ if there is an English sentence containing } x \text{ words, then there is an English sentence containing } x+5 \text{ words. Therefore,} \]
\[(Cr1) \text{ There is an English sentence containing } 15 \text{ words} \]

Premiss \((Pr1)\) is obviously true. Premiss \((Prx)\) is also clearly true, if one takes into account, for example, the prefixing of the words 'It is not true that' to any English sentence. The conclusion \((Cr1)\) is logically entailed by the premisses, and it is also obviously true.

We can now reconstruct the paradox, leading us successively to such conclusions as

\[(Cr998) \text{ There is an English sentence containing } 5000 \text{ words.} \]
\[(Cr3998) \text{ There is an English sentence containing } 20000 \text{ words.} \]
\[(Cr199998) \text{ There is an English sentence containing } 1000000 \text{ words.} \]

But now we are not compelled by logical grounds alone to the truth of \((Cr199998)\). For now we understand what we may demand, namely, assurance that the sentences composing the arguments all satisfy the bivalence requirement. That is, we are not being irrational in assenting to the truth of \((Pr1)\) and \((Prx)\) and yet questioning the truth of \((Cr199998)\) or even \((Cr3998)\). For as we have seen, this method of chaining arguments is paradoxical.

Of course to say that the reasoning just illustrated is paradoxical does not mean that the conclusions need be false. Our assurance that \((Cr199998)\), for example, is false must rest on independent grounds.
I do think, however, that the considerations mentioned as the first intuition are compellingly persuasive regarding the finiteness of natural language, providing only that 'is a sentence of a natural language' is being used in an empirical sense. And there is some evidence that linguists whose theories appear to commit them to the existence of infinitely many sentences of natural language are sensitive to this point. I would note for example the distinction drawn in Bach 1974 (25) between the infinitely many grammatical sentences of English as opposed to the finitely many acceptable sentences of English.

Although I would quarrel with Bach's grounds for making his distinction, I find no fault with his conclusion, which I take to be that there need be no exact fit between an abstract theory and the empirical domain to which it is intended to be applied. In particular, the abstract theory may well concern an infinite domain.

At the risk of simplifying the formal linguist's task beyond all recognition, let me attempt to explain a bit the point just raised. Suppose, following Quine 1961 (54), we take the task of the formal linguist to be to construct a purely formal device which generates precisely the sentences of English. Reasons of theoretical economy may well lead him to produce an infinitely large set S. But this will not matter very much, as long as S corresponds reasonably well to the examined sentences of English. For problems of applying the theory to sequences of unmanageable length are by hypothesis not going to arise.

What I have tried to show is that the puzzle concerning the finiteness or otherwise of natural language is not grounded in special considerations involving only linguists. It is just one manifestation of a logical paradox which may pop up almost anywhere.

Where the presumption of bivalence obtains, as in arithmetic, we may take a proof of the claim.
(1) There is no largest prime.

to be equivalent to the claim

(2) There are infinitely many primes

This is so because the series of arguments constituting the paradox may not pass from true premisses to a false conclusion, and since the only alternative to being false is being true, the arguments all have true conclusions.

But in a domain where bivalence may not be presumed, as in any empirical domain, presumably including linguistics, the claim

(3) There is no longest English sentence

is not strictly equivalent to the claim

(4) There are infinitely many English sentences.

For here the series of arguments constituting the paradox may pass from true premisses to conclusions which are neither true nor false, and from these to conclusions which are false.

We may if we wish coherently avoid identifying sentences (3) and (4) on the analogy of the natural equivalence between sentences (1) and (2). We may even assert, as I believe to be the case, that sentence (3) is true and sentence (4) is false.

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