Convergence Properties of Hausdorff Closed Spaces

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Abstract

The purpose of this work is to study the topological property of Hausdorff closedness as a purely convergence theoretic property. It is the author's opinion that this perspective proves to be a natural one from which to study H-closedness.

Chapter 1 provides a brief introduction to and history of the subject matter. Chapter 2 and the first section of Chapter 3 are mainly preliminary. Here the fundamental facts and definitions needed in the study of H-closed spaces, convergence spaces and especially pretopological spaces are given. In Chapter 2 most proofs are omitted for the sake of brevity, however in Section 3.1, many proofs are given in hopes of helping the reader gain an intuitive feel for pretopologies. Original work begins in Section 3.2, where a study of perfect maps between pretopological spaces is given.

Chapters 4 and 5 make up the heart of this work. In Chapter 4, we take an in-depth look at the pretopology θ. This convergence, which can be defined for any topological space, frames both H-closedness and the related property of being an H-set as convergence properties. Upon noting this fact, in Section 4.1, we immediately see the benefits of this framing. Of particular interest to those who have studied H-closed spaces are Theorems 4.1.5 and 4.1.8. Later in this chapter, so-called relatively θ-compact filters, which are defined using

the convergence θ, are used to obtain a new characterization of countable Katětov spaces in terms of multifunctions.

Chapter 5 provides an analogue of H-closedness which can be defined for any pretopological space. The definition of the so-called PHC spaces is due to the author. In Section 5.1, the PHC spaces are defined and their basic properties are investigated. In Section 5.2, we use the earlier work on perfect maps between pretopological spaces to generate new PHC spaces. Lastly, in Section 5.3, we study the PHC extensions of a pretopological space. In this section we have a construction which is analogous to the Katětov extension of a topological space. Theorem 5.3.6 points to an interesting difference between the usual Katětov extension and our pretopological version.

We finish this work with an investigation of the cardinal invariants of pretopological spaces. We are particularly interested in obtaining cardinality bounds for compact Hausdorff pretopological spaces in terms of their cardinal invariants. Throughout the paper we seek to highlight results which distinguish pretopologies from topologies and this chapter features several results of this flavor.

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Contents

Chapter 1

Introduction

The purpose of the study of topology can be seen from several distinct vantage points. In fact, this is the core from which topology derives much of its strength. The language provided by topology is sufficiently general to be applied to various different fields. In particular, this language captures the essential components needed in the study of continuous mappings in any setting.

In 1906, M. Fréchet [15] introduced the concept of a metric space (though not the actual term, which was coined later by Hausdorff). As in the study of subsets of $\mathbb{R}$, Fréchet noticed the importance of sequences in capturing the the structure of subsets of metric spaces. In both the study of $\mathbb{R}$ and the study of metric spaces generally, sequences and limit points can be used to define the closure operator. That is, for metric spaces, to be closed is to be sequentially closed. The axioms which define the open sets of a topology, originated by F. Hausdorff [17] among others, can be seen as having grown out of the study of subsets of $\mathbb{R}$ and the study of metric spaces. As K. Kuratowski noted in 1922, [24] the lattice of open subsets τ of a topological space is uniquely determined by the closure operator cl_τ. This is captured in the following theorem:

Theorem 1.0.1. *Let X be a set and $c : 2^X \to 2^X$ be a function satisfying:*

(K1) $c(\varnothing) = \varnothing$,

(K2) $A \subseteq c(A)$ *for* $A \in 2^X$,

(K3) $c(A \cup B) = c(A) \cup c(B)$ *for* $A, B \in 2^X$, *and*

(K4) $c(c(A)) \subseteq c(A)$ *for* $A \in 2^X$.

Then $\tau_c = \{X \setminus c(A) : A \in 2^X\}$ *is a topology on* X. *Moreover, if* (X, τ) *is a topological space and* $c = \mathrm{cl}_\tau$, *then* $\tau_c = \tau$.

However, this framework is too general to guarantee that sequences in the set X fully capture the function c. At the same time, more general types of spaces were being considered. In [5], E. Čech studies the so-called *closure spaces*, which are defined by a function satisfying (K1–K3), but not necessarily (K4). These spaces would go on to be known as *pretopological spaces* and will be central to this dissertation.

To regain a characterization of the closure operator in the parlance of convergence, the concept of the *filter* is required. In 1948, G. Choquet [6] axiomatized the *convergence spaces*, in which the filters do all of the heavy lifting in capturing the structure of the space. In this work Choquet laid out the axioms for convergence spaces, pretopological spaces and the newly-defined *pseudotopological spaces*. Topological spaces are then seen as a particular case of each of these more general types of spaces. Upon further inspection, important topological properties, and in particular compactness, are best captured in the pseudotopological setting.

Along purely topological lines, in 1924 P. Alexandroff and P. Urysohn [2] defined the *Hausdorff closed* spaces. Often abbreviated as H-closed spaces, these topological spaces are defined by the property of being closed in every Hausdorff space. As such, the H-closed spaces are a variation of compact Hausdorff spaces. H-closedness proved itself to be worthy of much study, in particular by those interested in studying Hausdorff topological spaces which are not regular. In 1968, N.V. Veličko [34] framed H-closedness as a new type of convergence property. This type of convergence went on to be widely studied, in particular by R.F. Dickman and J.R. Porter under the name "almost convergence" in [8]. The central work of this paper is to unite this line of inquiry with the pretopological spaces of Čech and Choquet.

A new vantage point is provided by the aforementioned unification, and one quickly sees that H-closedness, along with the related property of being an H-set, are more naturally framed as pretopological properties than as topological ones. This allows us to tackle two interesting tasks. First, we can revisit conjectures about H-closed spaces and H-sets which were shown to have negative answers when working in the category of topological spaces to see if these conjectures prove to be true when working with pretopological spaces. Secondly, we can investigate questions which have long remained open in general topology in hopes that our shift in perspective will allow for solutions.

Having made ourselves comfortable in the realm of pretopological spaces, it seems natural to follow this path further, defining an analogue of H-closedness for all pretopological spaces. We formulate the *pretopologically H-closed* (or *PHC*) spaces and investigate their fundamental properties, including a study of their cardinal invariants. It is our hope that the reader will find interest in this course of study and in particular we hope that these new spaces can be used to gain new insight into the structure of H-closed spaces and H-sets.

Chapter 2

Preliminaries

In this chapter we will list preliminary facts and definitions required in the rest of the paper. We will also use this chapter as an opportunity to standardize several notations that will be used throughout. Our focus here is two-fold. First, while we assume the reader has knowledge of several basic notions from topology – separation axioms, compactness, etc. – we will give the basics needed for the study of H-closed spaces. Once these preliminaries have been given, we end this chapter by laying out the framework of convergence spaces. We do not assume any prior knowledge of convergence spaces and hope that the parts of the paper pertaining to this subject matter will be largely self-contained.

2.1 Topological Notations

Since we will be shifting frequently between talk of convergence spaces and that of topological spaces, we will be explicit about what type of space is being discussed. To this end, we reserve the letter τ to stand for a topology on a set X and will usually write (X, τ) when discussing a topological space, dropping neither the X nor the τ. If (X, τ) is a topological space and $x \in X$, then $\mathcal{N}_\tau(x)$ is the family of neighborhoods of x. The usual topological closure operator on subsets of X will be written cl_τ.

2.2 Filters

In real analysis, we learn that the topological properties of $\mathbb{R}$ can be defined in terms of sequences. When dealing with general topological spaces, sequences are not powerful enough to capture these properties. The filter is the tool which allows us to reframe topological properties in terms of convergence. Filters are of central importance in the field of general topology and in the theory of convergence spaces filters are the essential object of study.

Definition 2.2.1. Let X be a set and 2^X denote the family of all subsets of X. Suppose that $\mathcal{L} \subseteq 2^X$ is a lattice ordered by $\subseteq$.

(a) A nonempty family $\mathcal{F} \subseteq \mathcal{L}$ with the finite intersection property is called an $\mathcal{L}$-*filter subbase.*

(b) A nonempty family $\mathcal{F} \subseteq \mathcal{L}$ is a $\mathcal{L}$-*filter base* if

(F1) $F \in \mathcal{F}$ implies $F \neq \varnothing$

(F2) $F, G \in \mathcal{F}$ implies $H \subseteq F \cap G$ for some $H \in \mathcal{F}$

(c) A nonempty family $\mathcal{F} \subseteq \mathcal{L}$ is an $\mathcal{L}$-*filter* if $\mathcal{F}$ satisfies (F1), (F2) and

(F3) If $F \in \mathcal{F}$, $G \in \mathcal{L}$ and $F \subseteq G$, then $G \in \mathcal{F}$.

We will make use of two choices of $\mathcal{L}$. If $\mathcal{L} = 2^X$, then $\mathcal{F}$ is simply called a *filter subbase*, a *filter base* or a *filter*. If (X, τ) is a topological space and $\mathcal{L} = \tau$, then $\mathcal{F}$ is called an *open filter subbase*, an *open filter base*, or an *open filter*.

If $\mathcal{F}$ is an $\mathcal{L}$-filter subbase or an $\mathcal{L}$-filter base, let $\langle\mathcal{F}\rangle$ be the $\mathcal{L}$-filter generated by $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{L}$-filter bases, we write $\mathcal{F} \leq \mathcal{G}$ if $\langle\mathcal{F}\rangle \subseteq \langle\mathcal{G}\rangle$. In this case we say that $\mathcal{G}$ is *finer* than $\mathcal{F}$ and that $\mathcal{F}$ is *coarser* than $\mathcal{G}$. The relation $\leq$ is a partial order on the $\mathcal{L}$-filter bases on X.

Definition 2.2.2. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{L}$-filters on a set X. We say that $\mathcal{F}$ *meets* $\mathcal{G}$ if $F \cap G \neq \varnothing$ for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

Definition 2.2.3. A filter $\mathcal{F}$ is called an $\mathcal{L}$-*ultrafilter* if $\mathcal{F} \leq \mathcal{G}$ implies $\mathcal{F} = \mathcal{G}$. (i.e. $\mathcal{F}$ is maximal with respect to $\leq$.)

The following well-known fact is a straight-forward consequence of Zorn's Lemma.

Proposition 2.2.4. *Every $\mathcal{L}$-filter on X is contained in some $\mathcal{L}$-ultrafilter on X.*

Let (X, τ) be a topological space. Filters are to topological spaces as sequences are to metric spaces or $\mathbb{R}$. That is to say that filters can be used to define a notion of convergence in (X, τ) and the usual closure operator cl_τ can be defined via convergent filters. Since cl_τ uniquely determines the topology τ, as noted in Theorem 1.0.1, the notion of convergence provided by filters determines the topology.

Definition 2.2.5. Let (X, τ) be a topological space and let $\mathcal{F}$ be a filter on X.

(a) We say that $\mathcal{F}$ *τ-converges to x*, written $x \in \lim_\tau \mathcal{F}$, if $\mathcal{N}_\tau(x) \subseteq \mathcal{F}$.

(b) The *τ-adherence* of $\mathcal{F}$ is defined to be $\bigcap_{F \in \mathcal{F}} cl_\tau F$.

If we say that a filter base $\mathcal{F}$ τ-converges to x, what we mean is that the filter $\langle\mathcal{F}\rangle$ τ-converges to x.

If $A \subseteq X$, then $\{A\}$ is a filter base on X. The usual closure operator can then be defined using the above definition since it is easily seen that $\mathrm{cl}_\tau A = \mathrm{adh}_\tau \langle\{A\}\rangle$. In 2.7.3, we will see a way of defining the adherence of a filter without reference to the closure operator.

The following well-known characterization of compactness is an example of the power provided by filters.

Theorem 2.2.6. *Let (X,τ) be a topological space. The following are equivalent:*

(a) (X,τ) *is compact*

(b) Every filter on X has nonempty τ-adherence

(c) Every ultrafilter on X τ-converges to some $x \in X$.

2.3 Variations of Compactness

A Hausdorff topological space (X,τ) is called *Hausdorff closed* (or *H-closed* for short) if it is closed in every Hausdorff space in which is it embedded. The following well-known characterizations of H-closed spaces are useful and will be used interchangeably as the definition of H-closed.

Theorem 2.3.1. *Let (X,τ) be a Hausdorff topological space. The following are equivalent.*

(a) (X,τ) *is H-closed,*

(b) Whenever $\mathcal{C}$ is an open cover of X, there exist $C_1,\ldots,C_n \in \mathcal{C}$ such that $X = \bigcup_{i=1}^{n} \mathrm{cl}_\tau C_i$,

(c) Every open filter on X has nonempty τ-adherence,

(d) Every open ultrafilter on X τ-converges to some point in X.

Notice that if τ is regular, then (X,τ) is H-closed if and only if (X,τ) is compact. Since every compact Hausdorff topolgical space is regular, the concept of H-closedness is often more useful than compactness when studying Hausdorff spaces which are not regular. Veličko [34] relativized the concept of H-closedness to subspaces in the following way:

Definition 2.3.2. If X is a Hausdorff topological space and $A \subseteq X$, we say that A is an *H-set in X* if whenever $\mathcal{C}$ is a cover of A by open subsets of X, there exist $C_1, \ldots, C_n \in \mathcal{C}$ such that $A \subseteq \bigcup_{i=1}^{n} \mathrm{cl}_\tau C_i$.

We note the following well-known characterizations of H-sets which mirror the characterizations of H-closedness in Theorem 2.3.1.

Proposition 2.3.3. *Let X be a topological space and $A \subseteq X$. The following are equivalent.*

(a) A is an H-set in X,

(b) If $\mathcal{F}$ is an open filter on X which meets A, then $\mathrm{adh}_\tau \mathcal{F} \cap A \neq \varnothing$,

(c) If $\mathcal{U}$ is an open ultrafilter on X which meets A, then $\mathrm{adh}_\tau \mathcal{U} \cap A \neq \varnothing$.

It is important to note that the property of H-closeness is not closed-hereditary. Also, note that the definition of an H-set is dependent on the ambient space being considered. In particular, not every H-set is H-closed. The following example, due to Urysohn, points to this distinction. Recall that a space (X, τ) is *semiregular* if the regular-open subsets of X $\{\mathrm{int}_\tau \mathrm{cl}_\tau A : A \subseteq X\}$ form an open base for τ.

Example 2.3.4. Let $X = \mathbb{N} \times \mathbb{Z} \cup \{+\infty, -\infty\}$. Define $U \subseteq X$ to be open if

$+\infty \in U$ implies that there is $n_U \in \mathbb{N}$ such that

$$\{(n,k) \in \mathbb{N} \times \mathbb{Z} : n > n_U, k > 0\} \subseteq U$$

$-\infty \in U$ implies that there is $n_U \in \mathbb{N}$ such that

$$\{(n,k) \in \mathbb{N} \times \mathbb{Z} : n > n_U, k < 0\} \subseteq U$$

$(n,0) \in U$ implies that there is some $k_U \in \mathbb{N}$ such that

$$\{(n,k) \in \mathbb{N} \times \mathbb{Z} : |k| > k_U\} \subseteq U$$

Then X is H-closed and semiregular.

Let $A = \{(n,0) \in \mathbb{N} \times \mathbb{Z} : n \in \mathbb{N}\} \cup \{+\infty\}$. Notice that A is a closed discrete subset of X and that A is an H-set in X. To see that A is an H-set of X, suppose that $\mathcal{C}$ is a cover of A by open subsets of X. For some $C_\infty \in \mathcal{C}$, it must be that $+\infty \in C_\infty$. By definition, there exist $n_\infty \in \mathbb{N}$ such that $\{(n,k) \in \mathbb{N} \times \mathbb{Z} : n > n_\infty, k > 0\} \subseteq C_\infty$. Therefore, $\{(n,0) : n > n_\infty\} \subseteq \mathrm{cl}_\tau C_\infty$. Let $C_1, \ldots, C_{n_\infty} \in \mathcal{C}$ such that $(k,0) \in C_k$ for $k = 1, \ldots, n_\infty$. Then, $A \subseteq \mathrm{cl}_\tau C_1 \cup \ldots \cup \mathrm{cl}_\tau C_{n_\infty} \cup \mathrm{cl}_\tau C_\infty$. Notice however, that with the subspace topology, A is homeomorphic to $\mathbb{N}$ with the discrete topology, and thus is not H-closed.

When considering the H-sets of a topological space (X,τ), it is sometimes useful to consider (X,τ) as a subspace of a space (Y,σ), where σ has some nice properties. The following proposition says that we don't lose any information about the H-sets of X in this process.

Proposition 2.3.5. *If (X,τ) is a Hausdorff topological space, A is an H-set in X and $i : (X,\tau) \to (Y,\sigma)$ is an embedding, then $i[A]$ is an H-set in Y.*

This is particularly useful since every Hausdorff space can be embedded as a dense subspace of an H-closed space. We will make use of the following weakening of the definition of an H-set.

Definition 2.3.6. Let (X,τ) be a Hausdorff topological space. Then $A \subseteq X$ is *H-bounded* if whenever $\mathcal{C}$ is an open cover of X, there exists a finite subfamily $\mathcal{A} \subseteq \mathcal{C}$ such that $A \subseteq \bigcup_{C \in \mathcal{A}} \mathrm{cl}_\tau C$.

Notice that if (X,τ) is H-closed, then *every* subset of X is H-bounded. Therefore, being H-bounded is strictly weaker than being an H-set. For a detailed study of H-bounded sets, including a proof of the next theorem, see [29].

Theorem 2.3.7. *Let (X,τ) be a Hausdorff topological space and let $A \subseteq X$. The following are equivalent:*

(a) A is H-bounded,

(b) If $\mathcal{F}$ is an open filter on X and $\mathcal{F}$ meets A, then $\mathrm{adh}_\tau \mathcal{F} \neq \varnothing$,

(c) If $\mathcal{U}$ is an open ultrafilter on X and $\mathcal{U}$ meets A, then $\mathrm{adh}_\tau \mathcal{U} \neq \varnothing$.

We will make use of the following variations of compactness in Section 4.1.

Definition 2.3.8. Let (X,τ) be a Hausdorff topological space.

(a) We say that τ is *minimal Hausdorff* if for any topology σ on X such that $\sigma \subseteq \tau$, $\sigma \neq \tau$ implies σ is not Hausdorff.

(b) The space (X,τ) is called *Katětov* if there exists a topology $\sigma \subseteq \tau$ such that σ is minimal Hausdorff.

(c) We say that (X,τ) is *C-compact* if every closed subset of (X,τ) is an H-set.

Theorem 2.3.9. *Let (X,τ) be a Hausdorff topological space.*

(a) τ is minimal Hausdorff if and only if (X,τ) is H-closed and semiregular.

(b) If (X,τ) is C-compact, then τ is minimal Hausdorff.

2.4 θ-continuity

Definition 2.4.1. Let (X,τ) and (Y,σ) be topological spaces. A function $f:(X,\tau)\to(Y,\sigma)$ is *θ-continuous* if for each $x\in X$ and $V\in\mathcal{N}_\sigma(f(x))$, there exists $U\in\mathcal{N}_\tau(x)$ such that $f[\mathrm{cl}_\tau U]\subseteq \mathrm{cl}_\sigma V$.

Every continuous function between topological spaces is also θ-continuous. If σ is regular, then every θ-continuous function $f:(X,\tau)\to(Y,\sigma)$ is also continuous.

In the study of H-closed spaces and H-sets, θ-continuous functions are especially useful because they are precisely what is needed to preserve these properties, as witnessed by the following proposition.

Proposition 2.4.2. *Let (X,τ) and (Y,σ) be topological spaces and $f:(X,\tau)\to(Y,\sigma)$ a θ-continuous function.*

(a) If X is H-closed and f is a surjection, then Y is H-closed.

(b) If A is an H-set in X, then $f[A]$ is an H-set in Y.

The following construction also makes use of θ-continuity. Let (X,τ) be a Hausdorff topological space. Let EX be the family of open ultrafilters on (X,τ) with nonempty adherence. Let $E\tau$ be the topology on EX which has as a basis sets of the form

$$OU=\{\mathcal{U}\in EX : U\in\mathcal{U}\}.$$

Let $k_X:(EX,E\tau)\to(X,\tau)$ be defined to that $k_X(\mathcal{U})\in\mathrm{adh}_\tau\mathcal{U}$. Since τ is Hausdorff, for each $\mathcal{U}\in EX$, $|\mathrm{adh}_\tau\mathcal{U}|=1$ and this function is uniquely defined. The space $(EX,E\tau)$ is called the *absolute of X*.

Theorem 2.4.3. *The space $(EX, E\tau)$ is a zero-dimensional, extremally disconnected, Hausdorff topological space and the function k_X is a perfect, irreducible, θ-continuous surjection.*

For a full treatment of absolutes, including a proof of Theorem 2.4.3, see [30]. We will cite an application of the absolute in Section 4.1.

2.5 H-closed Extensions of (X, τ)

In this section we introduce two important H-closed extensions of a Hausdorff topological space (X, τ). The first, which is known as the Fomin extension and is denoted σX, will be used in Section 4.4. In Section 5.3, we will develop an analogue of the second H-closed extension, known as the Katětov extension, for pretopological spaces. More information on each of these extensions and on H-closed extensions in general can be found in [30].

If (X, τ) is a Hausdorff topological space, let X' be the collection of open ultrafilter on X such that if $\mathcal{U} \in X'$, then $\operatorname{adh}_\tau \mathcal{U} = \varnothing$. For an open set $U \in \tau$, let

$$oU = U \cup \{\mathcal{U} \in X' : U \in \mathcal{U}\}.$$

Let $\sigma X = X \cup X'$ and let σ_τ be the topology which has $\{oU : U \in \tau\}$ as a base.

Theorem 2.5.1. $(\sigma X, \sigma_\tau)$ *is an H-closed extension of* (X, τ). *That is, σX is H-closed and X is embedded as a dense subspace of σX.*

The point-set of the Katětov extension of X, κX, is also $X \cup X'$. However, the open sets of κX have sets of the form $\{\mathcal{U}\} \cup U$, where $U \in \mathcal{U}$, as a base. Call this topology κ_τ.

Theorem 2.5.2. $(\kappa X, \kappa_\tau)$ *is an H-closed extension of X. Moreover, if Y is an H-closed* *extension of X, then there exists a continuous function $f : \kappa X \to Y$ which fixes the points of X.*

2.6 Multifunctions

By a multifunction $F : X \multimap Y$, we mean a relation which assigns to each $x \in X$ a subset $F(x) \subseteq Y$. Analogues of the usual properties studied of function between topological spaces will be defined in this section.

Definition 2.6.1. Let (X,τ) and (Y,σ) be topological space and let $F : X \multimap Y$ be a multifunction.

(a) F is *upper-semicontinuous at x* if whenever $F(x) \subseteq V \in \sigma$, there exists $U \in \mathcal{N}_\tau(x)$ such that $F[U] = \bigcup_{x \in U} F(x) \subseteq V$.

(b) F is *θ-usc at x* if whenever $F(x) \subseteq V \in \sigma$, there exists $U \in \mathcal{N}_\tau(x)$ such that $F[\mathrm{cl}_\tau U] \subseteq \mathrm{cl}_\sigma V$.

If $F : X \multimap Y$ is a multifunction and $\mathcal{F}$ is a filter on X, then $F(\mathcal{F}) = \{F[H] : H \in \mathcal{F}\}$ is a filter base on Y. We will write $F(\mathcal{F})$ to denote the filter on Y generated by this base.

2.7 Convergence Spaces

For a basic reference on convergence theory, see [10]. For an all-you-can-eat treatment convergence theory which starts from scratch, see [13].

Given a relation ξ between filters on X and elements of X, we write $x \in \lim_\xi \mathcal{F}$ whenever $(\mathcal{F}, x) \in \xi$ and say that x is a *ξ-limit point* of $\mathcal{F}$. If $A \subseteq X$, let $\langle A \rangle$ be the principal filter generated by A. We abbreviate $\langle \{x\} \rangle$ by $\langle x \rangle$.

Definition 2.7.1. A *convergence* is a relation ξ between filters on X and points of X which satisfies

(C1) $x \in \lim_\xi \langle x \rangle$, and

(C2) if $\mathcal{F} \subseteq \mathcal{G}$ and $x \in \lim_\xi \mathcal{F}$, then $x \in \lim_\xi \mathcal{G}$.

The pair (X, ξ) is called a *convergence space*.

Notice that, thanks to (C1), the range of the relation ξ is the whole works, X. Therefore, the underlying set is determined by the convergence. We will make use of this fact when it is important to note the particular convergence being discussed, but writing (X, ξ) is too cumbersome.

Clearly, a topological space (X, τ) paired with the usual topological notion of convergence in which $x \in \lim_\tau \mathcal{F}$ if and only if $\mathcal{N}_\tau(x) \subseteq \mathcal{F}$ is an example of a convergence space. Since topological convergence is determined by the topology τ, we will abuse notation and use the symbol τ for both the family of open subsets of X and the convergence determined by τ.

The class of convergence structures on a set X can be given a lattice structure. We say that σ is *coarser than* ξ, written $\sigma \leq \xi$ if $\lim_\sigma \mathcal{F} \supseteq \lim_\xi \mathcal{F}$ for each filter $\mathcal{F}$ on X. In this case we also say that ξ is *finer* than σ.

Given a convergence space (X, ξ), there exists a topology $T\xi$ such that $T\xi$ is the finest topology which as a convergence is coarser than ξ. This is done as follows: We say that $U \subseteq X$ is ξ-open if and only if $U \cap \lim_\xi \mathcal{F} \neq \varnothing$ implies that $U \in \mathcal{F}$ for each filter $\mathcal{F}$ on X. If $T\xi$ is the collection of ξ-open subsets of X, then $T\xi$ is the finest topology on X which is coarser than ξ. We say that a convergence ξ is *topological* if $\xi = T\xi$. The following is Example 14 of [10] and is a convergence which is not topological.

Example 2.7.2. A filter $\mathcal{F}$ is said to be *sequential* if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that for each $F \in \mathcal{F}$, there exists $N \in \mathbb{N}$ with $\{x_n : n > N\} \subseteq F$. In other word, the tails of the sequence form a filter base and the filter generated by this base is $\mathcal{F}$.

Now, we define a convergence ξ on $\mathbb{R}$. Let τ be the standard topology on $\mathbb{R}$. If $\mathcal{F}$ is a filter on $\mathbb{R}$, then $x \in \lim_\xi \mathcal{F}$ if and only if there exists a sequential filter $\mathcal{G}$ on $\mathbb{R}$ such that $\mathcal{G} \subseteq \mathcal{F}$ and $x \in \lim_\tau \mathcal{G}$. Notice that $0 \notin \lim_\xi \mathcal{N}_\tau(0)$. In particular, for each $U \in \mathcal{N}_\tau(0)$, $|U| = 2^{\aleph_0}$. Therefore, there cannot exist a sequential filter $\mathcal{G}$ such that $\mathcal{G} \subseteq \mathcal{N}_\tau(0)$.

Since the topology τ is determined entirely by sequences, it can be seen that $T\xi = \tau$. Since $0 \in \lim_\tau \mathcal{N}_\tau(0)$ and $0 \notin \lim_\xi \mathcal{N}_\tau(0)$, $T\xi \neq \xi$ and ξ is not topological.

Definition 2.7.3. Let (X,ξ) be a convergence space, $\mathcal{F}$ a filter on X and $A \subseteq X$.

(a) The *adherence of* $\mathcal{F}$ is defined to be

$$\operatorname{adh}_\xi \mathcal{F} = \bigcup\{\lim_\xi \mathcal{G} : \mathcal{G}\#\mathcal{F}\}.$$

We write $\operatorname{adh}_\xi A$ to abbreviate $\operatorname{adh}_\xi \langle A\rangle$.

(b) The *inherence of* A is

$$\operatorname{inh}_\xi A = X \setminus \operatorname{adh}_\xi (X \setminus A).$$

For $A \subseteq X$, $\operatorname{adh}_\xi A$ and $\operatorname{inh}_\xi A$ will function as versions of topological closure and interior generalized to convergence spaces.

A convergence ξ is *Hausdorff* if every filter has at most one limit point.

Topological spaces, as we have seen, are particular instances of convergence spaces. In fact, if (X,τ) is a topological space, then $\operatorname{adh}_\tau A = \operatorname{cl}_\tau A$ for any $A \subseteq X$ and $\operatorname{adh}_\tau \mathcal{F} = \bigcap_{F\in\mathcal{F}} \operatorname{cl}_\tau F$. Two other important classes of convergences are *pseudotopologies* and *pretopologies*. If $\mathcal{F}$ is a filter on X, let $\beta\mathcal{F}$ denote the set of all ultrafilters on X containing

$\mathcal{F}$. A convergence ξ is a pseudotopology if $\lim_{\xi} \mathcal{F} \supseteq \bigcap\{\lim_{\xi} \mathcal{U} : \mathcal{U} \in \beta\mathcal{F}\}$. In [18], Herrlich, Lowen-Colebunders and Schwartz discuss the categorical advantages of working in the category of pseudotopological spaces. We will discuss the usefulness of working with pretopological spaces to characterize H-closed spaces and H-sets in the Chapter 4.

A convergence space (X, ξ) is *compact* if every filter on X has nonempty adherence. The following notions of compactness for filters will allow us to get at compactness of subspaces.

Definition 2.7.4. Let (X, ξ) be a convergence space, $\mathcal{F}$ a filter on X and $A \subseteq X$. We say that $\mathcal{F}$ is *ξ-compact at A* if whenever $\mathcal{G}$ is a filter on X and $\mathcal{G}\#\mathcal{F}$, $\operatorname{adh}_{\xi} \mathcal{G} \cap A \neq \varnothing$. In particular, a filter $\mathcal{F}$ is *relatively ξ-compact* if it is compact at X.

If $\mathcal{B}$ is a family of subsets of X, then $\mathcal{F}$ is *ξ-compact at* $\mathcal{B}$ if whenever $\mathcal{G}\#\mathcal{F}$, $\operatorname{adh}_{\xi} \mathcal{G}\#\mathcal{B}$. A filter is *$\xi$-compact* if $\mathcal{F}$ is ξ-compact at itself. If we can be sure that there will be no confusing, we may drop "ξ-" from each of the above definitions.

Using this definition, $A \subseteq X$ is *compact* if whenever $\mathcal{G}$ is a filter on X which meets A, we have that $\operatorname{adh}_{\xi} \mathcal{G} \cap A \neq \varnothing$. Notice that for topological spaces this also characterizes the compact subspaces.

Definition 2.7.5. Let (X, ξ) and (Y, σ) be convergence spaces. A function $f : (X, \xi) \to (Y, \sigma)$ is *continuous* if

$$f[\lim_{\xi} \mathcal{F}] \subseteq \lim_{\sigma} f(\mathcal{F})$$

for each filter $\mathcal{F}$ on X, where $f(\mathcal{F})$ is the filter generated by $\{f[F] : F \in \mathcal{F}\}$.

Given $A \subseteq X$ and a convergence ξ on X, we can define the *subconvergence* on A as follows: If $\mathcal{F}$ is a filter on A, let $\langle\mathcal{F}\rangle$ be the filter on X generated by $\mathcal{F}$. Define $\lim_{\xi|_A} \mathcal{F} = \lim_{\xi} \langle\mathcal{F}\rangle \cap A$. This is also the initial convergence of A generated by the inclusion map

$i : A \to (X, \xi)$; that is, the coarsest convergence making the inclusion map continuous. Thus, A is a compact subset of (X, ξ) is equivalent to $(A, \xi|_A)$ is a compact convergence space.

Chapter 3

Pretopologies and Perfect Maps

Convergence spaces as defined in the previous chapter are members of a very general category of objects. As seen in Example 2.7.2, a convergence can behave quite differently from our usual topological notion of convergence. If we restrict ourselves to the pretopologies, a subcategory of the category of convergence spaces, many of our topological intuitions can be regained. In this chapter we begin by giving many of the important facts and definitions needed in the study of pretopological spaces. Results which stress the ways in which pretopologies behave similarly to topologies are highlighted. In several places we give proofs of well-known facts in hopes of helping the reader get a feel to these spaces. We conclude this chapter with a study of the so-called perfect maps. In Section 5.2, perfect maps will be used to construct examples of a new type of pretopological space.

3.1 Pretopologies

Let (X,ξ) be a convergence space. For $x \in X$, the *vicinity filter of* ξ *at* x is defined to be

$$\mathcal{V}_\xi(x) = \bigcap\{\mathcal{F} : x \in \lim_\xi \mathcal{F}\}.$$

Definition 3.1.1. Let (X,π) be a convergence space. We say that π is a *pretopology* if $x \in \lim_\pi \mathcal{V}_\pi(x)$ for each $x \in X$. In this case we say that (X,π) is a *pretopological space*.

As in this definition, we will reserve the Greek letter π to stand for a pretopology. Example 2.7.2 is a convergence which is not a pretopology. Example 3.1.7 is an example of a pretopology which is not in general a topology. Chapter 4 is an detailed study of the pretopology seen in Example 3.1.7.

When (X,π) is a pretopological space, the adherence of a filter can be defined similarly to 2.2.5(b).

Proposition 3.1.2. *Let (X,π) be a pretopological space and let $\mathcal{F}$ be a filter on X. Then $x \in \operatorname{adh}_\pi \mathcal{F}$ if and only if $\mathcal{F}\#\mathcal{V}_\pi(x)$.*

Proof. Suppose that $x \in \operatorname{adh}_\pi \mathcal{F}$. Then, there exists some filter $\mathcal{G}$ such that $x \in \lim_\pi \mathcal{G}$ and $\mathcal{G}\#\mathcal{F}$. By definition, since $x \in \lim_\pi \mathcal{G}$, we know that $\mathcal{V}_\pi(x) \subseteq \mathcal{G}$. Therefore it follows that $\mathcal{F}\#\mathcal{V}_\pi(x)$ as well. Now suppose that $\mathcal{F}\#\mathcal{V}_\pi(x)$. Since π is a pretopology, $x \in \lim_\pi \mathcal{V}_\pi(x)$. Thus $x \in \operatorname{adh}_\pi \mathcal{F}$ by definition. $\square$

Corollary 3.1.3. *Let (X,π) be a topological space and $A \subseteq X$. Then $x \in \operatorname{adh}_\pi A$ if and only if $V \cap A \neq \varnothing$ for each $V \in \mathcal{V}_\pi(x)$.*

We noted in the comments following Definition 2.7.3 that when we view a topological space (X,τ) as a convergence space, $\operatorname{adh}_\tau A = \operatorname{cl}_\tau A$ for each $A \subseteq X$. Corollary 3.1.3 shows that when working with pretopological spaces, the adherence operator can be defined similarly to the closure operator in a topological space, with $\mathcal{V}_\pi(x)$ taking the place of $\mathcal{N}_\tau(x)$. The following proposition is often useful when determining the vicinities of a point x and further extends the analogy between $\mathcal{V}_\pi(x)$ in pretopological space and $\mathcal{N}_\tau(x)$ in topological spaces.

Proposition 3.1.4. *Let (X,π) be a pretopological space. Then $V \in \mathcal{V}_\pi(x)$ if and only if $x \in \operatorname{inh}_\pi V$.*

Proof. Recall that $\operatorname{inh}_\pi V = X \setminus \operatorname{adh}_\pi(X \setminus V)$. By Corollary 3.1.3, $x \in \operatorname{inh}_\pi V$ if and only if there exists $U \in \mathcal{V}_\pi(x)$ such that $U \cap X \setminus V = \varnothing$.

Let $V \in \mathcal{V}_\pi(x)$. Then $V \cap X \setminus V = \varnothing$ and $x \in \operatorname{inh}_\pi V$. If $x \in \operatorname{inh}_\pi V$, let $U \in \mathcal{V}_\pi(x)$ such that $U \cap X \setminus V = \varnothing$. Therefore, $U \subseteq V$. Since $\mathcal{V}_\pi(x)$ is a filter, it follows that $V \in \mathcal{V}_\pi(x)$. $\square$

We will also need to define vicinities for subsets of X. We will use Proposition 3.1.4 to streamline this definition.

Definition 3.1.5. Let (X,π) be a pretopological space and $A \subseteq X$. We define the *vicinity filter of π at A* to be $\mathcal{V}_\pi(A) = \{V : A \subseteq \operatorname{inh}_\pi V\}$.

Proposition 3.1.6. *If (X,π) is a pretopological space, then the adherence operator satisfies each of the following*

(a) $\operatorname{adh}_\pi \varnothing = \varnothing$,

(b) $A \subseteq \operatorname{adh}_\pi A$ *for each* $A \subseteq X$,

(c) $\operatorname{adh}_\pi(A \cup B) = \operatorname{adh}_\pi A \cup \operatorname{adh}_\pi B$ *for any* $A, B \subseteq X$.

Notice that (a), (b) and (c) in Proposition 3.1.6 are equivalent to (K1–K3) in Theorem 1.0.1 with $c = \operatorname{adh}_\pi$. Thus, Proposition 3.1.6 provides us with an intuitive way to think about pretopological spaces. The adherence operator on subsets of a pretopological space functions as an analogue for the closure operator on a topological space. However, adh_π is not necessarily idempotent.

Example 3.1.7. Let (X,τ) be a topological space. Throughout this paper, given a topological space, let θ be the convergence on X defined by

$$x \in \lim_\theta \mathcal{F} \Longleftrightarrow \{\mathrm{cl}_\tau U : U \in \mathcal{N}_\tau(x)\} \subseteq \mathcal{F}.$$

If there is any possibility for confusion, we will write θ_X. This type of convergence was studied extensively under the name "almost convergence" in [8].

The space (X,θ) is a pretopological space and $\mathcal{V}_\theta(x)$ is the filter generated by $\{\mathrm{cl}_\tau U : U \in \mathcal{N}_\tau(X)\}$. For $A \subseteq X$, $\mathrm{adh}_\theta A$ is the well-known θ-closure, as seen in [34]. Explicitly,

$$x \in \mathrm{adh}_\theta A \Longleftrightarrow \forall (U \in \mathcal{N}_\tau(X))\, \mathrm{cl}_\tau U \cap A \neq \varnothing.$$

Example 3.1.7 is a pretopology which is not in general a topology. Chapter 4 is an in-depth study of this pretopology, and in particular of its important relation to H-closedness. To see why this convergence is not in general a topology, we make use of the following proposition which is Proposition 22 from [10].

Proposition 3.1.8. *Let (X,ξ) be a convergence space. If ξ is a topology, then the adherence operator adh_ξ is idempotent on subsets of X.*

Example 3.1.9. Let X be the topological space defined in Example 2.3.4 equipped with the pretopology θ described in Example 3.1.7 above. Consider the following subset B of X :

$$B = \{(n,m) \in \mathbb{N} \times \mathbb{Z} : n \in \mathbb{N}, m > 0\}.$$

Then

$$\mathrm{adh}_\theta B = B \cup \{(n,0) \in \mathbb{N} \times \mathbb{Z} : n \in \mathbb{N}\} \cup \{+\infty\}.$$

However,

$$\operatorname{adh}_\theta \operatorname{adh}_\theta B = \operatorname{adh}_\theta B \cup \{-\infty\}$$

and by Proposition 3.1.8, (X,θ) is not a topological space.

The following two propositions show that when working in the framework of Hausdorff pretopological spaces, many of our intuitions about Hausdorff topological spaces remain applicable.

Proposition 3.1.10. *If (X,π) is a pretopological space, then X is Hausdorff if and only if whenever x_1, $x_2 \in X$ and $x_1 \neq x_2$, there exists $U_i \in \mathcal{V}_\pi(x_i)$ $(i = 1,2)$ such that $U_1 \cap U_2 = \varnothing$.*

Proof. Let (X,π) be a Hausdorff pretopological space and fix $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Suppose that $\mathcal{V}_\pi(x_1) \# \mathcal{V}_\pi(x_2)$. Then, there exists a filter $\mathcal{F}$ on X such that $\mathcal{F} \supseteq \mathcal{V}_\pi(x_1) \cup \mathcal{V}_\pi(x_2)$. However, since π is a pretopology, it would then be the case that $\{x_1, x_2\} \subseteq \lim_\pi \mathcal{F}$. This contradicts that (X,π) is Hausdorff.

Now, let $\mathcal{F}$ be a filter on X and suppose that $|\lim_\pi \mathcal{F}| > 1$. Then, there exist $x_1, x_2 \in \lim_\pi \mathcal{F}$ such that $x_1 \neq x_2$. By assumption, we can find $U_1 \in \mathcal{V}_\pi(x_1)$ and $U_2 \in \mathcal{V}_\pi(x_2)$ such that $U_1 \cap U_2 = \varnothing$. Since π is a pretopology, $\mathcal{V}_\pi(x_i) \subseteq \mathcal{F}$ for $i = 1,2$. However, this implies that $U_1, U_2 \in \mathcal{F}$. Since all filters $\mathcal{F}$ have the finite intersection property, this is a contradiction. □

Proposition 3.1.11. *If (X,π) is a compact Hausdorff pretopological space and $A \subseteq X$ is compact, then* $\operatorname{adh}_\pi A = A$.

Proof. Let $A \subseteq X$ be compact. Since $A \subseteq \operatorname{adh}_\pi A$, we show that $\operatorname{adh}_\pi A \subseteq A$. Let $x \in \operatorname{adh}_\pi A$. Then $\mathcal{V}_\pi(x) \# A$. By compactness, $\operatorname{adh}_\pi \mathcal{V}_\pi(x) \cap A \neq \varnothing$. By Proposition 3.1.10, $\operatorname{adh}_\pi \mathcal{V}_\pi(x) = \{x\}$. Therefore, $x \in A$, as needed. □

Furthermore, when working with pretopological spaces, many of the usual characterizations of continuous functions between topological spaces are still applicable, as displayed by the following proposition.

Proposition 3.1.12. *Let* (X,π) *and* (Y,σ) *be pretopological spaces and* $f:(X,\pi)\to(Y,\sigma)$. *The following are equivalent*

(a) f *is continuous*

(b) $f[\mathrm{adh}_\pi \mathcal{F}] \subseteq \mathrm{adh}_\sigma f(\mathcal{F})$ *for each filter* $\mathcal{F}$ *on* X

(c) $f[\mathrm{adh}_\pi A] \subseteq \mathrm{adh}_\sigma f[A]$ *for each* $A \subseteq X$

(d) $f^{\leftarrow}[\mathrm{inh}_\sigma B] \subseteq \mathrm{inh}_\pi f^{\leftarrow}[B]$ *for each* $B \subseteq Y$

(e) For each $x \in X$, *if* $V \in \mathcal{V}_\sigma(f(x))$, *there exists* $U \in \mathcal{V}_\pi(x)$ *such that* $f[U] \subseteq V$.

Definition 3.1.13. A collection $\mathcal{C}$ of subsets of a pretopological space (X,π) is a π-*cover* (or simply *cover* if there is no possible confusion) if for each $x \in X$, $\mathcal{C} \cap \mathcal{V}_\pi(x) \neq \varnothing$. For $A \subseteq X$, we say that $\mathcal{C}$ is a *cover of* A if for each $x \in A$, $\mathcal{C} \cap \mathcal{V}_\pi(x) \neq \varnothing$.

Proposition 3.1.14. *Let* (X,π) *be a pretopological space,* $\mathcal{F}$ *a filter on* X *and* $A \subseteq X$. *Then* $\mathcal{F}$ *is compact at* A *if and only if whenever* $\mathcal{C}$ *is a* π*-cover of* A, *there exists* $F \in \mathcal{F}$ *and* $C_1,\dots,C_n \in \mathcal{C}$ *such that* $F \subseteq \bigcup_{i=1}^n C_i$.

Proof. Suppose that $\mathcal{F}$ is compact at A. By definition, if $\mathcal{G}$ is a filter on X which meets $\mathcal{F}$, $\mathrm{adh}_\pi \mathcal{G} \cap A \neq \varnothing$. Let $\mathcal{C}$ be a π-cover of A. Suppose that for any finite family $\mathcal{A} \subseteq \mathcal{C}$ and any $F \in \mathcal{F}$, $F \not\subseteq \bigcup_{C\in\mathcal{A}} C$. Then, $\{X \setminus C : C \in \mathcal{C}\}$ is a filter subbase on X which meets A. Let $\mathcal{G}$ be the filter generated by this subbase. It follows that $\mathcal{G}$ meets $\mathcal{F}$ and so $\mathrm{adh}_\pi \mathcal{G} \cap A \neq \varnothing$. Since $\mathcal{C}$ is a π-cover of A, we know that $A \subseteq \bigcup_{C\in\mathcal{C}} \mathrm{inh}_\pi C$. Notice that $\mathrm{adh}_\pi \mathcal{G} = X \setminus \bigcup_{C\in\mathcal{C}} \mathrm{inh}_\pi C$ and so it must be that $A \cap \mathrm{adh}_\pi \mathcal{G} = \varnothing$ a contradiction.

Now let $\mathcal{G}$ be a filter which meets $\mathcal{F}$ and suppose that $\mathrm{adh}_\pi \mathcal{G} \cap A = \varnothing$. Then, $\{X \setminus G : G \in \mathcal{G}\}$ is a π-cover of A since $\bigcup_{G \in \mathcal{G}} \mathrm{inh}_\pi(X \setminus G) = X \setminus \bigcap_{G \in \mathcal{G}} \mathrm{adh}_\pi \mathcal{G}$. By assumption, there exist $F \in \mathcal{F}$ and $G_1, ..., G_n \in \mathcal{G}$ such that $F \subseteq \bigcup_{i=1}^n X \setminus G_i$. Since $\bigcup_{i=1}^n X \setminus G_i = X \setminus \bigcap_{i=1}^n G_i$, this says that $F \cap G_1 \cap ... \cap G_n = \varnothing$. Since $G_1 \cap ... \cap G_n \in \mathcal{G}$, this contradicts that $\mathcal{F}$ meets $\mathcal{G}$. $\square$

The notion of covers has been studied before (see, for example, [10]) and it is well known that this definition of a pretopological cover is a specific case of the more general notion for convergence spaces. For pretopological spaces – and more generally for convergence spaces – the characterization of compactness in terms of covers in Proposition 3.1.14 is weaker than the notion of *cover-compactness* found in Definition 3.2.3. These characterizations coincide for topological spaces.

3.2 Perfect Maps

Much of the following can be seen as generalizing the results of [8] to pretopological spaces. Throughout this section, let (X, π) and (Y, σ) be pretopological spaces. The results below will be used in the construction of the θ-quotient convergence in Section 5.2. If $f : X \to Y$ if a function and $\mathcal{F}$ is a filter on Y, then $f^{\leftarrow}(\mathcal{F}) = \{f^{\leftarrow}[F] : F \in \mathcal{F}\}$ is a filter on X.

Definition 3.2.1. A function $f : (X, \pi) \to (Y, \sigma)$ is *perfect* if $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$ for each $y \in \lim_\sigma \mathcal{F}$.

In the case of topological spaces, this definition was shown by Whyburn [37] to be equivalent to the usual definition of a perfect function for topological spaces; that is, a function which is closed and has compact fibers.

Proposition 3.2.2. *A function $f:(X,\pi)\to(Y,\sigma)$ is perfect if and only if $f(\mathrm{adh}_\pi \mathcal{F}) \supseteq \mathrm{adh}_\sigma f(\mathcal{F})$ for each filter $\mathcal{F}$ on X.*

Proof. Suppose that f is perfect. Let $\mathcal{F}$ be a filter on X and let $y \in \mathrm{adh}_\sigma f(\mathcal{F})$. By way of contradiction, suppose that $f^{\leftarrow}(y)\cap \mathrm{adh}_\pi \mathcal{F} = \varnothing$. Since σ is a pretopology, $y \in \lim_\sigma \mathcal{V}_\sigma(y)$ and since f is perfect, it follows that $f^{\leftarrow}(\mathcal{V}_\sigma(y))$ is compact at $f^{\leftarrow}(y)$. Since $y \in \mathrm{adh}_\sigma f(\mathcal{F})$, $\mathcal{V}_\sigma(y)\# f(\mathcal{F})$. It follows that $f^{\leftarrow}(\mathcal{V}_\sigma(y))\#\mathcal{F}$. Thus, it must be that $\mathrm{adh}_\pi \mathcal{F}\cap f^{\leftarrow}(y)\neq\varnothing$, a contradiction. Hence, $y\in f(\mathrm{adh}_\pi \mathcal{F})$.

Conversely, suppose $\mathcal{F}$ is a filter on Y and $y\in\lim_\sigma \mathcal{F}$. Let $\mathcal{G}$ be a filter on X such that $\mathcal{G}\# f^{\leftarrow}(\mathcal{F})$. Then $f(\mathcal{G})\#\mathcal{F}$. Since $y\in\lim_\sigma\mathcal{F}$, it follows that $y\in \mathrm{adh}_\sigma f(\mathcal{G})\subseteq f[\mathrm{adh}_\pi \mathcal{G}]$. So, we can find $x\in \mathrm{adh}_\pi\mathcal{G}$ such that $f(x)=y$. In other words, $\mathrm{adh}_\pi \mathcal{G}\cap f^{\leftarrow}(y)\neq\varnothing$, and $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$. $\square$

To get a similar characterization to that of perfect functions between topological spaces for perfect functions between pretopological spaces we need the concept of *cover-compact* sets, a strengthening of compact sets. This characterization can be found in [9], but we feel it is worthwhile to lay out the details in this less technical setting.

Definition 3.2.3. Let (X,π) be a pretopological space and $A\subseteq X$. Then A is *cover-compact* if whenever $\mathcal{C}$ is a cover of A, there exist $C_1,\ldots,C_n\in\mathcal{C}$ such that $A\subseteq \mathrm{inh}_\pi\left(\bigcup_{i=1}^n C_i\right)$.

Proposition 3.2.4. *Let (X,π) be a pretopological space and $A\subseteq X$. The following are equivalent,*

1. *For any filter $\mathcal{F}$ on X, $\mathrm{adh}_\pi \mathcal{F}\cap A=\varnothing$ implies that there exists some $F\in\mathcal{F}$ such that $\mathrm{adh}_\pi F\cap A=\varnothing$,*
2. *A is cover-compact,*

3. $\mathrm{adh}_\pi \mathcal{F} \cap A = \varnothing$ *implies there exists* $V \subseteq X$ *and* $F \in \mathcal{F}$ *such that* $A \subseteq \mathrm{inh}_\pi V$ *and* $V \cap F = \varnothing$ *for any filter* $\mathcal{F}$ *on* X.

Proof. Let $\mathcal{C}$ be a π-cover of A. Suppose that no finite subcollection exists as needed by the definition of cover-compact. Then $\mathcal{F} = \{X \setminus (C_1 \cup \ldots \cup C_n) : C_i \in \mathcal{C}, i \in \mathbb{N}\}$ is a filter base on X. Note that $\mathrm{adh}_\pi \mathcal{F} \subseteq X \setminus \bigcup_{C \in \mathcal{C}} \mathrm{inh}_\pi C$ and as such $\mathrm{adh}_\pi \mathcal{F} \cap A = \varnothing$. By assumption, we can find $F \in \mathcal{F}$ such that $\mathrm{adh}_\pi F \cap A = \varnothing$. However, $F = X \setminus (C_1 \cup \ldots \cup C_n)$ for some $C_1, \ldots, C_n \in \mathcal{C}$, so we have that $A \subseteq \mathrm{inh}_\pi(C_1 \cup \ldots \cup C_n)$, a contradiction.

Suppose that $\mathcal{F}$ is a filter on X and $\mathrm{adh}_\pi \mathcal{F} \cap A = \varnothing$. Then, for each $x \in A$, fix $V_x \in \mathcal{V}_\pi(x)$ and $F_x \in \mathcal{F}$ such that $V_x \cap F_x = \varnothing$. Then $\{V_x : x \in A\}$ is a cover of A. By assumption, we can choose $x_1, \ldots, x_n \in A$ such that $A \subseteq \mathrm{inh}_\pi \left(\bigcup_{i=1}^n V_{x_i}\right)$. Therefore, $V = \bigcup_{i=1}^n V_{x_i} \in \mathcal{V}_\pi(A)$ and $V \cap (F_{x_1} \cap \ldots \cap F_{x_n}) = \varnothing$. Since $F_{x_1} \cap \ldots \cap F_{x_n} \in \mathcal{F}$, we have shown that (c) holds.

Lastly, let $\mathcal{F}$ be a filter on X such that $\mathrm{adh}_\pi \mathcal{F} \cap A = \varnothing$. By assumption, we can find $V \in \mathcal{V}_\pi(A)$ and $F \in \mathcal{F}$ such that $V \cap F = \varnothing$. For each $x \in A$, $V \in \mathcal{V}_\pi(x)$, so $x \notin \mathrm{adh}_\pi F$. It follows immediately that $A \cap \mathrm{adh}_\pi F \neq \varnothing$. □

It is useful to note that if $A \subseteq X$ is cover-compact, then $\mathrm{adh}_\pi A = A$.

Theorem 3.2.5. *Let* $f : (X, \pi) \to (Y, \sigma)$ *be a map between pretopological spaces satisfying (a)* $f[\mathrm{adh}_\pi A] \supseteq \mathrm{adh}_\sigma f[A]$ *for any* $A \subseteq X$ *and (b)* $f^{\leftarrow}(y)$ *is cover-compact for each* $y \in Y$. *Then* f *is perfect.*

Proof. Let $\mathcal{F}$ be a filter on Y which σ-converges to some $y \in Y$. Let $\mathcal{G}$ be a filter on X which meets $f^{\leftarrow}(\mathcal{F})$. Then $f(\mathcal{G})$ meets $\mathcal{F}$. Since $y \in \lim_\sigma \mathcal{F}$, $\mathcal{F}$ is compact at y. Therefore, $y \in \mathrm{adh}_\sigma f(\mathcal{G}) = \bigcap_{G \in \mathcal{G}} \mathrm{adh}_\sigma f[G]$. By assumption (a), for each $G \in \mathcal{G}$, $f[\mathrm{adh}_\pi G] \supseteq \mathrm{adh}_\sigma f[G]$. Therefore, $\mathrm{adh}_\pi G \cap f^{\leftarrow}(y) \neq \varnothing$ for each $G \in \mathcal{G}$. By assumption (b), $f^{\leftarrow}(y)$ is cover-compact, so $\mathrm{adh}_\pi \mathcal{G} \cap f^{\leftarrow}(y) \neq \varnothing$. In other words, $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$ and f is perfect. □

Theorem 3.2.6. *Let $f : (X,\pi) \to (Y,\sigma)$ be perfect and continuous. Then f satisfies (a) and (b) of 3.17.*

Proof. By Proposition 3.1.12(c) and Proposition 3.2.2, $f[\mathrm{adh}_\pi A] = \mathrm{adh}_\sigma f[A]$ for each $A \subseteq X$. Thus, a property stronger than (a) holds. To see that (b) holds, fix $y \in Y$ and let $\mathcal{F}$ be a filter on X such that $\mathrm{adh}_\pi \mathcal{F} \cap f^{\leftarrow}(y) = \varnothing$. By Proposition 3.14, $y \notin f[\mathrm{adh}_\pi \mathcal{F}] \supseteq \mathrm{adh}_\sigma f(\mathcal{F})$. Thus, we can find $V \in \mathcal{V}_\sigma(y)$ and $F \in \mathcal{F}$ such that $V \cap f(F) = \varnothing$. It follows that $f^{\leftarrow}[V] \cap F = \varnothing$. Since f is a continuous function, for each $x \in f^{\leftarrow}(y)$, fix $U_x \in \mathcal{V}_\pi(x)$ such that $f[U_x] \subseteq V$. Then $\bigcup_{x \in f^{\leftarrow}(y)} U_x \subseteq f^{\leftarrow}[V]$ and thus $\bigcup_{x \in f^{\leftarrow}(y)} U_x \cap F = \varnothing$. So, $\mathrm{adh}_\pi F \cap f^{\leftarrow}(y) = \varnothing$, as needed. $\square$

Corollary 3.2.7. *A continuous function $f : (X,\pi) \to (Y,\sigma)$ is perfect if and only if it satisfies (a) and (b) of 3.17.*

Chapter 4

θ-convergence

Throughout this chapter, let (X,τ) and (Y,σ) be a Hausdorff topological spaces and let θ_X and θ_Y be the pretopologies on X and Y described in Example 3.1.7. This chapter will be an in-depth study of the pretopological convergence θ. As such, we will not need to worry about whether the spaces under discussion are topological, pretopological, etc. Because of this we will often write simply X and Y, knowing that in this chapter they will always be Hausdorff topological spaces and that θ is the only non-topological convergence being considered.

4.1 H-closedness is Pretopological

In this section we characterize both H-closed spaces and H-sets in the terms of the pretopological convergence θ. The following theorem is well-known. The first part is due to Veličko [34] and the second can be found in [8]. As this fact is central to our project, we include the proof for the sake of completeness.

Theorem 4.1.1. *Let (X,τ) be a Hausdorff topological space and $A \subseteq X$. Then* *(a) X is H-closed if and only if* $\operatorname{adh}_\theta \mathcal{F} \neq \varnothing$ *for every filter $\mathcal{F}$ on X.*

(b) A is an H-set in X if and only if $\operatorname{adh}_\theta \mathcal{F} \cap A \neq \varnothing$ *for each filter $\mathcal{F}$ which meets A.*

Proof. We begin by proving (b). Notice that once (b) is proved, part (a) becomes a special case where $A = X$. Let A be an H-set in X and let $\mathcal{F}$ be a filter on X which meets A. Suppose that $\operatorname{adh}_\theta \mathcal{F} \cap A = \varnothing$. Then for each $x \in A$ fix $U_x \in \mathcal{N}_\tau(x)$ and $F_x \in \mathcal{F}$ such that $\operatorname{cl}_\tau U_x \cap F_x = \varnothing$. The family $\{U_x : x \in A\}$ is a τ-cover of A. Therefore, there exist $x_1, \ldots, x_n \in A$ such that $A \subseteq \operatorname{cl}_\tau(U_{x_1} \cup \ldots \cup U_{x_n})$. Let $F = F_{x_1} \cap \ldots \cap F_{x_n}$. Then $F \in \mathcal{F}$ and $F \cap A = \varnothing$, contradicting that $\mathcal{F}$ meets A.

Now, let $\mathcal{C}$ be a cover of A by open subsets of X. Suppose that for every finite collection $\mathcal{A} \subseteq \mathcal{C}$, $A \not\subseteq \operatorname{cl}_\tau \bigcup \mathcal{A}$. Then $\mathcal{F} = \{X \setminus \operatorname{cl}_\tau C : C \in \mathcal{C}\}$ is a filter subbase on X and $\langle \mathcal{F} \rangle$ meets A. By assumption, $\operatorname{adh}_\theta \mathcal{F} \cap A \neq \varnothing$. Since

$$\operatorname{adh}_\theta \mathcal{F} = X \setminus \bigcup_{C \in \mathcal{C}} \operatorname{inh}_\theta \operatorname{cl}_\tau C \subseteq X \setminus \bigcup \mathcal{C}$$

this implies that $A \not\subseteq \bigcup \mathcal{C}$, a contradiction. □

We now restate Theorem 4.1.1 using Definition 2.7.4. The following then characterizes both H-closed spaces and H-sets as pretopological notions.

Theorem 4.1.2. *Let X be a Hausdorff topological space and $A \subseteq X$.*

(a) X is H-closed if and only if (X, θ_X) is a compact pretopological space.

(b) A is an H-set in X if and only if A is a compact subset of (X, θ_X).

In particular, A is a compact subset of (X, θ) if and only if $(A, \theta|_A)$ is a compact pretopological space. Thus, 4.1.2(b) can be seen as a framing of the property of A being an H-set in X as property belonging to A without reference to the ambient space.

We will make use of the following lemma in the next section. Recall that a topological space X is Urysohn if distinct points have disjoint closed neighborhoods. A set A is θ-closed if $\mathrm{adh}_\theta A = A$.

Lemma 4.1.3. *Let (X,τ) be a Hausdorff topological space.*

(a) If $\{A_\alpha\}_{\alpha\in\Delta}$ is a family of θ-closed subsets of (X,τ), then $\bigcap_{\alpha\in\Delta} A_\alpha$ is θ-closed in X.

(b) If H is an H-set in X and $A \subseteq H$ is θ-closed, then A is an H-set.

(c) If X is Urysohn and H is an H-set in X, then H is θ-closed.

Proof. (a) Suppose $x \notin \bigcap_{\alpha\in\Delta} A_\alpha$. Then for some $\alpha \in \Delta$, $x \notin A_\alpha$. Since A_α is θ-closed, there exists $U \in \mathcal{N}_\tau(x)$ such that $\mathrm{cl}_\tau U \cap A_\alpha = \varnothing$. Therefore, $\mathrm{cl}_\tau U \cap \bigcap_{\alpha\in\Delta} A_\alpha = \varnothing$ and $\bigcap_{\alpha\in\Delta} A_\alpha$ is θ-closed.

(b) Let $\mathcal{C}$ be an open cover of A by open subsets of X. For each $x \in H\setminus A$, let $U_x \in \mathcal{N}_\tau(x)$ such that $\mathrm{cl}_\tau U_x \cap A = \varnothing$. Then, $\mathcal{C} \cup \{U_x : x \in H\setminus A\}$ is an open cover of H. Since H is an H-set, there exists $C_1,\ldots,C_n \in \mathcal{C}$ and $x_1,\ldots,x_m \in H\setminus A$ such that $H \subseteq \mathrm{cl}_\tau(C_1\cup\ldots\cup C_n) \cup \mathrm{cl}_\tau(U_{x_1}\cup\ldots U_{x_m})$. Since $\mathrm{cl}_\tau U_{x_i} \cap A = \varnothing$ for $i = 1,\ldots,m$, it follows that $A \subseteq \mathrm{cl}_\tau(C_1\cup\ldots\cup C_n)$, as needed.

(c) Suppose that $x \notin H$. Since X is Urysohn, for each $p \in H$ we can find $V_p \in \mathcal{N}_\tau(p)$ and $U_p \in \mathcal{N}_\tau(x)$ such that $\mathrm{cl}_\tau V_p \cap \mathrm{cl}_\tau U_p = \varnothing$. Since H is an H-set, we can find $p_1,\ldots,p_n \in P$ such that $H \subseteq \mathrm{cl}_\tau V_{p_1} \cup\ldots\cup \mathrm{cl}_\tau V_{p_n}$. Let $U = U_{p_1}\cap\ldots\cap U_{p_n}$. Notice that $U \in \mathcal{N}_\tau(x)$ and $\mathrm{cl}_\tau U \cap H = \varnothing$. Therefore, H is θ-closed. $\square$

Just as immediate as Theorem 4.1.2, but perhaps more interesting, is the case of H-sets in Urysohn spaces. As we saw in Theorem 2.4.3, for every Hausdorff topological space X, there exists an extremally disconnected, Tychonoff space EX and a perfect, irreducible,

θ-continuous map $k_X : EX \to X$. The following theorem is due to Vermeer [35] and makes use of this construction.

Theorem 4.1.4. *Let X be H-closed and Urysohn and let $A \subseteq X$. Then A is an H-set if and only if $k_X^{\leftarrow}[A]$ is a compact subset of EX.*

In the same paper, Vermeer gives an example of an H-closed non-Urysohn space X which has an H-set which is not the image under k_X of any compact subspace of EX. A more general phrasing of Theorem 4.1.4 of Vermeer is that if A is an H-set in an H-closed Urysohn space, then there exists a compact Hausdorff topological space K and a θ-continuous function $f : K \to X$ such that $f[K] = A$. Vermeer then asked if this was true for an H-set in any Hausdorff topological space; i.e. if X is a Hausdorff topological space and A is an H-set in X, does there exist a compact, Hausdorff topological space K and a θ-continuous function $f : K \to X$ such that $f[K] = A$? The answer, it turns out, is no. This was shown first by Bella and Yaschenko in [4]. Later, in [26], McNeill showed that it is in addition possible to construct a Urysohn space containing an H-set which is not the θ-continuous image of a compact, Hausdorff topological space. This makes the following observation interesting.

Theorem 4.1.5. *Let X be a Urysohn topological space. Then A is an H-set if and only if $(A, \theta|_A)$ is a compact, Hausdorff pretopological space, where $\theta|_A$ is the subconvergence on A inherited from (X, θ). In particular, if X is a Urysohn topological space and $A \subseteq X$ is an H-set, then there exists a compact, Hausdorff pretopological space (K, π) and a continuous function $f : (K, \pi) \to (X, \theta)$ such that $f[K] = A$.*

The question remains – if X is a Hausdorff topological space and A is an H-set in X, is there a compact, Hausdorff pretopological space (K, π) and a continuous function

$f:(K,\pi)\to(X,\theta)$ such that $f[K]=A$? More broadly, is there a pretopological version of the absolute?

Recall from Definition 2.3.9, a topology τ on a set X is *minimal Hausdorff* if any topology on X which is strictly coarser than τ fails to be Hausdorff. A topological space (X,τ) is *Katětov* if there exists a topology σ on X such that $\sigma\subseteq\tau$ and σ is minimal Hausdorff. In [14] the following is shown.

Theorem 4.1.6. *Let (X,τ) be a topological space. The following are equivalent*

(a) (X,τ) *is Katětov*

(b) There exists a discrete space D and an H-closed extension αD of D such that (X,τ) is homeomorphic to $\alpha D\setminus D$.

It can be seen that 4.1.6(b) implies that (X,τ) can be embedded as an H-set in some H-closed space. Another question which appears in [35] is this: If (X,τ) is a Hausdorff topological space which can be embedded as an H-set in some H-closed space (Y,σ), is (X,τ) Katětov? The following is Proposition 5.2 in [35] and gives a partial answer:

Theorem 4.1.7. *Let (X,τ) be a Hausdorff topological space. The following are equivalent:*

(a) τ contains a coarser compact Hausdorff topology on X

(b) There exists a discrete space D and an H-closed Urysohn extension αD of D such that (X,τ) is homeomorphic to $\alpha D\setminus D$

(c) (X,τ) can be embedded as an H-set in some H-closed Urysohn space.

It is important to note, however, that even if (X,τ) is a Urysohn topological space, it cannot necessarily be embedded in an H-closed Urysohn space. For example $\mathbb{Q}$ with the usual topology as a subspace of $\mathbb{R}$ cannot be embedded in an H-closed Urysohn space. Using pretopologies, we can remove the H-closed assumption in 4.1.7(c) and a similar result.

Theorem 4.1.8. *If (X,τ) is a topological space which can be embedded as an H-set in a Urysohn topological space, then there exists a compact Hausdorff pretopology π on X such that $\pi \leq \tau$.*

Proof. If (Y,σ) is a Urysohn topological space and $i:(X,\tau)\to(Y,\sigma)$ is an embedding such that X is an H-set in (Y,σ), let θ be the convergence on Y described in Example 3.1.7. Then, $\theta|_X \leq \tau$ and $\theta|_X$ is a compact Hausdorff pretopology. □

4.2 Relatively θ-compact Filters

Recall from Definition 2.7.4 that if (X,ξ) is a convergence space and $\mathcal{F}$ is a filter on X, then $\mathcal{F}$ is called *relatively ξ-compact* if $\mathcal{F}$ is compact at X. The relatively τ-compact filters for a topological space (X,τ) have been studied, mostly under the name *compactoid filters*; see [12], [25]. In this section, we investigate the properties of the relatively θ-compact filters on a topological space (X,τ). We begin by giving several characterizations of a relatively θ-compact filter.

Theorem 4.2.1. *Let (X,τ) be a topological space and let θ be the usual θ-convergence on X determined by τ. If $\mathcal{F}$ is a filter on X, then the following are equivalent:*

(a) $\mathcal{F}$ is relatively θ-compact

(b) If $\mathcal{U}$ is an ultrafilter on X and $\mathcal{U}\geq\mathcal{F}$, then $\mathrm{adh}_\theta\,\mathcal{U}=\varnothing$

(c) If $\mathcal{G}$ is a filter on X which meets $\mathcal{F}$, then $\mathrm{adh}_\theta\,\mathcal{G}\cap\mathrm{adh}_\theta\,\mathcal{F}\neq\varnothing$

(d) If $\mathcal{O}$ is an open filter on X which meets $\mathcal{F}$, then $\operatorname{adh}_\tau \mathcal{O} \neq \varnothing$

(e) For any open cover $\mathcal{C}$ of X, there exists a finite subfamily $\mathcal{A} \subseteq \mathcal{C}$ and there exists $F \in \mathcal{F}$ such that $F \subseteq \bigcup_{C \in \mathcal{A}} \operatorname{cl}_\tau C$

An immediate consequence of these characterizations is that if $\mathcal{F}$ is a filter on X and $A \in \mathcal{F}$ is H-bounded, then $\mathcal{F}$ is relatively θ-compact. However, this does not characterize the realtively θ-compact filters, as the following example shows.

Example 4.2.2. Let σ be the usual topology on $\mathbb{Q}$ and let $\mathcal{F} = \mathcal{N}_\sigma(0)$. Since $0 \in \lim_\sigma \mathcal{N}_\sigma(0)$, if $\mathcal{U}$ is an ultrafilter on X and $\mathcal{U} \geq \mathcal{N}_\sigma(0)$, then $\operatorname{adh}_\theta \mathcal{U} \neq \varnothing$. Therefore, $\mathcal{N}_\sigma(0)$ is relatively θ-compact.

Suppose that $A \in \mathcal{N}_\sigma(0)$. Then we can find $U \subseteq A$ a neighborhood of 0 which is both open and closed in $(\mathbb{Q}, \sigma)$. Since $\mathbb{Q}$ is regular, U is H-closed if and only if U is compact. Thus, U is not H-closed. If A is H-bounded and $B \subseteq A$ is such that $B = \operatorname{cl}_\tau \operatorname{int}_\tau B$, then B is H-closed. Thus, A is not H-bounded and $\mathcal{N}_\sigma(0)$ contains no H-bounded set.

If we set $\mathcal{F} = \langle A \rangle$, then Theorem 4.2.1 gives the well-known characterizations of H-bounded sets. However, 4.2.1(c) gives a characterization of H-bounded sets which heretofore has gone unnoticed.

Corollary 4.2.3. *Let (X, τ) be a Hausdorff topological space. A subset A of X is H-bounded if and only if* $\operatorname{adh}_\theta \mathcal{F} \cap \operatorname{adh}_\theta A \neq \varnothing$ *for any filter $\mathcal{F}$ on X which meets A.*

In addition, we can improve on 4.2.1(e) in the following way.

Proposition 4.2.4. *A filter $\mathcal{F}$ on X is relatively θ-compact if and only if for every τ-cover $\mathcal{C}$ of* $\operatorname{adh}_\theta \mathcal{F}$*, there exists a finite subfamily $\mathcal{A} \subseteq \mathcal{C}$ and there exists $F \in \mathcal{F}$ such that $F \subseteq \bigcup_{C \in \mathcal{A}} \operatorname{cl}_\tau C$.*

Proof. Suppose, by way on contradiction, that $\mathcal{C}$ is an open cover of $\mathrm{adh}_\theta \mathcal{F}$ such that for all $F \in \mathcal{F}$ and for each finite subcollection $\mathcal{A}$, $F \not\subseteq \bigcup_{C\in\mathcal{A}} \mathrm{cl}_\tau C$. Let $\mathcal{G} = \{X \setminus \mathrm{cl}_\tau C : C \in \mathcal{C}\}$. If $C_1, ..., C_n \in \mathcal{C}$, then $X \setminus \mathrm{cl}_\tau C_1 \cap ... \cap X \setminus \mathrm{cl}_\tau C_n = X \setminus \mathrm{cl}_\tau(C_1 \cup ... \cup C_n)$. By assumption, $\mathrm{cl}_\tau(C_1 \cup ... \cup C_n) \neq X$ and thus $\mathcal{G}$ is an open filter subbase on X. Similarly, by assumption it must be that $\mathcal{G}$ meets $\mathcal{F}$. Thus, $\mathrm{adh}_\tau \mathcal{G} \cap \mathrm{adh}_\theta \mathcal{F} \neq \varnothing$, by 4.2.1(d). However, $\mathrm{adh}_\tau \mathcal{G} = X \setminus (\bigcup_{C\in\mathcal{C}} \mathrm{int}_\tau \mathrm{cl}_\tau C) \subseteq X \setminus \mathrm{adh}_\theta \mathcal{F}$, a contradiction.

The converse follows easily since every open cover of X is also an open cover of $\mathrm{adh}_\theta \mathcal{F}$. □

Corollary 4.2.5. *A subset A of X is H-bounded if and only if for every τ-cover $\mathcal{C}$ of $\mathrm{adh}_\theta A$, there exists a finite subfamily $\mathcal{A} \subseteq \mathcal{C}$ such that $A \subseteq \bigcup_{C\in\mathcal{A}} \mathrm{cl}_\tau C$.*

Corollary 4.2.5 implies the following interesting fact.

Corollary 4.2.6. *Let $A \subseteq X$ be closed and H-bounded. Suppose that for each $x \in \mathrm{adh}_\theta A \setminus A$, there exists $y \in A$ and $U \in \mathcal{N}_\tau(x)$ such that for each $V \in \mathcal{N}_\tau(y)$ $(\mathrm{cl}_\tau U \setminus \mathrm{cl}_\tau V) \cap A$ is finite. Then A is an H-set.*

Proof. For each $x \in \mathrm{adh}_\theta A \setminus A$, fix $y_x \in A$ and $U_x \in \mathcal{N}_\tau(x)$ such that for every $V \in \mathcal{N}_\tau(y_x)$, $(\mathrm{cl}_\tau U_x \setminus \mathrm{cl}_\tau V) \cap A$ is finite. Let $\mathcal{C}_1 = \{C_y : y \in A\}$ be an open cover of A. Let $\mathcal{C}_2 = \{U_x : x \in \mathrm{adh}_\theta A \setminus A\}$. Then $\mathcal{C}_1 \cup \mathcal{C}_2$ is an open cover of $\mathrm{adh}_\theta A$. By Corollary 4.2.5, there exist $x_1, ..., x_n \in A$ and $x_{n+1}, ..., x_m \in \mathrm{adh}_\theta A \setminus A$ such that $A \subseteq \mathrm{cl}_\tau(C_{x_1} \cup ... \cup C_{x_n}) \cup \mathrm{cl}_\tau U_{x_{n+1}} \cup ... \cup \mathrm{cl}_\tau U_{x_m}$. Recall that by assumption, for $i = n+1, ..., m$, $(\mathrm{cl}_\tau U_{x_i} \setminus \mathrm{cl}_\tau C_{y_{x_i}}) \cap A$ is finite. Let $y_{x_i} = y_i$. Thus, if

$$a \in A \setminus (\mathrm{cl}_\tau(C_{x_1} \cup ... \cup C_{x_n}) \cup \mathrm{cl}_\tau C_{y_{n+1}} \cup ... \cup \mathrm{cl}_\tau U_{x_m})$$

then $a \in \mathrm{cl}_\tau U_{x_{n+1}} \setminus \mathrm{cl}_\tau C_{y_{n+1}}$. In other words, $a \in (\mathrm{cl}_\tau U_{x_{n+1}} \setminus \mathrm{cl}_\tau C_{y_{n+1}}) \cap A$ and $A \setminus (\mathrm{cl}_\tau(C_{x_1} \cup ... \cup C_{x_n}) \cup \mathrm{cl}_\tau C_{y_{n+1}} \cup ... \cup \mathrm{cl}_\tau U_{x_m})$ is finite. Inductively, $A \setminus \mathrm{cl}_\tau(C_{x_1} \cup ... \cup C_{x_n} \cup ... \cup C_{y_m})$ is finite. Thus, choose $C_{m+1},\dots,C_l \in \mathcal{C}_1$ covering the remaining finitely many points and we have

$$A \subseteq \left(\bigcup_{i=1}^{n} \mathrm{cl}_\tau C_{x_i}\right) \cup \left(\bigcup_{j=n+1}^{m} \mathrm{cl}_\tau C_{y_j}\right) \cup \left(\bigcup_{k=m+1}^{l} \mathrm{cl}_\tau C_k\right)$$

and A is an H-set, as needed. $\square$

Proposition 4.2.4 seems very close to saying: $\mathcal{F}$ is relatively θ-compact if and only if $\mathrm{adh}_\theta \mathcal{F}$ is H-bounded. The following example shows that this is not the case.

Example 4.2.7. For each $r \in \mathbb{N}$, let X_r be a copy of the topological space from Example 2.3.4. Let $A = \{(n,0) \in \mathbb{N} \times \mathbb{Z} : n \in \mathbb{N}\}$ and let $X = \left(\bigcup_{r\in\mathbb{N}} X_r\right)/A$ with the quotient topology. We will write $X = \{(n,m,r) : n,r \in \mathbb{N}, m \in \mathbb{Z}\} \cup \mathbb{N} \cup \{+\infty_r, -\infty_r : r \in \mathbb{N}\}$. Let $B = \{+\infty_1\} \cup \mathbb{N}$. Notice that B is an H-set. However, $\mathrm{adh}_\theta B = B \cup \{+\infty_r, -\infty_r : r \in \mathbb{N}\}$ is not even H-bounded in X. Thus, if we let $\mathcal{F} = \langle B \rangle$, then $\mathcal{F}$ is realtively θ-compact, but $\mathrm{adh}_\theta \mathcal{F}$ is not H-bounded.

Proposition 4.2.8. *A filter $\mathcal{F}$ is relatively θ-compact if and only if the filter $\{\mathrm{cl}_\tau F : F \in \mathcal{F}\}$ is relatively θ-compact.*

Proof. Let $\mathcal{C}$ be an open cover of X. If $\mathcal{F}$ is relatively θ-compact, there exists a finite subfamily $\mathcal{A}$ and $G \in \mathcal{F}$ such that $G \subseteq \bigcup_{C\in\mathcal{A}} \mathrm{cl}_\tau C$. Thus, $\mathrm{cl}_\tau G \subseteq \bigcup_{C\in\mathcal{A}} \mathrm{cl}_\tau C$. Since $\mathrm{cl}_\tau G \in \{\mathrm{cl}_\tau F : F \in \mathcal{F}\}$, this filter is relatively θ-compact. The converse follows since $\mathcal{F} \geq \{\mathrm{cl}_\tau F : F \in \mathcal{F}\}$. $\square$

Proposition 4.2.9. *If X is a Urysohn topological space and $\mathcal{F}$ is a filter subbase such that each $F \in \mathcal{F}$ is an H-set, then $\bigcap \mathcal{F} \neq \varnothing$. Moreover, $\bigcap \mathcal{F}$ is an H-set in X.*

Proof. If each $F \in \mathcal{F}$ is an H-set, then clearly $\mathcal{F}$ is relatively θ-compact and thus $\mathrm{adh}_\theta \mathcal{F} \neq \varnothing$. By Lemma 4.1.3(c), for each $F \in \mathcal{F}$, $\mathrm{adh}_\theta F = F$ and thus $\mathrm{adh}_\theta \mathcal{F} = \bigcap \mathcal{F} \neq \varnothing$. Since each $F \in \mathcal{F}$ is θ-closed, $\bigcap \mathcal{F}$ is θ-closed. By Lemma 4.1.3(b), since $\bigcap \mathcal{F}$ is a θ-closed subset of $F \in \mathcal{F}$ and F is an H-set, $\bigcap \mathcal{F}$ is an H-set as well. $\square$

The next example shows that if we drop the Urysohn assumption in Propositon 4.2.9, the statement is no longer true.

Example 4.2.10. Let X be a C-compact, non-compact Hausdorff space as in Example 2 of [36]. As X is not compact, there exists a family $\mathcal{F}$ of closed subsets of X with the finite intersection property such that $\bigcap \mathcal{F} = \varnothing$. Since X is C-compact, each $F \in \mathcal{F}$ is an H-set. Thus Corollary 4.2.9 is not in general true for Hausdorff spaces.

In [28] the following is proved: $A \subseteq X$ is H-bounded if and only if $\mathrm{cl}_\tau A$ is closed in every Hausdorff topological space in which X is embedded. This inspires the last result of this subsection.

Theorem 4.2.11. *A filter base $\mathcal{F}$ on X is relatively θ-compact if and only if* $\mathrm{adh}_\sigma \mathcal{F} \subseteq X$ *for any Hausdorff topological space (Y, σ) in which X is embedded.*

Proof. Let Y be a Hausdorff topological space in which X is embedded and let $p \in Y \setminus X$. Then $O^p = \{U \cap X : U \in \mathcal{N}_\sigma(p)\}$ is a free open filter base on X. It follows that O^p does not meet $\mathcal{F}$ since $\mathcal{F}$ is relatively θ-compact. Thus, $p \notin \mathrm{adh}_\sigma \mathcal{F}$. Since $p \in Y \setminus X$ was arbitrarily chosen, it follows that $\mathrm{adh}_\sigma \mathcal{F} = \mathrm{adh}_\tau \mathcal{F} \subseteq X$.

Suppose that $\mathcal{F}$ is not a relatively θ-compact filter base on X. Then, there is an open filter base $\mathcal{O}$ on X such that $\mathcal{O}$ meets $\mathcal{F}$ and $\mathrm{adh}_\tau \mathcal{O} = \varnothing$. Let $Y = X \cup \{p\}$ where $\mathcal{O}$ is a neighborhood base at p. Since $\mathcal{O}$ is a free open filter base on X, Y is Hausdorff. Also, notice that since $\mathcal{O}$ meets $\mathcal{F}$, $p \in \mathrm{adh}_\sigma \mathcal{F}$. Thus, $\mathrm{adh}_\sigma \mathcal{F} \not\subseteq X$, as needed. $\square$

4.3 θ-subcontinuity

In [33], R. Smithson extends the well-known concept of subcontinuity to multifunctions. The definition of a subcontinuous multifunction is given in terms of a convergence. Because of this, it is straight-forward to extend the definition to the convergence θ. In [21], J.E. Joseph used both subcontinuity and θ-subcontinuity to characterize properties of the graphs of multifunctions. The purpose of this section is to define and characterize θ-subcontinuous multifunctions. In the next section an application of subcontinuity will be given. Throughout this section, let (X,τ) and (Y,σ) be Hausdorff topological spaces and let θ_X and θ_Y be the θ-convergences on X and Y with respect to their topologies.

Definition 4.3.1. A multifunction $F : X \multimap Y$ is *θ-subcontinuous at x* if $F(\mathcal{F})$ is relatively θ_Y-compact for each filter $\mathcal{F}$ on X with $x \in \lim_{\theta_X} \mathcal{F}$. We say that F is *θ-subcontinuous* if F is θ-subcontinuous at each $x \in X$.

Theorem 4.3.2. *Let $F : X \multimap Y$ be a multifunction. The following are equivalent:*

(a) F is θ-subcontinuous,

(b) $F(\mathcal{V}_{\theta_X}(x))$ is relatively θ_Y-compact for each $x \in X$,

(c) If $\mathcal{U}$ is an ultrafilter on X and $\lim_{\theta_X} \mathcal{U} \neq \varnothing$, then $F(\mathcal{U})$ is relatively θ_Y-compact.

(d) $F(\mathcal{F})$ is relatively θ_Y-compact for each filter $\mathcal{F}$ on X which is relatively θ_X compact.

Proof. Since θ_X is a pretopology, $x \in \lim_{\theta_X} \mathcal{V}_{\theta_X}(x)$. Since F is θ-subcontinuous, it follows that $F(\mathcal{V}_{\theta_X}(x))$ is relatively θ_Y-compact.

Suppose that $\mathcal{U}$ is an ultrafilter on X and $x \in \lim_{\theta_X} \mathcal{U}$. It follows that $F(\mathcal{V}_{\theta_X}(x)) \leq F(\mathcal{U})$. By assumption, $F(\mathcal{V}_{\theta_X}(x))$ is relatively θ_Y-compact. Therefore, if $\mathcal{W}$ is an ultrafilter on Y

and $F(\mathcal{U}) \leq \mathcal{W}$, then $F(\mathcal{V}_{\theta_X}(x)) \leq \mathcal{W}$. By the relative θ_Y-compactness of $F(\mathcal{V}_{\theta_X}(x))$, it follows that $\mathrm{adh}_{\theta_Y} \mathcal{W} \neq \varnothing$. Hence, $F(\mathcal{U})$ is relatively θ_Y-compact.

Suppose that $\mathcal{F}$ is relatively θ_X-compact and let $\mathcal{W}$ be an ultrafilter on Y such that $F(\mathcal{F}) \leq \mathcal{W}$. Let $F^+(\mathcal{W})$ be the filter on X generated by $\{F^+(W) : W \in \mathcal{W}\}$, where $F^+(W) = \{x \in X : F(x) \cap W \neq \varnothing\}$. It follows that $F^+(\mathcal{W})$ meets $\mathcal{F}$. Since $\mathcal{F}$ is relatively θ_X-compact, $\mathrm{adh}_{\theta_X} F^+(\mathcal{W}) \neq \varnothing$. Fix $x \in \mathrm{adh}_{\theta_X} F^+(\mathcal{W})$ and fix an ultrafilter $\mathcal{U}$ on X such that $x \in \lim_{\theta_X} \mathcal{U}$ and $\mathcal{U} \geq F^+(\mathcal{W})$. Then $F(\mathcal{U})$ is relatively θ_Y-compact by assumption. Since $F(\mathcal{U})$ meets $\mathcal{W}$, we have that $\mathrm{adh}_{\theta_Y} \mathcal{W} \neq \varnothing$, as needed.

If $\mathcal{F}$ is a filter on X such that $\lim_{\theta_X} \mathcal{F} \neq \varnothing$, then $\mathcal{F}$ is relatively θ_X-compact. By assumption, $F(\mathcal{F})$ is relatively θ_Y-compact and so $\mathrm{adh}_{\theta_Y} F(\mathcal{F}) \neq \varnothing$, as needed. $\square$

Notice that since θ-subcontinuous multifunction preserve relatively θ-compact filters, if $A \subseteq X$ is H-bounded and $F : X \multimap Y$ is θ-subcontinuous, then $F(A)$ is H-bounded in Y.

The following fact related θ-subcontinuity to θ-upper semicontinuity.

Proposition 4.3.3. *Let (X,τ) and (Y,σ) be topological spaces and let $F : X \multimap Y$ be θ-upper semicontinuous such that $F(x)$ is compact for each $x \in X$. Then F is θ-subcontinuous.*

Proof. Fix $\mathcal{C}$ a σ-cover of Y. Since $F(x)$ is compact for each $x \in X$, we can find a finite subfamily $\mathcal{A}_x \subseteq \mathcal{C}$ such that $F(x) \subseteq \bigcup \mathcal{A}$. Since F is θ-usc, there exists $U \in \mathcal{N}_\tau(x)$ such that $F[\mathrm{cl}_\tau U] \subseteq \mathrm{cl}_\sigma \bigcup \mathcal{A}_x$. By Proposition 3.1.14, it follows that $\mathcal{V}_{\theta_X}(x)$ is relatively θ_Y-compact for each $x \in X$. $\square$

4.4 Čech g-spaces

In [20], subcontinuous multifunctions are used to give a characterization of Čech-complete topological spaces. Recall that Čech-complete spaces are by assumption Tychonoff. In [7], the so-called Čech g-spaces and Čech f-spaces are defined, which are analoguous to the Čech complete spaces for non-regular spaces. In [27] Čech g-spaces are used to give a characterization of countable Katětov spaces. In this section we will use θ-subcontinuous multifunctions to give a characterization of Čech g-spaces. As a corollary, we will have an additional characterization of countable Katětov spaces to add to the results of [27].

Definition 4.4.1. A topological space (X,τ) is called a *Čech g-space* if there exists a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of τ-covers of X such that whenever $\mathcal{O}$ is an open filter base on X and for each $n \in \mathbb{N}$, there exist $U_n \in \mathcal{O}$ and $C_n \in \mathcal{C}_n$ such that $U_n \subseteq C_n$, it follows that $\operatorname{adh}_\tau \mathcal{O} \neq \varnothing$.

The sequence of τ-covers in Definition 4.4.1 is called a *Čech g-sequence*. Next, we define what it means for a space X to have the *strong Čech g-property* which seems as though it is more suited to the θ convergence on (X,τ).

Definition 4.4.2. A topological space (X,τ) has the *strong Čech g-property* if there exists a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of τ-covers for X such that whenever $\mathcal{O}$ is an open filter base on X and for each $n \in \mathbb{N}$ there exists $U_n \in \mathcal{O}$ and $C_n \in \mathcal{C}_n$ such that $\operatorname{cl}_\tau U_n \subseteq \operatorname{cl}_\tau C_n$, it follows that $\operatorname{adh}_\tau \mathcal{O} \neq \varnothing$. In this case we call the sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ a *strong Čech g-sequence*.

Definition 4.4.3. A topological space (X,τ) is *regularly embedded* in (Y,σ) if whenever $x \in X$ and $X \subseteq V \in \sigma$, there exists $U \in \tau$ such that $x \in U$ and $\operatorname{cl}_\sigma U \subseteq V$.

Lemma 4.4.4. *Suppose that (Y,σ) has the strong Čech g-property. If (X,τ) is a regularly embedded, dense, G_δ subspace of Y, then X also has the strong Čech g-property.*

Proof. Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a Čech g-sequence for Y. Since X is a G_δ subspace of Y, fix $\{G_n : n \in \mathbb{N}\} \subseteq \sigma$ such that $X = \bigcap_{n\in\mathbb{N}} G_n$. For each $n \in \mathbb{N}$, we define $\mathcal{B}_n$, a family of open subsets of X as follows: $B \in \mathcal{B}_n$ if and only if $B = T \cap X$ for some $T \in \sigma$ such that

$\mathrm{cl}_\sigma T \subseteq G_n$ and $\mathrm{cl}_\sigma T \subseteq \mathrm{cl}_\sigma C_n$ for some $C_n \in \mathcal{C}_n$. Notice that each $\mathcal{B}_n$ is a τ-cover of X. To see this, if $x \in X$, then there is some $C_n \in \mathcal{C}_n$ such that $x \in C_n$. Since X is regularly embedded in Y, we can find $x \in V \in \sigma$ such that $\mathrm{cl}_\sigma V \subseteq U_n$. Thus $x \in (V \cap X) \in \mathcal{B}_n$.

We now claim that $(\mathcal{B}_n)_{n\in\mathbb{N}}$ is a strong Čech g-sequence for X. Let $\mathcal{O}$ be an open filter base on X such that for each $n \in \mathbb{N}$, there exists $U_n \in \mathcal{O}$ and $B_n \in \mathcal{B}_n$ such that $\mathrm{cl}_\tau U_n \subseteq \mathrm{cl}_\tau B_n$. Let $\mathcal{P} = \{V \in \sigma : V \cap X \in \mathcal{O}\}$, an open filter base on Y. It can be seen that for each $n \in \mathbb{N}$ there exists $V_n \in \mathcal{P}$ and $B_n \in \mathcal{B}_n$ such that $\mathrm{cl}_\tau(V \cap X) \subseteq \mathrm{cl}_\tau B_n$. By construction, $B_n = T_n \cap X$ for some $T_n \in \sigma$ such that $\mathrm{cl}_\sigma T_n \subseteq \mathrm{cl}_\sigma C_n$ for some $C_n \in \mathcal{C}_n$. Since X is dense in Y and V_n is open in Y, $\mathrm{cl}_\sigma(V_n \cap X) = \mathrm{cl}_\sigma V_n$. Then the following string of set inequalities holds:

$$
\begin{aligned}
\mathrm{cl}_\sigma V_n &= \mathrm{cl}_\sigma(V_n \cap X) \\
&\subseteq \mathrm{cl}_\sigma(\mathrm{cl}_\tau(V_n \cap X)) \\
&\subseteq cl_\sigma(cl_\tau B_n) \\
&= \mathrm{cl}_\sigma(\mathrm{cl}_\tau(T_n \cap X)) \\
&= \mathrm{cl}_\sigma(\mathrm{cl}_\sigma(T_n \cap X) \cap X) \\
&= \mathrm{cl}_\sigma((\mathrm{cl}_\sigma T_n) \cap X) \\
&\subseteq \mathrm{cl}_\sigma T_n \cap \mathrm{cl}_\sigma X \\
&= \mathrm{cl}_\sigma T_n \\
&\subseteq \mathrm{cl}_\sigma C_n
\end{aligned}
$$

Since $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is a strong Čech g-sequence for Y, $\mathrm{adh}_\sigma \mathcal{P} \neq \varnothing$. Notice that for any $V \in \mathcal{P}$ and any $n \in \mathbb{N}$, there is some $T_n \in \sigma$ such that $\mathrm{cl}_\sigma V_n \subseteq \mathrm{cl}_\sigma T_n \subseteq G_n$. Thus, $\bigcap\{\mathrm{cl}_\sigma V : V \in \mathcal{P}\} \subseteq \bigcap_{n\in\mathbb{N}} G_n = X$. Since X is dense in Y, notice that $\mathrm{adh}_\tau \mathcal{O} = \bigcap\{\mathrm{cl}_\tau(V \cap X) :$

$V \in \mathcal{P}\} = \bigcap\{\mathrm{cl}_\sigma V : V \in \mathcal{P}\} \cap X = \mathrm{adh}_\sigma \mathcal{P}$. Thus, $\mathrm{adh}_\tau \mathcal{O} \neq \varnothing$ and X has the strong Čech g-property. □

On its face, having the strong Čech g-property is stronger than being a Čech g-space. However, the next theorem provides several equivalent characterizations of Čech g-spaces, the strong Čech g-property among them.

Theorem 4.4.5. *Let (X, τ) be a Hausdorff topological space. The following are equivalent.*

(a) X is a Čech g-space,

(b) X is a G_δ set in σX,

(c) X has the strong Čech g-property,

(d) there exists a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of τ-covers of X such that whenever $\mathcal{O}$ is an open filter base on X such that for each $n \in \mathbb{N}$, there exists $U_n \in \mathcal{O}$ and $C_n \in \mathcal{C}_n$ such that $U_n \subseteq \mathrm{cl}_\tau C_n$, it follows that $\mathrm{adh}_\tau \mathcal{O} \neq \varnothing$,

(e) there exists a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of τ-covers of X such that whenever $\mathcal{F}$ is a filter base on X such that for each $n \in \mathbb{N}$, there exists $F_n \in \mathcal{F}$ and $C_n \in \mathcal{C}_n$ such that $F_n \subseteq \mathrm{cl}_\tau C_n$, it follows that $\mathrm{adh}_\theta \mathcal{F} \neq \varnothing$,

Proof. If X is a Čech g-space, it is well-known that X is a G_δ set in σX. See [7] for details.

Every H-closed space immediately has the strong Čech g-property. Since X is regularly embedded in σX and by assumption X is a G_δ subseteq of σX, we know that X has the strong Čech g-property by Lemma 4.4.4.

Since $U_n \subseteq \mathrm{cl}_\tau C_n$ implies that $\mathrm{cl}_\tau U_n \subseteq \mathrm{cl}_\tau C_n$, $(c \to d)$ is immediate.

Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be as hypothesized in (d). And let $\mathcal{F}$ be a filter base on X such that for each $n \in \mathbb{N}$ there exists $F_n \in \mathcal{F}$ and $C_n \in \mathcal{C}_n$ such that $F_n \subseteq \mathrm{cl}_\tau C_n$. Let $\mathcal{O} = \{U \in \tau : \exists F \in \mathcal{F}. F \subseteq \mathrm{cl}_\tau U\}$. Then, $\mathcal{O}$ is an open filter base on X and by assumption for each $n \in \mathbb{N}$ there exists $C_n \in \mathcal{O} \cap \mathcal{C}_n$. Thus, $\mathcal{O}$ satisfies the hypotheses of (d) and $\mathrm{adh}_\tau \mathcal{O} \neq \varnothing$. It is straight-forward to check that $\mathrm{adh}_\tau \mathcal{O} = \mathrm{adh}_\theta \mathcal{F}$ and thus $\mathrm{adh}_\theta \mathcal{F} \neq \varnothing$, as needed.

Lastly, let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be the sequence of open covers hypothesized by (e) and let $\mathcal{O}$ be an open filter base such that for each $n \in \mathbb{N}$ there exists $U_n \in \mathcal{O}$ and $C_n \in \mathcal{C}_n$ such that $U_n \subseteq C_n$. Notice that this implies the condition needed by (e) for $\mathrm{adh}_\theta \mathcal{O} \neq \varnothing$. Since $\mathrm{adh}_\tau \mathcal{O} = \mathrm{adh}_\theta \mathcal{O}$, we know that $\mathrm{adh}_\tau \mathcal{O} \neq \varnothing$. As such, X is a Čech g-space. $\square$

Next, we adapt Proposition 2.12 in [16] for Čech g-spaces.

Lemma 4.4.6. *If $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is a Čech g-sequence for (X,τ) and $\mathcal{F}$ is a filter base on X satisfying: for each $n \in \mathbb{N}$, there exists $F_n \in \mathcal{F}$ and a finite subfamily $\mathcal{A}_n \subseteq \mathcal{C}_n$ such that $F \subseteq \bigcup_{C\in\mathcal{A}_n} \mathrm{cl}_\tau C$, then $\mathrm{adh}_\theta \mathcal{F} \neq \varnothing$.*

Proof. Suppose that $\mathcal{F}$ is as described. Let $\mathcal{G}$ be an ultrafilter on X containing $\mathcal{F}$. Then $\mathcal{G}$ also satisfies the assumption: for each $n \in \mathbb{N}$, there exist $G_n \in \mathcal{G}$ and $\mathcal{A}_n \subseteq \mathcal{C}_n$ such that $\mathcal{A}_n$ is finite and $G_n \subseteq \bigcup_{C\in\mathcal{A}_n} \mathrm{cl}_\tau C$. However, since $\mathcal{G}$ is an ultrafilter, if tollows that for some $C \in \mathcal{A}_n$, $\mathrm{cl}_\tau C \in \mathcal{G}$. Therefore, because $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is a Čech g-sequence, it follows that $\mathrm{adh}_\theta \mathcal{G} \neq \varnothing$. Since $\mathrm{adh}_\theta \mathcal{G} \subseteq \mathrm{adh}_\theta \mathcal{F}$, we have that $\mathrm{adh}_\theta \mathcal{F} \neq \varnothing$, as needed. $\square$

If $F : X \multimap Y$ is a multifunction and Y is a Čech g-space, then we can weaken the criteria for F to be θ-subcontinuous.

Proposition 4.4.7. *Let $F : X \multimap Y$ be a multifunction and fix $x \in X$. Suppose that Y is a Čech g-space and let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a Čech g-sequence for Y. The following are equivalent.*

(a) F is θ-subcontinuous at x,

(b) For each $n \in \mathbb{N}$, there exists $U \in \mathcal{N}_\tau(x)$ and a finite subfamily $\mathcal{A}_n \subseteq \mathcal{C}_n$ such that $F[\mathrm{cl}_\tau U] \subseteq \bigcup_{C \in \mathcal{A}_n} \mathrm{cl}_\tau C$.

Proof. It is immediate that (a) implies (b). To see that (b) implies (a), we proceed by contradiction. Then there is an σ-cover $\mathcal{C}$ of Y such that for every $U \in \mathcal{N}_\tau(x)$ and each finite $\mathcal{A} \subseteq \mathcal{C}$, $F[\mathrm{cl}_\tau U] \subseteq \mathrm{cl}_\sigma \bigcup \mathcal{A} \neq \varnothing$. Therefore,

$$\mathcal{F} = \{F[\mathrm{cl}_\tau U \setminus \mathrm{cl}_\sigma \bigcup \mathcal{A}] : U \in \mathcal{N}_\tau(x) \text{ and } \mathcal{A} \subseteq \mathcal{C} \text{ is finite}\}$$

is a filter base on Y. By assumption, for each $n \in \mathbb{N}$ there exists $U_n \in \mathcal{N}_\tau(x)$ and a finite subfamily $\mathcal{A}_n \subseteq \mathcal{C}_n$ such that $F[\mathrm{cl}_\tau U_n] \subseteq \mathrm{cl}_\sigma \bigcup \mathcal{A}_n$. Thus, $\mathrm{adh}_\theta \mathcal{F} \neq \varnothing$. However, $\mathrm{adh}_\theta \mathcal{F} \subseteq Y \setminus \bigcup\{\mathcal{A} : \mathcal{A} \text{ is a finite subfamily of } \mathcal{C}\} = \varnothing$, a contradiction. □

Corollary 4.4.8. *Let Y be a Čech g-space and $F : X \multimap Y$ a multifunction. Then, the set of points of X at which F is θ-subcontinuous is a G_δ subset of X.*

Proof. Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a Čech g-sequence for Y. For each $n \in \mathbb{N}$, let

$$V_n = \{x \in X : \exists U \in \mathcal{N}_\tau(x) \exists \text{ a finite subfamily } \mathcal{A} \subseteq \mathcal{C} \text{ such that } F[\mathrm{cl}_\tau U] \subseteq \mathrm{cl}_\sigma \bigcup \mathcal{A}\}$$

Notice that each V_n is open in X and by Proposition 4.4.7, $\{x \in X : F \text{ is } \theta-\text{subcontinuous at } x\} = \bigcap_{n\in\mathbb{N}} V_n$. □

In fact, the converse to Corollary 4.4.8 holds as well.

Proposition 4.4.9. *If (Y,ρ) is a Hausdorff topological space such that for every Hausdorff topological space X and any multifunction $F : X \multimap Y$ the set of points at which F is θ-subcontinuous is a G_δ set in X, then Y is a Čech g-space.*

Proof. Fix a point $y_0 \in Y$. Let $F : \sigma Y \multimap Y$ be defined so that for each $\mathcal{O} \in \sigma Y \setminus Y$, $F(\mathcal{O}) = \{y_0\}$ and for $y \in Y$, $F(y) = \{y, y_0\}$. Let $\theta SC(F)$ denote the set of points of θ-subcontinuity of F. We claim that $\theta SC(F) = Y$. First, notice that F is compact-valued. By Proposition 4.3.3, if we show that F is θ-usc at each $y \in Y$, then it follows that F is θ-subcontinuous at y. Fix $V \in \rho$ such that $\{y, y_0\} \subseteq V$. Then, oV is open in σY and $y \in oV$. Since $F[\mathrm{cl}_{\sigma_\rho} oV] = F[oV \cup \mathrm{cl}_\rho V] \subseteq \mathrm{cl}_\rho V$, F is θ-usc at $y \in Y$.

We now show that F is not θ-subcontinuous at $\mathcal{O} \in \sigma Y \setminus Y$. If $\mathcal{O} \in \sigma Y \setminus Y$, then $\mathrm{adh}_\rho \mathcal{O} = \varnothing$. Notice that $\mathcal{O} \geq F(\mathcal{V}_{\theta_{\sigma Y}}(\mathcal{O}))$ since $F(\mathcal{V}_{\theta_{\sigma Y}}(\mathcal{O})) \supseteq \{F(oU) : U \in \mathcal{O}\} = \{\mathrm{cl}_\rho U \cup \{y_o\} : U \in \mathcal{O}\}$. Since $\mathrm{adh}_\rho \mathcal{O} = \mathrm{adh}_{\theta_Y} \mathcal{O} = \varnothing$, $F(\mathcal{V}_{\theta_{\sigma Y}}(\mathcal{O}))$ is not relatively θ_Y-compact. $\square$

Corollary 4.4.8 and Proposition 4.4.9 combine to generalize results in [20] to show the following.

Theorem 4.4.10. *A Hausdorff topological space Y is a Čech g-space if and only if whenever X is a Hausdorff topological space and $F : X \multimap Y$ is a multifunction, the set of points at which F is θ-subcontinuous is a G_δ set in X.*

Using results from [27] we get the following characterization of countable Katětov spaces.

Theorem 4.4.11. *Let (X, τ) be a countable Hausdorff topological space. The following are equivalent.*

(a) X is a Čech g-space

(b) X is Katětov and X_s is first-countable.

(c) Whenever Y is a Hausdorff topological space and $F : Y \multimap X$ is a multifunction, the set of points at which F is θ-subcontinuous is a G_δ set in Y.

Chapter 5

PHC Spaces

5.1 Definition and Basic Properties

In this section we will define a variation of H-closedness for pretopological spaces. After establishing some basic facts about the so-called *PHC spaces*, we will describe a method for constructing PHC pretopologies and PHC extensions. Much of this work is to appear in [31]. The following definition appears in [11].

Definition 5.1.1. Let (X,π) be a pretopological space. The *partial regularization* $r\pi$ of π is the pretopology determined by the vicinity filters $\mathcal{V}_{r\pi}(x) = \{\mathrm{adh}_\pi U : U \in \mathcal{V}_\pi(x)\}$.

Notice that if (X,τ) is a topological space, then $r\tau$ is the θ-convergence on X explored in Chapter 4. Thus, a Hausdorff topological space (X,τ) is H-closed if and only if $(X,r\tau)$ is compact. This inspires the following definition, aiming to generalize the notion of H-closed spaces to pretopological spaces.

Definition 5.1.2. Let (X,π) be a Hausdorff pretopological space. The pretopology π is *PHC (pretopologically H-closed)* if $(X,r\pi)$ is compact. Without the assumption of Hausdorff, we will use the term *quasi PHC*

For $n \in \mathbb{N}$ and $A \subseteq X$, let $\operatorname{inh}_\pi^n A$ be the n^{th} iteration of the inherence operator on A. Given a filter $\mathcal{F}$ on a pretopological space (X,π) let

$$i_\pi \mathcal{F} = \{F : \operatorname{inh}_\pi F \in \mathcal{F}\}.$$

Inductively, define

$$i_\pi^n \mathcal{F} = \{H : \operatorname{inh}_\pi^n H \in \mathcal{F}\},$$

and finally

$$i_\pi^\omega \mathcal{F} = \bigcap_{n\in\mathbb{N}} i_\pi^n \mathcal{F}.$$

We use the convention $i_\pi^0 \mathcal{F} = \mathcal{F}$ and $i_\pi^1 \mathcal{F} = i_\pi \mathcal{F}$. Notice then that $i_\pi^n \mathcal{F} = i_\pi \left(i_\pi^{n-1}\mathcal{F}\right)$ for each $n \in \mathbb{N}$.

Lemma 5.1.3. *Let (X,π) be a pretopological space and let $\mathcal{F}$ be a filter on X such that $i_\pi \mathcal{F} = \mathcal{F}$. Then* $\operatorname{adh}_\pi \mathcal{F} = \operatorname{adh}_{r\pi} \mathcal{F}$.

Proof. To begin, since $r\pi \leq \pi$, $\operatorname{adh}_\pi \mathcal{F} \subseteq \operatorname{adh}_{r\pi} \mathcal{F}$. Now, $x \notin \operatorname{adh}_\pi \mathcal{F}$ if and only if we can find $F \in \mathcal{F}$ and $U \in \mathcal{V}_\pi(x)$ such that $U \cap F = \varnothing$. Since $U \cap F = \varnothing$, if $y \in \operatorname{inh}_\pi F$, then $y \notin \operatorname{adh}_\pi U$. In other words, $\operatorname{adh}_\pi U \cap \operatorname{inh}_\pi F = \varnothing$. Since $\mathcal{F} = i_\pi \mathcal{F}$, $\operatorname{inh}_\pi F \in \mathcal{F}$ and by definition $x \notin \operatorname{adh}_{r\pi} \mathcal{F}$, as needed. □

Lemma 5.1.4. *Let (X,π) be a pretopological space and let $\mathcal{F}$ be a filter on X. Then* $\operatorname{adh}_{r\pi} i_\pi^n \mathcal{F} = \operatorname{adh}_\pi i_\pi^{n+1} \mathcal{F}$ *for each $n \in \mathbb{N}$.*

Proof. We begin by showing the lemma holds for $n = 0$. Recall that $i_\pi^0 \mathcal{F} = \mathcal{F}$. Suppose that $x \notin \operatorname{adh}_{r\pi} \mathcal{F}$. Then there exists $U \in \mathcal{V}_\pi(x)$ and there exists $F \in \mathcal{F}$ such that $\operatorname{adh}_\pi U \cap F = \varnothing$. So, $F \subseteq X \setminus \operatorname{adh}_\pi U = \operatorname{inh}_\pi(X \setminus U)$. By definition, it follows that $X \setminus U \in i_\pi \mathcal{F}$. Since $U \cap X \setminus U = \varnothing$, we have that $x \notin \operatorname{adh}_\pi i_\pi \mathcal{F}$. Conversely, if $x \notin \operatorname{adh}_\pi i_\pi \mathcal{F}$, then there exists

$U \in \mathcal{V}_\pi(x)$ and $F \in i_\pi \mathcal{F}$ such that $U \cap F = \varnothing$. As we have seen before, it follows that $\operatorname{adh}_\pi U \cap \operatorname{inh}_\pi F = \varnothing$. Since $F \in i_\pi F$, we know that $\operatorname{inh}_\pi F \in \mathcal{F}$. It follows that $x \notin \operatorname{adh}_{r\pi} \mathcal{F}$, as needed.

The remainder of the lemma follows easily by setting $\mathcal{F} = i_\pi^n \mathcal{F}$, in which case $i_\pi \mathcal{F} = i_\pi^{n+1} \mathcal{F}$. □

Definition 5.1.5. A filter $\mathcal{F}$ on a pretopological space is *inherent* if $\operatorname{inh}_\pi F \neq \varnothing$ for each $F \in \mathcal{F}$. If $\mathcal{U}$ is maximal with respect to the property of being inherent, we say that $\mathcal{U}$ is an *inherent ultrafilter*.

Theorem 5.1.6. *For a Hausdorff pretopological space (X, π), the following are equivalent.*

(a) X is PHC

(b) whenever $\mathcal{C}$ is a π-cover of X, there exists $C_1, ..., C_n \in \mathcal{C}$ such that $X = \bigcup_{i=1}^n \operatorname{adh}_\pi C_i$

(c) each inherent filter $\mathcal{F}$ on X has nonempty adherence

(d) $\operatorname{adh}_\pi i_\pi \mathcal{F} \neq \varnothing$ for each filter $\mathcal{F}$ on X.

Proof. Let $\mathcal{C}$ be a π-cover of X. Without loss of generality, assume that $\mathcal{C} = \{U_x : x \in X\}$ where each $U_x \in \mathcal{V}_\pi(x)$. Suppose no such finite subcollection exists. Then $\mathcal{A} = \{X \setminus \operatorname{adh}_\pi U_x : x \in X\}$ has the finite intersection property. Let $\mathcal{F}$ be the filter generated by $\mathcal{A}$. For each $x \in X$, $x \in \operatorname{inh}_{r\pi} \operatorname{adh}_\pi U$ if and only if there exists $V \in \mathcal{V}_\pi(x)$ such that $\operatorname{adh}_\pi V \subseteq \operatorname{adh}_\pi U$. Therefore, for each $x \in X$, $x \in \operatorname{inh}_{r\pi} \operatorname{adh}_\pi U_x$. Thus, $\operatorname{adh}_{r\pi} \mathcal{F} = X \setminus \bigcup_{x \in X} \operatorname{inh}_{r\pi} \operatorname{adh}_\pi U_x = \varnothing$, a contradiction.

Next, let $\mathcal{F}$ be a filter on X such that $\operatorname{inh}_\pi F \neq \varnothing$ for each $F \in \mathcal{F}$. Suppose that $\operatorname{adh}_\pi \mathcal{F} = \varnothing$. Then $\mathcal{C} = \{X \setminus F : F \in \mathcal{F}\}$ is a π-cover of X. By assumption, there exist $F_1, ..., F_n \in \mathcal{F}$

such that $\mathrm{adh}_\pi(X\setminus F_1\cup\ldots\cup X\setminus F_n)=X\setminus \mathrm{inh}_\pi(F_1\cap\ldots\cap F_n)=X$. However, $F_1\cap\ldots\cap F_n\in\mathcal{F}$ and thus by assumption $F_1\cap\ldots\cap F_n$ has nonempty inherence, a contradiction.

Let $\mathcal{F}$ be a filter on X. Notice that $i_\pi\mathcal{F}$ is a filter on X such that $\mathrm{inh}_\pi F\neq\varnothing$ for each $F\in i_\pi\mathcal{F}$. Then by assumption, $\mathrm{adh}_\pi\, i_\pi\mathcal{F}\neq\varnothing$.

Let $\mathcal{F}$ be a filter on X. Then $\mathrm{adh}_{r\pi}\,\mathcal{F}=\mathrm{adh}_\pi\, i_\pi\mathcal{F}\neq\varnothing$ by Lemma 5.1.4. Thus, we have shown that $(X,r\pi)$ is compact and the theorem is proven. $\square$

5.2 θ-quotient Convergence

Let (X,π) be a compact Hausdorff pretopological space, Y a set and $f:(X,\pi)\to Y$ a surjection such that $f^{\leftarrow}(y)$ is cover-compact for each $y\in Y$. For $A\subseteq X$, let $f^{\#}[A]=\{y\in Y: f^{\leftarrow}(y)\subseteq A\}$. Define the *$\theta$-quotient convergence* $f^{\#}\pi$ on Y as follows: a filter $\mathcal{F}$ on Y $f^{\#}\pi$-converges to y if and only if $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$.

Lemma 5.2.1. *Let $\mathcal{F}$ be a filter on Y. Then $y\in lim_{f^{\#}\pi}\mathcal{F}$ if and only if $f^{\leftarrow}(\mathcal{F})\supseteq\mathcal{V}_\pi(f^{\leftarrow}(y))$.*

Proof. Suppose that $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$. Then whenever $\mathcal{C}$ is a cover of $f^{\leftarrow}(y)$, there exists $F\in\mathcal{F}$ and $C_1,\ldots,C_n\in\mathcal{C}$ such that $f^{\leftarrow}[F]\subseteq\bigcup_{i=1}^n C_i$. Let $V\in\mathcal{V}_\pi(f^{\leftarrow}(y))$. By definition, $f^{\leftarrow}(y)\subseteq\mathrm{inh}_\pi V$. In other words, $\{V\}$ is a one-element cover of $f^{\leftarrow}(y)$. Thus, $V\in f^{\leftarrow}(\mathcal{F})$, as needed.

Conversely, let $\mathcal{C}$ be a cover of $f^{\leftarrow}(y)$. Since $f^{\leftarrow}(y)$ is cover-compact, we can find $C_1,\ldots,C_n\in\mathcal{C}$ such that $f^{\leftarrow}(y)\subseteq\mathrm{inh}_\pi\left(\bigcup_{i=1}^n C_i\right)$. By definition, $C=\bigcup_{i=1}^n C_i\in\mathcal{V}_\pi(f^{\leftarrow}(y))$. Thus, $C\in f^{\leftarrow}(\mathcal{F})$ and there exists $F\in\mathcal{F}$ such that $f^{\leftarrow}[F]\subseteq\bigcup_{i=1}^n C_i$ and $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$. $\square$

Lemma 5.2.2. *Let (X,π) be a Hausdorff pretopology. If $A,B\subseteq X$ are disjoint cover-compact subsets of X, then there exist disjoint vicinities $U\in\mathcal{V}_\pi(A)$, $V\in\mathcal{V}_\pi(B)$.*

Proof. First we show this holds for $B = \{x\}$. For each $z \in A$, choose disjoint $U_z \in \mathcal{V}_\pi(z)$ and $V_z \in \mathcal{V}_\pi(x)$. Since A is cover-compact, we can choose $z_1, ..., z_n \in A$ such that $A \subseteq \mathrm{inh}_\pi(\bigcup_{i=1}^n U_{z_i})$. Thus, $U = \bigcup_{i=1}^n U_{z_i} \in \mathcal{V}_\pi(A)$. Also, $V = \bigcap_{i=1}^n V_{z_i} \in \mathcal{V}_\pi(x)$ and $U \cap V = \varnothing$. It is a straight-forward exercise to now show this holds for disjoint cover-compact sets, A and B. $\square$

Proposition 5.2.3. *$(Y, f^\# \pi)$ is a Hausdorff pretopology. Furthermore, for each $y \in Y$,*

$$\mathcal{V}_{f^\#\pi}(y) = \langle \{f^\#[W] : W \in \mathcal{V}_\pi(f^\leftarrow(y))\}\rangle.$$

Proof. We first show that $f^\#\pi$ is indeed a pretopology. Notice that for $y \in Y$,

$$\bigcap\{\mathcal{F} : y \in \lim_{f^\#\pi} \mathcal{F}\} = \bigcap\{\mathcal{F} : V \in \mathcal{V}_\pi(f^\leftarrow(y)) \text{ implies } f^\#[V] \in \mathcal{F}\}.$$

It follows that $\mathcal{V}_{f^\#\pi}(y)$ is the filter generated by $\{f^\#[U] : U \in \mathcal{V}_\pi(f^\leftarrow(y))\}$. For any $A \subseteq X$, $f^\leftarrow[f^\#[A]] \subseteq A$. It follows easily that $f^\leftarrow(\mathcal{V}_{f^\#\pi}(y)) \supseteq \mathcal{V}_\pi(f^\leftarrow(y))$. By Lemma 5.2.1, then, $y \in \lim_{f^\#\pi} \mathcal{V}_{f^\#\pi}(y)$ and $f^\#\pi$ is a pretopology with the stated vicinity filters.

Now, if $y_1 \neq y_2$, by Lemma 5.2.2, for $i = 1, 2$, we can find $U_i \in \mathcal{V}_\pi(f^\leftarrow(y_i))$ such that $U_1 \cap U_2 = \varnothing$. It is immediate that $f^\#[U_1] \cap f^\#[U_2] = \varnothing$ and $f^\#\pi$ is Hausdorff. $\square$

Definition 5.2.4. Let (X, π) and (Y, σ) be pretopological spaces. A function $f : (X, \pi) \to (Y, \sigma)$ is *strongly irreducible* if there exists $y \in Y$ such that $f^\leftarrow(y) \subseteq U \cap V$ for any subsets U and V of X with nonempty inherence such that $U \cap V \neq \varnothing$.

The function f is *weakly θ-continuous ($w\theta$-continuous* for short) if $f : (X, \pi) \to (Y, r\sigma)$ is continuous.

Theorem 5.2.5. *If (X, π) is a compact, Hausdorff pretopological space, $f : (X, \pi) \to Y$ a strongly irreducible surjection such that $f^\leftarrow(y)$ is cover-compact for* *each $y \in Y$ and $f^{\#}\pi$ is the θ-quotient pretopology on Y, then $f : (X,\pi) \to (Y, f^{\#}\pi)$ is wθ-continuous and $(Y, f^{\#}\pi)$ is a PHC Hausdorff pretopological space.*

Proof. For $x \in X$, let $V \in \mathcal{V}_{f^{\#}\pi}(f(x))$. Without loss of generality, we can assume that $V = f^{\#}[W]$ for some $W \in \mathcal{V}_\pi(f^{\leftarrow}(f(x)))$. Note that in this case $x \in \mathrm{inh}_\pi W$. Supose that $w \in W$ and $f(w) \in f^{\#}[U]$ for some $U \in \mathcal{V}_\pi(f^{\leftarrow}(f(w)))$. Notice that $w \in W \cap U$, so $W \cap U \neq \varnothing$. Since f is strongly irreducible, we can find $y \in f^{\#}[U] \cap f^{\#}[W]$. Therefore, $f(w) \in \mathrm{adh}_\pi f^{\#}[W]$. In particular, $f[W] \subseteq \mathrm{adh}_\pi f^{\#}[W]$ and f is wθ-continuous.

Since the continuous image of a compact space is again compact, $(Y, rf^{\#}\pi)$ is compact and by definition $(Y, f^{\#}\pi)$ is PHC. $\square$

5.3 PHC Extensions of π

Let (X,π) be a pretopological space. By an *extension* of π, we mean a convergence ξ on a set Y such that (X,π) is a subspace of (Y,ξ) and $\mathrm{adh}_\xi X = Y$. There is an ordering on the family extensions of X. If ξ and ζ are extensions of π, we say that ξ *is projectively larger than* ζ, written $\xi \geq_\pi \zeta$ if there exists a continuous map $f : (Y,\xi) \to (Z,\zeta)$ which fixes the points of X. In the comment following the definition of a convergence, we noted that the underlying set of a convergence space is determined by the convergence itself. For coherence of notation, when discussing extensions of convergence spaces we will often refer to the convergence without reference to the underlying set. This is not a problem thanks to the aforementioned comment.

We borrow from topology the concepts of *strict* and *simple* extensions. If ξ is an extension of π, we define ξ^+ a new extension of π on the same underlying set as ξ. For $p \in Y$,

$$\mathcal{V}_{\xi^+}(p) = \langle \{\{p\} \cup U : \exists W \in \mathcal{V}_\xi(p), W \cap X = U\} \rangle.$$

If $\xi = \xi^+$, then we say ξ is a *simple extension* of π.

In a similar way, we define $\xi^\#$, an extension of π on the same set as ξ. If $A \subseteq X$, let

$$oA = \{p \in Y : \exists W \in \mathcal{V}_\xi(p), W \cap X = A\}.$$

If $p \in Y$, then $V_{\xi^\#}(p)$ is the filter generated by $\{oA : \exists V \in \mathcal{V}_\xi(p), V \cap X = A\}$. If $\xi = \xi^\#$, then we say that ξ is a *strict extension* of π.

Lemma 5.3.1. *If ξ is an extension of π, then $\xi^\# \leq \xi \leq \xi^+$.*

Proof. In both cases it is straight-forward to check that the identity map is continuous and fixes X. $\square$

Proposition 5.3.2. *Suppose that (X, π) is a Hausdorff pretopological space and ξ is a pretopology and a compactification of π. Then ξ^+ is PHC.*

Proof. Recall that by compactification, we mean a compact extension. Fix $p \in Y$ and let $\{p\} \cup U \in \mathcal{V}_{\xi^+}(p)$. Then $\mathrm{adh}_{\xi^+}(\{p\} \cup U) = oU \cup \mathrm{adh}_\pi U$. So, in the partial regularization of ξ^+, the vicinity filters are generated by sets of the form $oU \cup \mathrm{adh}_\pi U$ for $U \subseteq X$. In particular, this shows that $\mathcal{V}_{r\xi^+}(p) \subseteq \mathcal{V}_{\xi^\#}(p)$ for each $p \in Y$. Since $\xi^\#$ is a coarser pretopology than ξ, it follows that the partial regularization of ξ^+ is coarser than ξ. Since (Y, ξ) is compact, so is $(Y, r\xi^+)$ and by definition, ξ^+ is PHC. $\square$

For any Hausdorff convergence space (X, σ), Richardson [32] constructs a compact, Hausdorff convergence space (X^*, σ^*) in which X is densely embedded. It should be noted that Richardson's definition of a convergence includes the following third axiom in addition to the two in our definition:

$$\text{If } x \in \lim_\sigma \mathcal{F}, \text{ then } x \in \lim_\sigma(\langle x \rangle \cap \mathcal{F}). \tag{R}$$

We will make use of assumption (R) in Theorem 5.3.6. Note that if σ is a pretopology, then σ already satisfies (R). If σ is a pretopology, then so is σ^*. We will make use of Richardson's compactification only when σ is a pretopology. We detail this construction here.

Example 5.3.3 (Richardson's Compactification)**.** For a pretopological space (X,π), let X' be the family of ultrafilters on X such that $\lim_\pi \mathcal{U} = \varnothing$ for each $\mathcal{U} \in X'$. We define the pretopological space (X^*,π^*). The underlying set $X^* = X \cup X'$. For $A \subseteq X$, let $A^* = A \cup \{\mathcal{U} \in X' : A \in \mathcal{U}\}$. For a filter $\mathcal{F}$ on X, let $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$. We define π^* to be the pretopology defined on X^* by the following vicinity filters:

- For $x \in X$, $\mathcal{V}_{\pi^*}(x) = \mathcal{V}_\pi(x)^*$.
- For $\mathcal{U} \in X'$, $\mathcal{V}_{\pi^*}(\mathcal{U}) = \mathcal{U}^*$.

It can be shown that (X^*,π^*) is a compact Hausdorff pretopological space and that $\operatorname{adh}_{\pi^*} X = X^*$.

It is said that (X,ξ) is *regular* if $x \in \lim_\xi \mathcal{F}$ implies that $x \in \lim_\xi \{\operatorname{adh}_\xi F : F \in \mathcal{F}\}$. Richardson [32] proves the following:

Theorem 5.3.4. *If (X,σ) is a Hausdorff convergence space, (Y,ξ) is a compact, Hausdorff, regular convergence space and $f : (X,\sigma) \to (Y,\xi)$ is continuous, then there exists a unique continuous map $F : (X^*,\sigma^*) \to (Y,\xi)$ extending f.*

We seek to circumvent the assumption of regularity on (Y,ξ). For a Hausdorff pretopological space (X,π), let $(\kappa_\pi X, \kappa\pi) = (X^*,(\pi^*)^+)$. By the above proposition, $\kappa\pi$ is PHC. Additionally, $\kappa\pi$ has the following property.

Theorem 5.3.5. *Let (X,π) and (Y,ξ) be Hausdorff pretopological spaces spaces. If $f : (X,\sigma) \to (Y,\xi)$ is continuous, then there exists a continuous function $F : (\kappa_\pi X, \kappa\pi) \to (\kappa_\xi Y, \kappa\xi)$ which extends f.*

Proof. For each free ultrafilter $\mathcal{U}$ on X, $f(\mathcal{U})$ is an ultrafilter on Y. Define $F(\mathcal{U})$ as follows:

- If $y \in \lim_\xi f(\mathcal{U})$ for some $y \in Y$, let $F(\mathcal{U}) = y$.
- If $f(\mathcal{U})$ is free in (Y,ξ), let $F(\mathcal{U}) = f(\mathcal{U})$.

We show that F is continuous. Since f is continuous, if $x \in X$ and $F(x) \in \mathcal{V}_{\kappa\pi}(f(x)) = \langle \mathcal{V}_\pi(f(x)) \rangle$, then we can find $U \in \mathcal{V}_\pi(x)$ such that $f[U] \subseteq V$. Suppose $\mathcal{U} \in \kappa_\pi X \setminus X$. If $F(\mathcal{U}) \in Y$, let $V \in \mathcal{V}_\pi(F(\mathcal{U}))$. Since $y \in \lim_\xi f(\mathcal{U})$, $V \in f(\mathcal{U})$. Therefore, for some $U \in \mathcal{U}$, $f(U) \subseteq V$. It follows that $F[\{\mathcal{U}\} \cup U] \subseteq V$. Lastly, suppose that $F(\mathcal{U}) \in \kappa_\pi Y \setminus Y$ and fix $V \in F(\mathcal{U}) = f(\mathcal{U})$. Then for some $U \in \mathcal{U}$, $f[U] \subseteq V$. So, $F[\{U\} \cup U] \subseteq \{F(\mathcal{U})\} \cup V$ and F is continuous. $\square$

The pretopological space $\kappa_\pi X$ is a variation on the Katětov extension of a topological space. However, the corresponding version of Theorem 5.3.5 does not hold for topological spaces. What we mean to say is that it is possible to find topological spaces X and Y and a continuous function $f : X \to Y$ which does not extend continuously to the (topological) Katětov extensions of X and Y. See 5A in [30] for an example. Thus, Theorem 5.3.5 is surprising in much the same way as Theorem 4.1.5 and shows the value of broadening our perspective to include pretopological spaces when considering problems usually thought of as topological.

In [22], it is shown that a convergence ξ has a projective maximum compactification if and only if ξ has only finitely many free ultrafilters. In contrast with this, we have the following facts:

Theorem 5.3.6. *If (X,π) is a Hausdorff pretopological space, (Y,ξ) is a compact Hausdorff convergence space satisfying (R) and $f:(X,\pi)\to(Y,\xi)$ is continuous, then there exists a continuous map $F:(\kappa_\pi X,\kappa\pi)\to(Y,\xi)$ extending f.*

Proof. We define $F:(\kappa_\pi X,\kappa\pi)\to(Y,\xi)$ as we did in the proof of Theorem 5.3.5. However since (Y,ξ) is compact, for each free ultrafilter $\mathcal{U}$ on X, there exists $y_{\mathcal{U}}\in Y$ such that $y_{\mathcal{U}}\in\lim_\xi f(\mathcal{U})$. Let $F(\mathcal{U})=y_{\mathcal{U}}$. Since π is a pretopology, to show that F is continuous it is enough to show that for each $p\in\kappa_\pi X$, $F(p)\in\lim_\xi F(\mathcal{V}_{\kappa\pi}(p))$.

If $x\in X$, then $F(\mathcal{V}_{\kappa\pi}(x))\supseteq F(\mathcal{V}_\pi(x))=f(\mathcal{V}_\pi(x))$. Since f is continuous by assumption, we have that $F(x)\in\lim_\xi F(\mathcal{V}_{\kappa\pi}(x))$.

If $\mathcal{U}\in\kappa_\pi X\setminus X$, then $F(\mathcal{V}_{\kappa\pi}(\mathcal{U}))=\langle\{F(\mathcal{U})\cup F[H]:H\in\mathcal{U}\}\rangle=\langle F(\mathcal{U})\rangle\cap f(\mathcal{U})\rangle$. By construction, $F(\mathcal{U})\in\lim_\xi f(\mathcal{U})$ and thus by (R), $F(\mathcal{U})\in\lim_\xi F(\mathcal{V}_{\kappa\pi}(\mathcal{U}))$, as needed. $\square$

Corollary 5.3.7. *If (X,π) is a pretopological space, then $\kappa\pi\geq_\pi\xi$ for any Hausdorff compactification ξ of π.*

Chapter 6

Cardinal Invariants of Pretopologies

In this section we will employ several cardinal invariants of pretopological spaces which parallel well-known cardinal invariants, such as the character, pseudocharacter, Lindelöf degree and closed pseudocharacter, for topological spaces.

6.1 Background

We begin by defining the cardinal invariants of pretopologies which we will make use of.

Definition 6.1.1. Let (X,π) be a pretopological space. We define the following invariants of π:

(a) The *character* $\chi(\pi)$ is least cardinal κ such that for each $x \in X$, there exists a filter base $\mathcal{B}_x \subseteq \mathcal{V}_\pi(x)$ with $|\mathcal{B}_x| \leq \kappa$.

(b) The *pseudocharacter* $\psi(\pi)$ is least cardinal κ such that for each $x \in X$, there exists $\mathcal{F}_x \subseteq \mathcal{V}_\pi(x)$ such that $|\mathcal{F}_x| \leq \kappa$ and $\bigcap \mathcal{F}_x = \{x\}$

(c) The *closed pseudocharacter* $\overline{\psi}(\pi)$ is the least cardinal κ such that for each $x \in X$, there exists $\mathcal{F}_x \subseteq \mathcal{V}_\pi(x)$ such that $|\mathcal{F}_x| \leq \kappa$ and $\bigcap_{V \in \mathcal{F}_x} \mathrm{adh}_\pi V = \{x\}$.

(d) The *Linelöf degree* $L(\pi)$ is the least cardinal κ such that whenever $\mathcal{C}$ is a π-cover of X there exists $\mathcal{A} \subseteq \mathcal{C}$ such that $|\mathcal{A}| \leq \kappa$ and $X = \bigcup_{C \in \mathcal{A}} C$.

Notice that the closed pseudocharacter of π is only well-defined if π is Hausdorff. If π is a topology, then each of the above invariants agree with the usual notions for topological spaces.

In 1923, Alexandroff and Urysohn [1] asked the following question: If (X, τ) is a compact first-countable Hausdorff space, is $|X| \leq 2^{\aleph_0}$? In 1969, Arhangel'skii [3] proved the following famous theorem which answers their question.

Theorem 6.1.2. *Let (X, τ) be a topological space. Then $|X| \leq 2^{\chi(\tau)L(\tau)}$.*

In [19], R.E. Hodel provides an excellent survey of the results which followed Arhangel'skii's solution to Alexandroff's problem. In particular, Hodel isolates the techniques provided by Arhangel'skii's proof and gives them in purely set-theoretic terms. We will make use of the following result, which appears as Theorem 3.3 in [19], in the next section.

Theorem 6.1.3. *Let X be a set, let $Y \subseteq X$, and for each $x \in X$ let $\{V(\gamma, x) : \gamma < \kappa\}$ be a colleciton of subsets of X such that $x \in V(\gamma, x)$ for each $\gamma < \kappa$. Assume the following:*

(I) given $\alpha, \beta < \kappa$, there exists $\gamma < \kappa$ such that $V(\gamma, x) \subseteq V(\alpha, x) \cap V(\beta, x)$

(H) if $x \neq y$, then there exists $\alpha, \beta < \kappa$ such that $V(\alpha, x) \cap V(\beta, y) = \varnothing$

(C) if $f : X \to \kappa$, then there exists $A \subseteq X$ with $|A| \leq \kappa$ such that $Y \subseteq \bigcup_{x \in A} V(f(x), x)$.

Then $|Y| \leq 2^\kappa$.

6.2 Cardinality Bounds for Pretopologies

Our first task in this section is to prove that a corresponding version of Arhangel'skii's Theorem holds for Hausdorff pretopologies.

Theorem 6.2.1. *If X is a Hausdorff pretopological space, then $|X| \leq 2^{\chi(\pi)L(\pi)}$.*

Proof. We make use of the framework laid out in Theorem 3.3 of [19], given as Theorem 6.1.3 above. Let $\chi(\pi)L(\pi) = \kappa$ and for each $x \in X$ fix $\mathcal{B}_x$, a filter base for $\mathcal{V}_\pi(x)$ of cardinality $\leq \kappa$. Setting $\mathcal{B}_x = \{V(\gamma, x) : \gamma < \kappa\}$ and $Y = X$ in the afformentioned Theorem 3.3, it follows immediately that $|X| \leq 2^\kappa$. In particular, (I) is satisfied because $\mathcal{B}_x$ is a filter base, (H) is satisfied because π is Hausdoff and (C) is satisfied because $L(\pi) \leq \kappa$. □

Corollary 6.2.2. *If (X, π) is a compact Hausdorff pretopological space, then $|X| \leq 2^{\chi(\pi)}$.*

In [14], it is shown that if (X, τ) is H-closed, then $|X| \leq 2^{\overline{\psi}(\tau)}$. Since any compact Hausdorff topological space is also H-closed, for compact Hausdorff topological spaces we can improve the bound in Corollary 6.2.2 to $|X| \leq 2^{\overline{\psi}(\tau)}$. Further, since a compact Hausdorff topological space is regular, $\overline{\psi}(\tau) = \psi(\tau)$ and thus $|X| \leq 2^{\psi(\tau)}$ for compact Hausdorff topological spaces. However, the following example shows that this does not hold for pretopological spaces.

Example 6.2.3. In [26], a Urysohn topological space (X, τ) is constructed so that X has an H-set A where $|A| > 2^{\overline{\psi}(\tau)}$. We have seen in Theorem 4.1.5 that if A is an H-set in a Urysohn topological space (X, τ), then when A is viewed as a subspace of the pretopological space (X, θ), $(A, \theta|_A)$ is a compact Hausdorff pretopological space.

Let $\kappa = \overline{\psi}(\tau)$. Then, for each $x \in A$, there exists a family $\mathcal{F}_x$ of open neighborhoods of x such that $|\mathcal{F}_x| \leq \kappa$ and $\bigcap_{U \in \mathcal{F}_x} cl_\tau U = \{x\}$. Notice that if $x \in A$, then $\{cl_\tau U \cap A : U \in \mathcal{F}_x\}$

is a pseudobase for x in $(A,\theta|_A)$ of cardinality κ. Thus $(A,\theta|_A)$ is a compact, Hausdorff pretopological space, $\kappa=\psi(A,\theta|_A)$ and $|A|>2^{\psi(\theta|_A)}$.

The fact displayed by Example 6.2.3 is not entirely surprising in the following respect: A compact Hausdorff pretopological space (X,π) is not necessarily regular. In fact, if (X,π) is a compact, Hausdorff, regular pretopological space, then π is a topology. If (X,τ) is a Hausdorff topological space which is not regular, it is possible for $\psi(\tau)<\overline{\psi}(\tau)$, so we can make sense of the fact that the cardinality bound no longer holds. The next question, then, is this: If X is a compact, Hausdorff pretoplogical space, is $|X|\leq 2^{\overline{\psi}(\pi)}$? The answer is yes. We begin with a lemma.

Lemma 6.2.4. *Let (X,π) be a compact, Hausdorff pretopological space with $\overline{\psi}(\pi)=\kappa$. Let H be a subset of X such that whenever $\mathcal{F}$ is a filter base on H with $|\mathcal{F}|\leq\kappa$, $\mathrm{adh}_{\pi|_H}\mathcal{F}\neq\varnothing$. Then H is compact.*

Proof. In the statement of the lemma, $\mathrm{adh}_{\pi|_H}\mathcal{F}$ refers to the adherence of $\mathcal{F}$ in H when viewed as a subspace of (X,π). In particular, $\mathrm{adh}_{\pi|_H}\mathcal{F}=\mathrm{adh}_\pi\mathcal{F}\cap H$.

Let $\mathcal{F}$ be an ultrafilter on H. For each $x\in X$, fix $\mathcal{B}_x\subseteq\mathcal{V}_\pi(x)$ such that $|\mathcal{B}_x|\leq\kappa$ and $\bigcap_{B\in\mathcal{B}_x}\mathrm{adh}_\pi B=\{x\}$. Since X is compact, there exists $p\in\mathrm{adh}_\pi\mathcal{F}$. If $\langle\mathcal{F}\rangle$ is the filter on X generated by $\mathcal{F}$, then $\langle\mathcal{F}\rangle$ is an ultrafilter. It follows that for each $B\in\mathcal{B}_p$, there exists some $F_B\in\mathcal{F}$ such that $F_B\subseteq B$. The family $\mathcal{G}=\{F_B : B\in\mathcal{B}_p\}$ is a filter base on H of cardinality $\leq\kappa$. By assumption, $\mathrm{adh}_{\pi|_H}\mathcal{G}\neq\varnothing$. At the same time, $\mathrm{adh}_{\pi|_H}\mathcal{G}\subseteq\left(\bigcap_{B\in\mathcal{B}_p}\mathrm{adh}_\pi B\right)\cap H=\{p\}\cap H\subseteq\{p\}$. Thus, $p\in H$ and as such $p\in\mathrm{adh}_{\pi|_H}\mathcal{F}$, rendering H compact, as needed. $\square$

If a pretopological space (X,π) has the property ascribed to H in the above lemma, we say that (X,π) is *initially κ-compact*.

We make use of the following definition and lemma from [23] going forward.

Definition 6.2.5. A filter base $\mathcal{F}$ on a set X is of *type* κ if $|\mathcal{F}| \leq \kappa$ and $|F| = \kappa$ for each $F \in \mathcal{F}$.

Lemma 6.2.6. *If X is a set and $\mathcal{F}$ is a filter base on X of type κ, then there exists a filter base $\mathcal{H}$ on X of type κ such that $\mathcal{H}$ is finer than $\mathcal{F}$ and for each $A \subseteq X$, if $|A| < \kappa$, then there exists $H \in \mathcal{H}$ such that $H \cap A = \varnothing$.*

Notice that this lemma is purely set-theoretic, and thus applies in the setting of pretopological spaces.

Proposition 6.2.7. *Let (X, π) be a pretopological space. The following are equivalent.*

(a) Whenever $\mathcal{C}$ is a π-cover of X and $|\mathcal{C}| \leq \kappa$, there exists a finite $\mathcal{A} \subseteq \mathcal{C}$ such that $X = \bigcup \mathcal{A}$.

(b) If $\mathcal{F}$ is a filter base on X and $|\mathcal{F}| \leq \kappa$, then $\operatorname{adh}_\pi \mathcal{F} \neq \varnothing$ (i.e. X is initially κ-compact)

(c) If $A \subseteq X$ and $|A| \leq \kappa$, then there exists $p \in X$ such that for each $V \in \mathcal{V}_\pi(p)$, $|V \cap A| = |A|$ (i.e. p is a complete accumulation point of A)

Proof. Recall that $\mathcal{C}$ is a π-cover of X if for each $x \in X$, $\mathcal{V}_\pi(x) \cap \mathcal{C} \neq \varnothing$. By way of contradiction, suppose that $\mathcal{F}$ is a filter base with $|\mathcal{F}| \leq \kappa$ and that $\operatorname{adh}_\pi \mathcal{F} = \varnothing$. Then, $\{X \setminus F : F \in \mathcal{F}\}$ is a π-cover of X of cardinality $\leq \kappa$. By assumption, there exist $F_1, \ldots, F_n \in \mathcal{F}$ such that $X = \bigcup_{i=1}^n (X \setminus F_i)$. However, this implies that $F_1 \cap \ldots \cap F_n = \varnothing$, contradicting that $\mathcal{F}$ is a filter base.

Let $A \subseteq X$ such that $|A| = \lambda \leq \kappa$. Then $\{A\}$ is a filter base of type λ. Using Lemma 6.2.6, let $\mathcal{H}$ be a filter base of type λ with $A \in \mathcal{H}$ such that if $B \subseteq X$ and $|B| < \lambda$, then there exists some $H \in \mathcal{H}$ such that $H \cap B = \varnothing$. By assumption, $\operatorname{adh}_\pi \mathcal{H} \neq \varnothing$, so fix $p \in \operatorname{adh}_\pi \mathcal{H}$. Suppose that there exists some $V \in \mathcal{V}_\pi(p)$ such that $|V \cap A| < |A|$. Then we can find $H \in \mathcal{H}$

such that $H \cap V \cap A = (H \cap A) \cap V = \varnothing$. However, $H \cap A \in \mathcal{H}$ and this implies $p \notin \operatorname{adh}_\pi \mathcal{H}$, a contradiction. Therefore, A has a complete accumulation point, as needed.

Lastly, let $\mathcal{C}$ be a π-cover of X which does not contain a finite subcover. Further, choose $\mathcal{C}$ so that its cardinality is the least such that no finite subcover exists. Let $\lambda = |\mathcal{C}|$ and write $\mathcal{C} = \{V_\alpha : \alpha < \lambda\}$. If $\alpha < \lambda$, then $\bigcup_{\beta<\alpha} V_\beta \neq X$, by the minimality of λ (otherwise we'd be able to find a finite subcover of $\{V_\beta : \beta < \alpha\}$). Moreover, $|X \setminus \bigcup_{\beta<\alpha} V_\beta| \geq \lambda$, again by the minimality of λ. Thus, for each $\alpha < \lambda$, fix $x_\alpha \in X \setminus \bigcup_{\beta<\alpha} V_\beta$ such that $\alpha \neq \gamma$ implies $x_\alpha \neq x_\gamma$. Let $A = \{x_\alpha : \alpha < \lambda\}$. The set A has no complete accumulation point. To see this, recall that each for each $p \in X$, there is some $\alpha < \lambda$ such that $V_\alpha \in \mathcal{V}_\pi(p)$. Since $V_\alpha \cap A \subseteq \{x_\beta : \beta < \alpha\}$, it follows that for each $p \in X$, there exists $V \in \mathcal{V}_\pi(p)$ such that $|V \cap A| \leq |\{x_\beta : \beta < \alpha\}| < \lambda = |A|$. By assumption, each subset of X of cardinality $\leq \kappa$ has a complete accumulation point. It follows that $\lambda > \kappa$. Since λ was chosen to be the least cardinal such that there exists a π-cover of cardinality λ with no finite subcover, (a) holds. □

Before proceeding to the theorem, we need a lemma which generalizes a well-known fact about topological spaces to the pretopological setting. This lemma is used in the construction of the family $\{H_\alpha : \alpha < \kappa^+\}$ in the proof of Theorem 6.2.9. We omit the proof of the lemma as it is nearly identical to the topological proof.

Lemma 6.2.8. *If (X, π) is a compact pretopological space, then every infinite subset of X has a complete accumulation point.*

Theorem 6.2.9. *Let (X, π) be a compact, Hausdroff pretopological space with $\overline{\psi}(\pi) = \kappa$. Then $|X| \leq 2^\kappa$.*

Proof. We use a closure argument which parallels that of the proof of Theorem 4.1 in [19]. For each $x \in X$, let $\mathcal{B}_x$ be a family of vicinities of x witnessing the fact that $\overline{\psi}(\pi) = \kappa$. Let

$\{H_\alpha : \alpha < \kappa^+\}$ be a family of subsets of X satisfying:

1. $|H_\alpha| \le 2^\kappa$ for each $\alpha < \kappa^+$

2. If $A \subseteq \bigcup_{\beta<\alpha} H_\beta$ and $|A| \le \kappa$ then A has a complete accumulation point in H_α.

3. If $\mathcal{A} \subseteq \bigcup\{\mathcal{B}_x : x \in \bigcup_{\beta<\alpha} H_\beta\}$ is finite and $\bigcup\mathcal{A} \neq X$, then $H_\alpha \setminus \bigcup\mathcal{A} \neq \varnothing$.

How this construction works: Let H_0 be an nonempty subset of X of with $|H_0| \le 2^\kappa$. Suppose that $\{H_\beta : \beta < \alpha\}$ have been chosen which satisfying (1–3) above. Let $G_\alpha = \bigcup_{\beta<\alpha} H_\beta$. Notice that both $[G_\alpha]^{\le\kappa}$ and $F = \{\mathcal{A} \subseteq \bigcup_{x\in G_\alpha} \mathcal{B}_x : |\mathcal{A}| < \omega \text{ and } \bigcup\mathcal{A} \neq X\}$ have cardinality $\le 2^\kappa$. By Lemma 6.2.8, each $A \in [G_\alpha]^{\le\kappa}$ has a complete accumulation point $x_A \in X$. For each family $\mathcal{A} \in F$, let $x_\mathcal{A} \in X \setminus \bigcup\mathcal{A}$. Let $H_\alpha = G_\alpha \cup \{x_A : A \in [G_\alpha]^{\le\kappa}\} \cup \{x_\mathcal{A} : \mathcal{A} \in F\}$. By construction H_α satisfies (1–3).

Let $H = \bigcup_{\alpha<\kappa^+} H_\alpha$. By Lemma 6.2.4, Proposition 6.2.7 and Property (2), H is compact. By Property (3) and the fact that H is compact, $H = X$. Since $|H| \le 2^\kappa$ by construction, $|X| \le 2^\kappa$. □

Question 6.2.10. Two further questions in this direction present themselves.

1. If (X,ξ) is a compact Hausdorff convergence space which is not necessarily pretopological, is it possible that $|X| > 2^{\chi(\xi)}$?

2. If (X,π) is PHC, is it the case that $|X| \le 2^{\chi(\pi)}$?

The author is still investigating these questions. The end goal would be to use these cardinality bounds, as in [4] and [26], to answer the question: If (X,τ) is a Hausdorff topological space and A is an H-set in X, does there exists a compact Hausdorff pretopological space (K,π) and a continuous map $f : (K,\pi) \to (X,\theta)$ such that $f[K] = A$?

Chapter 7

Open Questions

The long and circuitous path to this finished dissertation began with an investigation of the so-called Katětov spaces – those topological spaces which admit a coarser minimal Hausdorff topology. In many ways the results of this dissertation seek to find ways to characterize these spaces, though this is not usually mentioned explicitly. The following question is asked by Vermeer in [35].

Question 7.0.1. If (X,τ) is a Hausdorff topological space which can be embedded as an H-set in some H-closed space, is (X,τ) Katětov?

The converse to this statement is known to be true (see [14]) but as far as the author is aware this problem is still open. Theorem 4.1.8 generalizes this question to pretopological spaces and provides a partial solution. The following question, however, is still open.

Question 7.0.2. If (X,τ) is a Hausdorff topological space which can be embedded as an H-set in another Hausdorff topological space, does there exists a compact Hausdorff pretopology π such that $\pi \leq \tau$?

A variation of this question would instead require π to be PHC. Recall from Chapter 5, if (K,π) is a compact Hausdorff pretopological space and $f:(K,\pi)\to(X,\tau)$ is a perfect strongly irreducible surjection, then $f^{\#}\pi$, the θ-quotient pretopology on X is a PHC pretopology and $f^{\#}\pi \leq \tau$. Using this as motivation, we pose the following question.

Question 7.0.3. If (X,τ) is a Hausdorff topological space and $A \subseteq X$ is an H-set, does there exists a compact Hausdorff pretopological space (K,π) and a continuous function $f:(K,\pi) \to (X,\theta)$ such that $f[K] = A$?

There are counterexamples to this question if (K,π) is instead required to be a compact Hausdorff topological space (see [4], [26]). These counterexamples rely on a cardinality argument. If there is a counterexample to Question 7.0.3, we believe that its construction will also rely on a cardinality argument. The author hopes a solution to the following problem will point us in the right direction in determining if constructing a counterexample is possible.

Question 7.0.4. If (X,π) is a PHC space, is $|X| \leq 2^{\chi(\pi)}$? Better yet, is $|X| \leq 2^{\overline{\psi}(\pi)}$?

We showed in Chapter 6 that is (X,π) is a compact Hausdorff pretopological space, then $|X| \leq 2^{\chi(\pi)}$. Is this still true if we replace π with a general convegence ξ? In particular:

Question 7.0.5. If (X,ξ) is a compact Hausdorff convergence space, is $|X| \leq 2^{\chi(\xi)}$?

The author expects the answer to this question is "No." However, at the time of writing we have not worked out the details. For a definition of charcater in the general convergence setting, we refer the reader to [13].

Outside of questions which pertain to Katětov spaces, this work has led to many interesting questions about the nature of pretopological spaces. In particular we give two open questions pertaining to pretopological extensions. It is shown in [22] that a convergence space has a maximal compactification if and only if the space has finitely many free ultrafilters. This fact allows for both the space and the extensions to be convergence spaces in the broadest sense. The author has not been able to find information which considers only those extensions which are themselves pretopologies. This leads to the following two questions.

Question 7.0.6. Which pretopological spaces have a one-point compactification which is itself a pretopology?

Question 7.0.7. Let (X, π) be a Hausdorff pretopological space. Under what assumptions, if any, is $\kappa_\pi X$ a maximal PHC extension of X?

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