Application of Stochastic Differential Equations to Option Pricing

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Abstract

The financial world is a world of random things and unpredictable events. Along with the innovative development of diversity and complexity in modern financial market, there are more and more financial derivative emerged in the financial industry in order to gain higher yields as well as hedge the risk . As a result, to price the derivative , indeed the future uncertainty, become an interesting topic in the field of mathematical finance and financial quantitative analysis.

In this thesis, I mainly focus on the application of stochastic differential equations to option pricing. Based on the arbitrage-free and risk-neutral assumption, I used the stochastic differential equations theory to solve the pricing problem for the European option of which underlying assets can be described by a geometric Brownian motion. The thesis explores the Black-Scholes model and forms an optimal control problem for the volatility that is an essential parameter in the Black-Scholes formula. Furthermore, the application of backward stochastic differential equations (BSDEs) has been discussed. I figured that BSDEs can model the pricing problem in a more clarifying and logical way. Also, based on the model discussed in the thesis, I provided a case study on pricing a Chinese option-like deposit product by using Mathematica, that shows the feasibility and applicability for the option pricing method based on stochastic differential equations.

Keyword: stochastic differential equations, BSDEs, optimal control, option pricing, Mathematica, Black-Scholes model

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Chapter 1

Introduction

In the field of mathematical finance, financial derivatives are not only a tool to reduce risk, but also a fancy chance to chase for high yield. As a result, pricing and measure of risk becomes more significant. The fundamental beginning of derivative pricing and risk measurement is widely considered as the publishing of *The Theory of Speculation* ;Bachellier [1900], which was accomplished by Louis Bachelier in 1900. His contribution was using a probabilistic description for price fluctuations on the financial market by introducing some stochastic analysis concepts, which generally involved others name working at considerably later decades. Bachelier first defined Brownian motion to model the change of stock prices and found that those were normal distribution data. In 1952, Harry Markowitz laid the foundation of the modern portfolio theory ;Markowitz [1952]. That theory inspired Lintner aroused the Capital Asset Pricing Model (CAPM) in 1965 ;Lintner [1965] that caused big change in both ideology and quantitative method of Wall Street. Based on CAPM model, Fischer Black and Myron Scholes developed the Black-Scholes model in 1973 ;Black and Scholes [1973a] ;Black and Scholes [1973b], which swept the Wall Street eventually. Meanwhile, more and more prosperous mathematical models like the binomial model ;Cox et al. [1979] and Monte Carlo model ;Boyle [1977] emerge and they all played important roles in the modern financial theory. After that, based on the Black-Scholes model, there are many revised models like martingale pricing model ;Harrison and Kreps [1979], jump diffusion model ;Merton [1976], stochastic volatility model ;Hull and White [1987] and Lévy process model ;Chan [1999].

Not like mathematics and physics that usually turn out to have convergent method for solving specific problems, there are divergent methods to be modelled and to make predictions for the financial market. In this thesis, we first introduces Black-Scholes (BS) model, an European option pricing formula based on arbitrage-free assumption and normal distribution. Although Black-Scholes model is a relatively mutual method and is widely used in industries, it was invented only for calculating the theoretical call option which is ignoring dividends and based on the constraint assumption such as risk-neutral market and arbitrage-free market.

The BSDE is the abbreviation of the Backward Differential Equation. Recently there has been an increasing interest in BSDE problems that arise in option pricing. BSDEs was first introduce by ;Bismut [1973], and after the general results given in ;Pardoux and Peng [1990], BSDEs was first been used in the financial field in ;El Karoui et al. [1997]. Compared with Black-Scholes model which is widely used in the financial industry, the BSDE is more robust to the uncertain probability model. In this thesis, we discuss the BSDE theory and the application to option pricing.

Moreover, the option pricing formulas firstly focused on European call option. However, the references we mention above have extended the model to American option pricing, Asian option pricing and many other pricing problems for contingent claims. More pricing method and formulas for various types of options can be found in the book "The complete guide to option pricing formulas" ;Haug [2007].

1.1 Modelling the Financial Market

Mainly inspired by ;Bachellier [1900], Itô calculus and geometric Brownian motion were introduced in 1944 ;Itô [1944]. Rapidly, geometric Brownian motion became an powerful model for the financial market. Thanks to the economic application for a geometric Brownian motion, Paul

A. Samuelson ;Samuelson [1965], Robert Merton ;Merton et al. [1971], Fishcher Black and Myron Scholes ;Black and Scholes [1973b], they all constructed fundamental theories in pricing problems which represent landmark development in mathematical finance. Those valuation formulas are still irreplaceable tools in today's financial market practice.

Let's illustrate what is a geometric Brownian motion. We first assume that we deposit S_0 in a financial institute at $t = 0$, and it has a risk-free interest rate r like the bank's interest rate. If the interest is paid at the end of time t, then the money we get in total is

$$S_t = S_0 + rtS_0 = S_0(1 + rt).$$

If it is paid twice, we get

$$S_t = [S_0 + \frac{rt}{2}S_0] + [S_0 + \frac{rt}{2}S_0]\frac{rt}{2} = S_0(1 + \frac{rt}{2})^2,$$

then if we have n payments in time t, we will get

$$S_t = S_0(1 + \frac{rt}{n})^n,$$

taking the limit we have

$$S_t = S_0(1 + \frac{rt}{n})^n \overset{n\to\infty}{\longrightarrow} S_0 exp(rt). \tag{1.1}$$

Similarly, we consider the price of an asset with stochastic fluctuations. It should contain two parts: the deterministic part and the random one.

The deterministic part is in some way analogous to (1.1)

$$dS_t = \mu S_t dt, \tag{1.2}$$

where μ is called drift and measures the average growth rate of the price. Usually μ is larger than

r, otherwise no one will take the risk of losing money if the asset yields the same return as the bank deposit.

The random one should be Markovian

$$dS_t = \sigma S_t dW_t, \tag{1.3}$$

which introduces the second parameter σ called volatility that measures the strength of the statistical price fluctuations. where W_t is a Wiener process or Brownian motion.

Then the price of the asset involved with stochastic fluctuations should satisfy

$$dS_t = \mu S_t dt + \sigma S_t dW(t) \qquad S_0 > 0 \tag{1.4}$$

This stochastic differential equation defines the geometric Brownian motion.

As demonstrated, the two parameters μ and σ, describe the expected gain and fluctuations around the average behavior. If the fluctuations are too huge, the investment to the asset is considered risky. Then how to avoid the risk and protect the investment from potential loss? The answer is to never put all the eggs in one basket!

To reduce the risk, people tend to have a portfolio which contains different assets from uncorrelated sectors of the market. For example, one could hold shares of the Marlboro and the Coca-Cola at the same time. Even if people do not smoke, they might still buy soft drink, then the loss from Marlboro will be offset by the gain from Coca-Cola. Theoretically, if the portfolio consists of enough uncorrelated assets, the risk indicated by the overall variance of the return should be largely reduced. This method is known as diversification.

However, there are still some fluctuations that do not come from specific risk, but are derived from the 'systematic error' of the market. Then we need to add in some sense anti-correlated assets to our portfolio. Those anti-correlated assets work as if one falls, another rises. For appropriate operation, this knock brings 'risk free' portfolio to people and we call that hedging in financial terminology. There is a kind of financial products, called derivatives which can be used as perfect tool for hedging.

1.2 Options Markets

Let's get to know the object we need to handle with-options.

Options are a kind of derivative securities, also called contingent claims, whose payoffs are contingent on the price of other securities usually called underlying assets ;Bodie et al. [2011]. When questioning the genesis of derivative securities, the answer is actually about risk management. Derivative securities, like options, is a contract enables people to trade risk instead of real merchandise or real assets, furthermore people can use a leverage effect to control risk and magnify the payoff. In other words, they can be powerful tools for both hedging and speculation.

A call option gives the holder the right to purchase an asset for a specified price K (strike price or exercise price), on or before specified expiration date T (maturity). The purchase price of the option is called the premium and the seller of call options receives premium income as payment for the right to exercise the call option. On the other hand, a put option gives the holder the right to sell an asset for a specified exercise price on or before the maturity T.

There are several types of options contracts and two most typical are European option and American option. An European options can exercise the option only on the expiration date while an American option allows the holder to exercise the option on or before the expiration date which add to the uncertainty and complexity of the option pricing problem.

Chapter 2

The Black-Scholes Model

In 1973, Fischer Black and Myron Scholes in the paper *The Pricing of Options and Corporate* first introduced the Black-Scholes model. It was a pioneer work to establish a theoretical formula for determining option price and led to a prosperous period in derivatives pricing including options, futures, forwards, etc.

Black-Scholes considers the option of underlying assets that have a risk-free interest rate. Then the option price is purely a function of the volatility of the stock. That is, the higher the volatility is, the higher the premium on the option would be.

The Black-Scholes model describes the behavior of prices in a continuous-time and assumes the following

1. The options are European call and no arbitrage opportunities.

2. Stocks pay no dividends during the life of the option.

3. Markets are efficient. This means a market movement cannot be predicted.

4. No commissions are charged, no transaction costs and no taxes.

5. Risk-free interest rates remain constant and are known and the economy is risk-neutral. This

means that all assets have expected return equal to the risk-free interest.

6. The percentage rate of change is normally distributed, and the level of the asset price at the expiration of the option is log-normally distributed

We denote the price process of the stock by S_t; expiration time is T; current price is S; option strike price is K; risk-free interest rate is r; the volatility of stock is σ ; the expected return rate is μ . Follow from the assumption, the distribution of stock price $S(t)$ is a normal distribution with the probability distribution function:

$$p(S_t) = \frac{1}{\sqrt{2\pi(T-t)}\sigma S_t} \exp\left[-\frac{(\log S_t - \mu)^2}{2(T-t)\sigma^2}\right] \qquad S_t > 0. \tag{2.1}$$

Then expected value can be expressed as:

$$E(S_t) = e^{\mu + \frac{T-t}{2}\sigma^2}. \tag{2.2}$$

The Black-Scholes formula is given by:

$$C(t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2). \tag{2.3}$$

Where

$$d_1 = \frac{\log\frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sqrt{T-t}\sigma}$$

$$d_2 = \frac{\log\frac{S_t}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sqrt{T-t}\sigma} = d_1 - \sqrt{T-t}\sigma$$

Where $N(\cdot)$ is the cumulative distribution function of normal distribution.

Example 2.0.1. Consider that there is a European call option with the six months to expiration. The current stock price $S_0 = 40$, and the strike price is $K = 49$. There is the risk-free interest rate $r = 0.1$, and the volatility is $\sigma = 0.2$. We can use the Mathematica code to calculate the option price :

```
In[66]:= NormD[x_] := 1/2*(1+Erf[x/Sqrt[2]])
         d1[s_, v_, k_, t_, r_, q_] :=
           ((r-q)*t+Log[s/k])/(v*Sqrt[t]) + (v*Sqrt[t])/2;
         d2[s_, v_, k_, t_, r_, q_] := d1[s, v, k, t, r, q] - v*Sqrt[t];

         BlackScholesCall[s_, k_, v_, r_, q_, t_] :=
           s*Exp[-q*t]*NormD[d1[s, v, k, t, r, q]] -
            k*Exp[-r*t]*NormD[d2[s, v, k, t, r, q]];

         BlackScholesCall[45, 49, 0.2, 0.1, 0, 0.5]

Out[70]= 1.85625
```

g:my_label

2.1 A Brief Derivation of Black-Scholes Formula

This brief derivation is a simple one not involved with the advanced mathematical tools like the background knowledge of stochastic process and *Itô* formula.

For a European call option, the value at expiration day is:

$$C_T = \max(S_T - K, 0). \tag{2.4}$$

Then the expected value is:

$$\begin{aligned} E(C_t) &= \int_{-\infty}^{+\infty} max(S_T - K, 0) p(S_t) dS_t \\ &= \int_{K}^{+\infty} max(S_T - K, 0) p(S_t) dS_t. \end{aligned} \tag{2.5}$$

From the risk-neutral assumption, the current price for the stock should be the result that we discount $E(S_t)$ at the risk-free rate of interest r, that is:

$$S_t = e^{-r(T-t)} E(S_T) = e^{\mu + \frac{T-t}{2}\sigma^2 - r(T-t)}. \tag{2.6}$$

So we have

$$e^{\mu + \frac{T-t}{2}\sigma^2} = S_t e^{r(t-t)}. \tag{2.7}$$

We take the logarithm function and get

$$\mu + \frac{T-t}{2}\sigma^2 = \log S_t + r(T-t), \tag{2.8}$$

and this result will be used in the following discussion.

Continually, we will see the current price of the option should be the result that we discount $E(C_t)$ at the risk-free rate of interest r, that is

$$\begin{aligned} C(t) &= e^{-r(T-t)}E(C_t) \\ &= e^{-r(T-t)}\int_K^{+\infty} \max(S_T - K, 0) p(S_t) dS_t \\ &= e^{-r(T-t)}\int_K^{+\infty} S_T p(S_t) dS_t - Ke^{-r(T-t)}\int_K^{+\infty} p(S_t) dS_t. \end{aligned} \tag{2.9}$$

We denote the two parts on the right hand side of (2.9) as C_1 and C_2, and let $w = \log S_t$, from (2.1) we get

$$\begin{aligned} C_1 &= e^{-r(T-t)}\int_K^{+\infty} S_t p(S_t) dS_t \\ &= e^{-r(T-t)}\frac{1}{\sqrt{2\pi(T-t)}\sigma}\int_K^{+\infty} \exp[-\frac{(\log S_t - \mu)^2}{2(T-t)\sigma^2}] dS_t \\ &= e^{-r(T-t)}\frac{1}{\sqrt{2\pi(T-t)}\sigma}\int_{\log K}^{+\infty} \exp[w - \frac{(w-\mu)^2}{2(T-t)\sigma^2}] dw. \end{aligned} \tag{2.10}$$

And from (2.8)

$$w - \frac{(w-\mu)^2}{2(T-t)\sigma^2} = -\frac{[w-\mu-(T-t)\sigma^2]^2}{2(T-t)\sigma^2} + \mu + \frac{T-t}{2}\sigma^2, \tag{2.11}$$

from (2.6)

$$C_1 = e^{-r(T-t)}\frac{1}{\sqrt{2\pi(T-t)}\sigma}\int_{\log K}^{+\infty} \exp[-\frac{[w-\mu-(T-t)\sigma^2]^2}{2(T-t)\sigma^2}] dw \tag{2.12}$$

$$= \frac{S}{\sqrt{2\pi(T-t)}\sigma}\int_{\log K}^{+\infty} \exp[\frac{[\mu+(T-t)\sigma^2-w]^2}{2(T-t)\sigma^2}] dw. \tag{2.13}$$

Let

$$h = \frac{1}{\sqrt{T-t}\sigma}(\mu + (T-t)\sigma^2 - w).$$

Then

$$C_1 = \frac{S_t}{\sqrt{2\pi}} \int_{\frac{\mu+(T-t)\sigma^2-\log K}{\sqrt{T-t}\sigma}}^{-\infty} \exp\left(-\frac{h^2}{2}\right)(-dh) = \frac{S_t}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\mu+(T-t)\sigma^2-\log K}{\sqrt{T-t}\sigma}} \exp\left(-\frac{h^2}{2}\right) \tag{2.14}$$

$$= S_t N[\frac{\mu + (T-t)\sigma^2 - \log K}{\sqrt{T-t}\sigma}] \tag{2.15}$$

$$= S_t N[\frac{\log S_t + r(T-t) + \frac{T-t}{2}\sigma^2 - \log K}{\sqrt{T-t}\sigma}] \qquad \text{from (2.8)} \tag{2.16}$$

$$= S_t N[\frac{\log \frac{S_t}{K} + [r + \frac{\sigma^2}{2}](T-t)}{\sqrt{T-t}\sigma}], \tag{2.17}$$

that is

$$C_1 = S_t N(d_1).$$

Similarly, we can prove that

$$C_2 = Ke^{-r(T-t)} \int_K^{+\infty} p(S_t) dS_t \tag{2.18}$$

$$= Ke^{-r(T-t)} N[\frac{\log \frac{S_t}{K} + [r - \frac{\sigma^2}{2}](T-t)}{\sqrt{T-t}\sigma}] \tag{2.19}$$

$$= Ke^{-r(T-t)} N(d_2). \tag{2.20}$$

Then the brief derivation of (2.3) is done.

2.2 The General Derivation of Black-Scholes Formula

First, some important terminologies need to be introduced to model the option.

Definition 2.2.1. Let (E, ε) be a measurable space. A continuous-time stochastic process with state space (E, ε) is a family $(X_t)_{t \in \mathbb{R}^+}$ of random variables defined on a probability space $(\Omega, \mathscr{A}, P)$ with values in (E, ε).

Definition 2.2.2. Consider a probability space $(\Omega, \mathscr{A}, \mathbb{P})$, a filtration on the space is an increasing family $(\mathscr{F}_t)_{t \geq 0}$ of σ-algebra included in $\mathscr{A}$. The σ-algebra $(\mathscr{F}_t)_{t \geq 0}$ can be seen as the information that will not been known until time t.

Definition 2.2.3. A Brownian motion is a continuous stochastic process $(X_t)_t \in \mathbb{R}^+$, with independent and stationary increment, that is ,

• continuity: $\mathbb{P}$ a.s. in the map $s \to X_s(\omega)$ is continuous.

• independent increments: if $s \leq t, X_t - X_s$ is dependent of

$$\mathscr{F}_s = \sigma(X_u, u < s).$$

• stationary increments: if $s \leq t, X_t - X_s$ *and* $X_{t-s} - X_0$ have the same probability distribution.

Definition 2.2.4. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space and $(W_t)_{t \geq 0}$ as $\mathscr{F}_t$ -Brownian motion. Then $(X_t)_{0 \leq t \leq T}$ is an Itô process if it can be written as:

$$\forall t \leq T \qquad X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s.$$

where

• X_0 *is* $\mathscr{F}_0$ measurable.

• $(K_t)_{0 \leq t \leq T}$ *and* $(H_t)_{0 \leq t \leq T}$ are $\mathscr{F}_t$-adapted process.

• $\int_0^T |K_s| ds < +\infty$ $\mathbb{P}$ a.s.

• $\int_0^T |H_s|^2 ds < +\infty$ $\mathbb{P}$ a.s.

Theorem 2.2.5. *(Itô formula) Let $(X_t)_{0 \le t \le T}$ be an Itô process*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s,$$

and f be a twice continuously differentiable function, then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X, X \rangle_s.$$

Where

$$\langle X, X \rangle_s = \int_0^t H_s^2 ds,$$

$$\int_0^t f'(X_s) dX_s = \int_0^t f'(X_s) K_s ds + \int_0^t f'(X_s) H_s dW_s.$$

The result can directly come from the idea of Taylor's expansion. And the complete proof can refer to the reference book ;Robert J. Elliott [1999].

Moreover, if there is a twice differentiable function $(t,x) \mapsto f(t,x)$ with respect to x and t, and these partial derivatives are continuous, the Itô formula give us,

$$f(t, X_t) = f(0, X_0) + \int_0^t f'_s(s, X_s) ds + \int_0^t f'_x(s, X_s) dX_s + \frac{1}{2} \int_0^t f''_{xx}(s, X_s) d\langle X, X \rangle_s.$$

Now we consider the price of a risky asset S_t. Let μ *and* $\sigma > 0$ be constants and $(W_t)_{t \ge 0}$ be a standard Brownian motion on $(\Omega, \mathscr{A}, \mathbb{P})$. Supppse S_t can be written in the form

$$S_t = S_0 + \int_0^t S_s(\mu ds + \sigma dW_s), \tag{2.21}$$

or in the form of

$$dS_t = S_t(\mu dt + \sigma dW_t). \tag{2.22}$$

To find the solution of S_t is actually looking for an adapted process $(S_t)_{t\geq 0}$ such that

$$S_t = S_0 + \int_0^t S_s \mu ds + \int_0^t \sigma S_s dW_s.$$

We suppose S_t is the solution of (2.14), we denote $Y_t = \log S_t$. Consider S_t is a Itô process with $K_s = \mu S_s$, $H_s = \sigma S_s$ then we apply the Itô Formula to $f(x) = \log(x)$, we have

$$\log S_t = \log S_0 + \int_0^t \frac{1}{S_s} dS_s + \frac{1}{2}\int_0^t (-\frac{1}{S_s^2})\sigma^2 S_s^2 ds.$$

From (2.22),

$$\log S_t = \log S_0 + \int_0^t (\mu ds + \sigma dW_s) - \frac{1}{2}\int_0^t \sigma^2 ds$$

$$\log S_t = \log S_0 + \int_0^t (\mu - \frac{\sigma^2}{2}) dt + \int_0^t \sigma dW_t$$

solving for S_t and get

$$S_t = S_0 \exp\left\{(\mu - \frac{\sigma^2}{2})t + \sigma W_t\right\}$$

is a solution of (2.13).

Moreover we can easily get

$$S_T = S_t \exp\{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}Z\} \tag{2.23}$$

where S_T is the stock price at maturity T, S_t is the current price at time t, μ is the expected return rate while σ is the standard deviation of the return. Z is a standard normal distribution since W_t is a standard Brownian motion.

From the risk-neutral assumption, we consider the expected return rate μ to be equal to the risk-free rate r. Then the call option price at time t, that is, $C(t)$ can be viewed as the discounted expected value of the option at maturity T, i.e.

$$
\begin{aligned}
C(t) &= e^{-r(T-t)}E[\max(S_T-K,0)] && (2.24)\\
&= e^{-r(T-t)}\{E[S_T|S_T>K]P[S_T>K]-KP[S_T>K]+E[0|S_T\le K]P[S_T\le K]\} && (2.25)\\
&= e^{-r(T-t)}E[S_T|S_T>K]P[S_T>K]-Ke^{-r(T-t)}P[S_T>K]. && (2.26)
\end{aligned}
$$

Follow from (2.23), we have

$$
\begin{aligned}
S_T>K &\Longrightarrow S_t\exp\{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Z\}>K, && (2.27)\\
&\Longrightarrow Z>\frac{\log\frac{K}{S_t}-(r-\frac{1}{2})\sigma^2(T-t)}{\sigma\sqrt{T-t}}, && (2.28)\\
&\Longrightarrow Z>\frac{-[\log\frac{S_t}{K}+(r-\frac{1}{2})\sigma^2(T-t)]}{\sigma\sqrt{T-t}}. && (2.29)\\
& && (2.30)
\end{aligned}
$$

recall the denotation $d_1=\frac{\log\frac{S_t}{K}+(r+\frac{1}{2})\sigma^2(T-t)}{\sigma\sqrt{T-t}}$, consecutively we have

$$
\begin{aligned}
&\Longrightarrow Z>-d_1+\sigma\sqrt{T-t}, && (2.31)\\
&\Longrightarrow Z<d_1-\sigma\sqrt{T-t}, && (2.32)\\
&\Longrightarrow Z<d_2. && (2.33)
\end{aligned}
$$

Then

$$P[S_T>K]=N(d_1-\sigma\sqrt{T-t})=N(d_2).$$

Where

$$N(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

is the cumulative distribution function of the standard normal distribution.

We evaluate

$$E[S_T|S_T > K]P[S_T > K] = \int_{-d_1+\sigma\sqrt{T-t}}^{+\infty} S_t e^{(\mu-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{2.34}$$

$$= S_t e^{r(T-t)} \int_{-d_1+\sigma\sqrt{T-t}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{T-t})^2} dz. \tag{2.35}$$

We do the substitution $u = z - \sigma\sqrt{T-t}$, then the lower bound $z > -d_1 + \sigma\sqrt{T-t} \Longrightarrow u > -d_1$, then the first part of (2.26) comes into being

$$= S_t e^{r(T-t)} \int_{-d_1}^{+\infty} \frac{1}{2\pi} e^{-\frac{1}{2}u^2} du \tag{2.36}$$

$$= S_t e^{r(T-t)} P[Z > -d_1] \tag{2.37}$$

$$= S_t e^{r(T-t)} P[Z < h] \tag{2.38}$$

$$= S_t e^{r(T-t)} N(d_1). \tag{2.39}$$

Thus (2.26) give us

$$C(t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2), \qquad \text{(The Black-Scholes Formula).}$$

Where

$$d_1 = \frac{\log \frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{2.40}$$

$$d_2 = \frac{\log \frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{T-t}\sigma} = d_1 - \sigma\sqrt{T-t}. \tag{2.41}$$

2.3 Call-Put Parity

Now, let us take a look at the European put option. The reason why we only concentrated on call option before is that the put option can be derived from call option. Once we got the answer for call option, we actually got the key to the solution for put option. It is important to assume that our market model rules out arbitrage, otherwise there would be no equilibrium existing in the market.

First, we denote that

$$x^+ = \begin{cases} x, & if\ x > 0, \\ 0, & if\ x \leq 0. \end{cases} \tag{2.42}$$

Then the payoff of the European call $C_T = (S_T - K)^+$ and the payoff of the European put $P_T = (K - S_T)^+$. For the maturity T, it is obvious that

$$C_T - P_T = (S_T - K)^+ - (K - S_T)^+ = S_T - K$$

But to avoid arbitrage, we actually require that $C_t - P_t = S_t - K$ holds at any time t.

Then we consider the put option price $P(t)$ is the discounted value of the option at maturity T, i.e.

$$P(t) = E[e^{-r(T-t)}(K - S_T)^+].$$

Since $(K - S_T)^+ = (S_T - K)^+ - S_T + K$, we have

$$\begin{aligned} E[e^{-r(T-t)}(K - S_T)^+] &= E[e^{r(T-t)}(S_T - K)^+ - e^{-r(T-t)}S_T + e^{-r(T-t)}K] && (2.43) \\ &= E[e^{r(T-t)}(S_T - K)^+] - E[e^{-r(T-t)}S_T] + E[e^{-r(T-t)}K] && (2.44) \\ &= C(t) - S_0 + Ke^{-r(T-t)}. && (2.45) \end{aligned}$$

So the price of European put option is

$$P(t) = C(t) - S_0 + Ke^{-r(T-t)}.$$

2.4 Discussion about the Black-Scholes Model with Dividends

From the above discussion, we assume that the underlying asset does not distribute dividends according to the second assumption of the Black-Scholes model. In the following discussion, a dividend paying stock case will be explored.

Obviously, by learning from the normal Black-Scholes model, and performing substitution, one can easily derive the Black-Scholes model with continuous dividends rate δ, i.e.

$$C'(t) = e^{-\delta(T-t)} S_t N(d_1') - K e^{-r(T-t)} N(d_2').$$

Where

$$d_1' = \frac{\log \frac{S_t}{K} + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2' = \frac{\log \frac{S_t}{K} + (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{T-t}\sigma} = d_1' - \sigma\sqrt{T-t}.$$

We can dig deeper to establish interesting symmetry relation between call and put price. To better describe the situation, we introduce some notations and definitions first.

We define a strategy by a process $\Phi = (\Phi_t)_{0 \le t \le T} = (H_t^0, H_t)_{0 \le t \le T}$ whose values define in $\mathbb{R}^2$. H_t^0 and H_t represent the the quantities of riskless asset and the risky asset respectively and both are adapted process to filtration $(\mathscr{F}_t)$ of the Brownian motion. Then the value of the portfolio at time t is

$$V_t(\Phi) = H_t^0 S_t^0 + H_t S_t.$$

Moreover, we can use stochastic differential equation to model the continuous-time portfolio process by

$$dV_t(\Phi) = H_t^0 dS_t^0 + H_t dS_t.$$

Definition 2.4.1. A self-financial strategy is a pair of adapted process $(H_t^0)_{0\leq t\leq T}$ and $(H_t)_{0\leq t\leq T}$ denoted as $\Phi = (H_t^0, H_t)$ satisfying:

1.$\int_0^T |H_t^0|dt + \int_0^T H_t^2 dt < \infty \; a.s..$

2.$H_0^t S_0^t + H_t S_t = H_0^0 S_0^0 + \int_0^t H_u^0 dS_u + H_0 S_0 + \int_0^t H_u dS_u \; a.s., t \in [0,T].$

We denote by $\hat{S}_t = e^{-rt} S_t$ the discounted value of the risky asset and also have $\hat{V}_t(\Phi) = e^{-rt} V_t(\Phi)$ be the discounted value of the portfolio. When there are dividends being taken into account, we assume the dividends are paid in continuous-time with a constants rate q. Then the self-financing condition is in the form

$$dV_t = H_t^0 dS_t^0 + H_t(dS_t + qS_t dt),$$

Originally the discounted form without dividends is

$$d\hat{V}_t = -r\hat{V}_t dt + e^{-rt} d\hat{V}_t \tag{2.46}$$

$$= -re^{-rt}(H_t^0 e^{rt} + H_t S_t) + e^{-rt} H_t^0 d(e^{rt}) + e^{-rt} H_t dS_t, \tag{2.47}$$

$$\text{(since } dS_t^0 = rS_t^0 dt \text{ , if setting } S_0^0 = 1\text{, then } S_t^0 = e^{rt}\text{)}$$

$$= H_t(-re^{-rt} S_t dt + e^{-rt} dS_t) \tag{2.48}$$

$$= H_t d\hat{S}_t, \tag{2.49}$$

and we have

$$d\hat{S}_t = -re^{-rt} S_t \hat{S}_t dt + e^{-rt} dS_t \tag{2.50}$$

$$= \hat{S}_t[(\mu - r)dt + \sigma dW_t]. \qquad (W_t \text{ is a standard Brownian motion}) \tag{2.51}$$

Then the discounted form comes into being

$$d\hat{V}_t = H_t(d\hat{S}_t + q\hat{S}_t dt) = H_t \sigma \hat{S}_t dW_t^q.$$

Where $W_t^q = W_t + \frac{(\mu+q-r)t}{\sigma}$. Now the pricing is measured in the probability $\mathbb{P}^q$ such that $(W_t^q)_{0\le t\le T}$ is a standard Brownian motion and the discounted value $(e^{(q-r)t}S_t))_{0\le t\le T}$ is a martingale from the Girsanov theorem (details see ;Lamberton and Lapeyre [2007]). Actually, from ;Harrison and Pliska [1981], the Girsanov theorem was used to depict the risk-neutral measure or equivalent martingale measure which allows one use risk-free interest rate r replace the drift μ in further discussion.

Then, we denote $C(t,S_t,K,r,q)$ the price of a European call option with the maturity T, and strike price K, interest rate r, and dividend rate q. Similarly, $P(t,S_t,K,r,q)$ is the price of put option. From the previous discussion, consider the discount value

$$C(t,S_t,K,r,q) = E^q[e^{-rt}(S_t e^{(r-q-\frac{\sigma^2}{2})(T-r)+\sigma(W_T^q - W_t^q)} - K)^+].$$

Then we have the following theorem.

Theorem 2.4.2. $C(t,S_t,K,r,q)$=$P(t,S_t,K,r,q)$.

Proof. To simplify the notation we denote the time till expiration by τ, i.e. $\tau = T-t$.

$$C(t,S_t,K,r,q) = E^q[e^{-r\tau}(S_t e^{(r-q-\frac{\sigma^2}{2})\tau+\sigma W_\tau^q} - K)^+]$$

we denote by $\hat{W}_t^q = W_t^q - \sigma t$ and $\hat{\mathbb{P}}^q$ with the density given by $d\hat{\mathbb{P}}^q = e^{\sigma W_T^q - \frac{\sigma^2}{2}T} d\mathbb{P}^q$, Then

$$\begin{aligned} C(t,S_t,K,r,q) &= E^q[e^{-r\tau}(S_t e^{(r-q-\frac{\sigma^2}{2})\tau+\sigma W_\tau^q} - K)^+] \\ &= E^q[e^{-q\tau} e^{\sigma W_\tau^q - \frac{\sigma^2}{2}\tau}(S_t - K e^{(q-r+\frac{\sigma^2}{2})\tau - \sigma W_\tau^q})^+] \\ &= E^q[e^{-q\tau} e^{\sigma W_\tau^q - \frac{\sigma^2}{2}\tau}(S_t - K e^{(q-r-\frac{\sigma^2}{2})\tau - \sigma \hat{W}_\tau^q})^+] \\ &= \hat{E}^q[e^{-q\tau}(S_t - K e^{(q-r-\frac{\sigma^2}{2})\tau - \sigma \hat{W}_\tau^q})^+]. \end{aligned}$$

The last equality comes from the fact that $(e^{\sigma W_T^q - \frac{\sigma^2}{2}T})_{t\ge 0}$ is a martingale. We know from the Girsanov theorem, $(\hat{W}_t^q)_{0\le t\le T}$ is a standard Brownian motion in the probability $\hat{\mathbb{P}}^q$, so is $(-\hat{W}_t^q)_{0\le t\le T}$

from the symmetry property. Therefore,

$$E^q\left[e^{-r\tau}(S_t e^{(r-q\frac{\sigma^2}{2})\tau+\sigma W^q_\tau}-K)^+\right]=\hat{E}^q\left[e^{-q\tau}(S_t-Ke^{(q-r-\frac{\sigma^2}{2})\tau-\sigma\hat{W}^q_\tau})^+\right],$$

i.e.

$$C(t,S_t,K,r,q)=P(t,S_t,K,r,q).$$

$\square$

2.5 Identification for the Volatility

There does not exist a perfect thing in the world.

Even though the Black-Scholes model seems powerful enough to deal with the pricing problem involved with European options pricing, American options pricing, situation with dividends and transaction costs or some more complex extensions, the Black-Scholes model is not impeccable.

We go back and observe the Black-Scholes formula. It is generally based on the following parameters: current stock price S_t (or other underlying asset), strike price K, maturity T, risk-free interest rate r and volatility σ.

$$\begin{aligned} C(t) &= S_t N(d_1) - Ke^{-r(T-t)}N(d_2). \\ d_1 &= \frac{\log\frac{S_t}{K}+(r+\frac{\sigma^2}{2})(T-t)}{\sqrt{T-t}\sigma}, \\ d_2 &= \frac{\log\frac{S_t}{K}+(r-\frac{\sigma^2}{2})(T-t)}{\sqrt{T-t}\sigma} = d_1-\sqrt{T-t}\sigma. \end{aligned} \tag{2.52}$$

One can write the expire time and the strike price in the option contract and read the interest rate and stock price from the newspaper while the only non-observable parameter is volatility. Given that the Black-Scholes model assuming that the volatility is a constant, we need to pick up the volatility with cautiou since it reflects the intrinsic dynamics of the underlying asset. In practice, there are three main methods to estimate σ: historical volatility, implied volatility and stochastic volatility.

2.5.1 Historical Volatility

A natural way to estimate σ is to use time series analysis.

Recall the result when we deriving the expression of the stock price

$$\log(S_t) = \log(S_0) + \int_0^t (\mu - \frac{\sigma^2}{2})dt + \int_0^t \sigma dW_t,$$

since the stock price follows a log-normal distribution, we can easily get

$$\log\frac{S_T}{S_t} \sim N((\mu - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)).$$

And we know that $\sigma^2 t$ is the variance of $\log S_t$. If we have $N+1$ observations of the stock price $(S_t^k)_{k=0,1,\ldots,N}$ divided by constant time interval Δt. In practice, we denote the return of it by

$$u_i = \log\frac{S_t^i}{S_t^{i-1}}$$

instead of the form

$$\frac{S_t^i - S_t^{i-1}}{S_t^{i-1}}.$$

Then the varibles

$$\log\frac{S_t^1}{S_t^0}, \log\frac{S_t^2}{S_t^1}, \ldots, \log\frac{S_t^N}{S_t^{N-1}}$$

are independent identically distributed. And we denote the average return in NΔt by

$$\overline{u} = \frac{1}{N\Delta t}\sum_{k=0}^{N-1}\log\frac{S_t^{k+1}}{S_t^k}.$$

Then, we are provided a good estimate for σ by the variance of the time series

$$\hat{\sigma} = \sqrt{\frac{1}{(N-1)\Delta t}\sum_{k=0}^{N-1}[\log\frac{S_t^{k+1}}{S_t^k} - \bar{u}]^2}. \tag{2.53}$$

This is often called historical volatility, but it changes with time instead of a constant. Besides the consistency is no good because the fluctuations would be dramatically big within long period.

2.5.2 Implied Volatility

Compared with historical volatility that only takes the information from the past, implied volatility is a estimate of the volatility based on both the current value and future value of the option. Generally, it increases when the market is bearish and decreases when it is bullish, because people generally believe there is more risky in bearish markets than in bullish markets.

How can we get the implied volatility? We can actually calculate that by inverting the Black-Scholes formula. This is because that the option is traded over all the time which means that the current market price of the call is known, then we can infer σ_{implied} by

$$\text{current market price of call option } C(S_t) \equiv C(S_t, K, T, r, \sigma_{\text{implied}}) \quad \Longrightarrow \quad \sigma_{\text{implied}}.$$

Since

$$\frac{\partial C(S_t, K, T, \tau, \sigma_{\text{implied}})}{\partial \sigma_{\text{implied}}} > 0,$$

there exists a unique solution for the implied volatility σ_{implied}. It reflects what the market expects the volatility to be. From the empirical research in ;MacBeth and Merville [1979], it reveals the facts:

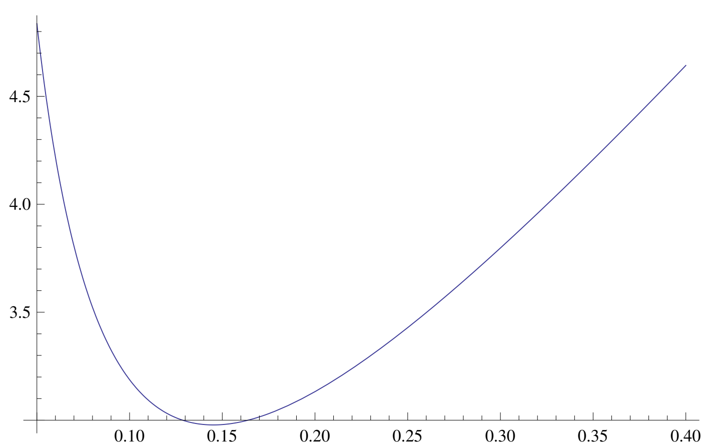
• The implied volatility depends on both the strike price K and the time till to maturity τ, that is $\sigma_{\text{implied}}(K, \tau)$.

• The implied volatility is almost constant for $\tau \geq 90$ days and when $S_t = K$ (at-the-money), which persuades people that the Black-Scholes model prices the option correctly and that the implied volatility is considered to be 'true'. So σ_{implied} can be used for the Black-Scholes formula.

• Using σ_{implied} to predict the Black-Scholes (BS) price for the case $S_t > K$ (in-the-money) and $S_t < K$ (out-of-the-money) and find that

$$BS\ price \begin{cases} \text{less than the market price} & S_t > K \\ \text{greater than the market price} & S_t < K \end{cases}$$

However, there is still one inconsistency that violates the assumption of the Black-Scholes formula. The implied volatility depends on different strike prices and maturities instead of a constant. We will be given convex and parabolic shape curves called "volatility smile" (see Fig.2.1) and "volatility skew" when the implied volatility versus different strike price K(details see ;Cont [1998]). As a result, a more consistent way to model the market data is to replace the constant volatility σ by a stochastic process $\sigma(S_t, t)$.



1 Figure 2.1: Volatility smile

2.5.3 Stochastic Volatility with Optimal Control Problem

The stochastic volatility model was well given by ;Hull and White [1987]. Suppose σ is a function of t and S, i.e. $\sigma(S,t)$, follow the stochastic process:

$$dS = \mu S dt + \sigma(S) S dW_t.$$

Where S is the market value of the underlying asset, μ is the drift, and W_t is a standard Brownian motion. Since we have the call-put parity, without loss of generality, we denote the price of the call option by $C = C(S,t,K,T)$ and assume the dividend rate is q. Based on the risk-neutral assumption, the corresponding Black-Scholes formula in partial differential equation form is given by

$$\begin{cases} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC = 0 & (0 \le S < \infty,\ 0 \le t < T) \\ C(S,T) = \max(S-K,0) = (S-K)^+ & (0 \le S < \infty) \end{cases} \tag{2.54}$$

with the boundary condition $C(0,T) = 0$ which indicates that the option is worthless if the underlying asset is valued at nothing.

We are interested in the inverse problem of option pricing, and we would like to figure out the pair of C(S,t,K,T) and $\sigma(S,t)$ satisfy (2.20) such that

$$C(S^*,t^*,K,T) = C^*(K,T).$$

Here, S^* is the current price of the underlying asset at current time t^*.

This is an inverse parabolic problem and was first discussed by ;Dupire et al. [1994]. Depending on different strike price K and maturity T, Dupire solved the inverse parabolic problem analytically and got

$$\sigma(K,T) = \sqrt{\frac{\frac{\partial C}{\partial T} + (r-q)K\frac{\partial C}{\partial K} + qC}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}}.$$

However, the identification of coefficient of the inverse problem is ill-posed since it is very sensitive to the change of variables, especially for the second derivative.

Based on Dupire's method and inspired by some results in ;Lishang and Youshan [2001], we modify the volatility $\sigma(S,t) := \sigma(S)$, i.e., σ is independent of t. We set

$$G(S,t,K,T) = \frac{\partial^2 U}{\partial K^2}(S,t,K,T),$$

such that G satisfies

$$\begin{cases} \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 G}{\partial S^2} + (r-q)S\frac{\partial G}{\partial S} - rG = 0 & (0 \leq S < \infty,\ 0 \leq t < T) \\ G(S,T) = \delta(K-S) & (0 \leq S < \infty). \end{cases} \tag{2.55}$$

$\delta(K-S)$ is the Dirac delta function mass at K. We integrate (2.55) twice with respect to K and formulating the problem

$$\begin{cases} \frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2(K)K^2\frac{\partial^2 C}{\partial K^2} + (r-q)K\frac{\partial C}{\partial K} - qC & 0 < K < \infty \\ C(K,0) = (S-K)^+. \end{cases} \tag{2.56}$$

Where $\tau = T - t$.

Here we want to find the $\sigma(K)$ which is changing with different strike price K by setting the current market price for the option equals to the option price, i.e.

$$C(S^*,t^*,K) \equiv C^*(K).$$

. We change the variables by

$$y = \log\frac{K}{S^*}, \qquad v(y,\tau) = \frac{C}{S^*}e^{q\tau}.$$

then (2.56) changes into an inverse parabolic problem with terminal observation:

$$\begin{cases} \frac{\partial v}{\partial \tau} = a(y)(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}) - (r-q)\frac{\partial v}{\partial y} & y \in \mathbb{R} \qquad 0 < \tau \leq \tau^* \\ v(y,0) = (1-e^y)^+ & y \in \mathbb{R} \end{cases} \tag{2.57}$$

where $a(y) = \frac{1}{2}\sigma^2(K)$ and with the terminal observation $v(y,\tau^*) = v^*(y) = \frac{U^*(K)}{S^*}e^{q\tau^*}$.

Here comes the optimal control problem also called the variational problem : we want to find $\bar{a} \in \mathscr{A}$, such that

$$J(\bar{a}) = \inf_{a \in \mathscr{A}} J(a). \tag{2.58}$$

Where

$$J(a) = \int_{\mathbb{R}} |v(y,\tau^*) - v^*(y)|^2 dy + N \int_{\mathbb{R}} |\nabla a|^2 dy \tag{2.59}$$

is the cost functional.

$$\mathscr{A} = \{a(y)|\ 0 < a_0 \leq a(y) \leq a_1,\ |\nabla a| \in L^2(R)\} \tag{2.60}$$

is the admissible set of the variational problem, and a_0, a_1 are two known constants as the boundaries.

$v(y,\tau)$ is the solution to (2.57), N is the regularization parameter, and $\bar{a}(y)$ is the optimal control or so called minimizer.

We use the following theorems to prove the solution for the optimal control problem exists. The proof generally comes from ;Lishang and Youshan [2001].

Theorem 2.5.1. *There exists a minimizer $\bar{a}(y) \in \mathscr{A}$ such that*

$$J(\bar{a}) = \inf_{a \in \mathscr{A}} J(a).$$

Proof. We infer that there exists a minimizing sequence $w_n = (v_n, a_n)$ such that

$$\lim_{n\to\infty} J(w_n) = \inf_{w\in\mathscr{A}} J(w)$$

since $|\nabla a| \in L^2(\mathbb{R})$, we have

$$||\nabla a_n||_{L^2(\mathbb{R})} \leq C, \qquad C \text{ denote a constant}$$

By the Sobolev's embedding theorem ;Adams and Fournier [2003],

$$||a_n||_{C^{\frac{1}{2}}(\mathbb{R})} \leq C,$$

Moreover, from the theorem for parabolic equations (see ;Ladyzenskaya [1967]), it guarantees that there exists a unique solution $v(y,\tau) \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}\times[0,\tau^*]) \cap C^{2+\alpha,1+\frac{\alpha}{2}}((\mathbb{R}/\{0\})\times[0,\tau^*])$, thus

$$||v_n(y,\tau)||_{C^{\frac{1}{2},\frac{1}{4}}(\mathbb{R}\times[0,\tau^*])} \leq C$$

$$||v_n(y,\tau)||_{C^{2+\frac{1}{2},1+\frac{1}{4}}(\mathbb{R}\times[0,\tau^*])} \leq C$$

then we can pick a subsequence of w_n, denoted by $w_{n_k} = (v_{n_k}, a_{n_k})$, such that

$$a_{n_k}(y) \Longrightarrow \bar{a}(y) \qquad \text{uniformly in } C^{\alpha}(\mathbb{R}). \qquad (0 \leq \alpha \leq \frac{1}{2})$$

$$v_{n_k}(y,\tau) \Longrightarrow \bar{v}(y,\tau) \qquad \text{uniformly in } C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}\times[0,\tau^*]) \cap C^{2+\alpha,1+\frac{\alpha}{2}}((\mathbb{R}/\{0\})\times[0,\tau^*]).$$

J is convex and continuous, thus by the Lebesgue control convergence theorem and the weak lower semi-continuity of the L^2 norm we get

$$J(\bar{a}) \leq \lim_{n\to\infty} \inf J(a_n) = \inf_{a\in\mathscr{A}} J(a).$$

□

Theorem 2.5.2. *Let a be the solution of the optimal control problem (2.58), then there exists a triple of functions* (v, φ, a) *satisfying the system:*

$$\begin{cases} \frac{\partial v}{\partial \tau} = a(y)(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}) - (r-q)\frac{\partial v}{\partial y} & (y,\tau) \in \mathbb{R} \times (0, \tau^*] \\ v(y,0) = (1-e^y)^+ & y \in \mathbb{R} \end{cases}$$

$$\begin{cases} \frac{\partial \varphi}{\partial \tau} = -a(y)(\frac{\partial^2 v}{\partial y^2} - \frac{\partial \varphi}{\partial y}) - (r-q)\frac{\partial \varphi}{\partial y} & (y,\tau) \in \mathbb{R} \times (0, \tau^*] \\ \varphi(y,0) = v(y,\tau^*) - v^*(y) & y \in \mathbb{R} \end{cases}$$

and

$$\int_0^{\tau^*} \int_{\mathbb{R}} \varphi(h-a)(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}) dy d\tau + N \int_{\mathbb{R}} \nabla a \nabla (h-a) dy \geq 0.$$

for any $h \in \mathscr{A}$

Proof. Since $\mathscr{A}$ is a convex set, then for any $h \in \mathscr{A}$, $0 \leq \lambda \leq 1$, define

$$a_\lambda = (1-\lambda)a + \lambda h \in \mathscr{A}.$$

Obviously, it reaches the minimum as $\lambda = 0$ since a is an optimal solution, let v_λ be the solution to (2.57) with $a = a_\lambda$, from (2.59) we have

$$\frac{d}{d\lambda} J(a_\lambda)\Big|_{\lambda=0} = 2\int_{\mathbb{R}} (v_\lambda - v^*) \frac{\partial v_\lambda}{\partial \lambda}\Big|_{\lambda=0} dy + 2N \int_{\mathbb{R}} \nabla a \nabla (h-a) dy \geq 0. \tag{2.61}$$

Denote $v'_\lambda = \frac{\partial v_\lambda}{\partial \lambda}$, plug into (2.23), we get

$$\frac{\partial v'_\lambda}{\partial \tau} = a_\lambda (\frac{\partial^2 v'_\lambda}{\partial y^2} - \frac{\partial v'_\lambda}{\partial y}) - (r-q)\frac{\partial v'_\lambda}{\partial y} + (h-a)(\frac{\partial^2 v'_\lambda}{\partial y^2} - \frac{\partial v'_\lambda}{\partial y}).$$

Continually, denote $\xi = v'_\lambda|_{\lambda=0}$, then we have

$$\xi_\tau = a(y)(\xi_{yy} - \xi_y) - (r-q)\xi_y + (h-a)(v_{yy} - v_y).$$

Use (2.61), we have

$$\int_{\mathbb{R}} (v(y,\tau^*) - v^*(y))\xi(y,\tau) + N\int_{\mathbb{R}} \nabla a \nabla (h-a) dy \geq 0. \tag{2.62}$$

Denote $\mathscr{L}\xi = \xi_\tau - a(y)(\xi_{yy} - \xi_y) - (r-q)\xi_y$ as an operator, and $\mathscr{L}^*$ is the adjoint operator that satisfying $\mathscr{L}^*\varphi = \varphi_\tau - a(y)(\varphi_{yy} - \varphi_y) - (r-q)\varphi_y = 0$, where φ is the solution of the equation above. Then we have

$$\begin{aligned} \int_0^{\tau^*}\int_{\mathbb{R}} \mathscr{L}^*\varphi\xi dyd\tau &= \int_{\mathbb{R}}\int_0^{\tau^*} -(\varphi\xi)_\tau d\tau dy + \int_0^{\tau^*}\int_{\mathbb{R}} \varphi\mathscr{L}\xi dyd\tau \\ &= -\int_{\mathbb{R}} (v(y,\tau^*) - v^*(y))\xi(y,\tau) + \int_0^{\tau^*}\int_{\mathbb{R}} \varphi(h-a)(v_{yy} - v_y) dyd\tau \end{aligned} \tag{2.63}$$

Combine (2.62), (2.63), Theorem 2.5.2 is proved. □

Once the existence has been verified, we can discuss the uniqueness of the optimal control problem. (For more details see ;Lishang and Youshan [2001])

However, it's hard to find the analytical solution for the optimal problem. To find the numerical solution, one can refer to the method of iteration in ;Jiang and Li [2005]

Chapter 3

Option Pricing with Backward Stochastic Differential Equations

We have explored the option pricing problem by the traditional stochastic differential equations model like the Black-Scholes model which is a mutual method that widely used in industry. Nowadays, there are still people making efforts on those pricing model to fit in different types of financial derivatives while other people are on their way to search for new tools.

Since the founder paper ;Pardoux and Peng [1990] considered the general existence and uniqueness results of backward stochastic differential equations (BSDEs) in 1990, more and more BSDEs have been applied to financial area. It is a robust tool that better accounts for uncertainty in financial derivative pricing and risk management .

3.1 Comparison to Forward Differential Equations

Recall the optimal control problem in section 2.5.3. We formed an inverse parabolic problem with terminal observation, that is we know the final state seeking for a minimizer that solve the optimal control problem. Indeed, it was the first time we came down to the idea "backward".

Before we introduce the the main results of BSDEs, let us try to feel difference between BSDEs and forward stochastic differential equations by the some moderate discussion. First, we take a look at the normal differential equation instead of the technical details.

$$\begin{cases} \dot{X}(t) = a(X(t)), \quad 0 \leq t \leq T \\ X(0) = X_0. \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{Y}(t) = b(Y(t)), \quad 0 \leq t \leq T \\ Y(T) = Y_T. \end{cases} \tag{3.2}$$

$a(\cdot)$ and $b(\cdot)$ are given functions, X_0 and Y_T are the data. We can easily tell that system (3.1) is forward while system (3.2) is backward. Under some certain conditions like the Lipshitz condition, both of the systems have unique solutions from the basic differential equations theories. However, the inside meaning between forward and backward are different. The existence and uniqueness of (3.1) depends on the certain initial state X_0 so that one can calculate the whole phase of the system once knowing the initial value. The existence and uniqueness of (3.2) consider how to get a initial value that can reach the desire object Y_T.

Obviously, those systems are based on the assumption that there is no random disturbance (the deterministic system). If the fluctuations have been taken into account, that is for a stochastic system, we replace the normal forward differential system by the general stochastic differential system:

$$\begin{cases} dX(t) = a(X(t))dt + \sigma(X(t))dW_t \\ X(0) = X_0. \end{cases} \tag{3.3}$$

W_t is a d-dimension Brownian motion that represents d independent disturbances. We can use $(X_i(t))_{i=1,2,\ldots,d}$ to represent the price of i th stock at time t, then W_t depicts the random fluctuations of those stocks.

Let us think about the inner understanding of the solution for (3.3) intuitively: the system starts from X_0 and follows the behavior that system (3.3) described. The future state $X(T)$ is a random variable which is so called an adaptive process that not be figure out till time T.

The same way, we consider to generalize the backward system (3.2) for the uncertain situation: we would like to use the given object $Y_T = \xi$ go back to figure out the proper initial value $Y(0)$ then get to know the whole states of the system. This process seems unrealistic in a normal stochastic system.

Is this process really unrealistic? let us start from a simple discrete time example: suppose there are two types of securities in the market and they form a portfolio: one is a bond which is risk-free with the return rate 10%. Another is a stock with the situation: when purchase 1 dollar and get good luck, the value is 1.2 dollar, for bad luck, it's worth 0.8 dollar. If someone's investment object is to get α dollars when has a good fortune and to get β dollars when bad fortune. His or Her investment in total is y dollars today and z dollars among y use to purchase the stock, then the model is:

$$\begin{cases} 1.1y + 0.1z = \alpha \\ 1.1y - 0.3z = \beta. \end{cases} \tag{3.4}$$

Obviously, the investment strategy contains two part (Y,Z). We calculate the problem and get a unique solution (y,z) :

$$\begin{cases} y = \dfrac{3\alpha + \beta}{4.4} \\ z = \dfrac{\alpha + \beta}{0.4}. \end{cases} \tag{3.5}$$

However, in real life, there is no way that can get to know the future earning for most case. It is actually the substance problem that BSDEs used to deal with. BSDEs can help to figure out today's investment portfolio for a future revenue object. That is to find out what the today's portfolio (Y,Z) should be based on the future investment goal.

3.2 General Results for BSDEs

BSDEs were first introduced by ;Bismut [1973]. In 1990, Pardoux and Peng gave the existence and uniqueness of the general BSDEs in ;Pardoux and Peng [1990]. Here, we give some denotation, they generally inherited from ;Pardoux and Peng [1990] and ;El Karoui et al. [1997]:

- For $X_t \in \mathbb{R}$, $\mathbb{L}^2(\mathbb{R}^d)$ denote the space of all $\mathscr{F}_t$-measuable random variables X_t: $\Omega \mapsto \mathbb{R}^d$ such that $||X||^2 = E|X_t|^2 dt < +\infty$, where $|\cdot|$ is the Euclidian norm.
- $\mathbb{H}^2(\mathbb{R}^d)$ denote the space of all predictable process ϕ: $\Omega \times [0,T] \mapsto \mathbb{R}^d$ such that

$$|\phi||^2 = E\int_0^T |\phi|^2 dt < +\infty.$$

- Eulidean norm is given by $|y| = \sqrt{\text{trace}(yy^*)}$ and denote $< y,z >= \text{trace}(yz^*)$.

We consider Y_t is the value of the replicating portfolio and Z_t is the hedging portfolio. (Y_t, Z_t) is a pair of unknown process. Then the general BSDE is given in following way:

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t$$
$$Y_t = \xi \in L^2(\mathscr{F}_t).$$

or, in stochastic equation form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{3.6}$$

Where

- W_t is a d-dimentional Brownian motion defined on probability space $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\})$ and $\mathscr{F}_t = \sigma\{W_t, \quad 0 \le t \le T\}$.
- $Y_t = \xi$ is the terminal condition (final wealth we want to attain at time T), and ξ is $\mathscr{F}_t$-measurable random variable.

- The generator f maps $\Omega \times \mathbb{R}^+ \times (\mathbb{R}^d \times \mathbb{R}^{d\times r})$ onto $\mathbb{R}^d$ and is $\mathscr{P} \otimes \mathscr{B}^d \otimes \mathscr{B}^{n\times d}$- measurable.

A solution is a pair (Y,Z). When $\{Y_t; t \in [0,T]\}$ is a continuous $\mathbb{R}^d$-valued adapted process and $\{Z_t; t \in [0,T]\}$ is an $\mathbb{R}^{n\times d}$ -valued predictable satisfying

$$\int_0^T |Z_s|^2 ds < +\infty,\ \mathbb{P}\ a.s..$$

From ;El Karoui et al. [1997], we are given a pair (ξ, f) called standard parameters for BSDE if it satisfying:

(A) $\xi \in \mathbb{L}^2(\mathbb{R})$.

(B) $\forall w$, $f(w,t,y,z)$ simply written in $f(t,y,z)$, $f(\cdot,0,0) \in \mathbb{H}^2(\mathbb{R}^d)$, and f is uniformly Lipschitz in (y,z), i.e. there exists a constant C, such that

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|). \qquad \forall (y_1,z_1),(y_2,z_2). \tag{3.7}$$

Theorem 3.2.1. *(;Pardoux and Peng [1990]) Given a standard parameters (ξ, f), there exists a unique solution (Y,Z) solves for (3.6).*

There are many versions of proof can be found in the work of predecessors like ;Pardoux and Peng [1990] ;El Karoui et al. [1997]. By learning from those ideas and referring to ;Pham [2009] we can give a brief proof for theorem 3.2.1.

Proof. We consider a function Φ on $\mathbb{L}^2(0,T)^m \times \mathbb{H}^2(0,T)^d$: $(U,V) \mapsto (Y,Z)$ given by

$$Y_t = \xi + \int_t^T f(s,U_s,V_s)ds - \int_t^T Z_s dW_s. \tag{3.8}$$

That is the pair (Y,Z) is constructed by the martingale $M_t = E[\xi + \int_t^T f(s,U_s,V_s)ds|\mathscr{F}_t]$, which is a square-integrable martingale under the assumptions on (ξ, f). We apply the Itô martingale representation theorem for Brownian motion (see ;Lamberton and Lapeyre [2007]), there gives the existence and uniqueness of $Z \in \mathbb{H}^2(0,T)^d$ satisfying:

$$M_t = M_0 + \int_0^t Z_s dW_s. \tag{3.9}$$

Then

$$M_t = M_T - \int_t^T Z_s ds.$$

Moreover

$$\begin{aligned} Y_t &= E[\xi + \int_t^T f(s,U_s,V_s)ds|\mathscr{F}_t] \\ &= M_T - \int_0^t f(s,U_s,V_s)ds - \int_t^T Z_s ds \\ &= M_t - \int_0^t f(s,U_s,V_s)ds. \end{aligned}$$

Noting that $Y_T = \xi$, we can verify that Y satisfies (3.8). By Doob's martingale inequality , we have

$$\begin{aligned} E[\sup_{0\leq t\ T}|\int_t^T Z_s dW_s|^2] &\leq 2E[|\int_0^T Z_s dW_s|^2] + 2E[\sup_{0\leq t\ T}|\int_0^t Z_s dW_s|^2] \\ &\leq 4E[\int_0^T |Z_s|^2 ds] < +\infty. \end{aligned}$$

Since the assumption of (ξ, f) implies Y is in $\mathbb{H}^d(0,T)^m$. Therefore Φ is a well-defined function maps $\mathbb{L}^2(0,T)^m \times \mathbb{H}^2(0,T)^d$ to itself. Then only thing we need to prove is that Φ is a contraction. Let $(U_1,\ V_1),\ (U_2,V_2) \in \mathbb{L}^2(0,T)^m \times \mathbb{H}^2(0,T)^d$, and $\Phi(U_1,\ V_1) = (Y_1,\ Z_1), \Phi(U_2,V_2) = (Y_2,Z_2)$. We denote that

$$(\bar{U},\bar{V}) = (U_1 - U_2, V_1 - V_2),$$

$$(\bar{Y},\bar{Z}) = (Y_1 - Y_2,\ Z_1 - Z_2),$$

and

$$\bar{f}_t = f(t,U_1(t),V_1(t)) - f(t,U_2(t),V_2(t)).$$

For any fixed $\beta > 0$, we apply Itô formula to $e^{\beta s}|\bar{Y}_t|^2$ from $s = 0$ to $s = T$. We have

$$||\bar{Y}_0||_\beta^2 = -\int_0^T e^{\beta s}(\beta|\bar{Y}_s|^2 - <\bar{Y}_s, \bar{f}_s>)ds - \int_0^T e^{\beta s}|\bar{Z}_s|^2 ds - 2\int_0^T e^{\beta s} <\bar{Y}_s, \bar{Z}_s dW_s> \quad (3.10)$$

Where $||\cdot||_\beta^2 := E\left\{\int_0^T e^{\beta t}|\cdot|^2 dt\right\}$.

Since

$$E[(\int_0^T e^{2\beta s}|\bar{Y}_s|^2|\bar{Z}_s|^2 ds \leq \frac{e^{\beta T}}{2}E[\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Z_s|^2 dt] < +\infty.$$

Then

$$\int_0^T e^{\beta s} <\bar{Y}_s, \bar{Z}_s dW_s>$$

is an uniformly integrable martingale by using Burkholder-Davis-Gundy inequalities (see ;Karatzas and Shreve [2012] p166 theorem 3.28).

We take expectation for (3.10), then

$$\begin{aligned} E||\bar{Y}_0||^2| + E[\int_0^T e^{\beta s}(\beta|\bar{Y}_s|^2 - <\bar{Y}_s, \bar{f}_s>)ds] &= 2E[\int_0^T e^{\beta s} <\bar{Y}_s, \bar{f}_s> ds] \\ &\leq 2CE[\int_0^T]e^{\beta s} <\bar{Y}_s, (|\bar{U}_s| + |\bar{V}_s|) > ds] \\ &\leq 4C^2 E[\int_0^T e^{\beta s}|\bar{Y}_s|^2 ds] + \frac{1}{2}E[\int_0^T e^{\beta s}(|\bar{U}_s|^2 + |\bar{V}_s|^2)ds] \end{aligned}$$

Now we take $\beta = 1 + 4C^2$ and have

$$E\left[\int_0^T e^{\beta s}(|\bar{Y}_s|^2 + |\bar{Z}_s|^2)ds\right] \leq \frac{1}{2}E\left[\int_0^T e^{\beta s}(|\bar{U}_s|^2 + |\bar{V}_s|^2)ds\right].$$

which shows Φ is a contraction on $\mathbb{L}^2(0,T)^m \times \mathbb{H}^2(0,T)^d$ endowed with the norm

$$||(Y,Z)||_\beta = \left(E\left[\int_0^T e^{\beta s}(|\bar{Y}_s|^2 + |\bar{Z}_s|^2)ds\right]\right)^{\frac{1}{2}}.$$

So there is a unique fixed point that is the solution to the BSDE (3.6). □

Even giving the existence and uniqueness theorem of BSDEs, usually there is no explicit analytical solution for BSDEs. Sometimes, we need to rely on numerical method. However, there do have analytical solution for some special case like linear BSDEs.

Consider the linear BSDE, that is the generator f is linear in y and z. We write the BSDE in the form

$$\begin{cases} -dY_t = (A_t Y_t + Z_t B_t + C_t)dt - Z_t dW_t \\ \quad Y_T = \xi \end{cases} \tag{3.11}$$

Where $A \in \mathbb{R}$, $B \in \mathbb{R}^d$ are bounded measurable process and $C \in \mathbb{H}^2(0,T)$.

Theorem 3.2.2. *The unique solution (Y,Z) to the linear BSDE (3.11) is given by*

$$\Gamma_t Y_t = E[\Gamma_T \xi + \int_t^T \Gamma_s C_s ds | \mathbb{F}_t]. \tag{3.12}$$

Where $(\Gamma_s)_{s\geq 0}$ is the solution to the SDE

$$\begin{cases} d\Gamma_s = \Gamma_s (A_t dt + B_t dW_t) \\ \Gamma_0 = 1 \end{cases}$$

Proof. We use the Itô calculus to $\Gamma_t Y_t$ get

$$d(\Gamma_t Y_t) = -\Gamma_t C_t dt + \Gamma_t (Z_t + Y_t B_t) dW_t.$$

We may also represent in the form

$$\Gamma_t Y_t + \int_0^t \Gamma_s C_s ds = Y_0 + \int_0^t \Gamma_s (Z_s + Y_s B_s) dW_t \tag{3.13}$$

Since A and B are bounded, $E[\sup_{0\le t\le T} |\Gamma_t|^2] < +\infty$, we denote the upper bound of B by b, using the Burkholder-Davis-Gundy inequality again, we have

$$E[(\int_0^T \Gamma_s^2 |Z_s + Y_s B_s|^2 ds)^{\frac{1}{2}}] \le \frac{1}{2} E[\sup_{0\le t\le T} |\Gamma_t|^2 + 2\int_0^T |Z_t| + 2b^2 \int_0^T |Y_t|^2 dt < +\infty.$$

Hence the local martingale $(\Gamma_t Y_t + \int_0^t \Gamma_s C_s ds)$ is a uniformly integrable martingale. By taking the expectation, find

$$\Gamma_t Y_t + \int_0^t \Gamma_s C_s ds = E[\Gamma_T \xi + \int_0^T \Gamma_s C_s ds | \mathbb{F}_t],$$

Then (3.12) is proved. □

There are more discussions and results like comparison theorem and nonlinear Feynman-Kac formula that can be found in ;El Karoui et al. [1997] and ;Peng [1992]. Here we emit those discussions.

3.3 Application in Finance

Here we are giving an one dimension BSDEs option pricing example. As for solving high dimension BSDEs problems which evolved with advanced parallel computer algorithm is beyond our discussion.

Suppose there is a portfolio with one risk-free asset and a stock denoting as following:

$$S_0(t) = e^{rt}, \qquad \text{risk-free asset}$$
$$S(t) = S(0)\exp[\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t]. \qquad \text{stock}$$

Where r is the risk-free expected return rate, $S(0)$ is the current price of the stock, μ is the expected return rate of the stock while σ is the volatility. W_t is an one-dimensional Brownian motion.

Without loss of generality, we set $\sigma = 1$, and assume the market is frictionless.

Suppose one has a self-financial strategy on $[0,T]$, i.e., the wealth at time t is y_t, invest z_t on the stock and put $y_t - z_t$ on the risk-free asset. Then y_t satisfying:

$$dy_t = f(y_t, z_t)dt - z_t dW_t, \qquad t \in [0,T] \tag{3.14}$$

Where $f(y_t, z_t) = ry + (\mu - r)z$. If the investor have an object

$$y_T = \xi \tag{3.15}$$

(3.14) (3.15) formed a linear BSDEs problem. We can find the unique solution (y_t, z_t), that gives the investor a investment strategy to tell them that if they want their wealth to reach ξ at time T, they should y_0 at time 0.

Moreover, we can consider the option pricing problem with BSDEs. If we consider that $C(S(t),t)$ is the value of the option. By using the results from nonlinear Feynman-Kac formula ;Peng [1992], $C(S(t),t)$ is the solution of the parabolic equation

$$\begin{cases} \frac{\partial C}{\partial t} + \mathbb{L}C + f = 0 \\ C(S(T),T) = (S(T) - K)^+ \end{cases}$$

Where $\mathbb{L}u$ is a second-order elliptic operator. K is the strike price.

Followed by a beautiful result, the analytical solution is just the Black-Scholes formula that we discussed in chapter 2.

Chapter 4

A Case Study on a Chinese Structured Deposit Pricing Problem

On November 30^{th} 2015, the International Monetary Fund agreed to add the Chinese Yuan to its reserve currency basket, which was a landmark in China's global economic emergence and in turn will boost the growth of the global economy as well as sustain its stability. Since 2005, China has executed a managed floating exchange-rate regime, and more and more structured deposit linked to the foreign stock index appeared. In this chapter, we will carry on a practical pricing problem using the application to stochastic differential equations from previous chapters to verify the feasibility and accuracy.

4.1 Problem Description

There is a Chinese structured deposit product linked to S&P 500 index. r_0 is the risk-free return rate, $\hat{S}$ is the average closing during the deposit period, K is the first closing index, θ is the participation rate which means one can gain from the rate of return for S&P 500, here θ is 0.5. The product can not be exercised until the end of deposit period and do not have dividend during the deposit period just like a European option. The product also has the following property:

Deposit Period	Principal-guaranteed	Return
24 months	100%	$1+min(r_0,\ max(0,\ (\frac{\hat{S}-K)}{K})\theta)$

Table 4.1: Product Instruction

4.2 Mathematical Modelling and Pricing Formula

There are some assumptions:

1. The S&P 500 index S_t follows the geometric Brownian motion, recall (2.14),i.e.,

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

2. There is a risk-free interest rate $r > 0$.

3. The market is frictionless. That is there are no arbitrage opportunities and no transaction cost, no taxes.

4. The bank only pay at time $t = T$. There is no chance to end the contract in advance.

5. The product does not affect by the exchange rate.

6. The face value for the deposit product is F.

To make the model more simple, we consider the problem within only one observation period, i.e. $[0,T]$. Suppose that $S_t = \max_{0\leq\tau\leq t} S_\tau$ is the highest S&P 500 index during time period, then S_T is the maximum index till expiration T. K, as the first closing index, is actually S_0. The if we denote $C = C(S_t,t)$ is the value of the product given by

$$C|_{t=T} = 1 + \min\{r_0, \frac{(S_t - S_0)^+}{2S_0}\}.$$

From previous discussion, the pricing model define as below

$$\begin{cases} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 & (0 \le S < \infty,\ 0 \le t < T) \\ \frac{\partial C}{\partial S_T}|_{t=T} \\ C|_{t=T} = 1 + \min\{r_0, \frac{(S_t - S_0)^+}{2S_0}\}. \end{cases}$$

Moreover, if we have N observation period: $[T_0, T_1], [T_1, T_2], ..., [T_{N-1}, T_N]$, and $0 = T_0 < T_1 < \cdots < T_{N-1} < T_N = T$. We denote that $C_i = C|_{T_{i-1} \le t \le T_i}$, and let

$$\mathscr{L}C_i = \frac{\partial C_i}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_i}{\partial S^2} + rS\frac{\partial C_i}{\partial S} - rC_i,$$

$S_{T_i} = \max_{T_{i-1} \le \tau \le T_i} S_\tau$, then we can build the model for every period, and use the continuity of $t = T_i$ to build the model for $[0, T]$.

First, we start with time interval $[T_{N-1}, T_N]$, and $S_{T_1}, S_{T_2}, ..., S_{T_{N-1}}, S_{T_N}$ are known observation, then $C_N = C(S_t, S_{T_N}, t)$ satisfying:

$$\begin{cases} \mathscr{L}C_N = 0 & (0 \le S < \infty,\ T_{N-1} \le t < T_N) \\ \frac{\partial C_N}{\partial S_{T_N}}|_{S_t = S_{T_N}} = 0 \\ C_N|_{t=T_N} = 1 + \min\{r_0, \frac{(\frac{S_{T_1} + ... + S_{T_{N-1}} + S_{T_N}}{N} - S_0)^+}{2S_0}\}. \end{cases}$$

We can get the solution for $C_N(S_t, S_{T_N}, t) := \hat{C}_{N-1}(S_t, S_{T_{N-1}}, t)$ and since the continuity of $t = T_{N-1}$, we have $C_{N-1}(S_t, S_{T_{N-1}}, t)$ on $[T_{N-2}, T_{N-1}]$ satisfying:

$$\begin{cases} \mathscr{L}C_{N-1} = 0 & (0 \le S < \infty,\ T_{N-2} \le t < T_{N-1}) \\ \frac{\partial C_{N-1}}{\partial S_{T_{N-1}}}|_{S_t = S_{T_{N-1}}} = 0 \\ C_{N-1}|_{t=T_{N-1}} = 1 + \min \hat{C}_{N-1}(S_t, S_{T_{N-1}}, t)\}. \end{cases}$$

We will get the value of the product C_{N-2}. Finally, by induction we will get the initial value $C_1|_{t=0}$.

4.3 Monte Carlo Simulation and Numerical Result

Monte Carlo methods can generate random numbers to simulate the sampling come from certain processes that following a certain distribution. It is very applicable and sometimes can be applied for complicated problem, like modelling financial systems. I will use Mathematica to monitor the stock market, carry on a Monte Carlo simulation to estimate the volatility. Then I will give the pricing the product base on the risk-neutral assumption and Black-Scholes formula.

Monitor the Stock Market

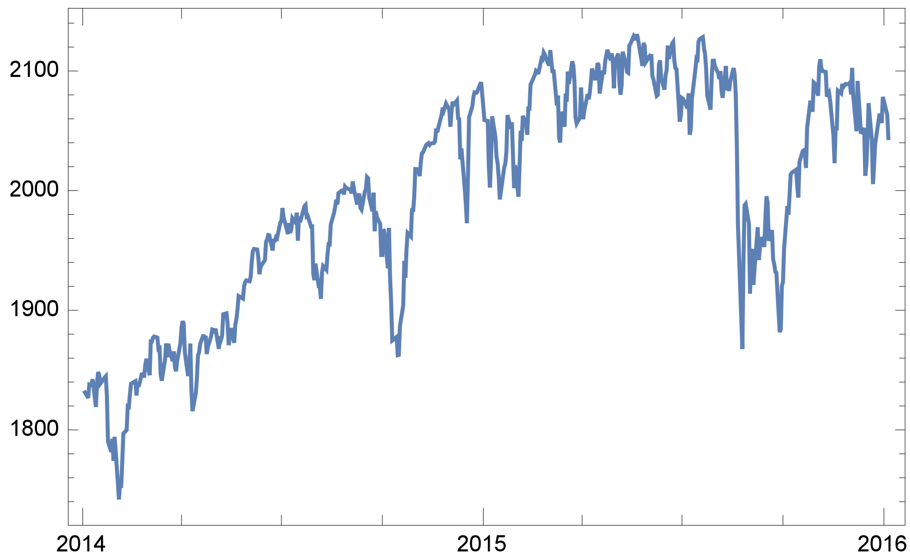
We use the built-in Wolfram Lauguage "FinancialData" to retrieve the close index of S&P 500 from Jan 3,2014 to Dec 31,2015, and we change the data into a time series, plot it:

```
in: data=FinancialData["SP500", {{2014, 1, 3}, {2015, 12, 3}}]

sp500 = TimeSeries[ data[[All, 2]], {data[[1, 1]], Automatic, "BusinessDay"}]

DateListPlot[sp500]
```

Fig. 4.1 shows that the fluctuation of S&P 500 for the recent two years. It seems random, however, the index can be modelled by geometric Brownian motion.



2 Figure 4.1: S&P 500 Index from Jan 2014 to Dec 2015

Estimate the Stock Market

We assume that the stock price followed a geometric Brownian motion, that is in the form

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

Then we use the "EstimatedProcess" to estimate $(\mu, \sigma,\ S_0)$:

in: process = EstimatedProcess[sp500["Values"], GeometricBrownianMotionProcess[μ, σ,S_0]]

out: GeometricBrownianMotionProcess$[0.000255487, 0.00857113, 1831.37]$

Here $\sigma = 0.0085713$ is not the annual volatility that we want. Since there are almost 252 business days for stock market, We annualised the volatility by

$$\text{annual volatility } \hat{\sigma} = \frac{\sigma}{\sqrt{\frac{1}{252}}} = 0.136063$$

Monte Carlo Simulation

From the former discussion in Chapter 2, the stock price follows the logarithm normal distribution. We estimate the parameter and generate the random value base on the parameter for 24 observations to see the random walks:

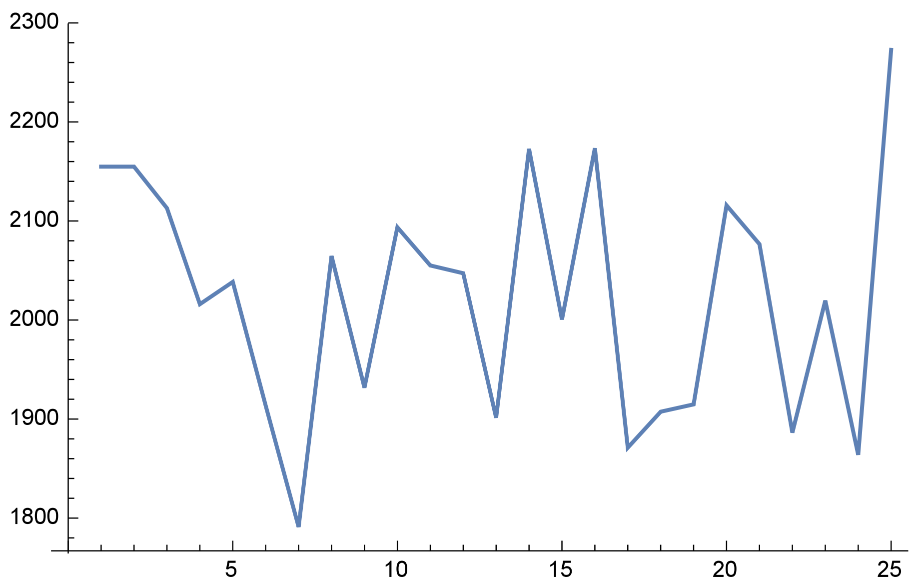
in: ditribution = EstimatedDistribution[data[[All, 2]], LogNormalDistribution[mu, vol]]

out: LogNormalDistribution[7.59805, 0.0477576]

in: rnorm1 = RandomVariate[LogNormalDistribution[7.598045472370625‘, 0.047757585864018494], 24]

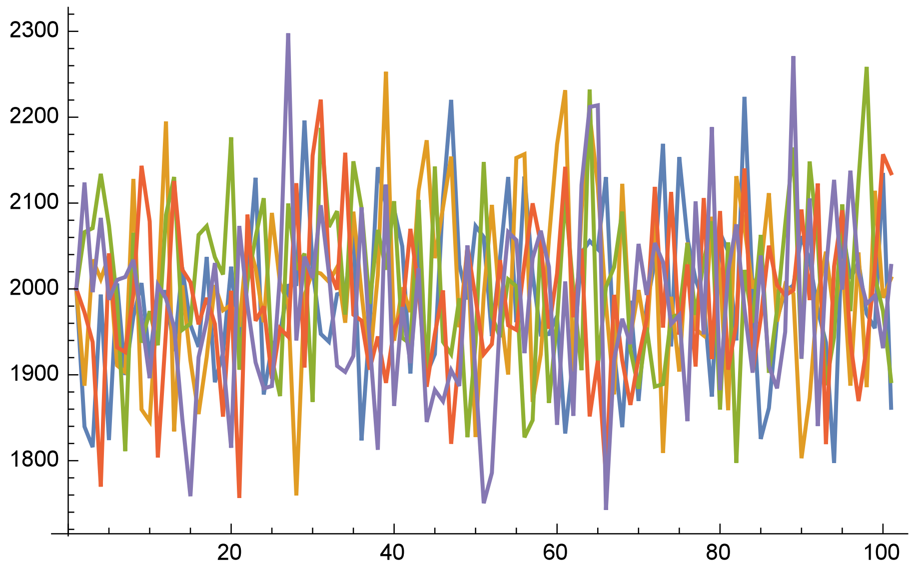
out: {2154.85, 2112.94, 2015.97, 2038.48, 1913.49, 1791.03, 2064.65, 1931.65, 2093.62, 2055.09, 2047.23, 1901.32, 2172.81, 2000.47, 2173.46, 1871.11, 1907.43, 1914.81, 2115.78, 2076.61, 1886.27, 2019.63, 1863.88, 2272.74}

We plot it and get:



3 Figure 4.2: Random Walk for 24 Observations

Also we can see the simulation for multiple random walks paths to verify the consistence with geometric Brownian motion:



4 Figure 4.3: 5 Random Walks Simulation

Then we fix an sample by generating 504 (2 years) value based on the estimated logarithm distribution. We divide the 504 value into 24 months and find the highest value during per month $(S_i)_{1\leq i\leq 24}$ and take the mean by $\bar{S} = \frac{1}{24}\sum_{i=1}^{24} S_i = 2064.32$

By the risk-neutral assumption and the modification of the Black-Scholes model. We get the pricing formula should be:

$$C = F \times e^{-r(T-t)} + \frac{F \times \theta \times \bar{S}}{S_0} N(d_1) - Fe^{-r(T-t)} N(d_2). \tag{4.1}$$

where

$$d_1 = \frac{\log(\bar{S}/S_0) + (r + \frac{\sigma^2}{2})(T-t)}{\sqrt{T-t}\sigma}, \tag{4.2}$$

$$d_2 = \frac{\log(\bar{S}/S_0) + (r - \frac{\sigma^2}{2})(T-t)}{\sqrt{T-t}\sigma} = d_1 - \sqrt{T-t}\sigma. \tag{4.3}$$

From the simulation value, we have

$S_0 = 1914.7$

$\bar{S} = 2064.32$

$r = 2.25\%$ is the interest rate for 2 years

$\hat{\sigma} = 0.136063$ is the annual volatility

Usually, the face value should be higher than the price from common sense. Otherwise, the product is not profitable for the bank. From the results running in Mathematica, if we set the Face value $F = 1$, we have the theoretical price for the deposit product should be $C = 0.70$, which verifies our assumption. Given that there assumes no arbitrage opportunities and no transaction cost, the result is acceptable.

Chapter 5

Conclusion and Future Work

In this thesis, we focus on the application of stochastic differential equations to option pricing. We use the a geometric Brownian motion to model the price of underlying asset. With the arbitrage-free and risk-neutral assumption, Black-Scholes model is first to be used to analyse the option price. We can see that the Black-Scholes model is very applicable and feasible. We have discussed the property of Black-Scholes model and talked about the important parameter volatility which developed into an optimal control problem.

Moreover, as a result of the uncertainty of the future earning, it is a natural thought that we apply backward stochastic differential equations (BSDEs) to the pricing problem. We gave the general results for BSDEs theory. We saw that how the BSDEs describe and solve for the financial problems.

At last, a case study on a real European option-like Chinese deposit pricing problem was proposed . Thanks to the powerful software, Mathematica, we use the results of Monte Carlo simulation to figure out an estimated acceptable price. The case study shows that all of our topics discussed in the thesis is not only confine to the book, but can be carried out and apply to the real financial world.

Fettered by the limit time to work on the thesis and also restricted by my current knowledge reserve, the thesis just gives my exploration for the application to stochastic differential equations on option pricing. There are much more works can be done in the future based on this thesis:

1. The pricing for the financial derivative type can be extend to American option, Asian option and other derivative whose price are related to the underlying assets.

2. The numerical method for solving BSDEs could be developed and improved. That could involved with various parallel computing.

3. Different types of model could be discussed other than the Black-Scholes model. And the convergence and asymptotic analysis could be further discussed.

4. The pricing problem for higher dimensions portfolio could be defined and the corresponding ways to solve that waiting to be clarified.

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