Depth and Associated Primes of Modules over a Ring

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Abstract

This thesis consists of three main topics. In the first topic, we let R be a commutative Noetherian ring, I, J ideals of R, M a finitely generated R-module and F an R-linear covariant functor. We ask whether the sets $\operatorname{Ass}_R F(M/I^nM)$ and the values $\operatorname{depth}_J F(M/I^nM)$ become independent of n for large n.

In the second topic, we consider rings of the form $R = k[x^a, x^{p_1}y^{q_1}, \ldots, x^{p_t}y^{q_t}, y^b]$, where k is a field and x, y are indeterminates over k. We will try to formulate simple criteria to determine whether or not R is Cohen-Macaulay.

Finally, in the third topic we introduce and study basic properties of two types of modules over a commutative Noetherian ring R of positive prime characteristic. The first is the category of modules of finite F-type. They include reflexive ideals representing torsion elements in the divisor class group. The second class is what we call F-abundant modules. These include, for example, the ring R itself and the canonical module when R has positive splitting dimension. We prove many facts about these two categories and how they are related. Our methods allow us to extend previous results by Patakfalvi-Schwede, Yao and Watanabe. They also afford a deeper understanding of these objects, including complete classifications in many cases of interest, such as complete intersections and invariant subrings.

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Contents

Chapter 1

Introduction

This Thesis consists of three projects in Commutative Algebra. In this Introduction, we will first describe the subject of Commutative Algebra, define the key terms in this Thesis, and then describe the three projects one by one.

1.1 The subject of Commutative Algebra

A classical problem in mathematics is to solve a system of equations. We may think of the subject of Commutative Algebra as an effort to tackle this problem. Let us begin with a field k, such as the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. Let us consider the set k^n where n is a natural number. Let $x_1, x_2, \ldots, x_n$ be variables over k and let $f_1, f_2, \ldots, f_j$ be polynomials in $x_1, x_2, \ldots, x_n$. A natural question to ask is then: What are the solutions of the system of equations $f_1 = 0$, $f_2 = 0$, $\ldots$, $f_j = 0$ in k^n? In general, this question is difficult to answer. Commutative Algebra seeks to understand and answer the question, perhaps indirectly, as follows. We consider the polynomial ring $S = k[x_1, x_2, \ldots, x_n]$ and the ideal I of S generated by $f_1, f_2, \ldots, f_j$. Let $Z(I)$ be the set of solutions of the system of equations $f_1 = f_2 = \cdots = f_j = 0$ and let R be the quotient ring S/I. We can then think of R as the set of polynomial functions over $Z(I)$. Studying the ring R could then hopefully yield information about the solution set $Z(I)$ of $f_1 = f_2 = \cdots = f_j = 0$.

More generally, a main object of study in Commutative Algebra is the class of commutative Noetherian rings that contain the element 1. A ring R is commutative if and only if multiplication in R is commutative, i.e. $ab = ba$ for all $a, b \in R$. A commutative ring is Noetherian if and only if all of its ideals are finitely generated. The polynomial ring $S = k[x_1, x_2, \ldots, x_n]$ and all of its quotient rings are Noetherian. In particular, suppose that we have an arbitrary number of polynomials $\{f_j\}_{j\in J}$ in S and we want to solve the system of equations $\{f_j = 0 \mid j \in J\}$. The fact that S is Noetherian means that there always exists a finite subset $\{f_{j_1}, f_{j_2}, \ldots, f_{j_m}\}$ of $\{f_j\}_{j\in J}$ such that the solution sets of the systems of equations $f_{j_1} = f_{j_2} = \cdots = f_{j_m} = 0$ and $\{f_j = 0 \mid j \in J\}$ are identical. This means that if we are solving polynomial equations in the variables $x_1, x_2, \ldots, x_n$, we may always assume that we only have finitely many equations to begin with.

1.2 Key definitions and known results

All our rings will be commutative with 1. Let R be a Noetherian ring unless specified otherwise, let $\mathrm{Mod}(R)$ denote the category of all R-modules and let $\mathrm{mod}(R)$ denote the category of finitely generated R-modules. We will now introduce some important notions concerning R that will appear in this Thesis and recall some related results.

The first notion that we will consider is about factorization. Over the ring of integers $\mathbb{Z}$, any positive number can be factored uniquely into a product of its (positive) prime factors. Given a general commutative Noetherian ring R and an ideal I of R with $I \neq R$, one may also ask if such a factorization exists for I as well. It turns out that if we change the notion of factorization slightly, we do have a similar result, which can be phrased more generally in terms of modules. In this result, the factorization of an element is replaced by an intersection of submodules.

Definition 1.2.1. Let $M \in \mathrm{mod}(R)$. We say that a prime ideal P of R is an **associated prime** of M, written $P \in \mathrm{Ass}_R(M)$, if there is $m \in M$ such that $P = \{r \in R \mid rm = 0\}$.

Equivalently, $P \in \mathrm{Ass}_R(M)$ iff there is an injective R-homomorphism $(R/P) \hookrightarrow M$.

Definition 1.2.2. Let $M \in \mathrm{Mod}(R)$ and $N \subseteq M$ a submodule. If $\mathrm{Ass}_R(M/N) = \{P\}$, then we say that N is a **primary submodule** or P-primary submodule of M.

Theorem 1.2.3 (Primary Decomposition)**.** *Let $M \in \mathrm{mod}(R)$ and $N \subsetneq M$ be a submodule. Then there is a primary decomposition*

$$N = N_1 \cap \cdots \cap N_j \tag{$*$}$$

of N, where $N_1, \ldots, N_j$ are primary submodules of M. If $(*)$ is a shortest primary decomposition and each N_i is P_i-primary, then the P_i are distinct and $\mathrm{Ass}_R(M/N) = \{P_1, \ldots, P_j\}$.

In Theorem 1.2.3, if $M = R$ and $N = I$ is an ideal of R, then we may think of $\mathrm{Ass}_R(R/I)$ as the set of "prime factors" of I.

Our second notion is about size (in an algebraic sense). Let I be an ideal of R and $M \in \mathrm{Mod}(R)$. We may ask how to measure the "size" of M with respect to I. One method in Commutative Algebra uses the notion of depth, which involves the concept of a regular sequence.

Definition 1.2.4. Let R be a commutative ring, $M \in \mathrm{Mod}(R)$ and $\vec{x} = x_1, \ldots, x_d$ be a sequence of elements in R. We say that $\vec{x}$ is an M**-(regular) sequence**, or a **regular sequence on** M, of length d if $M \neq (x_1, \ldots, x_d)M$ and the multiplication maps $M/(x_1, x_2, \ldots, x_{j-1})M \xrightarrow{x_j} M/(x_1, x_2, \ldots, x_{j-1})M$ are injective for $1 \leqslant j \leqslant d$.

Definition 1.2.5. Let R be a Noetherian ring, I an ideal of R and $M \in \mathrm{mod}(R)$.

(i) The I-**depth** of M, denoted $\mathrm{depth}_I M$, is defined to be the maximum length of an M-regular sequence in I if $M \neq IM$. If $M = IM$, then $\mathrm{depth}_I M = +\infty$ by convention.

(ii) If $(R, \mathfrak{m})$ is a local ring, then we sometimes use $\mathrm{depth}\, M$ to denote $\mathrm{depth}_{\mathfrak{m}} M$.

Theorem 1.2.6. *Let I be an ideal of R and $M \in \mathrm{mod}(R)$. Then* $\mathrm{depth}_I M = \min\{j \mid$

$\operatorname{Ext}_R^j(R/I, M) \neq 0\}$. *In particular, if $M \neq IM$, then all maximal M-regular sequences have the same length.*

Finally, our third notion is also about size (in a geometric sense).

Definition 1.2.7. Let $(R, \mathfrak{m})$ be a local ring.

(i) We let $\dim R = \max\{n \mid \text{there is a chain of prime ideals } \mathfrak{m} = P_n \supsetneq P_{n-1} \supsetneq \cdots \supsetneq P_1 \supsetneq P_0 \text{ in } R\}$, called the **dimension** of R.

(ii) If $M \in \operatorname{Mod}(R)$, then we let the **dimension** of M be $\dim M = \dim(R/\operatorname{ann}(M))$.

(iii) If $M \in \operatorname{mod}(R)$, then we say that M is **Cohen-Macaulay**, abbreviated as CM, if either $M = 0$ or $M \neq 0$ and $\operatorname{depth} M = \dim M$. We say that M is **maximal Cohen-Macaulay** if $\operatorname{depth} M = \dim R$.

In other words, we say that M is Cohen-Macaulay if the two notions of "size" that we have seen coincide. We think of such an M as being "nice." The terminology "maximal Cohen-Macaulay" comes from the fact that $\operatorname{depth} M \leqslant \dim M \leqslant \dim R$ for any $0 \neq M \in \operatorname{mod}(R)$. One may think of maximal Cohen-Macaulay modules as the analogue of nonzero finite dimensional vector spaces over a field.

Definition 1.2.8. Suppose that R is not necessarily local. Let m-Spec(R) denote the set of all maximal ideals of R.

(i) We let $\dim R = \max\{\dim R_{\mathfrak{m}} \mid \mathfrak{m} \in \text{m-Spec}(R)\}$.

(ii) If $M \in \operatorname{Mod}(R)$, then we still have $\dim M = \dim(R/\operatorname{ann}(M))$.

(iii) If $M \in \operatorname{mod}(R)$, then we say that M is (maximal) Cohen-Macaulay if $M_{\mathfrak{m}}$ is (maximal) Cohen-Macaulay for all $\mathfrak{m} \in \text{m-Spec}(R)$.

(iv) We say that R is a **Cohen-Macaulay ring** if R is a Cohen-Macaulay R-module.

Theorem 1.2.9. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded Noetherian ring such that R_0 is a field. Let $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$ be the homogeneous maximal ideal and let $M \in \operatorname{mod}(R)$ be a graded R-module. Then M is Cohen-Macaulay if and only if $M_{\mathfrak{m}}$ is Cohen-Macaulay.*

1.3 Outline of contents

We continue to use our conventions in Section 1.2. In Chapter 2 we seek to extend the following results from Brodmann, especially the ones on associated primes.

Theorem 1.3.1. *[9, 10] Let I, J be ideals of R, $M \in \operatorname{mod}(R)$ and n denote a natural number. Then the sets $\operatorname{Ass}_R(M/I^nM)$, $\operatorname{Ass}_R(I^{n-1}M/I^nM)$ and the values $\operatorname{depth}_J(M/I^nM)$, $\operatorname{depth}_J(I^{n-1}M/I^nM)$ become independent of n for large n.*

We are able to generalize the results using the following notion.

Definition 1.3.2. [15] Let $F\colon \operatorname{mod}(R) \to \operatorname{mod}(R)$ be an R-linear covariant functor. For $M \in \operatorname{Mod}(R)$, we let $h_M(-)$ denote the functor $\operatorname{Hom}_R(M, -)$. We say that F is coherent if there exist $N, L \in \operatorname{mod}(R)$ and an exact sequence $h_L \to h_N \to F \to 0$.

Example 1.3.3. [15] For any $N \in \operatorname{mod}(R)$ and $i \geqslant 0$, the functors $\operatorname{Tor}_i^R(N, -)$ and $\operatorname{Ext}_R^i(N, -)$ are coherent.

We will prove the following result, a special case of which appears as Theorem 2.1.10.

Theorem 1.3.4. *Let I, J be ideals of R, $M \in \operatorname{mod}(R)$, M' be a submodule of M and F be a coherent functor. Then the sets $\operatorname{Ass}_R F(M/I^nM')$, $\operatorname{Ass}_R F(I^nM/I^nM')$ and the values $\operatorname{depth}_J F(M/I^nM')$, $\operatorname{depth}_J F(I^nM/I^nM')$ become independent of n for large n.*

The rest of Chapter 2 investigates whether the results in Theorem 1.3.4 about associated primes still hold when F is not coherent. The investigation considers some special cases. For example, we consider some familiar functors F such as $F = \Gamma_I$ or τ_S, where I is an ideal of R, S is a multiplicatively closed subset of R and τ_S is the torsion functor. Alternatively, we may consider special classes of rings R such as Dedekind domains. We may also restrict both F and R. For example, we are able to prove a positive result when F arises from what we call a middle finite complex, and R is a one-dimensional domain. Still one can expect much more work to be done beyond these special cases.

Chapter 3 follows the thoughts of Gröbner in [14] and asks how to determine whether a (positive affine) semigroup ring, which we now define, is Cohen-Macaulay. Let k be a field, $x_1, x_2 \ldots, x_d$ indeterminates over k and S a finite set of monomials in $x_1, x_2, \ldots, x_d$. A semigroup ring is then a k-algebra of the form $R = k[S]$, i.e. the subring of the polynomial ring $k[x_1, x_2, \ldots, x_d]$ generated by the monomials in S. As explained in [39], these rings are of interest because they are simple enough to study but general enough to illustrate problems in Algebraic Geometry. We recall from Theorem 1.2.9 that $R = k[S]$ is Cohen-Macaulay if and only if $R_{\mathfrak{m}}$ is Cohen-Macaulay, where $\mathfrak{m}$ is the maximal ideal generated by the monomials in S.

The first main result regarding whether a semigroup ring $R = k[S]$ is Cohen-Macaulay is due to Hochster.

Theorem 1.3.5. *[17] If $k[S]$ is normal, then it is Cohen-Macaulay.*

In Chapter 3 we will approach the problem from a different direction. We will consider a special case where R is what is called a projective monomial curve in $\mathbb{P}^3$. We are able to find a simple numerical criterion to determine whether R is Cohen-Macaulay. The following result that contains the numerical criterion and more precise information appears as Lemma 3.4.4 and Theorem 3.4.6.

Theorem 1.3.6. *Let $R = k[x^n, x^{n-\ell}y^\ell, x^{n-m}y^m, y^n]$ where $0 < \ell < m < n$.*
Let c_1 be the smallest integer such that there are $m/\gcd(\ell, m) \geqslant a_1 > 0$ and $b_1 > 0$ with

$$a_1\ell + b_1 m = c_1 n \tag{1.3.1}$$

Let b_2 be the smallest integer such that there are $n/\gcd(\ell, n) > a_2 \geqslant 0$ and $c_2 > 0$ with

$$-a_2\ell + b_2 m = c_2 n \tag{1.3.2}$$

Let a_3 be the smallest positive integer such that there are $n/\gcd(m, n) > b_3 \geqslant 0$ and $c_3 \geqslant 0$

with

$$a_3\ell - b_3 m = c_3 n \tag{1.3.3}$$

Let $d = \gcd(\ell, m, n)$ *and* H *be the subgroup of* $\mathbb{Z}/n\mathbb{Z}$ *generated by* ℓ *and* m. *Then:*

(i) (1.3.1) = (1.3.2) + (1.3.3)

(ii) $n = d \begin{vmatrix} a_3 & -b_3 \\ -a_2 & b_2 \end{vmatrix} = d \begin{vmatrix} a_3 & -b_3 \\ a_1 & b_1 \end{vmatrix} = d \begin{vmatrix} a_1 & b_1 \\ -a_2 & b_2 \end{vmatrix}$

(iii) *The* $n/d = |H|$ *monomials* $\mathcal{B} = \{(x^{n-\ell}y^\ell)^a(x^{n-m}y^m)^b \mid (a,b) \in B\}$ *are linearly independent over* (x^n, y^n), *where*

$$\begin{aligned} B &= \{(a,b) \mid a < a_1 \text{ and } b < b_2\} \cup \{(a,b) \mid a < a_3 \text{ and } b < b_1\} \\ &= \{(a,b) \mid a < a_3 \text{ and } b < b_2\} \setminus \{(a,b) \mid a \geqslant a_1 = a_3 - a_2 \text{ and } b \geqslant b_1 = b_2 - b_3\} \end{aligned}$$

(iv) *The ring* R *is Cohen-Macaulay iff* $\mathcal{B}$ *is a basis of* $R/(x^n, y^n)$ *over* k *iff* $b_2 \geqslant a_2 + c_2$.

A related question to (iv) is: What is a vector space basis of $R/(x^n, y^n)$ over k? We have the following result from Theorem 3.4.10.

Theorem 1.3.7. *There is an algorithm that uses the equations* (1.3.1) *and* (1.3.2) *only to generate a basis of* $R/(x^n, y^n)$ *over* k.

In fact, we have similar results for slightly more general semigroup rings of the form $R = k[x^d, x^e y^\ell, x^f y^m, y^n]$ with $d, n > 0$, $e, f, \ell, m \geqslant 0$ and $(e, \ell) \neq (0, 0)$.
Another related question to (iv) is to find the multiplicity of the ideal $I = (x^n, y^n)$, denoted by $e(I)$, which can be calculated from the Hilbert polynomial of the graded ring $\bigoplus_{n=0}^{\infty} I^n/I^{n+1}$. This question is motivated by the following general fact.

Theorem 1.3.8. *Let* $(R, \mathfrak{m})$ *be a local ring. Then* R *is a Cohen-Macaulay ring if and only if* $\lambda(R/I) = e(I)$ *for some* I *generated by a system of parameters.*

In our case, $\lambda(R/I) = \dim_k R/(x^n, y^n)$. We are able to prove the following result from Theorem 3.2.7.

Theorem 1.3.9. *Let* $R = k[x^a, x^{p_1}y^{q_1}, x^{p_2}y^{q_2}, \ldots, x^{p_t}y^{q_t}, y^b]$ *and* $H \subseteq (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$ *be the subgroup generated by* $(p_1, q_1), (p_2, q_2), \ldots, (p_t, q_t)$. *Then we can find constants* $t_{p,q}, s_{p,q}, n_{p,q}$ *for* $(p, q) \in H$ *such that the Hilbert polynomial of* (x^a, y^b) *is* $P(n) = |H|(n+1) + \sum_{(p,q)\in H}(t_{p,q} + s_{p,q})$ *and the Hilbert function equals the Hilbert polynomial for* $n \geqslant \max_{(p,q)\in H}(n_{p,q})$. *In particular,* $e((x^a, y^b)) = |H|$.

In Chapter 4 we turn to characteristic p methods, which is a subject area that has applications to Algebraic Geometry. Let us give some background information for the Chapter. For a commutative ring R with prime characteristic p, the map $\varphi\colon R \to R$ with $\varphi(r) = r^p$ is a ring homomorphism and is called the Frobenius endomorphism. The map φ is widely used in characteristic p methods. It is hard to keep track of all subjects that use characteristic p methods, but some areas that are related to our work include the study of: the Hilbert-Kunz multiplicity, tight closure theory, rings of differential operators, singularities of R, F-purity and F-regularity, finite F-representation type and F-regularity, and the F-signature.

In our situation, $(R, \mathfrak{m}, k)$ will be a local ring and all R-modules will be finitely generated. If M is an R-module and $e \geqslant 0$, we let ${}^eM$ denote the abelian group M viewed as an R-module via φ^e, i.e. through restriction of scalars. We will further assume that R is reduced and F-finite, i.e. ${}^1R \in \mathrm{mod}(R)$, and we let $\alpha(R) = \log_p[k : k^p]$. Since R is reduced, we may identify ${}^eR$ with R^{1/p^e}, the ring of p^eth roots of R, and the map $\varphi^e\colon R \to R$ with the inclusion map $R \hookrightarrow R^{1/p^e}$. We let $F^e\colon \mathrm{mod}(R) \to \mathrm{mod}(R)$ denote the eth Peskine-Szpiro functor given by $F^e(M) = M \otimes_R {}^eR$, i.e. $F^e(M)$ is given by extension of scalars. Given $S \subseteq \mathrm{mod}(R)$, we use $\mathrm{add}_R(S)$ to denote the additive subcategory of $\mathrm{mod}(R)$ generated by S. We have the following definition from 4.1.1.

Definition 1.3.10.

(1) Let M be an R-module such that $\mathrm{Supp}(M) = \mathrm{Spec}\, R$ and M is locally free in codimension 1. We write $M(e) = F^e_R(M)^{**}$, the reflexive hull of $F^e_R(M)$, viewed as an R-module by identifying ${}^eR$ with R. We say that M is of finite F-type if $\{M(e)\}_{e\geqslant 0} \subseteq \mathrm{add}_R(X)$ for some R-module X. We let $\mathcal{FT}(R)$ denote the category of R-modules of finite F-type.

(2) Let N, L be R-modules. Let b_e be maximum such that ${}^eN = L^{\oplus b_e} \oplus N_e$ for some N_e. We say that (N, L) is an abundant pair if $\liminf_{e\to\infty} p^{e\alpha(R)}/b_e = 0$.

(3) Let L be an R-module. We say that L is an F-abundant module if (N, L) is an abundant pair for some N.

Example 1.3.11.

(1) Examples of modules of finite F-type include torsion elements of the divisor class group of a normal domain R (without any assumption about the order of the element), finite integral extensions that are étale in codimension one, or F-periodic vector bundles on the punctured spectrum of R and of the corresponding projective variety X when R is a local cone of some embedding of X.

(2) For F-abundant modules, R has positive F-splitting dimension if and only if (R, R) is an abundant pair.

(3) A good source of examples in both cases is the rings of invariants of a finite group.

Our main technical result below, which appears as Theorem 4.3.10, says roughly that under various extra conditions, if M is of finite F-type and L is F-abundant, then $\mathrm{Hom}_R(M, L)$ is maximal Cohen-Macaulay. Under such conditions, our result gives a strong generalization of Yao's result [42] on Cohen-Macaulayness of F-contributors.

Theorem 1.3.12. *Suppose that R is (S_2) and equidimensional. Let M, N be R-modules such that $M \in \mathcal{FT}(R)$ and N is (S_2). Assume that for every $P \in \operatorname{Spec} R$ such that $\mathrm{ht}(P) \geqslant 3$, (N_P, L_P) is an abundant pair. Assume further that for every $P \in \operatorname{Spec} R$ such that $3 \leqslant \mathrm{ht}(P) < d$, we have $N_P \in \mathrm{add}(L_P)$. Then $\mathrm{Hom}_R(M(e), L)$ is maximal Cohen-Macaulay for all $e \geqslant 0$.*

The following result, which appears as Corollary 4.3.12, extends a result by Watanabe [41].

Corollary 1.3.13. *Suppose that R is strongly F-regular and I is a reflexive ideal such that $[I]$ is torsion in the divisor class group $\mathrm{Cl}(R)$. Then I is maximal Cohen-Macaulay.*

The next theorem, which appears as Theorem 4.6.3, extends a result by Patakfalvi-Schwede [32].

Theorem 1.3.14. *Let R be an F-finite normal domain with perfect residue field and $X = \operatorname{Spec} R$. Let Δ be a $\mathbb{Q}$-divisor on X such that the pair (X, Δ) is strongly F-regular. Let D be an integral divisor such that $rD \sim r\Delta'$ for some integer $r > 0$ and $0 \leqslant \Delta' \leqslant \Delta$. Then $\mathscr{O}_X(-D)$ is Cohen-Macaulay.*

We also obtain a complete classification of $\mathcal{FT}(R)$ in many cases of interest. Some examples include the following which appear as Theorem 4.4.14, Lemma 4.4.10 and Corollary 4.4.13 respectively.

Theorem 1.3.15. *Suppose that R is a complete intersection and M is an R-module that is free in codimension 2. Then $M \in \mathcal{FT}(R)$ if and only if M^{**} is free.*

Lemma 1.3.16. *Suppose that R is regular. Consider the following statements:*

(a) $M \in \mathcal{FT}(R)$

(b) M^* *is free.*

(c) M^{**} *is free.*

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). If M is free in codimension 1, then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Corollary 1.3.17. *Suppose that k is algebraically closed. Let $S = k[[x_1, \ldots, x_d]]$. Let G be a finite subgroup of $GL(d, k)$ that contains no pseudo-reflections such that the order of G is coprime to p. Let $R = S^G$. Then $\mathcal{FT}(R) = \operatorname{add}_R S$.*

Chapter 2

Covariant Functors and Asymptotic Stability

2.1 Introduction

In this Chapter, we will extend two results on asymptotic stability by M. Brodmann. Let us begin by fixing some terminology. A ring will mean a commutative ring with unity, unless specified otherwise. For a ring R, we let $\mathrm{Mod}(R)$ denote the category of R-modules and $\mathrm{mod}(R)$ the category of finitely generated R-modules. A functor will mean a covariant functor. For a nonempty set X and a sequence of elements $\{x_n\}_{n\geqslant k}$ of X, we say that asymptotic stability holds for the elements x_n, or that the elements x_n stabilize, if the sequence $\{x_n\}_{n\geqslant k}$ is eventually constant.

For the rest of this section, we will let R be a Noetherian ring unless specified otherwise, $L, M, N \in \mathrm{mod}(R)$ and I, J be ideals of R. The background of our project can be traced back to one of Ratliff's papers.

Question 2.1.1. [33, Introduction] Suppose that R is a domain and P is a prime ideal of R. If $P \in \mathrm{Ass}_R(R/I^k)$ for some $k \geqslant 1$, is $P \in \mathrm{Ass}_R(R/I^n)$ for all large n ?

Brodmann [9, (9)] gave a negative answer to the question, but at the same time, he proved a related, by now well-known result. Using the notation established so far, we will state his first result that we are interested in.

Theorem 2.1.2. *[9, page 16] The sets* $\operatorname{Ass}_R(M/I^nM)$ *and* $\operatorname{Ass}_R(I^{n-1}M/I^nM)$ *stabilize.*

The second result that we are interested in is as follows.

Theorem 2.1.3. *[10, Theorems 2(i) and 12(i)] Asymptotic stability holds for the values* $\operatorname{depth}_J(M/I^nM)$ *and* $\operatorname{depth}_J(I^{n-1}M/I^nM)$.

Most of this Chapter will be related to Theorem 2.1.2. There have been numerous generalizations of the theorem over the years. Here are a few of them[1].

Theorem 2.1.4. *[29, Theorem 1] The sets* $\operatorname{Ass}_R \operatorname{Tor}_i^R(N, R/I^n)$ *and* $\operatorname{Ass}_R \operatorname{Tor}_i^R(N, I^{n-1}/I^n)$ *stabilize for any* $i \geqslant 0$.

Theorem 2.1.5. *[20, Proposition 3.4] Let* $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ *be a complex. Suppose that* $L' \subseteq L$, $M' \subseteq M$ *and* $N' \subseteq N$ *are submodules such that* $\alpha(L') \subseteq M'$ *and* $\beta(M') \subseteq N'$. *For* $n \geqslant 0$, *let* $\mathrm{H}(n)$ *denote the homology of the induced complex*

$$\frac{L}{I^nL'} \xrightarrow{\alpha_n} \frac{M}{I^nM'} \xrightarrow{\beta_n} \frac{N}{I^nN'}$$

Then the sets $\operatorname{Ass}_R \mathrm{H}(n)$ *stabilize.*

Corollary 2.1.6. *[20, Corollary 3.5] Let* $M' \subseteq M$ *be a submodule. Then for any* $i \geqslant 0$, *the sets* $\operatorname{Ass}_R \operatorname{Tor}_i^R(N, M/I^nM')$ *and* $\operatorname{Ass}_R \operatorname{Ext}_R^i(N, M/I^nM')$ *stabilize.*

A rather extensive introduction to results related to Theorems 2.1.2 and 2.1.3 can be found in [27]. However, we will proceed in a different direction. Our main goal is to relate the theorems to the following notions.

[1] Although the theorems quoted here are related, the authors of [20] and [29] did not seem to know about the results of each other.

Notation 2.1.7. Let R be a commutative ring and $M \in \mathrm{Mod}(R)$. Then we let h_M denote the functor $\mathrm{Hom}_R(M, -)$. We let $\mathscr{F}$ denote the category of R-linear covariant functors F from $\mathrm{mod}(R)$ to itself.

Definition 2.1.8. [15, page 53] Let R be a Noetherian ring and $F \in \mathscr{F}$. We say that:

(1) F is representable if $F \cong h_M$ for some $M \in \mathrm{mod}(R)$;

(2) F is coherent if there exist $M, N \in \mathrm{mod}(R)$ and an exact sequence $h_N \to h_M \to F \to 0$;

(3) F is finitely generated if there exist $M \in \mathrm{mod}(R)$ and an exact sequence $h_M \to F \to 0$.

Remark 2.1.9. Representable $\Rightarrow$ coherent $\Rightarrow$ finitely generated $\Rightarrow$ R-linear

We can now state our main result, which will be proved in several steps in Section 2.2.

Theorem 2.1.10. *Let R be a Noetherian ring, I, J ideals of R, $M \in \mathrm{mod}(R)$ and F be a coherent functor. Then the sets $\mathrm{Ass}_R\, F(M/I^nM)$, $\mathrm{Ass}_R\, F(I^{n-1}M/I^nM)$ and the values $\mathrm{depth}_J\, F(M/I^nM)$, $\mathrm{depth}_J\, F(I^{n-1}M/I^nM)$ stabilize.*

Remark 2.1.11. Theorem 2.1.10 gives an extension of Theorem 2.1.5 in the following sense. Using the notation in Theorem 2.1.5, let $L = L'$, $M = M'$ and $N = N'$. Then Theorem 2.1.5 is an instance of Theorem 2.1.10 by Lemma 2.2.3(b) (cf. proof of Theorem 2.5.6). However, by [15, Example 5.5], not all coherent functors are of the form given by Lemma 2.2.3(b). A technical generalization of Theorem 2.1.5 is given by Corollary 2.2.2.

A summary of the rest of the Chapter is as follows. In Section 2.3, we consider two covariant R-linear functors, the zeroth local cohomology functor Γ_I where I is an ideal of R, and the torsion functor τ_S where S is a multiplicatively closed subset of R. We show that in most cases, the functors id/Γ_I and id/τ_S are finitely generated but not coherent, while the functors Γ_I and τ_S are not even finitely generated. However, if $F = \mathrm{id}/\Gamma_I$, id/τ_S, Γ_I or τ_S, then whether or not F is coherent, the sets $\mathrm{Ass}_R\, F(M/I^nM)$ and $\mathrm{Ass}_R\, F(I^{n-1}M/I^nM)$ always stabilize. In Section 2.4, we consider the case where R is a Dedekind domain. We show that if F is a finitely generated functor, then the sets $\mathrm{Ass}_R\, F(M/I^nM)$ stabilize. We give a family of non-finitely generated functors F such that the sets $\mathrm{Ass}_R\, F(M/I^nM)$ do not stabilize. In Section 2.5, we consider a complex $\mathscr{S}\colon A \to B \to C$ of R-modules where $B \in \operatorname{mod}(R)$ and the functor $F(-) = \mathrm{H}(\mathscr{S} \otimes -)$, an example of which is the zeroth local cohomology functor. We show that if R is a one-dimensional Noetherian domain, then the sets $\operatorname{Ass}_R F(M/I^n M)$ stabilize.

2.2 Proof of stability results

In the section, we let R be a Noetherian ring. All R-modules will be finitely generated unless specified otherwise. We will prove our main result, Theorem 2.1.10, which will follow from Corollary 2.2.4, Corollary 2.2.9 and Corollary 2.2.13. First, we need a slightly more general result than Theorem 2.1.5. We recall that the Theorem follows from an even more general result.

Theorem 2.2.1. *[20, Proof of Proposition 3.4] Let $I \subseteq R$ be an ideal, $T \in \operatorname{mod}(R)$ and U, V, W submodules of T such that $W \subseteq V$. Then the sets $\operatorname{Ass}_R((U + I^n V)/I^n W)$ stabilize.*

Corollary 2.2.2. *Consider the situation as in Theorem 2.1.5. Let $c \in \mathbb{N}$ and L_1, L_2 be submodules of L such that $I^c L' \subseteq L_2$. For $n \geqslant c$, let $\mathrm{H}(n)$ denote the homology of the induced complex*

$$\frac{L_1 + I^{n-c} L_2}{I^n L'} \xrightarrow{\alpha_n} \frac{M}{I^n M'} \xrightarrow{\beta_n} \frac{N}{I^n N'}$$

Then the sets $\operatorname{Ass}_R \mathrm{H}(n)$ stabilize.

Proof. We follow [20, Proof of Proposition 3.4]. By the Artin-Rees Lemma, there is $d \geqslant c$ such that for all $n \geqslant d$, $\beta(M) \cap I^n N' = I^{n-d}(\beta(M) \cap I^d N')$. Then for $n \geqslant d$, we have

$$\begin{aligned} \mathrm{H}(n) &= \frac{\ker(\beta_n)}{\operatorname{im}(\alpha_n)} \\ &= \frac{\ker(\beta) + I^{n-d}(\beta^{-1}(I^d N'))}{\alpha(L_1) + I^{n-d}(I^{d-c}\alpha(L_2) + I^d M')} \ . \end{aligned}$$

The result then follows from Theorem 2.2.1 by letting

$$T = \frac{M}{\alpha(L_1)}\ , \qquad U = \frac{\ker(\beta)}{\alpha(L_1)}\ ,$$
$$V = \frac{\beta^{-1}(I^d N') + \alpha(L_1)}{\alpha(L_1)} \quad \text{and} \quad W = \frac{I^{d-c}\alpha(L_2) + I^d M' + \alpha(L_1)}{\alpha(L_1)}\ . \qquad \square$$

Next, we recall some results from [15].

Lemma 2.2.3. *[15, Lemma 1.2, Examples 2.1–2.5]*

(a) For any $M \in \operatorname{mod}(R)$ and $F \in \mathscr{F}$, there is a natural isomorphism $\operatorname{Nat}_{\mathscr{F}}(h_M, F) \cong F(M)$ given by $T \mapsto T_M(\operatorname{id}_M)$.

(b) Let $P_\bullet$ be a complex of finitely generated R-modules. Then for any $i \in \mathbb{Z}$, the functor $\operatorname{H}_i(P_\bullet \otimes -)$ is coherent.

(c) Let $M \in \operatorname{mod}(R)$. Then for any $i \geqslant 0$, the functors $\operatorname{Tor}_i^R(M, -)$ and $\operatorname{Ext}_R^i(M, -)$ are coherent.

We then obtain the following generalization of the first half of Theorem 2.1.2. By Lemma 2.2.3(c), Corollary 2.2.4 may also be viewed as a generalization of Corollary 2.1.6.

Corollary 2.2.4. *Let F be a coherent functor, $M \in \operatorname{mod}(R)$, M' be a submodule of M and $I \subseteq R$ an ideal. Then the sets $\operatorname{Ass}_R F(M/I^n M')$ stabilize.*

Proof. Let F be given by $h_L \to h_K \to F \to 0$. By Lemma 2.2.3(a), the map $h_L \to h_K$ arises from a map $f\colon K \to L$. Choose free resolutions of K and L and a lift of f such that the following diagram commutes.

$$\begin{array}{ccc} R^{\oplus k_1} & \longrightarrow & R^{\oplus \ell_1} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ R^{\oplus k_0} & \xrightarrow{\alpha} & R^{\oplus \ell_0} \\ \downarrow & & \downarrow \\ K & \xrightarrow{f} & L \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{2.2.1}$$

Apply $\operatorname{Hom}_R(-, M/I^nM')$ to get the commutative diagram

$$\begin{array}{ccccccc}
\dfrac{M^{\oplus \ell_1}}{I^n\left((M')^{\oplus \ell_1}\right)} & \longrightarrow & \dfrac{M^{\oplus k_1}}{I^n\left((M')^{\oplus k_1}\right)} & & & & \\
\uparrow{\scriptstyle \gamma_n^*} & & \uparrow{\scriptstyle \beta_n^*} & & & & \\
\dfrac{M^{\oplus \ell_0}}{I^n\left((M')^{\oplus \ell_0}\right)} & \xrightarrow{\alpha_n^*} & \dfrac{M^{\oplus k_0}}{I^n\left((M')^{\oplus k_0}\right)} & & & & \\
\uparrow & & \uparrow & & & & \\
h_L\left(\dfrac{M}{I^nM'}\right) & \xrightarrow{f_n^*} & h_K\left(\dfrac{M}{I^nM'}\right) & \longrightarrow & F\left(\dfrac{M}{I^nM'}\right) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & & & \\
0 & & 0 & & & &
\end{array}$$

where $f_n^*, \alpha_n^*, \beta_n^*, \gamma_n^*$ are induced by f, α, β, γ respectively. Then we have

$$F\left(\frac{M}{I^nM'}\right) \cong \frac{\ker \beta_n^*}{\alpha_n^*\left(\ker \gamma_n^*\right)} .$$

Similarly, we apply $\operatorname{Hom}_R(-, M)$ to (2.2.1) to get maps $\alpha^*, \beta^*, \gamma^*$ induced by α, β, γ respectively. Let $A = M^{\oplus \ell_0}$, $A' = (M')^{\oplus \ell_0}$ and $B' = (M')^{\oplus \ell_1}$. As in the proof of Corollary 2.2.2, there is $c \in \mathbb{N}$ such that $\gamma^*(A) \cap I^nB' = I^{n-c}(\gamma^*(A) \cap I^cB')$ for all $n \geqslant c$, and hence

$$\ker(\gamma_n^*) = \frac{\ker(\gamma^*) + I^{n-c}\left((\gamma^*)^{-1}\left(I^cB'\right)\right)}{I^nA'} .$$

The result then follows from Corollary 2.2.2. □

We next generalize the first half of Theorem 2.1.3 along similar lines.

Notation 2.2.5. Let T, U, V, W be as in Theorem 2.2.1. We let $T_n = (T, U, V, W)_n = (U + I^nV)/I^nW$.

Remark 2.2.6. Let L be an ideal of R. For a submodule S of T, we let $\overline{S}$ be the image of S

under the natural projection $T \to T/LU$. Then we have

$$\begin{aligned}\frac{T_n}{LT_n} &= \frac{U + I^nV}{LU + LI^nV + I^nW} \\ &= \frac{\overline{U} + I^n\overline{V}}{LI^n\overline{V} + I^n\overline{\overline{W}}} \\ &= (\,\overline{T}, \overline{U}, \overline{V}, \overline{LV + W}\,)_n\end{aligned}$$

Theorem 2.2.7. *The values* $\operatorname{depth}_J T_n$ *stabilize.*

Proof. First, suppose that $T_n/JT_n = (\,\overline{T}, \overline{U}, \overline{V}, \overline{JV + W}\,)_n = 0$ for infinitely many n. Then by Theorem 2.2.1, we see that $\operatorname{Ass}_R \overline{T}_n = \emptyset$ for large n. So for all large n, we have $T_n/JT_n = 0$ and hence $\operatorname{depth}_J T_n = \infty$. Hence we may assume that $T_n \neq JT_n$ for large n.

The rest of the proof is the same as that in [10, Theorem 2(i)]. We let $h_T = \liminf_{n\to\infty} \operatorname{depth}_J(T_n)$, $\ell_T = \lim_{n\to\infty} \operatorname{depth}_J(T_n)$ if such exists, and prove by induction on h_T that $\ell_T = h_T$. Suppose that $h_T = 0$. Then $J \subseteq \{r \in P \mid P \in \operatorname{Ass}_R T_n\}$ for infinitely many n. By Theorem 2.2.1, we have $J \subseteq \{r \in P \mid P \in \operatorname{Ass}_R T_n\}$ for all large n, so $\ell_T = h_T = 0$.

Now suppose that $h_T > 0$. Then by Theorem 2.2.1, there is $x \in J$ such that $x \notin \{r \in P \mid P \in \operatorname{Ass}_R T_n\}$ for all large n. Writing $T_n/xT_n = (\,\overline{T}, \overline{U}, \overline{V}, \overline{xV + W}\,)_n$, we have $\operatorname{depth}_J \overline{T}_n = \operatorname{depth}_J T_n - 1$ for all large n. Hence $h_{\overline{T}} = h_T - 1$. By induction, we have $\ell_{\overline{T}} = h_{\overline{T}}$, so $\ell_T = \ell_{\overline{T}} + 1 = h_T$. □

Corollary 2.2.8. *Let $J \subseteq R$ be an ideal. Consider the situation as in Corollary 2.2.2 with the complexes*

$$\frac{L_1 + I^{n-c}L_2}{I^nL'} \xrightarrow{\alpha_n} \frac{M}{I^nM'} \xrightarrow{\beta_n} \frac{N}{I^nN'}$$

and $\mathrm{H}(n)$ *denoting the homology of the complex. Then the values* $\operatorname{depth}_J \mathrm{H}(n)$ *stabilize.*

Corollary 2.2.9. *Let F be a coherent functor, $M \in \operatorname{mod}(R)$, M' be a submodule of M and I, J be ideals of R. Then the values* $\operatorname{depth}_J F(M/I^nM')$ *stabilize.*

In order to generalize the rest of Theorems 2.1.2 and 2.1.3, we let $S = \bigoplus_{n\geqslant 0} R_n$ be a Noetherian R-algebra generated in degree 1 with $R_0 = R$. We will use a result from [29].

Theorem 2.2.10. *[29, Lemma 2.1] Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded S-module. Then the sets $\operatorname{Ass}_R M_n$ stabilize.*

Corollary 2.2.11. *Let $L \to M \to N$ be a complex of $\mathbb{Z}$-graded S-modules, where the maps are homogeneous and $M \in \operatorname{mod}(S)$. Let $H = \bigoplus_{\mathbb{Z}} H_n$ be the homology of the complex. Then the sets $\operatorname{Ass}_R H_n$ stabilize.*

Corollary 2.2.12. *Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded S-module, for example the module H as in Corollary 2.2.11. Let J be an ideal of R. Then the values $\operatorname{depth}_J M_n$ stabilize.*

Proof. The proof of Theorem 2.2.7 works, since $M/JM = \bigoplus_{n\in\mathbb{Z}}(M_n/JM_n)$ and M/xM are also finitely generated graded S-modules. $\square$

Corollary 2.2.13. *Let I be an ideal of R, $M \in \operatorname{mod}(R)$ and $M' \subseteq M$ be a submodule. Then the sets $\operatorname{Ass}_R F(I^nM/I^nM')$ and the values $\operatorname{depth}_J F(I^nM/I^nM')$ stabilize.*

Proof. As in Corollary 2.2.4, we apply $\operatorname{Hom}_R(-, I^nM/I^nM')$ to (2.2.1) to get

$$\begin{array}{ccccccc}
\dfrac{I^n\left(M^{\oplus \ell_1}\right)}{I^n\left((M')^{\oplus \ell_1}\right)} & \longrightarrow & \dfrac{I^n\left(M^{\oplus k_1}\right)}{I^n\left((M')^{\oplus k_1}\right)} & & & & \\
\uparrow{\scriptstyle \gamma_n^*} & & \uparrow{\scriptstyle \beta_n^*} & & & & \\
\dfrac{I^n\left(M^{\oplus \ell_0}\right)}{I^n\left((M')^{\oplus \ell_0}\right)} & \xrightarrow{\alpha_n^*} & \dfrac{I^n\left(M^{\oplus k_0}\right)}{I^n\left((M')^{\oplus k_0}\right)} & & & & \\
\uparrow & & \uparrow & & & & \\
h_L\left(\dfrac{I^nM}{I^nM'}\right) & \xrightarrow{f_n^*} & h_K\left(\dfrac{I^nM}{I^nM'}\right) & \longrightarrow & F\left(\dfrac{I^nM}{I^nM'}\right) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & & & \\
0 & & 0 & & & &
\end{array}$$

Again we have $F\left(\dfrac{I^nM}{I^nM'}\right) \cong \dfrac{\ker \beta_n^*}{\alpha_n^*\left(\ker \gamma_n^*\right)}$, where $\alpha_n^*, \beta_n^*, \gamma_n^*$ are the maps induced by α, β, γ in (2.2.1) respectively, so the result follows from Corollaries 2.2.11 and 2.2.12 by letting $S = \bigoplus_{n\geqslant 0} I^n$. $\square$

A coherent functor F given by $h_L \to h_K \to F \to 0$ can be considered as a functor $\operatorname{Mod}(R) \to \operatorname{Mod}(R)$ since h_L and h_K are (cf. [15, Remark 3.3]). So the proof of Corollary 2.2.13 gives the next result.

Corollary 2.2.14. *Let F be a coherent functor, $M \in \operatorname{mod}(R)$, $M' \subseteq M$ be a submodule, I be an ideal of R, $S = \mathscr{R}(I) = \bigoplus_{n\geqslant 0} I^n$ and $\operatorname{gr}(I) = \bigoplus_{n\geqslant 0} I^n/I^{n+1}$. Then:*

(a) $F\left(\bigoplus_{n\geqslant 0} I^nM/I^nM'\right) = \bigoplus_{n\geqslant 0} F(I^nM/I^nM')$ *is a finitely generated graded S-module.*

(b) *When $M' = IM$, $F\left(\bigoplus_{n\geqslant 0} I^nM/I^{n+1}M\right) = \bigoplus_{n\geqslant 0} F(I^nM/I^{n+1}M)$ is a finitely generated graded $\operatorname{gr}(I)$-module.*

(c) *The module structures over S and $\operatorname{gr}(I)$ in (a) and (b) respectively correspond to the multiplication maps given by applying F to $I^nM/I^nM' \xrightarrow{x} I^{n+m}M/I^{n+m}M'$, where $x \in I^m$.*

Remark 2.2.15. Instead of studying asymptotic stability properties of covariant coherent functors, one may want to consider contravariant coherent functors as well. Unfortunately, as stated in [20, Remark 3.6], the sets $\operatorname{Ass}_R \operatorname{Ext}^i_R(R/I^n, R)$ do not stabilize in general, so our main focus will be on covariant functors. See [35, Introduction] and [36, Proposition 2.1] for further details.

2.3 Examples of non-coherent functors with asymptotic stability

In view of the results in Section 2.2, one may be interested in knowing whether or not a R-linear covariant functor is coherent. Some important examples of coherent functors are given in Lemma 2.2.3. In this section, we will study the zeroth local cohomology functor $\Gamma_I = \mathrm{H}^0_I$ where I is an ideal of R, and the torsion functor τ_S where S is a multiplicatively closed subset of R. It turns out that if $F = \Gamma_I$, τ_S, $\operatorname{id}/\Gamma_I$ or id/τ_S, then the functor F is usually not coherent. However, we will see that whether or not F is coherent, the sets $\operatorname{Ass}_R F(M/I^nM)$ and $\operatorname{Ass}_R F(I^{n-1}M/I^nM)$ always stabilize.

Lemma 2.3.1 (Yoneda's Lemma)**.** *Let R be a Noetherian ring and F be a finitely generated functor given by $h_M \xrightarrow{T} F \to 0$. Then for any $N \in \operatorname{mod}(R)$ and $x \in F(N)$, there is $f \in \operatorname{Hom}_R(M, N)$ such that $x = (F(f) \circ T_M)(\mathrm{id}_M)$. In particular, $x \in \operatorname{im} F(f)$.*

Proof. If $x \in F(N)$, then we let $f \in \operatorname{Hom}_R(M, N)$ be such that $T_N(f) = x$. The result follows from the commutative diagram

$$\begin{array}{ccccc} \operatorname{Hom}_R(M,M) & \xrightarrow{T_M} & F(M) & \longrightarrow & 0 \\ {\scriptstyle h_M(f)}\downarrow & & \downarrow{\scriptstyle F(f)} & & \\ \operatorname{Hom}_R(M,N) & \xrightarrow{T_N} & F(N) & \longrightarrow & 0 \end{array}$$

□

Corollary 2.3.2. *Let R be a Noetherian ring and $\{F_\lambda\}_{\lambda \in \Lambda}$ be a direct system of functors in $\mathscr{F}$. Let $F = \varinjlim_{\lambda \in \Lambda} F_\lambda$ be given by $\{\, T_\lambda \colon F_\lambda \to F \,\}_{\lambda \in \Lambda}$. If $F \in \mathscr{F}$ and is finitely generated, then $F = \operatorname{im} T_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. In particular, if T_λ is injective for all $\lambda \in \Lambda$, then $F = F_\lambda$ for all $\lambda \geqslant \lambda_0$.*

Proof. Let F be given by $h_M \to F \to 0$. Since $F(M) \in \operatorname{mod}(R)$, there is $\lambda_0 \in \Lambda$ such that $F(M) = \operatorname{im}(T_{\lambda_0})_M$. Let $N \in \operatorname{mod}(R)$ and $x \in F(N)$. By Lemma 2.3.1, there is $f \in \operatorname{Hom}_R(M, N)$ such that $x \in \operatorname{im} F(f) \subseteq \operatorname{im}(T_{\lambda_0})_N$.

$$\begin{array}{ccc} F_{\lambda_0}(M) & \xrightarrow{(T_{\lambda_0})_M} & F(M) \\ {\scriptstyle F_{\lambda_0}(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ F_{\lambda_0}(N) & \xrightarrow{(T_{\lambda_0})_N} & F(N) \end{array}$$

Therefore $F = T_{\lambda_0}(F_{\lambda_0})$. □

In the following, we will consider two applications of Corollary 2.3.2.

Corollary 2.3.3. *Let I be an ideal of a Noetherian ring R. The following are equivalent:*

(a) Γ_I *is representable.*

(b) Γ_I *is finitely generated.*

(c) $I^n = I^{n+1}$ *for some* $n \geqslant 0$.

Proof. For all $M \in \operatorname{Mod}(R)$, we have $\Gamma_I(M) = \varinjlim_n \operatorname{Hom}_R(R/I^n, M) = \varinjlim_n (0 :_M I^n)$. So by Corollary 2.3.2, Γ_I is finitely generated iff there exists $n \geqslant 0$ such that $\Gamma_I(M) = \operatorname{Hom}_R(R/I^n, M)$ for all $M \in \operatorname{mod}(R)$ iff $I^n = I^{n+1}$ for some $n \geqslant 0$ by considering $M = R/I^{n+1}$ for "only if". □

The relationship between our result and Section 2.2 is as follows.

Theorem 2.3.4. *[15, Theorem 1.1(a)] Let* F, G *be coherent functors and* $T\colon F \to G$ *be a natural transformation. Then* $\ker(T)$, $\operatorname{coker}(T)$ *and* $\operatorname{im}(T)$ *are also coherent.*

Lemma 2.3.5. *Let* R *be a Noetherian ring,* $I \subseteq R$ *be an ideal and* $M \in \operatorname{Mod}(R)$. *Then* $\operatorname{Ass}_R \Gamma_I(M) = \operatorname{Ass}_R(M) \cap V(I)$ *and* $\operatorname{Ass}_R(M/\Gamma_I(M)) = \operatorname{Ass}_R(M) \setminus V(I)$, *where* $V(I) = \{P \in \operatorname{Spec}(R) \mid P \supseteq I\}$.

Corollary 2.3.6. *Let* $\{M_n\}_{n \geqslant 0}$ *be a sequence of modules in* $\operatorname{mod}(R)$ *such that the sets* $\operatorname{Ass}_R(M_n)$ *stabilize. Let* $I \subseteq R$ *be an ideal. If* $I^n \neq I^{n+1}$ *for any* n, *then the functor* $\operatorname{id}/\Gamma_I$ *is finitely generated but not coherent, and* Γ_I *is not finitely generated. However, whether or not* $I^n = I^{n+1}$ *for any* n, *the sets* $\operatorname{Ass}_R(M_n/\Gamma_I(M_n))$ *and* $\operatorname{Ass}_R \Gamma_I(M_n)$ *always stabilize.*

Now we consider our second example.

Lemma 2.3.7. *Let* R *be a ring, possibly noncommutative, with 1. Let* $S \subseteq R$ *and* $f\colon S \times S \to R$ *be a function. The following are equivalent:*

(a) *For every* $r, s \in S$, *left* R*-module* M *and* $m \in M$, *if* $rm = 0$, *then* $f(r,s)m = f(s,r)m = 0$.

(b) *For every* $r, s \in S$ *we have* $f(r,s) \in Rr \cap Rs$.

Proof. (a) ⇒ (b): Let $r, s \in S$ and $M = R/Rr$. Then $r\overline{1} = \overline{0}$. By assumption, we have $f(r,s)\overline{1} = \overline{0}$, so $f(r,s) \in Rr$. Similarly, with $M = R/Rs$ we have $f(r,s) \in Rs$, so that $f(r,s) \in Rr \cap Rs$.

(b) $\Rightarrow$ (a): Let $r, s \in S$ and $m \in M$. By assumption, $f(r,s), f(s,r) \in Rr$. So if $rm = 0$, then $f(r,s)m, f(s,r)m \in Rrm = 0$. $\square$

Example 2.3.8. Let R be a UFD, $S = R$ and $f\colon R \times R \to R$. Then f satisfies the conditions in Lemma 2.3.7 iff for all $r, s \in R$ we have $f(r,s) \in (\operatorname{lcm}(r,s))$.

Definition 2.3.9. Let R be a commutative ring with 1.

(1) We say that a subset $S \subseteq R$ is common multiplicatively closed if $S \neq \emptyset$ and there is a function $f\colon S \times S \to S$ satisfying any condition in Lemma 2.3.7, or equivalently, for any $r, s \in S$ there is $f(r,s) \in S$ that satisfies any condition in Lemma 2.3.7.

(2) We say that a (nonempty) subset $S \subseteq R$ is coprincipal if there is $s \in S$ such that $s \in \bigcap_{r \in S} Rr$. Such an s is called a cogenerator of S.

(3) For any $S \subseteq R$ and $M \in \operatorname{Mod}(R)$, we let $\tau_S(M) = \{m \in M \mid rm = 0 \text{ for some } r \in S\}$. If S is common multiplicatively closed, then $\tau_S(M)$ is a submodule of M.

Example 2.3.10. (1) Any singleton subset of R is common multiplicatively closed.

(2) In general, any coprincipal subset $S \subseteq R$ is common multiplicatively closed, since if $s \in S$ is a cogenerator, then we can let $f(r,t) = s$ for all $r, t \in S$.

(3) Conversely, if $S = \{s_1, \ldots, s_n\} \subseteq R$ is common multiplicatively closed, then S has a cogenerator $f(\cdots f(f(s_1, s_2), s_3), \ldots, s_n)$.

(4) Any multiplicatively closed subset of R is common multiplicatively closed.

(5) If $r, s \in \mathbb{Z}$ and $(0) \neq (s) \subsetneq (r)$, then the subset $\{r, s\}$ of $\mathbb{Z}$ is common multiplicatively closed and coprincipal but not multiplicatively closed.

(6) Let $a \in \mathbb{Z}$ such that $a \neq 0, \pm 1$. Let $S = \{a^2\} \cup \{a^{8+12n} \mid n \geqslant 0\}$. Then S is a common multiplicatively closed subset of $\mathbb{Z}$ by the function $f(s,t) = (st)^2$, and S is neither multiplicatively closed nor coprincipal.

(7) Let $a \in \mathbb{Z}$ such that $a \neq 0, \pm 1$. Then the infinite multiplicatively closed subset $S = \{a^{-n} \mid n \geqslant 0\}$ of $\mathbb{Z}_a$ is coprincipal with 1 as a cogenerator; the subset $\{a^n \mid n \geqslant 0\}$ of $\mathbb{Z}$ is not. If $i \geqslant 0$ and $i \neq 1$, then $S \setminus \{a^{-i}\} \subseteq \mathbb{Z}_a$ is coprincipal but not multiplicatively closed.

(8) Let R_1, R_2 be rings and u be a unit in R_1. Let $S \subseteq R_1 \times R_2$ be the subset $\{(u^n, r) \mid n \geqslant 1\} \cup \{(1,1)\}$. If $u^n \neq 1$ for any $n \geqslant 1$, or if R_2 is infinite, then S is infinite, multiplicatively closed and coprincipal with cogenerator $(u, 0)$.

Remark 2.3.11. We have now seen that:

- Coprincipal $\Rightarrow$ common multiplicatively closed
- If S is finite, then S is coprincipal $\Leftrightarrow$ S is common multiplicatively closed
- Multiplicatively closed $\Rightarrow$ common multiplicatively closed
- Coprincipal and multiplicatively closed do not imply or refute each other
- Common multiplicatively closed $\not\Rightarrow$ coprincipal
- Common multiplicatively closed $\not\Rightarrow$ multiplicatively closed

Corollary 2.3.12. *Let R be a Noetherian ring and S be a common multiplicatively closed subset of R. The following are equivalent:*

(a) τ_S is representable.

(b) τ_S is finitely generated.

(c) S is coprincipal.

Proof. First, for all $M \in \mathrm{Mod}(R)$ we have $\tau_S(M) = \varinjlim_{Rs} \mathrm{Hom}_R(R/(s), M) = \varinjlim_{Rs}(0 :_M s) = \bigcup_s (0 :_M s)$, where s runs through S and $(s) \geqslant (t)$ iff $(s) \subseteq (t)$. So by Corollary 2.3.2, τ_S is finitely generated iff there exists $s \in S$ such that $\tau_S(M) = \mathrm{Hom}_R(R/(s), M)$ for all $M \in \mathrm{mod}(R)$ iff there exists $s \in S$ such that $(s) \subseteq (r)$ for all $r \in S$ by considering $M = R/(r)$ for "only if". □

Notation 2.3.13. We let $R^\times$ denote the set of units of a ring R.

Lemma 2.3.14. *Let R be a ring and S be a subset of R. Consider the following statements.*

(a) S is coprincipal.

(b) There are rings R_1, R_2 such that $R = R_1 \times R_2$, $S \cap (R_1)^\times \neq \emptyset$ and for all $s \in S$ we have $s(1,0) \in (R_1)^\times$.

Then (b) $\Rightarrow$ (a). If S is furthermore multiplicatively closed, then (a) $\Rightarrow$ (b).

Proof. (b) $\Rightarrow$ (a): Let $(u,0) \in S \cap (R_1)^\times$ and $s \in S$. Since $s(1,0) \in (R_1)^\times$, $(u,0) \in Rs$. Therefore $(u,0)$ is a cogenerator of S.

Now suppose that S is multiplicatively closed and coprincipal with cogenerator e. Since S is multiplicatively closed, $e^2 \in S$. Since e is a cogenerator of S, $e = re^2$ for some $r \in R$. Then $(re)^2 = r(re^2) = re$, so re is idempotent. Let $R_1 = R(re)$ and $R_2 = R(1-re)$, so that $R = R_1 \times R_2$. Then $e(re) = re^2 = e$, so $e \in R_1$, and $e(r^2e) = (re)^2 = re$, so $e \in S \cap (R_1)^\times$. Finally, let $s \in S$. Then $e = r's$ for some $r' \in R$, and $(r'r^2e)(sre) = (re)^3 = re$, so $sre \in (R_1)^\times$. □

Lemma 2.3.15. *Let R be a ring, $S \subseteq R$ and $M \in \mathrm{Mod}(R)$. If $\tau_S(M)$ is a submodule of M, then $\mathrm{Ass}_R(\tau_S(M)) = \{P \in \mathrm{Ass}_R(M) \mid P \cap S \neq \emptyset\}$. If R is Noetherian and S is a multiplicatively closed subset of R, then $\mathrm{Ass}_R(M/\tau_S(M)) = \{P \in \mathrm{Ass}_R(M) \mid P \cap S = \emptyset\}$.*

Remark 2.3.16. The second half of Lemma 2.3.15 is false if S is not multiplicatively closed. For example, let $R = \mathbb{Z}$, $S = \{p\}$ where p is prime, and $M = \mathbb{Z}/(p^2)$. Then $\mathrm{Ass}_R(M/\tau_S(M))$ $= \{(p)\}$, but $(p) \cap S \neq \emptyset$.

Corollary 2.3.17. *Let R be a Noetherian ring, S be a multiplicatively closed subset of R and $\{M_n\}_{n\geqslant 0}$ be a sequence of modules in $\mathrm{mod}(R)$ such that the sets $\mathrm{Ass}_R(M_n)$ stabilize. If S is not coprincipal, then the functor id/τ_S is finitely generated but not coherent, and τ_S is not finitely generated. However, whether or not S is coprincipal, the sets $\mathrm{Ass}_R(M_n/\tau_S(M_n))$ and $\mathrm{Ass}_R(\tau_S(M_n))$ always stabilize.*

2.4 Covariant functors over a Dedekind domain

In Section 2, we saw that the sets $\mathrm{Ass}_R\, F(M/I^nM)$ stabilize whenever F is a coherent functor. One may ask whether such asymptotic stability still holds when F is not coherent. In this section, we consider the case where R is a Dedekind domain. We will see that if F is a finitely generated functor over R, then the sets $\mathrm{Ass}_R\, F(M/I^nM)$ stabilize. We then construct a family of examples of R-linear covariant functors F such that the sets $\operatorname{Ass}_R F(R/I^n)$ do not stabilize.

Lemma 2.4.1. *Let R be a ring, F be an R-linear functor from $\operatorname{Mod}(R)$ to itself and $M \in \operatorname{Mod}(R)$. Then $\operatorname{ann}_R(M) \subseteq \operatorname{ann}_R(F(M))$.*

Theorem 2.4.2. *Let R be a Dedekind domain, I be an ideal of R, $M \in \operatorname{mod}(R)$ and F be a finitely generated functor. Then the sets $\operatorname{Ass}_R F(M/I^nM)$ stabilize.*

Proof. The proof will proceed in several steps.

Step 1. First, we will make some reductions. Since F is R-linear, it preserves finite direct sums. By the structure theorem for finitely generated modules over a Dedekind domain, we may assume that $M = J$ is an ideal of R or $M = R/P^i$ for some maximal ideal P of R and $i \geqslant 1$. If $M = R/P^i$, then either $M/I^nM = 0$ for all n or $M/I^nM = M$ for all $n \geqslant i$. If $0 \neq M = J \subseteq R$ and $I \neq 0$, then $M/I^nM \cong R/I^n$ for all $n \geqslant 1$. But R/I^n is again a direct sum of modules of the form R/P^{ni}. So it suffices to show that asymptotic stability holds for $\operatorname{Ass}_R F(R/P^n)$, where P is a maximal ideal of R. Furthermore, by Lemma 2.4.1, $\operatorname{Ass}_R F(R/P^n) = \{P\}$ or $\emptyset$ for all $n \geqslant 1$. So we only need to show that $F(R/P^n)$ is either always 0 or always nonzero for all large n.

Step 2. Let F be given by the surjection $h_L \to F$, where $L \in \operatorname{mod}(R)$. First we consider the case where $L = J$ is an ideal of R. Suppose that $F(R/P^n) = 0$ for infinitely many n. We will show that in fact $F(R/P^n) = 0$ for all n, which will conclude this case. So fix $n \geqslant 1$. Let $N \geqslant n$ be such that $F(R/P^N) = 0$. Let $\pi\colon R/P^N \to R/P^n$ be the natural projection map. Since J is a projective R-module, the map $h_J(\pi)\colon h_J(R/P^N) \to h_J(R/P^n)$ is surjective. From the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_R\left(J, \frac{R}{P^N}\right) & \twoheadrightarrow & F\left(\frac{R}{P^N}\right) = 0 \\ \downarrow{\scriptstyle h_J(\pi)} & & \downarrow{\scriptstyle F(\pi)} \\ \operatorname{Hom}_R\left(J, \frac{R}{P^n}\right) & \twoheadrightarrow & F\left(\frac{R}{P^n}\right) \end{array}$$

we see that $F(\pi)$ is surjective and therefore $F(R/P^n) = 0$.

Step 3. Next, we consider the case where $L = R/Q^i$ such that Q is a maximal ideal of R and $i \geqslant 1$. We may assume that $Q = P$. Suppose that $F(R/P^N) = 0$ for some $N \geqslant i$. We will show that in fact $F(R/P^n) = 0$ for all $n \geqslant N$, concluding this case. We recall the following facts. For any $n_1 \geqslant 1$, R/P^{n_1} is a principal ideal ring. Choose an element $p \in P \setminus P^2$. Then P^{n_2}/P^{n_1} is generated by p^{n_2} for all $0 \leqslant n_2 \leqslant n_1$. Now fix $n \geqslant N$. Let $p^{n-N}\colon R/P^N \to R/P^n$ denote multiplication by p^{n-N}. Again from the commutative diagram

$$\begin{array}{ccccc}
\frac{R}{P^i} & \xrightarrow[p^{N-i}]{\cong} & \frac{P^{N-i}}{P^N} = \operatorname{Hom}_R\left(\frac{R}{P^i}, \frac{R}{P^N}\right) & \twoheadrightarrow & F\left(\frac{R}{P^N}\right) = 0 \\
\downarrow \mathrm{id} & & \cong \downarrow p^{n-N} = h_J\left(p^{n-N}\right) & & \downarrow F\left(p^{n-N}\right) \\
\frac{R}{P^i} & \xrightarrow[p^{n-i}]{\cong} & \frac{P^{n-i}}{P^n} = \operatorname{Hom}_R\left(\frac{R}{P^i}, \frac{R}{P^n}\right) & \twoheadrightarrow & F\left(\frac{R}{P^n}\right)
\end{array}$$

we see that $F(p^{n-N})$ is surjective and therefore $F(R/P^n) = 0$.

Step 4. Finally, we consider the general case where $L = J_1 \oplus \cdots \oplus J_k \oplus R/Q_1^{i_1} \oplus \cdots \oplus R/Q_\ell^{i_\ell}$ such that $J_1, \ldots, J_k \subseteq R$ are ideals, $Q_1, \ldots, Q_\ell$ are maximal ideals of R and $i_1, \ldots, i_\ell \geqslant 1$. Again we may assume that $Q_1 = \cdots = Q_\ell = P$. Suppose that $F(R/P^n) = 0$ for infinitely many n. Fix $N \geqslant \max\{1, i_1, \ldots, i_\ell\}$ such that $F(R/P^N) = 0$. Then repeating Steps 2 and 3, we see that for all $n \geqslant N$, each direct summand of $h_L(R/P^n) = h_{J_1}(R/P^n) \oplus \cdots \oplus h_{J_k}(R/P^n) \oplus h_{R/P^{i_1}}(R/P^n) \oplus \cdots \oplus h_{R/P^{i_\ell}}(R/P^n)$ is mapped to 0 in $F(R/P^n)$. Therefore $F(R/P^n) = 0$ for all $n \geqslant N$. $\square$

Lemma 2.4.3. *Let R be a Dedekind domain, I be an ideal of R and $M \in \operatorname{mod}(R)$. Then the modules $I^n M/I^{n+1}M$ are all isomorphic for large n. In particular, let F be an R-linear functor from $\operatorname{Mod}(R)$ to itself. Then the sets $\operatorname{Ass}_R F(I^n M/I^{n+1}M)$ stabilize.*

Proof. As in Step 1 of Theorem 2.4.2, we may assume that $M = J$ is an ideal of R or $M = R/P^i$ for some maximal ideal P of R and $i \geqslant 1$. If $M = J \neq 0$ and $I \neq 0$, then $I^n M/I^{n+1}M \cong R/I$ for all $n \geqslant 0$. If $M = R/P^i$, then $I^n M/I^{n+1}M = 0$ for all $n \geqslant i$. $\square$

Theorem 2.4.4. *Let R be a Dedekind domain and $I \neq 0$ be an ideal of R. Then there exists $F \in \mathscr{F}$ such that the sets $\operatorname{Ass}_R F(R/I^n)$ do not stabilize. In fact, we may construct F such that $\operatorname{Ass}_R F(R/I^n)$ is given by any sequence of subsets of $\operatorname{Ass}_R(R/I) = V(I)$.*

Proof. First, let $\mathscr{T} \subseteq \operatorname{mod}(R)$ be the subcategory of finitely generated torsion R-modules. Then the torsion functor $\tau\colon \operatorname{mod}(R) \to \mathscr{T}$ is R-linear. Next, we recall from category theory that any category is naturally equivalent to any skeleton of itself. In particular, given a skeleton $\mathscr{T}_0$ of $\mathscr{T}$, there is an R-linear functor $\pi\colon \mathscr{T} \to \mathscr{T}_0$. Therefore it suffices to construct $F\colon \mathscr{T}_0 \to \mathscr{T}_0$ as in our Theorem.

We will define $\mathscr{T}_0$ as follows. Fix a linear ordering $\preceq$ of the nonzero prime ideals R, and let the objects of $\mathscr{T}_0$ be modules of the form $R/P_1^{e_1} \oplus \cdots \oplus R/P_j^{e_j}$, where $P_1 \preceq \cdots \preceq P_j$ and $e_i \leqslant e_{i+1}$ whenever $P_i = P_{i+1}$. For each maximal ideal P we choose a subset S_P of $\mathbb{N}_{>0}$. Then we define $F(R/P^e) = R/P$ if $e \in S_P$, and 0 otherwise. We let $F(R/P_1^{e_1} \oplus \cdots \oplus R/P_j^{e_j}) = \oplus_{i=1}^{k} R/P$, where k is the number of $F(R/P_i^{e_i})$ that are nonzero. Next we define $F(f)$ for $f\colon M \to N$, where $M, N \in \mathscr{T}_0$. It suffices to consider the case where M, N are both P-torsion for some maximal ideal P of R. Fix an element $p \in P \backslash P^2$. Then $\operatorname{Hom}_R(R/P^{n_1}, R/P^{n_2}) = P^{n_2-n_1}/P^{n_2}$ is generated by $p^{n_2-n_1}$ if $n_2 \geqslant n_1 \geqslant 1$, and $\operatorname{Hom}_R(R/P^{n_1}, R/P^{n_2}) = R/P^{n_2}$ if $n_1 \geqslant n_2 \geqslant 1$. So we can identify f with a square matrix with entries in R (more precisely, in R/P^{e_i} for suitable e_i) viewed as multiplication maps, adding rows or columns of zeroes if necessary. If M, N are both direct sums of copies of $R/P^{e_1}, \ldots, R/P^{e_j}$ with $e_1 < \cdots < e_j$, then we define

$$F(f) = F\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ p* & & & A_j \end{pmatrix} = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_j \end{pmatrix},$$

where the entries in the lower diagonal of the matrix on the left are multiples of p, and $A_1, A_2, \ldots, A_j$ are the square blocks that correspond to $R/P^{e_1}, \ldots, R/P^{e_j}$ respectively. Since $F(R/P^e) =$ either R/P or 0, the definition of $F(f)$ does not depend on the choice of coset representatives in the entries of f. It is then immediate that F preserves identity maps and is R-linear. Finally, if $f\colon M \to N$ and $g\colon N \to L$ where M, N, L are P-torsion, then

$$
\begin{aligned}
F(g \circ f) &= F\left(\begin{pmatrix} B_1 & & & * \\ & B_2 & & \\ & & \ddots & \\ p* & & & B_j \end{pmatrix}\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ p* & & & A_j \end{pmatrix}\right) \\
&= F\begin{pmatrix} B_1A_1 + p* & & & * \\ & B_2A_2 + p* & & \\ & & \ddots & \\ p* & & & B_jA_j + p* \end{pmatrix} \\
&= \begin{pmatrix} B_1A_1 + p* & & & 0 \\ & B_2A_2 + p* & & \\ & & \ddots & \\ 0 & & & B_jA_j + p* \end{pmatrix} \\
&= \begin{pmatrix} B_1A_1 & & & 0 \\ & B_2A_2 & & \\ & & \ddots & \\ 0 & & & B_jA_j \end{pmatrix} = F(g)F(f)
\end{aligned}
$$

Therefore F respects composition. □

Corollary 2.4.5. *The functors constructed in Theorem 2.4.4 are not finitely generated.*

Question 2.4.6. Is there a finitely generated non-coherent functor F such that the sets $\operatorname{Ass}_R F(R/I^n)$ do not stabilize?

2.5 Functors arising from middle finite complexes

In this section, we will study a class of R-linear covariant functors F which arise naturally and are non-finitely generated in general. An example of such kind of functor is the zeroth local cohomology functor. We will obtain results that are related to all the previous sections. Our main result is that over a one-dimensional Noetherian domain R, the sets $\operatorname{Ass}_R F(M/I^nM)$ stabilize.

Definition 2.5.1. Let R be a ring and $\mathscr{S}\colon A \to B \to C$ be a complex of R-modules.

(1) We say that an R-linear functor $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ arises from $\mathscr{S}$ if $F(-) = \operatorname{H}(\mathscr{S} \otimes -)$.

(2) We say that $\mathscr{S}$ is middle finite if $B \in \operatorname{mod}(R)$.

Example 2.5.2. Let R be a ring and $I = (x_1, \ldots, x_n)$ be an ideal of R. Then the functor Γ_I arises from the middle finite complex

$$0 \to R \to R_{x_1} \oplus \cdots \oplus R_{x_n}$$

Remark 2.5.3. Let R be a Noetherian ring. By Corollary 2.3.3, a functor that arises from a middle finite complex of R-modules is not finitely generated in general.

Lemma 2.5.4. *Let R be a Noetherian ring. Let F be a functor that arises from the middle finite complex $A \xrightarrow{\partial_A} B \xrightarrow{\partial_B} C$. Then F is coherent iff it is finitely generated.*

Proof. Suppose that F is finitely generated and is given by the surjection $h_M \to F$. Let K, I denote the functors given by $K(-) = \ker(\partial_B \otimes -)$ and $I(-) = \operatorname{im}(\partial_A \otimes -)$. Let $N \in \operatorname{mod}(R)$ and $n \in K(N)$ be such that $n + I(N) \in F(N)$. By Lemma 2.3.1, there is $f \in \operatorname{Hom}_R(M, N)$ such that $n + I(N) \in \operatorname{im} F(f)$. That is, there are $m \in K(M)$ and $x \in A \otimes N$ such that $n = (\operatorname{id}_B \otimes f)(m) + (\partial_A \otimes \operatorname{id}_N)(x)$. Now $C \otimes M = \varinjlim_D (D \otimes M)$, where D ranges over all finitely generated submodules of C. Since $B \otimes M \in \operatorname{mod}(R)$, there is a finitely generated submodule

C_0 of C that contains $\operatorname{im}\partial_B$ such that $\ker(B\otimes M\to C\otimes M)=\ker(B\otimes M\to C_0\otimes M)$. From the commutative diagram

$$\begin{array}{ccccc} A\otimes M & \xrightarrow{\partial_A\otimes\mathrm{id}_M} & B\otimes M & \xrightarrow{\partial_B\otimes\mathrm{id}_M} & C_0\otimes M \\ \downarrow{\scriptstyle \mathrm{id}_A\otimes f} & & \downarrow{\scriptstyle \mathrm{id}_B\otimes f} & & \downarrow{\scriptstyle \mathrm{id}_{C_0}\otimes f} \\ A\otimes N & \xrightarrow{\partial_A\otimes\mathrm{id}_N} & B\otimes N & \xrightarrow{\partial_B\otimes\mathrm{id}_N} & C_0\otimes N \end{array}$$

we see that in fact $n\in\ker(B\otimes N\to C_0\otimes N)$. Finally, let A_0 be a finitely generated submodule of A such that $\partial_A(A_0)=\partial_A(A)$. Then F arises from the complex $A_0\to B\to C_0$. Therefore F is coherent by Lemma 2.2.3. $\square$

Lemma 2.5.5. *Let R be a Noetherian ring, I,J be ideals of R, $M\in\operatorname{mod}(R)$, M' be a submodule of M and F be a functor that arises from the middle finite complex $A\to B\to C$. Then the sets $\operatorname{Ass}_R F(I^nM/I^nM')$ and the values $\operatorname{depth}_J F(I^nM/I^nM')$ stablize.*

Proof. The module $\bigoplus_{n\geqslant 0} B\otimes(I^nM/I^nM')$ is finitely generated graded over $S=\bigoplus_{n\geqslant 0} I^n$, and the maps in the induced complex

$$\bigoplus_{n\geqslant 0} A\otimes\frac{I^nM}{I^nM'}\to\bigoplus_{n\geqslant 0} B\otimes\frac{I^nM}{I^nM'}\to\bigoplus_{n\geqslant 0} C\otimes\frac{I^nM}{I^nM'}$$

are homogenous of degree 0. The result then follows from Corollaries 2.2.11 and 2.2.12. $\square$

Theorem 2.5.6. *Let R be a one-dimensional Noetherian domain, I be an ideal of R, $M\in\operatorname{mod}(R)$ and F be a functor that arises from the middle finite complex $\mathscr{S}\colon A\overset{\alpha}{\to}B\overset{\beta}{\to}C$. Then the sets $\operatorname{Ass}_R F(M/I^nM)$ stabilize.*

Proof. First, since $\mathscr{S}\otimes(M/I^nM)=(\mathscr{S}\otimes M)\otimes(R/I^n)$, it suffices to show that the sets $\operatorname{Ass}_R F(R/I^n)$ stabilize. We have $\mathscr{S}\otimes(R/I^n)\colon A/I^nA\xrightarrow{\alpha_{n-1}}B/I^nB\xrightarrow{\beta_{n-1}}C/I^nC$, so

$$F(R/I^n)=\frac{\ker\beta_{n-1}}{\operatorname{im}\alpha_{n-1}}=\frac{\beta^{-1}(I^nC)}{\alpha(A)+I^nB}=F'(R/I^n),$$

where F' arises from the complex $0\to B/\alpha(A)\to C$. So we may assume that $A=0$.

Furthermore, since localization is flat, we may assume that R is local of dimension one. So it remains to show that $F(R/I^n)$ is either always 0 or always nonzero for all large n.

Now let $S = \bigoplus_{n\geqslant 0} I^n$ and $\gamma\colon \bigoplus_{n\geqslant 0}(I^nB/I^{n+1}B) \to \bigoplus_{n\geqslant 0}(I^nC/I^{n+1}C)$ be the map induced by β with graded components γ_n. By Corollary 2.2.11, there is N so large such that the sets $\operatorname{Ass}_R(\ker\gamma_n)$ are equal for all $n > N$. Again we have $\beta_n\colon B/I^{n+1}B \to C/I^{n+1}C$, so that $F(R/I^n) = \ker\beta_{n-1}$. Suppose that there is $m > N$ such that $\ker\beta_{m-1} = 0$ but $\ker\beta_m \neq 0$. Then $I^{m+1}B \subsetneq \beta^{-1}(I^{m+1}C) \subseteq \beta^{-1}(I^mC) = I^mB$, so that $0 \neq \ker\beta_m \subseteq \ker\gamma_m$, and hence $\ker\gamma_n \neq 0$ for all $n > N$. But $\ker\beta_n \supseteq \ker\gamma_n$ always holds. Therefore we have $\ker\beta_n \neq 0$ for all $n > N$. $\square$

Chapter 3

The Cohen-Macaulay Property of Affine Semigroup Rings in Dimension 2

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3.1 Introduction

We investigate a criterion for determining whether monomial rings of the form

$$R = k[x^a, x^{p_1}y^{q_1}, \dots, x^{p_t}y^{q_t}, y^b]$$

are Cohen-Macaulay (or CM). An important special case of this problem is the projective monomial curve: $k[x^n, x^{n-a_1}y^{a_1}, \dots, x^{n-a_t}y^{a_t}, y^n]$ for integers $0 < a_1 < \dots < a_t < n$. The study of such rings is inspired by the original example of Macaulay [1, p. 98], $k[x^4, x^3y, xy^3, y^4]$. This is a domain with system of parameters (x^4, y^4) in which $\lambda(R/(x^4, y^4))$ $= 5 \neq e((x^4, y^4)) = 4$. It is observed that $\dim(R) = 2$ and $\operatorname{depth}(R) = 1$.

One essential manner of viewing this problem is by considering the semigroup of monomials which lie in the ring. For a polynomial ring with n variables, we let each monomial m be a point in $\mathbb{Z}^n$ corresponding to its exponent vector $\log(m)$. These points form a semigroup inside $\mathbb{Z}^n$ whose generators correspond to the monomials generating R over k.

One of the most important breakthroughs in the study of the CM property of these rings was made by Hochster.

Theorem 3.1.1 ([17, Theorem 1]). *If M is a monomial semigroup in the variables $x_1, \ldots, x_n$ and $k[M] \subset k[x_1, \ldots, x_n]$ is normal, then $R[M] \subset R[x_1, \ldots, x_n]$ is Cohen-Macaulay for any Cohen-Macualay ring R.*

While this settles a great number of cases, there are plenty of monomial semigroups for which $k[M]$ is not normal. In particular, the projective monomial curves described in the first paragraph are never normal unless $t = n - 1$. To see this, note that we may assume $\gcd(a_1, \ldots, a_t, n) = 1$, so that $\frac{x}{y}$ is in the fraction field of R. Hence each monomial of the form $x^i y^{n-i}$ is in the fraction field of R. In a similar way, many rings of the form $k[x^a, x^{p_1}y^{q_1}, \ldots, x^{p_t}y^{q_t}, y^b]$ are not normal. For example, the fraction field of $R = k[x^2, x^{11}y, xy^{11}, y^3]$ contains $\frac{x^{11}y}{(x^2)^5} = xy$ and $(xy)^6 \in R$.

In the case of simplicial affine semigroups, Goto, Suzuki and Watanbe give another criterion by which to evaluate CM. A semigroup is affine if it may be embedded in $\mathbb{Z}^n$ for some n. For any affine semigroup, we may consider the cone of S: $C(S) = \{\alpha \in \mathbb{R}^n | k \cdot \alpha \in S \text{ for some } 0 \leqslant k \in \mathbb{R}\}$. The semigroup is said to be simplicial if the cone may be generated in $\mathbb{R}^n$ by $rank(S)$-many linearly independent elements of S as $\mathbb{R}^n$ vectors.

Theorem 3.1.2 ([13, Theorem 5.1]; [38, Theorem 6.4]). *Let S be a simplicial affine semigroup. Let $e_1, \ldots, e_s$ be elements which span C_S. Then $k[S]$ is CM if and only if*

$$\{x \in G | x + e_i \in S \text{ and } x + e_j \in S \text{ for some } i \neq j\} = S$$

Goto and Watanabe defined a similar extension S' for a general affine semigroup S. Trung and Hoa [39, Theorem 4.1] identify a topological criterion which, together with $S = S'$, is necessary and sufficient for CM.

The semigroup defined by elements of $R = k[x^a, x^{p_1}y^{q_1}, \ldots, x^{p_t}y^{q_t}, y^b]$ is an affine semigroup by log(-). Furthermore, any element of $\mathbb{R}^2$ with nonnegative entries may be written as a combination of $(a, 0)$ and $(0, b)$. This includes every element of S, so S is a simplicial affine semigroup.

The criterion of theorem 3.1.2 is straightforward to check for a single ring, but does not lend itself to analyzing classes of rings. Reid and Roberts [34] introduce a related notion of a maximal projective monomial curve in order to demonstrate a large class of CM curves. The special case of projective monomial curves continues to be studied.

In this Chapter, we consider affine semigroup rings in dimension 2. The framework we find helpful emphasizes the congruence classes of the exponent vectors. This allows us to calculate the Hilbert polynomial of (x^a, y^b) in section 3.2. Note that the constants $t_{p,q}, s_{p,q}, n_{p,q}$ in the following theorem are described in 3.2.5 and 3.2.6.

Theorem (3.2.7)**.** *Let $R = k[x^a, x^{p_1}y^{q_1}, x^{p_2}y^{q_2}, \ldots, x^{p_t}y^{q_t}, y^b]$. The Hilbert polynomial of (x^a, y^b) is $P(n) = |H|(n+1) + \sum_{(p,q)\in H}(t_{p,q} + s_{p,q})$ where $H \subset (\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z})$ is the subgroup generated by $(p_1, q_1), (p_2, q_2), \ldots, (p_t, q_t)$. In particular, $e((x^a, y^b)) = |H|$, and the Hilbert function equals the Hilbert polynomial for $n \geqslant \max_{(p,q)\in H}(n_{p,q})$.*

These details allow us to manipulate the construction of a ring to achieve specific coefficients and level of stabilization for the Hilbert polynomial.

Our calculations also highlight an interesting set of k-independent monomials over $R/(x^a, y^b)$. In section 3.3, we give more specific calculations for rings with 4 monomial generators. Theorem 3.3.12 provides a simple criterion for determining the CM property and Theorem 3.3.15 presents an algorithm that identifies a monomial k-basis for $R/(x^a, y^b)$. Section 3.4 highlights the application of this work to projective monomial curves in $\mathbb{P}^3$. The CM condition for such rings is as follows.

Theorem (3.4.6)**.** *Let $R = k[x^n, x^{n-\ell}y^\ell, x^{n-m}y^m, y^n]$ with $0 < \ell < m < n$. Let b be the* *smallest integer such that there exist integers $a \geqslant 0$ and $c > 0$ with $bm = a\ell + cn$. Choose a such that $n/\gcd(n, \ell) > a$. Then R is CM if and only if $b \geqslant a + c$.*

There are several previous results which bear some resemblence to our conclusions. [24] provides an algorithm identifying 'basis points' in an affine semigroup. These are not the same as our basis elements of $R/(x^a, y^b)$ and the results assume seminormality of the semigroup. Among a study of the defining ideals of monomial rings, [25, Remark 2.17] provides a geometric condition on the monomial basis for a projective monomial curve in $\mathbb{P}^3$ to have a CM ring. In contrast, Theorem 3.4.6 is a clearly numerical condition on the exponents. In [30] we find another numerical criterion for a projective monomial curve in $\mathbb{P}^3$ to be CM, whereas our results in Section 3.3 do not require the monomial generators of R to have the same degree.

3.2 Asymptotic behavior of the system of parameters

One important window into the CM property is the asymptotic lengths $R/(x^a, y^b)^n$ for a system of parameters $(x^a, y^b) \in R$. The main result of this section, Theorem 3.2.7, gives the Hilbert polynomial for (x^a, y^b) in the ring $R = k[x^a, x^{p_1}y^{q_1}, x^{p_2}y^{q_2}, \ldots, x^{p_t}y^{q_t}, y^b]$. We demonstrate a method by which rings with a specific Hilbert polynomial may be constructed, then return to the implications for evaluating the CM property. A classic characterization of Cohen-Macaulay local rings links this property to multiplicity and length. The following theorem is taken from the text by Matsumura.

Theorem 3.2.1 ([2, Theorem 17.11])**.** *Let $(R, \mathfrak{m})$ be a local ring. The following are equivalent:*

(i) *R is a Cohen-Macaulay ring.*

(ii) *$\lambda(R/I) = e(I)$ for every I generated by a system of parameters.*

(iii) *$\lambda(R/I) = e(I)$ for some I generated by a system of parameters.*

In the following discussion, we assign x to have weight b and y to have weight a, so that $\deg(x^\alpha y^\beta) = b\alpha + a\beta$. This is so that x^a and y^b will have equal degree. When we wish to specifically highlight the exponent vector, we will use log: $\log(x^\alpha y^\beta) = (\alpha, \beta) \in \mathbb{Z}^2$.

Remark 3.2.2. It may be noted that there exists a ring isomorphism $\varphi : R \to R'$ with $\varphi(x) = x^b$, $\varphi(y) = y^a$ and $R' = k[x^{ab}, x^{bp_1}y^{aq_1}, \ldots, y^{ab}]$. Without loss of generality, we might have assumed that $a = b$. On the other hand, for $a \neq b$ we may freely assume $\gcd(a, p_1, p_2, \ldots, p_t) = \gcd(b, q_1, \ldots, q_t) = 1$.

Notation 3.2.3. We consider (x^a, y^b) and its powers and find it convenient to denote $X := x^a$ and $Y := y^b$. We sort monomials in R into congruence classes based on their exponents in $H \subset (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$. For each $(p, q) \in H$, we choose $\alpha_{p,q}$ to be one monomial of minimal (weighted) degree in $R \setminus (X, Y)$. We will approximate $\lambda\left(\frac{(X,Y)^n}{(X,Y)^{n+1}}\right)$ by the size of

$$A_n := \cup_{(p,q)\in H}\{\alpha_{p,q}X^n, \alpha_{p,q}X^{n-1}Y, \ldots, \alpha_{p,q}Y^n\}$$

The remaining monomials of $(X, Y)^n$ outside both $(X, Y)^{n+1}$ and A_n will be denoted B_n. Set $\alpha_{p,q} = x^{\ell a+p}y^{mb+q} = {}_0\beta = \beta_0$. For $i < n$, let ${}_i\beta := x^{(\ell-i)a+p}y^{(m+j)b+q}$ with j the least possible integer such that ${}_i\beta \in R$. It may be that there is no such monomial, in which case we do not consider ${}_i\beta$ to be defined. Similarly, $\beta_i := x^{(\ell+j)a+p}y^{(m-i)b+q}$.

Lemma 3.2.4. *Let $i > j$. If $\deg(\beta_i) \leqslant \deg(\beta_h)$ for all $h > j$ such that β_h is well-defined, then $\beta_i X^{n-i+h}Y^{i-h} \notin (X, Y)^{n+1}$ for $i \geqslant h > j$.*

Proof. If $\beta_i X^{n-i+h}Y^{i-h} \in (X, Y)^{n+1}$, then there exist $\gamma \in R$ and $c \in \mathbb{N}$ with $\gamma X^{n+1-c}Y^c = \beta_i X^{n-i+h}Y^{i-h}$. By the minimality of the β's, β_{h+c} divides γ, so that $\deg(\beta_{h+c}) < \deg(\beta_i)$. □

Lemma 3.2.5. *Fix $(p, q) \in H$. Let $s_{p,q} = s$ be the highest integer such that β_s is defined. Let u be the maximum value of $i - j - 1$, $s \geqslant i > j \geqslant 0$ such that $\deg(\beta_i) < \deg(\beta_h)$ for all $i > h > j$. Let $\mathcal{U}_{p,q,n} = \mathcal{U}_n = \{\beta_i X^{n-c}Y^c | 0 < i \leqslant s, 0 \leqslant c \leqslant n\}$. Then $|B_n \cap \mathcal{U}_n| \leqslant s$ with equality if and only if $n \geqslant u$.*

Proof. We first claim $|B_n \cap \mathcal{U}_n| \leqslant s$. $B_n \subset (X,Y)^n \setminus (X,Y)^{n+1}$ which means at most one monomial in $\mathcal{U}_n$ of a given y-exponent may lie in B_n. If $c \geqslant i$, then $\alpha_{p,q} X^{n-c+i} Y^{c-i}$ divides $\beta_i X^{n-c} Y^c$, so $\beta_i X^{n-c} Y^c$ cannot be in B_n. There are only s-many other y-exponents the monomials in $\mathcal{U}_n$ might take.

Let $s = i_v > i_{v-1} > \ldots > i_0 = 0$ be integers such that $\deg(\beta_{i_v}) \geqslant \deg(\beta_{i_{v-1}}) \geqslant \ldots \geqslant \deg(\beta_0)$ and $\deg(\beta_{i_z}) < \deg(\beta_h)$ for all $i_z > h > i_{z-1}$.

Note that each pair i_z, i_{z-1} satisfies the defining condition of u. Moreover, since $\deg(\beta_{i_z}) \leqslant \deg(\beta_h)$ for any $h > i_z$, any pair i, j with $i > i_z > j$ fails the condition that defines u. Hence $u = \max_z(i_z - i_{z-1} - 1)$.

Suppose $n \geqslant u$ and consider the following monomials:

$$\begin{aligned}
&\beta_s X^n, \beta_s X^{n-1} Y, \ldots, \beta_s X^{n-(s-i_{v-1}-1)} Y^{s-i_{v-1}-1}, \\
&\beta_{i_{v-1}} X^n, \ldots, \beta_{i_{v-1}} X^{n-(i_{v-1}-i_{v-2}-1)} Y^{i_{v-1}-i_{v-2}-1}, \\
&\ldots, \\
&\beta_{i_1} X^n, \ldots, \beta_{i_1} X^{n-(i_1-1)} Y^{i_1-1}
\end{aligned}$$

By Lemma 3.2.4, each of these monomials lies outside $(X,Y)^{n+1}$, so $|B_n \cap \mathcal{U}_n| = s$.

Suppose $n < u$ and let i, j be indices satisfying $\deg(\beta_i) < \deg(\beta_h)$ for all $i > h > j$ and $i - j - 1 = u$. Consider the set of monomials $\beta_h X^{i-h-1} Y^{h-i+n+1}$ with $i > h \geqslant i - n - 1$. By the assumption on n, $\deg(\beta_i) < \deg(\beta_h)$ for $i > h \geqslant i - n - 1$ so $\beta_i Y^{n+1}$ divides each $\beta_h X^{i-h-1} Y^{h-i+n+1}$. Moreover, these $\beta_h X^{i-h-1} Y^{h-i+n+1}$ are the only monomials in $\mathcal{U}_n$ with the same y-exponent as $\beta_i Y^{n+1}$. Hence $B_n \cap \mathcal{U}_n$ does not contain a monomial with this y-exponent and $|B_n \cap \mathcal{U}_n| < s$. □

Symmetry allows us to apply this result to ${}_t\beta, \ldots, {}_0\beta$. Define u' for ${}_t\beta, \ldots, {}_0\beta$ as u is defined for $\beta_s, \ldots, \beta_0$. For $n \geqslant n_{p,q} := \max(u, u')$, this yields $|B_n \cap \{\gamma | \log(\gamma) \equiv (p,q) \in H\}| = s + t$.

Proposition 3.2.6. *$|B_n| \leqslant \sum_{(p,q)\in H}(t_{p,q} + s_{p,q})$ with equality iff $n \geqslant \max_{(p,q)\in H}(n_{p,q})$.*

Proof. B_n is the disjoint union of $\mathcal{U}_{p,q,n} \cap B_n$ as (p,q) varies in H. By 3.2.5, $|\cup_{(p,q)\in H} (\mathcal{U}_{p,q,n} \cap B_n)| = \sum_{(p,q)\in H} |\mathcal{U}_{p,q,n} \cap B_n| = \sum_{(p,q)\in H}(t_{p,q} + s_{p,q})$ if and only if $n \geqslant \max_{(p,q)\in H} n_{p,q}$. □

Not only is $|B_n|$ constant for large values of n, it determines the constant of the Hilbert polynomial $P(n) = \lambda((X,Y)^n/(X,Y)^{n+1})$. We demonstrate this by calculating the multiplicity from the growth of $|A_n|$.

Theorem 3.2.7. *Let $R = k[x^a, x^{p_1}y^{q_1}, x^{p_2}y^{q_2}, \ldots, x^{p_t}y^{q_t}, y^b]$. The Hilbert polynomial of (x^a, y^b) is $P(n) = |H|(n+1) + \sum_{(p,q)\in H}(t_{p,q} + s_{p,q})$ where $H \subset (\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z})$ is the subgroup generated by $(p_1,q_1),(p_2,q_2),\ldots,(p_t,q_t)$. In particular, $e((x^a,y^b)) = |H|$, and the Hilbert function equals the Hilbert polynomial for $n \geqslant \max_{(p,q)\in H}(n_{p,q})$.*

Proof. $(X,Y)^n$ is generated by the $n+1$ monomials $X^n, X^{n-1}Y, \ldots, Y^n$. R has dimension 2, so $\lambda\left(\frac{(X,Y)^n}{(X,Y)^{n+1}}\right)$ is given by a linear polynomial, $P(n)$, for sufficiently high n. Applying the notation from 3.2.3, we have $P(n) = |A_n| + |B_n|$. But $A_n = \cup_{(p,q)\in H}\{\alpha_{p,q}X^{n-i}Y^i | 0 \leqslant i \leqslant n\}$, so $|A_n| = |H|(n+1)$. Together with Lemma 3.2.5, we have $P(n) = |A_n| + |B_n| = |H|(n+1) + \sum_{(p,q)\in H}(t_{p,q} + s_{p,q})$. □

Taken together, these results allow us to construct rings with arbitrary conditions on the Hilbert polynomial and the level of its stabilization.

Corollary 3.2.8. *Given any subgroup $0 \neq H \subset (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$ and integers $C, m \geqslant 0$, there exists R such that (x^a,y^b) has Hilbert polynomial $P(n) = |H|(n+1) + C$ which equals the Hilbert function exactly for $n \geqslant m$.*

Proof. Fix $(0,0) \neq (p_0,q_0) \in H$ and an integer $N \geqslant C+m+1$. For any $(0,0) \neq (p,q) \in H$ we let $\alpha_{p,q} = x^p y^q X^N Y^{N+C+m}$. Let $\beta_{p_0,q_0,j} = x^{p_0}y^{q_0}X^{N+j}Y^{N+C+m-j}$ for $j = 0,1,2,\ldots,C-1, C+m$. Let $S' = \{\alpha_{p,q} \mid (0,0) \neq (p,q) \in H\} \cup \{\beta_{p_0,q_0,j} \mid j = 0,1,2,\ldots,C-1,C+m\}$, $S = S' \cup \{x^a,y^b\}$ and $R = k[S]$. We will show that $(x^a,y^b) \subset R$ has the required Hilbert polynomial.

Let $\gamma,\delta \in S'$, and let $\log_x, \log_y$ denote the respective exponents of a monomial. Then $\log_x(\gamma\delta) \geqslant 2Na \geqslant (N+C+m+1)a > (N+C+m)a + p$ and $\log_y(\gamma\delta) \geqslant 2Nb >$

$(N + C + m)b + q$ for any $0 \leqslant p < a$ and $0 \leqslant q < b$. In particular, $\log_x(\gamma\delta) > \log_x(\alpha_{p,q})$ and $\log_y(\gamma\delta) > \log_y(\alpha_{p,q})$ for any $\alpha_{p,q}$, so there is no ${}_{p,q,i}\beta$ for any $(p, q) \in H$ or $\beta_{p,q,j}$ for any $(p, q) \neq (p_0, q_0)$ or for $(p, q) = (p_0, q_0)$ with $j > C+m$. Similarly, $\log_x(\gamma\delta) > \log_x(\beta_{p_0,q_0,j})$ and $\log_y(\gamma\delta) > \log_y(\beta_{p_0,q_0,j})$ for any $j = 0, 1, 2, \ldots, C-1, C+m$, so $\deg(\beta_{p_0,q_0,j}) > \deg(\beta_{p_0,q_0,C+m})$ for all $j = C, C+1, \ldots, C+m-1$. In particular, $\beta_{p_0,q_0,j} = \beta_{p_0,q_0,C+m}Y^{C+m-j}$. This gives the maximum u as in Lemma 3.2.5 as $(C + m) - (C - 1) - 1 = m$. Therefore $(x^a, y^b) \subset R$ has Hilbert polynomial $P(n) = |H|(n + 1) + C$ which equals the Hilbert function exactly at $n \geqslant m$ by Proposition 3.2.6. □

Let us return to consideration of the CM property. In general, $\lambda(R/(X, Y)) \geqslant e((X, Y))$ and equality implies CM by 3.2.1.

Proposition 3.2.9. *The following are equivalent:*

(i) *R is CM*

(ii) $B_n = \emptyset$ *for all* n.

(iii) $B_i = \emptyset$ *for some* i.

Proof. $(i) \Rightarrow (ii)$ If R is CM, then by Theorems 3.2.1 and 3.2.7, $\lambda(R/(X, Y)) = |H|$. Then every monomial of R may be written as $\alpha_{p,q}X^iY^j$ for some $i, j \in \mathbb{N}$. Hence $B_n = \emptyset$ for all n.
$(iii) \Rightarrow (i)$ If R is not CM, then $\lambda(R/(X, Y)) > |H|$. By the pigeonhole principle, some congruence class $(p, q) \in H$ must be associated with two monomials in $R \setminus (X, Y)$. That is, for some $(p, q) \in H$, there is ${}_i\beta$ or β_j. If $s > 0$ is the highest integer such that β_s is defined, then $\beta_s X^i \in B_i$ for all i. □

An alternative manner of viewing this result helps to motivate the calculations in the following sections. Impose the reverse lexicographic order on monomials in R. Let $\mu_{p,q}$ be the least monomial in this order such that $\log(\mu_{p,q}) \equiv (p, q)$. We use $\mathcal{B}_0$ to indicate the collection of $\mu_{p,q}$ for all $(p, q) \in H$. Alternatively, we may use the lexicographic order of the monomials, and form a set $\mathcal{B}'_0$ of elements $\mu'_{p,q}$.

Proposition 3.2.10. *The following are equivalent:*

(i) *R is CM*

(ii) $\mathcal{B}_0 = \mathcal{B}'_0$

(iii) $\mu_{p,q} = \mu'_{p,q}$ *for all* $(p,q) \in H$.

Proof. $(i) \Rightarrow (ii)$: If R is CM, then $\lambda(R/(x^a, y^b)) = e((x^a, y^b)) = |\mathcal{B}_0| = |\mathcal{B}'_0|$ by Theorem 3.2.7. Elements of $\mathcal{B}_0$ are outside (X, Y) by construction, so $\mathcal{B}_0$ is a k-basis for $R/(X, Y)$. But the same is true for $\mathcal{B}'_0$ and there is only one k-basis consisting of monomials.

$(ii) \Rightarrow (iii)$: Suppose $\mathcal{B}_0 = \mathcal{B}'_0$. For each congruence class in H, $\mathcal{B}_0$ and $\mathcal{B}'_0$ contain exactly one element whose log lies in that class. Since $\mu_{p,q} \in \mathcal{B}'_0$, it must be that $\mu_{p,q} = \mu'_{p,q}$.

$(iii) \Rightarrow (i)$: Suppose $\mu_{p,q} = \mu'_{p,q}$, so that $\mu_{p,q}$ has the smallest x-exponent and the smallest y-exponent of any monomial in its congruence class. $B_n \cap \{\beta | \log(\beta) = (p,q)\} = \emptyset$. Since this holds for all $(p,q) \in H$, $B_n = \emptyset$ and R is CM by Proposition 3.2.9. □

3.3 Semigroup rings with four generators

In this section, we will consider semigroup rings of the form $R = k[x^d, x^e y^\ell, x^f y^m, y^n]$ with $d, n > 0$, $e, f, \ell, m \geqslant 0$ and $(e, \ell) \neq (0, 0)$. Our first main result in this section is Theorem 3.3.12, which gives a simple criterion to determine whether R is Cohen-Macaulay. The second main result is Theorem 3.3.15, which gives an algorithm to generate a k-basis of $R/(x^d, y^n)$. As noted in Remark 3.2.2, one may assume that $d = n$ for most results in this section, whereas Corollary 3.3.18 is probably best stated without assuming $d = n$.

Notation 3.3.1. Given a group G and an element $g \in G$, we write $\operatorname{ord}(g, G)$ to denote the order of g in G. For elements $(g, h), (g', h') \in \mathbb{Z}^2$ we let $(g, h) \prec (g', h')$ if and only if either $h' > h$ or $h' = h$ but $g' > g$.

Throughout this section, we fix $a_i, b_i \in \mathbb{N}$ and $(g_i, h_i) \in d\mathbb{Z} \oplus n\mathbb{Z}$, $i = 1, 2, 3$ as follows. Let $(g_1, h_1) \in d\mathbb{Z} \oplus n\mathbb{Z}$ be the smallest element with respect to $\prec$ such that there are positive integers a_1, b_1 with $b_2 \geqslant b_1$ (b_2 to be defined below) and

$$a_1(e, \ell) + b_1(f, m) = (g_1, h_1) \tag{3.3.1}$$

Let b_2 be the smallest positive integer such that there exist $a_2 \geqslant 0$ and $(g_2, h_2) \in d\mathbb{Z} \oplus n\mathbb{Z}$ with either at least one of g_2, h_2 being positive or $(g_2, h_2) = (0, 0)$ such that

$$-a_2(e, \ell) + b_2(f, m) = (g_2, h_2) \tag{3.3.2}$$

We choose $a_2 < \operatorname{ord}((e, \ell), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}))$.

Let a_3 be the smallest positive integer such that there exist $b_3 \geqslant 0$ and $(g_3, h_3) \in d\mathbb{Z} \oplus n\mathbb{Z}$ with $g_3, h_3 \geqslant 0$ and g_3, h_3 not both 0 such that

$$a_3(e, \ell) - b_3(f, m) = (g_3, h_3) \tag{3.3.3}$$

We choose $b_3 < \operatorname{ord}((f, m), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}))$.

Lemma 3.3.2. *We have $a_3 > a_2$ and $b_2 > b_3$.*

Proof. If $a_3 \leqslant a_2$, then (3.3.2) + (3.3.3) gives

$$(b_2 - b_3)(f, m) = (g_2 + g_3, h_2 + h_3) + (a_2 - a_3)(e, \ell)$$

By the definitions of b_2 and a_3, at least one of $g_2 + g_3$ or $h_2 + h_3$ is positive, so $b_2 - b_3 > 0$. If $b_3 = 0$, then the definition of a_3 gives $\operatorname{ord}((e, \ell), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) = a_3 \leqslant a_2$, contradicting the choice of a_2, so $b_3 > 0$. But then $b_2 > b_2 - b_3 > 0$ contradicts the minimality of b_2 in (3.3.2). Therefore $a_3 > a_2$.

If $b_2 \leqslant b_3$, then (3.3.2) + (3.3.3) gives

$$(a_3 - a_2)(e, \ell) = (g_2 + g_3, h_2 + h_3) + (b_3 - b_2)(f, m)$$

Again at least one of $g_2 + g_3$ or $h_2 + h_3$ is positive, so $a_3 - a_2 > 0$. If $a_2 = 0$, then the definition of b_2 gives $\operatorname{ord}((f,m), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) = b_2 \leqslant b_3$, contradicting the choice of b_3, so $a_2 > 0$. If $g_2 + g_3$ and $h_2 + h_3$ are both nonnegative, then $a_3 > a_3 - a_2 > 0$ contradicts the minimality of a_3 in (3.3.3). So without loss of generality, suppose that $h_2 + h_3 > 0$ and $g_2 + g_3 < 0$, so $g_2 < 0$ and $h_2 > 0$. Let $q \in \mathbb{Z}$, $q > 1$ be such that $a_3 - (q-1)a_2 > 0$ but $a_3 - qa_2 \leqslant 0$. Then $(q-1)(3.3.2) + (3.3.3)$ gives

$$(a_3 - (q-1)a_2)(e,\ell) = ((q-1)g_2 + g_3, (q-1)h_2 + h_3) + (b_3 - (q-1)b_2)(f,m)$$

Then $(q-1)g_2 + g_3 < 0$ gives $b_3 - (q-1)b_2 > 0$. Next, $q(3.3.2) + (3.3.3)$ gives

$$(qb_2 - b_3)(f,m) = (qg_2 + g_3, qh_2 + h_3) + (qa_2 - a_3)(e,\ell)$$

Since $qh_2 + h_3 > 0$ and $qa_2 - a_3 \geqslant 0$ we have $qb_2 - b_3 > 0$. But then $qb_2 - b_3 = b_2 - ((b_3 - (q-1)b_2) < b_2$ contradicts the minimality of b_2 in (3.3.2). Therefore $b_2 > b_3$. □

Lemma 3.3.3. *If $u, v \in \mathbb{Z}$ are such that $a_3 > u \geqslant 0$, $b_2 > v \geqslant 0$ and $(u,v) \neq (0,0)$, then $u(e,\ell) - v(f,m) \notin d\mathbb{Z} \oplus n\mathbb{Z}$.*

Proof. Suppose that $u(e,\ell) = (g,h) + v(f,m)$ for some $(g,h) \in d\mathbb{Z} \oplus n\mathbb{Z}$. If $g, h \geqslant 0$ and g, h are not both 0, then $u > 0$, contradicting the minimality of a_3. Otherwise we have $v(f,m) = (-g,-h) + u(e,\ell)$ with $-g > 0$, $-h > 0$ or $(-g,-h) = (0,0)$, contradicting the minimality of b_2. Therefore such (g,h) does not exist. □

Lemma 3.3.4. *Suppose that $a, b \in \mathbb{N}$ and $(g,h) \in d\mathbb{Z} \oplus n\mathbb{Z}$ are such that*

$$a(e,\ell) + b(f,m) = (g,h) \tag{3.3.4}$$

(i) *If $a_2 = a = 0$, then $b_2 \mid b$. If $b_3 = b = 0$, then $a_3 \mid a$.*

(ii) *If $a \geqslant a_3$ and $b > b_2$, then $g \geqslant g_2 + g_3$ and $h \geqslant h_2 + h_3$. If $a > a_3 - a_2$ or $(f,m) \neq (0,0)$, then $(g,h) \neq (g_2 + g_3, h_2 + h_3)$.*

(iii) *Suppose that* $a \geqslant a_3$, $b \leqslant b_2$ *and* $(a,b) \neq (a_3 - a_2, b_2 - b_3)$. *If either* $b \geqslant b_2 - b_3$ *or* $b_2 - b_3 > b$ *and* b_3, b *are not both 0, then* $g \geqslant g_2 + g_3$, $h \geqslant h_2 + h_3$ *and* $(g,h) \neq (g_2 + g_3, h_2 + h_3)$.

(iv) *If* $a \leqslant a_3$, $b \leqslant b_2$, a_2, a *are not both 0 and* b_3, b *are not both 0, then* $(a,b) = (a_3 - a_2, b_2 - b_3)$ *or* $(0,0)$, *or* $a > a_3 - a_2$ *and* $b > b_2 - b_3$. *In fact, there exists* $q \in \mathbb{N}$ *such that* $a = q(a_3 - a_2)$ *and* $b = q(b_2 - b_3)$.

In particular, we have (3.3.1) = (3.3.2) + (3.3.3), *and one may use either lexicographic or reverse lexicographic ordering in the definition of* (3.3.1).

Proof. (i): If $a_2 = 0$, then $b_2 = \mathrm{ord}((f,m), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}))$, so if $a = 0$, then $b_2 \mid b$. Similarly, if $b_3 = b = 0$, then $a_3 = \mathrm{ord}((e,\ell), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) \mid a$.
(ii) and (iii): Now since $a_3 > a_2$ and $b_2 > b_3$, (3.3.2) + (3.3.3) gives

$$(a_3 - a_2)(e,\ell) + (b_2 - b_3)(f,m) = (g_2 + g_3, h_2 + h_3) \tag{3.3.5}$$

Suppose that $a \geqslant a_3 \geqslant a_3 - a_2$ and $b \geqslant b_2 - b_3$. Then (3.3.4) – (3.3.5) gives $g \geqslant g_2 + g_3$ and $h \geqslant h_2 + h_3$. Suppose furthermore that $(a,b) \neq (a_3 - a_2, b_2 - b_3)$. If $a > a_3 - a_2$, then $g > g_2 + g_3$ or $h > h_2 + h_3$. If $a = a_3 - a_2$ so that $b > b_2 - b_3$, then $b_2 \leqslant \mathrm{ord}((f,m), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) \mid b - (b_2 - b_3)$, so $b > b_2$. So suppose that $a \geqslant a_3$, $b < b_2 - b_3$ and b_3, b are not both 0. If $g < g_2 + g_3$ or $h < h_2 + h_3$ or $(g,h) = (g_2 + g_3, h_2 + h_3)$, then (3.3.5) – (3.3.4) gives

$$(b_2 - b_3 - b)(f,m) = (g_2 + g_3 - g, h_2 + h_3 - h) + (a - a_3 + a_2)(e,\ell),$$

contradicting the minimality of b_2. So $g \geqslant g_2 + g_3$, $h \geqslant h_2 + h_3$ and $(g,h) \neq (g_2 + g_3, h_2 + h_3)$.
(iv): Now suppose that $a_3 \geqslant a$, $b_2 \geqslant b$, a_2, a are not both 0, b_3, b are not both 0, $(a,b) \neq (a_3 - a_2, b_2 - b_3)$ and $(a,b) \neq (0,0)$. Then by Lemma 3.3.3, we cannot have $a \geqslant a_3 - a_2$ and $b \leqslant b_2 - b_3$, or $a \leqslant a_3 - a_2$ and $b \geqslant b_2 - b_3$. So suppose that $a < a_3 - a_2$ and $b < b_2 - b_3$. If

$a \neq 0$, then (3.3.3) − (3.3.4) gives

$$(a_3 - a)(e, \ell) - (b + b_3)(f, m) = (g_3 - g, h_3 - h),$$

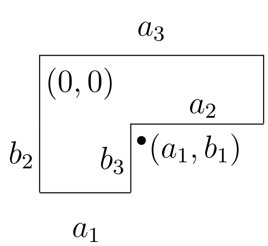
contradicting Lemma 3.3.3. Similarly, $b \neq 0$ and (3.3.4) − (3.3.2) gives a contradiction. Therefore $a > a_3 - a_2$ and $b > b_3 - b_2$. In such a case, let $q \in \mathbb{N}$ be such that $a - (q-1)(a_3 - a_2), b - (q-1)(b_2 - b_3) > 0$ but one of $a - q(a_3 - a_2)$ or $b - q(b_2 - b_3)$ is nonpositive. By what we just proved, we have $(a - (q-1)(a_3 - a_2), b - (q-1)(b_2 - b_3)) = (a_3 - a_2, b_2 - b_3)$, so $a = q(a_3 - a_2)$ and $b = q(b_2 - b_3)$. □

Notation 3.3.5. Let a, b denote natural numbers. We let

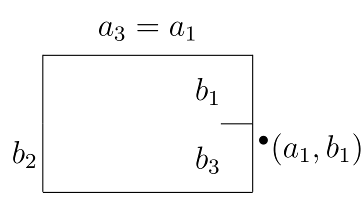
$$\begin{aligned} B_0 &= \{(a,b) \mid a < a_1 \text{ and } b < b_2\} \cup \{(a,b) \mid a < a_3 \text{ and } b < b_1\} \\ &= \{(a,b) \mid a < a_3 \text{ and } b < b_2\} \setminus \{(a,b) \mid a \geqslant a_1 = a_3 - a_2 \text{ and } b \geqslant b_1 = b_2 - b_3\} \end{aligned}$$

Let us write $\langle a, b \rangle = a(e, \ell) + b(f, m)$. We write $\langle a, b \rangle \equiv \langle a', b' \rangle$ to mean $\langle a, b \rangle - \langle a', b' \rangle \in d\mathbb{Z} \oplus n\mathbb{Z}$. We let H be the subgroup of $(\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})$ generated by $(e, \ell) = \langle 1, 0 \rangle$ and $(f, m) = \langle 0, 1 \rangle$.

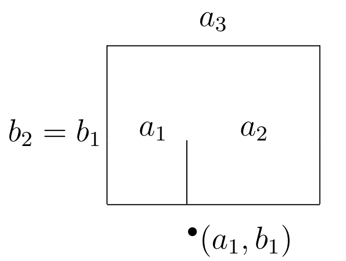
Remark 3.3.6. We may visualize the set B_0 as follows. For $(a, b) \in \mathbb{N} \times \mathbb{N}$, the first coordinate a increases to the right and the second coordinate b increases downwards.



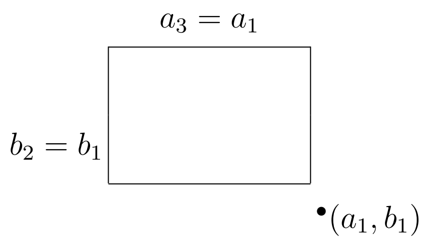
General case



Case $a_2 = 0$, $b_3 \neq 0$



Case $a_2 \neq 0$, $b_3 = 0$



Case $a_2 = b_3 = 0$

Lemma 3.3.7. *We have* $|B_0| = |H|$.

Proof. $|B_0| \geqslant |H|$: We will show that for every $\langle a', b' \rangle$ with $a', b' \in \mathbb{N}$, there exists $(a, b) \in B_0$ such that $\langle a, b \rangle \equiv \langle a', b' \rangle$. First, we show that there exist $a'', b'' \in \mathbb{N}$ such that $\langle a', b' \rangle \equiv \langle a'', b'' \rangle$ and $b'' < b_2$. Let $q, r \in \mathbb{N}$ be such that $b' = qb_2 + r$ as in the Euclidean algorithm. Then from (3.3.2) we have $\langle 0, b_2 \rangle \equiv \langle a_2, 0 \rangle$, so $\langle a', b' \rangle \equiv \langle a' + qa_2, r \rangle$ with $b_2 > r \geqslant 0$.

So assume that $b' < b_2$. We will now reduce to the case that $a' < a_3$. It suffices to show that if $a' \geqslant a_3$, then there exist $a'', b'' \in \mathbb{N}$ such that $\langle a', b' \rangle \equiv \langle a'', b'' \rangle$, $a'' < a'$ and $b'' < b_2$.

Case 1: $b' \geqslant b_1$. From (3.3.1) we have $\langle a_1, b_1 \rangle \equiv \langle 0, 0 \rangle$, so $\langle a', b' \rangle \equiv \langle a' - a_1, b' - b_1 \rangle$ with $a' > a' - a_1 \geqslant a' - a_3 \geqslant 0$ and $b_2 > b' > b' - b_1 \geqslant 0$.

Case 2: $b' < b_1$ and $b' + b_3 < b_2$. From (3.3.3) we have $\langle a_3, 0 \rangle \equiv \langle 0, b_3 \rangle$, so $\langle a', b' \rangle \equiv \langle a' - a_3, b' + b_3 \rangle$.

Case 3: $b' < b_1$ and $b' + b_3 \geqslant b_2$. From (3.3.3) and (3.3.2) we have $\langle a', b' \rangle \equiv \langle a' - a_3 + a_2, b' + b_3 - b_2 \rangle$ with $a' > a' - a_1 = a' - a_3 + a_2$ and $b_2 > b' > b' - b_1 = b' + b_3 - b_2 \geqslant 0$.

So suppose that $a' < a_3$ and $b' < b_2$ but $a' \geqslant a_1$ and $b' \geqslant b_1$. Let $q \in \mathbb{N}$ be such that $a' - qa_1, b' - qb_1 \geqslant 0$ but $a' - (q+1)a_1$ or $b' - (q+1)b_1$ is negative, so that $a' - qa_1 < a_1$ or $b' - qb_1 < b_1$. Then $\langle a', b' \rangle \equiv \langle a' - qa_1, b' - qb_1 \rangle$ and $(a' - qa_1, b' - qb_1) \in B_0$.

$|B_0| \leqslant |H|$: Suppose that $(a, b), (a', b') \in B_0$ and $(a, b) \neq (a', b')$. If $a' - a \geqslant 0$ and $b' - b \leqslant 0$ then $\langle a, b \rangle \not\equiv \langle a', b' \rangle$ by Lemma 3.3.3. If $a' - a, b' - b \geqslant 0$ then $\langle a, b \rangle \not\equiv \langle a', b' \rangle$ by Lemma 3.3.4. Therefore $|B_0| \leqslant |H|$. $\square$

Notation 3.3.8. Given $a, b \in \mathbb{N}$, we define the monomial

$$\vec{x}^{\langle a,b\rangle} = (x^e y^\ell)^a (x^f y^m)^b = x^{ae+bf} y^{a\ell+bm}$$

We also define the set of monomials $\mathcal{B}_0 = \{\vec{x}^{\langle a,b\rangle} \mid (a,b) \in B_0\}$.

Remark 3.3.9. Let $a, b, a', b' \in \mathbb{N}$.

(i) If $a' \geqslant a$, $b' \geqslant b$ and $\langle a', b'\rangle - \langle a, b\rangle = (g, h)$, then $g, h \geqslant 0$.

(ii) Equations (3.3.1) and (3.3.3) show that $\vec{x}^{\langle a_1,b_1\rangle} \in (x^d, y^n)$ and $\vec{x}^{\langle a_3,0\rangle} \in \vec{x}^{\langle 0,b_3\rangle}(x^d, y^n)$. Hence $\vec{x}^{\langle a',b'\rangle} \in (x^d, y^n)$ if $a' \geqslant a_3$, or $a' \geqslant a_1$ and $b' \geqslant b_1$.

(iii) If $a' \leqslant a$, $b' \leqslant b$ and $(a,b) \in B_0$, then $(a', b') \in B_0$.

Lemma 3.3.10. *Given a set $S \subseteq \mathbb{N} \times \mathbb{N}$, the set of monomials $\{\vec{x}^{\langle a,b\rangle} \mid (a,b) \in S\}$ is linearly independent in $R/(x^d, y^n)$ over k if and only if:*

(i) if $(a,b) \in S$, $a', b' \in \mathbb{N}$ and $\langle a, b\rangle - \langle a', b'\rangle = (g,h) \in d\mathbb{Z} \oplus n\mathbb{Z}$, then $g < 0$ or $h < 0$ or $(g,h) = (0,0)$, and

(ii) if $(a,b), (a',b') \in S$ and $(a,b) \neq (a',b')$, then $\langle a,b\rangle \neq \langle a',b'\rangle$.

Proof. Every monomial in (x^d, y^n) can be written as a scalar multiple of $x^g \vec{x}^{\langle a,b\rangle} y^h$ for some $a, b \in \mathbb{N}$ and $(g,h) \in d\mathbb{Z} \oplus n\mathbb{Z}$ with $g, h \geqslant 0$ and $(g,h) \neq (0,0)$. □

Lemma 3.3.11. *The set $\mathcal{B}_0$ is linearly independent in $R/(x^d, y^n)$ over k.*

Proof. By Lemma 3.3.7, we only need to verify (i) in Lemma 3.3.10 for $(a,b) \in B_0$ and $(a',b') \notin B_0$. By Remark 3.3.9, we may assume that $a' < a$ or $b' < b$. If $a' < a$, then by Remark 3.3.9 we have $b' > b$, so $\langle a - a', 0\rangle = \langle 0, b' - b\rangle + (g,h)$. By the minimality of a_3 we have $g < 0$, $h < 0$ or $(g,h) = (0,0)$. Similarly, if $b' < b$, then $a' > a$ and $\langle 0, b - b'\rangle = \langle a' - a, 0\rangle + (g,h)$ and the result follows from the minimality of b_2. □

Theorem 3.3.12. *For the ring $R = k[x^d, x^e y^\ell, x^f y^m, y^n]$, we have:*

(i) $|\mathcal{B}_0| = |H| = \begin{vmatrix} a_3 & -b_3 \\ -a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_3 & -b_3 \\ a_1 & b_1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ -a_2 & b_2 \end{vmatrix}$

(ii) $\dim_k R/(x^d, y^n) \geqslant |H| = |\mathcal{B}_0|$

(iii) *The ring R is Cohen-Macaulay if and only if $\mathcal{B}_0$ is a basis of $R/(x^d, y^n)$ over k if and only if $g_2, h_2 \geqslant 0$.*

Proof. (i): We have $|\mathcal{B}_0| = |B_0| = |H|$ by Lemma 3.3.7. The definition of B_0 gives

$$|B_0| = a_3b_2 - (a_3 - a_1)(b_2 - b_1) = a_3b_2 - a_2b_3$$

The rest again follows from (3.3.1) = (3.3.2) + (3.3.3).

(ii): By Lemma 3.3.11, the set $\mathcal{B}_0$ is linearly independent in $R/(x^d, y^n)$ over k.

(iii): If $g_2 < 0$ or $h_2 < 0$, then $(a, b) = (0, b_2)$ and $(a', b') = (a_2, 0)$ satisfy Lemma 3.3.10 by (3.3.2). Let us verify Lemma 3.3.10(i) for $(0, b_2)$ and $(a', b') \notin B_0$. By Remark 3.3.9 we may assume that $b' < b_2$. If $b' > 0$, then $\langle 0, b_2 - b' \rangle = \langle a', 0 \rangle + (g, h)$ and (i) is satisfied by the linear independence of $\mathcal{B}_0$ in $R/(x^d, y^n)$ over k. If $b' = 0$, then $a' \geqslant a_3 > a_2$. If $\langle 0, b_2 \rangle = \langle a', 0 \rangle + (g, h)$ with $g, h \geqslant 0$, then $g_2, h_2 \geqslant 0$ in (3.3.2), contradicting our assumption. Therefore Lemma 3.3.10(i) holds for $(a, b) = (0, b_2)$ and hence $\mathcal{B}_0 \cup \{\vec{x}^{\langle 0, b_2 \rangle}\}$ is linearly independent in $R/(x^d, y^n)$ over k.

If $g_2, h_2 \geqslant 0$, then $\vec{x}^{\langle 0, b_2 \rangle} = \vec{x}^{\langle a_2, 0 \rangle}$ or $\vec{x}^{\langle 0, b_2 \rangle} \in \vec{x}^{\langle a_2, 0 \rangle}(x^d, y^n)$. In the first case, for $a, b \in \mathbb{N}$ we have $\vec{x}^{\langle a, b \rangle} = \vec{x}^{\langle a + qa_2, b - qb_2 \rangle}$ for any $q \in \mathbb{Z}$. So by the definition of B_0 and Remark 3.3.9 we see that for all $(a', b') \notin B_0$ either $\vec{x}^{\langle a', b' \rangle} \in (x^d, y^n)$ or $\vec{x}^{\langle a', b' \rangle} = \vec{x}^{\langle a, b \rangle}$ for some $(a, b) \in B_0$. Therefore $\mathcal{B}_0$ is a basis of $R/(x^d, y^n)$ over k.

Finally, Theorems 3.2.1 and 3.2.7 show that R is Cohen-Macaulay iff $\dim_k R/(x^d, y^n) = |H|$. By (ii), $\dim_k R/(x^d, y^n) = |H|$ if and only if $\mathcal{B}_0$ is a basis of $R/(x^d, y^n)$ over k if and only if $g_2, h_2 \geqslant 0$. □

Remark 3.3.13. In part (iii) of Theorem 3.3.12, instead of using Theorems 3.2.1 and 3.2.7, one can also prove the result using the fact that R is Cohen-Macaulay if and only if x^d, y^n is a regular sequence.

Corollary 3.3.14. *The ring $k[x^d, x^e y^\ell, y^n]$ is Cohen-Macaulay, where $d, n > 0$ and $(e, \ell) \neq (0, 0)$.*

Proof. Take $(f, m) = u_1(d, 0) + u_2(e, \ell) + u_3(0, n)$ for any $u_1, u_2, u_3 \in \mathbb{N}$. □

Theorem 3.3.15. *We can use the following algorithm to obtain a basis of $R/(x^d, y^n)$ over k.*

1 Let $B = B_0$.

2 Let $base = a_1$, $a^* = a_2$, $b^* = b_2$, $g^* = g_2$ and $h^* = h_2$.

3 While $g^* < 0$ or $h^* < 0$, do the following steps.

4 If $a^* \geqslant a_1$, then:

Replace B by $B \cup \{(0, b^*) + (u, v) \mid u < base \text{ and } v < b_1\}$.

Replace a^* by $a^* - a_1$, b^* by $b^* + b_1$, g^* by $g^* + g_1$ and h^* by $h^* + h_1$.

5 If $a^* \leqslant a_1 - base$, then:

Replace B by $B \cup \{(0, b^*) + (u, v) \mid u < base \text{ and } v < b_2\}$.

Replace a^* by $a^* + a_2$, b^* by $b^* + b_2$, g^* by $g^* + g_2$ and h^* by $h^* + h_2$.

6 If $a_1 - base < a^* < a_1$, then:

Replace B by $B \cup \{(0, b^*) + (u, v) \mid (u < base \text{ and } v < b_1) \text{ or } (u < a_1 - a^* \text{ and } v < b_2)\}$.

Replace a^* by $a^* + a_2$, b by $b^* + b_2$, g^* by $g^* + g_2$, h^* by $h^* + h_2$ and $base$ by $a_1 - a^*$.

After the algorithm stops, the set of monomials $\mathcal{B} = \{\vec{x}^{\langle a,b \rangle} \mid (a, b) \in B\}$ forms a basis of $R/(x^d, y^n)$ over k.

Remark 3.3.16. Theorem 3.3.15 only needs to use information from (3.3.1) and (3.3.2), or equivalently, from (3.3.2) and (3.3.3). Given the equation

$$-a^*(e, \ell) + b^*(f, m) = (g^*, h^*), \tag{3.3.6}$$

Step 4 corresponds to (3.3.6) + (3.3.1) and Steps 5 and 6 correspond to (3.3.6) + (3.3.2). Furthermore, in each iteration of the algorithm, the new elements added to the set B are in one-to-one correspondence with those in $\{(a, b) \in B_0 \mid a^* \leqslant a < a^* + base\}$.

Proof of Theorem 3.3.15. First, we note by induction that throughout the algorithm,

(a) $a^* + base \leqslant a_3$,

(b) the value of *base* is always positive and weakly decreasing, and

(c) if $a, b, a', b' \in \mathbb{N}$, $a' \leqslant a$, $b' \leqslant b$ and $(a, b) \in B$, then $(a', b') \in B$.

Let ^{u} denote the updated value of a variable after an iteration of the algorithm. We note also that in Steps 4, 5 and 6:

(d) Let $C = B^u \setminus B$ and $(a, b) \in C$. Then $\langle a, b\rangle \equiv \langle a + a^*, b - b^*\rangle$ and $(a + a^*, b - b^*) \in B_0$. Hence if $(a', b') \in C$ such that $(a, b) \neq (a', b')$, then $\langle a, b\rangle \not\equiv \langle a', b'\rangle$.

We will now prove by induction on the number of iterations that after each iteration of the algorithm,

(e) the set $\mathcal{B}^u = \{\vec{x}^{\langle a,b\rangle} \mid (a, b) \in B^u\}$ is linearly independent in $R/(x^d, y^n)$ over k, and

(f) $\vec{x}^{\langle a',b'\rangle} \in (x^d, y^n)$ for all $(a', b') \notin B^u$ such that $b' < b^{*u}$.

The base case of $B = \emptyset$, i.e. $B^u = B_0$, is given by Theorem 3.3.12(ii) and Remark 3.3.9. In the induction step, we will first show (e) by using Lemma 3.3.10.

Let $(a, b) \in C = B^u \setminus B$ and $(a', b') \in B$ such that $\langle a, b\rangle \equiv \langle a', b'\rangle$. If $(a', b') \in B_0$, then $(a', b') = (a+a^*, b-b^*)$ and $\langle a, b\rangle - \langle a+a^*, b-b^*\rangle = (g^*, h^*)$. By assumption, $g^* < 0$ or $h^* < 0$, so (a, b) and (a', b') satisfy Lemma 3.3.10. If $(a', b') \notin B_0$, then we have $\langle a, b-b_2\rangle \equiv \langle a', b'-b_2\rangle$ and by the linear independence of $\mathcal{B}$, (a, b) and (a', b') again satisfy Lemma 3.3.10.

So suppose that $(a', b') \notin B$ and $\langle a, b\rangle - \langle a', b'\rangle = (g, h) \in d\mathbb{Z} \oplus n\mathbb{Z}$. Let us verify Lemma 3.3.10(i). By the proof of Lemma 3.3.11, we may assume that $b' < b$.

Case 1: $b' \geqslant b_2$. Lemma 3.3.10(i) holds from $\langle a, b - b'\rangle - \langle a', 0\rangle = (g, h)$ and the linear independence of $\mathcal{B}$.

Case 2: $b_2 > b' \geqslant b_1$. By the definition of B_0 we have $a' \geqslant a_1$. Let $q \in \mathbb{N}$ be such that $a' - qa_1, b' - qb_1 \geqslant 0$ but one of $a' - (q + 1)a_1$ or $b' - (q + 1)b_1$ is negative, so that $\langle a, b\rangle - \langle a' - qa_1, b' - qb_1\rangle = \langle a, b\rangle - \langle a', b'\rangle + q\langle a_1, b_1\rangle = (g + qg_1, h + qh_1)$. If $(a' - qa_1, b' - qb_1) \in B_0$, then $g + qg_1 = g^* < 0$ or $h + qh_1 = h^* < 0$, so Lemma 3.3.10(i) holds for (a, b) and (a', b'). Otherwise, replacing $(a' - qa_1, b' - qb_1)$ by (a', b'), we are reduced to the case where

$b' < b_1$.

Case 3: $b_1 > b'$. By the definition of B_0 we have $a' \geqslant a_3$. Then $\langle a', b'\rangle - \langle a + a^*, b - b^*\rangle = \langle a', b'\rangle - \langle a, b\rangle + \langle a, b\rangle - \langle a + a^*, b - b^*\rangle = (g^* - g, h^* - h)$. If $b' \geqslant b - b^*$, then $0 > g^* \geqslant g$ or $0 > h^* \geqslant h$ and Lemma 3.3.10(i) is satisfied. If $b' < b - b^*$, then $\langle 0, b - b^* - b'\rangle = \langle a' - (a + a^*), 0\rangle + (g - g^*, h - h^*)$. By the minimality of b_2 we have $g - g^*, h - h^* \leqslant 0$ and again we are done.

Now we verify (f). Let $(a', b') \notin B^u$ with $b' < b^{*u}$. By induction, we may assume that $b' \geqslant b^*$ and by Remark 3.3.9 we may assume that $a' < base$, so we only need to consider Step 6 with $a' \geqslant a_1 - a^*$ and $b' \geqslant b^* + b_1$. We have $\langle a_1 - a^*, b^* + b_1\rangle = \langle a_1, b_1\rangle + (g^*, h^*) \in d\mathbb{Z} \oplus n\mathbb{Z}$, so $\vec{x}^{\langle a_1 - a^*, b^* + b_1\rangle} \in (x^d, y^n)$ and the result follows from Remark 3.3.9.

Finally, the algorithm must stop at or before $b^* = \operatorname{ord}((f, m), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}))$. After the algorithm stops, we already know that $\mathcal{B}$ is linearly independent by (e). By (3.3.6) we have $\vec{x}^{\langle 0, b^*\rangle} = \vec{x}^{\langle a^*, 0\rangle}$ or $\vec{x}^{\langle 0, b^*\rangle} \in \vec{x}^{\langle a^*, 0\rangle}(x^d, y^n)$. By (f) and Remark 3.3.9 we see that for all $(a', b') \notin B$ either $\vec{x}^{\langle a', b'\rangle} \in (x^d, y^n)$ or $\vec{x}^{\langle a', b'\rangle} = \vec{x}^{\langle a, b\rangle}$ for some $(a, b) \in B$. Therefore $\mathcal{B}$ is a basis of $R/(x^d, y^n)$ over k. □

Remark 3.3.17. Each iteration of the algorithm gives $\langle 0, b^*\rangle \equiv \langle a^*, 0\rangle$. Since the algorithm must stop at or before $b^* = \operatorname{ord}((f, m), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z}))$, we cannot have $\langle 0, b^{1*}\rangle \equiv \langle 0, b^{2*}\rangle$ for different values b^{1*}, b^{2*} of b^*. So the number of iterations of the algorithm is at most $a_3 \leqslant \operatorname{ord}((e, \ell), (\mathbb{Z}/d\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})) \leqslant |H| \leqslant dn$.

Corollary 3.3.18. *We have* $\dim_k R/(x^d, y^n) \leqslant |H|(|H| + 1)/2 \leqslant dn(dn + 1)/2$.

Proof. Let us set $a^* = 0$ in Step 1. In each iteration of the algorithm, at most $i_{a^*} = |\{(a, b) \in B_0 \mid a \geqslant a^*\}|$ elements are added to the set B by Remark 3.3.16. We have $1 \leqslant i_{a^*} \leqslant |B_0| = |H| \leqslant dn$ and that the map $a^* \mapsto i_{a^*}$ is injective. Since there exist at most $|H|$ possible values of a^* before the stopping criterion is reached, we have $\dim_k R/(x^d, y^n) = |B| \leqslant \sum_{i=1}^{|H|} i = |H|(|H| + 1)/2 \leqslant dn(dn + 1)/2$. □

Example 3.3.19. Here we give examples showing that the algorithm is "best possible", in the sense that the maximum number of iterations can be attained. In the second example, we will show that the upper bound in Corollary 3.3.18 is also attained. Let p, q be distinct prime numbers.

(a) Let $R = k[x^p, x^{jpq-1}y, xy^{jpq-1}, y^q]$ with $j \in \mathbb{N}$, $j > 1$. The successive values of $\langle 0, b^* \rangle$ are

$$(1, jpq-1),\ (2, 2(jpq-1)),\ \ldots,\ (pq-1, (pq-1)(jpq-1)),\ (pq, pq(jpq-1))$$

and those of $\langle a^*, 0 \rangle$ are

$$((pq-1)(jpq-1), pq-1),\ ((pq-2)(jpq-1), pq-2),\ \ldots,\ (jpq-1, 1),\ (0,0)$$

When $j = 2$, $p = 2$ and $q = 3$ (or $p = 3$ and $q = 2$), we display the elements of $\langle B \rangle = \{\langle a, b \rangle \mid (a, b) \in B\} = \log(\mathcal{B})$ as follows.

(0,0)	(11,1)	(22,2)	(33,3)	(44,4)	(55,5)
(1,11)					
(2,22)					
(3,33)					
(4,44)					
(5,55)					

$$R = k[x^2, x^{11}y, xy^{11}, y^3]$$

(b) Let $R = k[x^p, x^{jpq+1}y, xy^{jpq+1}, y^q]$ with $j \in \mathbb{N}$, $j > 0$. The successive values of $\langle 0, b^* \rangle$ are

$$(1, jpq+1),\ (2, 2(jpq+1)),\ \ldots,\ (pq-1, (pq-1)(jpq+1)),\ (pq, pq(jpq+1))$$

and those of $\langle a^*, 0 \rangle$ are

$$(jpq+1, 1),\ (2(jpq+1), 2),\ \ldots,\ ((pq-1)(jpq+1), pq-1),\ (0,0)$$

When $j = 1$, $p = 2$ and $q = 3$, we display the elements of $\langle B \rangle$ as follows.

(0,0)	(7,1)	(14,2)	(21,3)	(28,4)	(35,5)
(1,7)	(8,8)	(15,9)	(22,10)	(29,11)	
(2,14)	(9,15)	(16,16)	(23,17)		
(3,21)	(10,22)	(17,23)			
(4,28)	(11,29)				
(5,35)					

$$R = k[x^2, x^7y, xy^7, y^3]$$

Remark 3.3.20. Having found the basis $\mathcal{B}$ of $R/(x^a, y^b)$ as in Theorem 3.3.15, one may sort the monomials in $\mathcal{B}$ and find the Hilbert polynomial $P(n)$ for $(x^a, y^b) \subseteq R$ and the least integer m such that the Hilbert polynomial equals the Hilbert function for all $n \geqslant m$ by Theorem 3.2.7.

3.4 Projective monomial curves in $\mathbb{P}^3$

In this section, we will consider rings of the form $R = k[x^n, x^{n-\ell}y^\ell, x^{n-m}y^m, y^n]$ with $0 < \ell < m < n$. We will apply the results from Section 3.3 to obtain stronger results for such rings R. In particular, Theorem 3.4.6 gives a simple criterion to determine whether R is Cohen-Macaulay and Theorem 3.4.10 gives a simple algorithm to generate a k-basis of $R/(x^n, y^n)$.

Notation 3.4.1. In this section, we fix $a_i, b_i, c_i, h_i \in \mathbb{N}$, $i = 1, 2, 3$ as follows. Let c_1 be the smallest integer such that there are $m/\gcd(\ell, m) \geqslant a_1 > 0$ and $b_1 > 0$ with

$$a_1\ell + b_1m = c_1n = h_1 \tag{3.4.1}$$

Let b_2 be the smallest integer such that there are $n/\gcd(\ell, n) > a_2 \geqslant 0$ and $c_2 > 0$ with

$$-a_2\ell + b_2m = c_2n = h_2 \tag{3.4.2}$$

Let a_3 be the smallest positive integer such that there are $n/\gcd(m,n) > b_3 \geqslant 0$ and $c_3 \geqslant 0$ with

$$a_3\ell - b_3 m = c_3 n = h_3 \tag{3.4.3}$$

Remark 3.4.2. We recall from Section 3.1 that for any $d \in \mathbb{Z}$ we have $\operatorname{ord}((n-d,d),(\mathbb{Z}/n\mathbb{Z})\oplus(\mathbb{Z}/n\mathbb{Z})) = \operatorname{ord}((-d,d),(\mathbb{Z}/n\mathbb{Z})\oplus(\mathbb{Z}/n\mathbb{Z})) = n/\gcd(d,n)$.

Lemma 3.4.3. *Let $a,b,c,d \in \mathbb{N}$.*

(i) *If $-a(n-\ell,\ell)+b(n-m,m) = (cn,dn)$, $b>0$ and $c \geqslant 0$, then $d>0$.*

(ii) *If $a(n-\ell,\ell)-b(n-m,m) = (cn,dn)$, $a>0$ and $d \geqslant 0$, then $c>0$.*

Proof. (i): Since $b>0$ we have $(a,c) \neq (0,0)$. If $c \geqslant 0$, then $b(n-m) = cn + a(n-\ell) > (c+a)(n-m)$, so $b > c+a$. Hence $dn = -a\ell + bm = (b-a-c)n > 0$.
(ii): Replace a by b, b by a, ℓ by $n-m$, m by $n-\ell$, c by d and d by c in (i). □

Lemma 3.4.4. *The definitions of a_i, b_i, h_i, $i = 1,2,3$ in Notation 3.3.1 and 3.4.1 coincide. In particular, we have* (3.4.1) = (3.4.2) + (3.4.3).

Proof. Let us temporarily write a_i', b_i', h_i', $i = 1,2,3$ for the definitions of a_i, b_i, h_i in this section. We have $e = n-\ell$ and $f = n-m$. Let us first consider (3.3.2). Suppose that $g_2 \geqslant 0$. By Lemma 3.4.3(i) we have $h_2 > 0$, so the conditions that $g_2 > 0$ or $(g_2,h_2) = (0,0)$ become redundant. Hence $b_2 = b_2'$, $a_2 = a_2'$ and $h_2 = h_2'$.
Similarly, in (3.3.3) suppose that $h_3 \geqslant 0$. By Lemma 3.4.3(ii) the conditions $g_3 \geqslant 0$ and $(g_3,h_3) \neq (0,0)$ are redundant. Hence $a_3 = a_3'$, $b_3 = b_3'$ and $h_3 = h_3'$.
Now (3.4.2) + (3.4.3) gives

$$(a_3-a_2)\ell + (b_2-b_3)m = (c_2+c_3)n = h_2+h_3$$

Let us show that $c_1 = c_2+c_3$. First we have $a_3 - a_2 \leqslant a_3 \leqslant m/\gcd(\ell,m)$. Now suppose that $a,b,c \in \mathbb{N}$ are such that $a,b>0$, $a\ell + bm = cn$ and $(a,b) \neq (a_3-a_2, b_2-b_3)$. By Remark 3.3.9 we may assume that $a < a_3-a_2$ or $b < b_2-b_3$. If $b < b_2-b_3$, then $c \geqslant c_2+c_3$ by Lemma 3.3.4.

If $c = c_2 + c_3$, then $(a - (a_3 - a_2))\ell = (b_2 - b_3 - b)m$, so $(m/\gcd(\ell, m)) \mid a - (a_3 - a_2)$ and $a > m/\gcd(\ell, m)$. If $a < a_3 - a_2$, then $b > b_2 - b_3$ by Lemma 3.3.4 and hence $c > c_2 + c_3$ by the minimality of a_3. Therefore $h_1 = h_1'$, $a_1 = a_1'$ and $b_1 = b_1'$. □

Notation 3.4.5. We let $B_0, H, \vec{x}^{\langle a,b \rangle}, \mathcal{B}_0$ be as in Notation 3.3.5 and 3.3.8.

Theorem 3.4.6. *Let $R = k[x^n, x^{n-\ell}y^\ell, x^{n-m}y^m, y^n]$ and $d = \gcd(\ell, m, n)$. Then:*

(i) $n = d\begin{vmatrix} a_3 & -b_3 \\ -a_2 & b_2 \end{vmatrix} = d\begin{vmatrix} a_3 & -b_3 \\ a_1 & b_1 \end{vmatrix} = d\begin{vmatrix} a_1 & b_1 \\ -a_2 & b_2 \end{vmatrix}$

(ii) $m = d\begin{vmatrix} a_3 & c_3 \\ -a_2 & c_2 \end{vmatrix} = d\begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} = d\begin{vmatrix} a_1 & c_1 \\ -a_2 & c_2 \end{vmatrix}$

(iii) $\ell = d\begin{vmatrix} c_3 & -b_3 \\ c_2 & b_2 \end{vmatrix} = d\begin{vmatrix} c_3 & -b_3 \\ c_1 & b_1 \end{vmatrix} = d\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$

(iv) $\dim_k R/(x^n, y^n) \geqslant n/d = |\mathcal{B}_0|$

(v) *The ring R is Cohen-Macaulay iff $\mathcal{B}_0$ is a basis of $R/(x^d, y^n)$ over k iff $b_2 \geqslant a_2 + c_2$.*

Proof. We may identify H with the subgroup of $\mathbb{Z}/n\mathbb{Z}$ generated by ℓ and m, so $|H| = n/d$. Therefore (iv) and (i) follow from (i) of Theorem 3.3.12, and (ii) and (iii) follow from Cramer's rule. Using (iii) of Theorem 3.3.12, we have $g_2 \geqslant 0$ iff $-a_2(n - \ell) + b_2(n - m) \geqslant 0$ iff $(b_2 - a_2 - c_2)n \geqslant 0$ iff $b_2 \geqslant a_2 + c_2$, so (v) follows from Lemma 3.4.3(i). □

Corollary 3.4.7. *Let $\ell = 1$ and $n = qm + r$ as in the Euclidean algorithm. If $r = 0$, then R is Cohen-Macaulay. If $r \neq 0$, then R is Cohen-Macaulay if and only if $q + r \geqslant m$.*

Proof. If $r = 0$, then $a_2 = 0$, $b_2 = q$ and $c_2 = 1$, so $b_2 = q \geqslant 1 = a_2 + c_2$. If $r \neq 0$, then $a_2 = m - r$, $b_2 = q + 1$ and $c_2 = 1$, so R is Cohen-Macaulay iff $q + 1 \geqslant m - r + 1$ iff $q + r \geqslant m$. □

Corollary 3.4.8. *If $\gcd(\ell, m) = 1$ and $\ell + m = n$, then R is Cohen-Macaulay if and only if $m = \ell + 1$.*

Proof. Since $\gcd(\ell, m) = 1$ and $\ell + m = n$ we have $\gcd(m, n) = 1$. From $b_2 m - a_2(n-m) = c_2 n$ we get $(b_2 + a_2)m = (c_2 + a_2)n$. By the minimality of b_2 we get $b_2 + a_2 = n$ and $c_2 + a_2 = m$, so $c_2 = 1$, $a_2 = m - 1$ and $b_2 = n - (m-1)$. Then $b_2 \geqslant a_2 + c_2$ iff $n - (m-1) \geqslant m - 1 + 1$ iff $n \geqslant 2m - 1$. But $n = m + \ell \leqslant m + m - 1 = 2m - 1$, so R is Cohen-Macaulay iff $n = 2m - 1$ iff $m = \ell + 1$. □

Remark 3.4.9. We therefore recover Macaulay's result that $k[x^4, x^3y, xy^3, x^4]$ is not Cohen-Macaulay.

Theorem 3.4.10. *We can use the following algorithm to obtain a basis of $R/(x^n, y^n)$ over k.*

1 Let $B = B_0$.

2 Let base $= a_1$, $a^* = a_2$, $b^* = b_2$ and $c^* = c_2$.

3 While $b^* < a^* + c^*$, do the following steps.

4 If $a^* \geqslant a_1$, then:

Replace B by $B \cup \{(0, b^*) + (u, v) \mid u < base$ and $v < b_1\}$.

Replace a^* by $a^* - a_1$, b^* by $b^* + b_1$ and c^* by $c^* + c_1$.

5 If $a^* \leqslant a_1 - base$, then:

Replace B by $B \cup \{(0, b^*) + (u, v) \mid u < base$ and $v < b_2\}$.

Replace a^* by $a^* + a_2$, b^* by $b^* + b_2$ and c^* by $c^* + c_2$.

6 If $a_1 - base < a^* < a_1$, then:

Replace B by $B \cup \{(0, b^*) + (u, v) \mid (u < base$ and $v < b_1)$ or $(u < a_1 - a^*$ and $v < b_2)\}$.

Replace a^* by $a^* + a_2$, b by $b^* + b_2$, c^* by $c^* + c_2$ and base by $a_1 - a^*$.

After the algorithm stops, the set of monomials $\mathcal{B} = \{\vec{x}^{\langle a,b \rangle} \mid (a, b) \in B\}$ forms a basis of $R/(x^n, y^n)$ over k. □

Example 3.4.11. Let $R = k[x^{23}, x^{21}y^2, x^5y^{18}, y^{23}]$. We will use Theorem 3.4.10 to calculate the size of the monomial k-basis $\mathcal{B}$ of $R/(x^{23}, y^{23})$ and find the elements of $\mathcal{B}$.

Step	$\lvert B\rvert$	*base*	Equation		Remark
			$2 \times 18 = -5 \times 2 + 2 \times 23$	(3.4.1)	
1	23	5	$3 \times 18 = 4 \times 2 + 2 \times 23$	(3.4.2)	
6	34	1	$6 \times 18 = 8 \times 2 + 4 \times 23$		Add equation (3.4.2) (to itself).
4	36	1	$8 \times 18 = 3 \times 2 + 6 \times 23$		Add equation (3.4.1).
5	39	1	$11 \times 18 = 7 \times 2 + 8 \times 23$		Add equation (3.4.2).
4	$\boxed{41}$	1	$13 \times 18 = 2 \times 2 + 10 \times 23$		Add equation (3.4.1) and stop.

We display the second coordinates of $\langle B\rangle = \log(\mathcal{B})$, i.e. the y-degrees of elements of $\mathcal{B}$, as follows.

0	2	4	6	8	10	12	14	16
18	20	22	24	26	28	30	32	34
36	38	40	42	44				
54	56	58	60	62				
72	74	76	78	80				
90								
108								
126								
144								
162								
180								
198								
216								

Question 3.4.12. Can we find a “sharp” bound on $\dim_k R/(x^n, y^n)$ as in Corollary 3.3.18?

Chapter 4

Finite F-type and F-abundant Modules

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4.1 Introduction

Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring of dimension d and prime characteristic $p > 0$. Let $\alpha(R) = \log_p[k : k^p]$. All modules are in $\mathrm{mod}(R)$, the category of finitely generated R-modules.

In this Chapter we prove some connections between two types of objects defined using the Frobenius endomorphism $R \to R$ with $r \mapsto r^p$.

These objects generalize some well-studied concepts. Let us start with the definitions. Fix an R-module M. For $e \in \mathbb{Z}_{\geqslant 0}$, let ${}^eM$ denote the abelian group M viewed as an R-module via the eth iteration of the Frobenius map. We let $F^e\colon \mathrm{mod}(R) \to \mathrm{mod}(R)$ denote the eth Peskine-Szpiro functor given by

$$F^e(M) := M \otimes_R {}^eR.$$

Given $S \subseteq \operatorname{mod}(R)$, we use $\operatorname{add}_R(S)$ to denote the additive subcategory of $\operatorname{mod}(R)$ generated by S.

Definition 4.1.1. (1) Let M be an R-module such that $\operatorname{Supp}(M) = \operatorname{Spec} R$ and is locally free in codimension 1. We let $M(e) = F_R^e(M)^{**}$, the reflexive hull of $F_R^e(M)$, viewed as an R-module by identifying ${}^eR$ with R. We say that M is of finite F-type if $\{M(e)\}_{e\geqslant 0} \subseteq \operatorname{add}_R(X)$ for some R-module X (see Lemma 4.4.3). We let $\mathcal{FT}(R)$ denote the category of R-modules of finite F-type.

(2) Let N, L be R-modules. Let b_e be maximum such that ${}^eN = L^{\oplus b_e} \oplus N_e$ for some N_e. We say that (N, L) is an abundant pair if $\liminf_{e\to\infty} p^{e\alpha(R)}/b_e = 0$.

(3) Let L be an R-module. We say that L is an F-abundant module if (N, L) is an abundant pair for some N.

Examples of modules of finite F-type include torsion elements of the divisor class group of a normal domain R (without any assumption about the order of the element), finite integral extensions that are étale in codimension one (see section 4.4), or F-periodic vector bundles on the punctured spectrum of R (and of the corresponding projective variety X when R is a local cone of some embedding of X; see section 4.6). For F-abundant modules, R has positive splitting dimension if and only if (R, R) is an abundant pair (see section 4.5). A good source of examples in both cases are the rings of invariants of a finite group, see Example 4.5.3.

Our main technical results (collected in section 4.3) say roughly that under various extra conditions, if M is of finite F-type and N is F-abundant, then $\operatorname{Hom}_R(M, N)$ is maximal Cohen-Macaulay. We shall give plenty of examples to show that the technical conditions are necessary. Our approach yields a strengthening of some well-known results as well as many new ones:

(1) An extension of results by Patakfalvi-Schwede ([32, Theorem 3.1]) and Watanabe ([41, Corollary 2.9]) on depth of divisor classes associated to F-regular singularities. See Theorem 4.6.3.

(2) Under certain conditions, a strong generalization of Yao's result ([42, Lemma 2.2]) on Cohen-Macaulayness of F-contributors. See Theorems 4.3.8 and 4.3.10.

(3) A complete classification of the categories of finite F-type R-modules and abundant F-modules in many cases of interest, such as complete intersections and invariant subrings (Corollary 4.4.13, Theorem 4.4.14 and 4.5).

4.2 Notation and preliminary results

Definition 4.2.1. ([5, Definition 2.4] and [7, Definition 4.5]) Let $a_e = a_e(R)$ be maximum such that ${}^eR = R^{\oplus a_e} \oplus R_e$ for some R_e. The largest integer k such that

$$\lim_{e\to\infty} \frac{a_e}{p^{e(k+\alpha(R))}} > 0$$

is called the F-splitting dimension of R, and is denoted by $\operatorname{sdim} R$.

Notation 4.2.2. (a) We use (S_k) to denote Serre's criteria for $k \geqslant 0$.

(b) We use $\mathrm{H}^i_{\mathfrak{m}}(M)$ to denote the ith local cohomology module of an R-module M supported on $\mathfrak{m}$.

(c) For an R-module M, we let M^* denote $\operatorname{Hom}_R(M,R)$.

(d) Suppose that R has a canonical module ω_R. Then for an R-module M, we let $M^\vee$ denote $\operatorname{Hom}_R(M,\omega_R)$.

(e) For $S \subseteq R$, we let ${}^eS$ to denote S viewed as a subset of ${}^eR$. Then for $P \in \operatorname{Spec} R$, ${}^eR_{{}^eP}$ is an R_P-module, and ${}^eR_{{}^eP} = {}^e(R_P)$.

(f) For an ideal $I \subseteq R$, an integer $e \geqslant 0$ and $q = p^e$, we let $I^{[q]} = \{x^q \mid x \in I\}$, the qth Frobenius power of I.

(g) We use CM(R) to denote the subcategory of mod(R) consisting of all C-M R-modules and Ref(R) the subcategory of all reflexive modules.

Remark 4.2.3. We would like to remind the reader of another characterization of $\operatorname{sdim} R$. We consider the splitting prime $\mathcal{P}(R)$ of R, as defined in [5, Definition 3.2]. By [5, Theorem 3.3 and Corollary 3.4], $\mathcal{P}(R)$ is a prime ideal if $\operatorname{sdim}(R) \neq -\infty$ or the unit ideal otherwise. Corollary 4.3 of [7] shows that $\operatorname{sdim} R = \dim(R/\mathcal{P}(R))$ when $\operatorname{sdim}(R) \neq -\infty$.

Proposition 4.2.4. *Assume the following for R:*

(1) R is equidimensional;

(2) R_P is C-M for all $P \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$; and

(3) $\operatorname{sdim} R > 0$.

Then R is C-M.

Proof. Since R is F-finite, it is a homomorphic image of a Gorenstein ring by [18, Remark before Lemma A.2]. By (1) and (2), for $0 \leqslant i < d$ we have that $\mathrm{H}^i_{\mathfrak{m}}(R)$ has finite length by [4, Proposition 21.24]; see also [3, Theorem 9.5.2 (Grothendieck's Finiteness Theorem)]. So

$$\begin{aligned} p^{e\alpha(R)}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) &= \lambda_R({}^e\mathrm{H}^i_{\mathfrak{m}}) \\ &= \lambda_R(\mathrm{H}^i_{\mathfrak{m}}({}^eR)) \\ &= \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R^{\oplus a_e} \oplus M_e)) \\ &= a^e\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) + \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(M_e)) \\ \frac{1}{p^e}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) &= \frac{a_e}{p^{e(1+\alpha(R))}}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) + \frac{1}{p^{e(1+\alpha(R))}}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(M_e)) \\ 0 = \lim_{e\to\infty}\frac{1}{p^e}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) &\geqslant \lim_{e\to\infty}\frac{a_e}{p^{e(1+\alpha(R))}}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) \end{aligned}$$

Since $\operatorname{sdim} R \geqslant 1$, we have $\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) = 0$ and hence $\mathrm{H}^i_{\mathfrak{m}}(R) = 0$ for $0 \leqslant i < d$. Then by [4, Theorem 10.36], R is C-M. $\square$

Corollary 4.2.5. *If R is F-split, is C-M on* $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ *(e.g. R is an isolated singularity) but not C-M, then* $\operatorname{sdim} R = 0$. □

Example 4.2.6. Let X be an ordinary Abelian variety of dimension at least two and R be the coordinate ring of an embedding of X with respect to some polarization. It is well-known, as in [28, 37], that R is F-split but not C-M. So by Corollary 4.2.5, $\operatorname{sdim} R = 0$.

Lemma 4.2.7. *Let M, N be R-modules such that ${}^eM = N^{b_e} \oplus M_e$ and $\liminf_{e\to\infty} \frac{b_e}{p^{e(k+\alpha(R))}} > 0$. Then $\operatorname{depth} N \geqslant k$. In particular, if $k = \dim(M)$, then N is C-M.*

Proof. We use the same proof as in [42, Lemma 2.2]. For $0 \leqslant i < k$, we have

$$0 = \lim_{e\to\infty} \frac{\lambda_R(\mathrm{H}^i_R(\underline{x}^{p^e}, M))}{p^{ek}} = \lim_{e\to\infty} \frac{\lambda_R(\mathrm{H}^i_R(\underline{x}, {}^eM))}{p^{e(k+\alpha(R))}} \geqslant \liminf_{e\to\infty} \frac{b_e}{p^{e(k+\alpha(R))}} \lambda_R(\mathrm{H}^i_R(\underline{x}, N))$$

as in Proposition 4.2.4. So $\mathrm{H}^i_R(\underline{x}, N) = 0$ for all $0 \leqslant i < k$, and hence $\operatorname{depth} N \geqslant k$. □

Remark 4.2.8. It is already known that $\operatorname{sdim} R = d \Rightarrow R$ is strongly F-regular $\Rightarrow R$ is C-M.

Remark 4.2.9. Let ${}^eM = R^{b_e} \oplus M_e$ with b_e the largest possible. The largest integer k such that

$$\liminf_{e\to\infty} \frac{b_e}{p^{e(k+\alpha(R))}} > 0$$

was defined in [5, Definition 5.4] to be $\operatorname{sdim}(M)$, the s-dimension of M.

Definition 4.2.10. We will let (sdim_n) to denote the statement: $\operatorname{sdim} R_P > 0$ for all $P \in \operatorname{Spec} R$ such that $\operatorname{ht} P \geqslant n$.

Lemma 4.2.11. *If R satisfies (sdim_n), then $\operatorname{ht} \mathcal{P}(R) < n$.*

Proof. Suppose that R satsifies (sdim_n). By [5, Proposition 3.6], we have $\mathcal{P}(R_{\mathcal{P}(R)}) = \mathcal{P}(R)R_{\mathcal{P}(R)}$. Since R is F-finite and reduced, so is $R_{\mathcal{P}(R)}$. By [5, Corollary 3.4], since $\mathcal{P}(R_{\mathcal{P}(R)})$ is the maximal ideal of $R_{\mathcal{P}(R)}$, we have $\operatorname{sdim} R_{\mathcal{P}(R)} = 0$, so $\operatorname{ht} \mathcal{P}(R) < n$. □

4.3 Main technical results

Lemma 4.3.1. *Let $f\colon M \to N$ be a homomorphism of R-modules, where M is (S_2) and N is (S_1). Suppose that f is an isomorphism in codimension 1. Then f is an isomorphism.*

Remark 4.3.2. In many of the results below, we can replace the assumption of a module being locally free in codimension 1 by R being quasinormal.

Corollary 4.3.3. *Let L, N be R-modules such that L is free in codimension 1 and N is (S_2). Then*

$$\mathrm{Hom}_R(L, N) \cong \mathrm{Hom}_R(L^{**}, N)$$

Proof. Let $f\colon L \to L^{**}$ be the canonical map and $\overline{f}\colon \mathrm{Hom}_R(L^{**}, N) \to \mathrm{Hom}_R(L, N)$ the induced map. Then $\overline{f}$ is an isomorphism in codimension 1. By Lemma 4.3.1, it suffices to show that $\mathrm{Hom}_R(M, N)$ is (S_2) for every R-module M. Given M, let $F_1 \to F_0 \to M \to 0$ be a finite presentation of M. Applying $\mathrm{Hom}_R(-, N)$ gives

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$$

Since N is (S_2), so are $\mathrm{Hom}_R(F_i, N)$ for $i = 1, 2$ and $\mathrm{Hom}_R(M, N)$, as required. □

Corollary 4.3.4. *Suppose that R is C-M with a canonical module ω_R. Suppose that M is (S_2) and $M^\vee$ is MCM. Then M is MCM.*

Proof. Consider the natural map $f\colon M \to M^{\vee\vee}$. In codimension 1, M is MCM since it is (S_2), so f is an isomorphism. Since $M^\vee$ is MCM, so is $M^{\vee\vee}$. Hence f is an isomorphism by Lemma 4.3.1. □

Remark 4.3.5. Let M be an R-module, $f\colon R \to S$ be a ring homomorphism, U be a multiplicative subset of S and $T = f^{-1}(U)$. Then we have $(M \otimes_R S)_U = M \otimes_R S \otimes_S S_U = M \otimes_R S_U = M \otimes_R R_T \otimes_{R_T} S_U = M_T \otimes_{R_T} S_U$.

Lemma 4.3.6. *Let $R \xrightarrow{f_1} R_1 \xrightarrow{f_2} R_2$ be a sequence of ring homomorphisms where R_2 is (S_2). Let $f = f_2 \circ f_1$. Let M be an R-module. Suppose that M_P is free for every $P \in \operatorname{Spec} R$ of height 1 and for every $P = f^{-1}(Q)$ such that $Q \in \operatorname{Spec} R_2$ of height 1. Let $M_i = (M \otimes_R R_i)^{**}$ (over R_i) for $i = 1, 2$. Then $(M_1 \otimes_{R_1} R_2)^{**} \cong M_2$.*

Proof. Let $N = M \otimes_R R_1$. The natural map $N \to M_1 = N^{**}$ gives rise to a map $g \colon M \otimes_R R_2 = M \otimes_R R_1 \otimes_{R_1} R_2 = N \otimes_{R_1} R_2 \to N^{**} \otimes_{R_1} R_2 = M_1 \otimes_{R_1} R_2$. Let $Q \in \operatorname{Spec} R_2$, $P_1 = (f_2)^{-1}(Q)$ and $P = f^{-1}(Q)$. Then $(M \otimes_R R_2)_Q = M_P \otimes_{R_P} (R_2)_Q$ and $(M_1 \otimes_{R_1} R_2)_Q = (M \otimes_R R_1)^{**}_{P_1} \otimes_{(R_1)_{P_1}} (R_2)_Q = (M_P \otimes_{R_P} (R_1)_{P_1})^{**} \otimes_{(R_1)_{P_1}} (R_2)_Q$, so g is an isomorphism in codimension 1. Applying ** gives the map $M_2 = (M \otimes_R R_2)^{**} \to (M_1 \otimes_{R_1} R_2)^{**}$, which is an isomorphism by Lemma 4.3.1 since R_2 is (S_2). □

Corollary 4.3.7. *("Index shifting") Let R be (S_2) and M be as in Definition 4.1.1. Let e, f be nonnegative integers. Then $[M(e)](f) \cong M(e + f)$.*

Proof. Let $R_1 = {}^eR$ and $R_2 = {}^{e+f}R$ as in Lemma 4.3.6. Then we have $M(e) = M_1$, and so $[M(e)](f) = (M_1 \otimes_{R_1} R_2)^{**} \cong M_2 = M(e + f)$. □

Theorem 4.3.8. *Suppose that R is (S_2) and equidimensional. Let M, N be R-modules such that $M \in \mathcal{FT}(R)$ and N is (S_2). Assume that (N, L) is an abundant pair and that $N_P \in \operatorname{add} L_P$ for all $P \in \operatorname{Spec} R$ such that $3 \leqslant \operatorname{ht}(P) < d$. Assume further that for every $P \in \operatorname{Spec} R$ such that $3 \leqslant \operatorname{ht}(P) < d$ and $e \geqslant 0$, $(\operatorname{Hom}_R(M(e), L))_P$ is MCM. Then $\operatorname{Hom}_R(M(e), L)$ is MCM for all $e \geqslant 0$.*

Proof. If $d \leqslant 2$, then $\operatorname{Hom}_R(M(e), L)$ is MCM since L is (S_2). So we may assume that $d \geqslant 3$. By assumption, for $P \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, we have $N_P \in \operatorname{add} L_P$ for $\operatorname{ht}(P) \geqslant 3$, so $(\operatorname{Hom}_R(M(e), N))_P$ is MCM for $e \geqslant 0$. So for $0 \leqslant i < d$ and $e \geqslant 0$, $\mathrm{H}^i_{\mathfrak{m}}(\operatorname{Hom}_R(M(e), N))$ has finite length — see Proposition 4.2.4. By Corollary 4.3.3, we have

$$
\begin{aligned}
\operatorname{Hom}_{{}^eR}({}_{{}^eR}[F^e_R(M)^{**}], {}_{{}^eR}N) &= \operatorname{Hom}_{{}^eR}({}_{{}^eR}[(M \otimes_R {}^eR)^{**}], {}_{{}^eR}N) \\
&= \operatorname{Hom}_{{}^eR}({}_{{}^eR}[M \otimes_R {}^eR], {}_{{}^eR}N)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Hom}_R(M, \mathrm{Hom}_{{}^eR}({}^eR, {}_{{}^eR}N)) \\
&= \mathrm{Hom}_R(M, {}_{{}^eR}N) \\
&= \mathrm{Hom}_R(M, {}^eN) \\
&= \mathrm{Hom}_R(M, L^{\oplus b_e} \oplus N_e) \\
&= \mathrm{Hom}_R(M, L)^{\oplus b_e} \oplus \mathrm{Hom}_R(M, N_e)
\end{aligned}
$$

Apply $\mathrm{H}^i_{\mathfrak{m}}$ for $0 \leqslant i < d$ to get

$$
\begin{aligned}
b_e \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(M, L))) &\leqslant \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_{{}^eR}({}_{{}^eR}[F^e_R(M)^{**}], {}_{{}^eR}N))) \\
&= \lambda_R(\mathrm{H}^i_{\mathfrak{m}({}^eR)}(\mathrm{Hom}_{{}^eR}({}_{{}^eR}[F^e_R(M)^{**}], {}_{{}^eR}N))) \quad \text{(base change)} \\
&= \lambda_R(\mathrm{H}^i_{{}^e\mathfrak{m}}(\mathrm{Hom}_{{}^eR}({}_{{}^eR}[F^e_R(M)^{**}], {}_{{}^eR}N))) \\
&= \lambda_R({}^e\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(M(e), {}_RN))) \\
&= p^{e\alpha(R)} \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(M(e), {}_RN))) \\
\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(M, L))) &\leqslant \frac{p^{e\alpha(R)}}{b_e} \max\{\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(M(e), {}_RN)))\}
\end{aligned}
$$

Taking $\liminf$ shows that $\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(M, L)) = 0$ for $0 \leqslant i < d$, so $\mathrm{Hom}_R(M, L)$ is MCM. By Corollary 4.3.7, we have $[M(e)](f) \cong M(e + f)$. So we may replace M by $M(e)$ to conclude that $\mathrm{Hom}_R(M(e), L)$ is MCM. □

Remark 4.3.9. By Lemma 4.4.3, we only need to check that $(\mathrm{Hom}_R(M_i, L))_P$ is MCM for $3 \leqslant \mathrm{ht}(P) < d$ for the finitely many indecomposable modules M_i that appear among $\{M(e)\}_{e \geqslant 0}$.

Theorem 4.3.10. *Let R be as in Theorem 4.3.8. Let M, N be R-modules such that $M \in \mathcal{FT}(R)$ and N is (S_2). Assume that for every $P \in \operatorname{Spec} R$ such that $\mathrm{ht}(P) \geqslant 3$, (N_P, L_P) is an abundant pair. Assume further that for every $P \in \operatorname{Spec} R$ such that $3 \leqslant \mathrm{ht}(P) < d$, we have $N_P \in \operatorname{add} L_P$. Then $\mathrm{Hom}_R(M(e), L)$ is MCM for all $e \geqslant 0$.*

Proof. First, for every $P \in \operatorname{Spec} R$, we have the following.

$$\begin{aligned}
{}_{{}^e(R_P)}[F^e_{R_P}(M_P)^{**}] &= \operatorname{Hom}_{{}^e(R_P)}\left(\operatorname{Hom}_{{}^e(R_P)}\left(M_P \otimes_{{}^e(R_P)} {}^e(R_P), {}^e(R_P)\right), {}^e(R_P)\right) \\
&= \operatorname{Hom}_{{}^e(R_P)}\left(\operatorname{Hom}_{{}^e(R_P)}\left(M \otimes_R R_P \otimes_{{}^e(R_P)} {}^e(R_P), {}^e(R_P)\right), {}^e(R_P)\right) \\
&= \operatorname{Hom}_{{}^e(R_P)}\left(\operatorname{Hom}_{{}^e(R_P)}\left(M \otimes_R {}^e(R_P), {}^e(R_P)\right), {}^e(R_P)\right) \\
&= \operatorname{Hom}_{{}^e(R_P)}\left(\operatorname{Hom}_{{}^eR_{^eP}}\left(M \otimes_R {}^eR \otimes_{{}^eR_{^eP}} {}^eR_{^eP}, {}^eR_{^eP}\right), {}^e(R_P)\right) \\
&= \operatorname{Hom}_{{}^e(R_P)}\left(\operatorname{Hom}_{{}^eR}\left(M \otimes_R {}^eR, {}^eR\right)_{^eP}, {}^e(R_P)\right) \\
&= \left(\operatorname{Hom}_{{}^eR}\left(\operatorname{Hom}_{{}^eR}\left(M \otimes_R {}^eR, {}^eR\right), {}^eR\right)\right)_{^eP} \\
&= (F^e_R(M)^{**})_P
\end{aligned}$$

So we can prove by induction on d that $\operatorname{Hom}_R(M(e), L)$ is MCM for all $e \geqslant 0$. We may assume that $d \geqslant 3$. Let $P \in \operatorname{Spec} R$ such that $3 \leqslant \operatorname{ht}(P) < d$. By induction, $(\operatorname{Hom}_R(M(e), L))_P$ is MCM. So by Theorem 4.3.8, $\operatorname{Hom}_R(M(e), L)$ is MCM. □

Corollary 4.3.11. *Suppose that R is C-M and $M \in \mathcal{FT}(R)$ is (S_2). Assume that:*

(a) *either* $\operatorname{sdim} R > 0$ *and $M(e)_P$ is MCM for every $P \in \operatorname{Spec} R$ such that $3 \leqslant \operatorname{ht}(P) < d$ and $e \geqslant 0$; or*

(b) *R is* (sdim_3).

Then M is MCM.

Proof. Since R is a homomorphic image of a Gorenstein ring, it has a canonical module ω_R by [3, 12.1.3(iii)]. Since R is C-M, it is equidimensional, and ω_R is reflexive and hence (S_2). We will show that $\operatorname{Hom}_R(M, \omega_R) = M^\vee$ is MCM, so that by Corollary 4.3.4, M is MCM. We may assume that $d > 2$. Since $\operatorname{sdim} R > 0$, (R, R) is an abundant pair by Example 4.5.1. Then (ω_R, ω_R) is also an abundant pair, since $\omega_{{}^eR} \cong \operatorname{Hom}_R({}^eR, \omega_R) = \operatorname{Hom}_R(R^{\oplus a_e} \oplus R_e, \omega_R) = \omega_R^{\oplus a_e} \oplus \operatorname{Hom}_R(R_e, \omega_R)$. If (a) holds, then $(\operatorname{Hom}_R(M(e), \omega_R))_P$ is MCM for every $P \in \operatorname{Spec} R$ such that $3 \leqslant \operatorname{ht}(P) < d$ and $e \geqslant 0$. By Theorem 4.3.8, $\operatorname{Hom}_R(M, \omega_R) = M^\vee$ is MCM. If (b) holds, then $(\omega_{R_P}, \omega_{R_P})$ is an abundant pair for $P \in \operatorname{Spec} R$ such that $\operatorname{ht}(P) \geqslant 3$, so

$\mathrm{Hom}_R(M, \omega_R)$ is MCM by Theorem 4.3.10. ☐

Corollary 4.3.12. *Suppose that R is strongly F-regular and $M \in \mathcal{FT}(R)$ is (S_2). Then M is MCM. In particular, if I is a reflexive ideal such that $[I]$ is torsion in* $\mathrm{Cl}(R)$*, then I is MCM.*

Proof. Since R is strongly F-regular, it is (sdim_3). Corollary 4.3.11 shows that M is MCM. By Example 4.4.1, we have $I \in \mathcal{FT}(R)$. ☐

Remark 4.3.13. Corollary 4.3.12 generalizes [32, Corollary 3.3].

4.4 The category of finite F-type modules

In this section we study the category of finite F-type in more detail. We completely classify this category when R is a quotient singularity with a finite group whose order is coprime to p (Corollary 4.4.12), or when R is a complete intersection that is regular in codimension 2 (Theorem 4.4.14).

Example 4.4.1. The following are R-modules M of finite F-type.

(a) ([41, paragraph 2.3]) R is a normal domain and $M = I$, where I is a divisorial ideal. Then $M(e) \cong I^{(e)}$, so $M \in \mathcal{FT}(R)$ iff $[I]$ is torsion in $\mathrm{Cl}(R)$.

(a') M is a free R-module.

(b) ([41, Theorem 2.7]) Let $R \to S$ be a finite homomorphism of normal domains which is étale in codimension 1 and $M = S$. From the natural map $S \otimes_R {}^eR \to {}^eS$ we get $(S \otimes_R {}^eR)^{**} \cong S^{**}$, i.e. $M(e) \cong S$.

(c) See also Lemma 4.6.2.

Lemma 4.4.2. *$\mathcal{FT}(R)$ is closed under direct sums and direct summands.*

Proof. Obvious. ☐

Lemma 4.4.3. *Let $S \subseteq \operatorname{mod} R$. Then $\operatorname{add}_R(S)$ has finitely many indecomposable objects iff $S \subseteq \operatorname{add}_R(X)$ for some R-module X. Hence for an R-module M, $M \in \mathcal{FT}(R)$ if and only if only finitely many indecomposable direct summands appear among $\{M(e)\}_{e \geqslant 0}$.*

Proof. For the "only if" part, take X to be the direct sum of the indecomposable objects of $\operatorname{add}_R(S)$. For the "if" part, we consider the endomorphism ring $E = \operatorname{End}_R(X)$. Consider the category $\operatorname{Proj} E \subseteq \operatorname{mod}(E)$ of left projective modules over E. Then $F\colon \operatorname{add}_R(X) \to \operatorname{Proj} E$ given by $F(L) = \operatorname{Hom}_R(L, X)$ for $L \in \operatorname{add}_R(X)$ is an equivalence of categories. Since E is finitely generated over the local ring R, it is semilocal. By [12, Theorem 9], $\operatorname{Proj} E$ has only finitely many isomorphism classes of indecomposable objects, and hence so does $\operatorname{add}_R(X)$. □

Corollary 4.4.4. *Let R, M be as in Corollary 4.3.7. Then $M \in \mathcal{FT}(R)$ iff there are $e \geqslant 0$ and $f > 0$ such that $M(e) \cong M(e+f)$.*

Proof. "If": We note that $M(e+g) \cong [M(e)](g) \cong [M(e+f)](g) \cong M(e+f+g)$ by Corollary 4.3.7, so there are only finitely many isomorphism classes of $\{M(e)\}_{e \geqslant 0}$.
"Only if": Let X be an R-module such that $\{M(e)\}_{e \geqslant 0} \subseteq \operatorname{add}_R(X)$. Let S be the set of indecomposable direct summands of $\{M(e)\}_{e \geqslant 0}$. Then $S \subseteq \operatorname{add}_R(X)$, and S is finite by Lemma 4.4.3. By index shifting, it suffices to prove that $N(e) \cong N(e+f)$ for some $e \geqslant 0$ and $f > 0$ for all $N \in S$. By assumption, R is local, so we can prove the claim by induction on the minimum number of generators of N. Suppose first that $N(e)$ is indecomposable for all e. Then the claim holds since S is finite. So suppose that $N(e_0)$ is not indecomposable for some e_0 and has indecomposable direct summands $N_i \in S$. By induction, the claim holds for the N_i, and hence for N by index shifting. □

Proposition 4.4.5. *Let R be (S_2). Let $\mathcal{FT} = \mathcal{FT}(R)$. Then:*

(a) if $M \in \mathcal{FT}$, then $M^{**} \in \mathcal{FT}$.

(b) if $M, N \in \mathcal{FT}$, then $M \otimes N \in \mathcal{FT}$.

Proof. (a) By Corollary 4.3.7, we have $[M(0)](e) \cong M(e)$, that is, $(M^{**})(e) \cong M(e)$.

(b) For $P \in \operatorname{Spec} R$, we have $(M \otimes_R N)_P = M_P \otimes_{R_P} N_P$, so $M \otimes N$ is locally free in codimension 1. Next, we have

$$\begin{aligned} M \otimes_R N \otimes_R {}^eR &= M \otimes_R N \otimes_R {}^eR \otimes_{{}^eR} {}^eR \\ &= M \otimes_R {}^eR \otimes_{{}^eR} {}^eR \otimes_R N \\ &= (M \otimes_R {}^eR) \otimes_{{}^eR} (N \otimes_R {}^eR) \\ (M \otimes_R N \otimes_R {}^eR)^{**} &= [(M \otimes_R {}^eR) \otimes_{{}^eR} (N \otimes_R {}^eR)]^{**} \end{aligned}$$

The natural maps $M \otimes_R {}^eR \to (M \otimes_R {}^eR)^{**}$ and $N \otimes_R {}^eR \to (N \otimes_R {}^eR)^{**}$ give rise to the map $f \colon [(M \otimes_R {}^eR) \otimes_{{}^eR} (N \otimes_R {}^eR)]^{**} \to [(M \otimes_R {}^eR)^{**} \otimes_{{}^eR} (N \otimes_R {}^eR)^{**}]^{**}$. Since R is (S_2), Lemma 4.3.1 shows that f is an isomorphism. So we have an isomorphism

$$(M \otimes_R N)(e) \cong (M(e) \otimes_R N(e))^{**}$$

By Lemma 4.4.3, only finitely many indecomposable direct summands $\{K_i\}_{i=1}^m$ appear in $\{M(e)\}_{e \geqslant 0}$ and $\{L_j\}_{j=1}^n$ in $\{N(e)\}_{e \geqslant 0}$. Let $X = \sum_{i,j} (K_i \otimes_R L_j)^{**}$. Then $\{(M \otimes_R N)(e)\}_{e \geqslant 0} \subseteq \operatorname{add}_R(X)$. □

Question 4.4.6. If R is Gorenstein, is $M^* \in \mathcal{FT}(R)$?

Lemma 4.4.7. *Let $f \colon R \to S$ be a ring homomorphism. Suppose that S is (S_2). Let M be an R-module. Suppose that:*

(a) f is flat; or

(b) M_P is free for every $P = f^{-1}(Q)$ such that $Q \in \operatorname{Spec} S$ and $\operatorname{ht}(Q) = 1$.

If $M \in \mathcal{FT}(R)$, then $M \otimes_R S \in \mathcal{FT}(S)$.

Proof. Let $N = M \otimes_R S$. First, we consider (b) and suppose that $Q \in \operatorname{Spec} S$, $\operatorname{ht}(Q) = 1$ and $M \in \mathcal{FT}(R)$. Let $P = f^{-1}(Q)$. Then $N_Q = M_P \otimes_{R_P} S_Q$, which is free over S_Q by assumption. Next, we consider the modules $F_S^e(N)^{**} = (M \otimes_R S \otimes_S {}^eS)^{**} = (M \otimes_R {}^eS)^{**} = (M \otimes_R {}^eR \otimes_{{}^eR} {}^eS)^{**}$.

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \varphi \downarrow & & \downarrow \varphi \\ {}^eR & \xrightarrow{f} & {}^eS \end{array}$$

Consider the maps $R \xrightarrow{\varphi} {}^eR \xrightarrow{f} {}^eS$, where φ is the Frobenius map, and let $R_1 = {}^eR$, $R_2 = {}^eS$. Then Lemma 4.3.6 shows that $F_S^e(N)^{**} = (M \otimes_R {}^eS)^{**} \cong (F_R^e(M)^{**} \otimes_{{}^eR} {}^eS)^{**}$, so $N(e) \cong (M(e) \otimes_R S)^{**} \subseteq \mathrm{add}_S((X \otimes_R S)^{**})$.

Now suppose that f is flat. Let Q, P and M be as above. Then we have $\mathrm{ht}(Q) = \mathrm{ht}(P) + \dim(S_Q/PS_Q)$, so $\mathrm{ht}(P) \leqslant 1$ and M_P is free, giving (b). □

Lemma 4.4.8. *Suppose that R is regular. Let M be an R-module. Then M^* is reflexive.*

Proof. Consider the canonical map $f\colon M^* \to M^{***}$. In codimension 1, R is a principal ideal domain, so M^* is free, and f is an isomorphism. Since R is (S_2), so are M^* and M^{***}. By Lemma 4.3.1, f is an isomorphism. □

Lemma 4.4.9. *Let $R \to S$ be a flat ring extension. Suppose that M is a reflexive R-module. Then $M \otimes_R S$ is a reflexive S-module.*

Proof. Since $R \to S$ is flat, we have $M \otimes_R S = M^{**} \otimes_R S = (M \otimes_R S)^{**}$. □

Lemma 4.4.10. *Suppose that R is regular. Consider the following statements:*

(a) $M \in \mathcal{FT}(R)$

(b) M^* is free.

(c) M^{**} is free.

Then (a) ⇒ (b) ⇔ (c). If M is free in codimension 1, then (a) ⇔ (b) ⇔ (c).

Proof. (b) ⇒ (c): Obvious.

(c) ⇒ (b): If M^{**} is free, then so is $M^{***} = (M^*)^{**}$. So M^* is free by Lemma 4.4.8.

(a) $\Rightarrow$ (c): Suppose that $M \in \mathcal{FT}(R)$. By Proposition 4.4.5(a) and Lemma 4.3.1, we may replace M by M^{**} and assume that M is reflexive. So we need to show that M is free. Since R is regular, the ring extension $R \to {}^eR$ is flat. By Lemma 4.4.9, we have $M(e) = (M \otimes_R {}^eR)^{**} = M \otimes_R {}^eR$ for all $e \geqslant 0$. By Corollary 4.4.4, we have $M(e') \cong M(e'+e)$ for some $e' \geqslant 0$ and $e > 0$. First, suppose that $e' = 0$, so that $M \cong M(e)$. Consider a minimal free resolution

$$\cdots \to F_1 \xrightarrow{A} F_0 \to M \to 0$$

of M given by the matrix A. Let $I(M) \subseteq \mathfrak{m}$ be the Fitting ideal of M generated by the entries of A and let $q = p^e$. Tensoring with ${}^eR$ gives a free resolution

$$\cdots \to F_1 \xrightarrow{A^{[q]}} F_0 \to M(e) \to 0, \qquad (*)$$

so $I(M(e)) = I(M)^{[q]}$. Since $M \cong M(e)$, we have $I(M) = I(M(e)) = I(M)^{[q]}$, so $I(M) = I(M)^q$. By Nakayama's Lemma, we have $I(M) = 0$. So $A = 0$, and $M^{**} = M$ is free.

In general, the above shows that $M(e)$ is free for some $e \geqslant 0$ by Corollary 4.3.7. Suppose that $e > 0$. Since $(*)$ is a free resolution of the free module $M(e)$, it is a direct sum of a trivial complex and the resolution $0 \to F \to M(e) \to 0$. So the entries in $A^{[q]}$ are either 1 or 0. Since A has entries in $\mathfrak{m}$ and R is a domain, we must have $A = 0$, so again M is free.

(c) $\Rightarrow$ (a): Suppose that M is free in codimension 1. If M^{**} is free, then $M^{**} \in \mathcal{FT}(R)$, so $M \in \mathcal{FT}(R)$ by Proposition 4.4.5(a). □

Theorem 4.4.11. *Let $\varphi\colon R \to S$ be a finite homomorphism of normal domains such that φ is étale in codimension 1 and splits as a map of R-modules. Let N, L be R-modules such that N is (S_2) and that locally, (N, L) is an abundant pair and $N \in \operatorname{add}(L)$. Then $L \in \operatorname{add}_R(\mathrm{CM}(S))$.*

Proof. Since $S \in \mathcal{FT}(R)$, $\operatorname{Hom}_R(S, L)$ is C-M by Theorem 4.3.10, so $\operatorname{Hom}_R(S, L) \in \mathrm{CM}(S)$. Since φ splits, L is a direct summand of $\operatorname{Hom}_R(S, L)$. □

Corollary 4.4.12. *Let $\varphi\colon R \to S$ be a finite homomorphism of normal domains such that φ is étale in codimension 1 and splits as a map of R-modules and that S is regular (for example, if R is a quotient singularity as in example 4.5.3). If $M \in \mathcal{FT}(R)$, then $M^* \in \operatorname{add}_R(S)$. If $S^* = \operatorname{Hom}_R(S,R) \cong S$ (again, for example if R is a quotient singularity as in 4.5.3) then $\mathcal{FT}(R) = \operatorname{add}_R(S)$.*

Proof. Let $N = M \otimes_R S$. If $M \in \mathcal{FT}(R)$, then $N \in \mathcal{FT}(S)$ by Lemma 4.4.7. Then N^* is free by Lemma 4.4.10. We have $N^* = \operatorname{Hom}_S(M \otimes_R S, S) = \operatorname{Hom}_R(M, \operatorname{Hom}_S(S,S)) = \operatorname{Hom}_R(M,S)$. Since f splits, $\operatorname{Hom}_R(M,R)$ is a direct summand of $\operatorname{Hom}_R(M,S)$, so $M^* \in \operatorname{add}_R(S)$. If $S^* \cong S$, then it follows that $\mathcal{FT}(R) \subseteq \operatorname{add}_R(S)$, and $S \in \mathcal{FT}(R)$ by Example 4.4.1. □

Corollary 4.4.13. *Suppose that k is algebraically closed. Let $S = k[[x_1, \ldots, x_d]]$. Let G be a finite subgroup of $GL(d,k)$ that contains no pseudo-reflections such that the order of G is coprime to p. Let $R = S^G$. Then $\mathcal{FT}(R) = \operatorname{add}_R S$.*

Proof. Use Corollary 4.4.12. □

Theorem 4.4.14. *Suppose that R is a complete intersection and M is an R-module that is free in codimension 2. Then $M \in \mathcal{FT}(R)$ if and only if M^{**} is free.*

Proof. The proof of "if" is given by (c) $\Rightarrow$ (a) in Lemma 4.4.10. For the "only if" part, as in the proof of Lemma 4.4.10 (a) $\Rightarrow$ (c), it suffices to assume that M is reflexive and show that M is free. The proof is by induction on d. For $d \geqslant 3$, M is free on $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ by induction. Let $r = \operatorname{depth} M \geqslant 2$. Suppose that $r < d$. Then by [11, Proposition 4.14], we have $\mathrm{H}^r_{\mathfrak{m}}(F^e_R(M)) \cong \mathrm{H}^r_{\mathfrak{m}}(F^e_R(M)^{**}) = {}^e\mathrm{H}^r_{\mathfrak{m}}(M(e))$ as in Theorem 4.3.8, so

$$\lim_{e\to\infty} \frac{\lambda_R(\mathrm{H}^r_{\mathfrak{m}}(F^e_R(M))}{p^{e(d+\alpha(R))}} = \lim_{e\to\infty} \frac{\lambda_R({}^e\mathrm{H}^r_{\mathfrak{m}}(M(e)))}{p^{e(d+\alpha(R))}} = \lim_{e\to\infty} \frac{p^{e\alpha(R)}\lambda_R(\mathrm{H}^r_{\mathfrak{m}}(M(e)))}{p^{e(d+\alpha(R))}} = 0$$

by the assumption that $M \in \mathcal{FT}(R)$. Then [11, Theorem 4.12] gives $\operatorname{pd}_R M < d - r$. By the Auslander-Buchsbaum formula, we have $\operatorname{depth} M > r$, a contradiction. So $r = d$, and again [11, Theorem 4.12] with $k = r - 1$ shows that $\operatorname{pd}_R M = 0$, so M is free. □

Example 4.4.15. Let $R = k[[x, y, z]]/(xy - z^2)$ and M be the ideal (x, z). Then $[M]$ has order 2 in $\mathrm{Cl}(R)$ and so $M \in \mathcal{FT}(R)$ by Example 4.4.1, but M^{**} is not free, so the condition that M is locally free in codimension 2 is necessary in Theorem 4.4.14.

4.5 F-abundant pairs and modules

In this section we give many examples of F-abundant pairs and modules.

Example 4.5.1. (a) If $\operatorname{sdim} R \geqslant 1$, in particular if R is strongly F-regular of dimension $\geqslant 1$, then (R, R) is an abundant pair.

(b) [23, Proposition 2.3] shows that $\alpha(R_P) = \alpha(R) + \dim(R/P)$. Let N, L be as in Definition 4.1.1 and $P \in \operatorname{Spec} R$. If $\liminf_{e\to\infty} p^{e(\alpha(R)+\dim(R/P))}/b_e = 0$, then (N_P, L_P) is an abundant pair.

(c) F-contributors for modules of finite F-representation type, as in [42, Section 2].

Example 4.5.2. ([11, Example 6.1]) Let k be an algebraically closed field of characteristic $p > 2$. Consider the hypersurface $R = k[[x, y, u, v]]/(xy - uv)$. Then every MCM R-module is F-abundant.

Example 4.5.3. Let k be an algebraically closed field of characteristic $p > 0$ and V be a k-vector space of dimension d. Let S be the symmetric algebra of V. Let G be a finite subgroup of $GL(V)$ without pseudo-reflections such that $|G|$ is coprime to p. Let $R = S^G$ be the ring of invariants. Let $V_0, \cdots, V_n$ be a complete set of irreducible representations of G over k. Let $M_i = (S \otimes_k V_i)^G$. It is classical that $\mathrm{add}_R(S) = \{M_0, M_1, \ldots, M_n\}$. Also, by the main results of [16]:

1. S, and hence all of $M_0, M_1, \ldots, M_n$, are modules of finite F-type. Note that $\operatorname{rank} M_i = \dim_k V_i$, so we have many examples of modules of finite F-type which are not ideals.

2. (M_i, M_j) is an F-abundant pair for all $0 \leqslant i, j \leqslant n$.

Corollary 4.5.4. *Let R, N be as in Theorem 4.3.8. Suppose that* $\liminf_{e\to\infty} p^{e(\alpha(R)+d-3)}/b_e = 0$ *as in Example 4.5.1 with $N = L$ (for example, when $d = 3$ and (N, N) is an abundant pair). Then N is MCM. In particular, if R is regular, then N is free.*

Proof. Theorem 4.3.10 with $M = R$ and $e = 0$ shows that N is MCM. □

Example 4.5.5. Let k be a perfect field, $R = k[[x, y, z]]$ and M be the ideal (x, y). Then (M, M) is an abundant pair, but M is not (S_2). So the assumption in Corollary 4.5.4 for N to be (S_2) cannot be weakened.

Proof. Let $C = k[[z]]$. Consider the exact sequence

$$0 \to M \to R \to C \to 0$$

Let $q = p^e$. Since ${}^e-$ is an exact functor, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^eM & \longrightarrow & {}^eR & \longrightarrow & {}^eC & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ & & & & R^{\oplus q^3} & \longrightarrow & C^{\oplus q} & \longrightarrow & 0 \end{array}$$

It follows that ${}^eM$ has exactly q copies of M, so (M, M) is an abundant pair. Since $1 = \operatorname{depth} M_M < \dim M_M = 2$, M is not (S_2). □

Example 4.5.6. Let k be a perfect field, $R = k[[x_1, x_2, \ldots, x_{d-3}, u, v, w]]$, $C = R/(u, v, w)$ and $M = \Omega^2 C$, the second syzygy of C. Then M is (S_2) and (M, M) is an abundant pair. So the assumptions in Corollary 4.5.4 cannot be weakened.

Proof. Let $q = p^e$. Then as in Example 4.5.5, we have

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & {}^eM & \longrightarrow & {}^e(R^3) & \longrightarrow & {}^eR & \longrightarrow & {}^eC & \longrightarrow & 0 \\ & & & & \| & & \| & & \| & & \\ & & & & R^{\oplus 3q^d} & \longrightarrow & R^{\oplus q^d} & \longrightarrow & C^{\oplus q^{d-3}} & \longrightarrow & 0 \end{array}$$

It follows that ${}^{e}M$ has exactly q^{d-3} copies of M. Then as in Corollary 4.5.4, we have $\liminf_{e\to\infty} q^{(\alpha(R)+d-3)}/b_e = q^{d-3}/q^{d-3} = 1$. Since M is a second syzygy, it is (S_2). Since $\operatorname{depth} C = d-3$, we have $\operatorname{depth} M = d-1 \neq d$, so M is not free. $\square$

4.6 Geometric applications

Discussion 4.6.0. Let $(A, \mathfrak{m})$ be a standard graded ring that is (S_2) of dimension at least 2. We first fix some notation and record some results from [19, 26]. Let $R = A_{\mathfrak{m}}$. Let $X = \mathbf{Proj}(A)$, $Y = \mathbf{Spec}(R) \setminus \{\mathfrak{m}_{\mathfrak{m}}\}$ and $Z = \mathbf{Spec}(R)$. Let $\iota\colon Y \to Z$ be the inclusion morphism. There is also an affine surjective morphism $\pi\colon Y \to X$ ([26, Proposition I.5]). Corollary I.6 of [26] states that for every A-graded module M we have sheaf cohomology isomorphisms

$$\oplus_{d\in\mathbb{Z}} H^i(X, \widetilde{M}(d)) \xrightarrow{\cong} H^i(Y, \widetilde{M_{\mathfrak{m}}}|_Y)$$

for $i \geqslant 0$. For $1 \leqslant i \leqslant d-1$, we have $H^i(Y, \widetilde{M_{\mathfrak{m}}}|_Y) \cong H^{i+1}_{\mathfrak{m}}(M_{\mathfrak{m}})$.

Let $V(\cdot)$ denote the category of vector bundles over a scheme. Let Γ be the global section functor. If $\mathscr{G} \in V(Y)$, then $\iota_*\mathscr{G}$ is a coherent $\mathscr{O}_Z$-module. Let $\psi = \Gamma \circ \pi^*$. Then we have maps

$$V(X) \xrightarrow{\pi^*} V(Y) \xrightarrow{\Gamma} \{M \in \operatorname{Ref}(R) \mid M \text{ is locally free on } Y\}$$

By [19, Theorem 1.3], $\operatorname{depth}\Gamma(\mathscr{G}) \geqslant 2$. Γ satisfies the property $\Gamma(\mathscr{G}_1 \otimes \mathscr{G}_2) = (\Gamma(\mathscr{G}_1) \otimes_R \Gamma(\mathscr{G}_2))^{**}$. Let $\mathscr{F}_i \in V(X)$, $i = 1, 2$, be indecomposable such that $\psi(\mathscr{F}_i)$ are isomorphic up to free summand. Then by [19, Proposition 9.5], $\mathscr{F}_1 \cong \mathscr{F}_2(m) = \mathscr{F}_2 \otimes \mathscr{O}_X(m)$ for some m.

Definition 4.6.1. ([8, Introduction]) Let X be a smooth projective variety defined over a field k of characteristic $p > 0$. Let $\varphi\colon X \to X$ be the absolute Frobenius morphism. Then a vector bundle $\mathscr{F} \in V(X)$ is (e, f)-Frobenius periodic, or (e, f)-F periodic in short, if there are $e < f$ such that $(\varphi^e)^*(\mathscr{F}) \cong (\varphi^f)^*(\mathscr{F})$.

Lemma 4.6.2. *Suppose that $\mathscr{F} \in V(X)$ is (e,f)-F periodic. Let X, Y be as in 4.6.0. Let $M = \psi(\mathscr{F})$. Then $\mathscr{F}$ is (e_1, e_2)-F periodic iff $M(e_1) \cong M(e_2)$.*

Proof. For "only if", we have $\psi\big((\varphi^e)^*(\mathscr{F})\big) = \psi(\mathscr{F} \otimes (\varphi^e)^*(X)) = \big(\psi(\mathscr{F}) \otimes (\varphi^e)^*(X)\big)^{**} = M(e)$ as in 4.6.0. The "if" part comes from sheafification. □

Next, we discuss a generalization of one of the results in [32, Theorem 3.1]. For that we need to recall some notation. Let R be a F-finite normal domain and let D be a $\mathbb{Q}$-divisor.
In the next result we are able to remove the condition that the characteristic p is coprime to r as in [32, Theorem 3.1]. We follow the same trick as in [32], with a crucial difference suggested by our approach in this Chapter: the reflexive module representing a torsion element in the class group has finite F-type.

Theorem 4.6.3. *Let R be an F-finite normal domain with perfect residue field and $X = \operatorname{Spec} R$. Let Δ be a $\mathbb{Q}$-divisor on X such that the pair (X, Δ) is strongly F-regular. Let D be an integral divisor such that $rD \sim r\Delta'$ for some integer $r > 0$ and $0 \leqslant \Delta' \leqslant \Delta$. Then $\mathscr{O}_X(-D)$ is Cohen-Macaulay.*

Proof. Since we also have that the pair (X, Δ') is strongly F-regular, one may assume $\Delta' = \Delta$. Now, this assumption implies that there is a decomposition of R-modules ($q = p^e$) ([7, Lemma 3.5]):

$$F^e_* \mathscr{O}_X((q-1)\Delta) = \mathscr{O}_X^{n_e} \oplus N_e$$

such that $\liminf_{e \to \infty} \frac{n_e}{q^d} > 0$. Twisting by $\mathscr{O}_X(-D)$, reflexifying, we get a decomposition:

$$F^e_* \mathscr{O}_X((q-1)(\Delta - D) - D) = \mathscr{O}_X(-D)^{n_e} \oplus N'_e$$

The key point now is that as $r(\Delta - D) \sim 0$, there are only finitely many isomorphism classes of the modules $\mathscr{O}_X((q-1)(\Delta - D) - D)$. Let M be the direct sum of all these modules and

$I = \mathscr{O}_X(-D)$, what we have is precisely:

$$^eM \cong I^{n_e} \oplus P_e$$

with $\liminf_{e\to\infty} \frac{n_e}{q^d} > 0$. Lemma 4.2.7 now forces I to be Cohen-Macaulay. $\square$

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