ALIGNMENT ARRANGEMENTS IN A MODULAR PLANE.

by

Ellis R. Ott

A. B., Southwestern College

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Approved by:

W. M. Mitchell

Instructor in charge.

June, 1929

C. H. Ashby

Head of Dept.
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I. Introduction.

In 1894, Gabriel Arnaux published "Arithmetique Graphique," a book which contains ideas closely related to modular geometries.

In 1906, Veblen and Bussey published in the Transactions of the American Mathematical Society, Vol. 7, pp. 241-59, a paper defining finite (or modular) geometries. They gave both synthetic and analytic definitions. A brief summary of these definitions is given below.

A. Synthetic Definition.

By a finite projective geometry is meant a set of elements, which for suggestiveness, are called points, subject to the five conditions:

I. The set contains a finite number (greater than two) of points. It contains subsets called lines, each of which contains at least three points.

II. If A and B are distinct points, there is one and only one line that contains A and B.

III. If A, B, and C are non-collinear points, and if a line, m, contains a point D of the line AB and a point E of the line BC, but does not contain A, B, or C, then the line m contains a point F of the line CA.

Definition of Plane.

A plane ABC (A, B, and C being non-collinear points) is defined as the set of all points collinear with a point A and any point of the line BC. It may be proved by III that a plane so defined has the usual projective properties. For example, a plane is uniquely determined by any three of its points which are non-collinear, and the line joining any two points of a plane is contained wholly in the plane.

Definition of a $k$-space.

A $k$-space is defined by the following inductive definition. A point is an 0-space. If $A_1, A_2, \ldots A_{k+1}$ are points not all in the same $(k-1)$ space, the set of all points collinear with the point $A_{k+1}$ and any point of the $(k-1)$ space $(A_1, A_2, \ldots A_k)$ is the $k$-space $(A_1, A_2, \ldots A_{k+1})$. Thus a line is a 1-space, and a plane is a 2-space. From this definition it can be proved that spaces (if they exist) satisfy the following theorem.

In a $k$-space, an $n$-space and an $m$-space have a point in common if $n+m \geq k$. They have at least an $r$-space if $n+m-k = r$.

(Axioms of extension and closure.)

IV$_k$. If $n$ is an integer less than $k$, not all of the points considered are in the same $n$-space.

V$_k$. If IV$_k$ is satisfied, there exists in the set of points considered no $(k+1)$ space.
B. Analytic Definition.

If $x_1, x_2, \ldots x_{k+1}$ are marks of a Galois field of order $s = p^n$, there are $s^k + s^{k-1} + \ldots + s + 1$ elements of the form $(x_1, x_2, \ldots x_{k+1})$ provided that the elements $(x_1, x_2, \ldots x_{k+1})$ and $(lx_1, lx_2, \ldots lx_{k+1})$ are thought of as the same element, where $l$ is any mark not zero, and provided that the element $(0,0,\ldots,0)$ is excluded from consideration. These elements constitute a finite projective geometry of $k$-dimensions when arranged according to the following scheme. The equation,

$$u_1x_1 + u_2x_2 + \ldots + u_{k+1}x_{k+1} = 0,$$

(where the coefficients and variables must be in the GF ($s$)), is said to be the equation of a $(k - 1)$-space except when $u_1 = u_2 = \ldots = u_{k+1} = 0$. It is denoted by the symbol, $(u_1, u_2, \ldots u_{k+1})$. The points of the $(k - 1)$-space are those points of the finite geometry which satisfy its equation. A $(k - 2)$-space is represented by two equations of type (1) and a $(k - n)$ space by $n$ equations of type (1). There are $s^k + s + 1$ points in a two-space and $s + 1$ points on a line.

The finite projective $k$-dimensional geometry obtained in this way from the GF($s$) is denoted by the symbol $PG(k,s)$. Since there is a Galois field of order $s$ for every $s$ of the form $s = p^n$, it follows that there is a $PG(k,p^n)$ for every pair of integers $k$ and $n$ and for every prime $p$. 
C. Purpose of this paper and references.

It is the purpose of this paper to investigate inductively the possible ways of exhibiting, both analytically and synthetically, the alignments of some of the simplest of such plane modular geometries.


5. Madison Colloquim Lectures, Lectures IV and V.
II. Analytic Arrangements.

Arrangements for finite geometries, using homogeneous coordinates, can be obtained by the use of some transformations of the general form,

\[ \rho x'_i = a_{ii} x_i + a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 \]

where the determinant,

\[ D = \begin{vmatrix} a_{ii} & a_{i1} & a_{i2} \\ a_{i1} & a_{i2} & a_{i3} \\ a_{i2} & a_{i3} & a_{i3} \end{vmatrix} \]

is not zero.

The result so obtained can be exhibited graphically as in plates I-XI.

For example, consider the rational points of the plane for mod. 2. They have these coordinates:

(100), (110), (101), (011), (010), (001), and (111).

The transformation,

\[ \begin{align*}
\rho x'_1 &= x_1 + x_2 + x_3 \\
\rho x'_2 &= x_2 + x_3 \\
\rho x'_3 &= x_1 + x_3
\end{align*} \]

permutes these points as follows:

(100) → (101) → (010) → (110) → (011) → (001) → (111) → (100).

If we consider these as the points of the plane, the coordinates of the lines will be transformed by the induced*

*Veblen and Young, "Projective Geometry", Vol. I, p. 188.
transformation,
\[ \sigma u'_1 = u_1 + u_2 + u_3 \]
\[ \sigma u'_2 = u_2 + u_3 \]
\[ \sigma u'_3 = u_3 + u_3 \]

The points (100) and (101) determine the line [010] and the above transformation gives the following order for the coordinates of the lines:

\[ [010] \rightarrow [101] \rightarrow [001] \rightarrow [111] \rightarrow [100] \rightarrow [110] \rightarrow [011] \rightarrow [010] \]

Here, as for the points, every line is transformed into every other line, and the coordinates of the last line are transformed back into the coordinates of the first line.

If the coordinates of the seven points are arranged along the top of the diagram in the order determined by the transformation (1), starting with (100), and if the line coordinates are arranged along the side in the order determined by (2), we have the arrangement shown in Plate I. In this diagram, the x's indicate which points are on each line, or conversely, which lines are on each point. For any modulus, there are \((p + 1)\) points on every line and \((p + 1)\) lines on every point. Thus, the point whose coordinates are (110) is on the three lines whose symbols are read directly to the left of the three x's below (110)—that is, [001], [111], and [110], since (110) reduces the three equations, \(u_3 = 0\), \(u_1 - u_2 - u_3 = 0\), and \(u_1 + u_2 = 0\), either to zero or to a constant which is congruent to zero, mod. 2.

The two points, (100) and (101) determine the line \(u_2 = 0\).
PLATE I - MOD. 2

<table>
<thead>
<tr>
<th>100</th>
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</table>

\[
\rho \chi_i' = \chi_i + \chi_3 \\
\rho \chi_2' = \chi_2 \\
\rho \chi_3' = \chi_3 
\]
or [010] and it is at once seen that (001) is the third point
on this line. The remaining x's may be immediately written by
marking diagonally across the squares. When a diagonal gets to
the right hand column, it is found to begin again at the extreme
left, one space lower than at the end point on the right.
Similarly, when a diagonal reaches the bottom row, it is found
to begin again one space to the right at the top of the diagram.
The only requirement placed on a transformation so that it
will give an alignment for a finite geometry, PG(2,p), is that
it shall be of the period, \( p^2 + p + 1 \).

Consider the transformation:

\[
\begin{align*}
\rho x' &= ax + bx + cx \\
\rho x' &= x \\
\rho x' &= x
\end{align*}
\]

where a, b, and c are any marks of the field, and the deter-
minant,

\[
D = \begin{vmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = c \text{ is not zero.}
\]

The three values of \( \rho \) obtained from the characteristic equation
must lie in the GF(p^3) and not in the GF(p^2)* if the transfor-
mation is to have the desired period.

\[
\begin{vmatrix} a - \rho & b & c \\ 1 & -\rho & 0 \\ 0 & 1 & -\rho \end{vmatrix} = 0
\]

This gives the characteristic equation \( \rho^3 - a \rho^2 - b \rho - c = 0 \).

The transformation (3), then, is one from which all possible characteristic equations may be obtained by proper choices of \( a, b, \) and \( c \). It will be possible, then, to obtain characteristic equations whose three roots lie outside the \( GF(p^r) \); the alignment for a finite geometry will then be determined by the transformation (3).

A notation commonly used is to represent a transformation by the matrix of its coefficients: that is, the transformation,

\[
\begin{align*}
\rho x'_1 &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\
\rho x'_2 &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \\
\rho x'_3 &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3
\end{align*}
\]

is represented by,

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

Some transformations which give an alignment for modular geometries are here given.
Modulus 3.

The transformations

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

with their induced transformations give the alignments for modular geometries as shown in Fig. 1 and Fig. 2, respectively, of Plate II.

In addition, other transformations which give the same alignments, but different arrangements of point and line coordinates are:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

\[
(100) \rightarrow (001) \rightarrow (011) \rightarrow (11-1) \rightarrow (1-11) \rightarrow (-111) \rightarrow (111) \rightarrow (110) \rightarrow (-101) \rightarrow (010) \rightarrow (101) \rightarrow (011) \rightarrow (001) \rightarrow (100)
\]

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

\[
(100) \rightarrow (101) \rightarrow (11-1) \rightarrow (1-10) \rightarrow (010) \rightarrow (111) \rightarrow (011) \rightarrow (110) \rightarrow (101) \rightarrow (011) \rightarrow (001) \rightarrow (100)
\]

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

\[
(100) \rightarrow (111) \rightarrow (110) \rightarrow (1-11) \rightarrow (01-1) \rightarrow (1-10) \rightarrow (011) \rightarrow (100)
\]

\[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]

\[
(100) \rightarrow (011) \rightarrow (10-1) \rightarrow (1-10) \rightarrow (110) \rightarrow (001) \rightarrow (101) \rightarrow (111) \rightarrow (110) \rightarrow (01-1) \rightarrow (100)
\]

\[
\begin{pmatrix}
-1 & -1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
(100) \rightarrow (110) \rightarrow (0-11) \rightarrow (101) \rightarrow (010) \rightarrow (-101) \rightarrow (110) \rightarrow (01-1) \rightarrow (100)
\]

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
\]

\[
(100) \rightarrow (101) \rightarrow (11-1) \rightarrow (-101) \rightarrow (010) \rightarrow (110) \rightarrow (011) \rightarrow (111) \rightarrow (011) \rightarrow (11-1) \rightarrow (100)
\]
Fig. 1
\[ \rho x_1 = x_2 + x_3 \]
\[ \rho x_2 = x_1 \]
\[ \rho x_3 = x_2 \]

Fig. 2
\[ \rho x_1 = x_4 + x_2 + x_3 \]
\[ \rho x_2 = x_1 \]
\[ \rho x_3 = x_2 \]

Fig. 3
Modulus 5.

The transformations
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
2 & 0 & -2 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -2 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & -1 & 1
\end{pmatrix}
\]

with their induced transformations give the alignments for modular geometries as shown in Plates III to VII, respectively.

In addition, other transformations which give the same alignments, but different arrangements of point and line coordinates are:

\[
\begin{pmatrix}
1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{array}{c}
(100)\rightarrow(110)\rightarrow(111)\rightarrow(211)\rightarrow(-211)\rightarrow(12-2)\rightarrow(-112)\rightarrow
\end{array}
\]

\[
\begin{pmatrix}
2 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{array}{c}
(100)\rightarrow(210)\rightarrow(-121)\rightarrow(11-2)\rightarrow(011)\rightarrow(101)\rightarrow
\end{array}
\]

\[
\begin{pmatrix}
1 & -1 & 1 \\
0 & -1 & 1 \\
1 & 0 & 1
\end{pmatrix}
\begin{array}{c}
(1-12)\rightarrow(112)\rightarrow(12-1)\rightarrow(210)\rightarrow(-210)\rightarrow
\end{array}
\]
Mod. 5 (continued).

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

\[
(100) \rightarrow (110) \rightarrow (211) \rightarrow (-121) \rightarrow (2-12) \rightarrow (-22-1) \rightarrow (12-2) \rightarrow \\
(112) \rightarrow (-111) \rightarrow (1-11) \rightarrow (11-1) \rightarrow (111) \rightarrow (-211) \rightarrow (0-21) \rightarrow \\
(102) \rightarrow (-210) \rightarrow (12-1) \rightarrow (212) \rightarrow (021) \rightarrow (-101) \rightarrow (010) \rightarrow \\
(101) \rightarrow (210) \rightarrow (-221) \rightarrow (1-22) \rightarrow (11-2) \rightarrow (011) \rightarrow (201) \rightarrow \\
(-110) \rightarrow (0-11) \rightarrow (001) \rightarrow (100) \\
\]

\[
\begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

\[
(100) \rightarrow (010) \rightarrow (-101) \rightarrow (1-10) \rightarrow (11-1) \rightarrow (-211) \rightarrow (0-21) \rightarrow \\
(101) \rightarrow (-111) \rightarrow (0-11) \rightarrow (20-1) \rightarrow (-120) \rightarrow (21-2) \rightarrow (221) \rightarrow \\
(-122) \rightarrow (0-12) \rightarrow (201) \rightarrow (120) \rightarrow (-212) \rightarrow (1-21) \rightarrow (2-12) \rightarrow \\
(2-21) \rightarrow (1-11) \rightarrow (21-1) \rightarrow (-221) \rightarrow (12-2) \rightarrow (112) \rightarrow (111) \rightarrow \\
(011) \rightarrow (001) \rightarrow (100) \\
\]

\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

\[
(100) \rightarrow (-210) \rightarrow (0-21) \rightarrow (102) \rightarrow (010) \rightarrow (101) \rightarrow (-110) \rightarrow \\
(112) \rightarrow (121) \rightarrow (112) \rightarrow (111) \rightarrow (111) \rightarrow (201) \rightarrow (110) \rightarrow (-111) \rightarrow \\
(111) \rightarrow (-211) \rightarrow (1-21) \rightarrow (21-2) \rightarrow (021) \rightarrow (-101) \rightarrow (210) \rightarrow \\
(221) \rightarrow (-122) \rightarrow (1-12) \rightarrow (1-11) \rightarrow (2-11) \rightarrow (12-1) \rightarrow (-112) \rightarrow \\
(0-11) \rightarrow (002) \rightarrow (100) \\
\]

\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

\[
(100) \rightarrow (110) \rightarrow (-211) \rightarrow (1-21) \rightarrow (2-12) \rightarrow (22-1) \rightarrow \\
(011) \rightarrow (-201) \rightarrow (120) \rightarrow (012) \rightarrow (-101) \rightarrow (010) \rightarrow (201) \rightarrow (-110) \rightarrow \\
(1-11) \rightarrow (01-1) \rightarrow (101) \rightarrow (210) \rightarrow (-121) \rightarrow (11-2) \rightarrow (111) \rightarrow \\
(-111) \rightarrow (2-11) \rightarrow (12-1) \rightarrow (-112) \rightarrow (21-1) \rightarrow (-221) \rightarrow (11-1) \rightarrow \\
(211) \rightarrow (021) \rightarrow (001) \rightarrow (100) \\
\]

\[
\begin{pmatrix}
-2 & 0 & 2 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

\[
(100) \rightarrow (-210) \rightarrow (12-1) \rightarrow (112) \rightarrow (211) \rightarrow (-221) \rightarrow (1-22) \rightarrow \\
(21-2) \rightarrow (221) \rightarrow (-111) \rightarrow (1-11) \rightarrow (111) \rightarrow (011) \rightarrow (201) \rightarrow (-110) \rightarrow \\
(2-11) \rightarrow (2-21) \rightarrow (1-11) \rightarrow (01-1) \rightarrow (-201) \rightarrow (1-20) \rightarrow (2-12) \rightarrow \\
(0-21) \rightarrow (10-1) \rightarrow (110) \rightarrow (-211) \rightarrow (1-21) \rightarrow (01-2) \rightarrow (101) \rightarrow (010) \rightarrow \\
(001) \rightarrow (100) \\
\]
\[ p_1' = x_1 + x_2 + x_3 \]
\[ \sigma u_1' = 2x_1 + x_2 - x_3 \]
\[ p_2' = x_2 + x_3 \]
\[ \sigma u_2' = -2x_1 + x_2 + x_3 \]
\[ p_3' = x_1 + 2x_3 \]
\[ \sigma u_3' = -x_2 + x_3 \]
\[\rho_1 = -2x_2 + x_3 \quad \sigma u_1 = u_3\]
\[\rho_4 = x_1 \quad \sigma u_4 = u_1\]
\[\rho_5 = x_2 \quad \sigma u_5 = u_6 + 2u_3\]
\[ \begin{align*}
\rho'_{y_1} &= y_1 + x_2 \\
\rho'_{y_2} &= y_1 + x_3 \\
\rho'_{y_3} &= y_1 + x_2 + x_3 \\
\sigma u_1' &= -u_1 + u_3 \\
\sigma u_2' &= -u_1 + u_3 \\
\sigma u_3' &= u_1 - u_2 - u_3
\end{align*} \]
\[
\begin{align*}
\rho \chi_r &= -\chi_r - x_2 + x_3 \\
\sigma u_r &= u_3 \\
\rho \chi_2 &= x_1 \\
\sigma u_2 &= u_1 + u_3 \\
\rho \chi_3 &= x_2 \\
\sigma u_3 &= u_2 + u_3
\end{align*}
\]
Modulus 7.

The transformations
\[
\begin{pmatrix}
-3 & 0 & 3 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 3
\end{pmatrix}
\]

with their induced transformations give the alignments for modular geometries as shown in Plates VIII and IX.

In addition, the following transformation will give a similar alignment:

\[
\begin{pmatrix}
1 & -3 & 1 \\
0 & -3 & -3 \\
1 & 0 & 1
\end{pmatrix}
\]

\[
(101) \rightarrow (2-32) \rightarrow (1-33) \rightarrow (103) \rightarrow (323) \rightarrow (011)
\]
\[
(-211) \rightarrow (21-1) \rightarrow (102) \rightarrow (111) \rightarrow (112) \rightarrow (22-1)
\]
\[
(231) \rightarrow (21-3) \rightarrow (21-2) \rightarrow (110) \rightarrow (331)
\]
\[
(32-2) \rightarrow (201) \rightarrow (111) \rightarrow (-101) \rightarrow (010) \rightarrow (110) \rightarrow (23-1)
\]
\[
(-111) \rightarrow (-310) \rightarrow (-133) \rightarrow (032) \rightarrow (0-12) \rightarrow (23-2) \rightarrow (230) \rightarrow (01-1)
\]
\[
(-301) \rightarrow (232) \rightarrow (-213) \rightarrow (-221) \rightarrow (021) \rightarrow (-221) \rightarrow (233) \rightarrow (33-2)
\]
\[
(13-1) \rightarrow (210) \rightarrow (-232) \rightarrow (210) \rightarrow (13-2) \rightarrow (331) \rightarrow (22-3) \rightarrow (03-1)
\]
\[
(3-11) \rightarrow (001) \rightarrow (1-31) \rightarrow (31-2) \rightarrow (-231) \rightarrow (3-21) \rightarrow (11-1)
\]
\[
(100) \rightarrow (101)
\]

Modulus 11.

The transformations
\[
\begin{pmatrix}
5 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

with their induced transformations give the alignments for modular geometries as shown in Plate XI.
\[ \rho \chi_1 = -3\chi_1 + 3\chi_3 \]
\[ \sigma \chi_2 = \chi_1 \]
\[ \rho \chi_3 = \chi_2 \]
\[ \sigma u_1 = u_3 \]
\[ \sigma u_2 = 3u_1 + 3u_3 \]
\[ \sigma u_3 = 3u_2 \]
\[ \begin{align*}
\rho_x &= x + y_2 + y_3 \\
\rho y_2 &= x + x_3 \\
\rho y_3 &= x_1 + 3x_3
\end{align*} \]

\[ \begin{align*}
\sigma u_1 &= 3u_1 + u_2 - u_3 \\
\sigma u_2 &= -3u_1 + 2u_2 + u_3 \\
\sigma u_3 &= -u_2 + u_3
\end{align*} \]
III. Synthetic Arrangements.

In the synthetic arrangements which are here considered, certain restrictions have been placed. In every case, the first and second rows are started with the natural numbers, 1 and 2, respectively. In addition, the succeeding elements of the rows are obtained by writing the natural numbers in succession from 1 to \((p^2 + p + 1)\), (which we shall, hereafter, designate by \(m\)) and then beginning again with unity. These restrictions in the synthetic arrangement correspond to the arrangement of a double line down the main diagonal in the analytic arrangement.

For the modulus 2, there is only one arrangement of this type possible, neglecting the matter of notation:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 \\
2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\
4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 \\
\end{array}
\]

This arrangement is consistent with the five assumptions previously listed.

For modulus 3, only two distinct arrangements of this nature occur:

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 1 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 1 & 2 & 3 \\
10 & 11 & 12 & 13 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

The other arrangement possible is obtained by using 1, 2, 5, 7, respectively, as the four numbers of the first column.
For the modulus 5, at least five distinct solutions are possible. One of these is:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \cdots & m \\
2 & 3 & 4 & 5 & 6 & 7 & \cdots & m & 1 \\
4 & 5 & 6 & 7 & 8 & 9 & \cdots & m & 1 & 2 & 3 \\
11 & 12 & 13 & 14 & 15 & 16 & \cdots & m & 1 & \cdots & 9 & 10 \\
15 & 16 & 17 & 18 & 19 & \cdots & m & 1 & \cdots & 13 & 14 \\
27 & 28 & 29 & 30 & m & 1 & 2 & \cdots & \cdots & 25 & 26
\end{array}
\]

Four other arrangements can be obtained by using the combinations: \{(1, 2, 4, 9, 13, 19), (1, 2, 5, 7, 14, 22), (1, 2, 9, 12, 14, 18)\}, and \{(1, 2, 5, 11, 12, 18)\} for the numbers of the first column.

Two arrangements for mod. 7 can be secured by writing the numbers, \{(1, 2, 5, 10, 21, 23, 35, 52)\}, and \{(1, 2, 4, 14, 33, 37, 44, 53)\} as members of the first column; and similarly for mod. 11 with the numbers \{(1, 2, 13, 15, 23, 29, 55, 61, 64, 91, 111, 130)\}, and \{(1, 2, 6, 25, 45, 72, 75, 81, 106, 113, 121, 123)\}. The rows will be written as previously indicated.

It can be seen that this plan of arrangement of a finite geometry depends only upon the selection of the members of the first column.

Consider the numbers, 1, 2, 4, 10, which were used in the arrangement for mod. 3. An essential point to be noticed in the choice of numbers for any modulus is the matter of differences between the numbers. No two differences between any two numbers,
taken in any order, can be the same, remembering that the numbers from 1 to \((p^2 + p + 1)\) are adjacent numbers. The numbers may be thought of as on a circle; for example, the arrangement of the elements for mod. 3 would be:

![Circular Arrangement](image)

If the difference between any two of the numbers were the same as the difference between another pair, Assumption II would be violated. For suppose the first three numbers proposed for an arrangement for any modulus were 1, 3, 5. The arrangement would be:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & \ldots & \ldots & m \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots & m & 1 & 2 \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & \ldots & m & 1 & 4 \\
\end{array}
\]

An immediate consequence of such an arrangement is that no pairs of numbers occur in each of two different columns, starting with the two pair indicated—contrary to our previous Assumption II.

Plate I, Fig. 3, and Plate X are tables of alignment for the arrangement of a modular geometry, mod. 3 and mod. 5, respectively, as determined by J. T. Rosson.

   "Finite Projective Geometries."

2. Dickson, L. E.
   "Linear Groups."
   B. G. Teubner, Leipzig, 1901.

   "Geometry and Collineation Groups of the Finite Projective Plane, PG(2, 2)." Dissertation, Princeton University, 1913.


5. Reagan, Chas. A.
   "Distribution of Prime Points in Certain Modular Spaces."
   Master's Thesis, University of Kansas, 1926.


7. Arnoux, Gabriel.
   "Graphique Arithmétique." Paris, 1894.

8. Rosson, J. T.
   "The Irreducibility of certain Sets of Assumptions."
   Master's Thesis, University of Kansas, 1925.