Numerical solutions of rough differential equations and stochastic differential equations

By

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Abstract

In this dissertation, we investigate time-discrete numerical approximation schemes for rough differential equations and stochastic differential equations (SDE) driven by fractional Brownian motions (fBm). The dissertation is organized as follows.

In Chapter 1, we introduce the basic settings and define time-discrete numerical approximation schemes.

In Chapter 2, we consider the Euler scheme for SDEs driven by fBms. For a SDE driven by a fBm with Hurst parameter $H > \frac{1}{2}$ it is known that the existing (naive) Euler scheme has the rate of convergence n^{1-2H}. Since the limit $H \to \frac{1}{2}$ of the SDE corresponds to a Stratonovich SDE driven by standard Brownian motion, and the naive Euler scheme is the extension of the classical Euler scheme for Itô SDEs for $H = \frac{1}{2}$, the convergence rate of the naive Euler scheme deteriorates for $H \to \frac{1}{2}$. The new (modified Euler) approximation scheme we are introducing in this chapter is closer to the classical Euler scheme for Stratonovich SDEs for $H = \frac{1}{2}$ and it has the rate of convergence γ_n^{-1}, where $\gamma_n = n^{2H-\frac{1}{2}}$ when $H < \frac{3}{4}$, $\gamma_n = n/\sqrt{\log n}$ when $H = \frac{3}{4}$ and $\gamma_n = n$ if $H > \frac{3}{4}$. Furthermore, we study the asymptotic behavior of the fluctuations of the error. More precisely, if $\{X_t, 0 \le t \le T\}$ is the solution of a SDE driven by a fBm and if $\{X_t^n, 0 \le t \le T\}$ is its approximation obtained by the new modified Euler scheme, then we prove that $\gamma_n(X^n - X)$ converges stably to the solution of a linear SDE driven by a matrix-valued Brownian motion, when $H \in (\frac{1}{2}, \frac{3}{4}]$. In the case $H > \frac{3}{4}$, we show the L^p convergence of

$n(X_t^n - X_t)$ and the limiting process is identified as the solution of a linear SDE driven by a matrix-valued Rosenblatt process. The rate of weak convergence is also deduced for this scheme. We also apply our approach to the naive Euler scheme.

In Chapter 3, we consider the Crank-Nicolson method for a SDE driven by a m-dimensional fBm. We consider the Crank-Nicolson method in three cases: (i) $m > 1$; (ii) $m = 1$ and and the drift term is equal to non-zero; and (iii) $m = 1$ and the drift term is equal to zero. We will show that the convergence rate of the Crank-Nicolson method is $n^{1/2-2H}$, $n^{-1/2-H}$ and n^{-2H}, respectively, in these three cases, and these convergence rates are exact in the sense that the error process for the Crank-Nicolson method converges to the solution of a linear SDE. Our main tools are the fractional calculus and the fourth moment theorem.

In Chapter 4, we study two variations of the time-discrete Taylor schemes for rough differential equations and for stochastic differential equations driven by fractional Brownian motions. One is the incomplete Taylor scheme which excludes some terms of an Taylor scheme in its recursive computation so as to reduce the computation time. The other one is to add some deterministic terms to an incomplete Taylor scheme to improve the mean rate of convergence. Almost sure rate of convergence and L_p-rate of convergence are obtained for the incomplete Taylor schemes. Almost sure rate is expressed in terms of the Hölder exponents of the driving signals and the L_p-rate is expressed by the Hurst parameters. Our explicit expressions of the convergence rates allow us to compare different incomplete Taylor schemes, and then help us construct the best incomplete schemes, depending on that one needs the almost sure convergence or one needs L_p-convergence. As in the smooth case, general Taylor schemes are always complicated to deal with. The incomplete Taylor scheme is even more sophisticated to analyze. A new feature of our approach is the explicit expression of the error functions which will be easier to study. Estimates for multiple integrals and formulas for the iterated vector fields are obtained to analyze the error functions and then to obtain the rates of convergence.

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Contents

Chapter 1

Introduction

1.1 Motivations

A fractional diffusion process has the following form

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t V(X_s)dB_s, \quad t \in [0,T]. \tag{1.1}$$

It consists of an initial value $X_0 = x$, a slowly varying continuous component $\int_0^t b(X_s)ds$ called the drift and a rapidly varying continuous random component $\int_0^t V(X_s)dB_s$ called the diffusion. The second integral in (1.1) is a pathwise Riemann-Stieltjes integrals with respect to the m-dimensional fractional Brownian motion (fBm) $B = \{B_t, t \geq 0\}$ with Hurst parameter $H > 1/2$. The integral equation (1.1) is often written in the differential form

$$dX_t = b(X_t)dt + V(X_t)dB_t$$

and is then called a stochastic differential equation (SDE).

Unfortunately explicitly solvable SDEs are rare in practical applications, and the gap between the well developed theory of stochastic differential equations and its application is still wide. A crucial task in bridging this gap is the development of efficient numerical methods for SDEs, a task to which this dissertation is addressed.

Here we shall introduce various time discrete numerical methods which are appropriate for the simulation of sample paths or functionals of fractional diffusions.

1.2 Euler scheme

The simplest heuristic time-discrete approximation is the stochastic generalization of the Euler approximation. For the SDE (1.1) it has the form

$$X_{k+1}^n = X_k^n + b(X_k^n)\frac{T}{n} + V(X_k^n)(B_{t_{k+1}} - B_{t_k}), \tag{2.1}$$

for $k = 0, 1, \ldots, n-1$, and $X_0^n = x$, where for simplicity of the presentation we consider uniform partitions of the interval $[0,T]$, $t_k = \frac{kT}{n}$, $k = 0, \ldots, n$, and we restrict attention to a 1-dimensional driving fractional Brownian motion B.

It was proved by Mishura [28] that for any real number $\varepsilon > 0$ there exists a random variable C_ε such that almost surely,

$$\sup_{0\le k\le n} |X_k^n - X_{t_k}| \le C_\varepsilon n^{1-2H+\varepsilon}.$$

Moreover, the convergence rate n^{1-2H} is sharp for this scheme, in the sense that $n^{2H-1}[X_t^n - X_t]$ converges almost surely to a finite and nonzero limit. This has been proved in the one-dimensional case by Nourdin and Neuenkirch in [30] using the Doss representation of the solution and generalized to the multi-dimensional case in [13]; see also Theorem 2.10.1 below.

Notice that when H tends to $\frac{1}{2}$, the convergence rate $2H-1$ of the numerical scheme (2.1) deteriorates, and so it is not a proper extension of the Euler-Maruyama scheme for the case $H=\frac{1}{2}$ (see for example [11, 22]). This is not surprising because the limit $H\to\frac{1}{2}$ of the SDE (1.1) corresponds to a Stratonovich SDE driven by standard Brownian motion, while the Euler scheme (2.1) is the extension of the classical Euler scheme for the Itô SDEs. It is then natural to ask the following question: Can we find a numerical scheme that generalizes the Euler scheme to the fBm case?

We introduce a new approximation scheme that we call *modified Euler scheme*:

$$\begin{aligned} X^n_{k+1} &= X^n_k + b(X^n_k)\frac{T}{n} + V(X^n_k)(B_{t_{k+1}} - B_{t_k}) \\ &\quad + \frac{1}{2}(V'V)(X^n_k)(\frac{T}{n})^{2H}, \end{aligned} \tag{2.2}$$

for $X^n_0 = x$.

Notice that if we formally set $H=\frac{1}{2}$ and replace B by a standard Brownian motion W, this is the classical Euler scheme for the Stratonovich SDE

$$\begin{aligned} X_t &= x + \int_0^t b(X_s)ds + \int_0^t V(X_s)dW_s \\ &= x + \int_0^t b(X_s)ds + \int_0^t V(X_s)\delta W_s + \frac{1}{2}\int_0^t (V'V)(X_s)ds. \end{aligned}$$

In the above equation, d denotes the Stratonovich integral and δ denotes the Itô (or Skorohod) integral.

For this new numerical scheme, we shall consider the strong and weak convergence, as well as the asymptotic error distribution. Our results suggest that the modified Euler scheme should be viewed as an authentic modified version of the Euler-Maruyama scheme (2.1).

1.3 Crank-Nicolson method

In the previous section, we see that to simulate the solutions of SDEs, one can not simply use a deterministic numerical method for ordinary differential equations. In this section, we consider the Crank-Nicolson (or Trapezoidal) method. It provides an example of numerical schemes which have different convergence rates for SDEs driven by 1-dimensional fBms and multi-dimensional fBms.

The Crank-Nicolson method is defined as follows:

$$\begin{aligned} X_{k+1}^n =& X_k^n + \frac{1}{2}\left[b(X_{k+1}^n) + b(X_k^n)\right]\frac{T}{n} \\ &+ \frac{1}{2}\left[V(X_{k+1}^n) + V(X_k^n)\right](B_{t_{k+1}} - B_{t_k}), \\ X_0^n =& x, \end{aligned} \tag{3.1}$$

for $k = 0, 1, \ldots, n-1$.

In the scalar SDE case, that is, assuming that $m = 1$ and the drift term $V_0 \equiv 0$, we will show that the convergence rate of the Crank-Nicolson method is n^{-2H}. This result coincides with the deterministic ordinary differential equation case. By taking $H = 1/2$, we obtain the convergence rate n^{-1}, which coincides that of the Crank-Nicolson method for scalar SDE's driven by Brownian motion (see [30, 29]).

In the multi-dimensional case, due to the appearance of the weighted Lévy area term in the error process, the Crank-Nicolson method has very different properties. We will show that if $m > 1$, then the Crank-Nicolson method has convergence rate $n^{1/2-2H}$ for $H > 1/2$, and if $m = 1$, then its convergence rate is $n^{-1/2-H}$. By considering the weak convergence of the Lévy area term, we also obtain the asymptotic error distributions for the Crank-Nicolson method.

1.4 Taylor scheme

The Taylor scheme is obtained by truncating the stochastic Taylor formula. For example, by chain rule we have

$$X_t = X_s + V(X_s)(B_t - B_s) + \int_s^t \int_s^u (V'V)(X_v) dB_v dB_u, \tag{4.1}$$

where for simplicity we assume that B is a 1-dimensional fBm and $b \equiv 0$. By ignoring the double integral in (4.1), we obtain the first-order Taylor scheme, that is, the Euler scheme (2.1).

Applying chain rule again to (4.1), we obtain

$$X_t = X_s + V(X_s)(B_t - B_s) + (V'V)(X_s) \int_s^t \int_s^u dB_v dB_u + \int_s^t \int_s^u ((V'V)'V)(X_v) dB_v dB_u.$$

By truncating the above identity, we obtain the second-order Taylor scheme.

$$X_{k+1}^n = X_k^n + V(X_k)(B_{t_{k+1}} - B_{t_k}) + (V'V)(X_{t_k}) \int_{t_k}^{t_{k+1}} \int_{t_k}^u dB_v dB_u.$$

In Chapter 4, we shall consider the convergence of Taylor schemes and their generalizations.

Chapter 2

Euler scheme for stochastic differential equations

2.1 Introduction

Consider the following stochastic differential equation (SDE) on $\mathbb{R}^d$

$$X_t = x + \int_0^t b(X_s)ds + \sum_{j=1}^{m} \int_0^t V^j(X_s)dB_s^j, \quad t \in [0,T], \tag{1.1}$$

where $x \in \mathbb{R}^d$, $B = (B^1, \ldots, B^m)$ is an m-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $b, V^1, \ldots, V^m : \mathbb{R}^d \to \mathbb{R}^d$ are continuous functions. The above stochastic integrals are pathwise Riemann-Stieltjes integrals. If $\sigma^1, \ldots, \sigma^m$ are continuously differentiable and their partial derivatives are bounded and locally Hölder continuous of order $\delta > \frac{1}{H} - 1$ and b is Lipschitz, then equation (1.1) has a unique solution which is Hölder continuous of order γ for any $0 < \gamma < H$. This result was first proved by Lyons in [24] using Young integrals (see [48]) and p-variation estimates, and later by Nualart and Rascanu in [36] using fractional calculus (see [49]).

We are interested in numerical approximations for the solution to equation (1.1). For simplicity of the presentation we consider uniform partitions of the interval $[0,T]$, $t_i = \frac{iT}{n}$, $i = 0, \ldots, n$. For every positive integer n, we define $\eta(t) = t_i$ when $t_i \leq t <$

$t_i + \frac{T}{n}$. The following naive Euler numerical approximation scheme has been previously studied

$$X_t^n = x + \int_0^t b(X_{\eta(s)}^n)ds + \sum_{j=1}^m \int_0^t V^j(X_{\eta(s)}^n)dB_s^j, \quad t \in [0,T]. \quad (1.2)$$

This scheme can also be written as

$$X_t^n = X_{t_k}^n + b(X_{t_k}^n)(t - t_k) + \sum_{j=1}^m V^j(X_{t_k}^n)(B_t^j - B_{t_k}^j),$$

for $t_k \le t \le t_{k+1}$, $k = 0, 1, \ldots, n-1$, and $X_0^n = x$. It was proved by Mishura [28] that for any real number $\varepsilon > 0$ there exists a random variable C_ε such that almost surely,

$$\sup_{0 \le t \le T} |X_t^n - X_t| \le C_\varepsilon n^{1-2H+\varepsilon}.$$

Moreover, the convergence rate n^{1-2H} is sharp for this scheme, in the sense that $n^{2H-1}[X_t^n - X_t]$ converges almost surely to a finite and nonzero limit. This has been proved in the one-dimensional case by Nourdin and Neuenkirch in [30] using the Doss representation of the solution (see also Theorem 2.10.1 below). Notice that while H tends to $\frac{1}{2}$, the convergence rate $2H - 1$ of the numerical scheme (1.2) deteriorates, and so it is not a proper extension of the Euler-Maruyama scheme for the case $H = \frac{1}{2}$ (see for example [11, 22]). This is not surprising because the limit $H \to \frac{1}{2}$ of the SDE (1.1) corresponds to a Stratonovich SDE driven by standard Brownian motion, while the Euler scheme (1.2) is the extension of the classical Euler scheme for the Itô SDEs. It is then natural to ask the following question: Can we find a numerical scheme that generalizes the Euler-Maruyama scheme to the fBm case?

In this chapter we introduce a new approximation scheme that we call *modified Euler scheme*:

$$\begin{aligned} X_t^n &= x+\int_0^t b(X_{\eta(s)})ds+\sum_{j=1}^m\int_0^t V^j(X_{\eta(s)}^n)dB_s^j \\ &\quad +H\sum_{j=1}^m\int_0^t(\nabla V^jV^j)(X_{\eta(s)}^n)(s-\eta(s))^{2H-1}ds, \end{aligned} \tag{1.3}$$

or

$$\begin{aligned} X_t^n &= X_{t_k}^n+b(X_{t_k}^n)(t-t_k)+\sum_{j=1}^m V^j(X_{t_k}^n)(B_t^j-B_{t_k}^j) \\ &\quad +\frac{1}{2}\sum_{j=1}^m(\nabla V^jV^j)(X_{t_k}^n)(t-t_k)^{2H}, \end{aligned}$$

for any $t\in[t_k,t_{k+1}]$ and $X_0^n=x$. Here ∇V^j denotes the $d\times d$ matrix $\left(\frac{\partial V^{j,i}}{\partial x_k}\right)_{1\le i,k\le d}$, and $(\nabla V^jV^j)^i=\sum_{k=1}^d\frac{\partial V^{j,i}}{\partial x_k}V^{j,k}$.

Notice that if we formally set $H=\frac{1}{2}$ and replace B by a standard Brownian motion W, this is the classical Euler scheme for the Stratonovich SDE

$$\begin{aligned} X_t &= x+\int_0^t b(X_s)ds+\sum_{j=1}^m\int_0^t V^j(X_s)dW_s^j \\ &= x+\int_0^t b(X_s)ds+\sum_{j=1}^m\int_0^t V^j(X_s)\delta W_s^j+\frac{1}{2}\int_0^t\sum_{j=1}^m(\nabla V^jV^j)(X_s)ds. \end{aligned}$$

In the above and throughout this chapter, d denotes the Stratonovich integral and δ denotes the Itô (or Skorohod) integral.

For our new modified Euler scheme (1.3) we shall prove the following estimate

$$\sup_{0\le t\le T}(\mathbb{E}|X_t-X_t^n|^p)^{\frac{1}{p}}\le C\gamma_n^{-1}, \tag{1.4}$$

for any $p \geq 1$, where

$$\gamma_n = \begin{cases} n^{2H-\frac{1}{2}} & \text{if} \quad \frac{1}{2} < H < \frac{3}{4}, \\ \frac{n}{\sqrt{\log n}} & \text{if} \quad H = \frac{3}{4}, \\ n & \text{if} \quad \frac{3}{4} < H < 1\,. \end{cases} \tag{1.5}$$

Note that in (1.4), if we formally set $H = \frac{1}{2}$, then the convergence rate is $n^{-\frac{1}{2}}$, which is exactly the convergence rate of the classical Euler-Maruyama scheme in the Brownian motion case. This suggests that the modified Euler scheme should be viewed as an authentic modified version of the Euler-Maruyama scheme (1.2). The cutoff of the convergence rate for the Euler scheme has already been observed in a simpler context in [31]. The Lévy area corresponds to the simple SDE with $b = 0$, $V^1(x,y) = (1,0)$, $V^2(x,y) = (0,x)$. In particular, one has $\nabla V^j V^j = 0$, $j = 1,2$ here, i.e. no diagonal noise.

The proof of this result combines the techniques of Malliavin calculus with classical fractional calculus. We also make use of uniform estimates for the moments of all orders of the processes X, X^n and their first and second order Malliavin derivatives, which can be obtained using techniques of fractional calculus, following the approach used, for instance, by Hu and Nualart in [16]. The idea of the proof is to properly decompose the error $X_t - X_t^n$ into a weighted quadratic variation term plus a higher order term, that is,

$$X_t - X_t^n = \sum_{i,j=1}^{m} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} f^{i,j}(t_k) \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \delta B_u^i \delta B_s^j + R_t^n, \tag{1.6}$$

where $\lfloor x \rfloor$ denotes the integer part of a real number x. The weighted quadratic variation term provides the desired rate of convergence in L^p.

To further study this new scheme and compare it to the classical Brownian motion case, it is natural to ask the questions: Is the above rate of convergence (1.4) exact or not? Namely, does the quantity $\gamma_n(X_t - X_t^n)$ have a non-zero limit? If yes, how do we identify the limit, and is there a similarity to the classical Brownian motion case (see [19, 23])? In the second part of the chapter, we give a complete answer to these questions. The weighted variation term in (1.6) is still a key ingredient in our study of the scheme. As it happens in the Breuer-Major theorem, there is a different behavior in the cases $H \in (\frac{1}{2}, \frac{3}{4}]$ and $H \in (\frac{3}{4}, 1)$. If $H \in (\frac{1}{2}, \frac{3}{4}]$, we show that $\gamma_n(X_t - X_t^n)$ converges stably to the solution of a linear stochastic differential equation driven by a matrix-valued Brownian motion W independent of B. The main tools in this case are Malliavin calculus and the fourth moment theorem. We will also make use of a recent limit theorem in law for weighted sums proved in [4]. In the case $H \in (\frac{3}{4}, 1)$, we show the convergence of $\gamma_n(X_t - X_t^n)$ in L^p to the solution of a linear stochastic differential equation driven by a matrix-valued Rosenblatt process. Again we use the technique of Malliavin calculus and the convergence in L^p of weighted sums, which is obtained applying the approach introduced in [4]. We refer to [32] for a discussion on the asymptotic behavior of some weighted Hermite variations of one-dimensional fBm, which are related with the results proved here.

We also consider a weak approximation result for our new numerical scheme. In this case, the rate is n^{-1} for all values of H. More precisely, we are able to show that $n\,[\mathbb{E}(f(X_t)) - \mathbb{E}(f(X_t^n))]$ converges to a finite non zero limit which can be explicitly computed. This extends the result of [45] to $H > \frac{1}{2}$. Let us mention that the techniques of Malliavin calculus also allow us to provide an alternative and simpler proof of the fact that the rate of convergence of the numerical scheme (1.2) is of the order n^{1-2H} and this rate is optimal, extending to the multidimensional case the results by Neuenkirch and Nourdin in [30].

If the driven process is a standard Brownian motion, similar problems have been studied in [19, 23] and the references therein. See also [3] for the precise L^2-limit and also for a discussion on the "best" partition. In the case $\frac{1}{4} < H < \frac{1}{2}$ the SDE (1.1) can be solved using the theory of rough paths introduced by Lyons (see [26]). There are also a number of results on the rate of convergence of Euler-type numerical schemes in this case (see, for instance, the chapter by Deya, Neuenkirch and Tindel [5] for a Milstein-type scheme without Lévy area in the case $\frac{1}{3} < H < \frac{1}{2}$, the paper by Friz and Riedel [7] for the N-step Euler scheme without involving iterated integrals, and the monograph by Friz and Victoir [8]).

The chapter is organized as follows. The next section contains basic materials on fractional calculus and Malliavin calculus that will be used along the chapter, and introduces a matrix-valued Brownian motion and a generalized Rosenblatt process, both of which are key ingredients in our results on the asymptotic behavior of the error (see Section 2.6 and Section 2.8). In Section 3, we derive the necessary estimates for the uniform norms and Hölder seminorms of the processes X, X^n and their Malliavin derivatives. In Section 4, we prove our result on the rate of convergence in L^p for the numerical scheme (1.3). In Section 2.5, we prove a central limit theorem for weighted quadratic sums, and then in Section 2.6 we apply this result to the study of the asymptotic behavior of the error $\gamma_n(X_t - X_t^n)$ in case $H \in (\frac{1}{2}, \frac{3}{4}]$. In Section 2.7, we study the L^p-convergence of some weighted random sums. In Section 2.8, we apply the results of Section 7 to establish the L^p-limit of $n(X_t - X_t^n)$ in case $H \in (\frac{3}{4}, 1)$. The weak approximation result is discussed in Section 9. In Section 10, we deal with the numerical scheme (1.2). In Section 11, we prove some auxiliary results.

2.2 Preliminaries and notations

Throughout the chapter we consider a fixed time interval $[0,T]$. To simplify the presentation we only deal with the uniform partition of this interval, that is, for each $n \geq 1$ and $i = 0,1,\ldots,n$ we set $t_i = \frac{iT}{n}$. We use C and K to represent constants that are independent of n and whose values may change from line to line.

2.2.1 Elements of fractional calculus

In this subsection we introduce the definitions of the fractional integral and derivative operators and we review some properties of these operators.

Let $a,b \in [0,T]$ with $a < b$ and let $\beta \in (0,1)$. We denote by $C^{\beta}(a,b)$ the space of β-Hölder continuous functions on the interval $[a,b]$. For a function $x : [0,T] \to \mathbb{R}$, $\|x\|_{a,b,\beta}$ denotes the β-Hölder seminorm of x on $[a,b]$, that is,

$$\|x\|_{a,b,\beta} = \sup\left\{\frac{|x_u - x_v|}{(v-u)^{\beta}}; a \leq u < v \leq b\right\}.$$

We will also make use of the following seminorm:

$$\|x\|_{a,b,\beta,n} = \sup\left\{\frac{|x_u - x_v|}{(v-u)^{\beta}}; a \leq u < v \leq b, \eta(u) = u\right\}. \tag{2.1}$$

Recall that for each $n \geq 1$ and $i = 0,1,\ldots,n$, $t_i = \frac{iT}{n}$, and $\eta(t) = t_i$ when $t_i \leq t < t_i + \frac{T}{n}$.

We will denote the uniform norm of x on the interval $[a,b]$ as $\|x\|_{a,b,\infty}$. When $a = 0$ and $b = T$, we will simply write $\|x\|_{\infty}$ for $\|x\|_{0,T,\infty}$ and $\|x\|_{\beta}$ for $\|x\|_{0,T,\beta}$.

Let $f \in L^1([a,b])$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined, for almost all $t \in (a,b)$, by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

and

$$I_{b-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds,$$

respectively, where $(-1)^{\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^{\infty} r^{\alpha-1} e^{-r} dr$ is the Gamma function. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p([a,b])$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$, then the fractional Weyl derivatives are defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}} ds \right) \tag{2.2}$$

and

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t)-f(s)}{(s-t)^{\alpha+1}} ds \right), \tag{2.3}$$

where $a < t < b$.

Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Then, according to [48], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives. We refer to [49] for additional details.

Proposition 2.2.1. *Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Let $\lambda > \alpha$ and $\mu > 1-\alpha$. Then the Riemann-Stieltjes integral $\int_a^b f dg$ exists and it can be* *expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt, \tag{2.4}$$

where $g_{b-}(t) = \mathbf{1}_{(a,b)}(t)(g(t) - g(b-))$.

The notion of Hölder continuity and the above result on the existence of Riemann-Stieltjes integrals can be generalized to functions taking values in some normed spaces. We fix a probability space $(\Omega, \mathscr{F}, P)$ and denote by $\|\cdot\|_p$ the norm in the space $L^p := L^p(\Omega)$, where $p \geq 1$.

Definition 2.2.1. *Let* $f = \{f(t), t \in [0,T]\}$ *be a stochastic process such that* $f(t) \in L^p$ *for all* $t \in [0,T]$. *We say that* f *is Hölder continuous of order* $\beta > 0$ *in* L^p *if*

$$\|f(t) - f(s)\|_p \leq C|t-s|^{\beta}, \tag{2.5}$$

for all $s, t \in [0,T]$.

The following result shows that with proper Hölder continuity assumptions on f and g the Riemann-Stieltjes integral $\int_0^T f dg$ exists and equation (2.4) holds.

Proposition 2.2.2. *Let the positive numbers* p_0, λ, μ, p, q *satisfy* $p_0 \geq 1$, $\lambda + \mu > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p_0 p > \frac{1}{\mu}$, $p_0 q > \frac{1}{\lambda}$. *Assume that* $f = \{f(t), t \in [0,T]\}$ *and* $g = \{g(t), t \in [0,T]\}$ *are Hölder continuous stochastic processes of order* μ *and* λ *in* $L^{p_0 p}$ *and* $L^{p_0 q}$, *respectively, and* $f(0) \in L^{p_0 p}$. *Let* $\pi : 0 = t_0 < t_1 < \cdots < t_N = T$ *be a partition on* $[0,T]$, *and* $\xi_i : t_{i-1} \leq \xi_i \leq t_i$. *Then the sum* $\sum_{i=1}^N f(\xi_i)[g(t_i) - g(t_{i-1})]$ *converges in* L^{p_0} *to the Riemann-Stieltjes integral* $\int_0^T f dg$ *as* $|\pi|$ *tends to zero, where* $|\pi| = \max_{1 \leq i \leq N} |t_i - t_{i-1}|$, *and equation (2.4) holds.*

Proposition 2.2.2 can be proved through a slight modification of the proof in the real-valued case done in [49] using Hölder's inequality.

2.2.2 Elements of Malliavin Calculus

We briefly recall some basic facts about the stochastic calculus of variations with respect to a fBm. We refer the reader to [33] for further details. Let $B = \{(B_t^1, \dots, B_t^m), t \in [0,T]\}$ be an m-dimensional fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, defined on some complete probability space $(\Omega, \mathscr{F}, P)$. Namely, B is a mean zero Gaussian process with covariance

$$\mathbb{E}(B_t^i B_s^j) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})\delta_{ij}, \quad i,j = 1, \dots, m,$$

for all $s,t \in [0,T]$, where δ_{ij} is the Kronecker symbol.

Let $\mathscr{H}$ be the Hilbert space defined as the closure of the set of step functions on $[0,T]$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathscr{H}} = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

It is easy to see that the covariance of fBm can be written as

$$\alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv,$$

where $\alpha_H = H(2H-1)$. This implies that

$$\langle \psi, \phi \rangle_{\mathscr{H}} = \alpha_H \int_0^T \int_0^T \psi_u \phi_v |u-v|^{2H-2} du dv$$

for any pair of step functions ϕ and ψ on $[0,T]$.

The Hilbert space $\mathscr{H}$, or more generally, the space $\mathscr{H}^{\otimes l}$ may contain distributions that are not functions (see [40] and [41]). We can find a linear space of functions contained in $\mathscr{H}^{\otimes l}$ in the following way. Let $|\mathscr{H}|^{\otimes l}$ be the linear space of measurable functions ϕ on $[0,T]^l \subset \mathbb{R}^l$ such that

$$\|\phi\|^2_{|\mathscr{H}|^{\otimes l}} := \alpha_H^l \int_{[0,T]^{2l}} |\phi_{\mathbf{u}}||\phi_{\mathbf{v}}||u_1 - v_1|^{2H-2} \cdots |u_l - v_l|^{2H-2} d\mathbf{u} d\mathbf{v} < \infty,$$

where $\mathbf{u} = (u_1, \ldots, u_l), \mathbf{v} = (v_1, \ldots, v_l) \in [0,T]^l$. Suppose $\phi \in L^{\frac{1}{H}}([0,T]^l)$. The following estimate holds

$$\|\phi\|_{|\mathscr{H}|^{\otimes l}} \leq b_{H,l} \|\phi\|_{L^{\frac{1}{H}}([0,T]^l)} \tag{2.6}$$

for some constant $b_{H,l} > 0$ (the case $l = 1$ was proved in [27] and the extension to general case is easy, see [17, equation (2.5)].

The mapping $\mathbf{1}_{[0,t_1]} \times \cdots \times \mathbf{1}_{[0,t_m]} \mapsto (B^1_{t_1}, \ldots, B^m_{t_m})$ can be extended to a linear isometry between $\mathscr{H}^m$ and the Gaussian space spanned by B. We denote this isometry by $h \mapsto B(h)$. In this way, $\{B(h), h \in \mathscr{H}^m\}$ is an isonormal Gaussian process indexed by the Hilbert space $\mathscr{H}^m$.

Let $\mathscr{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B_{s_1}, \ldots, B_{s_N}),$$

where $N \geq 1$ and $f \in C_b^\infty(\mathbb{R}^{m \times N})$. For each $j = 1, \ldots, m$ and $t \in [0,T]$, the derivative operator $D^j F$ on $F \in \mathscr{S}$ is defined as the $\mathscr{H}$-valued random variable

$$D^j_t F = \sum_{i=1}^N \frac{\partial f}{\partial x^j_i}(B_{s_1}, \ldots, B_{s_N}) \mathbf{1}_{[0,s_i]}(t),\ t \in [0,T].$$

We can iterate this procedure to define higher order derivatives $D^{j_1,\dots,j_l}F$ which take values on $\mathscr{H}^{\otimes l}$. For any $p \geq 1$ and any integer $k \geq 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of $\mathscr{S}$ with respect to the norm

$$\|F\|_{k,p}^p = \mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\sum_{l=1}^{k}\left(\sum_{j_1,\dots,j_l=1}^{m}\left\|D^{j_1,\dots,j_l}F\right\|_{\mathscr{H}^{\otimes l}}^2\right)^{\frac{p}{2}}\right].$$

If V is a Hilbert space, $\mathbb{D}^{k,p}(V)$ denotes the corresponding Sobolev space of V-valued random variables.

For any $j = 1,\dots,m$ we denote by δ^j the adjoint of the derivative operator D^j. We say $u \in \mathrm{Dom}\delta^j$ if there is a $\delta^j(u) \in L^2(\Omega)$ such that for any $F \in \mathbb{D}^{1,2}$ the following duality relationship holds

$$\mathbb{E}\left(\langle u, D^j F\rangle_{\mathscr{H}}\right) = \mathbb{E}(\delta^j(u)F). \tag{2.7}$$

The random variable $\delta^j(u)$ is also called the Skorohod integral of u with respect to the fBm B^j and we use the notation $\delta^j(u) = \int_0^T u_t \delta B_t^j$.

Let $F \in \mathbb{D}^{1,2}$ and u be in the domain of δ^j such that $Fu \in L^2(\Omega;\mathscr{H})$. Then (see [34]) Fu belongs to the domain of δ^j and the following equality holds

$$\delta^j(Fu) = F\delta^j(u) - \langle D^j F, u\rangle_{\mathscr{H}}, \tag{2.8}$$

provided the right-hand side of (2.8) is square integrable.

Suppose that $u = \{u_t, t \in [0,T]\}$ is a stochastic process whose trajectories are Hölder continuous of order $\gamma > 1 - H$. Then, for any $j = 1,\dots,m$, the Riemann-Stieltjes integral $\int_0^T u_t dB_t^j$ exists. On the other hand, if $u \in \mathbb{D}^{1,2}(\mathscr{H})$ and the derivative $D_s^j u_t$ exists and satisfies almost surely

$$\int_0^T \int_0^T |D_s^j u_t| \, |t-s|^{2H-2} ds dt < \infty,$$

and $\mathbb{E}\left(\|D^j u\|^2_{L^{\frac{1}{H}}([0,T]^2)} \right) < \infty$, then (see Proposition 5.2.3 in [34]) $\int_0^T u_t \delta B_t^j$ exists and we have the following relationship between these two stochastic integrals

$$\int_0^T u_t dB_t^j = \int_0^T u_t \delta B_t^j + \alpha_H \int_0^T \int_0^T D_s^j u_t |t-s|^{2H-2} ds dt. \tag{2.9}$$

The following result is Meyer's inequality for the Skorohod integral (see, for example, Proposition 1.5.7 of [34]). Given $p > 1$ and an integer $k \geq 1$, there is a constant $c_{k,p}$ such that

$$\|\delta^k(u)\|_p \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathscr{H}^{\otimes k})} \qquad \text{for all } u \in \mathbb{D}^{k,p}(\mathscr{H}^{\otimes k}). \tag{2.10}$$

Applying (2.6) and then the Minkowski inequality to the right-hand side of (2.10) yields

$$\|\delta^k(u)\|_p \leq \quad C \left\| \|u\|_p \right\|_{L^{\frac{1}{H}}([0,T]^p)} + C \sum_{l=1}^{k} \sum_{j_1,\ldots,j_l=1}^{m} \left\| \|D^{j_1,\ldots,j_l} u\|_p \right\|_{L^{\frac{1}{H}}([0,T]^{p+l})} \tag{2.11}$$

for all $u \in \mathbb{D}^{k,p}(\mathscr{H}^{\otimes k})$ provided $pH \geq 1$.

2.2.3 Stable convergence

Let Y_n, $n \in \mathbb{N}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ with values in a Polish space $(E, \mathscr{E})$. We say that Y_n *converges stably* to the limit Y, where Y is defined on an extension of the original probability space $(\Omega', \mathscr{F}', P')$, if and only if for any bounded $\mathscr{F}$-measurable random variable Z it holds that

$$(Y_n, Z) \Rightarrow (Y, Z)$$

as $n \to \infty$, where $\Rightarrow$ denotes the convergence in law.

Note that stable convergence is stronger than weak convergence but weaker than convergence in probability. We refer to [20] and [1] for more details on this concept.

2.2.4 A matrix-valued Brownian motion

The aim of this subsection is to define a matrix-valued Brownian motion that will play a fundamental role in our central limit theorem. First, we introduce two constants Q and R which depend on H.

Denote by μ the measure on $\mathbb{R}^2$ with density $|s-t|^{2H-2}$. Define, for each $p \in \mathbb{Z}$,

$$Q(p) = T^{4H} \int_0^1 \int_p^{p+1} \int_0^t \int_p^s \mu(dvdu)\mu(dsdt)$$

and

$$R(p) = T^{4H} \int_0^1 \int_p^{p+1} \int_t^1 \int_p^s \mu(dvdu)\mu(dsdt).$$

It is not difficult to check that for $\frac{1}{2} < H < \frac{3}{4}$ the series $\sum_{p\in\mathbb{Z}} Q(p)$ and $\sum_{p\in\mathbb{Z}} R(p)$ are convergent and for $H = \frac{3}{4}$, they diverge at the rate $\log n$. Then we set (we omit the explicit dependence of Q and R on H to simplify the notation)

$$Q = \sum_{p\in\mathbb{Z}} Q(p), \quad R = \sum_{p\in\mathbb{Z}} R(p), \tag{2.12}$$

for the case $H \in (\frac{1}{2}, \frac{3}{4})$, and

$$Q = \lim_{n\to\infty} \frac{\sum_{|p|\le n} Q(p)}{\log n} = \frac{T^{4H}}{2}, \quad R = \lim_{n\to\infty} \frac{\sum_{|p|\le n} R(p)}{\log n} = \frac{T^{4H}}{2},$$

for the case $H = \frac{3}{4}$.

Lemma 2.2.1. *The constants Q and R satisfy $R \le Q$.*

Proof: If $H = \frac{3}{4}$, we see from (2.12) that these two constants are both equal to $\frac{T^{4H}}{2}$. Suppose $H \in (\frac{1}{2}, \frac{3}{4})$. Consider the functions on $\mathbb{R}^2$ defined by $\varphi_p(v,s) = \mathbf{1}_{\{p\le v\le s\le p+1\}}$, $\psi_p(v,s) = \mathbf{1}_{\{p\le s\le v\le p+1\}}$, $p \in \mathbb{Z}$. Then

$$\begin{aligned}
\frac{1}{n}\left\|\sum_{p=0}^{n-1}(\varphi_p - \psi_p)\right\|^2_{L^2(\mathbb{R}^2,\mu)} &= \frac{2}{n}\sum_{p,q=0}^{n-1}\Big(\big\langle \mathbf{1}_{\{p\le v\le s\le p+1\}}, \mathbf{1}_{\{q\le v\le s\le q+1\}}\big\rangle_{L^2(\mathbb{R}^2,\mu)} \\
&\quad - \big\langle \mathbf{1}_{\{p\le v\le s\le p+1\}}, \mathbf{1}_{\{q\le s\le v\le q+1\}}\big\rangle_{L^2(\mathbb{R}^2,\mu)}\Big) \\
&= \frac{2}{n}\sum_{p,q=0}^{n-1}\left(Q(p-q) - R(p-q)\right).
\end{aligned}$$

It is easy to see that the above is equal to

$$\frac{2}{n}\sum_{j=0}^{n-1}\sum_{k=-j}^{j}(Q(k) - R(k)).$$

It then follows from a Cesàro limit argument that the quantity in the right hand side of the above converges to $2(Q-R)$ as n tends to infinity. Therefore, $Q \ge R$. □

Let $\widetilde{W}^{0,ij} = \{\widetilde{W}^{0,ij}_t, t \in [0,T]\}$, $i \le j, i,j = 1,\dots,m$ and $\widetilde{W}^{1,ij} = \{\widetilde{W}^{1,ij}_t, t \in [0,T]\}$, $i,j = 1,\dots,m$ be independent standard Brownian motions. When $i > j$, we define $\widetilde{W}^{0,ij}_t = \widetilde{W}^{0,ji}_t$. The matrix-valued Brownian motion $(W^{ij})_{1\le i,j\le m}$, $i,j = 1,\dots,m$ is defined as follows:

$$W^{ii} = \frac{\alpha_H}{\sqrt{T}} \left(\sqrt{Q+R}\, \widetilde{W}^{1,ii} \right)$$

and

$$W^{ij} = \frac{\alpha_H}{\sqrt{T}} \left(\sqrt{Q-R}\, \widetilde{W}^{1,ij} + \sqrt{R}\, \widetilde{W}^{0,ij} \right) \quad \text{when } i \neq j.$$

Notice that this definition makes sense because $R \leq Q$. The random matrix W_t is not symmetric when $H < \frac{3}{4}$ (see the plot and table below). For $i, j, i', j' = 1, \ldots, m$, the covariance $\mathbb{E}(W_t^{ij} W_s^{i'j'})$ is equal to

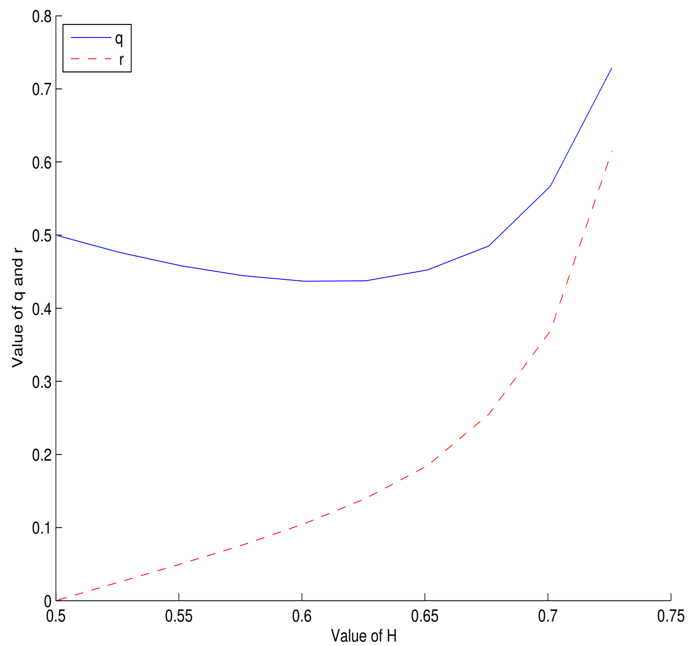
$$\frac{\alpha_H^2 (t \wedge s)}{T} (R\delta_{ji'}\delta_{ij'} + Q\delta_{jj'}\delta_{ii'}),$$

where δ is the Kronecker function.

In the following plot and table, we consider two quantities for $H \in (\frac{1}{2}, \frac{3}{4})$,

$$q = \frac{\alpha_H^2}{T^{4H}} Q, \text{ and } r = \frac{\alpha_H^2}{T^{4H}} R.$$

We see that the values of q and r approach 0.5 and 0 as H tends to $\frac{1}{2}$, respectively, and both of them tend to infinity when H gets closer to $\frac{3}{4}$.



H	0.5010	0.5260	0.5510	0.6010	0.6260	0.6510	0.7010	0.7260
q	0.4990	0.4763	0.4580	0.4369	0.4375	0.4522	0.5669	0.7290
r	9.9868×10^{-4}	0.0256	0.0503	0.1053	0.1400	0.1845	0.3689	0.6149

2.2.5 A matrix-valued generalized Rosenblatt process

In this subsection we introduce a generalized Rosenblatt process which will appear in the limiting result proved in Section 2.8 when $H > \frac{3}{4}$. Consider an m-dimensional fBm $B_t = (B_t^1, \ldots, B_t^m)$ with Hurst parameter $H \in (\frac{3}{4}, 1)$. Define for $i_1, i_2 \in 1, \ldots, m$

$$Z_n^{i_1,i_2}(t) := n \sum_{j=1}^{\lfloor \frac{nt}{T} \rfloor} \int_{t_j}^{t_{j+1}} (B_s^{i_1} - B_{t_j}^{i_1}) \delta B_s^{i_2}.$$

When $i_1 = i_2 = i$ we can write

$$Z_n^{i,i}(t) = \frac{T^{2H}}{2n^{2H-1}} \sum_{j=1}^{\lfloor \frac{nt}{T} \rfloor} H_2(\xi_j^{n,i}),$$

where $H_2(x) = x^2 - 1$ is the second degree Hermite polynomial and $\xi_j^{n,i} = T^{-H} n^H (B^i_{t_{j+1}} - B^i_{t_j})$. It is well known (see [32]) that for each $i = 1, \dots, m$, the process $Z_n^{i,i}(t)$ converges in L^2 to the *Rosenblatt process* $R(t)$. We refer the reader to [43] and [47] for further details on the Rosenblatt process.

When $i_1 \neq i_2$, the stochastic integral $\int_{t_j}^{t_{j+1}} (B_s^{i_1} - B_{t_j}^{i_1}) \delta B_s^{i_2}$ cannot be written as the second Hermite polynomial of a Gaussian random variable. Nevertheless, the process $Z_n^{i_1,i_2}(t)$ is still convergent in L^2. Indeed, for any positive integers n and n', we have

$$\begin{aligned}
&\mathbb{E}\left(Z_n^{i_1 i_2}(t) Z_{n'}^{i_1 i_2}(t) \right) \\
&= nn' \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \sum_{k'=0}^{\lfloor \frac{n't}{T} \rfloor} \mathbb{E} \left[\int_{\frac{k}{n}T}^{\frac{k+1}{n}T} (B_s^{i_1} - B^{i_1}_{\frac{k}{n}T}) \delta B_s^{i_2} \int_{\frac{k}{n'}T}^{\frac{k+1}{n'}T} (B_s^{i_1} - B^{i_1}_{\frac{k}{n'}T}) \delta B_s^{i_2} \right] \\
&= nn' \alpha_H^2 \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \sum_{k'=0}^{\lfloor \frac{n't}{T} \rfloor} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} \int_{\frac{k'}{n'}T}^{\frac{k'+1}{n'}T} \int_{\frac{k}{n}T}^{t} \int_{\frac{k'}{n'}T}^{s} \mu(dvdu) \mu(dsdt) \\
&\to \frac{T^2 \alpha_H^2}{4} \int_0^t \int_0^t |u-v|^{4H-4} dudv \\
&= c_H t^{4H-2},
\end{aligned}$$

as $n', n \to +\infty$, where $c_H = \frac{T^2 H^2 (2H-1)}{4(4H-3)}$. This allows us to conclude that $Z_n^{i_1 i_2}(t)$ is a Cauchy sequence in L^2. We denote by $Z_t^{i_1 i_2}$ the L^2-limit of $Z_n^{i_1 i_2}(t)$. Then $Z_t^{i_1 i_2}$ can be considered as a *generalized Rosenblatt process*.

It is easy to show that

$$\mathbb{E}[|Z_t^{i_1 i_2} - Z_s^{i_1 i_2}|^2] \le C|t-s|^{4H-2},$$

and by the hypercontractivity property, we deduce

$$\mathbb{E}[|Z_t^{i_1 i_2} - Z_s^{i_1 i_2}|^p] \leq C_p |t-s|^{p(2H-1)} \tag{2.13}$$

for any $p \geq 2$ and $s,t \in [0,T]$. By the Kolmogorov continuity criterion this implies that $Z^{i_1 i_2}$ has a Hölder continuous version of exponent λ for any $\lambda < 2H-1$.

2.3 Estimates for solutions of some SDE's

The purpose of this section is to provide upper bounds for the Hölder seminorms of solutions of two types of SDE's. The first type (see (3.1)) covers equation (1.1) and its Malliavin derivatives, as well as all the other SDE's involving only continuous integrands which we will encounter in this chapter. The second type (see (3.13)) deals with the case where the integrands are step processes. These SDE's arise from the approximation schemes such as (1.2) and (1.3).

For any integers $k, N, M \geq 1$, we denote by $C_b^k(\mathbb{R}^M; \mathbb{R}^N)$ the space of k times continuously differentiable functions $f : \mathbb{R}^M \to \mathbb{R}^N$ which are bounded together with their first k partial derivatives. If $N = 1$ we simply write $C_b^k(\mathbb{R}^M)$.

In order to simplify the notation we only consider the case when the fBm is one-dimensional, that is, $m = 1$. All results of this section can be generalized to the case $m > 1$. Throughout the remaining part of the chapter we let β be any number satisfying $\frac{1}{2} < \beta < H$. The first two lemmas are path-wise results and they will still hold when B is replaced by general Hölder continuous functions of index $\gamma > \beta$. The constants appearing in the lemmas depend on β, H, T, and the uniform and Hölder seminorms of the coefficients. We fix a time interval $[\tau, T]$, and to simplify we omit the dependence on τ and T of the uniform norm and β-Hölder seminorm on the interval $[\tau, T]$.

Lemma 2.3.1. *Fix $\tau \in [0,T)$. Let $V = \{V_t, t \in [\tau,T]\}$ be an $\mathbb{R}^M$-valued processes satisfying*

$$V_t = S_t + \int_\tau^t [g_1(V_u) + U_u^1 V_u]du + \int_\tau^t [g_2(V_u) + U_u^2 V_u]dB_u, \tag{3.1}$$

where $g_1 \in C_b(\mathbb{R}^M;\mathbb{R}^M), g_2 \in C_b^1(\mathbb{R}^M;\mathbb{R}^M)$ and $U^i = \{U_t^i, t \in [\tau,T]\}$, $i = 1,2$, and $S = \{S_t, \in [\tau,T]\}$ are $\mathbb{R}^{M\times M}$-valued and $\mathbb{R}^M$-valued processes, respectively. We assume that S has β-Hölder continuous trajectories, and the processes U^i, $i = 1,2$, are uniformly bounded by a constant C.

(i) If $U^1 = U^2 = 0$, then we can find constants K and K' such that $(t-s)^\beta \|B\|_\beta \le K$, $\tau \le s < t \le T$, implies

$$\|V\|_{s,t,\beta} \le K'(\|B\|_\beta + 1) + 2\|S\|_\beta.$$

(ii) Suppose that there exist constants K_0 and K_0' such that $(t-s)^\beta \|B\|_\beta \le K_0$, $\tau \le s < t \le T$, implies

$$\|U^2\|_{s,t,\beta} \le K_0'(\|B\|_\beta + 1). \tag{3.2}$$

Then there exists a positive constant K such that

$$\max\{\|V\|_\infty, \|V\|_\beta\} \le K e^{K\|B\|_\beta^{1/\beta}} (|S_\tau| + \|S\|_\beta + 1). \tag{3.3}$$

Proof: The proof follows the approach used, for instance, by Hu and Nualart in [16]. Let $\tau \le s < t \le T$. By the definition of V,

$$V_t - V_s = S_t - S_s + \int_s^t [g_1(V_u) + U_u^1 V_u]du + \int_s^t [g_2(V_u) + U_u^2 V_u]dB_u. \tag{3.4}$$

Applying Lemma 2.11.1(ii) to the vector valued function $f:(u,v)\to g_2(v)+uv$ and the integrator $z=B$ and taking $\beta'=\beta$ yield

$$
\begin{aligned}
|V_t-V_s| \quad \leq \quad & \|S\|_\beta(t-s)^\beta+(\|g_1\|_\infty+C\|V\|_{s,t,\infty})(t-s) \\
& +K_1(\|g_2\|_\infty+C\|V\|_{s,t,\infty})\|B\|_\beta(t-s)^\beta \\
& +K_2(\|\nabla g_2\|_\infty+C)\|V\|_{s,t,\beta}\|B\|_\beta(t-s)^{2\beta} \\
& +K_2\|V\|_{s,t,\infty}\|U^2\|_{s,t,\beta}\|B\|_\beta(t-s)^{2\beta}.
\end{aligned} \tag{3.5}
$$

Step 1. In the case $U^1=U^2=0$ (which means that we can take $C=0$ and $\|U^2\|_{s,t,\beta}=0$), dividing both sides of (3.5) by $(t-s)^\beta$ and taking the Hölder seminorm on the left-hand side, we obtain

$$
\|V\|_{s,t,\beta}\leq\|S\|_\beta+c_1(t-s)^{1-\beta}+K_1c_1\|B\|_\beta+K_2c_1\|V\|_{s,t,\beta}\|B\|_\beta(t-s)^\beta, \tag{3.6}
$$

where and throughout this section we denote

$$
c_1=\max\{C,\|g_1\|_\infty,\|g_2\|_\infty,\|\nabla g_2\|_\infty\}. \tag{3.7}
$$

Take $K=\frac{1}{2}(K_2c_1)^{-1}$. Then for any $\tau\leq s<t\leq T$ such that $(t-s)^\beta\|B\|_\beta\leq K$, we have

$$
\|V\|_{s,t,\beta}\leq 2\|S\|_\beta+2c_1(t-s)^{1-\beta}+2K_1c_1\|B\|_\beta,
$$

which implies (i).

Step 2. As in Step 1, we divide (3.5) by $(t-s)^\beta$ and then take the Hölder seminorm on the left-hand side to obtain

$$\begin{aligned} \|V\|_{s,t,\beta} \leq & \|S\|_\beta + c_1(1+\|V\|_{s,t,\infty})(t-s)^{1-\beta} \\ & +K_1c_1(1+\|V\|_{s,t,\infty})\|B\|_\beta \\ & +2K_2c_1\|V\|_{s,t,\beta}\|B\|_\beta(t-s)^\beta \\ & +K_2\|V\|_{s,t,\infty}\|U^2\|_{s,t,\beta}\|B\|_\beta(t-s)^\beta. \end{aligned} \tag{3.8}$$

If $(t-s)^\beta\|B\|_\beta \leq \frac{1}{4}(K_2c_1)^{-1}$, then the coefficient of $\|V\|_{s,t,\beta}$ on the right-hand side of (3.8) is less or equal than $\frac{1}{2}$. Thus, we obtain

$$\begin{aligned} \|V\|_{s,t,\beta} \leq & 2\|S\|_\beta + 2c_1(1+\|V\|_{s,t,\infty})(t-s)^{1-\beta} \\ & +2K_1c_1(1+\|V\|_{s,t,\infty})\|B\|_\beta \\ & +2K_2\|V\|_{s,t,\infty}\|U^2\|_{s,t,\beta}\|B\|_\beta(t-s)^\beta. \end{aligned}$$

On the other hand, assuming $(t-s)^\beta\|B\|_\beta \leq K_0$ and applying (3.2), we obtain

$$\|V\|_{s,t,\beta} \leq 2\|S\|_\beta + C_1(1+\|B\|_\beta)(1+\|V\|_{s,t,\infty}), \tag{3.9}$$

for some constant C_1. This implies

$$\|V\|_{s,t,\infty} \leq |V_s| + 2(t-s)^\beta\|S\|_\beta + C_1(t-s)^\beta(1+\|B\|_\beta)(1+\|V\|_{s,t,\infty}).$$

Assuming $(t-s)^\beta\|B\|_\beta \leq \frac{1}{4C_1}$ and $(t-s)^\beta \leq \frac{1}{4C_1} \wedge \frac{1}{2}$ we obtain

$$\|V\|_{s,t,\infty} \leq 2|V_s| + 2\|S\|_\beta + 1. \tag{3.10}$$

Take $\Delta = \left[\|B\|_\beta^{-1} \min\left(\frac{1}{4K_2c_1}, K_0, \frac{1}{4C_1}\right)\right]^{1/\beta} \wedge \left(\frac{1}{4C_1} \wedge \frac{1}{2}\right)^{1/\beta}$. We divide the interval $[\tau, T]$ into $N = \lfloor \frac{T-\tau}{\Delta} \rfloor + 1$ subintervals and denote by $s_1, s_2, \ldots, s_N$ the left endpoints of these intervals and $s_{N+1} = T$. Applying the inequality (3.10) to each interval $[s_i, s_{i+1}]$ for $i = 1, \ldots, N$ yields

$$\|V\|_\infty \le 2^{N+1} \left(|S_\tau| + 2\|S\|_\beta + 1\right). \tag{3.11}$$

From the definition of Δ we get

$$N \le 1 + \frac{T}{\Delta} \le 1 + T \max\left(C_2, C_3 \|B\|_\beta^{1/\beta}\right), \tag{3.12}$$

for some constants C_2 and C_3. From inequalities (3.11) and (3.12) we obtain the desired estimate for $\|V\|_\infty$.

If $t, s \in [\tau, T]$ satisfy $0 \le t - s \le \Delta$ then from (3.9) and from the upper bound of $\|V\|_\infty$ we can estimate $\frac{V_t - V_s}{(t-s)^\beta}$ by the right-hand side of (3.3) for some constant K. On the other hand, if $t - s > \Delta$, then

$$\frac{\left|V_t - V_s\right|}{(t-s)^\beta} \le 2\|V\|_\infty \Delta^{-1}.$$

We can obtain a similar estimate from the upper bound of $\|V\|_\infty$ and from the definition of Δ. This gives then the desired estimate for $\|V\|_\beta$ and hence we complete the proof of (ii). □

For the second lemma we fix n and consider the partition of $[0, T]$ given by $t_i = i\frac{T}{n}$, $i = 0, 1, \ldots, n$. Define $\eta(t) = t_i$ if $t_i \le t < t_i + \frac{T}{n}$ and $\varepsilon(t) = t_i + \frac{T}{n}$ if $t_i < t \le t_i + \frac{T}{n}$.

Lemma 2.3.2. *Suppose that S, g_i, U^i, $i=1,2$, are the same as in Lemma 2.3.1. Let $g \in C([0,T])$. Let $V=\{V_t, t\in[\tau,T]\}$ be an $\mathbb{R}^M$-valued processes satisfying the equation*

$$\begin{aligned} V_t &= S_t + \int_{\varepsilon(\tau)}^{t\vee\varepsilon(\tau)} [g_1(V_{\eta(u)}) + U^1_{\eta(u)} V_{\eta(u)}] g(u-\eta(u)) du \\ &\quad + \int_{\varepsilon(\tau)}^{t\vee\varepsilon(\tau)} [g_2(V_{\eta(u)}) + U^2_{\eta(u)} V_{\eta(u)}] dB_u. \end{aligned} \tag{3.13}$$

(i) If $U^1=U^2=0$, then we can find constants K and K' such that $(t-s)^\beta \|B\|_\beta \le K$, $\tau \le s < t \le T$, implies

$$\|V\|_{s,t,\beta,n} \le K'(\|B\|_\beta + 1) + 2\|S\|_\beta.$$

(ii) Suppose that there exist constants K_0 and K_0' such that $(t-s)^\beta \|B\|_\beta \le K_0$, $\tau \le s < t \le T$, implies

$$\|U^2\|_{s,t,\beta,n} \le K_0'(\|B\|_\beta + 1). \tag{3.14}$$

Then, there exists a constant K such that

$$\max\{\|V\|_\infty, \|V\|_\beta\} \le K e^{K\|B\|_\beta^{1/\beta}} (|S_\tau| + \|S\|_\beta + 1).$$

Remark 2.3.1. *The proof of this result is similar to that of Lemma 2.3.1. Nevertheless, since the integral is discrete, we need to replace the Hölder seminorm $\|\cdot\|_{s,t,\beta}$ by the seminorm $\|\cdot\|_{s,t,\beta,n}$ introduced in (2.1).*

Proof: Let $s,t \in [\tau,T]$ be such that $s<t$ and $s=\eta(s)$. This implies $s \ge \varepsilon(\tau)$. As in

the proof of (3.5), applying Lemma 2.11.1(i) (instead of Lemma 2.11.1(ii)) yields

$$
\begin{aligned}
|V_t - V_s| \leq & \|S\|_\beta (t-s)^\beta + (\|g_1\|_\infty + C\|V\|_{s,t,\infty})\|g\|_\infty (t-s) \\
& + K_1(\|g_2\|_\infty + C\|V\|_{s,t,\infty})\|B\|_\beta (t-s)^\beta \\
& + K_3[(\|\nabla g_2\|_\infty + C)\|V\|_{s,t,\beta,n} + \|V\|_{s,t,\infty}\|U^2\|_{s,t,\beta,n}]\|B\|_\beta (t-s)^{2\beta}.
\end{aligned}
$$

Dividing both sides of the above inequality by $(t-s)^\beta$ and taking the Hölder seminorm on the left-hand side we obtain

$$
\begin{aligned}
\|V\|_{s,t,\beta,n} \leq \; & \|S\|_\beta + (\|g_1\|_\infty + C\|V\|_{s,t,\infty})\|g\|_\infty (t-s)^{1-\beta} \\
& + K_1(\|g_2\|_\infty + C\|V\|_{s,t,\infty})\|B\|_\beta \\
& + K_3(\|\nabla g_2\|_\infty + C)\|V\|_{s,t,\beta,n}\|B\|_\beta (t-s)^\beta \\
& + K_3\|V\|_{s,t,\infty}\|U^2\|_{s,t,\beta,n}\|B\|_\beta (t-s)^\beta.
\end{aligned}
\tag{3.15}
$$

Step 1. In the case $U^1 = U^2 = 0$, (3.15) becomes

$$
\|V\|_{s,t,\beta,n} \leq \|S\|_\beta + c_1\|g\|_\infty (t-s)^{1-\beta} + K_1 c_1 \|B\|_\beta + K_3 c_1 \|V\|_{s,t,\beta,n}\|B\|_\beta (t-s)^\beta,
$$

where c_1 is defined in (3.7). Taking $K = \frac{1}{2}(K_3 c_1)^{-1}$, for any $\tau \leq s < t \leq T$ such that $(t-s)^\beta \|B\|_\beta \leq K$, we have

$$
\|V\|_{s,t,\beta,n} \leq 2\|S\|_\beta + 2c_1\|g\|_\infty (t-s)^{1-\beta} + 2K_1 c_1 \|B\|_\beta.
$$

This completes the proof of (i).

Step 2. In the general case, we follow the proof of Lemma 2.3.1, except that we assume $s = \eta(s)$ and use the seminorm $\|\cdot\|_{s,t,\beta,n}$ instead of $\|\cdot\|_{s,t,\beta}$. We also apply (3.14) instead of (3.2). In this way we obtain the inequality (3.9) with $\|V\|_{s,t,\beta}$ replaced by $\|V\|_{s,t,\beta,n}$, that is,

$$\|V\|_{s,t,\beta,n} \quad \leq \quad 2\|S\|_\beta + C_1(1+\|B\|_\beta)(1+\|V\|_{s,t,\infty}) \tag{3.16}$$

for some constant C_1. The inequality (3.10) remains the same

$$\|V\|_{s,t,\infty} \quad \leq \quad 2|V_s| + 2\|S\|_\beta + 1, \tag{3.17}$$

provided $s = \eta(s)$ and both $t-s$ and $(t-s)^\beta \|B\|_\beta$ are bounded by some constant C_4.

Take $\Delta = (C_4^{1/\beta}\|B\|_\beta^{-1/\beta}) \wedge C_4$. We are going to consider two cases depending on the relation between Δ and $\frac{2T}{n}$.

If $\Delta > \frac{2T}{n}$, we take $N = \lfloor \frac{2(T-\varepsilon(\tau))}{\Delta} \rfloor$ and divide the interval $[\varepsilon(\tau), \varepsilon(\tau) + N\frac{\Delta}{2}]$ into N subintervals of length $\frac{\Delta}{2}$. Since the length of each of these subintervals is larger than $\frac{T}{n}$, we are able to choose N points $s_1, s_2, \ldots, s_N$ from each of these intervals such that $s_1 = \varepsilon(\tau)$ and $\eta(s_i) = s_i$, $i = 1, 2, \ldots, N$. On the other hand, we have $s_{i+1} - s_i \leq \Delta$ for all $i = 1, \ldots, N-1$. Applying the inequality (3.17) to each of the intervals $[s_1, s_2]$, $[s_2, s_3]$, $\ldots$, $[s_{N-1}, s_N]$, $[s_N, T]$ yields

$$\|V\|_{\varepsilon(\tau),T,\infty} \quad \leq \quad 2^{N+1}\left(|S_{\varepsilon(\tau)}| + 2\|S\|_\beta + 1\right). \tag{3.18}$$

From the definition of Δ we have

$$N \leq \frac{2T}{\Delta} \leq K + K\|B\|_\beta^{1/\beta}, \tag{3.19}$$

for some constant K depending on T and C_4. From (3.18) and (3.19) and taking into account that

$$\|V\|_{\tau,\varepsilon(\tau),\infty} = \|S\|_{\tau,\varepsilon(\tau),\infty} \leq |S_\tau| + T^\beta \|S\|_\beta, \tag{3.20}$$

we obtain the desired estimate for $\|V\|_\infty$.

If $\Delta \leq \frac{2T}{n}$, that is, when $n \leq \frac{2T}{\Delta} \leq K + K\|B\|_\beta^{1/\beta}$, then by equation (3.13) we have

$$\begin{aligned} |V_t| &\leq |V_{\eta(t)}| + |S_t - S_{\eta(t)}| + (c_1 + C|V_{\eta(t)}|)\|g\|_\infty (T/n) \\ &\quad + (c_1 + C|V_{\eta(t)}|)\|B\|_\beta (T/n)^\beta \\ &\leq A_n + B_n |V_{\eta(t)}|, \end{aligned}$$

for any $t \in [\tau, T]$, where

$$A_n = \|S\|_\beta (T/n)^\beta + c_1 \|g\|_\infty (T/n) + c_1 \|B\|_\beta (T/n)^\beta$$

and

$$B_n = 1 + C\|g\|_\infty (T/n) + C\|B\|_\beta (T/n)^\beta.$$

Iterating this estimate, we obtain

$$\begin{aligned} \|V\|_{\varepsilon(\tau),T,\infty} &\leq |S_{\varepsilon(\tau)}| B_n^n + n A_n B_n^{n-1} \\ &\leq K(|S_{\varepsilon(\tau)}| + \|S\|_\beta + 1) e^{K\|B\|_\beta^{1/\beta}}, \end{aligned} \tag{3.21}$$

for some constant K independent of n, where we have used the inequality

$$B_n^n \leq e^{K(1+\|B\|_\beta) n^{1-\beta}},$$

and the fact that $n \leq K + K\|B\|_\beta^{1/\beta}$ for some constant K. Taking into account (3.20), we obtain the desired upper bound for $\|V\|_\infty$.

In order to show the upper bound for $\|V\|_{\tau,T,\beta}$, we notice that if $0 \leq t-s \leq \Delta$, then from (3.16) and from the upper bound of $\|V\|_{\tau,T,\infty}$ we have

$$\|V\|_{\varepsilon(s),t,\beta,n} \leq K(|S_\tau| + \|S\|_\beta + 1)e^{K\|B\|_\beta^{1/\beta}},$$

for some constant K. Thus

$$\begin{aligned} \frac{|V_t - V_s|}{(t-s)^\beta} &\leq \|V\|_{\varepsilon(s),t,\beta,n} + \frac{|V_{\varepsilon(s)} - V_s|}{(\varepsilon(s)-s)^\beta} \\ &\leq K(|S_\tau| + \|S\|_\beta + 1)e^{K\|B\|_\beta^{1/\beta}}. \end{aligned}$$

If $t-s \geq \Delta$, we can obtain the upper bound of $\|V\|_\beta$ by a similar argument as in the proof of Lemma 2.3.1. The proof of (ii) is now complete. □

The following result gives upper bounds for the norm of Malliavin derivatives of the solutions of the two types of SDE's (3.1) and (3.13). Given a process $P = \{P_t, t \in [\tau, T]\}$ such that $P_t \in \mathbb{D}^{N,2}$ for each t and some $N \geq 1$, we denote by $\mathscr{D}_N^* P$ the maximum of the supnorms of the functions P_{r_0}, $D_{r_1}P_{r_0}$, ..., $D^N_{r_1,\dots,r_N}P_{r_0}$ over $r_0, \dots, r_N \in [\tau, T]$, and denote by $\mathscr{D}_N P$ the maximum of the random variable $\mathscr{D}_N^* P$ and the supnorms of $\|P\|_\beta$, $\|D_{r_1}P\|_{r_1,T,\beta}$, ..., $\|D^N_{r_1,\dots,r_N}P\|_{r_1\vee\cdots\vee r_N,T,\beta}$ over $r_0, \dots, r_N \in [\tau, T]$. If $N = 0$ we simply write $\mathscr{D}_0^* P = \|P\|_\infty$ and $\mathscr{D}_0 P = \max\left(\|P\|_\infty, \|P\|_\beta\right)$.

Lemma 2.3.3. *(i) Let V be the solution of equation (3.1). Assume that $g_1 = g_2 = 0$. Suppose that U^1 are U^2 are uniformly bounded by a constant C and assume that there exist constants K_0 and K_0' such that $(t-s)^\beta \|B\|_\beta \leq K_0$, $\tau \leq s < t \leq T$, implies*

$$\|U^2\|_{s,t,\beta} \leq K_0'(\|B\|_\beta + 1). \tag{3.22}$$

Suppose that $S, U^1, U^2 \in \mathbb{D}^{N,2}$, where $N \geq 0$ is an integer, and $D_r S_t = D_r U_t^i = 0$, i=1,2, if $0 \leq t < r \leq T$, and suppose that there exists a constant $K > 0$ such that the random variables $\mathscr{D}_N S$, $\mathscr{D}_N^* U^1$, $\mathscr{D}_N U^2$, are less than or equal to $Ke^{K\|B\|_\beta^{1/\beta}}$. Then there exists a constant $K' > 0$ such that $\mathscr{D}_N V$ is less than $K' e^{K'\|B\|_\beta^{1/\beta}}$.

(ii) Let V be the solution of the equation (3.13). Then the conclusion in (i) still holds true under the same assumptions except that in (3.22) we replace $\|U^2\|_{s,t,\beta}$ by $\|U^2\|_{s,t,\beta,n}$.

Proof : We first show point (i). The upper bounds of $\|V\|_\infty$ and $\|V\|_\beta$ follow from Lemma 2.3.1(ii). The Malliavin derivative $D_r V_t$ satisfies the equation (see Proposition 7 in [37])

$$D_r V_t = S_t^{(1)} + \int_r^t U_u^1 D_r V_u du + \int_r^t U_u^2 D_r V_u dB_u$$

while $t \in [r \vee \tau, T]$ and $D_r V_t = 0$ otherwise, where

$$S_t^{(1)} := D_r S_t + U_r^2 V_r + \int_r^t [D_r U_u^1] V_u du + \int_r^t [D_r U_u^2] V_u dB_u \tag{3.23}$$

for $t \in [r \vee \tau, T]$. Lemma 2.3.1(ii) applied to the time interval $[r, T]$, where $r \geq \tau$, implies that

$$\max\left\{\|D_r V\|_{r,T,\infty}, \|D_r V\|_{r,T,\beta}\right\} \leq Ke^{K\|B\|_\beta^{1/\beta}} (|S_r^{(1)}| + \|S^{(1)}\|_{r,T,\beta} + 1).$$

Therefore, to obtain the desired upper bound it suffices to show that there exists a constant K independent of r such that both $\|S^{(1)}\|_{r,T,\infty}$ and $\|S^{(1)}\|_{r,T,\beta}$ are less than or equal to $Ke^{K\|B\|_\beta^{1/\beta}}$. Applying Lemma 2.11.1(ii) to the second integral in (3.23) and noticing that $\|D_r U^2\|_\infty$, $\|D_r U^2\|_{r,T,\beta}$, $\|V\|_\infty$, $\|V\|_{r,T,\beta}$ are bounded by $Ke^{K\|B\|_\beta^{1/\beta}}$, we see that the upper bound of $\|S^{(1)}\|_\infty$ is bounded by $Ke^{K\|B\|_\beta^{1/\beta}}$. On the other hand, in order to show the upper bound for $\|S^{(1)}\|_{r,T,\beta}$, we calculate $\frac{S_t^{(1)}-S_s^{(1)}}{(t-s)^\beta}$ using (3.23) to obtain

$$\begin{aligned}\frac{S_t^{(1)}-S_s^{(1)}}{(t-s)^\beta} &\leq \|D_rS\|_{r,T,\beta}+(t-s)^{-\beta}\int_s^t [D_rU_u^1]V_u du \\ &\quad +(t-s)^{-\beta}\int_s^t [D_rU_u^2]V_u dB_u\,.\end{aligned}$$

Now we can estimate each term of the above right-hand side as before. Taking the supremum over $s,t\in[r,T]$ yields the upper bound of $\|S^{(1)}\|_{r,T,\beta}$.

We turn to the second derivative. As before, we are able to find the equation of $D^2_{r_1,r_2}V_t$(see Proposition 7 in [37]). The estimates of $D^2_{r_1,r_2}V_t$ can then be obtained in the same way as above by applying Lemma 2.3.1(ii) and the estimates that we just obtained for V_t and D_sV_t, as well as the assumptions on S and U^i. The estimates of the higher order derivatives of V can be obtained analogously.

The proof of (ii) follows the same lines except that we use Lemma 2.3.2(ii) and Lemma 2.11.1(i) instead of Lemma 2.3.1(ii) and Lemma 2.11.1(ii). □

Remark 2.3.2. *Since $\beta>\frac{1}{2}$, from Fernique's theorem we know that $Ke^{K\|B\|_\beta^{1/\beta}}$ has finite moments of any order. So Lemma 2.3.3 implies that the uniform norms and Hölder seminorms of the solutions of (3.1) and (3.13) and their Malliavin derivatives have finite moments of any order. We will need this fact in many of our arguments.*

The next proposition is an immediate consequence of Lemma 2.3.3. Recall that the random variables $\mathscr{D}_N^*P$ and $\mathscr{D}_NP$ are defined in Section 2.3.

Proposition 2.3.1. *Let X be the solution of equation (1.1) and let X^n be the solution of the Euler scheme (1.2). Fix $N\geq 0$ and suppose that $b\in C_b^N(\mathbb{R}^d,\mathbb{R}^d), V\in C_b^{N+1}(\mathbb{R}^d,\mathbb{R}^d)$ (recall that we assume $m=1$). Then there exists a positive constant K such that the random variables $\mathscr{D}_NX$ and $\mathscr{D}_NX^n$ are bounded by $Ke^{K\|B\|_\beta^{1/\beta}}$ for all $n\in\mathbb{N}$. If we* *further assume $V \in C_b^{N+2}(\mathbb{R}^d, \mathbb{R}^d)$, then the same upper bound holds for the modified Euler scheme (1.3).*

Proof : We first consider the process X, the solution to equation (1.1). The upper bounds for $\|X\|_\infty$ and $\|X\|_\beta$ follow from Lemma 2.3.1(ii). The Malliavin derivative D_rX_t satisfies the following linear stochastic differential equation

$$D_rX_t = V(X_r) + \int_r^t \nabla b(X_u) D_rX_u du + \int_r^t \nabla V(X_u) D_rX_u dB_u\,, \tag{3.24}$$

while $0 < r \le t \le T$, and $D_rX_t = 0$ otherwise. Then, it suffices to show that

$$\sup_{r\in[0,T]} \mathscr{D}_M(D_rX) \le Ke^{K\|B\|_\beta^{1/\beta}}, \tag{3.25}$$

for $M = N-1$. We can prove estimate (3.25) by induction on $N \ge 1$. Set $S_t = V(X_r)$, $U_t^1 = \nabla b(X_t)$ and $U_t^2 = \nabla V(X_t)$. Applying Lemma 2.3.1(i) to X we obtain that U^2 satisfies (3.22). Therefore, Lemma 2.3.3 implies that (3.25) holds for $M = 0$. Now we assume that

$$\sup_{r\in[0,T]} \mathscr{D}_M(D_rX) \le Ke^{K\|B\|_\beta^{1/\beta}}$$

for some $0 \le M \le N-2$. It is then easy to see that

$$\mathscr{D}^*_{M+1}(U^1) \vee \mathscr{D}_{M+1}(U^2) \vee \mathscr{D}_{M+1}(S) \le Ke^{K\|B\|_\beta^{1/\beta}},$$

taking into account that $b \in C_b^N(\mathbb{R}^d;\mathbb{R}^d)$, $V \in C_b^{N+1}(\mathbb{R}^d;\mathbb{R}^d)$, which enables us to apply Lemma 2.3.3 to (3.24) to obtain the upper bound of the quantity $\sup_{r\in[0,T]} \mathscr{D}_{M+1}(D_rX)$.

The estimates of the Euler scheme and the modified Euler scheme and their derivatives can be obtained in the same way. We omit the proof and we only point out that one more derivative of V is needed for the modified Euler scheme because the function ∇V is involved in its equation. □

2.4 Rate of convergence for the modified Euler scheme and related processes

The main result of this section is the convergence rate of the scheme defined by (1.3) to the solution of the SDE (1.1). Recall that γ_n is the function of n defined in (1.5).

Theorem 2.4.1. *Let X and X^n be solutions to equations (1.1) and (1.3), respectively. We assume $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $V \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$. Then for any $p \geq 1$ there exists a constant C independent of n (but dependent on p) such that*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|X_t^n - X_t|^p\right]^{\frac{1}{p}} \leq C\gamma_n^{-1}.$$

Proof: Denote $Y := X - X^n$. Notice that Y depends on n, but for notational simplicity we shall omit the explicit dependence on n for Y and some other processes when there is no ambiguity. The idea of the proof is to decompose Y into seven terms (see (4.7) below), and then study their convergence rate individually.

Step 1. By the definitions of the processes X and X^n, we have

$$\begin{aligned} Y_t &= \int_0^t \left[b(X_s) - b(X_s^n) + b(X_s^n) - b(X_{\eta(s)}^n)\right] ds \\ &\quad + \sum_{j=1}^m \int_0^t \left[V^j(X_s) - V^j(X_s^n) + V^j(X_s^n) - V^j(X_{\eta(s)}^n)\right] dB_s^j \\ &\quad - H\sum_{j=1}^m \int_0^t (\nabla V^j V^j)(X_{\eta(s)}^n)(s-\eta(s))^{2H-1} ds. \end{aligned}$$

By denoting

$$
\begin{aligned}
V_0^j(s) &= (\nabla V^j V^j)(X^n_{\eta(s)}), \quad b_1(s) = \int_0^1 \nabla b(\theta X_s + (1-\theta)X_s^n) d\theta, \\
V_1^j(s) &= \int_0^1 \nabla V^j(\theta X_s + (1-\theta)X_s^n) d\theta \,,
\end{aligned}
$$

we can write

$$
\begin{aligned}
Y_t &= \int_0^t b_1(s) Y_s ds + \sum_{j=1}^m \int_0^t V_1^j(s) Y_s dB_s^j + \int_0^t \left[b(X_s^n) - b(X^n_{\eta(s)}) \right] ds \\
&\quad + \sum_{j=1}^m \int_0^t \left[V^j(X_s^n) - V^j(X^n_{\eta(s)}) \right] dB_s^j - H \sum_{j=1}^m \int_0^t V_0^j(s)(s-\eta(s))^{2H-1} ds.
\end{aligned}
$$

Let $\Lambda^n = \{\Lambda_t^n, t \in [0,T]\}$ be the $d \times d$ matrix-valued solution of the following linear SDE

$$
\Lambda_t^n = I + \int_0^t b_1(s) \Lambda_s^n ds + \sum_{j=1}^m \int_0^t V_1^j(s) \Lambda_s^n dB_s^j \,, \tag{4.1}
$$

where I is the $d \times d$ identity matrix. Applying the chain rule for the Young integral to $\Gamma_t^n \Lambda_t^n$, where Γ_t^n, $t \in [0,T]$ is the unique solution of the equation

$$
\Gamma_t^n = I - \int_0^t \Gamma_s^n b_1(s) ds - \sum_{j=1}^m \int_0^t \Gamma_s^n V_1^j(s) dB_s^j \,, \tag{4.2}
$$

for $t \in [0,T]$, we see that $\Gamma_t^n \Lambda_t^n = \Lambda_t^n \Gamma_t^n = I$ for all $t \in [0,T]$. Therefore, $(\Lambda_t^n)^{-1}$ exists and coincides with Γ_t^n.

We can express the process Y_t in terms of Λ_t^n as follows

$$\begin{aligned} Y_t &= \int_0^t \Lambda_t^n \Gamma_s^n \Big[b(X_s^n) - b(X_{\eta(s)}^n) \Big] ds \\ &\quad + \sum_{j=1}^m \int_0^t \Lambda_t^n \Gamma_s^n \Big[V^j(X_s^n) - V^j(X_{\eta(s)}^n) \Big] dB_s^j \\ &\quad - H \sum_{j=1}^m \int_0^t \Lambda_t^n \Gamma_s^n V_0^j(s)(s-\eta(s))^{2H-1} ds. \end{aligned} \tag{4.3}$$

The first two terms in the right-hand side of equation (4.3) can be further decomposed as follows:

$$\begin{aligned} &\int_0^t \Lambda_t^n \Gamma_s^n \left[V^j(X_s^n) - V^j(X_{\eta(s)}^n) \right] dB_s^j \\ &= \int_0^t \Lambda_t^n \Gamma_s^n b_2^j(s)(s-\eta(s)) dB_s^j + \sum_{i=1}^m \int_0^t \Lambda_t^n \Gamma_s^n V_2^{j,i}(s)(B_s^i - B_{\eta(s)}^i) dB_s^j \\ &\quad + \int_0^t \Lambda_t^n \Gamma_s^n V_3^j(s)(s-\eta(s))^{2H} dB_s^j \\ &:= I_{2,j}(t) + \sum_{i=1}^m I_{3,j,i}(t) + I_{4,j}(t), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} b_2^j(s) &= \int_0^1 \nabla V^j(\theta X_s^n + (1-\theta) X_{\eta(s)}^n) b(X_{\eta(s)}^n) d\theta, \\ V_2^{j,i}(s) &= \int_0^1 \nabla V^j(\theta X_s^n + (1-\theta) X_{\eta(s)}^n) V^i(X_{\eta(s)}^n) d\theta, \\ V_3^j(s) &= \frac{1}{2} \int_0^1 \nabla V^j(\theta X_s^n + (1-\theta) X_{\eta(s)}^n) \sum_{l=1}^m V_0^l(s) d\theta, \end{aligned}$$

and

$$\Lambda_t^n \int_0^t \Gamma_s^n \left[b(X_s^n) - b(X_{\eta(s)}^n) \right] ds$$
$$= \Lambda_t^n \int_0^t \Gamma_s^n b_3(s) \left[b(X_{\eta(s)}^n)(s-\eta(s)) + \sum_{j=1}^m V^j(X_{\eta(s)}^n)(B_s^j - B_{\eta(s)}^j) \right.$$
$$\left. + \frac{1}{2} \sum_{j=1}^m V_0^j(s)(s-\eta(s))^{2H} \right] ds$$
$$:= I_{11}(t) + \sum_{j=1}^m I_{12,j}(t) + I_{13}(t), \tag{4.5}$$

where $b_3(s) = \int_0^1 \nabla b(\theta X_s^n + (1-\theta) X_{\eta(s)}^n) d\theta$. We also denote

$$I_{5,j}(t) = -H\Lambda_t^n \int_0^t \Gamma_s^n V_0^j(s)(s-\eta(s))^{2H-1} ds. \tag{4.6}$$

Substituting equations (4.4), (4.5) and (4.6) into (4.3) yields

$$Y = I_{11} + \sum_{j=1}^m I_{12,j} + I_{13} + \sum_{j=1}^m I_{2,j} + \sum_{j,i=1}^m I_{3,j,i} + \sum_{j=1}^m I_{4,j} + \sum_{j=1}^m I_{5,j}\,. \tag{4.7}$$

Step 2. Denote by $(\Lambda^n)_i$, $i = 1, \ldots d$, the i-th columns of Λ^n. We claim that $(\Lambda^n)_i$ satisfy the conditions in Lemma 2.3.3 with $M = d$, $\tau = 0$, $U_t^1 = b_1(t)$, $U_t^2 = V_1^j(t)$ and $N = 2$. We first show that U^2 satisfies (3.22). Taking into account that $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $V \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$, it suffices to show that both X and X^n satisfy (3.22). This is clear for X because of Lemma 2.3.1 (i). It follows from Lemma 2.3.2 (i) that there exist constants K and K' such that $(t-s)^\beta \|B\|_\beta \le K$, $0 \le s < t \le T$, implies

$$\|X^n\|_{s,t,\beta,n} \le K'(\|B\|_\beta + 1).$$

Notice that

$$\begin{aligned}\frac{|X_t^n - X_s^n|}{(t-s)^\beta} &\leq \frac{|X_t^n - X_{\varepsilon(s)}^n|}{(t-\varepsilon(s))^\beta} + \frac{|X_{\varepsilon(s)}^n - X_s^n|}{(\varepsilon(s)-s)^\beta} \\ &\leq \|X^n\|_{s,t,\beta,n} + \frac{|X_{\varepsilon(s)}^n - X_s^n|}{\varepsilon(s)-s}\end{aligned}$$

for $t,s : t \geq \varepsilon(s)$, where we recall that $\varepsilon(s) = t_{k+1}$ when $s \in (t_k, t_{k+1}]$. Therefore, to verify (3.22) for X^n it suffices to show that

$$\|X^n\|_{s,t,\beta} \leq K'(\|B\|_\beta + 1)$$

for $s,t \in [t_k, t_{k+1}]$ for some k. But this follows immediately from (1.3). On the other hand, the fact that $\mathscr{D}_2^* U^1$ and $\mathscr{D}_2 U^2$ are less than $Ke^{K\|B\|_\beta^{1/\beta}}$ for some K follows from Proposition 2.3.1 and the assumption that $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $V \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$, where $\mathscr{D}_2^*$ and $\mathscr{D}_2$ are defined in Section 2.3.

In the same way we can show that the columns of Γ^n satisfy the assumptions of Lemma 2.3.3. As a consequence, it follows from Lemma 2.3.3 that

$$\mathscr{D}_2 \Lambda^n \vee \mathscr{D}_2 \Gamma^n \leq Ke^{K\|B\|_\beta^{1/\beta}}. \tag{4.8}$$

Step 3. From (4.8) and from the fact that $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$ and $V \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$, it follows

$$\mathbb{E}(|I_{11}(t)|^p)^{\frac{1}{p}} \leq Cn^{-1} \quad \text{and} \quad \mathbb{E}(|I_{13}(t)|^p)^{\frac{1}{p}} \leq Cn^{-2H}. \tag{4.9}$$

Notice that n^{-1} and n^{-2H} are bounded by γ_n^{-1}. Applying estimates (11.4) and (11.5), inequality (4.8), and Proposition 2.3.1, we have for any j

$$\mathbb{E}(|I_{12,j}(t)|^p)^{\frac{1}{p}} \le Cn^{-1}, \quad \mathbb{E}(|I_{2,j}(t)|^p)^{\frac{1}{p}} \le Cn^{-1}, \quad \mathbb{E}(|I_{4,j}(t)|^p)^{\frac{1}{p}} \le Cn^{-2H}. \tag{4.10}$$

Now to complete the proof of the theorem it suffices to show that for any j, $\mathbb{E}(|\sum_{i=1}^m I_{3,j,i}(t) + I_{5,j}(t)|^p)^{\frac{1}{p}} \le C\gamma_n^{-1}$. For any fixed j we make the decomposition

$$\sum_{i=1}^m I_{3,j,i} + I_{5,j} = E_{1,j} + E_{2,j} + E_{3,j}, \tag{4.11}$$

where

$$\begin{aligned} E_{1,j}(t) &= \Lambda_t^n \sum_{i=1}^m \int_0^t \left[\Gamma_s^n V_2^{j,i}(s) - \Gamma_{\eta(s)}^n (\nabla V^j V^i)(X_{\eta(s)}^n)\right] (B_s^i - B_{\eta(s)}^i) dB_s^j, \\ E_{2,j}(t) &= \Lambda_t^n \sum_{i=1}^m \int_0^t \Gamma_{\eta(s)}^n (\nabla V^j V^i)(X_{\eta(s)}^n)(B_s^i - B_{\eta(s)}^i) dB_s^j \\ &\quad - H\Lambda_t^n \int_0^t \Gamma_{\eta(s)}^n V_0^j(s)(s-\eta(s))^{2H-1} ds, \\ E_{3,j}(t) &= H\Lambda_t^n \int_0^t (\Gamma_{\eta(s)}^n - \Gamma_s^n) V_0^j(s)(s-\eta(s))^{2H-1} ds. \end{aligned}$$

Applying (4.8) for the quantities $\|\Lambda^n\|_\infty$ and $\|\Gamma^n\|_\beta$, it is easy to see that $\mathbb{E}(|E_{3,j}(t)|^p)^{\frac{1}{p}} \le Cn^{1-2H-\beta}$ for any $\frac{1}{2} < \beta < H$. On the other hand, applying estimate (11.15) from Lemma 2.11.5 to $E_{1,j}$ we obtain $\mathbb{E}(|E_{1,j}(t)|^p)^{\frac{1}{p}} \le Cn^{1-3\beta}$ for any $\frac{1}{2} < \beta < H$. Notice that the exponents $n^{1-2H-\beta}$ and $n^{1-3\beta}$ are bounded by γ_n^{-1} if β is sufficiently close to H.

Taking into account the relationship between the Skorohod and path-wise integral, we can express the term $E_{2,j}$ as follows

$$E_{2,j}(t) = \Lambda_t^n \sum_{i=1}^{m} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} F_{t_k}^{n,i,j} \int_{t_k}^{t_{k+1}\wedge t} \int_{t_k}^{s} \delta B_u^i \delta B_s^j, \tag{4.12}$$

for $t \in [0,T]$, where $F_t^{n,i,j} = \Gamma_t^n(\nabla V^j V^i)(X_t^n)$, and we define $t_{n+1} = (n+1)\frac{T}{n}$. From (4.8) and Proposition 2.3.1, we have

$$\max\{|F_t^{n,i,j}|, |D_{r_1} F_t^{n,i,j}|, |D_{r_2} D_{r_1} F^{n,i,j}|\} \leq K e^{K\|B\|_\beta^{1/\beta}}. \tag{4.13}$$

Hence, applying estimate (11.8) from Lemma 2.11.4 to $E_{2,j}(t)$ we obtain $\mathbb{E}(|E_{2,j}(t)|^p)^{\frac{1}{p}} \leq C\gamma_n^{-1}$. The proof is now complete. □

The following result provides a rate of convergence for the Malliavin derivatives of the modified scheme and some related processes. Recall that β satisfies $\frac{1}{2} < \beta < H$.

Lemma 2.4.1. *Let X and X^n be the processes defined by (1.1) and (1.3), respectively. Suppose that $V \in C_b^5(\mathbb{R}^d;\mathbb{R}^{d\times m})$, $b \in C_b^4(\mathbb{R}^d;\mathbb{R}^d)$. Let $p \geq 1$. Then,*

(i) There exists a constant C such that the quantities $\|D_s X_t - D_s X_t^n\|_p$, $\|D_r D_s X_t - D_r D_s X_t^n\|_p$, $\|D_u D_r D_s X_t - D_u D_r D_s X_t^n\|_p$ are less than $Cn^{1-2\beta}$ for all $u, r, s, t \in [0,T]$ and $n \in \mathbb{N}$,

(ii) Let V and V^n be d-dimensional processes satisfying the equations

$$\begin{aligned} V_t &= V_0 + \int_0^t f_1(X_u, X_u) V_u du + \sum_{j=1}^{m} \int_0^t f_2^j(X_u, X_u) V_u dB_u^j, \\ V_t^n &= V_0 + \int_0^t f_1(X_u, X_u^n) V_u^n du + \sum_{j=1}^{m} \int_0^t f_2^j(X_u, X_u^n) V_u^n dB_u^j, \end{aligned}$$

where $f_1 \in C_b^3(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^{d\times d})$ and $f_2^j \in C_b^4(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^{d\times d})$. Then there exists a constant C such that the quantities $\|V_t - V_t^n\|_p$, $\|D_sV_t - D_sV_t^n\|_p$, $\|D_rD_sV_t - D_rD_sV_t^n\|_p$ are less than $Cn^{1-2\beta}$ for all $r,s,t \in [0,T]$ and $n \in \mathbb{N}$.

Remark 2.4.1. *The above results still hold when the approximation process X^n is replaced by the one defined by the recursive scheme (1.2). The proof follows exactly the same lines.*

Proof. (i) Taking the Malliavin derivative in both sides of (4.3), we obtain

$$\begin{aligned} D_r(X_t - X_t^n) &= \int_0^t D_r\Big[\Lambda_t^n\Gamma_s^n\big(b(X_s^n) - b(X_{\eta(s)}^n)\big)\Big]ds \\ &\quad + \sum_{j=1}^m \int_0^t D_r\Big[\Lambda_t^n\Gamma_s^n\big(V^j(X_s^n) - V^j(X_{\eta(s)}^n)\big)\Big]dB_s^j \\ &\quad + \sum_{j=1}^m \Lambda_t^n\Gamma_r^n\big(V^j(X_r^n) - V^j(X_{\eta(r)}^n)\big) \\ &\quad - H\sum_{j=1}^m \int_0^t D_r\big[\Lambda_t^n\Gamma_s^n V_0^j(s)\big](s-\eta(s))^{2H-1}ds. \end{aligned}$$

Proposition 2.3.1 and equation (4.8) imply that the first, third and last term of the above right-hand side have L^p-norms bounded by Cn^{1-2H}. Applying estimate (11.16) from Lemma 2.11.5 to the second term and noticing that $\|X\|_\beta$ and $\sup_{r\in[0,T]}\|D_rX\|_\beta$ have finite moments of any order, we see that its L^p-norm is also bounded by $Cn^{1-2\beta}$.

Similarly, we can take the second derivative in (4.3) and then estimate each term individually as before to obtain that the upper bound of $\|D_rD_sX_t - D_rD_sX_t^n\|_p$ is bounded by $Cn^{1-2\beta}$.

(ii) Using the chain rule for Young's integral we derive the following explicit expression for $V_t - V_t^n$

$$\begin{aligned} V_t - V_t^n &= \int_0^t \Upsilon_t \Upsilon_s^{-1} \left(f_1\left(X_s, X_s\right) - f_1\left(X_s, X_s^n\right)\right) V_s^n ds \\ &\quad + \sum_{j=1}^m \int_0^t \Upsilon_t \Upsilon_s^{-1} \left(f_2^j\left(X_s, X_s\right) - f_2^j\left(X_s, X_s^n\right)\right) V_s^n dB_s^j, \end{aligned} \tag{4.14}$$

where $\Upsilon = \{\Upsilon_t, t \in [0,T]\}$ is the $\mathbb{R}^{d\times d}$-valued process that satisfies

$$\Upsilon_t = I + \int_0^t f_1\left(X_s, X_s\right) \Upsilon_t ds + \sum_{j=1}^m \int_0^t f_2^j\left(X_s, X_s\right) \Upsilon_t dB_s^j.$$

Lemma 2.3.3 implies that there exists a constant K such that for all $n \in \mathbb{N}$, $u, r, s, t \in [0,T]$, we have

$$\max\{\Upsilon_t, D_s\Upsilon_t, D_rD_s\Upsilon_t, D_uD_rD_s\Upsilon_t\} \le Ke^{K\|B\|_\beta^{1/\beta}}. \tag{4.15}$$

Therefore, applying estimate (11.4) to the second integral in (4.14) with $\nu = 0$ and taking into account the estimate of Lemma 2.4.1(i), we obtain

$$\|V - V^n\|_p \le Cn^{1-2\beta}.$$

Taking the Malliavin derivative on both sides of (4.14), and then applying estimates (11.4) from Lemma 2.11.3 and Lemma 2.4.1(i) as before, we can obtain the desired estimate for $\|D_sV_t - D_sV_t^n\|_p$. The estimate for $\|D_rD_sV_t - D_rD_sV_t^n\|_p$ can be obtained in a similar way. □

We define $\{\Lambda_t\,,\,t\in[0,T]\}$ as the solution of the limiting equation of (4.1), that is,

$$\Lambda_t = I + \int_0^t \nabla b(X_s)\Lambda_s ds + \sum_{j=1}^m \int_0^t \nabla V^j(X_s)\Lambda_s dB_s^j. \tag{4.16}$$

The inverse of the matrix Λ_t, denoted by Γ_t, exists and satisfies

$$\Gamma_t = I - \int_0^t \Gamma_t \nabla b(X_s) ds - \sum_{j=1}^m \int_0^t \Gamma_t \nabla V^j(X_s) dB_s^j.$$

It follows from Lemma 2.4.1 that if we assume that $V\in C_b^5(\mathbb{R}^d;\mathbb{R}^{d\times m})$ and $b\in C_b^4(\mathbb{R}^d;\mathbb{R}^d)$, then the estimate in Lemma 2.4.1(ii) holds with the pair (V,V^n) being replaced by (Γ_i,Γ_i^n) or (Λ_i,Λ_i^n), $i=1,\dots,d$, where the subindex i denotes the i-th column of each matrix.

2.5 Central limit theorem for weighted sums

Our goal in this section is to prove a central limit result for weighted sums (see Proposition 2.5.5 below) that will play a fundamental role in the proof of Theorem 2.6.1 in the next section. This result has an independent interest and we devote this entire section to it.

We recall that $B=\{B_t, t\in[0,T]\}$ is an m-dimensional fBm and we assume that the Hurst parameter satisfies $H\in\left(\frac{1}{2},\frac{3}{4}\right]$. For any $n\geq 1$ we set $t_j=\frac{jT}{n}$, $j=0,\dots,n$. Recall that $\eta(s)=t_k$ if $t_k\leq s<t_{k+1}$. Consider the $d\times d$ matrix-valued process

$$\Xi_t^{n,i,j} = \gamma_n \sum_{k=0}^{\{t\}} \int_{t_k}^{t_{k+1}} (B_s^i - B_{\eta(s)}^i)\delta B_s^j, \qquad 1\leq i,j\leq m,$$

where we denote $\{t\}=\lfloor \frac{nt}{T}\rfloor$ for $t\in[0,T)$ and $\{T\}=t_{n-1}$.

Proposition 2.5.1. *The following stable convergence holds as n tends to infinity*

$$(\Xi^n, B) \to (W, B)$$

where $W = \{W_t, t \in [0,T]\}$ *is the matrix-valued Brownian motion, introduced in Section 2.2.4, and* W *and* B *are independent.*

Proof. From the inequality (11.8) in Lemma 2.11.4 it follows

$$\mathbb{E}\left(\left|\Xi^n_{t_k} - \Xi^n_{t_j}\right|^4\right) \le C\left(\frac{k-j}{n}\right)^2, \tag{5.1}$$

for any $j \le k$. This implies the tightness of (Ξ^n, B).

Then, it remains to show the convergence of the finite dimensional distributions of (Ξ^n, B) to that of (W, B). To do this, we fix a finite set of points $r_1, \dots, r_{L+1} \in [0,T]$ such that $0 = r_1 < r_2 < \cdots < r_{L+1} \le T$ and define the random vectors $B_L = (B_{r_2} - B_{r_1}, \dots, B_{r_{L+1}} - B_{r_L})$, $\Xi^n_L = (\Xi^n_{r_2} - \Xi^n_{r_1}, \dots, \Xi^n_{r_{L+1}} - \Xi^n_{r_L})$ and $W_L = (W_{r_2} - W_{r_1}, \dots, W_{r_{L+1}} - W_{r_L})$. We claim that as n tends to infinity the following convergence in law holds

$$(\Xi^n_L, B_L) \Rightarrow (W_L, B_L). \tag{5.2}$$

For notational simplicity, we add one term to each component of Ξ^n_L and we define

$$\Theta^n_l(i,j) := \Xi^{n,i,j}_{r_{l+1}} - \Xi^{n,i,j}_{r_l} + \zeta^{i,j}_{\{r_l\},n} = \gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \zeta^{i,j}_{k,n}, \tag{5.3}$$

for $1 \le l \le L$, $1 \le i, j \le d$,, where

$$\zeta^{i,j}_{k,n} = \int_{t_k}^{t_{k+1}} (B^i_s - B^i_{t_k})\delta B^j_s.$$

Then Slutsky's lemma implies that the convergence in law in (5.2) is equivalent to

$$(\Theta_l^n(i,j), 1 \le i,j \le d, 1 \le l \le L, B_L) \Rightarrow (W_L, B_L).$$

According to [39] (see also Theorem 6.2.3 in [38]), to show the convergence in law of (Θ_L^n, B_L), it suffices to show the convergence of each component of (Θ_L^n, B_L) to the correspondent component of (W_L, B_L) and the convergence of the covariance matrix.

The convergence of the covariance matrix of Θ_L^n follows from Propositions 2.5.2 and 2.5.3 below. The convergence in law of each component to a Gaussian distribution follows from Proposition 2.5.4 below and the fourth moment theorem (see [35] and also Theorem 5.2.7 in [38]). This completes the proof. □

In order to show the convergence of the covariance matrix and the fourth moment of Θ_n we first introduce the following notation.

$$\begin{aligned} \mathscr{D}_k &= \{(s,t,v,u):\ t_k \le v \le s \le t_{k+1}, u,t \in [0,T]\}, \\ \mathscr{D}_{k_1,k_2} &= \left\{(s,t,v,u): t_{k_2} \le v \le s \le t_{k_2+1}, t_{k_1} \le u \le t \le t_{k_1+1}\right\}. \end{aligned} \tag{5.4}$$

The next two propositions provide the convergence of the covariance $\mathbb{E}[\Theta_{l'}^n(i',j')\Theta_l^n(i,j)]$ in the cases $l = l'$ and $l \neq l'$, respectively. We denote $\beta_{\frac{k}{n}}(s) = \mathbf{1}_{[t_k,t_{k+1}]}(s)$.

Proposition 2.5.2. *Let $\Theta_l^n(i,j)$ be defined in (5.3). Then*

$$\mathbb{E}[\Theta_l^n(i',j')\Theta_l^n(i,j)] \ \to \ \alpha_H^2 \frac{r_{l+1}-r_l}{T}(R\delta_{ji'}\delta_{ij'} + Q\delta_{jj'}\delta_{ii'}), \tag{5.5}$$

as $n \to +\infty$. Here $\delta_{ii'}$ is the Kronecker function, $\alpha_H = H(2H-1)$ and Q and R are the constants defined in (2.12).

Proof: The proof will be done in several steps.

Step 1. Applying twice the integration by parts formula (2.7), we have

$$\mathbb{E}[\Theta_l^n(i',j')\Theta_l^n(i,j)] \tag{5.6}$$
$$= \alpha_H^2 \gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathscr{D}_k} D_u^i D_t^j \Theta_l^n(i',j')\mu(dvdu)\mu(dsdt),$$

where we recall that $\{t\} = \lfloor \frac{nt}{T} \rfloor$ for $t \in [0,T)$ and $\{T\} = t_{n-1}$, and $\mathscr{D}_k$ is defined in (5.4). Since

$$D_u^i D_t^j \Theta_l^n(i',j') \tag{5.7}$$
$$= \gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \left(\mathbf{1}_{[t_k,t]}(u)\beta_{\frac{k}{n}}(t)\delta_{jj'}\delta_{ii'} + \mathbf{1}_{[t_k,u]}(t)\beta_{\frac{k}{n}}(u)\delta_{ji'}\delta_{ij'} \right),$$

the left-hand side of (5.5) equals

$$\alpha_H^2\gamma_n^2 \sum_{k,k'=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathscr{D}_k} \Big\{ \mathbf{1}_{[t_{k'},t]}(u)\beta_{\frac{k'}{n}}(t)\delta_{jj'}\delta_{ii'}$$
$$+\mathbf{1}_{[t_{k'},u]}(t)\beta_{\frac{k'}{n}}(u)\delta_{ji'}\delta_{ij'} \Big\} \mu(dvdu)\mu(dsdt)$$
$$:= \alpha_H^2\gamma_n^2 \left(G_1\delta_{jj'}\delta_{ii'} + G_2\delta_{ji'}\delta_{ij'} \right).$$

In the next two steps, we compute the limits of $\gamma_n^2 G_1$ and $\gamma_n^2 G_2$ as n tends to infinity in the case $H \in (\frac{1}{2}, \frac{3}{4})$ and in the case $H = \frac{3}{4}$ separately.

Step 2. In this step, we consider the case $H \in (\frac{1}{2}, \frac{3}{4})$. Recall that

$$\begin{aligned} Q(p) &= T^{4H} \int_0^1 \int_p^{p+1} \int_0^t \int_p^s \mu(dvdu)\mu(dsdt) \\ &= n^{4H} \int_{\mathscr{D}_{k',k'+p}} \mu(dvdu)\mu(dsdt), \end{aligned}$$

which is independent of n, where the set $\mathscr{D}_{k_1,k_2}$ is defined in (5.4). We can express $\gamma_n^2 G_1$ in terms of $Q(p)$ as follows

$$\begin{aligned}\gamma_n^2 G_1 &= n^{4H-1}\sum_{k,k'=\{r_l\}}^{\{r_{l+1}\}}\int_{\mathscr{D}_{k',k}}\mu(dvdu)\mu(dsdt)\\ &= \frac{1}{n}\sum_{p=\{r_l\}-\{r_{l+1}\}}^{\{r_{l+1}\}-\{r_l\}}\ \sum_{k'=(\{r_l\}-p)\vee\{r_l\}}^{(\{r_{l+1}\}-p)\wedge\{r_{l+1}\}} Q(p)\\ &= \sum_{p=-\infty}^{\infty}\Psi_l^n(p)Q(p),\end{aligned}$$

where

$$\Psi_l^n(p) = \frac{(\{r_{l+1}\}-p)\wedge\{r_{l+1}\}-(\{r_l\}-p)\vee\{r_l\}}{n}\mathbf{1}_{[\{r_l\}-\{r_{l+1}\},\{r_{l+1}\}-\{r_l\}]}(p).$$

The term $\Psi_l^n(p)$ is uniformly bounded and converges to $\frac{r_{l+1}-r_l}{T}$ as n tends to infinity for any fixed p. Therefore, taking into account that $\sum_{p=-\infty}^{\infty}Q(p)=Q<\infty$, the dominated convergence theorem implies

$$\lim_{n\to\infty}\gamma_n^2 G_1 = \frac{r_{l+1}-r_l}{T}Q.$$

Similarly, we can show that

$$\lim_{n\to\infty}\gamma_n^2 G_2 = \frac{r_{l+1}-r_l}{T}R.$$

Step 3. In the case $H = \frac{3}{4}$, we can write

$$\begin{aligned}
\gamma_n^2 G_1 &= \frac{n^2}{\log n} \sum_{k,k'=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathscr{D}_{k',k}} \mu(dvdu)\mu(dsdt) \\
&= \frac{1}{n\log n} \sum_{p=\{r_l\}-\{r_{l+1}\}}^{\{r_{l+1}\}-\{r_l\}} \sum_{k'=\{r_l\}}^{\{r_{l+1}\}} Q(p) \\
&\quad - \frac{1}{n\log n} \left\{ \sum_{p=\{r_l\}-\{r_{l+1}\}}^{0} \sum_{k'=\{r_l\}}^{\{r_l\}-p-1} + \sum_{p=1}^{\{r_{l+1}\}-\{r_l\}} \sum_{k'=\{r_{l+1}\}-p+1}^{\{r_{l+1}\}} \right\} Q(p) \\
&:= G_{11} + G_{12}.
\end{aligned}$$

Taking into account that $Q(p)$ behaves like $1/|p|$ as $|p|$ tends to infinity, it is then easy to see that G_{12} converges to zero. On the other hand, recall that $Q = \lim_{n\to+\infty} \frac{\sum_{|p|\le n} Q(p)}{\log n}$. This implies that G_{11} converges to $\dfrac{Q}{T}(r_{l+1} - r_l)$. This gives the limit of $\gamma_n^2 G_1$. The limit of $\gamma_n^2 G_2$ can be obtained similarly. □

Proposition 2.5.3. *Let $l, l' \in \{1, \ldots, L\}$ be such that $l \neq l'$. Let Θ^n be defined as in (5.3). Then*

$$\lim_{n\to\infty} \mathbb{E}[\Theta_{l'}^n(i',j')\Theta_l^n(i,j)] = 0. \tag{5.8}$$

Proof: Without any loss of generality, we assume $l' < l$. As in (5.6) we have

$$\mathbb{E}[\Theta_{l'}^n(i',j')\Theta_l^n(i,j)] = \alpha_H^2 \gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathscr{D}_k} D_u^i D_t^j \Theta_{l'}^n(i',j')\mu(dvdu)\mu(dsdt).$$

Taking into account (5.7), we can write

$$\begin{aligned}
\mathbb{E}[\Theta^n_{l'}(i',j')\Theta^n_l(i,j)] &= \alpha_H^2\gamma_n^2 \sum_{k=\{r_l\}}^{\{r_{l+1}\}}\sum_{k'=\{r_{l'}\}}^{\{r_{l'+1}\}} \int_{\mathscr{D}_k} \Big\{ \mathbf{1}_{[t_{k'},t]}(u)\beta_{\frac{k'}{n}}(t)\delta_{jj'}\delta_{ii'} \\
&\quad + \mathbf{1}_{[t_{k'},u]}(t)\beta_{\frac{k'}{n}}(u)\delta_{ji'}\delta_{ij'} \Big\} \mu(dvdu)\mu(dsdt) \\
&:= \alpha_H^2\gamma_n^2 \left(\widetilde{G}_1 \delta_{jj'}\delta_{ii'} + \widetilde{G}_2 \delta_{ji'}\delta_{ij'} \right).
\end{aligned}$$

In the case $H \in (\frac{1}{2}, \frac{3}{4})$ we have

$$\begin{aligned}
\gamma_n^2 \widetilde{G}_1 &= n^{4H-1} \sum_{k=\{r_l\}}^{\{r_{l+1}\}}\sum_{k'=\{r_{l'}\}}^{\{r_{l'+1}\}} \int_{\mathscr{D}_k} \mathbf{1}_{[t_{k'},t]}(u)\beta_{\frac{k'}{n}}(t)\mu(dvdu)\mu(dsdt) \\
&= \frac{1}{n} \sum_{p=\{r_l\}-\{r_{l'+1}\}}^{\{r_{l+1}\}-\{r_{l'}\}} \sum_{k'=(\{r_l\}-p)\vee\{r_{l'}\}}^{\{r_{l'+1}\}\wedge(\{r_{l+1}\}-p)} Q(p) \\
&= \sum_{p=-\infty}^{\infty} \Phi^n_l(p) Q(p),
\end{aligned}$$

where $\Phi^n_l(p)$ is equal to

$$\frac{\max\{(\{r_{l'+1}\}-p)\wedge\{r_{l+1}\}-(\{r_l\}-p)\vee\{r_l'\},0\}}{n}\mathbf{1}_{[\{r_l\}-\{r_{l'+1}\},\{r_{l+1}\}-\{r_{l'}\}]}(p).$$

The term $\Phi^n_l(p)$ is uniformly bounded and converges to 0 as n tends to infinity for any fixed p because $l < l'$. Therefore, taking into account that $\sum_{p=-\infty}^{\infty} Q(p) = Q < \infty$, the dominated convergence theorem implies that $\gamma_n^2 \widetilde{G}_1$ converges to zero as n tends to infinity. Similarly, we can show that $\gamma_n^2 \widetilde{G}_2$ converges to zero as n tends to infinity.

In the case $H = \frac{3}{4}$, since

$$\begin{aligned}\gamma_n^2 \widetilde{G}_1 =& \frac{n^2}{\ln n} \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \sum_{k'=\{r_{l'}\}}^{\{r_{l'+1}\}} \int_{\mathscr{D}_{k',k}} \mu(dvdu)\mu(dsdt) \\ =& \frac{1}{n \ln n} \sum_{p=\{r_l\}-\{r_{l'+1}\}}^{\{r_{l+1}\}-\{r_{l'}\}} \sum_{k'=(\{r_l\}-p)\vee\{r_{l'}\}}^{\{r_{l'+1}\}\wedge(\{r_{l+1}\}-p)} Q(p),\end{aligned}$$

we have

$$\begin{aligned}\gamma_n^2 \widetilde{G}_1 \leq& \frac{1}{n \ln n} \sum_{p=\{r_l\}-\{r_{l'+1}\}}^{\{r_{l+1}\}-\{r_{l'}\}} \sum_{k'=\{r_{l'}\}}^{\{r_{l+1}\}-p} Q(p) \\ \leq& \frac{1}{n \ln n} \sum_{p=-n}^{0} (p+1) Q(p).\end{aligned}$$

Noticing that $Q(p) = O(\frac{1}{|p|})$, so we conclude that $\gamma_n^2 \widetilde{G}_1 \leq \frac{C}{\ln n}$. This shows that $\gamma_n^2 \widetilde{G}_1$ converges to zero as n tends to infinity. In the same way we can show that $\gamma_n^2 \widetilde{G}_2$ converges to zero. □

The following estimate is needed in the calculation of the fourth moment of $\Theta_l^n(i,j)$ in Proposition 2.5.4.

Lemma 2.5.1. *Let $H \in (\frac{1}{2}, \frac{3}{4}]$. We have the following estimate:*

$$\sum_{k_1,k_2,k_3,k_4=0}^{n-1} \langle \beta_{\frac{k_1}{n}}, \beta_{\frac{k_2}{n}} \rangle_{\mathscr{H}} \langle \beta_{\frac{k_2}{n}}, \beta_{\frac{k_3}{n}} \rangle_{\mathscr{H}} \langle \beta_{\frac{k_3}{n}}, \beta_{\frac{k_4}{n}} \rangle_{\mathscr{H}} \langle \beta_{\frac{k_1}{n}}, \beta_{\frac{k_4}{n}} \rangle_{\mathscr{H}} \leq C n^{-2} \gamma_n^{-2}.$$

Proof: Since the indices k_1, k_2, k_3, k_4 are symmetric, it suffices to consider the case $k_1 \le k_2 \le k_3 \le k_4$. By definition of the inner product we have

$$\begin{aligned}
&\sum_{0\le k_1\le k_2\le k_3\le k_4\le n-1} \langle \beta_{\frac{k_1}{n}}, \beta_{\frac{k_2}{n}} \rangle_{\mathscr{H}} \langle \beta_{\frac{k_2}{n}}, \beta_{\frac{k_3}{n}} \rangle_{\mathscr{H}} \langle \beta_{\frac{k_3}{n}}, \beta_{\frac{k_4}{n}} \rangle_{\mathscr{H}} \langle \beta_{\frac{k_1}{n}}, \beta_{\frac{k_4}{n}} \rangle_{\mathscr{H}} \\
&= \frac{T^{8H}}{2^4 n^{8H}} \sum_{k_1=0}^{n-1} \sum_{k_2=k_1}^{n-1} \sum_{k_3=k_2}^{n-1} \sum_{k_4=k_3}^{n-1} (|k_2-k_1+1|^{2H} + |k_2-k_1-1|^{2H} - 2|k_2-k_1|^{2H}) \\
&\quad \times (|k_3-k_2+1|^{2H} + |k_3-k_2-1|^{2H} - 2|k_3-k_2|^{2H}) \\
&\quad \times (|k_4-k_3+1|^{2H} + |k_4-k_3-1|^{2H} - 2|k_4-k_3|^{2H}) \\
&\quad \times (|k_4-k_1+1|^{2H} + |k_4-k_1-1|^{2H} - 2|k_4-k_1|^{2H}).
\end{aligned}$$

Denote $p_i = k_{i+1} - k_i$, $i = 1,2,3$. Then the above sum is bounded by

$$Cn^{1-8H} \sum_{p_1,p_2,p_3=1}^{n-1} p_1^{2H-2} p_2^{2H-2} p_3^{2H-2} (p_1+p_2+p_3)^{2H-2},$$

which is again bounded by

$$Cn^{1-8H} \sum_{p_1,p_2,p_3=1}^{n-1} p_1^{2H-2} p_2^{2H-2} p_3^{4H-4}.$$

In the case $H \in (\frac{1}{2}, \frac{3}{4})$, the series $\sum_{p_3=1}^{n-1} p_3^{4H-4}$ is convergent. When $H = \frac{3}{4}$, it is bounded by $C\log n$. So the above sum is bounded by Cn^{-4H-1} if $\frac{1}{2} < H < \frac{3}{4}$ and bounded by $Cn^{-4}\log n$ if $H = \frac{3}{4}$. The proof is complete. □

The following proposition contains a result on the convergence of the fourth moment of $\Theta_n^l(i,j)$.

Proposition 2.5.4. *The fourth moment of $\Theta_l^n(i,j)$ and $3\mathbb{E}(|\Theta_l^n(i,j)|^2)^2$ converge to the same limit as $n \to \infty$.*

Proof: Applying the integration by parts formula (2.7) yields

$$\begin{aligned}
\mathbb{E}[\Theta_l^n(i,j)^4] &= \alpha_H^2\gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathscr{D}_k} \mathbb{E}\left[D_u^i D_t^j \left[\Theta_l^n(i,j)^3\right]\right] \mu(dvdu)\mu(dsdt) \\
&= \alpha_H^2\gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathscr{D}_k} \mathbb{E}\left[\left\{3\Theta_l^n(i,j)^2 D_u^i D_t^j \left[\Theta_l^n(i,j)\right]\right.\right. \\
&\qquad \left.\left. +6\Theta_l^n(i,j) D_t^j \left[\Theta_l^n(i,j)\right] D_u^i \left[\Theta_l^n(i,j)\right]\right\}\right] \mu(dvdu)\mu(dsdt) \\
&:= \overline{G}_1 + \overline{G}_2.
\end{aligned}$$

Since $D_u^i D_t^j \left[\Theta_l^n(i,j)\right]$ is deterministic, it is easy to see that $\overline{G}_1 = 3\mathbb{E}(|\Theta_l^n(i,j)|^2)^2$. We have shown the convergence of $\mathbb{E}(|\Theta_n^l(i,j)|^2)$ in Proposition 2.5.2. It remains to show that $\overline{G}_2 \to 0$ as $n \to \infty$.

Applying again the integration by parts formula (2.7) yields

$$\begin{aligned}
\overline{G}_2 =& 6\alpha_H^4\gamma_n^2 \sum_{k,k'=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathscr{D}_k\times\mathscr{D}_{k'}} D_{u'}^i D_{t'}^j \left\{D_t^j \left[\Theta_l^n(i,j)\right] D_u^i \left[\Theta_l^n(i,j)\right]\right\} \\
& \cdot \mu(dv'du')\mu(ds'dt')\mu(dvdu)\mu(dsdt).
\end{aligned}$$

Using equation (5.7) we can derive the inequalities

$$\begin{aligned}
\overline{G}_2 &\leq 6\alpha_H^4\gamma_n^4 \sum_{k,k',h,h'=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathscr{D}_k\times\mathscr{D}_{k'}} \left\{\beta_{\frac{h}{n}}(t)\beta_{\frac{h}{n}}(t')\beta_{\frac{h'}{n}}(u)\beta_{\frac{h'}{n}}(u')\right. \\
&\quad \left. +\beta_{\frac{h}{n}}(t)\beta_{\frac{h}{n}}(u')\beta_{\frac{h'}{n}}(u)\beta_{\frac{h'}{n}}(t')\right\}\mu(dv'du')\mu(ds'dt')\mu(dvdu)\mu(dsdt) \\
&\leq 12\alpha_H^4\gamma_n^4 \sum_{k,k',h,h'=\{r_l\}}^{\{r_{l+1}\}} \langle\beta_{\frac{h}{n}},\beta_{\frac{k}{n}}\rangle_{\mathscr{H}} \langle\beta_{\frac{h'}{n}},\beta_{\frac{k}{n}}\rangle_{\mathscr{H}} \langle\beta_{\frac{h}{n}},\beta_{\frac{k'}{n}}\rangle_{\mathscr{H}} \langle\beta_{\frac{h'}{n}},\beta_{\frac{k'}{n}}\rangle_{\mathscr{H}}.
\end{aligned}$$

The convergence of $\overline{G}_2$ to zero now follows from Lemma 2.5.1. □

We can now establish a central limit theorem for weighted sums based on the previous proposition. Recall that $\zeta_{k,n}^{i,j} = \int_{t_k}^{t_{k+1}} (B_s^i - B_{t_k}^i) \delta B_s^j$, $k = 0, \ldots, n-1$ and $\zeta_{n,n}^{i,j} = 0$.

Proposition 2.5.5. *Let $f = \{f_t, t \in [0,T]\}$ be a stochastic process with values on the space of $d \times d$ matrices and with Hölder continuous trajectories of index greater than $\frac{1}{2}$. Set, for $i, j = 1, \ldots, m$,*

$$\Psi_n^{i,j}(t) = \sum_{k=0}^{\{t\}} f_{t_k}^{i,j} \zeta_{k,n}^{i,j}.$$

Then, the following stable convergence in the space $D([0,T])$ holds as n tends to infinity,

$$\{\gamma_n \Psi_n(t), t \in [0,T]\} \to \left\{ \left(\int_0^t f_s^{i,j} dW_s^{ij} \right)_{1 \le i,j \le m}, t \in [0,T] \right\},$$

where W is a matrix-valued Brownian motion independent of B with the covariance introduced in Section 2.4.

Proof: This proposition is an immediate consequence of the central limit result for weighted random sums proved in [4]. In fact, the process $\Psi_n^{i,j}(t)$ satisfies the required conditions due to Proposition 2.5.1 and the estimate (5.1). □

2.6 CLT for the modified Euler scheme in the case $H \in (\frac{1}{2}, \frac{3}{4}]$

The following central limit type result shows that in the case $H \in (\frac{1}{2}, \frac{3}{4}]$, the process $\gamma_n(X - X^n)$ converges stably to the solution of a linear stochastic differential equation driven by a matrix-valued Brownian motion independent of B as n tends to infinity.

Theorem 2.6.1. *Let $H \in (\frac{1}{2}, \frac{3}{4}]$ and let X, X^n be the solutions of the SDE (1.1) and recursive scheme (1.3), respectively. Let $W = \{W_t, t \in [0,T]\}$ be the matrix-valued Brown-*

ian motion introduced in Section 2.2.4. Assume $V \in C_b^5(\mathbb{R}^d;\mathbb{R}^{d\times m})$ and $b \in C_b^4(\mathbb{R}^d;\mathbb{R}^d)$. Then the following stable convergence in the space $C([0,T])$ holds as n tends to infinity,

$$\{\gamma_n(X_t - X_t^n), t \in [0,T]\} \to \{U_t, t \in [0,T]\}, \tag{6.1}$$

where $\{U_t, t \in [0,T]\}$ is the solution of the linear d-dimensional SDE

$$\begin{aligned} U_t &= \int_0^t \nabla b(X_s)U_s ds + \sum_{j=1}^m \int_0^t \nabla V^j(X_s)U_s dB_s^j \\ &\quad + \sum_{i,j=1}^m \int_0^t (\nabla V^j V^i)(X_s) dW_s^{ij}. \end{aligned} \tag{6.2}$$

Remark 2.6.1. *It follows from [23] that when B is replaced by a standard Brownian motion the process $\sqrt{n}(X - X^n)$ converges in law to the unique solution of the d-dimensional SDE:*

$$dU_t = \nabla b(X_t)U_s dt + \sum_{j=1}^m \nabla V^j(X_t)U_t dB_t^j + \sqrt{\frac{T}{2}} \sum_{j,i=1}^m (\nabla V^j V^i)(X_t) dW_t^{ij}. \tag{6.3}$$

with $U_0 = 0$. Here W^{ij}, $i,j = 1,\dots,m$, are independent one-dimensional Brownian motions, independent of B. To compare our theorem 2.6.1 with this result, we let the Hurst parameter H converge to $\frac{1}{2}$. Then the constant R will converge to 0 and $\frac{\alpha_H}{\sqrt{T}}\sqrt{Q-R}$ converges to $\sqrt{\frac{T}{2}}$. This formally recovers equation (6.3).

Remark 2.6.2. *The process U defined in (6.2) is given by*

$$U_t = \sum_{i,j=1}^m \int_0^t \Lambda_t \Gamma_s (\nabla V^j V^i)(X_s) dW_s^{ij}, \quad t \in [0,T], \tag{6.4}$$

where we recall that Λ is defined in (4.16) and Γ is its inverse.

Proof of Theorem 2.6.1. Recall that $Y_t = X_t - X_t^n$. We would like to show that the process $\{\gamma_n Y_t, B_t, t \in [0,T]\}$ converges weakly in $C([0,T];\mathbb{R}^{d+m})$ to $\{U_t, B_t, t \in [0,T]\}$. To do this, it suffices to prove the following two statements:

(i) Convergence of the finite dimensional distributions of $\{\gamma_n Y_t, B_t, t \in [0,T]\}$;

(ii) Tightness of the process $\{\gamma_n Y_t, B_t, t \in [0,T]\}$.

We first show (i). Recall the decomposition of Y_t given in (4.7) and (4.11) and recall the estimates obtained for each term in the decomposition of Y_t. Since the other terms converge to zero in L^p for $p \geq 1$, from the Slutsky theorem it suffices to consider the convergence of the finite dimensional distributions of $\{\gamma_n \sum_{j=1}^m E_{2,j}(t), B_t, t \in [0,T]\}$, where $E_{2,j}$ is defined in Theorem 2.4.1 Step 3. Set

$$F_s^{i,j} := \Lambda_t^n \Gamma_s^n (\nabla V^j V^i)(X_s^n) - \Lambda_t \Gamma_s (\nabla V^j V^i)(X_s). \tag{6.5}$$

It follows from Lemma 2.4.1 and Remark 2.4.1 that

$$\sup_{r,s,t \in [0,T]} \left(\left\| F_t^{i,j} \right\|_p \vee \left\| D_s F_t^{i,j} \right\|_p \vee \left\| D_r D_s F_t^{i,j} \right\|_p \right) \leq C n^{1-2\beta}.$$

Denote

$$\widetilde{E}_{2,j}(t) = \Lambda_t \sum_{i=1}^m \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \Gamma_{t_k} (\nabla V^j V^i)(X_{t_k}) \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \delta B_u^i \delta B_s^j, \tag{6.6}$$

for $t \in [0,T)$, and $\widetilde{E}_{2,j}(T) = \widetilde{E}_{2,j}(T-)$. Then applying Lemma 2.11.4 (11.9) with $F^{i,j}$ defined by (6.5), we obtain that

$$\gamma_n \left\| E_{2,j}(t) - \widetilde{E}_{2,j}(t) \right\|_p \leq C \gamma_n n^{-H} n^{1-2\beta},$$

which converges to zero as $n \to \infty$ since β can be taken as close as possible to H. By Slutsky theorem again, it suffices to consider the convergence of the finite dimensional distributions of

$$\left\{ \gamma_n \sum_{j=1}^{m} \widetilde{E}_{2,j}(t), B_t, t \in [0,T] \right\}. \tag{6.7}$$

Applying Proposition 2.5.5 to the family of processes $f_t^{i,j} = \Gamma_t(\nabla V^j V^i)(X_t)$, we obtain the convergence of the finite dimensional distributions of

$$\{\gamma_n \sum_{j=1}^{m} \Gamma_t \widetilde{E}_{2,j}(t), B_t, t \in [0,T]\}$$

to those of $\{\Gamma_t U_t, B_t, t \in [0,T]\}$. This implies the convergence of the finite dimensional distributions of

$$\{\gamma_n \sum_{j=1}^{m} \widetilde{E}_{2,j}(t), B_t, t \in [0,T]\}$$

to those of $\{U_t, B_t, t \in [0,T]\}$.

To show (ii), we prove the following tightness condition

$$\sup_{n \geq 1} \mathbb{E}\left(|\gamma_n(X_t - X_t^n) - \gamma_n(X_s - X_s^n)|^4 \right) \leq C(t-s)^2. \tag{6.8}$$

Taking into account of (4.7) and (4.11), we only need to show the above inequality for $\gamma_n I_{11}$, $\gamma_n I_{12,j}$, $\gamma_n I_{13}$, $\gamma_n I_{2,j}$, $\gamma_n I_{4,j}$, $\gamma_n E_{1,j}$, $\gamma_n E_{2,j}$ and $\gamma_n E_{3,j}$. The tightness for the terms $\gamma_n I_{11}$, $\gamma_n I_{13}$ and $\gamma_n E_{3,j}$ is clear. Now we consider the tightness of the term $I_{2,j}$. We write

$$\begin{aligned} I_{2,j}(t) - I_{2,j}(s) =& (\Lambda_t^n - \Lambda_s^n) \int_0^t \Gamma_s^n b_2^j(s)(s - \eta(s)) dB_s^j \\ &+ \int_s^t \Lambda_s^n \Gamma_u^n b_2^j(u)(u - \eta(u)) dB_u^j. \end{aligned}$$

Then it follows from Lemma 2.11.3 (11.4) that

$$\mathbb{E}\left(\left|\gamma_n(\Lambda_t^n-\Lambda_s^n)\int_0^t \Gamma_s^n b_2^j(s)(s-\eta(s))dB_s^j\right|^4\right) \le C(t-s)^{4\beta}\left(\mathbb{E}\|\Lambda^n\|_\beta^8\right)^{\frac{1}{2}} \le C(t-s)^{4\beta}.$$

Lemma 2.11.3 (11.4) also implies that the fourth moment of the second term is bounded by $C(t-s)^{4H}$. The tightness for $\gamma_n I_{12,j}$, $\gamma_n I_{4,j}$, $\gamma_n E_{1,j}$, $\gamma_n E_{2,j}$ can be obtained in a similar way by applying the estimates (11.5) and (11.4) from Lemma 2.11.3, (11.15) from lemma 2.11.5, and (11.8) from Lemma 2.11.4, respectively. □

2.7 A limit theorem in L^p for weighted sums

Following the methodology used in [4], we can show the following limit result for random weighted sums. The proof uses the techniques of fractional calculus and the classical decompositions in large and small blocks.

Consider a double sequence of random variables $\zeta = \{\zeta_{k,n}, n\in\mathbb{N}, k=0,1,\dots,n\}$ and for each $t\in[0,T]$, we denote

$$g_n(t) := \sum_{k=0}^{\lfloor \frac{nt}{T}\rfloor} \zeta_{k,n}\,. \tag{7.1}$$

Proposition 2.7.1. *Fix $\lambda > 1-\beta$, where $0<\beta<1$. Let $p\ge 1$ and $p', q' > 1$ such that $\frac{1}{p'}+\frac{1}{q'}=1$ and $pp' > \frac{1}{\beta}$, $pq' > \frac{1}{\lambda}$. Let g_n be the sequence of processes defined in (7.1). Suppose that the following conditions hold true:*

(i) For each $t\in[0,T]$, $g_n(t)\to z(t)$ in $L^{pq'}$.

(ii) For any $j,k = 0,1,\dots,n$ we have

$$\mathbb{E}(|g_n(kT/n) - g_n(jT/n)|^{pq'}) \le C(|k-j|/n)^{\lambda pq'}.$$

Let $f = \{f(t), t \in [0,T]\}$ be a process such that $\mathbb{E}(\|f\|_\beta^{pp'}) \le C$ and $\mathbb{E}(|f(0)|^{pp'}) \le C$. Then for each $t \in [0,T]$,

$$F^n(t) := \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} f(t_k)\zeta_{k,n} \to \int_0^t f(s)dz(s) \quad \text{in } L^p \text{ as } n \to \infty. \tag{7.2}$$

Remark 2.7.1. *The integral $\int_0^t f(s)dz(s)$ is interpreted as a Young integral in the sense of Proposition 2.2.2, which is well defined because f and z, as functions on $[0,T]$ with values in $L^{pp'}$ and $L^{pq'}$, are Hölder continuous (conditions (i) and (ii) together imply the Hölder continuity of z) of order β and λ, respectively. Recall that the Hölder continuity of a function with values in L^p is defined in (2.5).*

Remark 2.7.2. *The convergence (7.2) still holds true if the condition $\mathbb{E}(\|f\|_\beta^{pp'}) \le C$ is weakened by assuming that f is Hölder continuous of order β in $L^{pp'}$. The proof will be similar to that of Proposition 2.7.1.*

Proof: Given two natural numbers $m < n$ we consider the associated partitions of the interval $[0,T]$ given by $t_k = \frac{kT}{n}$, $k = 0,1,\dots,n$ and $u_l = \frac{lT}{m}$, $l = 0,1,\dots,m$. Then, we have the decomposition

$$F^n(t) = \sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} f(u_l) \sum_{k \in I_m(l)} \zeta_{k,n} + \sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} \sum_{k \in I_m(l)} [f(t_k) - f(u_l)]\zeta_{k,n}, \tag{7.3}$$

where $I_m(l) := \{k : 0 \le k \le \lfloor \frac{nt}{T} \rfloor, t_k \in [u_l, u_{l+1})\}$.

Because of condition (i) and the assumption that $\mathbb{E}(|f(t)|^{pp'}) \le C$ for all $t \in [0,T]$, the first term on the right-hand side of the above expression converges in L^p, as n tends to infinity, to

$$\sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} f(u_l)[z(u_{l+1}) - z(u_l)].$$

Applying Proposition 2.2.2 to f and z we obtain that the above Riemann-Stieltjes sum converges to the Young integral $\int_0^t f(s)dz(s)$ in L^p as m tends to infinity. To show the convergence (7.2) it suffices to show that

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}\left(\left|\sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} \sum_{k\in I_m(l)} [f(t_k) - f(u_l)]\zeta_{k,n}\right|^p\right) = 0. \tag{7.4}$$

Notice that k belongs to $I_m(l)$ if and only if $u_l \le t_k < \varepsilon(u_{l+1})$ and $t_k \le \eta(t)$. Recall that $\varepsilon(u) = t_{k+1}$ if $t_k < u \le t_{k+1}$ and $\eta(u) = t_k$ if $t_k \le u < t_{k+1}$. As a consequence, we can write

$$\sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} \sum_{k\in I_m(l)} [f(t_k) - f(u_l)]\zeta_{k,n} = \sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} \int_{(a_l,b_l)} [f(s) - f(a_l)]dg_n(s),$$

where $a_l = u_l$ and $b_l = \varepsilon(u_{l+1}) \wedge \left(\eta(t) + \frac{T}{n}\right)$. By the fractional integration by parts formula,

$$\begin{aligned} &\int_{(a_l,b_l)} [f(s) - f(a_l)]dg_n(s) \\ &\quad = (-1)^\alpha \int_{a_l}^{b_l} D^\alpha_{a_l+}[f(s) - f(a_l)] D^{1-\alpha}_{b_l-}[g_n(s) - g_n(b_l-)]ds, \end{aligned} \tag{7.5}$$

where we take $\alpha \in (1-\lambda, \beta)$. By (2.2), it is easy to show that

$$\begin{aligned} |D^\alpha_{a_l+}[f(s) - f(a_l)]| &\le \frac{1}{\Gamma(1-\alpha)} \frac{\beta}{\beta-\alpha} \|f\|_\beta (s-a_l)^{\beta-\alpha} \\ &\le C\|f\|_\beta m^{\alpha-\beta}. \end{aligned} \tag{7.6}$$

On the other hand, by (2.3) we have

$$\begin{aligned}
&\left|D_{b_l-}^{1-\alpha}[g_n(s)-g_n(b_l-)]\right| \\
&=\frac{1}{\Gamma(\alpha)}\left|\frac{g_n(s)-g_n(b_l-)}{(b_l-s)^{1-\alpha}}+(1-\alpha)\int_s^{b_l}\frac{g_n(s)-g_n(u)}{(u-s)^{2-\alpha}}du\right|.
\end{aligned}\tag{7.7}$$

We can calculate the integral in the above equation explicitly.

$$\begin{aligned}
&\int_s^{b_l}\frac{g_n(s)-g_n(u)}{(u-s)^{2-\alpha}}du \\
&=\int_{\varepsilon(s)}^{b_l}\frac{g_n(s)-g_n(u)}{(u-s)^{2-\alpha}}du \\
&=\sum_{k:t_k\in[\varepsilon(s),b_l)}[g_n(s)-g_n(t_k)]\int_{t_k}^{t_{k+1}}(u-s)^{\alpha-2}du \\
&=\sum_{k:t_k\in[\varepsilon(s),b_l)}[g_n(s)-g_n(t_k)]\frac{1}{1-\alpha}[(t_k-s)^{\alpha-1}-(t_{k+1}-s)^{\alpha-1}].
\end{aligned}\tag{7.8}$$

Substituting (7.6), (7.7) and (7.8) into (7.5), we obtain

$$\begin{aligned}
&\left|\int_{(a_l,b_l)}[f(s)-f(a_l)]dg_n(s)\right| \\
&\leq C\|f\|_\beta m^{\alpha-\beta}\int_{a_l}^{b_l}\left|D_{b_l-}^{1-\alpha}[g_n(s)-g_n(b_l-)]\right|ds \\
&\leq C\|f\|_\beta m^{\alpha-\beta}\sum_{k:t_k\in[\eta(a_l),b_l)}\int_{t_k}^{t_{k+1}}\left|D_{b_l-}^{1-\alpha}[g_n(s)-g_n(b_l-)]\right|ds \\
&\leq C\|f\|_\beta m^{\alpha-\beta}\sum_{k:t_k\in[\eta(a_l),b_l)}|g_n(t_k)-g_n(b_l-)|\int_{t_k}^{t_{k+1}}(b_l-s)^{\alpha-1}ds \\
&\quad +C\|f\|_\beta m^{\alpha-\beta}\sum_{k,j:\eta(a_l)\leq t_k<t_j<b_l}|g_n(t_k)-g_n(t_j)| \\
&\qquad \cdot\int_{t_k}^{t_{k+1}}[(t_j-s)^{\alpha-1}-(t_{j+1}-s)^{\alpha-1}]ds.
\end{aligned}$$

We denote the first term in the right-hand side of the above expression by $A_{1,l}$ and the second one by $A_{2,l}$.

Applying the Minkowski inequality, we see that the quantity

$$\mathbb{E}\left(\left|\sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} A_{1,l}\right|^p\right)^{\frac{1}{p}} \tag{7.9}$$

is less than

$$Cm^{\alpha-\beta}\left|\sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} \sum_{k:t_k\in[\eta(a_l),b_l)} \mathbb{E}(\|f\|_\beta^p |g_n(t_k)-g_n(b_l-)|^p)^{\frac{1}{p}} \int_{t_k}^{t_{k+1}} (b_l-s)^{\alpha-1}ds\right|,$$

so by applying the Hölder inequality, the condition (ii) and the assumption $\mathbb{E}(\|f\|_\beta^{pp'}) \leq C$ to the above we can show that the quantity (7.9) is less than

$$Cm^{\alpha-\beta}\left|\sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} \sum_{k:t_k\in[\eta(a_l),b_l)} (b_l-t_k)^\lambda \int_{t_k}^{t_{k+1}} (b_l-s)^{\alpha-1}ds\right|. \tag{7.10}$$

Since

$$\begin{aligned}
&\sum_{k:t_k\in[\eta(a_l),b_l)} (b_l-t_k)^\lambda \int_{t_k}^{t_{k+1}} (b_l-s)^{\alpha-1}ds \\
&= \frac{1}{\alpha}\left(\frac{T}{n}\right)^{\lambda+\alpha} + \sum_{k:t_k\in\left[\eta(a_l),b_l-\frac{T}{n}\right)} (b_l-t_k)^\lambda \int_{t_k}^{t_{k+1}} (b_l-s)^{\alpha-1}ds \\
&\leq \frac{1}{\alpha}\left(\frac{T}{n}\right)^{\lambda+\alpha} + \frac{T}{n}\sum_{k:t_k\in\left[\eta(a_l),b_l-\frac{T}{n}\right)} (b_l-t_k)^\lambda (b_l-t_{k+1})^{\alpha-1} \\
&\leq \frac{1}{\alpha}\left(\frac{T}{n}\right)^{\lambda+\alpha} + C\frac{T}{n}\frac{n}{m}m^{-\alpha+1-\lambda} \\
&\leq Cm^{-\alpha-\lambda},
\end{aligned}$$

where in the second inequality we used the assumption that $\alpha > 1-\lambda$ and the fact that the number of partition points $\{t_k, k=0,1,\ldots,n\}$ in $\left[\eta(a_l), b_l-\frac{T}{n}\right)$ is bounded by $\frac{n}{m}$, the estimate (7.10) of (7.9) implies that

$$\mathbb{E}\left(\left|\sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} A_{1,l}\right|^p\right)^{\frac{1}{p}} \leq Cm^{\alpha-\beta} \sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} m^{-\alpha-\lambda} \leq Cm^{1-\beta-\lambda} \to 0 \tag{7.11}$$

as m tends to ∞.

Using an argument similar to the estimate of the quantity (7.9), it can be shown that the quantity

$$\mathbb{E}\left(\left|\sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} A_{2,l}\right|^p\right)^{\frac{1}{p}}$$

is less than

$$C\left|m^{\alpha-\beta} \sum_{l=0}^{\lfloor \frac{mt}{T} \rfloor} \sum_{k,j:\eta(a_l)\leq t_k<t_j<b_l} |t_k-t_j|^\lambda \int_{t_k}^{t_{k+1}} [(t_j-s)^{\alpha-1}-(t_{j+1}-s)^{\alpha-1}]ds\right|.$$

The summand in the above can be estimated as follows.

$$\begin{aligned}
&\sum_{k,j:\eta(a_l)\leq t_k<t_j<b_l} |t_k-t_j|^\lambda \int_{t_k}^{t_{k+1}} [(t_j-s)^{\alpha-1}-(t_{j+1}-s)^{\alpha-1}]ds \\
&\leq C\frac{n}{m}\left(\frac{T}{n}\right)^{\alpha+\lambda} + \sum_{k,j:\eta(a_l)\leq t_{k+1}<t_j<b_l} |t_k-t_j|^\lambda \int_{t_k}^{t_{k+1}} [(t_j-s)^{\alpha-1}-(t_{j+1}-s)^{\alpha-1}]ds \\
&\leq C\frac{n}{m}\left(\frac{T}{n}\right)^{\alpha+\lambda} + C\left(\frac{T}{n}\right)^2 \sum_{k,j:\eta(a_l)\leq t_{k+1}<t_j<b_l} |t_{k+1}-t_j|^\lambda (t_j-t_{k+1})^{\alpha-2} \\
&\leq C\frac{n}{m}\left(\frac{T}{n}\right)^{\alpha+\lambda} + Cn^{-2}n^{2-\lambda-\alpha} \sum_{k,j:\eta(a_l)\leq t_{k+1}<t_j<b_l} (j-k-1)^{\alpha-2+\lambda} \\
&\leq C\frac{n}{m}\left(\frac{T}{n}\right)^{\alpha+\lambda} + Cn^{-2}n^{2-\lambda-\alpha} \frac{n}{m} \sum_{p=2}^{n/m} (p-1)^{\alpha-2+\lambda} \\
&\leq Cm^{-\alpha-\lambda}.
\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left(\left|\sum_{l=0}^{\lfloor \frac{mt}{T}\rfloor} A_{2,l}\right|^p\right)^{\frac{1}{p}} = Cm^{\alpha-\beta}\left|\sum_{l=0}^{\lfloor \frac{mt}{T}\rfloor} m^{-\alpha-\lambda}\right| \to 0 \quad \text{as } m\to\infty.$$

The above convergence and the equality (7.11) together imply the convergence (7.4). The proof is now complete. □

This result has the following two consequences.

Corollary 2.7.1. *Let $B = \{B_t, t \in [0,T]\}$ be an m-dimensional fBm with Hurst parameter $H > 3/4$. Define*

$$\zeta_{k,n}^{ij} = n\int_{t_k}^{t_{k+1}} (B_s^i - B_{\eta(s)}^i)\delta B_s^j,$$

for $i, j = 1,\dots,m$ and $k = 0,\dots,n-1$, where we recall that $t_k = \frac{kT}{n}$. Set also $\zeta_{n,n} = 0$. Let $\lambda = \frac{1}{2}$, and β, p, p', q', f satisfy the assumptions in Proposition 2.7.1. Then

$$\sum_{k=0}^{\lfloor \frac{nt}{T}\rfloor} f(t_k)\zeta_{k,n}^{ij} \to \int_0^t f(s)dZ_s^{ij} \quad \text{in } L^p,$$

where Z^{ij} is the generalized Rosenblatt process defined in Section 2.2.5.

Proof: To prove the corollary, it suffices to show that the conditions in Proposition 2.7.1 are all satisfied here. We have shown in Section 2.2.5 the L^2 convergence of $g_n(t) = \sum_{k=0}^{\lfloor \frac{nt}{T}\rfloor} \zeta_{k,n}^{ij}$ to Z_t^{ij}. This convergence also holds in L^p due to the equivalence of all the L^p-norms in a finite Wiener chaos. Applying (11.8) in Lemma 2.11.4 with $F \equiv 1$ and taking into account that $\gamma_n = n$ when $H > \frac{3}{4}$, we obtain condition (ii) in Proposition 2.7.1 with $\lambda = \frac{1}{2}$. □

The following result will also be useful later.

Corollary 2.7.2. *Let $B = \{B_t, t \in [0,T]\}$ be one-dimensional fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$. Define*

$$\zeta_{k,n} = \int_{t_k}^{t_{k+1}} (s - \eta(s)) dB_s \,, \tag{7.12}$$

for $k = 0, \dots, n-1$. Set also $\zeta_{n,n} = 0$. Let $\lambda = H$, and β, p, p', q', f satisfy the assumptions in Proposition 2.7.1. Then for each $t \in [0,T]$,

$$n \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} f(t_k) \zeta_{k,n} \to \frac{T}{2} \int_0^t f(s) dB_s \,,$$

in L^p, as n tends to infinity. This convergence still holds true when we replace the above $\zeta_{k,n}$ by

$$\tilde{\zeta}_{k,n} = \int_{t_k}^{t_{k+1}} (B_s - B_{\eta(s)}) ds \,.$$

Proof: As before, to prove the corollary it suffices to show that the conditions in Proposition 2.7.1 are all satisfied here. Let us first consider the convergence for $\zeta_{k,n}$. Set

$$g_n(t) := n \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \zeta_{k,n},$$

where $\zeta_{k,n}$ is defined in (7.12). Condition (ii) follows from estimate (11.4) in Lemma 2.11.3 by taking $F \equiv 1$ and $\nu = 1$. The covariance of the process g_n is given by

$$\begin{aligned}
\mathbb{E}(g_n(t) g_{n'}(t)) &= \alpha_H n n' \int_0^t \int_0^t (u - \eta_n(u))(v - \eta_{n'}(v)) \mu(dudv) \\
&\to \frac{T^2}{4} \alpha_H \int_0^t \int_0^t |u - v|^{2H-2} dudv \\
&= \frac{T^2}{4} t^{2H}
\end{aligned}$$

as $n, n' \to \infty$, which implies that $g_n(t)$ is a Cauchy sequence in L^2. Here $\eta_n(t) = \frac{T}{n}i$ when $\frac{T}{n}i \le t < \frac{T}{n}(i+1)$ and $\eta_{n'}(t) = \frac{T}{n'}i$ when $\frac{T}{n'}i \le t < \frac{T}{n'}(i+1)$. In fact, we can also calculate the kernel of the limit of $z_n(t)$. Suppose that $\phi_n \in \mathscr{H}$ satisfies $g_n(t) = \delta(\phi_n(t))$. Then for any $\psi \in \mathscr{H}$

$$\begin{aligned}\langle n\phi_n, \psi\rangle_{\mathscr{H}} &= n\alpha_H \int_0^T \int_0^{\eta(t)} (u - \eta(u))\psi(v)|u-v|^{2H-2} du dv \\ &\to \frac{T}{2}\langle \psi, \mathbf{1}_{[0,t]}\rangle_{\mathscr{H}},\end{aligned}$$

as $n \to +\infty$. This implies that the kernel of the limit of $g_n(t)$ is $\frac{T}{2}\mathbf{1}_{[0,t]}$, in other words, the random variable $g_n(t)$ converges in L^2 to $\frac{T}{2}B_t$.

The convergence result for $\tilde{\zeta}_{k,n}$ can be shown by noticing that

$$\tilde{\zeta}_{k,n} = \int_{t_k}^{t_{k+1}} (B_s - B_{\eta(s)}) ds = \frac{T}{n}(B_{t_{k+1}} - B_{t_k}) - \int_{t_k}^{t_{k+1}} (s - \eta(s)) dB_s.$$

This completes the proof of the corollary. □

2.8 Asymptotic error of the modified Euler scheme in case $H \in (\frac{3}{4}, 1)$

The limit theorems for weighted sums proved in the previous section allow us to derive the L^p-limit of the quantity $n(X_t - X_t^n)$ in the case $H \in (\frac{3}{4}, 1)$.

Theorem 2.8.1. *Let $H \in (\frac{3}{4}, 1)$. Suppose that X and X^n are defined by (1.1) and (1.3), respectively. Let Z^{ij}, $i, j = 1, \dots, m$, be the matrix-valued generalized Rosenblatt process defined in Section 2.2.5. Assume $V \in C_b^5(\mathbb{R}^d; \mathbb{R}^{d\times m})$ and $b \in C_b^4(\mathbb{R}^d; \mathbb{R}^d)$. Then*

$$n(X_t - X_t^n) \to \overline{U}_t$$

in $L^p(\Omega)$ as n tends to infinity, where $\{\bar{U}_t, t \in [0,T]\}$ is the solution of the following linear stochastic differential equation

$$\begin{aligned}\overline{U}_t = &\int_0^t \nabla b(X_s)\overline{U}_s ds + \sum_{j=1}^m \int_0^t \nabla V^j(X_s)\overline{U}_s dB_s^j + \sum_{i,j=1}^m \int_0^t (\nabla V^j V^i)(X_s) dZ_s^{ij} \\ &+ \frac{T}{2}\int_0^t (\nabla bb)(X_s)ds + \frac{T}{2}\int_0^t (\nabla bV)(X_s)dB_s + \frac{T}{2}\sum_{j=1}^m \int_0^t (\nabla V^j b)(X_s)dB_s^j.\end{aligned} \tag{8.1}$$

Proof: Recall the decomposition $Y_t = X_t - X_t^n$ given in (4.7) and (4.11). We have shown that $nI_{13}(t)$, $nI_{4,j}(t)$, $nE_{1,j}(t)$ and $nE_{3,j}(t)$ converge in L^p to zero for each $t \in [0,T]$. It remains to show the L^p convergence of $nI_{11}(t)$, $nI_{12,j}(t)$, $nI_{2,j}(t)$ and $nE_{2,j}(t)$ and identify their limits.

Step 1. Recall $\widetilde{E}_{2,j}(t)$ is defined in (6.6). It has been shown in the proof of Theorem 2.6.1 that $n(E_{2,j}(t) - \widetilde{E}_{2,j}(t))$ converges to zero in L^p. On the other hand, applying Corollary 2.7.1 to $n\widetilde{E}_{2,j}(t)$ yields

$$n\widetilde{E}_{2,j}(t) \to \sum_{i=1}^m \int_0^t \Lambda_t \Gamma_s (\nabla V^j V^i)(X_s) dZ_s^{ij} \quad \text{in } L^p.$$

Therefore, $nE_{2,j}(t)$ converges in L^p and the limit is the same as $n\widetilde{E}_{2,j}(t)$.

Step 2. Denote

$$\tilde{I}_{2,j}(t) = \Lambda_t \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \Gamma_{t_k}(\nabla V^j b)(X_{t_k}) \int_{t_k}^{t_{k+1}\wedge t} (s - \eta(s)) dB_s^j,$$

for $t \in [0,T]$ (as before, we define $t_{n+1} = \frac{T}{n}(n+1)$). Applying Corollary 2.7.2 to $n\tilde{I}_{2,j}(t)$ yields

$$n\tilde{I}_{2,j}(t) \to \frac{T}{2}\int_0^t \Lambda_t \Gamma_s (\nabla V^j b)(X_s) dB_s^j \quad \text{in } L^p.$$

We want to show that $nI_{2,j}(t)$ and $n\tilde{I}_{2,j}(t)$ have the same limit in L^p. Write

$$
\begin{aligned}
& n(I_{2,j}(t)-\tilde{I}_{2,j}(t)) && (8.2)\\
&= n\int_0^t (\Lambda_t^n\Gamma_s^n b_2^j(s)-\Lambda_t\Gamma_s\tilde{b}_2^j(s))(s-\eta(s))dB_s^j \\
&\quad + n\int_0^t \Lambda_t(\Gamma_s\tilde{b}_2^j(s)-\Gamma_{\eta(s)}(\nabla V^j b)(X_{\eta(s)}))(s-\eta(s))dB_s^j,
\end{aligned}
$$

where $\tilde{b}_2^j(s)=\int_0^1 \nabla V^j(\theta X_s+(1-\theta)X_{\eta(s)})b(X_{\eta(s)})d\theta$. It suffices to show that the two terms in the right-hand side of (8.2) both converge to zero in L^p. The convergence of the second term follows from estimate (11.16) of Lemma 2.11.5. Lemma 2.4.1 implies that the L^p-norms of $\left[\Lambda_t^n\Gamma_s^n b_2^j(s)-\Lambda_t\Gamma_s\tilde{b}_2^j(s)\right]$ and its Malliavin derivative converge to zero as $n\to\infty$. So applying Lemma 2.11.3 (11.4) with $\nu=1$ and $F_s=\Lambda_t^n\Gamma_s^n b_2^j(s)-\Lambda_t\Gamma_s\tilde{b}_2^j(s)$ we obtain the convergence of the first term.

Step 3. Following the lines in Step 2 we can show that $nI_{12,j}(t)$ converges in L^p to

$$\frac{T}{2}\int_0^t \Lambda_t\Gamma_s(\nabla bV^j)(X_s)dB_s^j.$$

Instead of (11.4) and (11.16) in Step 2, we need to use the estimates (11.5) and (11.15) here.

Similarly, it can be shown that nI_{11} converges in L^p to

$$\frac{T}{2}\int_0^t \Lambda_t\Gamma_s(\nabla bb)(X_s)ds.$$

Step 4. We have shown that $n(X_t - X_t^n)$ converges in L^p to $\overline{U}_t$, where we define, for each $t \in [0,T]$,

$$\overline{U}_t = \sum_{i,j=1}^{m} \int_0^t \Lambda_t \Gamma_s (\nabla V^j V^i)(X_s) dZ_s^{ij} + \frac{T}{2} \int_0^t \Lambda_t \Gamma_s (\nabla bb)(X_s) ds$$
$$+ \frac{T}{2} \int_0^t \Lambda_t \Gamma_s (\nabla bV)(X_s) dB_s + \frac{T}{2} \sum_{j=1}^{m} \int_0^t \Lambda_t \Gamma_s (\nabla V^j b)(X_s) dB_s^j .$$

The theorem follows from the fact that the process $\overline{U}$ satisfies the equation (8.1). □

2.9 Weak approximation of the modified Euler scheme

The next result provides the weak rate of convergence for the modified Euler scheme (1.3).

Theorem 2.9.1. *Let X and X^n be the solution to equations (1.1) and (1.3), respectively. Suppose that $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $V \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$. Then for any function $f \in C_b^3(\mathbb{R}^d)$ there exists a constant C independent of n such that*

$$\sup_{0\le t\le T} \Big| \mathbb{E}\big[f(X_t)\big] - \mathbb{E}\big[f(X_t^n)\big] \Big| \le Cn^{-1}. \tag{9.1}$$

If we further assume that $b \in C^4$, $V \in C^5$ and $f \in C^4$, then for each $t \in [0,T]$, the sequence

$$n\Big\{\mathbb{E}\big[f(X_t)\big] - \mathbb{E}\big[f(X_t^n)\big]\Big\}, \ n \in \mathbb{N},$$

converges as n tends to infinity, and the limit is equal to the sum of the following two quantities,

$$\frac{\alpha_H^2 T}{2} \sum_{j,i=1}^m \int_0^t \int_0^t \int_0^t \mathbb{E}\Big\{ D_u^i D_r^j \Big[\nabla f(X_t) \Lambda_t \Gamma_s \big(\nabla V^j V^i \big)(X_s) \Big] \Big\} \\ \cdot |u-s|^{2H-2} |s-r|^{2H-2} du ds dr \tag{9.2}$$

and

$$\frac{T}{2} \mathbb{E}\Big\{ \nabla f(X_t) \Lambda_t \Big[\int_0^t \Gamma_s(\nabla b b)(X_s) ds + \int_0^t \Gamma_s(\nabla b V)(X_s) dB_s \\ + \sum_{j=1}^m \int_0^t \Gamma_s(\nabla V^j b)(X_s) dB_s^j \Big] \Big\}. \tag{9.3}$$

Proof: We use again the decompositions (4.7) and (4.11) of $Y_t = X_t - X_t^n$, $t \in [0,T]$ and we continue to use the notations there. Given a function $f \in C_b^3(\mathbb{R}^d)$, we can write

$$n\left\{ \mathbb{E}\left[f(X_t) \right] - \mathbb{E}\left[f(X_t^n) \right] \right\} = n \int_0^1 \mathbb{E}\Big[\nabla f(Z_t^\theta) Y_t \Big] d\theta,$$

where we denote $Z_t^\theta = \theta X_t + (1-\theta) X_t^n$, $0 \le t \le T$.

Step 1. In this step, we show that $\sup_{0 \le t \le T} \left| \mathbb{E}\left[\nabla f(Z_t^\theta) Y_t \right] \right| \le C n^{-1}$, which implies (9.1). From the estimates (4.9) and (4.10) it follows that this inequality is true when Y is replaced by $I_{11}, I_{13}, I_{12,j}, I_{2,j}$ or $I_{4,j}$. Therefore, it suffices to show that $\left| \mathbb{E}\left[\nabla f(Z_t^\theta) E_{i,j}(t) \right] \right| \le C n^{-1}$ for $i = 1,2,3$ and $j = 1, \dots, m$, where $E_{ij}(t)$ are defined in Theorem 2.4.1 Step 3. Consider first the term $i = 2$. The use of the expression (4.12) and an application of the integration by parts formula yield

$$
\begin{aligned}
&\mathbb{E}\left[\nabla f(Z_t^\theta)E_{2,j}(t)\right] \\
&= \mathbb{E}\left[\nabla f(Z_t^\theta)\Lambda_t^n \sum_{i=1}^m \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} F_{t_k}^{n,i,j} \int_{t_k}^{t_{k+1}\wedge t} \int_{t_k}^{s} \delta B_u^i \delta B_s^j \right] \\
&= \alpha_H^2 \sum_{i=1}^m \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \mathbb{E}\left[\int_0^t \int_{t_k}^{t_{k+1}\wedge t} \int_0^t \int_{t_k}^s D_v^i D_r^j \left[\nabla f(Z_t^\theta)\Lambda_t^n F_{t_k}^{n,i,j}\right] \mu(dudv)\mu(dsdr)\right],
\end{aligned}
\tag{9.4}
$$

where we recall that $F_t^{n,i,j} = \Gamma_t^n(\nabla V^j V^i)(X_t^n)$ (as before, in the above equation we set $t_{n+1} = \frac{T}{n}(n+1)$). Therefore,

$$
\begin{aligned}
\left|\mathbb{E}\left[\nabla f(Z_t^\theta)E_{2,j}(t)\right]\right| &\le C \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_0^t \int_{t_k}^{t_{k+1}} \int_0^t \mu(dudv)\mu(drds) \\
&\le Cn^{-1}.
\end{aligned}
$$

For the term containing $E_{1,j}$ we can write

$$
\mathbb{E}\left[\nabla f(Z_t^\theta)E_{1,j}(t)\right] = \sum_{i=1}^m \mathbb{E}\left[\int_0^t H_s^{n,i,j}(B_s^i - B_{\eta(s)}^i)dB_s^j\right],
$$

where $H_s^{n,i,j} = \nabla f(Z_t^\theta)\Lambda_t^n \left[\Gamma_s^n V_2^{j,i}(s) - \Gamma_{\eta(s)}^n(\nabla V^j V^i)(X_{\eta(s)}^n)\right]$. An application of the relation between the Skorohod and path-wise integral (2.9) yields

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t H_s^{n,i,j}(B_s^i - B_{\eta(s)}^i)dB_s^j\right] \\
&= \alpha_H \int_0^T \int_0^t \mathbb{E}\left[D_u^j(H_s^{n,i,j}(B_s^i - B_{\eta(s)}^i))\right] |s-u|^{2H-2} dsdu \\
&= \alpha_H \int_0^T \int_0^t \mathbb{E}\left[D_u^j H_s^{n,i,j}(B_s^i - B_{\eta(s)}^i)\right] |s-u|^{2H-2} dsdu \\
&\quad + \alpha_H \int_0^T \int_0^t \mathbb{E}\left[H_s^{n,i,j}\right] \mathbf{1}_{[\eta(s),s]}(u)\delta_{ij}|s-u|^{2H-2} dsdu \\
&:= A_1 + A_2.
\end{aligned}
$$

By the integration by parts we see that A_1 is equal to

$$\alpha_H^2 \int_0^T \int_0^t \int_0^T \int_0^s \mathbb{E}\left[D_r^i D_u^j H_s^{n,i,j}\right] \mathbf{1}_{[\eta(s),s]}(v)|v-r|^{2H-2}|s-u|^{2H-2} dv dr ds du\,.$$

Using $\sup_{r,u,s}\left|\mathbb{E}\left[D_r^i D_u^j H_s^{n,i,j}\right]\right| \le Cn^{-\beta}$ for any $\frac{1}{2} < \beta < H$ we obtain

$$|A_1| \le Cn^{-1-\beta}. \tag{9.5}$$

On the other hand, it is easy to show by the definitions of Γ^n, X^n and X that the quantity $\left[\Gamma_s^n V_2^{j,i}(s) - \Gamma_{\eta(s)}^n (\nabla V^j V^i)(X_{\eta(s)}^n)\right]$ can be expressed as the sum of integrals over the interval $[\eta(s),s]$. So by applying (2.9) and integration by parts we can show that $\left|\mathbb{E}\left[H_s^{n,i,j}\right]\right| \le Cn^{-1}$, which implies

$$|A_2| \le Cn^{-2H}. \tag{9.6}$$

From (9.5) and (9.6) we conclude that $\left|\mathbb{E}\left[\nabla f(Z_t^\theta)E_{1,j}(t)\right]\right| \le Cn^{-1}$. Finally, for the term containing $E_{3,j}$ we have

$$\mathbb{E}\left[\nabla f(Z_t^\theta)E_{3,j}(t)\right] = \int_0^t \mathbb{E}[J_s^{n,i,j}](s-\eta(s)^{2H-1} ds,$$

where $J_s^{n,i,j} = H\nabla f(Z_t^\theta)\Lambda_t^n\left(\Gamma_{\eta(s)}^n - \Gamma_s^n\right)V_0^j(s)$. By expressing the term $(\Gamma_{\eta(s)}^n - \Gamma_s^n)$ as the sum of integrals over the interval $[\eta(s),s]$ and then applying (2.9) and integration by parts we can show that $\sup_{s\in[0,T]}\mathbb{E}[J_s^{n,i,j}] \le Cn^{-1}$. This implies

$$\left|\mathbb{E}\left[\nabla f(Z_t^\theta)E_{3,j}(t)\right]\right| \le Cn^{-2H}\,, \tag{9.7}$$

which completes the proof of (9.1).

Step 2. Now we show the second part of the theorem. From the estimates (4.9), (4.10), (9.5), (9.6) and (9.7) we see that the expression $n\int_0^1 \mathbb{E}\left[\nabla f(Z_t^\theta)Y_t\right]d\theta$ converges to zero as n tends to infinity when Y_t is replaced by $I_{13}(t)$, $I_{4,j}(t)$, $E_{1,j}(t)$ or $E_{3,j}(t)$. Therefore, it suffices to consider $n\int_0^1 \mathbb{E}\left[\nabla f(Z_t^\theta)Y_t\right]d\theta$ when Y_t is replaced by the remaining terms in the decomposition of Y_t.

Consider first the term $E_{2,j}(t)$ and denote

$$G_{s,r,v}^{i,j} = D_v^i D_r^j \left[\nabla f(X_t)\Lambda_t \Gamma_s(\nabla V^j V^i)(X_s)\right].$$

It is clear that

$$\begin{aligned} &n\alpha_H^2 \sum_{i,j=1}^m \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \int_0^t \int_{t_k}^{t_{k+1}\wedge t} \int_0^t \int_{t_k}^s G_{t_k,r,v}^{i,j} \mu(dudv)\mu(dsdr) \\ &\quad \to \frac{\alpha_H^2 T}{2} \sum_{i,j=1}^m \int_0^t \int_0^t \int_0^t G_{s,r,v}^{i,j} |s-v|^{2H-2}|r-s|^{2H-2} dsdvdr, \end{aligned} \tag{9.8}$$

almost surely. Therefore, by the dominated convergence theorem, the expectation of the left-hand side of the above expression converges to the expectation of the right-hand side which is the term (9.2). From Lemma 2.4.1, we have

$$\left\| D_v^i D_r^j \left[\nabla f(Z_t^\theta)\Lambda_t^n F_{t_k}^{n,i,j}\right] - D_v^i D_r^j \left[\nabla f(X_t)\Lambda_t \Gamma_{t_k}(\nabla V^j V^i)(X_{t_k})\right] \right\|_p \le Cn^{1-2\beta}, \tag{9.9}$$

which, together with equation (9.4), implies that $n\sum_{j=1}^m \mathbb{E}\left[\nabla f(Z_t^\theta)E_{2,j}(t)\right]$ converges to the same limit as the expectation of the left-hand side of (9.8).

The results in Step 2 and 3 of the proof of Theorem 2.8.1 imply that the terms $n\mathbb{E}[\nabla f(Z_t^\theta)I_{11}(t)]$, $n\mathbb{E}[\nabla f(Z_t^\theta)I_{12,j}(t)]$ and $n\mathbb{E}[\nabla f(Z_t^\theta)\sum_{j=1}^m I_{2,j}(t)]$ converge to the second, third and fourth term in (9.3), respectively. For example let us consider $n\mathbb{E}[\nabla f(Z_t^\theta)\sum_{j=1}^m I_{2,j}(t)]$.

We have shown in Theorem 2.8.1 that

$$nI_{2,j}(t) \to \frac{T}{2}\Lambda_t \int_0^t \Gamma_s(\nabla V^j b)(X_s)dB_s^j$$

in L^p for any $p \geq 1$. So it follows from the Hölder inequality that

$$\left|\mathbb{E}\left[n\nabla f(Z_t^\theta)I_{2,j}(t) - \nabla f(X_t)\frac{T}{2}\Lambda_t \int_0^t \Gamma_s(\nabla V^j b)(X_s)dB_s^j\right]\right| \to 0$$

as $n \to \infty$. The other two terms can be studied in similar way. This completes the proof of the theorem. □

Remark 2.9.1. *Theorem 2.9.1 may be used to construct a Richard extrapolation scheme with error bound $o(n^{-1})$.*

2.10 Rate of convergence for the Euler scheme

In this section, we apply our approach based on Malliavin calculus developed in Section 4 to study the rate of convergence of the naive Euler scheme defined in (1.2). Our first result is the rate of the strong convergence of the naive Euler scheme. As we will see, the weak rate of convergence and the rate of strong convergence are the same for the naive Euler scheme. We still use X^n to represent the naive Euler scheme (1.2). This will not cause confusion since we will only deal with this scheme in this section.

Theorem 2.10.1. *Let X and X^n be the processes defined in (1.1) and (1.2), respectively. Suppose that $b \in C_b^1(\mathbb{R}^d;\mathbb{R}^d)$ and $V \in C_b^2(\mathbb{R}^d;\mathbb{R}^{d\times m})$. Then for each $p \geq 1$, we have*

$$n^{2H-1} \sup_{t\in[0,T]} \mathbb{E}(|X_t - X_t^n|^p)^{\frac{1}{p}} \leq C.$$

If we assume $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$ and $V \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$, then as n tends to infinity,

$$n^{2H-1}(X_t - X_t^n) \to \frac{T^{2H-1}}{2}\sum_{j=1}^m \int_0^t \Lambda_t \Gamma_s (\nabla V^j V^j)(X_s) ds,$$

where Λ is the solution to the linear equation (4.16) and $\Gamma_t = \Lambda_t^{-1}$, and the convergence holds in L^p for all $p \geq 1$.

Proof: We let $Y_t = X_t - X_t^n$, $t \in [0,T]$. Then as in the proof of Theorem 2.4.1, we can derive the decomposition of Y_t

$$\begin{aligned} Y_t =& \Lambda_t^n \int_0^t \Gamma_s^n b_3(s) \left[b(X_{\eta(s)}^n)(s-\eta(s)) + \sum_{l=1}^m V^l(X_{\eta(s)}^n)(B_s^l - B_{\eta(s)}^l) \right] ds \\ &+ \sum_{j=1}^m \int_0^t \Lambda_t^n \Gamma_s^n b_2^j(s)(s-\eta(s)) dB_s^j + \sum_{i,j=1}^m \int_0^t \Lambda_t^n \Gamma_s^n V_2^{j,i}(s)(B_s^i - B_{\eta(s)}^i) dB_s^j \\ =:& I_1(t) + I_2(t) + I_3(t) + I_4(t), \end{aligned}$$

where Λ^n, Γ^n, $b_2^j(s)$, $V_2^{j,i}(s)$ and $b_3(s)$ are the same terms as those defined in the proof of Theorem 2.4.1 with the scheme X^n replaced by the classical Euler scheme (1.2).

It is clear that $\|I_1(t)\|_p \leq Cn^{-1}$. On the other hand, estimates (11.4) and (11.5) of Lemma 2.11.3 imply that $\|I_2(t)\|_p \leq Cn^{-1}$ and $\|I_3(t)\|_p \leq Cn^{-1}$. Finally, as in the proof of (11.15) in Lemma 2.11.5 we obtain $\|I_4(t)\|_p \leq Cn^{1-2H}$. This completes the proof of the first part of the theorem.

Applying the integration by parts to $I_4(t)$ yields

$$
\begin{aligned}
&\int_0^t \Lambda_t^n \Gamma_s^n V_2^{j,i}(s)(B_s^i - B_{\eta(s)}^i) dB_s^j \\
&= \int_0^t \Lambda_t^n \Gamma_s^n V_2^{j,i}(s)(B_s^i - B_{\eta(s)}^i) \delta B_s^j \\
&\quad + \alpha_H \int_0^t \int_0^t D_r^j \left[\Lambda_t^n \Gamma_s^n V_2^{j,i}(s)\right](B_s^i - B_{\eta(s)}^i)\mu(dsdr) \\
&\quad + \delta_{ij}\alpha_H \int_0^t \int_0^t \Lambda_t^n \Gamma_s^n V_2^{j,i}(s) \mathbf{1}_{[\eta(s),s]}(r)\mu(dsdr) \\
&=: A_n^1(t) + A_n^2(t) + A_n^3(t).
\end{aligned}
$$

From (11.8) we have $\|A_n^1(t)\|_p \le C\gamma_n^{-1}$. Applying (11.5) with F_u replaced by

$$
\int_0^t D_r^j \left[\Lambda_t^n \Gamma_s^n V_2^{j,i}(s)\right] |r-u|^{2H-2} dr
$$

we obtain $\|A_n^2(t)\|_p \le Cn^{-1}$. So it suffices to identify the limit of $n^{2H-1}A_n^3(t)$ in L^p. It follows from Lemma 2.4.1 and Remark 2.4.1 that

$$
\left\| \Lambda_t^n \Gamma_s^n V_2^{j,j}(s) - \Lambda_t \Gamma_s (\nabla V^j V^j)(X_s) \right\|_p \le Cn^{1-2\beta}.
$$

Therefore, $n^{2H-1}A_n^3(t)$ and the quantity

$$
n^{2H-1} \int_0^t \int_0^t \Lambda_t \Gamma_s (\nabla V^j V^j)(X_s) \mathbf{1}_{[\eta(s),s]}(r) |r-s|^{2H-2} dsdr
$$

converges to the same value in L^p. The theorem now follows by noticing that

$$
\begin{aligned}
&n^{2H-1} \int_0^t \int_0^t \Lambda_t \Gamma_s (\nabla V^j V^j)(X_s) \mathbf{1}_{[\eta(s),s]}(r) |r-s|^{2H-2} dsdr \\
&= n^{2H-1} \int_0^t \Lambda_t \Gamma_s (\nabla V^j V^j)(X_s) \frac{(s-\eta(s))^{2H-1}}{2H-1} ds \\
&\to \frac{T^{2H-1}}{2\alpha_H} \int_0^t \Lambda_t \Gamma_s (\nabla V^j V^j)(X_s) ds,
\end{aligned}
$$

in L^p for all $p \geq 1$. □

As a consequence of the above theorem, we can deduce the following result.

Corollary 2.10.1. *Let X and X^n be the processes defined in (1.1) and (1.2), respectively. Suppose that $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $V \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$ and $f \in C_b^2(\mathbb{R}^d)$. Let Λ be defined in (4.16). Then we have the following L^p-convergence as $n \to \infty$ for all $p \geq 1$,*

$$n^{2H-1}\left[f(X_t^n) - f(X_t)\right] \to \frac{T^{2H-1}}{2}\sum_{j=1}^{m}\int_0^t \nabla f(X_t)\Lambda_t\Gamma_s(\nabla V^j V^j)(X_s)ds\,.$$

Proof: We can write

$$n^{2H-1}\left[f(X_t^n) - f(X_t)\right] = n^{2H-1}\left(\int_0^1 \nabla f(Z_t^\theta)d\theta\right)(X_t^n - X_t)\,,$$

where we denote $Z_t^\theta = \theta X_t + (1-\theta)X_t^n$, $t \in [0,T]$. Then the result follows from Theorem 2.10.1, the convergence of X_t^n to X_t and the assumption on f. □

The above corollary implies the following weak approximation result

$$\begin{aligned}&\lim_{n\to\infty} n^{2H-1}\left\{\mathbb{E}\left[f(X_t)\right] - \mathbb{E}[f(X_t^n)]\right\}\\ &= \frac{T^{2H-1}}{2}\sum_{j=1}^{m}\int_0^t \mathbb{E}[\nabla f(X_t)\Lambda_t\Gamma_s(\nabla V^j V^j)(X_s)]ds.\end{aligned}$$

2.11 Appendix

2.11.1 Estimates of a Young integral

In this section, we give an estimate on pathwise integral using fractional calculus.

Lemma 2.11.1. *Let $z = \{z_t, t \in [0,T]\}$ be a Hölder continuous function with index $\beta \in (0,1)$. Suppose that $f : \mathbb{R}^{l+m} \to \mathbb{R}$ is continuously differentiable. We denote by $\nabla_x f$ the l-dimensional vector with coordinates $\frac{\partial f}{\partial x_i}$, $i = 1,\dots,l$, and by $\nabla_y f$ the m-dimensional vector with coordinates $\frac{\partial f}{\partial x_{l+i}}$, $i = 1,\dots,m$. Consider processes $x = \{x_t, t \in [0,T]\}$ and $y = \{y_t, t \in [0,T]\}$ with dimensions l and m, respectively, such that $\|x\|_{0,T,\beta'}$ and $\|y\|_{0,T,\beta',n}$ are finite for each $n \geq 1$, where $\beta' \in (0,1)$ is such that $\beta' + \beta > 1$. Then, we have the following estimates:*

(i) For any $s,t \in [0,T]$ such that $s \leq t$ and $s = \eta(s)$ we have

$$\begin{aligned}\left|\int_s^t f(x_r, y_{\eta(r)}) dz_r\right| &\leq K_1 \sup_{r\in[s,t]} |f(x_r, y_{\eta(r)})| \|z\|_\beta (t-s)^\beta \\ &\quad + K_2 \sup_{r_1,r_2\in[s,t]} |\nabla_x f(x_{r_1}, y_{\eta(r_2)})| \|x\|_{s,t,\beta'} \|z\|_\beta (t-s)^{\beta+\beta'} \\ &\quad + K_3 \sup_{r_1,r_2\in[s,t]} |\nabla_y f(x_{r_1}, y_{r_2})| \|y\|_{s,t,\beta',n} \|z\|_\beta (t-s)^{\beta+\beta'},\end{aligned}$$

where the K_i, $i = 1,2,3$, are constants depending on β and β'.

(ii) If the function f only depends on the first l variables, then the above estimate holds for all $0 \leq s \leq t \leq T$.

Proof: Take α such that $\beta' > \alpha > 1 - \beta$. Let $s,t \in [0,T]$ be such that $s = \eta(s)$ and $s \leq t$. Applying the fractional integration by parts formula in Proposition 2.2.1 we obtain

$$\left|\int_s^t f(x_r, y_{\eta(r)}) dz_r\right| \leq \int_s^t \left|D_{s+}^\alpha f(x_r, y_{\eta(r)})\right| \left|D_{t-}^{1-\alpha}(z_r - z_t)\right| dr. \tag{11.1}$$

By the definition of fractional differentiation in (2.3) and taking into account that $\alpha + \beta - 1 > 0$, we can show that

$$\left| D_{t-}^{1-\alpha}(z_r - z_t) \right| \leq K_0 \|z\|_\beta (t-r)^{\alpha+\beta-1}, \qquad s \leq r \leq t, \tag{11.2}$$

where $K_0 = \frac{\beta}{(\beta+\alpha-1)\Gamma(\alpha)}$. On the other hand, using (2.2) we obtain

$$\begin{aligned}
&\left| D_{s+}^{\alpha} f(x_r, y_{\eta(r)}) \right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \left[\frac{|f(x_r, y_{\eta(r)})|}{(r-s)^\alpha} + \alpha \int_s^r \frac{|f(x_r, y_{\eta(r)}) - f(x_u, y_{\eta(u)})|}{(r-u)^{\alpha+1}} du \right] \\
&\leq \frac{1}{\Gamma(1-\alpha)} \Bigg[\sup_{r \in [s,t]} |f(x_r, y_{\eta(r)})| (r-s)^{-\alpha} \\
&\quad + \alpha \sup_{r_1, r_2 \in [s,t]} |\nabla_x f(x_{r_1}, y_{\eta(r_2)})| \|x\|_{s,t,\beta'} \int_s^r (r-u)^{\beta'-\alpha-1} du \\
&\quad + \alpha \sup_{r_1, r_2 \in [s,t]} |\nabla_y f(x_{r_1}, y_{r_2})| \|y\|_{s,t,\beta',n} \int_s^r \frac{|\eta(r) - \eta(u)|^{\beta'}}{(r-u)^{\alpha+1}} du \Bigg].
\end{aligned} \tag{11.3}$$

The inequalities (11.1), (11.3) and (11.2) together imply

$$\begin{aligned}
&\left|\int_s^t f(x_r, y_{\eta(r)}) dz_r\right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_s^t \left[\sup_{r\in[s,t]} |f(x_r, y_{\eta(r)})| (r-s)^{-\alpha} \right. \\
&\quad + \alpha \sup_{r_1, r_2 \in [s,t]} |\nabla_x f(x_{r_1}, y_{\eta(r_2)})| \|x\|_{s,t,\beta'} \int_s^r (r-u)^{\beta'-\alpha-1} du \\
&\quad \left. + \alpha \sup_{r_1, r_2 \in [s,t]} |\nabla_y f(x_{r_1}, y_{r_2})| \|y\|_{s,t,\beta',n} \int_s^r \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}} du \right] \\
&\qquad \cdot K_0 \|z\|_\beta (t-r)^{\alpha+\beta-1} dr \\
&\leq K_1 \sup_{r\in[s,t]} |f(x_r, y_{\eta(r)})| \|z\|_\beta (t-s)^\beta \\
&\quad + K_2 \sup_{r_1, r_2 \in [s,t]} |\nabla_x f(x_{r_1}, y_{\eta(r_2)})| \|x\|_{s,t,\beta'} \|z\|_\beta (t-s)^{\beta+\beta'} \\
&\quad + K_3 \sup_{r_1, r_2 \in [s,t]} |\nabla_y f(x_{r_1}, y_{r_2})| \|y\|_{s,t,\beta',n} \|z\|_\beta (t-s)^{\beta+\beta'},
\end{aligned}$$

where $K_1 = K_0 \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta+1)}$, $K_2 = K_0 \frac{\alpha\Gamma(\alpha+\beta)\Gamma(\beta'-\alpha+1)}{\Gamma(1-\alpha)\Gamma(\beta+\beta'+1)(\beta'-\alpha)}$, $K_3 = K_0 K_4 \frac{\alpha}{\Gamma(1-\alpha)}$ and K_4 is the constant in Lemma 2.11.2. This completes the proof. □

Lemma 2.11.2. *Let β, β' and α be such that $\beta' > \alpha > 1-\beta$. Then for any $s, t \in [0, T]$ such that $s < t$, $s = \eta(s)$, there exists a constant K_4 depending on α, β and T, such that*

$$\int_s^t (t-r)^{\alpha+\beta-1} \int_s^r \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}} du dr \leq K_4 (t-s)^{\beta+\beta'}.$$

Proof: Without loss of generality, we let $T = 1$. Note that when $\eta(s) = s < t \leq \eta(s) + \frac{1}{n}$, the double integral equals zero. In the following we will assume $t > \eta(s) + \frac{1}{n}$.

We first write

$$
\begin{aligned}
&\int_s^t (t-r)^{\alpha+\beta-1} \int_s^r \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}} du dr \\
&= \int_{\eta(s)+\frac{1}{n}}^t (t-r)^{\alpha+\beta-1} \int_{\eta(s)}^{\eta(r)} \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}} du dr \\
&= \int_{\eta(s)+\frac{1}{n}}^t (t-r)^{\alpha+\beta-1} \left(\int_{\eta(r)-\frac{1}{n}}^{\eta(r)} + \int_{\eta(s)}^{\eta(r)-\frac{1}{n}} \right) \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}} du dr \\
&:= J_1 + J_2.
\end{aligned}
$$

On one hand, notice that in the term J_2 we always have $r-u > \frac{1}{n}$, and thus $\eta(r) - \eta(u) \leq r-u+\frac{1}{n} \leq 2(r-u)$. Therefore,

$$
\begin{aligned}
J_2 &\leq \int_{\eta(s)+\frac{1}{n}}^t (t-r)^{\alpha+\beta-1} \int_{\eta(s)}^{\eta(r)-\frac{1}{n}} \frac{2^{\beta'}(r-u)^{\beta'}}{(r-u)^{\alpha+1}} du dr \\
&\leq K(t-s)^{\beta+\beta'}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
J_1 =& \int_{\eta(s)+\frac{1}{n}}^t (t-r)^{\alpha+\beta-1} \int_{\eta(r)-\frac{1}{n}}^{\eta(r)} \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}} du dr \\
\leq& K n^{-\beta'} (t-s)^{\alpha+\beta-1} \int_{\eta(s)+\frac{1}{n}}^t [\frac{1}{(r-\eta(r))^{\alpha}} - \frac{1}{(r-\eta(r)+\frac{1}{n})^{\alpha}}] dr \\
\leq& K n^{-\beta'} (t-s)^{\alpha+\beta-1} \int_{\eta(s)+\frac{1}{n}}^t \frac{1}{(r-\eta(r))^{\alpha}} dr \\
\leq& K n^{-\beta'} (t-s)^{\alpha+\beta-1} \frac{(\eta(t)+\frac{1}{n}) - (\eta(s)+\frac{1}{n})}{1/n} n^{\alpha-1} \\
\leq& K(t-s)^{\beta+\beta'}.
\end{aligned}
$$

The lemma is now proved. □

2.11.2 Estimates for some special Young and Skorohod integrals

In this section we derive estimates for some specific Young and Skorohod integrals. We fix $n \in \mathbb{N}$ and consider the uniform partition on $[0,T]$.

Lemma 2.11.3. *Let $B = \{B_t, t \in [0,T]\}$ be a one-dimensional fBm with Hurst parameter $H > \frac{1}{2}$. Fix $\nu \geq 0$ and $p \geq \frac{1}{H}$. Let $F = \{F_t, t \in [0,T]\}$ be a stochastic process whose trajectories are Hölder continuous of order $\gamma > 1 - H$ and such that $F_t \in \mathbb{D}^{1,q}$, $t \in [0,T]$, for some $q > p$. For any $\rho > 1$ we set*

$$F_{1,\rho} = \sup_{s,t \in [0,T]} \left(\|F_t\|_\rho \vee \|D_s F_t\|_\rho \right).$$

Then there exists a constant C (independent of F) such that the following inequalities hold for all $0 \leq s < t \leq T$

$$\left\| \int_s^t F_u (u - \eta(u))^\nu dB_u \right\|_p \leq Cn^{-\nu}(t-s)^H F_{1,p}, \tag{11.4}$$

$$\left\| \int_s^t F_u (B_u - B_{\eta(u)}) du \right\|_p \leq Cn^{-1}(t-s)^H F_{1,q}. \tag{11.5}$$

Proof of (11.4): Applying (2.9) we can decompose the Young's integral as the sum of a Skorohod integral plus a complementary term:

$$\int_s^t F_u (u-\eta(u))^\nu dB_u = \int_s^t F_u (u-\eta(u))^\nu \delta B_u + \alpha_H \int_s^t \int_0^T (u-\eta(u))^\nu D_r F_u |r-u|^{2H-2} dr du. \tag{11.6}$$

It follows from (2.11) that the L^p-norm of the first integral of the right-hand side of (11.6) is bounded by $Cn^{-\nu}(t-s)^H F_{1,p}$. On the other hand, from Minkowski's inequality it follows that the L^p-norm of the second integral is less than or equal to $Cn^{-\nu}(t-s)F_{1,p}$. These estimates imply (11.4) because $(t-s) \leq (t-s)^H T^{1-H}$.

Proof of (11.5): If $t-s \leq \frac{1}{n}$, we can write

$$\begin{aligned}
\left\| \int_s^t F_u(B_u - B_{\eta(u)})du \right\|_p &\leq \int_s^t \left\| F_u(B_u - B_{\eta(u)}) \right\|_p du \\
&\leq C \sup_{t\in[0,T]} \|F_t\|_q n^{-H}(t-s) \\
&\leq C \sup_{t\in[0,T]} \|F_t\|_q n^{-1}(t-s)^H,
\end{aligned}$$

where the first inequality follows from Minkowski's inequality and the second one from Hölder's inequality. Suppose that $t-s \geq \frac{1}{n}$. Applying Fubini's theorem for the Young integral we obtain

$$\int_s^t F_u(B_u - B_{\eta(u)})du = \int_{\eta(s)}^t \left(\int_v^{\varepsilon(v)} \mathbf{1}_{[s,t]}(u) F_u du \right) dB_v.$$

Applying (11.4) with $\nu = 0$ we obtain

$$\begin{aligned}
\left\| \int_{\eta(s)}^t \left(\int_v^{\varepsilon(v)} \mathbf{1}_{[s,t]}(u) F_u du \right) dB_v \right\|_p &\leq C(t-\eta(s))^H n^{-1} F_{1,p} \qquad (11.7) \\
&\leq C(t-s)^H n^{-1} F_{1,p}\,.
\end{aligned}$$

This completes the proof of (11.5).

Lemma 2.11.4. *Let $B = \{B_t, t\in[0,T]\}$ be an m-dimensional fBm with Hurst parameter $H > \frac{1}{2}$. Fix $p \geq \frac{1}{H}$. Let $F = \{F_t, t\in[0,T]\}$ be a stochastic process such that $F_t \in \mathbb{D}^{2,q}$, $t\in[0,T]$, for some $q > p$. For any $\rho > 1$ we set*

$$F_{2,\rho} = \sup_{r,s,t\in[0,T]} \left(\|F_t\|_\rho \vee \|D_s F_t\|_\rho \vee \|D_r D_s F_t\|_\rho \right).$$

Set also

$$F_* = \sup_{r,s,t\in[0,T]} \left(|F_t| \vee |D_sF_t| \vee |D_rD_sF_t|\right).$$

Then there exists a constant C (independent of F) such that the following holds for all $0 \le s < t \le T$, $i, j = 1, \dots, m$,

$$\left\| \sum_{k=\lfloor \frac{ns}{T} \rfloor}^{\lfloor \frac{nt}{T} \rfloor} F_{t_k} \int_{t_k \vee s}^{t_{k+1}\wedge t} \int_{t_k}^{u} \delta B_v^i \delta B_u^j \right\|_p \le C\gamma_n^{-1}(t-s)^{\frac{1}{2}} \|F_*\|_q, \tag{11.8}$$

$$\left\| \sum_{k=\lfloor \frac{ns}{T} \rfloor}^{\lfloor \frac{nt}{T} \rfloor} F_{t_k} \int_{t_k \vee s}^{t_{k+1}\wedge t} \int_{t_k}^{u} \delta B_v^i \delta B_u^j \right\|_p \le Cn^{-H}(t-s)^H F_{2,q}. \tag{11.9}$$

Proof: Using (2.8), we can write

$$\sum_{k=\lfloor \frac{ns}{T} \rfloor}^{\lfloor \frac{nt}{T} \rfloor} F_{t_k} \int_{t_k \vee s}^{t_{k+1}\wedge t} \int_{t_k}^{u} \delta B_v^i \delta B_u^j = \int_s^t F_{\eta(u)}(B_u^i - B_{\eta(u)}^i)\delta B_u^j \tag{11.10}$$
$$+\alpha_H \int_s^t \int_0^T D_r^j F_{\eta(u)}(B_u^i - B_{\eta(u)}^i)\mu(drdu).$$

Applying (11.5) to the second integral of the right-hand side of (11.10) with F_u replaced by $\int_0^T D_r^j F_{\eta(u)} |r-u|^{2H-2} dr$ (notice that here we do not need the Hölder continuity of the integrand for the Young integral to be well defined) yields

$$\left\| \int_s^t \int_0^T D_r^j F_{\eta(u)}(B_u^i - B_{\eta(u)}^i)\mu(drdu) \right\|_p \tag{11.11}$$
$$\le Cn^{-1}(t-s)^H F_{2,q} \sup_{u\in[0,T]} \int_0^T |r-u|^{2H-2} dr$$
$$\le Cn^{-1}(t-s)^H F_{2,q}.$$

This implies both the estimates (11.8) and (11.9).

Applying (2.8) to the first summand in the right-hand side of (11.10) yields

$$\begin{aligned}
&\int_s^t F_u(B_u^i - B_{\eta(u)}^i)\delta B_u^j \\
&= \int_s^t \int_{\eta(u)}^u F_u \delta B_v^i \delta B_u^j + \alpha_H \int_s^t \left\{ \int_0^T \int_{\eta(u)}^u D_v^i F_u \mu(drdv) \right\} \delta B_u^j .
\end{aligned} \tag{11.12}$$

Now we apply (2.11) to the second term of the right-hand side of (11.12) and we obtain

$$\begin{aligned}
&\left\| \int_s^t \left\{ \int_0^T \int_{\eta(u)}^u D_v^i F_u \mu(drdv) \right\} \delta B_u^j \right\|_p \\
&\leq CF_{2,p} \left\| \mathbf{1}_{[s,t]}(u) \int_0^T \int_{\eta(u)}^u \mu(drdv) \right\|_{L^{\frac{1}{H}}([0,T])} \\
&\leq CF_{2,p} n^{-1}(t-s)^H.
\end{aligned} \tag{11.13}$$

Again, this inequality implies both the estimates (11.8) and (11.9).

It remains to estimate the term $I_{s,t} := \int_s^t \int_{\eta(u)}^u F_u \delta B_v^i \delta B_u^j$. It follows from (2.11) that

$$\begin{aligned}
\|I_{s,t}\|_p &\leq CF_{2,p} \|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v)\|_{L^{\frac{1}{H}}([0,T]^2)} \\
&\leq CF_{2,p} n^{-H}(t-s)^H,
\end{aligned}$$

which completes the proof of (11.9).

To derive (11.8) we need a more accurate estimate.

Meyer's inequality implies that

$$\begin{aligned}
\|I_{s,t}\|_p &\leq C \Big[\left\| \|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v) F_u\|_{\mathscr{H}^{\otimes 2}} \right\|_p \\
&\quad + \left\| \|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v) D_r F_u\|_{\mathscr{H}^{\otimes 3}} \right\|_p \\
&\quad + \left\| \|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v) D_{r'} D_r F_u\|_{\mathscr{H}^{\otimes 4}} \right\|_p \Big] \\
&\leq C \|F_*\|_p \left\| \mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v) \right\|_{\mathscr{H}^{\otimes 2}} .
\end{aligned}$$

Therefore, to complete the proof, it suffices to show that

$$\begin{aligned}
&\|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v)\|^2_{\mathscr{H}^{\otimes 2}} \\
&= \alpha_H^2 \int_s^t \int_s^t \int_{\eta(u')}^{u'} \int_{\eta(u)}^{u} \mu(dvdv')\mu(dudu') \\
&\leq (t-s)\gamma_n^{-2}.
\end{aligned} \tag{11.14}$$

In the case $t-s \geq \frac{1}{n}$,

$$\begin{aligned}
&\int_s^t \int_s^t \int_{\eta(u')}^{u'} \int_{\eta(u)}^{u} \mu(dvdv'dudu') \\
&\leq \sum_{k,k'=\lfloor \frac{ns}{T} \rfloor}^{\lfloor \frac{nt}{T} \rfloor} \int_{t_{k'}}^{t_{k'+1}} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{u'} \int_{t_k}^{u} \mu(dvdv'dudu') \\
&\leq \sum_{k=\lfloor \frac{ns}{T} \rfloor}^{\lfloor \frac{nt}{T} \rfloor} \sum_{p=1-n}^{n-1} \int_{t_{k+p}}^{t_{k+p+1}} \int_{t_k}^{t_{k+1}} \int_{t_{k+p}}^{u'} \int_{t_k}^{u} \mu(dvdv'dudu') \\
&= n^{-4H} \sum_{k=\lfloor \frac{ns}{T} \rfloor}^{\lfloor \frac{nt}{T} \rfloor} \sum_{p=1-n}^{n-1} Q(p) \\
&\leq C(t-s)\gamma_n^{-2},
\end{aligned}$$

where we recall that $Q(p)$ is defined in Section 2.2.4, and the inequality (11.14) follows.

In the case $t-s \leq \frac{1}{n}$, we have the raw estimate

$$\int_s^t \int_s^t \int_{\eta(u')}^{u'} \int_{\eta(u)}^{u} \mu(drdr'dudu') \leq \frac{1}{n^{2H}} \int_s^t \int_s^t \mu(dudu') = \frac{1}{n^{2H}}(t-s)^{2H},$$

and

$$n^{-2H}(t-s)^{2H} \leq (t-s)\gamma_n^{-2}.$$

So (11.14) is also true for this case. The proof of the lemma is now complete. □

Lemma 2.11.5. *Let $B = \{B_t, t \in [0,T]\}$ be a one-dimensional fBm with Hurst parameter $H > \frac{1}{2}$. Suppose that $F = \{F_t, t \in [0,T\}$, $G = \{G_t, t \in [0,T]\}$ are processes that are Hölder continuous of order $\beta \in (\frac{1}{2}, H)$. Then, there exists a constant C (not depending on F or G) such that for all $0 \le s < t \le T$, $\nu \ge 0$,*

$$\left| \int_s^t F_u (G_u - G_{\eta(u)})(B_u - B_{\eta(u)}) dB_u \right| \le C(\|F\|_\infty + \|F\|_\beta) \|G\|_\beta \|B\|_\beta^2 n^{1-3\beta} (t-s)^\beta \,, \tag{11.15}$$

and

$$\left| \int_s^t F_u (G_u - G_{\eta(u)})(u - \eta(u))^\nu dB_u \right| \le C(\|F\|_\infty + \|F\|_\beta) \|G\|_\beta \|B\|_\beta n^{1-2\beta-\nu} (t-s)^\beta \,. \tag{11.16}$$

Proof of (11.15): We assume first that $s,t \in [t_k, t_{k+1}]$ for some $k = 0, 1, \dots, n-1$. By Lemma 2.11.1(ii),

$$\begin{aligned} &\left| \int_s^t F_u (G_u - G_{\eta(u)})(B_u - B_{\eta(u)}) dB_u \right| \\ &\le K_1 \sup_{u \in [s,t]} |F_u (G_u - G_{t_k})(B_u - B_{t_k})| \|B\|_\beta (t-s)^\beta \\ &\quad + K_2 \sup_{u \in [s,t]} \Big[|F_u (G_u - G_{t_k})| \, \|B\|_\beta^2 (t-s)^{2\beta} \\ &\quad + |F_u (B_u - B_{t_k})| \, \|G\|_\beta \|B\|_\beta (t-s)^{2\beta} \\ &\quad + |(G_u - G_{t_k})(B_u - B_{t_k})| \, \|F\|_\beta \|B\|_\beta (t-s)^{2\beta} \Big] \\ &\le C \kappa_\beta(F,G) n^{-2\beta} (t-s)^\beta, \end{aligned} \tag{11.17}$$

where $\kappa_\beta(F,G) = (\|F\|_\infty + \|F\|_\beta)\|G\|_\beta\|B\|_\beta^2$. In the general case, we can write

$$\begin{aligned}
&\left|\int_s^t F_u(G_u - G_{\eta(u)})(B_u - B_{\eta(u)})dB_u\right| \\
&= \left|\left(\int_s^{\varepsilon(s)} + \sum_{k=\lfloor \frac{ns}{T}\rfloor+1}^{\lfloor \frac{nt}{T}\rfloor} \int_{t_k}^{t_{k+1}} + \int_{\eta(t)}^t\right) F_u(G_u - G_{\eta(u)})(B_u - B_{\eta(u)})dB_u\right| \\
&\leq C\kappa_\beta(F,G)n^{-2\beta}\left[(\varepsilon(s)-s)^\beta + (t-\eta(t))^\beta + \sum_{k=\lfloor \frac{ns}{T}\rfloor+1}^{\lfloor \frac{nt}{T}\rfloor} (T/n)^\beta\right] \\
&\leq C\kappa_\beta(F,G)n^{-2\beta}[(\varepsilon(s)-s)^\beta + (t-\eta(t))^\beta + (\eta(t)-\varepsilon(s))n^{1-\beta}] \\
&\leq C\kappa_\beta(F,G)n^{1-3\beta}(t-s)^\beta,
\end{aligned}$$

where the first inequality follows from (11.17).

Proof of (11.16): This estimate can be proved by following the lines of the proof of (11.15) and noticing the fact that $(u-\eta(u))^\nu$ has finite ν-Hölder seminorm on (t_k, t_{k+1}) for each $k = 1, \ldots, n-1$. □

Chapter 3

Crank-Nicolson method for stochastic differential equtions

3.1 Introduction

We consider the following stochastic differential equation (SDE) on $\mathbb{R}^d$

$$X_t = x + \int_0^t V(X_s) dB_s\,, \quad t \in [0,T]\,, \tag{1.1}$$

where $V = (V_0, V_1, \dots, V_m)$ is a continuous mapping, $B = (B^0, B^1, \dots, B^m)^T$, and $(B^1, \dots, B^m)$ is a m-dimensional fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$. We take $B^0_t = t$ for $t \in [0,T]$ to include the drift term in (1.1). The integral in the right hand side of (1.1) is of Riemann-Stieltjes type. It is well-known (see [24, 34]) that if V is α-Lipschitz in the sense of [44] for $\alpha > \frac{1}{H} - 1$, then there exists a solution for equation (1.1), and if we further assume that $\alpha > \frac{1}{H}$, then this solution is also unique.

In this chapter, we are interested in the Crank-Nicolson (or Trapezoidal) method for (1.1):

$$\begin{aligned} X^n_{t_{k+1}} =& X^n_{t_k} + \frac{1}{2}\left[V(X^n_{t_{k+1}}) + V(X^n_{t_k})\right](B_{t_{k+1}} - B_{t_k}), \\ X^n_0 =& x, \end{aligned} \tag{1.2}$$

for $k = 0, 1, \ldots, n-1$. The Crank-Nicolson method has been considered in [30, 29] for scalar SDE's with the Hurst parameter $H \in (1/3, 1/2)$. It has been shown in [30] that the convergence rate of the Crank-Nicolson scheme in this case is $n^{1/2-3H}$, and in [29] that this convergence rate is exact in the sense that the scaled error process converges weakly to a non-zero limit.

In this chapter, we consider the Crank-Nicolson method for a multi-dimensional SDE with $H > 1/2$. We will consider the following situations.

In the scalar SDE case, that is, assuming that $m = 1$ and the drift term $V_0 \equiv 0$, we will show that the convergence rate of the Crank-Nicolson method is n^{-2H}. This result coincides with the deterministic ordinary differential equation case. By taking $H = 1/2$, we obtain the convergence rate n^{-1}, which coincides that of the Crank-Nicolson method for scalar SDE's driven by Brownian motion (see [30, 29]).

In the multi-dimensional case, due to the appearance of the weighted Lévy area term in the error process, the Crank-Nicolson method has very different properties. Precisely, we will derive the following explicit expression for the error:

$$X_{t_l} - X^n_{t_l} = \sum_{i,j=0}^{m} \sum_{k=0}^{l-1} F^{ij}_{t_k} \left(\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB^i_u dB^j_s - \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB^j_u dB^i_s \right) + R_{t_l}\,, \tag{1.3}$$

where $l = 1, \ldots, n$, $\{0 = t_0 < t_1 < \cdots < t_n\}$ is a partition on $[0, T]$, $F = (F)_{1 \le i,j \le m}$ is a process on $[0, T]$ and R is the remainder term whose L^p-norm is less than a constant times n^{-2H}. Note that in the scalar SDE case, the Lévy area term disappears, and so the error $X_{t_l} - X^n_{t_l}$ has convergence rate n^{-2H}. In the general case, the error is dominated by the Lévy area term. We will show that if $m > 1$, then the Crank-Nicolson method has convergence rate $n^{1/2-2H}$ for $H > 1/2$, and if $m = 1$, then its convergence rate is $n^{-1/2-H}$. By considering the weak convergence of the Lévy area term, we also obtain the asymptotic error distributions for the Crank-Nicolson method. Our main tools are fractional calculus and the fourth moment theorem. Our methods are similar to those in [13].

The chapter is organized as follows. In Section 3.2, we consider the weak convergence of the Lévy area term in (1.3). In Section 3.3 we consider the L^p-convergence and then in Section 3.4 consider the asymptotic error distribution for the Crank-Nicolson method. We prove some auxiliary results in the appendix.

3.2 Asymptotic distribution of some random sums

Define

$$Z_n(t) = \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \left(\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} d\widetilde{B}_u dB_s - \int_{t_k}^{t_{k+1}} \int_{s}^{t_{k+1}} d\widetilde{B}_u dB_s \right) \tag{2.1}$$

for $t \in [0,T]$. Here $B_t, t \geq 0$ is a one-dimensional fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$. In this section, we study the convergence rate and asymptotic distribution of this process in two cases: (i) $\widetilde{B}$ is an independent copy of B; (ii) $\widetilde{B}_t = t$, $t \geq 0$. We will need these results for the convergence of the Crank-Nicolson method.

3.2.1 Case (i)

In this subsection, we assume that $\widetilde{B}$ is an independent copy of B. Denote by μ the measure on $\mathbb{R}^2$ with density $H(2H-1)|s-t|^{2H-2}$. We define, for each $p \in \mathbb{Z}$

$$Q(p) = T^{4H} \int_0^1 \int_p^{p+1} \int_0^t \int_p^s \mu(dvdu)\mu(dsdt), \quad R(p) = T^{4H} \int_0^1 \int_p^{p+1} \int_t^1 \int_p^s \mu(dvdu)\mu(dsdt).$$

In the following, we denote $D = \{\frac{T}{n}k, k = 0, 1, \ldots, n\}$.

Proposition 3.2.1. *Let Z_n be the process defined in (2.1) and $\widetilde{B}$ be an independent copy of B. Then, there exists a constant K depending on H and T such that for $t, s \in D$ we have*

$$n^{4H-1}\mathbb{E}([Z_n(t) - Z_n(s)]^2) \leq K|t-s|. \tag{2.2}$$

Furthermore, the finite dimensional distribution of

$$\left(n^{2H-\frac{1}{2}} Z_n, B, \widetilde{B}\right)$$

converges weakly to that of

$$\left(\sqrt{\frac{2\kappa}{T}} W, B, \widetilde{B}\right)$$

as n tends to infinity, where $W = \{W_t, t \in [0,T]\}$ is a standard Brownian motion independent of $(B, \widetilde{B})$, and

$$\kappa := \sum_{p \in \mathbb{Z}} (Q(p) - R(p)). \tag{2.3}$$

Proof: Step 1. We first derive the limit of the covariance $n^{4H-1}\mathbb{E}(Z_n(t)Z_n(s))$. It is easy to show by integration by parts and then the change of variables and the exchange of integration orders that

$$\begin{aligned}
\mathbb{E}[Z_n(t)^2] &= 2n^{-4H}\sum_{k=0}^{\lfloor\frac{nt}{T}\rfloor}\sum_{k'=0}^{\lfloor\frac{nt}{T}\rfloor}[Q(k'-k)-R(k'-k)] \\
&= 2n^{-4H}\left(\sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}\sum_{k'=p}^{\lfloor\frac{nt}{T}\rfloor} + \sum_{p=-\lfloor\frac{nt}{T}\rfloor}^{-1}\sum_{k'=0}^{\lfloor\frac{nt}{T}\rfloor+p}\right)[Q(p)-R(p)] \\
&:= q_1+q_2\,.
\end{aligned} \tag{2.4}$$

We decompose q_1 as follows,

$$\begin{aligned}
q_1 &= 2n^{-4H}\sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}(\lfloor\frac{nt}{T}\rfloor-p+1)(Q(p)-R(p)) \\
&= 2n^{-4H}\left(\lfloor\frac{nt}{T}\rfloor\sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}(Q(p)-R(p))-\sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}(p-1)(Q(p)-R(p))\right). \\
&:= q_{11}+q_{12}.
\end{aligned}$$

Since $|Q(p)-R(p)|\le Kp^{4H-5}$ for $p>0$, it is easy to verify that the sum

$$\sum_{p=0}^{\infty}(Q(p)-R(p))$$

is convergent, and that

$$\left|\sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}(p-1)(Q(p)-R(p))\right|\le Kn^{4H-3}\vee K.$$

Here $a \vee b$ is the maximum of a and b. Therefore,

$$\begin{aligned}\lim_{n\to\infty} n^{4H-1} q_{11} &= \lim_{n\to\infty} n^{4H-1} 2n^{-4H} \lfloor \frac{nt}{T} \rfloor \sum_{p=0}^{\lfloor \frac{nt}{T} \rfloor} (Q(p)-R(p)) \\ &= 2\frac{t}{T} \sum_{p=0}^{\infty} (Q(p)-R(p))\end{aligned} \tag{2.5}$$

and

$$\lim_{n\to\infty} n^{4H-1} q_{12} = 0. \tag{2.6}$$

In summary of (2.5) and (2.6), we obtain

$$\lim_{n\to\infty} n^{4H-1} q_1 = 2\frac{t}{T} \sum_{p=0}^{\infty} (Q(p)-R(p)). \tag{2.7}$$

We can consider $n^{4H-1} q_1$ similarly to obtain the convergence for q_2,

$$\lim_{n\to\infty} n^{4H-1} q_2 = 2\frac{t}{T} \sum_{p=-\infty}^{-1} (Q(p)-R(p)). \tag{2.8}$$

Applying (2.7) and (2.8) to (2.4) yields

$$\lim_{n\to\infty} n^{4H-1} \mathbb{E}(Z_n(t)^2) = 2\frac{t}{T} \kappa, \tag{2.9}$$

where recall that κ is defined in (2.3).

Note that Z_n as a process on D has stationary increment, that is,

$$\mathbb{E}[|Z_n(t)-Z_n(s)|^2] = \mathbb{E}[Z_n(|t-s|)^2], \tag{2.10}$$

for $s,t \in D$. Therefore, we have

$$\begin{aligned}\mathbb{E}[Z_n(t)Z_n(s)] &= \mathbb{E}\left[Z_n(\eta(t))Z_n(\eta(s))\right] \\ &= \mathbb{E}\left[Z_n(\eta(t))^2 + Z_n(\eta(s))^2 - Z_n(|\eta(t)-\eta(s)|)^2\right] \\ &= \mathbb{E}\left[Z_n(t)^2 + Z_n(s)^2 - Z_n(|\eta(t)-\eta(s)|)^2\right]. \end{aligned} \tag{2.11}$$

By applying (2.9) to the right hand side of (2.11) it is easy to see that

$$\lim_{n\to\infty} n^{4H-1}\mathbb{E}[Z_n(t)Z_n(s)] = 2\frac{(t\wedge s)}{T}\kappa \tag{2.12}$$

for $s,t \in [0,T]$.

Step 2. In this step, we show the inequality (2.2). Take $t \in D : t > 0$. By the definition of q_1 we have

$$\begin{aligned} q_1 \le& 2n^{-4H}\sum_{p=0}^{\frac{nt}{T}}\left(\frac{nt}{T} - p + 1\right)|Q(p) - R(p)| \\ \le& 4n^{-4H}\frac{nt}{T}\sum_{p=0}^{\infty}|Q(p)-R(p)|. \end{aligned}$$

In the same way, we can show that

$$q_2 \le 4n^{-4H}\frac{nt}{T}\sum_{p=-\infty}^{-1}|Q(p)-R(p)|.$$

So by the decomposition in (2.4), we have

$$n^{4H-1}\mathbb{E}(Z_n(t)^2) \le Kt \tag{2.13}$$

for $t \in D : t > 0$, where K is a constant depending on H, T. The inequality (2.2) then follows by applying (2.13) to (2.10).

Step 3. In this step, we prove the weak convergence for the finite dimensional distribution of $(n^{2H-1/2}Z_n, B, \widetilde{B})$. Given $r_1, \dots, r_L \in [0,T]$, $L \in \mathbb{N}$, we need to show that the random vector

$$\Theta_L^n := \left(n^{2H-\frac{1}{2}}(Z_n(r_1), \dots, Z_n(r_L)), B_{r_1}, \dots, B_{r_L}, \widetilde{B}_{r_1}, \dots, \widetilde{B}_{r_L}\right)$$

converges in law to

$$\Theta_L := (\sqrt{\frac{2\kappa}{T}}W(r_1), \dots, \sqrt{\frac{2\kappa}{T}}W(r_L), B_{r_1}, \dots, B_{r_L}, \widetilde{B}_{r_1}, \dots, \widetilde{B}_{r_L})$$

as n tends to infinity. According to [39] (see also Theorem 6.2.3 in [38]), this is true if we can show the weak convergence of each component of Θ_L^n to the correspondent component of Θ_L and the convergence of its covariance matrix to that of Θ_L. The convergence of the covariance of $Z_n(r_i)$ and $Z_n(r_j)$ to that of $\sqrt{\frac{2\kappa}{T}}W(r_i)$ and $\sqrt{\frac{2\kappa}{T}}W(r_j)$ follows from (2.12). By the fourth moment theorem (see [35] and also Theorem 5.2.7 in [38]) and taking into account (2.12), to show the weak convergence of the components of Θ_L^n it suffices to show that the limits of their fourth moments exist and are equal to three times the square of that of their second moments. This will be done in the next step.

Step 4. In this step, we show that the limit of the fourth moment of $n^{2H-1/2}Z_n(t)$ exists and the following identity holds true

$$\lim_{n\to\infty} n^{8H-2}\mathbb{E}\left[Z_n(t)^4\right] = 3\lim_{n\to\infty} n^{8H-2}\left(\mathbb{E}[Z_n(t)^2]\right)^2. \tag{2.14}$$

We denote

$$\beta_{\frac{k}{n}}(s) = \mathbf{1}_{[t_k,t_{k+1}]}(s) \quad \text{and} \quad \gamma_{t_k,s}(u) = \mathbf{1}_{[t_k,s]}(u) - \mathbf{1}_{[s,t_{k+1}]}(u).$$

Applying integration by parts and taking into account the definition of Z_n we obtain

$$\mathbb{E}[Z_n(t)^4] = \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \mathbb{E} \int_0^T \int_0^T \int_0^T \int_0^T \widetilde{D}_{u'} D_{s'} \left[Z_n(t)^3 \right] \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u) \mu(dudu') \mu(dsds'), \tag{2.15}$$

where D and $\widetilde{D}$ are the differential operator associated with B and $\widetilde{B}$, respectively. We expand the second derivative $\widetilde{D}_{u'} D_{s'} \left[Z_n(t)^3 \right]$ as follows:

$$\widetilde{D}_{u'} D_{s'} \left[Z_n(t)^3 \right] = 3 Z_n(t)^2 \widetilde{D}_{u'} D_{s'} Z_n(t) + 6 Z_n(t) \widetilde{D}_{u'} Z_n(t) D_{s'} Z_n(t).$$

Substituting the above identity into (2.15) we obtain

$$\mathbb{E}[Z_n(t)^4] := d_1 + d_2 \,,$$

where

$$d_1 = 3 \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \mathbb{E}[Z_n(t)^2] \int_0^T \int_0^T \int_0^T \int_0^T \widetilde{D}_{u'} D_{s'} Z_n(t) \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u) \mu(dudu') \mu(dsds'),$$

and

$$d_2 = 6 \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \mathbb{E} \int_0^T \int_0^T \int_0^T \int_0^T Z_n(t) \widetilde{D}_{u'} Z_n(t) D_{s'} Z_n(t) \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u) \mu(dudu') \mu(dsds'). \tag{2.16}$$

It is easy to see that

$$d_1 = 3\left(\mathbb{E}[Z_n(t)^2]\right)^2. \tag{2.17}$$

On the other hand, it can be shown that (see Section 3.5.1)

$$\lim_{n\to\infty} n^{8H-2} d_2 = 0. \tag{2.18}$$

The identity (2.17) and the convergence (2.18) together implies the convergence in (2.14). This completes the proof of the proposition. □

3.2.2 Case (ii)

In this subsection, we consider the process Z_n defined in (2.1) such that $\widetilde{B}_t = t, t \in [0,T]$. We will denote $z_n := Z_n$ in the following proposition to distinguish it from Z_n in the previous section. For each $p \in \mathbb{Z}$, we define

$$\widetilde{Q}(p) = T^{2H+2} \int_0^1 \int_p^{p+1} \int_0^t \int_p^s dv du \mu(dsdt), \quad \widetilde{R}(p) = T^{2H+2} \int_0^1 \int_p^{p+1} \int_t^1 \int_p^s dv du \mu(dsdt),$$

where recall that μ is the measure on $\mathbb{R}^2$ with density $H(2H-1)|s-t|^{2H-2}$.

Proposition 3.2.2. *Let z_n be the process defined in (2.1) with $\widetilde{B}_t = t$, $t \in [0,T]$. Then, there exists a constant K depending on H and T such that for $t, s \in D$ we have*

$$n^{2H+1}\mathbb{E}([z_n(t) - z_n(s)]^2) \leq K|t-s|. \tag{2.19}$$

Furthermore, the finite dimensional distribution of

$$\left(n^{H+\frac{1}{2}}z_n, B\right)$$

converges weakly to that of

$$\left(\sqrt{\frac{2\rho}{T}}W, B\right),$$

as n tends to infinity, where $W = \{W_t, t \in [0,T]\}$ is a standard Brownian motion independent of B, and

$$\rho := \sum_{p\in\mathbb{Z}} (\widetilde{Q}(p) - \widetilde{R}(p)).$$

Proof: *Step 1.* We first calculate the second moment of $z_n(t)$. By the integration by parts we have

$$\begin{aligned}\mathbb{E}[z_n(t)^2] = \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \sum_{k'=0}^{\lfloor \frac{nt}{T} \rfloor} \Bigg\{ &\int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} (s-t_k)(s'-t_{k'})\mu(ds'ds) \\ &+ \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} (t_{k+1}-s)(t_{k'+1}-s')\mu(ds'ds) \\ &- \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} (s-t_k)(t_{k'+1}-s')\mu(ds'ds) \\ &- \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} (t_{k+1}-s)(s'-t_{k'})\mu(ds'ds) \Bigg\}. \end{aligned} \tag{2.20}$$

By the change of variables for the first term in the right hand side of the above equation it is easy to show that

$$\int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{t_{k'+1}} (s-t_k)(s'-t_{k'})\mu(ds'ds) = n^{-2H-2}\widetilde{Q}(k'-k).$$

Similarly, for the third term we have

$$\int_{t_k}^{t_{k+1}}\int_{t_{k'}}^{t_{k'+1}}(t_{k+1}-s)(s'-t_{k'})\mu(ds'ds)=n^{-2H-2}\widetilde{R}(k'-k),$$

and for the fourth term, we have

$$\begin{aligned}&\sum_{k=0}^{\lfloor\frac{nt}{T}\rfloor}\sum_{k'=0}^{\lfloor\frac{nt}{T}\rfloor}\int_{t_k}^{t_{k+1}}\int_{t_{k'}}^{t_{k'+1}}(s-t_k)(t_{k'+1}-s')\mu(ds'ds)\\ =&n^{-2H-2}\sum_{k=0}^{\lfloor\frac{nt}{T}\rfloor}\sum_{k'=0}^{\lfloor\frac{nt}{T}\rfloor}\widetilde{R}(k-k')\\ =&n^{-2H-2}\sum_{k=0}^{\lfloor\frac{nt}{T}\rfloor}\sum_{k'=0}^{\lfloor\frac{nt}{T}\rfloor}\widetilde{R}(k'-k),\end{aligned}$$

where the last equation follows by exchanging the notation of k and k' in the summation.

For the second term we have

$$\begin{aligned}&\int_{t_k}^{t_{k+1}}\int_{t_{k'}}^{t_{k'+1}}(t_{k+1}-s)(t_{k'+1}-s')\mu(ds'ds)\\ =&(\frac{T}{n})^{2H+2}\int_0^1\int_{k'-k}^{k'-k+1}(1-s)(1+k'-k-s')\mu(ds'ds)\\ =&(\frac{T}{n})^{2H+2}\int_0^1\int_{-(k'-k)}^{1-(k'-k)}s(s'+k'-k)\mu(ds'ds)\\ =&(\frac{T}{n})^{2H+2}\int_{k'-k}^{k'-k+1}\int_0^1(s-(k'-k))s'\mu(ds'ds)\\ =&n^{-2H-2}\widetilde{Q}(k'-k).\end{aligned}$$

In summary, we obtain

$$\begin{aligned}\mathbb{E}[z_n(t)^2] &= n^{-2H-2}\sum_{k=0}^{\lfloor \frac{nt}{T}\rfloor}\sum_{k'=0}^{\lfloor \frac{nt}{T}\rfloor}\Big(\widetilde{Q}(k'-k)+\widetilde{Q}(k'-k)-\widetilde{R}(k'-k)-\widetilde{R}(k'-k)\Big)\\ &= 2n^{-2H-2}\left(\sum_{p=0}^{\lfloor \frac{nt}{T}\rfloor}\sum_{k=0}^{\lfloor \frac{nt}{T}\rfloor-p}+\sum_{p=-\lfloor \frac{nt}{T}\rfloor}^{-1}\sum_{k=-p}^{\lfloor \frac{nt}{T}\rfloor}\right)(\widetilde{Q}(p)-\widetilde{R}(p)) \qquad (2.21)\\ &:= \widetilde{q}_1+\widetilde{q}_2.\end{aligned}$$

Step 2. In this step, we show the inequality (2.19). It is easy to see that the sum

$$\sum_{p\in\mathbb{Z}}|\widetilde{Q}(p)-\widetilde{R}(p)|$$

is convergent, so we have the estimate

$$\tilde{q}_1 \leq 2n^{-1-2H}\frac{t}{T}\sum_{p=1}^{\infty}|\widetilde{Q}(p)-\widetilde{R}(p)|$$

and

$$\tilde{q}_2 \leq 2n^{-1-2H}\frac{t}{T}\sum_{p=-\infty}^{0}|\widetilde{Q}(p)-\widetilde{R}(p)|.$$

These two estimates together implies that

$$n^{2H+1}\mathbb{E}(z_n(t)^2) \leq Kt. \qquad (2.22)$$

On the other hand, it is clear that z_n has stationary increment as a process on D, that is,

$$\mathbb{E}(|z_n(t)-z_n(s)|^2) = \mathbb{E}(z_n(t-s)^2) \qquad (2.23)$$

for $s,t \in D$. The inequality (2.19) then follows by applying the inequality (2.22) to (2.23).

Step 3. In this step, we show the weak convergence for the finite dimensional distribution of $(n^{2H-1/2}z_n, B)$. We write

$$\begin{aligned}
\tilde{q}_1 =& 2n^{-2H-2}\sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}(\lfloor\frac{nt}{T}\rfloor - p + 1)(\widetilde{Q}(p) - \widetilde{R}(p)) \\
=& 2n^{-2H-2}\left(\lfloor\frac{nt}{T}\rfloor\sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}(\widetilde{Q}(p) - \widetilde{R}(p)) - \sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}(p-1)(\widetilde{Q}(p) - \widetilde{R}(p))\right) \\
:=& \widetilde{q}_{11} + \widetilde{q}_{12}.
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}
\lim_{n\to\infty} n^{2H+1}\widetilde{q}_{11} =& \lim_{n\to\infty} 2n^{2H+1}n^{-2H-2}\lfloor\frac{nt}{T}\rfloor\sum_{p=0}^{\lfloor\frac{nt}{T}\rfloor}(\widetilde{Q}(p) - \widetilde{R}(p)) \\
=& 2\frac{t}{T}\sum_{p=0}^{\infty}(\widetilde{Q}(p) - \widetilde{R}(p)).
\end{aligned}$$

On the other hand, since $\sum_{p=0}^{\lfloor\frac{ns}{T}\rfloor}(p-1)(\widetilde{Q}(p) - \widetilde{R}(p)) \sim O(n^{2H-1})$ as n tends to infinity, we have

$$\lim_{n\to\infty} n^{2H+1}\widetilde{q}_{12} = 0.$$

In summary, we obtain

$$\lim_{n\to\infty} n^{2H+1}\widetilde{q}_1 = 2\frac{t}{T}\sum_{p=0}^{\infty}(\widetilde{Q}(p) - \widetilde{R}(p)). \tag{2.24}$$

Similarly, we can show that

$$\lim_{n\to\infty} n^{2H+1}\widetilde{q}_2 = 2\frac{t}{T}\sum_{p=-\infty}^{-1}(\widetilde{Q}(p)-\widetilde{R}(p)). \tag{2.25}$$

Substituting (2.24) and (2.25) into (2.21), we obtain

$$\lim_{n\to\infty} n^{2H+1}\mathbb{E}(z_n(t)^2) = 2\rho\frac{t}{T}. \tag{2.26}$$

The fact that z_n has stationary increments implies that

$$\begin{aligned}
\lim_{n\to\infty} n^{2H+1}\mathbb{E}(z_n(t)z_n(s)) &= \lim_{n\to\infty} n^{2H+1}\mathbb{E}(z_n(\eta(t))z_n(\eta(s))) \\
&= \lim_{n\to\infty} n^{2H+1}\mathbb{E}(z_n(\eta(t))^2+z_n(\eta(s))^2-z_n(\eta(t)-\eta(s))^2) \\
&= 2\rho\frac{(t\wedge s)}{T}.
\end{aligned}$$

Since the process $z_n(t)$, $t\in[0,T]$ is Gaussian, the above convergence of the covariance of z_n implies that the finite dimensional distribution of $(n^{H+\frac{1}{2}}z_n(t), B_t)$, $t\in[0,T]$, converges weakly to that of $(\sqrt{\frac{2\rho}{T}}W_t, B_t)$, $t\in[0,T]$, where W is a standard Brownian motion independent of B. □

3.3 The strong convergence

We consider the following interpolation of the Crank-Nicholson method:

$$\begin{aligned}
X_t^n &= X_{t_k}^n + \frac{1}{2}\left[V(X_{t_k}^n)+V(X_{t_{k+1}}^n)\right](B_t-B_{t_k}), \quad t\in[t_k,t_{k+1}], \\
X_0^n &= x,
\end{aligned} \tag{3.1}$$

for $k = 0, \dots, n-1$. Here $B_t = (t, B_t^1, \dots, B_t^m)$, and $(B^j, j = 1, \dots, m)$ is a standard m-dimensional fBm with Hurst parameter $H > 1/2$. We will consider the convergence of the Crank-Nicolson method in three cases: (1) $m > 1$; (2) $m = 1$; (3) $m = 1$ and $V_0 \equiv 0$, in other words, there is no drift term in the equation (1.1). We define the function γ_n in these three cases:

$$\gamma_n = \begin{cases} n^{2H-\frac{1}{2}}, & m > 1, \\ n^{H+\frac{1}{2}}, & m = 1, \\ n^{2H}, & m = 1 \text{ and } V_0 \equiv 0. \end{cases}$$

We have the following strong convergence result. Recall that $D = \{t_k = \frac{T}{n}k : k = 0, 1, \dots, n\}$, and we denote by K the constants independent of n.

Theorem 3.3.1. *Let X and X^n be the solution of the equation (1.1) and (3.1), respectively. Assume that $V \in C_b^4$. Then we have*

$$\sup_{t\in[0,T]} (\mathbb{E}|X_t - X_t^n|^p)^{1/p} \le K/\gamma_n. \tag{3.2}$$

Furthermore, the following tightness inequality

$$\mathbb{E}\left(|(X_t - X_t^n) - (X_s - X_s^n)|^p\right)^{1/p} \le \begin{cases} Kn^{-2H+\frac{1}{2}}(t-s)^{\frac{1}{2}}, & m > 1, \\ Kn^{-H-\frac{1}{2}}(t-s)^{\frac{1}{2}}, & m = 1, \end{cases} \tag{3.3}$$

holds true for $s, t \in D$.

Proof: *Step 1.* In this step, we derive a decomposition for the error function $Y_t := X_t - X_t^n$, $t \in [0,T]$. By the definition of X and X^n,

$$
\begin{aligned}
Y_t &= \int_0^t [V(X_s) - V(X_s^n)]\, dB_s + \frac{1}{2}\int_0^t \left[V(X_s^n) - V(X_{\eta(s)}^n)\right] dB_s \\
&\quad + \frac{1}{2}\int_0^t \left[V(X_s^n) - V(X_{\varepsilon(s)}^n)\right] dB_s \\
&:= \sum_{j=0}^m \sum_{i=1}^d \int_0^t V_{ji}(s) Y_s^i dB_s^j + \frac{1}{2} J_1(t) + \frac{1}{2} J_2(t)\,, \qquad (3.4)
\end{aligned}
$$

where we denote

$$
V_{ji}(s) = \int_0^1 \partial_i V_j(\theta X_s + (1-\theta) X_s^n) d\theta .
$$

Here ∂_i denotes the partial differential operator with respect to the ith variable, that is, $\partial_i f(x) = \frac{\partial f}{\partial x_i}(x)$ for $f \in C^1$.

By the chain rule we have

$$
\begin{aligned}
&V(X_s^n) - V(X_{\eta(s)}^n) \\
&= \sum_{i=1}^d \partial_i V(X_{\eta(s)}^n) X_{\eta(s),s}^{n,i} + \sum_{i,i'=1}^d \int_{\eta(s)}^s \int_{\eta(s)}^u \partial_{i'} \partial_i V(X_v^n) dX_v^{n,i'} dX_u^{n,i} .
\end{aligned}
$$

So for $J_1(t)$ we have the decomposition

$$
J_1(t) = R_0(t) + R_1(t),
$$

with

$$
R_1(t) = \int_0^t \left[\sum_{i,i'=1}^d \int_{\eta(s)}^s \int_{\eta(s)}^u \partial_{i'} \partial_i V(X_v^n) dX_v^{n,i'} dX_u^{n,i} \right] dB_s\,, \qquad (3.5)
$$

and

$$
\begin{aligned}
R_0(t) &= \int_0^t \left[\sum_{i=1}^{d} \partial_i V(X^n_{\eta(s)}) X^{n,i}_{\eta(s),s}\right] dB_s \\
&= \frac{1}{2}\sum_{j,j'=0}^{m} \int_0^t \partial V_j(X^n_{\eta(s)}) \left[V_{j'}(X^n_{\varepsilon(s)}) + V_{j'}(X^n_{\eta(s)})\right] \int_{\eta(s)}^{s} dB^{j'}_u dB^j_s \,. \qquad (3.6)
\end{aligned}
$$

Here $\partial = (\partial_1, \dots, \partial_d)$.

Similarly, for $J_2(t)$ we have the decomposition

$$
J_2(t) = -\widetilde{R}_0(t) + \widetilde{R}_1(t),
$$

with

$$
\begin{aligned}
\widetilde{R}_1(t) &= \int_0^t \left[\sum_{i,i'=1}^{d} \int_s^{\varepsilon(s)} \int_u^{\varepsilon(s)} \partial_{i'}\partial_i V(X^n_v) dX^{n,i'}_v dX^{n,i}_u\right] dB_s \,, \qquad (3.7) \\
\widetilde{R}_0(t) &= \frac{1}{2}\sum_{i=1}^{d} \int_0^t \partial_i V(X^n_{\varepsilon(s)}) \left[\left(V^i(X^n_{\varepsilon(s)}) + V^i(X^n_{\eta(s)})\right) B_{s,\varepsilon(s)}\right] dB_s \,.
\end{aligned}
$$

For $t \in D$, set

$$
\begin{aligned}
I_1(t) &= \sum_{j,j'=0}^{m} \sum_{k=0}^{nt/T-1} (\partial V_j V_{j'})(X^n_{t_k}) \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB^{j'}_u dB^j_s, \qquad (3.8) \\
I_2(t) &= \sum_{j,j'=0}^{m} \sum_{k=0}^{nt/T-1} (\partial V_j V_{j'})(X^n_{t_k}) \int_{t_k}^{t_{k+1}} \int_{s}^{t_{k+1}} dB^{j'}_u dB^j_s \,,
\end{aligned}
$$

and consider the following decomposition of $J_1 + J_2$:

$$
\begin{aligned}
J_1(t) + J_2(t) &= (I_1(t) - I_2(t)) + (R_0(t) - I_1(t)) + \left(I_2(t) - \widetilde{R}_0(t)\right) + R_1(t) + \widetilde{R}_1(t) \\
&:= E_1(t) + E_2(t) + E_3(t) + E_4(t) + E_5(t) \,. \qquad (3.9)
\end{aligned}
$$

Denote

$$\zeta_{s,t}^{ij} = \int_s^t \int_s^u dB_v^i dB_u^j - \int_s^t \int_u^t dB_v^i dB_u^j,$$

for $0 \le s \le t \le T$. Noticing that $\zeta^{ij} = 0$ if $i = j$. Therefore, for $t \in D$ we have

$$E_1(t) = \sum_{j \neq j'} \sum_{k=0}^{nt/T-1} \left(\partial V_j V_{j'}\right)(X_{t_k}^n)\zeta_{t_k,t_{k+1}}^{j'j}.$$

Meanwhile, by exchanging the order of the integrals in $\zeta_{s,t}^{ij}$ it is easy to see that $\zeta_{s,t}^{ij} = -\zeta_{s,t}^{ji}$, so the above expression becomes

$$E_1(t) = \sum_{j'<j} \sum_{k=0}^{nt/T-1} \phi_{jj'}(X_{t_k}^n)\zeta_{t_k,t_{k+1}}^{j'j}, \tag{3.10}$$

where

$$\phi_{jj'} = \partial V_j V_{j'} - \partial V_{j'} V_j.$$

Step 2. It follows from Proposition 5.4 in [14] that there exists a constant K such that

$$\|X^n\|_\infty \vee \|X^n\|_\beta \le K + K\|B\|_\beta^{1/\beta}. \tag{3.11}$$

Here recall that $a \vee b = \max\{a,b\}$. Furthermore, there exist constants K_0 and K_0' independent of n such that for $0 \le s < t \le T$, $(t-s)^\beta \|B\|_\beta \le K_0$, we have

$$\|X^n\|_{s,t,\beta} \le K_0'(\|B\|_\beta + 1). \tag{3.12}$$

Step 3. In this step, we derive the following L^p-estimates for E_e, $e = 1, \dots, 5$:

$$\|E_e(t) - E_e(s)\|_p \le Kn^{-2H}(t-s)^{1/2} \quad s,t \in D : s \le t. \tag{3.13}$$

We first consider E_2. Take $t \in D$. From (3.6) and (3.8) we have

$$E_2(t) = \frac{1}{4} \sum_{j,j'=0}^{m} \sum_{k=0}^{nt/T-1} \partial V_j(X^n_{t_k}) \int_{t_k}^{t_{k+1}} \partial V_{j'}(X^n_v) \left[V(X^n_{t_{k+1}}) + V(X^n_{t_k}) \right] dB_v \\ \cdot \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB^{j'}_u dB^j_s. \tag{3.14}$$

Applying Lemma 3.5.2 to (3.14) and taking into account the estimate (3.11) of X^n we obtain the estimate (3.13) for $e = 2$. Here $\|\cdot\|_p$ denotes L_p-norm. The L^p-estimate (3.13) still holds true for the case when $e = 3, 4, 5$. The proof is based on Lemma 3.5.2 and is similar to the case $e = 2$.

We turn to $E_1(t)$. For $t \in D$, we denote

$$E_1(t) = \sum_{0 \ne j' < j} \sum_{k=0}^{nt/T-1} \phi_{jj'}(X^n_{t_k}) \zeta^{j'j}_{t_k,t_{k+1}} + \sum_{0 = j' < j} \sum_{k=0}^{nt/T-1} \phi_{jj'}(X^n_{t_k}) \zeta^{j'j}_{t_k,t_{k+1}} \\ := E_{11}(t) + E_{12}(t).$$

Take

$$g_n(t) = n^{2H-1/2} \sum_{0 \ne j' < j} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \zeta^{j'j}_{t_k,t_{k+1}}.$$

It follows from (2.2) in Proposition 3.2.1 that g_n satisfies the assumptions in Lemma 3.5.4. By taking $f = \phi_{jj'}(X^n_\cdot)$ and applying Lemma 3.5.4 to E_{11} we obtain the estimate

$$\|E_{11}(t) - E_{11}(s)\|_p \le Kn^{-2H+\frac{1}{2}}(t-s)^{\frac{1}{2}}, \quad s,t \in D. \tag{3.15}$$

On the other hand, it follows from (2.19) in Proposition 3.2.2 that

$$\tilde{g}_n(t) = n^{1/2+H} \sum_{0=j'<j} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \zeta_{t_k,t_{k+1}}^{j'j}$$

satisfies the conditions in Lemma 3.5.4, So by Lemma 3.5.4 we have the estimate

$$\|E_{12}(t) - E_{12}(s)\|_p \leq K n^{-H-\frac{1}{2}} (t-s)^{\frac{1}{2}}, \quad s,t \in D. \tag{3.16}$$

In summary of the estimates (3.13) for $e = 2,3,4,5$, the estimates (3.15) and (3.16), and taking into account the fact that $E_{11} = 0$ when $m = 1$ and $E_{11} = E_{12} = 0$ when $m = 1$ and $V_0 \equiv 0$, we obtain that

$$\sum_{e=1}^{5} \|E_e(t) - E_e(s)\|_p \leq K(t-s)^{\frac{1}{2}}/\gamma_n, \quad s,t \in D. \tag{3.17}$$

Step 4. In this step, we define the fundamental solutions of X^n and X, and derive their estimates. Let $\Lambda^n = \left(\Lambda_{i'}^{n,i}\right)_{1\leq i,i'\leq d}$ be the solution of the equation:

$$\Lambda_{i'}^{n,i}(t) = \delta_{i'}^{i} + \sum_{j=0}^{m} \sum_{i''=1}^{d} \int_0^t V_{ji''}^{i}(s) \Lambda_{i'}^{n,i''}(s) dB_s^j, \qquad i,i' = 1,\dots,d, \;\; t \in [0,T]. \tag{3.18}$$

Here $\delta_{i'}^{i}$ is the Kronecker function, that is, $\delta_{i'}^{i} = 1$ when $i = i'$ and $\delta_{i'}^{i} = 0$ otherwise. The $d \times d$ matrix $\Lambda^n(t)$ is invertible. We denote its inverse by $\Gamma^n(t)$. It is easy to verify that Γ^n satisfies the equation

$$\Gamma_{i'}^{n,i}(t) = \delta_{i'}^{i} - \sum_{j=0}^{m} \sum_{i''=1}^{d} \int_0^t \Gamma_{i''}^{n,i}(s) V_{ji'}^{i''}(s) dB_s^j, \qquad i,i' = 1,\dots,d, \; t \in [0,T].$$

By the product rule of Young integral it is easy to verify that

$$Y_t = \frac{1}{2}\Lambda_t^n \sum_{i=1}^{2} \int_0^t \Gamma_s^n dJ_i(s), \quad t \in [0,T]. \tag{3.19}$$

Applying Lemma 3.2 in [13] and taking into account (3.12), we obtain the estimate

$$\|\Lambda^n\|_\infty \vee \|\Lambda^n\|_\beta \vee \|\Gamma^n\|_\infty \vee \|\Gamma^n\|_\beta \leq K e^{K\|B\|_\beta^{1/\beta}}. \tag{3.20}$$

Let $\Lambda = \left(\Lambda_{i'}^i\right)_{1\leq i,i'\leq d}$ be the solution of the following equation,

$$\Lambda_{i'}^i(t) = \delta_{i'}^i + \sum_{j=0}^{m} \sum_{i''=1}^{d} \int_0^t \partial_{i''} V_j^i(X_s) \Lambda_{i'}^{i''}(s) dB_s^j, \tag{3.21}$$

for $t \in [0,T]$, $i,i' = 1,\dots,d$, and denote by $\Gamma(t)$ the inverse of $\Lambda(t)$. It follows from Lemma 3.1 in [13] that the estimate (3.20) still holds true if we replace Λ^n and Γ^n by Λ and Γ.

Step 5. In this step, we consider

$$\sum_{i=1}^{2} \int_s^t \Gamma_u^n dJ_i(u) = \sum_{e=1}^{5} \int_s^t \Gamma_{\eta(u)}^n dE_e(u) + \sum_{i=1}^{2} \int_s^t \int_{\eta(u)}^u d\Gamma_v^n dJ_i(u) \tag{3.22}$$

for $s,t \in D$. Applying Lemma 3.5.4 to the first quantity in the right hand side of the above and taking into account the estimate (3.17) we obtain

$$\left\| \sum_{e=1}^{5} \int_s^t \Gamma_{\eta(u)}^n dE_e(u) \right\|_p \leq K(t-s)^{\frac{1}{2}}/\gamma_n \tag{3.23}$$

for $s,t \in D$.

We turn to the second term in (3.22). By the definition of Γ^n and J_1 we have

$$\int_{t_k}^{t_{k+1}} \int_{t_k}^{u} d\Gamma_v^n dJ_1(u) = \frac{1}{2} \sum_{j\,j'=0}^{m} \sum_{i,i'=1}^{d} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \left(-\Gamma_{i'}^n(v) V_{j,i}^{i'}(v) \right) dB_v^j \int_{t_k}^{u} \partial V_{j'}^{i}(X_r^n) dX_r^n dB_u^{j'}.$$

So by Lemma 3.5.2 we have the estimate

$$\left\| \int_s^t \int_{\eta(u)}^{u} d\Gamma_v^n dJ_i(u) \right\|_p \le K n^{-2H} (t-s)^{1/2} \tag{3.24}$$

for $i = 1$. This estimate still holds true in the case $i = 2$, and the proof is similar. In summary of (3.23) and (3.24) for $i = 1, 2$, we obtain the estimate

$$\left\| \sum_{i=1}^{2} \int_s^t \Gamma_u^n dJ_i(u) \right\|_p \le K (t-s)^{\frac{1}{2}} / \gamma_n \tag{3.25}$$

for $s, t \in D$.

It is easy to see that

$$\left\| \int_{t_k}^{t} \Gamma_u^n dJ_e(u) \right\|_p \le K n^{-2H}, \quad t \in [t_k, t_{k+1}].$$

Combining this estimate with (3.25) we obtain the inequality

$$\sup_{t \in [0,T]} \left\| \sum_{i=1}^{2} \int_0^t \Gamma_u^n dJ_i(u) \right\|_p \le K / \gamma_n. \tag{3.26}$$

Step 5. The tightness inequality (3.3) follows by applying the estimate in (3.25) to (3.19). The strong convergence result (3.2) follows by applying the estimate (3.26) to (3.19). This completes the proof of Theorem 3.3.1. □

We end the section with the following lemma.

Lemma 3.3.1. *Let the assumptions be as in Theorem 3.3.1. In the case $m > 1$, we have the estimate*

$$\sup_{s,t\in D}\left\|\sum_{i=1}^{2}\int_s^t \Gamma_u^n dJ_i(u) - \int_s^t \Gamma_{\eta(u)}^n dE_{11}(u)\right\|_p \leq Kn^{-1/2-H}. \tag{3.27}$$

In the case $m = 1$, we have the estimate

$$\sup_{s,t\in D}\left\|\sum_{i=1}^{2}\int_s^t \Gamma_u^n dJ_i(u) - \int_s^t \Gamma_{\eta(u)}^n dE_{12}(u)\right\|_p \leq Kn^{-2H}. \tag{3.28}$$

In the case $m = 1$ and $V_0 \equiv 0$, we have the estimate

$$\sup_{t\in[0,T]}\left\|\sum_{i=1}^{2}\int_0^t \Gamma_u^n dJ_i(u) - \sum_{e=2}^{5}\int_0^{\eta(t)} \Gamma_{\eta(u)}^n dE_e(u)\right\|_p \leq n^{1-4\beta}. \tag{3.29}$$

Proof: By subtracting $\int_s^t \Gamma_{\eta(u)}^n dE_{11}(u)$ from both sides of (3.22) we obtain

$$\begin{aligned}&\sum_{i=1}^{2}\int_s^t \Gamma_u^n dJ_i(u) - \int_s^t \Gamma_{\eta(u)}^n dE_{11}(u)\\ &= \int_s^t \Gamma_{\eta(u)}^n dE_{12}(u) + \sum_{e=2}^{5}\int_s^t \Gamma_{\eta(u)}^n dE_e(u) + \sum_{i=1}^{2}\int_s^t\int_{\eta(u)}^u d\Gamma_v^n dJ_i(u).\end{aligned} \tag{3.30}$$

Similar to the estimate in (3.23), we can show that the first and second terms in the right hand side of (3.30) are bounded by $Kn^{1/2+H}$ and Kn^{-2H}, respectively. On the other hand, we have shown in (3.24) that the third term is bounded by Kn^{-2H}. In summary, we obtain the estimate (3.27).

The estimate (3.28) can be shown similarly. See Section 3.5.4 for the proof of the estimate (3.29). □

3.4 Asymptotic error distribution

In this section, we consider the asymptotic error distribution of the Crank-Nicolson method. We first state the following lemma. As before, we denote by K the constants independent of n.

Lemma 3.4.1. *Let Λ^n and Λ be the solutions of equations (3.18) and (3.21), respectively, and Γ^n and Γ be their inverses. Then we have*

$$\|\Lambda^n - \Lambda\|_{\beta,p} + \|\Gamma^n - \Gamma\|_{\beta,p} \leq Kn^{1-2\beta}. \tag{4.1}$$

Proof: See Section 3.5.5. □

Define the process $\{W_t, t \in [0,T]\}$ such that in the case $m > 1$, $W = (W^{j'j})_{1 \leq j' < j \leq m}$ is a standard $\frac{m(m-1)}{2}$-dimensional Brownian motion, while in the case $m = 1$, W is a one-dimensional Brownian motion. We assume that W is independent of B. Define

$$\bar{X}_t = X_{t_k}, \quad \bar{X}^n_t = X^n_{t_k}, \quad \text{for } t \in [t_k, t_{k+1}), \quad k = 0, 1, \dots, n. \tag{4.2}$$

Following is our main result in this section.

Theorem 3.4.1. *Let $\bar{X}$ and $\bar{X}^n$ be the processes defined in (4.2), and assume that $V \in C^5_b$.*

Assume that $V_0 \neq 0$, then

$$(\gamma_n(\bar{X} - \bar{X}^n), B) \to (U, B) \tag{4.3}$$

in $D([0,T])$ as $n \to \infty$. In the case $m > 1$, U is the solution of the linear SDE

$$dU_t = \sum_{j=0}^{m} \partial V_j(X_t) U_t dB^j_t + \sqrt{\frac{\kappa}{2T}} \sum_{1 \leq j' < j \leq m} \phi_{jj'}(X_t) dW^{j'j}_t, \quad t \in [0,T]. \tag{4.4}$$

In the case $m = 1$, U is the solution of the linear SDE

$$dU_t = \sum_{j=0}^{1} \partial V_j(X_t) U_t dB_t^j + \sqrt{\frac{2\rho}{T}} \phi_{10}(X_t) dW_t\,, \quad t \in [0,T]. \tag{4.5}$$

Assume that $V_0 \equiv 0$ and $m = 1$, we have the following L^p-convergence

$$n^{2H}(\bar{X}_t - \bar{X}_t^n) \to U_t, \quad t \in [0,T], \tag{4.6}$$

where U is the solution of the linear SDE

$$dU_t = \partial V(X_t) U_t dB_t - \frac{T^{2H}}{4} \sum_{i,i'=1}^{d} (V^i V^{i'} \partial_i \partial_{i'} V)(X_t) dB_t\,, \quad t \in [0,T]. \tag{4.7}$$

Proof: *Step 1.* The inequality (3.3) implies the tightness of the scaled error function $\gamma_n(\bar{X} - \bar{X}^n)$. So to prove the weak convergence of $(\gamma_n(\bar{X} - \bar{X}^n), B)$ it suffices to show the convergence of its finite dimensional distribution (f.d.d.). It is clear that for $t \in [t_k, t_{k+1})$ we have

$$\begin{aligned} \bar{X}_t - \bar{X}_t^n &= X_{t_k} - X_{t_k}^n \\ &= \frac{1}{2} \Lambda_{t_k}^n \sum_{i=1}^{2} \int_0^{t_k} \Gamma_u^n dJ_i(u). \end{aligned}$$

Step 2. We assume that $m > 1$. According to the estimate (3.27), the f.d.d. limit of $(\gamma_n(\bar{X} - \bar{X}^n), B)$ is equivalent to that of

$$\left(\frac{1}{2} \gamma_n \Lambda_{\eta(t)}^n \int_0^{\eta(t)} \Gamma_{\eta(s)}^n dE_{11}(s),\ B_t\,, \quad t \in [0,T] \right). \tag{4.8}$$

Take $t \in D$ and set

$$S(t) = \Lambda_t^n \int_0^t \Gamma_{\eta(s)}^n dE_{11}(s) - \Lambda_t \sum_{0 \neq j' < j} \sum_{k=0}^{\frac{nt}{T}-1} \Gamma_{t_k} \phi_{jj'}(X_{t_k}) \zeta_{t_k,t_{k+1}}^{j'j}$$
$$= \sum_{0 \neq j' < j} \int_0^t \left[\Lambda_t^n \Gamma_{\eta(s)}^n \phi_{jj'}(X_{\eta(s)}^n) - \Lambda_t \Gamma_{\eta(s)} \phi_{jj'}(X_{\eta(s)}) \right] d\zeta_{\eta(s),s}^{j'j}.$$

It follows from Lemma 3.4.1 that

$$\|\Lambda_t^n \Gamma_\cdot^n \phi_{jj'}(X_\cdot^n) - \Lambda_t \Gamma_\cdot \phi_{jj'}(X_\cdot)\|_{\beta,p} \leq K n^{1-2\beta}.$$

So by applying Lemma 3.5.4 to $S(t)$ we obtain

$$\|S(t)\|_p \leq K n^{1-2\beta+1/2-2H}.$$

This implies that the f.d.d. convergence of (4.8) is the same as that of

$$\left(\frac{1}{2} n^{2H-1/2} \Lambda_{\eta(t)} \sum_{0 \neq j' < j} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \Gamma_{t_k} \phi_{jj'}(X_{t_k}) \zeta_{t_k,t_{k+1}}^{j'j}, B_t, \quad t \in [0,T] \right). \tag{4.9}$$

Applying Proposition 3.5.1 to the process (4.9) and taking into account the weak convergence result in Proposition 3.2.1, we see that its f.f.d. converges to that of

$$\left(\alpha_H \sqrt{\frac{\kappa}{2T}} \Lambda_t \sum_{1 \leq j' < j \leq m} \int_0^t \Gamma_s \phi_{jj'}(X_s) dW_s^{j'j}, \;\; B_t \right) \tag{4.10}$$

as $n \to \infty$. The convergence (4.3) follows from the fact that the first term in (4.10) solves the SDE (4.4).

Step 3. We assume $m = 1$. The estimate (3.28) implies that the f.d.d. convergence of $(\bar{X} - \bar{X}^n, B)$ is equal to that of

$$\left(\frac{1}{2}\gamma_n \Lambda^n_{\eta(t)} \int_0^{\eta(t)} \Gamma^n_{\eta(s)} dE_{12}(s),\ B_t, \quad t \in [0,T]\right). \tag{4.11}$$

As in the case $m > 1$, with the help of Lemma 3.4.1 we can show that the convergence of the finite dimensional distribution of the process (4.11) is the same as that of

$$\left(\frac{1}{2} n^{H+1/2} \Lambda(\eta(t)) \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \Gamma_{t_k} \phi_{10}(X_{t_k}) \zeta^{01}_{t_k, t_{k+1}}, B_t, \quad t \in [0,T]\right).$$

Applying Proposition 3.5.1 to the above above process and taking into account the weak convergence result in Proposition 3.2.2, we see that its f.d.d. converges to that of

$$\left(\sqrt{\frac{\alpha_H \rho}{2T}} \Lambda_t \int_0^t \Gamma_s \phi_{10}(X_s) dW_s, B_t, t \in [0,T]\right). \tag{4.12}$$

as $n \to \infty$. The convergence (4.3) follows from the fact that the first term in (4.12) solves the SDE (4.5).

Step 4. Assume that $m = 1$ and the drift term is equal to zero. Then we have $E_1 = 0$. The estimate (3.29) implies that the L_p-convergence of $n^{2H} Y^n_t$ is the same as that of

$$\frac{1}{2} n^{2H} \Lambda^n_t \sum_{e=1}^{5} \int_0^{\eta(t)} \Gamma^n_{\eta(s)} dE_e(s). \tag{4.13}$$

As in the case $m > 1$, with the help of Lemma 3.4.1 we can show that the L_p-convergence of (4.13) is the same as that of

$$\frac{1}{2} n^{2H} \Lambda_t \sum_{e=1}^{5} \int_0^{\eta(t)} \Gamma_{\eta(s)} dE_e(s).$$

Take $t \in D$. According to (3.14), we have

$$E_2^i(t) = \frac{1}{2}\sum_{i'=1}^{d}\sum_{k=0}^{nt/T-1} \partial_{i'} V^i(X_{t_k}^n) \int_{t_k}^{t_{k+1}} \partial V^{i'}(X_{s''}^n) \frac{1}{2}\left[V(X_{t_{k+1}}^n) + V(X_{t_k}^n)\right] dB_{s''} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB_{s'} dB_s.$$

Take

$$\begin{aligned}
\widetilde{E}_2^i(t) =& \frac{1}{2}\sum_{i'=1}^{d}\sum_{k=0}^{nt/T-1} (\partial_{i'} V^i \partial V^{i'} V)(X_{t_k}) \int_{t_k}^{t_{k+1}} dB_{s''} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB_{s'} dB_s \\
=& \frac{1}{4}\sum_{i'=1}^{d}\sum_{k=0}^{nt/T-1} (\partial_{i'} V^i \partial V^{i'} V)(X_{t_k}) (B_{t_k,t_{k+1}})^3,
\end{aligned}$$

then it is easy to show that

$$n^{2H}\left(E_2^i(t) - \widetilde{E}_2^i(t)\right) \to 0 \quad \text{in } L^p \quad \text{as } n \to \infty \tag{4.14}$$

for $t \in D$. Similarly, we take

$$\begin{aligned}
\widetilde{E}_3^i(t) =& -\frac{1}{4}\sum_{i'=1}^{d}\sum_{k=0}^{nt/T-1} \left(\partial(\partial_{i'} V^i V^{i'})V + V^{i'}\partial(\partial_{i'} V^i)V\right)(X_{t_k})(B_{t_k,t_{k+1}})^3, \\
\widetilde{E}_4^i(t) =& \widetilde{E}_5^i(t) = \frac{1}{6}\sum_{i',i''=1}^{d}\sum_{k=0}^{nt/T-1} (V^{i'} V^{i''} \partial_{i''}\partial_{i'} V^i)(X_{t_k})(B_{t_k,t_{k+1}})^3,
\end{aligned}$$

then it is easy to show that

$$n^{2H}(E_e^i(t) - \widetilde{E}_e^i(t)) \to 0 \quad \text{in } L^p \quad \text{as } n \to \infty \quad \text{for } e = 3,4,5. \tag{4.15}$$

In summary of (4.14) and (4.15), we obtain

$$n^{2H}\sum_{e=2}^{5} E_e^i(t) - n^{2H}\sum_{e=2}^{5} \widetilde{E}_e^i(t) \to 0 \quad \text{in } L^p \quad \text{as } n \to \infty, \tag{4.16}$$

for $t \in D$.

It is easy to see that

$$\sum_{e=2}^{5} \widetilde{E}_e^i(t) = -\frac{1}{6} \sum_{i',i''=1}^{d} \sum_{k=0}^{nt/T-1} (V^{i'} V^{i''} \partial_{i''} \partial_{i'} V^i)(X_{t_k})(B_{t_k,t_{k+1}})^3, \tag{4.17}$$

for $t \in D$.

Applying Proposition 3.5.2 to (4.17) and taking into account (4.16) and Lemma 3.5.1 (ii) we obtain the following convergence results in L^p:

$$n^{2H} \sum_{e=2}^{5} E_e^i(\eta(t)) \to -\frac{T^{2H}}{2} \sum_{i',i''=1}^{d} \int_0^t (V^{i'} V^{i''} \partial_{i''} \partial_{i'} V^i)(X_t) dB_s \tag{4.18}$$

for $t \in [0,T]$.

Recall that $\sum_{e=1}^{5} E_e$ satisfies the tightness inequality (3.17). So by applying Proposition 3.5.2 to $\frac{1}{2}\Lambda_t \sum_{e=1}^{5} \int_0^{\eta(t)} \Gamma_{\eta(s)} dE_e(s)$ and taking into account (4.18) we obtain the convergence

$$\frac{1}{2} n^{2H} \Lambda_t \sum_{e=1}^{5} \int_0^{\eta(t)} \Gamma_{\eta(s)} dE_e(s) \to -\frac{T^{2H}}{4} \Lambda_t \sum_{i',i''=1}^{d} \int_0^t \Gamma_s (V^{i'} V^{i''} \partial_{i''} \partial_{i'} V^i)(X_t) dB_s .$$

The convergence (4.6) follows from the fact that the right-hand side of the above solves the SDE (4.7). □

3.5 Appendix

3.5.1 Proof of (2.18)

The proof will be done in several steps.

Step 1. In this step, we derive a decomposition of d_2. It is clear that

$$\widetilde{D}_v D_r Z_n(t) = \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \beta_{\frac{k}{n}}(r)\gamma_{t_k,r}(v)$$

for $t \in [0,T]$. So by the integration by parts we obtain

$$\begin{aligned}
&\mathbb{E}\left[Z_n(t)\widetilde{D}_{u'}Z_n(t)D_{s'}Z_n(t)\right] \\
&= \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \int_0^T \int_0^T \int_0^T \int_0^T \left[D_{r'}\widetilde{D}_{u'}Z_n(t)\right]\left[\widetilde{D}_{v'}D_{s'}Z_n(t)\right] \\
&\qquad \cdot \beta_{\frac{k}{n}}(r)\gamma_{t_k,r}(v)\mu(dvdv')\mu(drdr') \\
&= \sum_{k,k_3,k_4=0}^{\lfloor \frac{nt}{T} \rfloor} \int_0^T \int_0^T \int_0^T \int_0^T \beta_{\frac{k_3}{n}}(r')\gamma_{t_{k_3},r'}(u')\beta_{\frac{k_4}{n}}(s')\gamma_{t_{k_4},s'}(v') \\
&\qquad \cdot \beta_{\frac{k}{n}}(r)\gamma_{t_k,r}(v)\mu(dvdv')\mu(drdr').
\end{aligned}$$

Substituting the above expression into (2.16) we obtain that d_2 equals

$$\begin{aligned}
6 \sum_{k_1,k_2,k_3,k_4=0}^{\lfloor \frac{nt}{T} \rfloor} \int_{t_{k_4}}^{t_{k_4+1}} \int_{t_{k_1}}^{t_{k_1+1}} \int_0^T \int_0^T \int_{t_{k_3}}^{t_{k_3+1}} \int_{t_{k_2}}^{t_{k_2+1}} \int_0^T \int_0^T \gamma_{t_{k_3},r'}(u')\gamma_{t_{k_4},s'}(v') \\
\cdot \gamma_{t_{k_2},r}(v)\gamma_{t_{k_1},s}(u)\mu(dvdv')\mu(drdr')\mu(dudu')\mu(dsds').
\end{aligned}$$

By the change of variables we obtain

$$d_2 = 6\left(\frac{T}{n}\right)^{8H} \sum_{k_1,k_2,k_3,k_4=0}^{\lfloor \frac{nt}{T} \rfloor} c(k_1,k_2,k_3,k_4), \tag{5.1}$$

where

$$c(k_1,k_2,k_3,k_4) = \int_{k_4}^{k_4+1}\int_{k_1}^{k_1+1}\int_{k_3}^{k_3+1}\int_{k_2}^{k_2+1}\int_0^n\int_0^n\int_0^n\int_0^n \varphi_{k_3,r'}(u')\varphi_{k_4,s'}(v') \\ \cdot \varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\mu(dvdv')\mu(dudu')\mu(drdr')\mu(dsds'),$$

and we denote

$$\varphi_{k,s}(u) = \varphi^0_{k,s}(u) - \varphi^1_{k,s}(u), \quad \varphi^0_{k,s}(u) = \mathbf{1}_{k,s}(u) \quad \text{and} \quad \varphi^1_{k,s}(u) = \mathbf{1}_{s,k+1}(u)\,.$$

We decompose the summation (5.1) into five terms based on the following decomposition of the set of the integers k_1,k_2,k_3,k_4:

$$I := \left\{k_1,k_2,k_3,k_4 = 0,1,\ldots,\lfloor\frac{nt}{T}\rfloor\right\} = \bigcup_{l=1}^{5} I_l\,,$$

where

$$I_1 = \big(\{|k_i-k_j|>2, i\neq j\}\cup\{|k_4-k_3|\leq 2\}\cup\{|k_2-k_1|\leq 2\}\big)\cap I;$$

$$I_2 = (\{|k_4-k_2|\leq 2\}\cup\{|k_3-k_1|\leq 2\})\cap I;$$

$$I_3 = (\{|k_4-k_1|\leq 2\}\cup\{|k_3-k_2|\leq 2\})\cap I;$$

$$I_4 = \cup_{l=1}^4\{|k_i-k_j|\leq 2, i\neq j, i,j\neq l\}\cap I;$$

$$I_5 = \{|k_i-k_j|\leq 2, i\neq j\}\cap I.$$

By symmetry of the expression (5.1), to consider the sets I_2, I_3 and I_4 it suffices to consider the their subsets:

$$J_2 = \{|k_3 - k_1| \le 2\} \cap I;$$

$$J_3 = \{|k_3 - k_2| \le 2\} \cap I;$$

$$J_4 = \{|k_i - k_j| \le 2,\, i, j = 1,2,3,\, i \ne j\} \cap I.$$

For notational convenience, we denote $J_1 = I_1$ and $J_5 = I_5$. Denote

$$d_{2l} := 6\left(\frac{T}{n}\right)^{8H} \sum_{(k_1,k_2,k_3,k_4)\in J_l} c(k_1,k_2,k_3,k_4). \tag{5.2}$$

Then to show that $n^{8H-2} d_2 \to 0$ as $n \to \infty$, it suffices to show that $n^{8H-2} d_{2l} \to 0$ for $l = 1, \ldots, 5$.

Step 2. We consider the decomposition

$$\varphi_{k_3,r'}(u')\varphi_{k_4,s'}(v')\varphi_{k_2,r}(v)\varphi_{k_1,s}(u) = \sum_{i,j=0,1} (-1)^{i+j} \varphi^i_{k_3,r'}(u')\varphi_{k_4,s'}(v')\varphi^j_{k_2,r}(v)\varphi_{k_1,s}(u). \tag{5.3}$$

Substituting (5.3) into (5.2), we obtain four terms, which we denote

$$d_{2l} := 6\left(\frac{T}{n}\right)^{8H} \sum_{(k_1,k_2,k_3,k_4)\in J_l} \sum_{i,j=0,1} (-1)^{i+j} c_{ij}. \tag{5.4}$$

In the remaining part of this step, we derive a more convenient expression for c_{ij}.

Fix $i_0, j_0 \in \{0,1\}$. We consider the following decomposition

$$\varphi^{i_0}_{k_3,r'}(u')\varphi_{k_4,s'}(v')\varphi^{j_0}_{k_2,r}(v)\varphi_{k_1,s}(u) = \sum_{i=0,1} (-1)^i \varphi^{i_0}_{k_3,r'}(u')\varphi^i_{k_4,s'}(v')\varphi^{j_0}_{k_2,r}(v)\varphi_{k_1,s}(u).$$

The above identity provides us a decomposition of $c_{i_0 j_0}$:

$$c_{i_0 j_0} = e_0 - e_1,$$

where

$$e_i = \int_{k_4}^{k_4+1} \int_{k_1}^{k_1+1} \int_{k_3}^{k_3+1} \int_{k_2}^{k_2+1} \int_0^n \int_0^n \int_0^n \int_0^n \varphi_{k_3,r'}^{i_0}(u') \varphi_{k_4,s'}^{i}(v') \\ \cdot \varphi_{k_2,r}^{j_0}(v) \varphi_{k_1,s}(u) \mu(dvdv') \mu(dudu') \mu(drdr') \mu(dsds'),$$

By exchanging the order of the integrals for the variables v' and s' in e_1, we obtain

$$e_1 = \int_{k_4}^{k_4+1} \int_{k_1}^{k_1+1} \int_{k_3}^{k_3+1} \int_{k_2}^{k_2+1} \int_{k_4}^{v'} \int_0^n |v - v'|^{2H-2} |s - s'|^{2H-2} \varphi_{k_2,r}^{j_0}(v) dvds' \\ \cdot \int_0^n \int_0^n \varphi_{k_1,s}(u) \varphi_{k_3,r'}^{i_0}(u') \mu(dudu') \mu(drdr') dsdv',$$

which, by exchanging the notations s' and v', is equal to

$$\int_{k_4}^{k_4+1} \int_{k_1}^{k_1+1} \int_{k_3}^{k_3+1} \int_{k_2}^{k_2+1} \int_{k_4}^{s'} \int_0^n |v - s'|^{2H-2} |s - v'|^{2H-2} \varphi_{k_2,r}^{j_0}(v) dvdv' \\ \cdot \int_0^n \int_0^n \varphi_{k_1,s}(u) \varphi_{k_3,r'}^{i_0}(u') \mu(dudu') \mu(drdr') dsds'.$$

Therefore, we obtain

$$e_0 - e_1 = \int_{k_3}^{k_3+1} \int_{k_2}^{k_2+1} \int_{k_4}^{k_4+1} \int_{k_1}^{k_1+1} \int_{k_4}^{s'} \int_0^n \phi(s, s', v, v') \varphi_{k_2,r}^{j_0}(v) dvdv' \\ \cdot \int_0^n \int_0^n \varphi_{k_1,s}(u) \varphi_{k_3,r'}^{i_0}(u') \mu(dudu') dsds' \mu(drdr'), \tag{5.5}$$

where

$$\phi(s,s',v,v') = |s-s'|^{2H-2}|v-v'|^{2H-2} - |v-s'|^{2H-2}|s-v'|^{2H-2}.$$

Recall that

$$\varphi_{k_1,s}(u) = \varphi^0_{k_1,s}(u) - \varphi^1_{k_1,s}(u).$$

This identity allows us to decompose the expression of $e_0 - e_1$ in (5.5) into two terms:

$$e_0 - e_1 := A_0 - A_1.$$

By exchanging the order of the integrals for the variables u and s, and then exchanging the notations u and s, we obtain

$$\begin{aligned} A_1 = &\int_{k_3}^{k_3+1}\int_{k_2}^{k_2+1}\int_{k_4}^{k_4+1}\int_{k_1}^{k_1+1}\int_{k_4}^{s'}\int_0^n \phi(s,s',v,v')\varphi^{j_0}_{k_2,r}(v)dvdv' \\ &\cdot \int_0^n\int_s^{k_1+1} \varphi^{i_0}_{k_3,r'}(u')\mu(dudu')dsds'\mu(drdr') \\ = &\int_{k_3}^{k_3+1}\int_{k_2}^{k_2+1}\int_{k_4}^{k_4+1}\int_{k_1}^{k_1+1}\int_0^n\int_{k_1}^{s}\int_{k_4}^{s'}\int_0^n \phi(u,s',v,v')\varphi^{j_0}_{k_2,r}(v)\varphi^{i_0}_{k_3,r'}(u') \\ &\cdot |u'-s|^{2H-2}dvdv'dudu'dsds'\mu(drdr'). \end{aligned}$$

Substituting the above identity into (5.5), we obtain

$$\begin{aligned} c_{i_0 j_0} = &e_0 - e_1 \\ = &\int_{k_3}^{k_3+1}\int_{k_2}^{k_2+1}\int_{k_4}^{k_4+1}\int_{k_1}^{k_1+1}\int_0^n\int_{k_1}^{s}\int_{k_4}^{s'}\int_0^n \widetilde{\phi}(u,u',v,v',s,s') \\ &\cdot \varphi^{j_0}_{k_2,r}(v)\varphi^{i_0}_{k_3,r'}(u')dvdv'dudu'dsds'\mu(drdr'). \end{aligned} \tag{5.6}$$

where

$$\widetilde{\phi}(u,u',v,v',s,s') = \phi(s,s',v,v')|u-u'|^{2H-2} - \phi(u,s',v,v')|u'-s|^{2H-2}.$$

Step 3. In this step, we consider d_{21}, that is, the summation in (5.2) corresponding to the integers k_1,k_2,k_3,k_4 such that $|k_i - k_j| > 2$ for $i \neq j$. The following three inequalities follows immediately from the mean value theorem:

$$\left|\phi(s,s',v,v')\right| \leq K(|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-2} + |k_4-k_2|^{2H-3}|k_4-k_1|^{2H-2}), \quad (5.7)$$

$$\begin{aligned}
&\left|\phi(s,s',v,v') - \phi(u,s',v,v')\right| \\
&\leq K(|k_4-k_1|^{2H-4}|k_4-k_2|^{2H-2} + |k_4-k_1|^{2H-3}|k_4-k_2|^{2H-3} \\
&\qquad + |k_4-k_2|^{2H-4}|k_4-k_1|^{2H-2}|),
\end{aligned}$$

and

$$
\begin{aligned}
&\left|\phi(s,s',v,v')|u-u'|^{2H-2}-\phi(u,s',v,v')|u'-s|^{2H-2}\right| \\
&\leq \left|\phi(s,s',v,v')-\phi(u,s',v,v')\right|\cdot|u'-s|^{2H-2} \\
&\qquad+\left|\phi(s,s',v,v')\right|\cdot\left||u-u'|^{2H-2}-|u'-s|^{2H-2}\right| \\
&\leq K\Big\{|k_4-k_1|^{2H-4}|k_4-k_2|^{2H-2}|k_1-k_3|^{2H-2} \\
&\qquad\quad+|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-3}|k_1-k_3|^{2H-2} \\
&\qquad\quad+|k_4-k_1|^{2H-2}|k_4-k_2|^{2H-4}|k_1-k_3|^{2H-2} \\
&\qquad\quad+|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-2}|k_1-k_3|^{2H-3} \\
&\qquad\quad+|k_4-k_2|^{2H-3}|k_4-k_1|^{2H-2}|k_1-k_3|^{2H-3}\Big\} \\
&:=\sum_{i=1}^{5}\psi_i(k_1,k_2,k_3,k_4). \qquad (5.8)
\end{aligned}
$$

Applying the inequality (5.8) to the expression of c_{ij} in (5.6) and taking into account the expression of d_{21} in (5.4), we obtain

$$
n^{8H-2}d_{21}\leq 6Kn^{-2}\sum_{(k_1,k_2,k_3,k_4)\in\tilde{I}_1}|k_2-k_3|^{2H-2}\sum_{i=1}^{5}\psi_i(k_1,k_2,k_3,k_4). \qquad (5.9)
$$

In the following, we show that the above five terms converges to zero as n tends to infinity. By the following change of variables

$$
p_1=k_1-k_3, \quad p_2=k_4-k_1, \quad p_3=k_2-k_4, \qquad (5.10)
$$

we obtain

$$\begin{aligned}
&\sum_{(k_1,k_2,k_3,k_4)\in\tilde{I}_1} |k_2-k_3|^{2H-2}(\varphi_2+\varphi_4+\varphi_5)(k_1,k_2,k_3,k_4) \\
&\le \sum_{k_3=1}^{n}\sum_{p_1=1}^{n}\sum_{p_2=1}^{n}\sum_{p_3=1}^{n}(p_1+p_2+p_3)^{2H-2}\Big(p_2^{2H-3}p_3^{2H-3}p_1^{2H-2} \\
&\qquad\qquad + p_2^{2H-3}p_3^{2H-2}p_1^{2H-3} + p_2^{2H-2}p_3^{2H-3}p_1^{2H-3}\Big) \\
&\le 3n\sum_{p_1=1}^{n}\sum_{p_2=1}^{n}\sum_{p_3=1}^{n}(p_1+p_2+p_3)^{2H-2}p_2^{2H-3}p_3^{2H-3}p_1^{2H-2} \\
&\le 3n\sum_{p_1=1}^{n}\sum_{p_2=1}^{n}\sum_{p_3=1}^{n}p_2^{4H-5}p_3^{2H-3}p_1^{2H-2}.
\end{aligned}$$

Since $\sum_{p_3=1}^{n} p_1^{2H-3}$ and $\sum_{p_3=1}^{n} p_2^{4H-5}$ are convergent, and $\sum_{p_1=1}^{n} p_2^{2H-2} \le Kn^{2H-1}$, the above is less than Kn^{2H}. This implies that the second, fourth and fifth terms in the right hand side of (5.9) converge to zero as n tends to infinity. The convergence of the first and third terms in (5.9) can be shown similarly. Instead of (5.10), we set

$$p_1 = k_1 - k_3, \quad p_2 = k_3 - k_2, \quad p_3 = k_2 - k_4,$$

for the first term of (5.9), and set

$$p_1 = k_1 - k_4, \quad p_2 = k_2 - k_3, \quad p_3 = k_3 - k_1,$$

for the third term. This concludes that $n^{8H-2}d_{21}$ converges to zero as n tends to infinity.

Step 4. In this step, we consider d_{22}. Note that for $(k_1,k_2,k_3,k_4) \in J_2$, the estimate in (5.7) still holds true. Applying (5.7) to (5.6) we obtain that $|c_{i_0 j_0}|$ is less than

$$2K\int_{k_3}^{k_3+1}\int_{k_2}^{k_2+1}\int_{k_4}^{k_4+1}\int_{k_1}^{k_1+1}\int_{k_3}^{k_3+1}\int_{k_1}^{k_1+1}\int_{k_4}^{k_4+1}\int_{k_2}^{k_2+1}\Big(|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-2} + |k_4-k_2|^{2H-3}|k_4-k_1|^{2H-2}|\Big)(|u-u'|^{2H-2}+|s-u'|^{2H-2}) \cdot dvdv'dudu'dsds'\mu(drdr').$$

Therefore,

$$\begin{aligned}|d_{22}| &\leq Kn^{-8H}\sum_{(k_1,k_2,k_3,k_4)\in J_2}\big(|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-2} \\ &\qquad + |k_4-k_2|^{2H-3}|k_4-k_1|^{2H-2}|\big)|k_2-k_3|^{2H-2} \\ &\leq Kn^{-8H}\sum_{k_1,k_2,k_4=0}^{n-1}\big(|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-2} \\ &\qquad + |k_4-k_2|^{2H-3}|k_4-k_1|^{2H-2}|\big)|k_2-k_1|^{2H-2} \\ &\leq Kn^{-8H}\sum_{k_1=0}^{n-1}\sum_{p_1,p_2=1}^{n}(p_1^{2H-3}p_2^{2H-2}+p_2^{2H-3}p_1^{2H-2})(p_1+p_2)^{2H-2}.\end{aligned}$$

It is now easy to show that $n^{8H-2}d_{22}$ approaches zero as n tends to infinity.

Step 5. We consider d_{23}. Note that for $(k_1,k_2,k_3,k_4) \in J_3$, the inequality (5.8) still holds true. Applying (5.8) to d_{23} we obtain

$$n^{8H-2}d_{23} \leq 6Kn^{-2}\sum_{(k_1,k_2,k_3,k_4)\in\tilde{I}_3}\sum_{i=1}^{5}\varphi_i(k_1,k_2,k_3,k_4).$$

In the similar way as in Step 1, we can show that $n^{8H-2}d_{23}$ approaches zero as n tends to infinity.

We turn to the quantity d_{24}. Note that for $(k_1,k_2,k_3,k_4) \in J_4$, the estimate in (5.7) still holds true. By applying (5.7) to (5.6) it is easy to see that

$$n^{8H-2} d_{24} \leq K n^{-2} \sum_{k_1,k_4=0}^{n-1} |k_4 - k_1|^{4H-5},$$

which converges to zero as n tends to infinity.

Finally, we consider the quantity d_{25}. Note that

$$\sup_{k_1,k_2,k_3,k_4} |c(k_1,k_2,k_3,k_4)| \leq K,$$

for some constant K. So we have

$$n^{8H-2} d_{25} \leq K n^{-2} \sum_{(k_1,k_2,k_3,k_4)\in J_5} K \leq n^{-1},$$

which approaches to zero as n tends to infinity. □

3.5.2 Estimates of some triple integrals

In this subsection, we derive estimates for some triple integrals.

Lemma 3.5.1. *(i) For $t \in [0,T]$, denote*

$$G_t = \sum_{k=0}^{\lfloor nt/T \rfloor - 1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \int_{t_k}^{u} dB_v^1 dB_u^2 dB_s^3, \tag{5.11}$$

where B_t^i, $t \in [0,T]$, $i = 1,2,3$ is either a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ or $B_t^i = t$. Then we have

$$\|G_t - G_s\|_p \leq K n^{-2H} |t-s|^{\frac{1}{2}}$$

for $s,t \in D$.

(ii) Take $p \geq 1$ *and* $t \in [0,T]$. *We have the following convergence in* L_p*:*

$$n^{2H} \sum_{k=0}^{\lfloor nt/T \rfloor - 1} (B_{t_{k+1}} - B_{t_k})^3 \to 3T^{2H} B_t.$$

Proof: The result in (i) follows from Proposition 5.10 in [14]. The convergence in (ii) follows immediately from results in [9] or [46]. □

We need the following technical lemma.

Lemma 3.5.2. *Let* f *and* g *be Hölder continuous stochastic processes of order* $\beta > 1/2$ *on* $[0,T]$ *such that* $\mathbb{E}[\|f\|_\beta^p] + \mathbb{E}[\|g\|_\beta^p] \leq K$ *for some* $p \geq 1$. *Let* h *be a process on* $[0,T]$ *such that*

$$\|h_t - h_s\|_p \leq K(t-s)^\beta, \quad s,t \in D : s \leq t.$$

Denote

$$\widetilde{G}_t = \sum_{k=0}^{\lfloor nt/T \rfloor - 1} h_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_3} \int_{t_k}^{s_2} f_{s_i} g_{s_j} dB_{s_1}^1 dB_{s_2}^2 dB_{s_3}^3,$$

where B^1, B^2 *and* B^3 *are the same as in Lemma 3.5.1, and* $i, j = 1,2,3$. *Then*

$$\|\widetilde{G}_t - \widetilde{G}_s\|_p \leq K n^{-2H} |t-s|^{\frac{1}{2}} \tag{5.12}$$

for $s,t \in D$.

Proof: We decompose $\widetilde{G}$ as follows:

$$\begin{aligned}\widetilde{G}_t = & \sum_{k=0}^{\lfloor nt/T \rfloor -1} f_{t_k} g_{t_k} h_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_3} \int_{t_k}^{s_2} dB^1_{s_1} dB^2_{s_2} dB^3_{s_3} \\ & + \sum_{k=0}^{\lfloor nt/T \rfloor -1} f_{t_k} h_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_3} \int_{t_k}^{s_2} \int_{t_k}^{s_j} dg_{s_4} dB^1_{s_1} dB^2_{s_2} dB^3_{s_3} \\ & + \sum_{k=0}^{\lfloor nt/T \rfloor -1} h_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_3} \int_{t_k}^{s_2} \int_{t_k}^{s_i} g_{s_j} df_{s_4} dB^1_{s_1} dB^2_{s_2} dB^3_{s_3}. \end{aligned} \quad (5.13)$$

Applying Proposition 5.10 in [14] to the second and third terms in the right hand side of (5.13), and applying Lemma 3.5.4 to the first term and taking into account the estimate in Lemma 3.5.1 (i), we obtain the inequality (5.12). □

3.5.3 Fractional integrals

In this section, we recall some recent results on discrete and continuous fractional integrals. As before, we denote by $\|\cdot\|_p$ the norm in the space $L^p := L^p(\Omega)$, where $p \geq 1$.

Definition 3.5.1. *Let $f = \{f(t), t \in (a,b)\}$ be a stochastic process such that $f(t) \in L^p$ for all $t \in (a,b)$. We say that f is Hölder continuous of order $\beta > 0$ in L^p if*

$$\|f(t) - f(s)\|_p \leq K|t-s|^\beta,$$

for all $t, s \in [a,b]$. We denote

$$\|f\|_{\beta,p} = \sup\left\{ \frac{\|f(t)-f(s)\|_p}{|t-s|^\beta} : t,s \in (a,b) \right\}.$$

The following lemma provides an L^p-estimate on a fractional Riemann-Stieltjes integral. The result follows from the fractional integration by parts formula; see [49]. The proof is similar to that of Lemma 11.1 in [13], and is omitted.

Lemma 3.5.3. *Take $p \geq 1$, $p', q' : \frac{1}{p'} + \frac{1}{q'} = 1$ and $\beta, \beta' \in (0,1) : \beta + \beta' > 1$. Let $f(x)$, $g(x)$, $x \in (a,b)$ be Hölder continuous functions of order β and β' in $L^{pp'}$ and $L^{pq'}$, respectively, and assume that $f(a+) \in L^{pp'}$. Then*

$$\left\| \int_a^b f dg \right\|_p \leq \left(K\|f\|_{\beta,pp'} + \|f(a+)\|_{pp'} \right) \|g\|_{\beta',pq'} (b-a)^{\beta'}, \tag{5.14}$$

where K is a constant depending only on the parameters.

Define a double sequence of random variables $\zeta = \{\zeta_{k,n}, n \in \mathbb{N}, k = 0,1,\ldots,n\}$ and for each $t \in [0,T]$ we denote

$$g_n(t) := \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \zeta_{k,n}. \tag{5.15}$$

The following lemma from [14] considers the weighted summations of g_n.

Lemma 3.5.4. *Let p, p', q', β, β' and f be the same as in Lemma 3.5.3, and assume that f is Hölder continuous of order β. Let g_n be the sequence of processes defined in (5.15), and for any $j,k = 0,1,\ldots,n$ we have*

$$\mathbb{E}(|g_n(kT/n) - g_n(jT/n)|^{pq'}) \leq K(|k-j|/n)^{\beta' pq'}. \tag{5.16}$$

Then for $i, j = 0,1,\ldots,n-1$, $i > j$,

$$\left\| \sum_{k=j+1}^{i} f(t_k)\zeta_{k,n} \right\|_p \leq K \left(\|f\|_{\beta,pp'} + \|f(t_j+)\|_{pp'} \right) \left(\frac{i-j}{n} \right)^{\beta'}.$$

We will need the following two convergence results on weighted random sums; see [4, 13].

Proposition 3.5.1. *Let g^n be defined as in (5.15) such that it satisfies the inequality*

$$\mathbb{E}(|g_n(kT/n) - g_n(jT/n)|^4) \quad \leq \quad K(|k-j|/n)^2.$$

for $j,k = 0,1,\ldots,n$. Assume that the finite dimensional distribution of $g_n(t)$, $t \in [0,T]$ converges stably to that of W_t, $t \in [0,T]$, where W is a standard Brownian motion independent of B. Let $f(t)$, $t \in [0,T]$ be a β-Hölder continuous process with the order $\beta > 1/2$. Then the finite dimensional distribution of $\sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} f(t_k)\zeta_{k,n}$ converges stably to that of $\int_0^t f(s)dW_s$.

Following is the L_p-convergence version of Proposition 3.5.1.

Proposition 3.5.2. *We take $\lambda > 1-\beta$ for $0 < \beta < 1$. Let $p \geq 1$ and $p', q' > 1$ such that $\frac{1}{p'} + \frac{1}{q'} = 1$ and $pp' > \frac{1}{\beta}$, $pq' > \frac{1}{\lambda}$. Let g_n be the sequence of processes defined in (5.15). Suppose that the following conditions hold true:*

(i) For each $t \in [0,T]$, $g_n(t) \to z(t)$ in $L^{pq'}$.

(ii) For any $j,k = 0,1,\ldots,n$ we have

$$\mathbb{E}(|g_n(kT/n) - g_n(jT/n)|^{pq'}) \leq K(|k-j|/n)^{\lambda pq'}.$$

Let $f = \{f(t), t \in [0,T]\}$ be a process such that $\mathbb{E}(\|f\|_\beta^{pp'}) \leq K$ and $\mathbb{E}(|f(0)|^{pp'}) \leq K$. Then for each $t \in [0,T]$,

$$F^n(t) := \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} f(t_k)\zeta_{k,n} \to \int_0^t f(s)dz(s) \qquad \text{in } L^p \text{ as } n \to \infty.$$

3.5.4 Proof of (3.29)

It is clear that

$$\begin{aligned}
&\sum_{i=1}^{2}\int_0^t \Gamma_u^n dJ_i(u) - \sum_{e=2}^{5}\int_0^{\eta(t)} \Gamma_{\eta(u)}^n dE_e(u) \\
&= \sum_{i=1}^{2}\int_0^t \Gamma_u^n dJ_i(u) - \sum_{i=1}^{2}\int_0^{\eta(t)} \Gamma_{\eta(u)}^n dJ_i(u) \\
&= \sum_{i=1}^{2}\int_0^t \int_{\eta(s)}^{s} d\Gamma_u^n dJ_i(s) + \sum_{i=1}^{2}\int_{\eta(t)}^{t} \Gamma_{\eta(u)}^n dJ_i(u). \qquad (5.17)
\end{aligned}$$

So it suffices to show that the L_p-norm of the two terms in the right hand side of (5.17) is less than $Kn^{1-4\beta}$.

For $t \in [0,T]$, we define

$$I(t) = \int_0^t (\partial VV)(X_{\eta(s)}^n) B_{\eta(s),s} dB_s,$$

It is clear that $I(t_k) = I_1(t_k) = I_2(t_k)$ for $k = 0,1,\dots,n$. As in (3.9), we have the decomposition

$$\begin{aligned}
J_1(t) + J_2(t) &= (R_0(t) - I(t)) + \left(I(t) - \widetilde{R}_0(t)\right) + R_1(t) + \widetilde{R}_1(t) \\
&:= E_2(t) + E_3(t) + E_4(t) + E_5(t)
\end{aligned}$$

for $t \in [0,T]$. Note that the E_2 and E_3 defined here are extensions of those in (3.9).

By applying Lemma 8.2 in [14] to (3.5) and (3.7) we obtain

$$\|E_e\|_{[t_k,t_{k+1}],\beta} \le K e^{K\|B\|_\beta^{1/\beta}} n^{-2\beta} \qquad (5.18)$$

for $e = 4, 5$. On the other hand, it is also easy to show that the inequality (5.18) holds for $e = 2, 3$. Therefore, we obtain

$$\begin{aligned} \|J_1 + J_2\|_{[t_k, t_{k+1}], \beta} &= \left\| \sum_{e=2}^{5} E_e \right\|_{[t_k, t_{k+1}], \beta} \\ &\leq K n^{-2\beta}. \end{aligned} \tag{5.19}$$

By applying Lemma 8.2 in [14] and taking into account the estimate (5.19) we obtain

$$\left\| \sum_{i=1}^{2} \int_{t'}^{t''} \int_{\eta(s)}^{s} d\Gamma_u^n dJ_i(s) \right\|_p \leq K n^{-4\beta}$$

for $t', t'' \in [t_k, t_{k+1}]$. Therefore, we have

$$\begin{aligned} \left\| \sum_{i=1}^{2} \int_{0}^{t} \int_{\eta(s)}^{s} d\Gamma_u^n dJ_i(s) \right\|_p &\leq \sum_{k=0}^{\lfloor nt/T \rfloor} \left\| \sum_{i=1}^{2} \int_{t_k}^{t_{k+1} \wedge t} \int_{\eta(s)}^{s} d\Gamma_u^n dJ_i(s) \right\|_p \\ &\leq K n^{1-4\beta}. \end{aligned}$$

By applying (5.19) to

$$\sum_{i=1}^{2} \int_{\eta(t)}^{t} \Gamma_{\eta(s)}^n dJ_i(s) = \sum_{i=1}^{2} \Gamma_{\eta(t)}^n (J_i(t) - J_i(\eta(t))),$$

we also obtain

$$\left\| \sum_{i=1}^{2} \int_{\eta(t)}^{t} \Gamma_{\eta(s)}^n dJ_i(s) \right\|_p \leq K n^{-3\beta} \leq K n^{1-4\beta}.$$

This completes the proof. □

3.5.5 Proof of Lemma 3.4.1

By the definition of J_1, we have

$$\begin{aligned}\int_s^t \Gamma_u^n dJ_1(u) &= \sum_{i=1}^d \int_s^t \Gamma_u^n \int_{\eta(u)}^u \partial_i V(X_v^n) dX_v^{n,i} dB_u \\ &= \sum_{i=1}^d \sum_{k=\lfloor \frac{ns}{T} \rfloor}^{\lfloor \frac{nt}{T} \rfloor} \int_{t_k \vee s}^{t_{k+1} \wedge t} \Gamma_u^n \int_{\eta(u)}^u \partial_i V(X_v^n) dX_v^{n,i} dB_u\end{aligned}$$

for $s,t \in [0,T]$. Applying Lemma 8.2 in [14] to the right hand side of the above equation we obtain the estimate

$$\begin{aligned}\left\| \int_s^t \Gamma_u^n dJ_i(u) \right\|_p &\leq \sum_{i=1}^d \sum_{k=\lfloor \frac{ns}{T} \rfloor}^{\lfloor \frac{nt}{T} \rfloor} n^{-\beta} (t_{k+1} \wedge t - t_k \vee s)^\beta \\ &\leq K|t-s|^\beta n^{1-2\beta}, \qquad (5.20)\end{aligned}$$

for $i = 1$. In the same way we can show estimate (5.20) for $i = 2$. Applying these two estimates to

$$Y_t = \frac{1}{2} \Lambda_t^n \sum_{i=1}^2 \int_0^t \Gamma_s^n dJ_i(s),$$

we obtain

$$\|Y\|_{\beta,p} \leq K n^{1-2\beta}. \qquad (5.21)$$

We denote $\Phi := \Lambda - \Lambda^n$. Subtracting (3.18) from (3.21) we obtain

$$\Phi^i_{i'}(t) = \sum_{j=0}^{m}\sum_{i''=1}^{d}\int_0^t \left[\partial_{i''}V^i_j(X_s)\Lambda^{i''}_{i'}(s) - V^i_{ji''}(s)\Lambda^{n,i''}_{i'}(s)\right] dB^j_s$$
$$= \sum_{j=0}^{m}\sum_{i''=1}^{d}\int_0^t \partial_{i''}V^i_j(X_s)\Phi^{i''}_{i'}(s)dB^j_s + \sum_{j=0}^{m}\sum_{i''=1}^{d}\int_0^t \left[\partial_{i''}V^i_j(X_s) - V^i_{ji''}(s)\right]\Lambda^{n,i''}_{i'}(s)dB^j_s.$$

By the product rule we can verify the following identity,

$$\Lambda(t) - \Lambda^n(t) = \sum_{i,i'=1}^{d}\sum_{j=0}^{m}\Lambda(t)\int_0^t \Gamma_{i'}(s)\left[\partial_i V^{i'}_j(X_s) - V^{i'}_{ji}(s)\right]\Lambda^{n,i}(s)dB^j_s. \tag{5.22}$$

We take

$$\widetilde{V}(X_s, X^n_s) = \sum_{i=1}^{d}\int_0^1\int_0^1 \partial_i\partial_{i'}V^{i''}_j(\lambda X_s + (1-\lambda)(\theta X_s + (1-\theta)X^n_s))(1-\theta)d\lambda d\theta,$$

then it is easy to verify that

$$\partial_{i'}V^{i''}_j(X_s) - V^{i''}_{ji'}(s) = \widetilde{V}(X_s, X^n_s)Y_s.$$

We denote

$$g_t = \sum_{i',i''=1}^{d}\sum_{j=0}^{m}\int_0^t \Gamma_{i''}(s)\widetilde{V}(X_s, X^n_s)\Lambda^{n,i'}_i(s)dB^j_s.$$

Then (5.22) becomes

$$\Lambda_i(t) - \Lambda^n_i(t) = \Lambda_t\int_0^t dg_s\cdot Y_s.$$

Applying Lemma 3.5.3 and taking into account the estimate (5.21), we obtain

$$\left\| \Lambda_t \int_s^t dg_s \cdot Y_s \right\|_p \leq K n^{1-2\beta} (t-s)^{\beta}.$$

This completes the proof of the estimate (4.1). The estimate for the quantity $\Gamma - \Gamma^n$ can be shown similarly. □

Chapter 4

Taylor schemes for rough differential equations and fractional diffussions

4.1 Introduction

Consider the d-dimensional differential equation

$$dy_t = V(y_t)dx_t, \quad t \in [0,T], \quad y_0 \in \mathbb{R}, \tag{1.1}$$

where $x = (x^1, \dots, x^m)$ is Hölder continuous of order $\beta > 1/2$ and $V = (V_j^i)_{1\le i\le d, 1\le j\le m}$ is a continuous mapping from $\mathbb{R}^d$ to $\mathbb{R}^{d\times m}$. It is well-known (see [24] and [36]) that if V is continuously differentiable and its partial derivatives are bounded and locally Hölder continuous of order $\delta > \frac{1}{\beta} - 1$, then equation (1.1) has a unique solution that is Hölder continuous of order β.

Our goal in this chapter is to study numerical approximations for the solution of equation (1.1). We briefly recall the way to obtain some general numerical approximation schemes for equation (1.1).

Assume that V has sufficient regularity. A simple Taylor expansion (iterated application of chain rule) leads, when t is sufficiently close to s, to the following approximation

$$y_t \approx y_s + \mathscr{E}^{(N)}_{s,t}(y_s), \tag{1.2}$$

where

$$\mathscr{E}^{(N)}_{s,t}(y) := \sum_{r=1}^{N} \sum_{(j_1,\dots,j_r)\in\Gamma_r} \mathscr{V}_{j_1}\cdots\mathscr{V}_{j_r}I(y)\int_s^t\int_s^{u_r}\cdots\int_s^{u_2} dx^{j_1}_{u_1}\cdots dx^{j_r}_{u_r}, \quad y\in\mathbb{R}^d.$$

In this expression, Γ_r is the collection of multi-indices of length r with elements in $\{1,2,\dots,m\}$, I is the identity function ($I(y)=y$) from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $\mathscr{V}_j$ is the vector field

$$\mathscr{V}_j f = \sum_{i=1}^{d} V^i_j \partial_i f, \quad j=1,\dots,m, \tag{1.3}$$

where ∂_i denotes the differential operator $\frac{\partial}{\partial y^i}$, $i=1,\dots,d$ (we refer the reader to [2, 6, 12] for more details; [12] gives a Taylor expansion with explicit form of the residual).

Let $0=t_0<t_1<\cdots<t_n=T$ be any partition of the interval $[0,T]$. On the interval $[t_k,t_{k+1}]$, we may use $y^n_{t_k}+\mathscr{E}^{(N)}_{t_k,t}(y^n_{t_k})$ to approximate y_t. We iterate this on each subinterval of the partition to obtain the following recursive scheme for (1.1),

$$y^n_t = y^n_{t_k} + \mathscr{E}^{(N)}_{t_k,t}(y^n_{t_k}), \quad t\in[t_k,t_{k+1}], \tag{1.4}$$

for $k=0,1,\dots,n-1$, with $y^n_0=y_0$, In this chapter, we shall take $t_k=\frac{T}{n}k$, $k=0,1\dots,n$. The recursive scheme (1.4) can also be written as

$$y^n_t = y_0 + \sum_{k=0}^{\lfloor \frac{nt}{T}\rfloor} \mathscr{E}^{(N)}_{t_k,t_{k+1}\wedge t}(y^n_{t_k}), \tag{1.5}$$

where $a \wedge b$ is the smaller of the numbers a and b and $\lfloor a \rfloor$ is the integer part of a. The recursive scheme (1.4) is usually called the *time discrete Taylor scheme* or simply *Taylor scheme*, of order N. Note that the interpolation on each interval $[t_k, t_{k+1}]$ used in (1.4) and (1.5) guarantees that the numerical scheme has the same convergence rate at non-discretization points $t \in [0,T] \setminus D$ as at the discretization points $t \in D = \{\frac{k}{n}T, k = 0, 1, \dots, n\}$.

Taylor scheme (1.5) has been considered in [8] when x is a *weak geometric p-rough path* (see Section 4.7), $p \geq 1$. It is proved that under some additional regularity assumptions on V, for $N \geq \lfloor p \rfloor$, the rate of convergence of y_t^n to y_t is $n^{1-(N+1)/p}$. Clearly, the larger the N in (1.5) is the higher will be the convergence rate. If $N = 1$, then (1.5) is reduced to the classical Euler scheme

$$y_t^n = y_{t_k}^n + V(y_{t_k}^n)(x_t - x_{t_k}), \quad t \in [t_k, t_{k+1}], \tag{1.6}$$

for $k = 0, 1, \dots, n-1$, with $y_0^n = y_0$. This classical Euler scheme has been studied, for instance, in [13, 28, 30].

Remark 4.1.1. *With an abuse of notation we shall use the same notation y_t^n to denote the approximation obtained by different schemes when there is no confusion.*

When one of the driving signals x is the time, say $x^1(t) = t$, and when the others are independent standard Brownian motions, an important scheme is the so-called Milstein scheme, which has the following form

$$y_t^n \quad = \quad y_{t_k}^n + \bar{\mathscr{E}}_{t_k,t}(y_{t_k}^n), \quad t \in [t_k, t_{k+1}], \tag{1.7}$$

for $k=0,1,\ldots,n-1$, with $y_0^n = y_0$, where

$$\bar{\mathscr{E}}_{s,t}(y) \quad := \quad \sum_{j\in\Gamma_1} \mathscr{V}_j I(y) \int_s^t dx_u^j + \sum_{(j,j')\in\bar{\Gamma}_2} \mathscr{V}_j \mathscr{V}_{j'} I(y) \int_s^t \int_s^{u'} dx_u^j dx_{u'}^{j'}$$

and $\bar{\Gamma}_2 = \{(j,j') \in \Gamma_2 : j, j' = 2,\ldots,m\}$. This scheme does not include the terms $\int_s^t \int_s^u dx_v^j dx_u^{j'}$, where one or both of the j and j' are 1. In the Brownian motion case, it is well-known that the Milstein scheme has the same rate of convergence as the order 2 Taylor scheme while it requires fewer computations.

This motivates us to ask the following question.

Question 1. How to eliminate as many terms as possible in $\mathscr{E}_{s,t}^{(N)}(y)$ while keeping the same rate of convergence? More precisely, we want to find subsets $\widetilde{\Gamma}_r \subseteq \Gamma_r$ so that $\widetilde{\Gamma}_r$ contains as few elements as possible and when we replace $\mathscr{E}_{t_k,t_{k+1}\wedge t}^{(N)}\left(y_{t_k}^n\right)$ in the Taylor scheme (1.5) by $\widetilde{\mathscr{E}}_{t_k,t_{k+1}\wedge t}^{(N)}\left(y_{t_k}^n\right)$, we have the same rate of convergence as that of the original one. Here

$$\widetilde{\mathscr{E}}_{s,t}^{(N)}(y) \quad := \quad \sum_{r=1}^{N} \sum_{(j_1,\ldots,j_r)\in\widetilde{\Gamma}_r} \mathscr{V}_{j_1} \cdots \mathscr{V}_{j_r} I(y) \int_s^t \int_s^{u_r} \cdots \int_s^{u_2} dx_{u_1}^{j_1} \cdots dx_{u_r}^{j_r}. \qquad (1.8)$$

We shall call such new Taylor scheme an *incomplete Taylor scheme*, which has the following form:

$$y_t^n \quad = \quad y_{t_k}^n + \widetilde{\mathscr{E}}_{t_k,t}^{(N)}(y_{t_k}^n), \quad t \in [t_k, t_{k+1}], \qquad (1.9)$$

for $k=0,1,\ldots,n-1$, with $y_0^n = y_0$.

We shall study the rate of convergence of y_t^n to y_t for any choice of $\widetilde{\Gamma}_r$ in (1.8). Two types of convergence will be studied in detail: almost sure convergence (when the x^j are Hölder continuous with exponents β_j) and the L_p-convergence (when the x^j are fractional Brownian motions of Hurst parameters H_j). The rates will be different for these two types of convergence. Fix a set

$$\widetilde{\Gamma} = \cup_{r=1}^{N} \widetilde{\Gamma}_r, \quad N = \max\{|\alpha| : \alpha \in \widetilde{\Gamma}\},$$

where throughout the chapter $|\alpha|$ denotes the length of the multi-index α. The almost sure rate $\theta_{\widetilde{\Gamma}}$ can be expressed in terms of β_j (see (4.2) below) and the L_p-rate $\rho_{\widetilde{\Gamma}}$ can be expressed in term of Hurst parameters H_j (see (6.3) below). These two expressions lead to the best choices of $\widetilde{\Gamma}_r$ in (1.8), depending on that one needs the almost sure convergence ($\widetilde{\Gamma}_r$ is given by (4.6) in Section 4) or one needs L_p-convergence ($\widetilde{\Gamma}_r$ is given by (6.12) in Section 6).

To motivate our second problem, let us recall that when the driving signals are fractional Brownian motions of Hurst parameter $H > 1/2$, the classical Euler scheme (1.6) has the exact convergence rate n^{1-2H} (see [13, 30]). When we formally equal H to $1/2$ (the standard Brownian motion case), we obtain no convergence! This demonstrates on one hand, that in dealing with the incomplete Taylor schemes we may not be able to use the same ideas from the Brownian motion case ([11, 22]). This is largely due to the lack of the martingale property of the driving signals. We will pay special attention to this fact. On the other hand, to improve the Euler scheme for the fractional Brownian motion case, a modified Euler scheme is proposed and investigated in [13]:

$$y_t^n = y_{t_k}^n + V(y_{t_k}^n)(x_t - x_{t_k}) + \frac{1}{2}\sum_{j=1}^{m}(\partial V_j V_j)(y_{t_k}^n)(t - t_k)^{2H}, \quad t \in [t_k, t_{k+1}], \qquad (1.10)$$

for $k = 0, 1, \dots, n-1$, with $y_0^n = y_0$. Here we denote $V = (V_1, \dots, V_m)$. It has been shown that this modified Euler scheme has a higher rate of convergence than the classical Euler scheme. In particular, it is proved in [13] that for any $t \in [0,T]$,

$$\mathbb{E}(|y_t - y_t^n|^p)^{1/p} \quad \leq \quad \begin{cases} Kn^{1/2-2H} & \text{if} \quad \frac{1}{2} < H < \frac{3}{4}\,, \\ Kn^{-1}\sqrt{\log n} & \text{if} \quad H = \frac{3}{4}\,, \\ Kn^{-1} & \text{if} \quad \frac{3}{4} < H < 1\,, \end{cases} \tag{1.11}$$

under proper regularity assumptions on V, where K is a constant independent of n.

The scheme (1.10) is obtained by adding to the classical Euler scheme (1.6) a deterministic term (note that for simplicity, we assume here that x is a standard m-dimensional fractional Brownian motion). The inclusion in (1.10) of the deterministic terms $\frac{1}{2}\sum_{j=1}^{m}(\partial V_j V_j)(y_{t_k}^n)(t-t_k)^{2H}$, as opposed to double integral terms as in (1.7), helps to save computation time due to the evaluation of double stochastic integrals. It is then natural to ask the following question:

Question 2. Can we add some deterministic terms to the incomplete Taylor scheme (1.9) so as to increase the rate of convergence?

We shall answer this question in the case when the x^j's are fractional Brownian motions or $x_t^j = t$ by introducing the following *modified Taylor scheme*:

$$y_t^n = y_{t_k}^n + \widetilde{\mathscr{E}}_{t_k,t}^{(N)}(y_{t_k}^n) + \sum_{(j_1,\dots,j_r)\in\Gamma'} \mathscr{V}_{j_1}\cdots\mathscr{V}_{j_r} I(y_{t_k}^n) D_{j_1,\dots,j_r}(t-t_k), \tag{1.12}$$

for $t \in [t_k, t_{k+1}]$, $k = 0,1,\dots,n-1$, $y_0^n = y_0$. The above set Γ' is a finite subset of $\Gamma \setminus \widetilde{\Gamma}$, where $\Gamma = \bigcup_{r=1}^{\infty}\Gamma_r$ and $\widetilde{\Gamma} = \bigcup_{r=1}^{N}\widetilde{\Gamma}_r$, that will be given explicitly in Section 4.6.2. The explicit form of $D_{j_1,\dots,j_r}(t)$, $t \in [0,T]$, is given in Remark 4.5.7.

The main tasks of this chapter are to establish the almost sure and the L_p-rate of convergence results for the incomplete Taylor scheme (1.9) and the modified Taylor scheme (1.12). It is worthy to emphasize that the modified Taylor scheme (1.12) has a higher L_p-rate of convergence than the incomplete Taylor scheme (1.9) (compare Theorem 4.6.3 with Theorem 4.6.1). We also point out that our result extends that of [13]: in the simplest case $N = 1$, our result recovers the upper bound estimate (1.11) (see Example 4.6.4).

The remainder of the Taylor expansion (1.2) has an involved expression (see [2]). If we throw some terms away, then the remainder is even more complicated. In the study of the convergence rate for the schemes (1.9) and (1.12) it is necessary to investigate this type of remainders. We shall express the error in the following form:

$$y_t - y_t^n = \Phi_t^{-1} \int_0^t \Phi_s dR_s, \tag{1.13}$$

where $\Phi \in \mathbb{R}^{d\times d}$ is the solution of a linear differential equation, Φ^{-1} is its inverse, that is, $\Phi\Phi^{-1} \equiv I$, and R_t is the remainder term, whose upper bound usually provides the desired convergence rate. The study of (1.13) is based on the algebraic properties of equation (1.1), which are interesting in its own right (see Section 4.2.1-4.2.3). It is well know that for $i_1, \dots, i_r, j_1, \dots, j_{r'} = 1, \dots, m$, the product $\int_s^t \cdots \int_s^{u_2} dx_{u_1}^{i_1} \cdots dx_{u_r}^{i_r}$ and $\int_s^t \cdots \int_s^{u_2} dx_{u_1}^{j_1} \cdots dx_{u_{r'}}^{j_{r'}}$ is equal to the summation of integrals of the form $\int_s^t \cdots \int_s^{u_2} dx_{u_1}^{l_1} \cdots dx_{u_{r+r'}}^{l_{r+r'}}$, where the summation runs over the multi-indices $(l_1, \dots, l_{r+r'})$ obtained by shuffling the two multi-indices $(i_1, \dots, i_r)$ and $(j_1, \dots, j_{r'})$. The study of the error function R_t needs an expansion of the multiple integral $\int_\tau^s \int_\tau^{s_p} \cdots \int_\tau^{s_2} dg_{\tau,s_1}^{\gamma^1} \cdots dg_{\tau,s_{p-1}}^{\gamma^{p-1}} dg_{\tau,s_p}^{\gamma^p}$, where each $g_{\tau,s}^{\gamma}$ is itself a multiple integral. This expansion of multiple integral of the multiple integrals can also be done by the shuffle-type permutations. A key ingredient in our proof is to establish a relation between these shuffle-type permutations with the permutations when we expand the iterated vector fields $\mathscr{V}_{j_1} \cdots \mathscr{V}_{j_r} I(y)$ through a generalized Leibniz rule (see Propositions 4.2.1, 4.2.2 and 4.2.3).

To obtain the rate of convergence for the modified Taylor scheme (1.12) we need some subtle L_2-estimates of a multiple Riemann-Stieltjes integral

$$J_r(\mathscr{A}) \;:=\; \int_0^T \cdots \int_0^T \mathbf{1}_{\mathscr{A}} dB^1_{s_1} \cdots dB^r_{s_r},$$

and its centralization

$$\widetilde{J}_r(\mathscr{A}) \;:=\; J_r(\mathscr{A}) - \mathbb{E}[J_r(\mathscr{A})],$$

for $\mathscr{A} = \bigcup_{k=0}^{n-1}\{(s_1,\ldots,s_r) : t_k \le s_1 < \cdots < s_r \le t_{k+1}\}$, where B^j, $j = 1,\ldots,r$ is either a fractional Brownian motion with Hurst parameter larger than $1/2$ or the identity function. Note that $J_r(\mathscr{A})$ is well defined as an integrated Riemann-Stieltjes integral. The L_2-estimates are made possible by a monotonicity property of the multiple integral obtained in Section 4.5.3; that is, for $\mathscr{A}' = \bigcup_{k=0}^{n-1}[t_k, t_{k+1}]^r$, the L_2-norms of $J_r(\mathscr{A})$ and $\widetilde{J}_r(\mathscr{A})$ are less than those of $J_r(\mathscr{A}')$ and $\widetilde{J}_r(\mathscr{A}')$, respectively (see Sections 4.5.3 and 4.5.4).

The chapter is organized as follows. In Section 4.2, we introduce some shuffle-type permutations and then apply them to expand the multiple integral of the multiple integrals and we also derive a generalized Leibniz rule for iterated vector fields. With these preparations we derive, in Section 4.3, an explicit expression for the error function for the scheme (1.9). In Section 4.4, we obtain the almost sure convergence rate for the scheme (1.9). In Section 4.5, we prove some L_p-estimate results. These estimates are applied to obtain the L_p-convergence rate for the incomplete scheme (1.9) in subsection 4.6.1 and the L_p-convergence rate for the modified Taylor scheme (1.12) in subsection 4.6.2. In Section 4.7, we generalize the results in Section 4.3 to the rough paths case.

In the appendix, we provide some necessary estimates of some multiple integrals and the solution of some differential equations.

Along the chapter we denote by C a generic constant, that may be different form line to line, and which might depend on T and the vector fields V_j^i.

4.2 Multiple integral of multiple integrals and generalized Leibniz rule

The primary aim of this section is to prove an identity on multiple integral of multiple integrals (see Proposition 4.2.2) and a generalized Leibniz rule (see Proposition 4.2.3). To do so, we need to introduce some shuffle-type permutations and their inverses (see Section 4.2.1).

4.2.1 Shuffle-type permutations and their inverses

Let $\alpha = (\alpha_1, \dots, \alpha_r) \in \Gamma_r$, where Γ_r is the collection of multi-indices of length r with elements in $\{1, \dots, m\}$. Take $\vec{l} = (l_1, \dots, l_p)$ such that $1 \le l_1 < \cdots < l_p = r$, $p \le r$. Assume that $f_j^i \in C^{r-p}(\mathbb{R})$, $i = 1, \dots, p$, $j = 1, \dots, m$. As a motivation, we first consider the following expression :

$$\mathscr{V}_{\alpha_{0,l_1}} \left(f^1_{\alpha_{l_1}} \cdots \mathscr{V}_{\alpha_{l_{p-2},l_{p-1}}} \left(f^{p-1}_{\alpha_{l_{p-1}}} \mathscr{V}_{\alpha_{l_{p-1},l_p}} f^p_{\alpha_{l_p}} \right) \right). \tag{2.1}$$

Here we denote $\alpha_{i,j} := (\alpha_{i+1}, \dots, \alpha_{j-1})$, $\mathscr{V}_{j_1,\dots,j_k} := \mathscr{V}_{j_1} \cdots \mathscr{V}_{j_k}$, and recall that $\mathscr{V}_j$ is the differential operator defined in (1.3). Note that the subindex of α in each element in (2.1), either an operator or a function, identifies the location of this element in (2.1). For example, $\mathscr{V}_{\alpha_j}$ is the jth element and $f^i_{\alpha_{l_i}}$ is the l_ith element.

It is easy to verify that $\mathscr{V}_j$ satisfies the product rule, that is, $\mathscr{V}_j(fg) = g\mathscr{V}_j f + f\mathscr{V}_j g$ for $f, g \in C^1$. By applying the product rule to (2.1), the operators $\mathscr{V}_{\alpha_j}$, $j \in \{1,\dots,r\} \setminus \{l_1,\dots,l_p\}$ act on functions $f^i_{\alpha_{l_i}}$, $i = 1,\dots,p$, in such a way that:

(i) for j such that $l_{i-1} < j < l_i$, the operator $\mathscr{V}_{\alpha_j}$ act on one of the functions $f^i_{\alpha_i}$, ..., $f^p_{\alpha_p}$;

(ii) if two operators $\mathscr{V}_i$ and $\mathscr{V}_j$ act on the same function $f^k_{\alpha_{l_k}}$ then their order in (2.1) is kept.

Note that we take $l_0 = 0$ in (i). The quantity (2.1) is then expanded into the summation of quantities of the following form,

$$\left(\mathscr{V}_{\alpha'_{0,\tau_1}} f^1_{\alpha'_{\tau_1}}\right) \cdots \left(\mathscr{V}_{\alpha'_{\tau_{p-1},\tau_p}} f^p_{\alpha'_{\tau_p}}\right), \tag{2.2}$$

where α' is some permutation of α such that $\alpha'_{\tau_i} = \alpha_{l_i}$, $i = 1,\dots,p$, and $(\tau_1,\dots,\tau_p)$ are constants such that $1 \le \tau_1 < \cdots < \tau_p = r$. Denote by $\mu(i)$ the new location of the ith element of (2.1) in (2.2), then μ is the permutation of $\{1,2,\dots,r\}$ such that $\alpha = \alpha' \circ \mu$. In particular, we have $\mu(l_i) = \tau_i$, $i = 1,\dots,p$. Each quantity of the form (2.2) obtained from applying the product rule to (2.1) is then identified with a permutation μ on $\{1,\dots,r\}$. It is easy to see that these permutations satisfy:

Rule 1. $\mu(l_i) < \mu(l_{i+1})$, $i = 1,\dots,p$;

Rule 2. $\mu(y) < \mu(y')$ *if* $y < y'$ *and* $\mu(y), \mu(y') \in I_i$ *for some* i*, where* $(I_i,\ i = 1,\dots,p)$ *is the partition of* $\{1,\dots,r\}$ *defined as follows*

$$I_1 = \{1,\dots,\mu(l_1)\};\;\; I_i = \{\mu(l_{i-1})+1,\dots,\mu(l_i)\},\;\; i = 2,\dots,p. \tag{2.3}$$

In fact, Rule 2 is the "translation" of condition (ii) in terms of μ and Rule 1 is required to fix the ordering of the terms $\left(\mathscr{V}_{\alpha'_{\tau_{i-1},\tau_i}} f^1_{\alpha'_{\tau_i}}\right)$, $i = 1,\dots,p$, in (2.2).

The "translation" of condition (i) in terms of μ is:

Rule 3. $\mu(l_{i-1}) < \mu(y)$ *if* $l_{i-1} < y < l_i$.

This rule is implied by Rule 1 and 2:

Lemma 4.2.1. *Assume that* μ *is a permutation of* $\{1,\dots,r\}$ *that satisfies Rule 1 and 2. Then* μ *also satisfies Rule 3.*

Proof We take y such that $l_{i-1} < y$. Since μ is a bijection, we have $\mu(l_{i-1}) \neq \mu(y)$. Suppose that $\mu(l_{i-1}) > \mu(y)$. Then there exists j such that $\mu(l_j), \mu(y) \in I_j$ and $\mu(l_{i-1}) \geq \mu(l_j) > \mu(y)$. By Rule 2 we have $l_j > y$. On the other hand, Rule 1 implies that $l_{i-1} \geq l_j$. So we obtain $l_{i-1} > y$, which contradicts the assumption.

We are ready to define the shuffle-type permutations.

Definition 4.2.1. *Take* $\vec{l} = (l_1,\dots,l_p)$ *such that* $1 \leq l_1 < \cdots < l_p = r$. *We define* $\Theta_r(\vec{l})$ *as the collection of all permutations of* $\{1,\dots,r\}$ *that satisfy Rule 1 and 2. Take* $\vec{\tau} = (\tau_1,\dots,\tau_p)$ *such that* $1 \leq \tau_1 < \cdots < \tau_p = r$. *We define* $\Theta_r(\vec{l};\vec{\tau})$ *as the collection of permutations in* $\Theta_r(\vec{l})$ *such that* $\mu(l_i) = \tau_i$, $i = 1,\dots,p$.

Note that the set $\Theta_r(\vec{l};\vec{\tau})$ could be empty for some $l_1,\dots,l_p$ and $\tau_1,\dots,\tau_p$.

According to the above discussion, we have the following result:

Lemma 4.2.2. *Take* $\alpha = (\alpha_1,\dots,\alpha_r) \in \Gamma_r$, $\vec{l} = (l_1,\dots,l_p)$ *such that* $1 \leq l_1 < \cdots < l_p = r$ *and* $f^i_j \in C^{r-p}$, $i = 1,\dots,p$, $j = 1,\dots,m$. *Then the quantity (2.1) is equal to*

$$\sum_{\mu\in\Theta_r(\vec{l})} \left(\mathscr{V}_{\alpha\circ\mu^{-1}(0,\mu(l_1))} f^1_{\alpha_{l_1}}\right) \cdots \left(\mathscr{V}_{\alpha\circ\mu^{-1}(\mu(l_{p-1}),\mu(l_p))} f^p_{\alpha_{l_p}}\right)$$

or

$$\sum_{1\leq\tau_1<\cdots<\tau_p=r} \sum_{\mu\in\Theta_r(\vec{l};\vec{\tau})} \left(\mathscr{V}_{\alpha\circ\mu^{-1}(0,\tau_1)} f^1_{\alpha\circ\mu^{-1}(\tau_1)}\right) \cdots \left(\mathscr{V}_{\alpha\circ\mu^{-1}(\tau_{p-1},\tau_p)} f^p_{\alpha\circ\mu^{-1}(\tau_p)}\right).$$

The proof of the lemma is omitted. Please note that in the above summation, when $\Theta_r(\vec{l};\vec{\tau}) = \emptyset$, we follow the convention that a summation over the empty set is 0.

We introduce another type of permutations, which will be the inverses of those in $\Theta_r(\vec{l})$. Let $\tau_0 = 0$ and $\vec{\tau} = (\tau_1, \ldots, \tau_p)$ be the same as in Definition 4.2.1, and define the partition on $\{1, \ldots, r\}$ given by

$$I_i = \{\tau_{i-1} + 1, \ldots, \tau_i\}, \ \ i = 1, \ldots, p. \tag{2.4}$$

Definition 4.2.2. *Let $\vec{l}$ and $\vec{\tau}$ be as in Definition 4.2.1. We define $\Xi_r(\vec{\tau})$ as the collection of permutations ρ on $\{1,2,\ldots,r\}$ such that ρ keeps the ordering of $\tau_1, \ldots, \tau_p$, i.e. the last elements of $I_1, \ldots, I_p$, and the order of the elements in each I_i. In other words, $\rho \in \Xi_r(\vec{\tau})$ iff ρ satisfies*

Rule 4. $\rho(\tau_i) < \rho(\tau_j)$ *if* $i < j$;

Rule 5. $\rho(y) < \rho(y')$ *if* $y, y' \in I_i$ *and* $y < y'$.

We define $\Xi_r(\vec{l};\vec{\tau})$ as the collection of permutations ρ in $\Xi_r(\vec{\tau})$ such that $\rho(\tau_i) = l_i$, $i = 1, \ldots, p$.

Note that for $\rho \in \Xi_r(\vec{\tau})$ we always have $\rho(\tau_p) = \rho(r) = r$.

The following proposition shows that the permutations introduced in Definitions 4.2.1 and 4.2.2 are inverses of each other.

Proposition 4.2.1. *Let $\vec{l}$ and $\vec{\tau}$ be as in Definition 4.2.1. Suppose that at least one of the two sets $\Xi_r(\vec{l};\vec{\tau})$ and $\Theta_r(\vec{l};\vec{\tau})$ is not empty. Then the following holds,*

$$\Xi_r(\vec{l};\vec{\tau}) = \{\rho : \rho^{-1} \in \Theta_r(\vec{l};\vec{\tau})\}. \tag{2.5}$$

Remark 4.2.1. *It follows from Proposition 4.2.1 that* $\Xi_r(\vec{l};\vec{\tau}) = \emptyset$ *if and only if* $\Theta_r(\vec{l};\vec{\tau}) = \emptyset$.

Remark 4.2.2. *Equation (2.5) is equivalent to the following,*

$$\Theta_r(\vec{l};\vec{\tau}) = \{\mu : \mu^{-1} \in \Xi_r(\vec{l};\vec{\tau})\}.$$

Proof of Proposition 4.2.1: We first note that the partitions $(I_i, i = 1, \ldots, p)$ defined in (2.4) and in (2.3) are the same. Take $\rho \in \Xi_r(\vec{l};\vec{\tau})$ and denote $\mu := \rho^{-1}$. We show that $\mu \in \Theta_r(\vec{l};\vec{\tau})$. It is clear that μ satisfies Rule 1. Take y, y' such that $y < y'$ and $\mu(y)$, $\mu(y') \in I_i$. We have

$$\rho(\mu(y)) = y < y' = \rho(\mu(y')).$$

So Rule 5 in the definition of $\Xi_r(\vec{\tau})$ implies that $\mu(y) < \mu(y')$. This shows that μ satisfies Rule 2. We conclude that μ belongs to the right-hand side of (2.5). We take ρ such that $\rho^{-1} =: \mu \in \Theta_r(\vec{l};\vec{\tau})$. Since

$$\rho(\tau_i) = \mu^{-1}(\tau_i) = l_i < l_j = \mu^{-1}(\tau_j) = \rho(\tau_j)$$

for $i < j$, ρ satisfies Rule 4. Take $y, y' \in I_i$ such that $y < y'$. From Rule 2, it is easy to see that

$$\mu^{-1}(y) < \mu^{-1}(y'),$$

that is, $\rho(y) < \rho(y')$. So ρ satisfies Rule 5. We conclude that $\rho \in \Xi_r(\vec{l};\vec{\tau})$. □

4.2.2 Multiple integrals

Let $g = (g^1, \ldots, g^m)$ be a Hölder continuous function on $[0,T]$ of order $\beta > 1/2$ with values in $\mathbb{R}^m$. Take $\alpha = (\alpha_1, \ldots, \alpha_r) \in \Gamma_r$. Recall that we denote by Γ_r the collection of multi-indices of length r with elements in $\{1,\ldots,m\}$. We also denote $\Gamma = \cup_{r=1}^{\infty}\Gamma_r$ the collection of all multi-indices with elements in $\{1,\ldots,m\}$. Recall that $|\gamma|$ is the length of the multi-index γ. Given a permutation ρ on $\{1,\ldots,r\}$, denote $\alpha\circ\rho = (\alpha_{\rho(1)},\ldots,\alpha_{\rho(r)}) \in \Gamma_r$.

In this subsection, we study multiple integrals, defined as iterated Riemann-Stieltjes integrals, of the form

$$g^{\alpha}_{\tau,s} \;\; := \;\; \int_{\tau}^{s}\int_{\tau}^{s_r}\cdots\int_{\tau}^{s_2} dg^{\alpha_1}_{s_1}\cdots dg^{\alpha_{r-1}}_{s_{r-1}}dg^{\alpha_r}_{s_r}, \tag{2.6}$$

where $0 \le \tau \le s \le T$. We define the differential of $g^{\alpha}_{\tau,s}$ by

$$dg^{\alpha}_{\tau,s} = \left(\int_{\tau}^{s}\cdots\int_{\tau}^{s_2} dg^{\alpha_1}_{s_1}\cdots dg^{\alpha_{r-1}}_{s_{r-1}}\right)dg^{\alpha_r}_{s}. \tag{2.7}$$

The following lemma gives a formula for the product of two such multiple integrals.

Lemma 4.2.3. *Let γ', γ'' be multi-indices in Γ and denote $r' = |\gamma'|$, $r'' = |\gamma''|$ and $r = |\gamma'|+|\gamma''|$. Denote $\gamma = (\gamma',\gamma'') \in \Gamma$. Then*

$$g^{\gamma'}_{\tau,s}g^{\gamma''}_{\tau,s} = \sum_{\rho\in Sh(\gamma',\gamma'')} g^{\gamma\circ\rho^{-1}}_{\tau,s},$$

where $Sh(\gamma',\gamma'')$ is the collection of permutations ρ on $\{1,\ldots,r\}$ such that ρ does not change the orderings of $(1,\ldots,r')$ and the orderings of $(r'+1,\ldots,r)$, that is, if $y,y' \in \{1,\ldots,r'\}$ or $y,y' \in \{r'+1,\ldots,r\}$, and $y < y'$, then we have $\rho(y) < \rho(y')$.

This result can be shown by the properties of shuffle product of words (see, for example [42]) and Fubini's theorem.

The following is an immediate corollary of the above result.

Lemma 4.2.4. *Let γ', γ'', r', r and γ be as in Lemma 4.2.3. Then*

$$\int_\tau^s \int_\tau^{s''} dg^{\gamma'}_{\tau,s'} dg^{\gamma''}_{\tau,s''} = \int_\tau^s g^{\gamma'}_{\tau,s''} dg^{\gamma''}_{\tau,s''} \quad = \sum_{\rho \in \Xi_r(r',r)} g^{\gamma\circ\rho^{-1}}_{\tau,s}. \tag{2.8}$$

Proof Using (2.7), we can rewrite the left-hand side of (2.8) as

$$\int_\tau^s g^{\gamma'}_{\tau,s''} g^{\gamma''-}_{\tau,s''} dg^{\gamma(r)}_{s''},$$

where we denote by $\gamma-$ the multi-index obtained by removing the last element of γ, that is, $\gamma- = (\gamma_1, \ldots, \gamma_{r-1})$, and recall that $\gamma(i) = \gamma_i$ denotes the ith element of γ. Applying Lemma 4.2.3 to $g^{\gamma'}_{\tau,s''} g^{\gamma''-}_{\tau,s''}$ yields

$$\int_\tau^s \int_\tau^{s''} dg^{\gamma'}_{\tau,s'} dg^{\gamma''}_{\tau,s''} \quad = \quad \int_\tau^s \sum_{\rho \in Sh(\gamma',\gamma''-)} g^{(\gamma-)\circ\rho^{-1}}_{\tau,s''} dg^{\gamma(r)}_{s''}. \tag{2.9}$$

Denote by $\widetilde{\Xi}_r(r',r)$ the collection of permutations ρ on $\{1,\ldots,r\}$ such that there exists $\rho' \in Sh(\gamma',\gamma''-)$ such that

$$\rho(j) = \begin{cases} \rho'(j), & j = 1,\ldots,r-1, \\ r, & j = r. \end{cases}$$

Then (2.9) becomes

$$\sum_{\rho \in \widetilde{\Xi}_r(r',r)} g^{\gamma\circ\rho^{-1}}_{\tau,s}.$$

Equation (2.8) then follows by noticing that $\widetilde{\Xi}_r(r',r) = \Xi_r(r',r)$. □

The following result is a generalization of Lemma 4.2.4.

Proposition 4.2.2. *Let γ^1, ..., γ^p be multi-indices in Γ, and denote $r = |\gamma^1| + \cdots + |\gamma^p|$ and $\tau_i = |\gamma^1| + \cdots + |\gamma^i|$, $i = 1, \ldots, p$. Denote $\gamma = (\gamma^1, \ldots, \gamma^p) \in \Gamma$. Then*

$$\int_\tau^s \int_\tau^{s_p} \cdots \int_\tau^{s_2} dg_{\tau,s_1}^{\gamma^1} \cdots dg_{\tau,s_{p-1}}^{\gamma^{p-1}} dg_{\tau,s_p}^{\gamma^p} = \sum_{\rho \in \Xi_r(\tau_1,\ldots,\tau_p)} g_{\tau,s}^{\gamma\circ\rho^{-1}}. \tag{2.10}$$

Proof We prove the proposition by induction on p. The proposition is clearly true when $p = 1$, and by Lemma 4.2.4 it is true when $p = 2$. Take $p \geq 3$. Assuming that (2.10) holds for $p - 1$, we can write

$$\int_\tau^s \int_\tau^{s_p} \cdots \int_\tau^{s_2} dg_{\tau,s_1}^{\gamma^1} \cdots dg_{\tau,s_{p-1}}^{\gamma^{p-1}} dg_{\tau,s_p}^{\gamma^p} = \sum_{\widetilde{\rho} \in \Xi_{\widetilde{r}}(\tau_1,\ldots,\tau_{p-1})} \int_\tau^s g_{\tau,s_p}^{\widetilde{\gamma}\circ\widetilde{\rho}^{-1}} dg_{\tau,s_p}^{\gamma^p}, \tag{2.11}$$

where $\widetilde{\gamma} = (\gamma^1, \ldots, \gamma^{p-1})$ and $\widetilde{r} = \tau_{p-1}$. Applying Lemma 4.2.4 to the right-hand side of (2.11) we have

$$\int_\tau^s \int_\tau^{s_p} \cdots \int_\tau^{s_2} dg_{\tau,s_1}^{\gamma^1} \cdots dg_{\tau,s_{p-1}}^{\gamma^{p-1}} dg_{\tau,s_p}^{\gamma^p} = \sum_{\widetilde{\rho} \in \Xi_{\widetilde{r}}(\tau_1,\ldots,\tau_{p-1})} \sum_{\widehat{\rho} \in \Xi_r(\tau_{p-1},\tau_p)} g_{\tau,s}^{(\widetilde{\gamma}\circ\widetilde{\rho}^{-1},\gamma^p)\circ\widehat{\rho}^{-1}} \tag{2.12}$$

For each $\widetilde{\rho} \in \Xi_{\widetilde{r}}(\tau_1, \ldots, \tau_{p-1})$ and $\widehat{\rho} \in \Xi_r(\tau_{p-1}, \tau_p)$, we define a permutation $\rho = \rho_{\widetilde{\rho},\widehat{\rho}}$ on $\{1, \ldots, r\}$ such that

$$\rho(j) = \begin{cases} \widehat{\rho}(j), & j = \tau_{p-1} + 1, \ldots, \tau_p, \\ \widehat{\rho}(\widetilde{\rho}(j)), & j = 1, \ldots, \tau_{p-1}. \end{cases}$$

Then (2.12) is equal to

$$\sum_{\widetilde{\rho} \in \Xi_{\widetilde{r}}(\tau_1,\ldots,\tau_{p-1})} \sum_{\widehat{\rho} \in \Xi_r(\tau_{p-1},\tau_p)} g_{\tau,s}^{\gamma\circ\rho_{\widetilde{\rho},\widehat{\rho}}^{-1}}. \tag{2.13}$$

We show that

$$\Xi_r(\vec{\tau}) \;=\; \left\{\rho_{\widetilde{\rho},\widehat{\rho}} : \widetilde{\rho} \in \Xi_{\widetilde{r}}(\tau_1,\ldots,\tau_{p-1}), \widehat{\rho} \in \Xi_r(\tau_{p-1},\tau_p)\right\}. \tag{2.14}$$

It is easy to see that $\Xi_r(\vec{\tau})$ is included on the right-hand side of (2.14), that is, for each $\rho \in \Xi_r(\vec{\tau})$, we can find $\widetilde{\rho} \in \Xi_{\widetilde{r}}(\tau_1,\ldots,\tau_{p-1})$ and $\widehat{\rho} \in \Xi_r(\tau_{p-1},\tau_p)$ such that $\rho = \rho_{\widetilde{\rho},\widehat{\rho}}$.

In the following, we show the other inclusion. We take $\rho = \rho_{\widetilde{\rho},\widehat{\rho}}$ from the right-hand side of (2.14). Rule 4 for $\widetilde{\rho}$ implies that $\widetilde{\rho}(\tau_i) < \widetilde{\rho}(\tau_j)$ for $i,j = 1,\ldots,p-1$ and $i < j$. This fact and Rule 5 for $\widehat{\rho}$ imply that

$$\rho(\tau_i) = (\widehat{\rho}\circ\widetilde{\rho})(\tau_i) < (\widehat{\rho}\circ\widetilde{\rho})(\tau_j) = \rho(\tau_j)$$

for $i,j = 1,\ldots,p-1$ and $i < j$. On the other hand, Rule 4 for $\widehat{\rho}$ implies that

$$\rho(\tau_{p-1}) = \widehat{\rho}(\widetilde{\rho}(\tau_{p-1})) = \widehat{\rho}(\tau_{p-1}) < \widehat{\rho}(\tau_p) = \rho(\tau_p).$$

Therefore, ρ satisfies Rule 4 in the definition of $\Xi_r(\vec{\tau})$. Take now $y < y'$ and $y,y' \in I_i$, $i = 1,\ldots,p-1$. By Rule 5 for $\widetilde{\rho}$, we have $\widetilde{\rho}(y) < \widetilde{\rho}(y')$, and thus $\rho(y) = \widehat{\rho}(\widetilde{\rho}(y)) < \widehat{\rho}(\widetilde{\rho}(y')) = \rho(y')$. On the other hand, if $y < y'$ and $y,y' \in I_p$, then by Rule 5 in the definition of $\widehat{\rho}$, we have $\rho(y) = \widehat{\rho}(y) < \widehat{\rho}(y') = \rho(y')$. We conclude that ρ satisfies Rule 5 in the definition of $\Xi_r(\vec{\tau})$. In summary, we have shown that $\rho \in \Xi_r(\vec{\tau})$. This proves identity (2.14).

It is easy to show that there is no duplicated element in the set on the right-hand side of (2.14), that is, whenever $\widetilde{\rho} \neq \widetilde{\rho}'$ or $\widehat{\rho} \neq \widehat{\rho}'$, we have $\rho_{\widetilde{\rho},\widehat{\rho}} \neq \rho_{\widetilde{\rho}',\widehat{\rho}'}$. This fact, together with identity (2.14), imply that (2.13) is equal to

$$\sum_{\rho\in\Xi_r(\tau_1,\ldots,\tau_p)} g_{\tau,s}^{\gamma\circ\rho^{-1}}.$$

This completes the proof.

4.2.3 A generalized Leibniz rule

Using the permutation set $\Xi_r(\vec{\tau})$, we can state the following Leibniz rule. Recall that given a multi-index $\alpha=(\alpha_1,\ldots,\alpha_r)$ we denote $\alpha_{i,j}=(\alpha(i+1),\ldots,\alpha(j-1))$. We also use $\mathscr{V}_{j_1,\ldots,j_k}:=\mathscr{V}_{j_1}\cdots\mathscr{V}_{j_k}$, where $\mathscr{V}_j$ is defined in (1.3).

Proposition 4.2.3. *Take $\alpha=(\alpha_1,\ldots,\alpha_r)\in\Gamma_r$, $\vec{l}=(l_1,\ldots,l_p)$ such that $1\le l_1<\cdots<l_p=r$, $p\le r$. Assume that $f_j^i\in C^{r-p}(\mathbb{R})$, $i=1,\ldots,p$, $j=1,\ldots,m$. Then the following Leibniz rule holds:*

$$\begin{aligned}&\mathscr{V}_{\alpha_{0,l_1}}\left(f^1_{\alpha_{l_1}}\cdots\mathscr{V}_{\alpha_{l_{p-2},l_{p-1}}}\left(f^{p-1}_{\alpha_{l_{p-1}}}\mathscr{V}_{\alpha_{l_{p-1},l_p}}f^p_{\alpha_{l_p}}\right)\right)\\&=\sum_{1\le\tau_1<\cdots<\tau_p=r}\sum_{\rho\in\Xi_r(\vec{l};\vec{\tau})}\left(\mathscr{V}_{\alpha\circ\rho(0,\tau_1)}f^1_{\alpha\circ\rho(\tau_1)}\right)\cdots\left(\mathscr{V}_{\alpha\circ\rho(\tau_{p-1},\tau_p)}f^p_{\alpha\circ\rho(\tau_p)}\right).\end{aligned}$$

Proof The above formula follows from Lemma 4.2.2 and Proposition 4.2.1.

4.3 The error function

The objective of this section is to derive an explicit expression for the remainder in the incomplete Taylor scheme (1.9). Let y be the solution of the differential equation

$$dy_t \;=\; V(y_t)dx_t, \quad y_0\in\mathbb{R}^d, \tag{3.1}$$

on $[0,T]$, where $x:[0,T]\to\mathbb{R}^m$ is Hölder continuous of order $\beta>1/2$ and $V=(V_j^i)_{1\le i\le d,1\le j\le m}$ is a continuous mapping from $\mathbb{R}^d$ to $\mathbb{R}^{d\times m}$. We consider the Taylor scheme

$$y_t^n \;=\; y_{t_k}^n+\mathscr{E}_{t_k,t}^{(N)}(y_{t_k}^n), \quad y_0^n=y_0, \tag{3.2}$$

for $t\in[t_k,t_{k+1}]$, $k=0,1,\dots,n-1$, where

$$\mathscr{E}_{s,t}^{(N)}(y) \;:=\; \sum_{r=1}^{N}\sum_{\gamma\in\Gamma_r}\mathscr{V}_\gamma I(y)x_{s,t}^{\gamma}, \quad y\in\mathbb{R}^d \tag{3.3}$$

is the order-N Taylor expansion and $\mathscr{V}_\gamma:=\mathscr{V}_{\gamma(1)}\cdots\mathscr{V}_{\gamma(r)}$ for $\gamma=(\gamma(1),\dots,\gamma(r))\in\Gamma_r$ (the collection of multi-indices of length r with elements in $\{1,\dots,m\}$). Recall that I is the identity function on $\mathbb{R}^d$, the vector field $\mathscr{V}_j$ is defined in (1.3) and for $\gamma\in\Gamma_r$ we denote (see (2.6))

$$x_{s,t}^{\gamma}:=\int_s^t\cdots\int_s^{t_2}dx_{t_1}^{\gamma(1)}\cdots dx_{t_r}^{\gamma(r)}.$$

By (3.1) we can write

$$y_t-y_t^n \;=\; \int_0^t[V(y_s)-V(y_s^n)]\,dx_s+R_t, \tag{3.4}$$

where $R_t:=\int_0^t V(y_s^n)dx_s-y_t^n$ is the remainder term. If we can find a function $\dot{V}_j(\xi,\xi')$ on $\mathbb{R}^d\times\mathbb{R}^d$ (see (3.27)) such that

$$V_j(y_t)-V_j(y_t^n)=\dot{V}_j(y_t,y_t^n)(y_t-y_t^n),$$

$j = 1, \ldots, m$, then equation (3.4) can be considered as a linear differential equation for the error function $y - y^n$. Our aim in this section is to derive an explicit expression for the remainder $R_t = \int_0^t V(y_s^n) dx_s - y_t^n$.

4.3.1 A linear differential equation for the error function

We first consider the differential operator $\mathscr{V}_\gamma$ appearing in (3.3). We denote by Υ_p the collection of multi-indices of length p with elements in $\{1, \ldots, d\}$ and $\Upsilon = \cup_{p=1}^\infty \Upsilon_p$. For $\zeta \in \Upsilon_p$, we denote

$$\partial_\zeta \;:=\; \partial_{\zeta_1} \partial_{\zeta_2} \cdots \partial_{\zeta_p} ,$$

where ζ_j is the jth element of ζ and recall that $\partial_i = \frac{\partial}{\partial y^i}$. For $\alpha \in \Gamma_r$ such that $p \leq r$, $\vec{\tau} = (\tau_1, \ldots, \tau_p)$ such that $1 \leq \tau_1 < \cdots < \tau_p = r$, and $y \in \mathbb{R}^d$ we introduce the function

$$\begin{aligned} H(\alpha, \zeta, \vec{\tau})(y) &= \left(\mathscr{V}_{\alpha_{0,\tau_1}} V_{\alpha_{\tau_1}}^{\zeta_1}(y) \right) \cdots \left(\mathscr{V}_{\alpha_{\tau_{p-1},\tau_p}} V_{\alpha_{\tau_p}}^{\zeta_p}(y) \right) \\ &= \left(\mathscr{V}_{\alpha_{0,\tau_1+1}} I_{\zeta_1}(y) \right) \cdots \left(\mathscr{V}_{\alpha_{\tau_{p-1},\tau_p+1}} I_{\zeta_p}(y) \right) , \end{aligned} \tag{3.5}$$

where $I_i(y) = y_i$, $i = 1, \ldots, d$ is the projection function, and recall that given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_r)$, we denote $\alpha_{i,j} = (\alpha_{i+1}, \ldots, \alpha_{j-1})$. Note that the second equation in (3.5) follows from the identity $\mathscr{V}_j I_i(y) = V_j^i(y)$.

Lemma 4.3.1. *Let $f \in C^r(\mathbb{R}^d)$ and $\alpha \in \Gamma_r$. Then the following identity holds true:*

$$\mathscr{V}_\alpha f \;=\; \sum_{p=1}^r \sum_{\zeta \in \Upsilon_p} \sum_{\substack{\vec{\tau}=(\tau_1,\ldots,\tau_p): \\ 1 \leq \tau_1 < \cdots < \tau_p = r}} \sum_{\rho \in \Xi_r(\vec{\tau})} H(\alpha \circ \rho, \zeta, \vec{\tau}) \partial_\zeta f . \tag{3.6}$$

Proof It is easy to show by induction and by the definition of the vector field $\mathscr{V}_j$ that

$$\mathscr{V}_\alpha f = \sum_{p=1}^{r} \sum_{\zeta \in \Upsilon_p} \sum_{1 \le l_1 < \cdots < l_p = r} \mathscr{V}_{\alpha_{0,l_1}} \left(V_{\alpha_{l_1}}^{\zeta_1} \cdots \mathscr{V}_{\alpha_{l_{p-2},l_{p-1}}} \left(V_{\alpha_{l_{p-1}}}^{\zeta_{p-1}} \mathscr{V}_{\alpha_{l_{p-1},l_p}} V_{\alpha_{l_p}}^{\zeta_p} \right) \right) \partial_\zeta f.$$

Applying Proposition 4.2.3 to the above expression yields

$$\mathscr{V}_\alpha f = \sum_{p=1}^{r} \sum_{\zeta \in \Upsilon_p} \sum_{\substack{1 \le l_1 < \cdots < l_p = r \\ 1 \le \tau_1 < \cdots < \tau_p = r}} \sum_{\rho \in \Xi_r(\vec{l};\vec{\tau})} H(\alpha \circ \rho, \zeta, \vec{\tau}) \partial_\zeta f \,.$$

Equation (3.6) is then obtained by noticing that the following two sets are identical:

$$\Xi_r(\vec{\tau}) = \bigcup_{\substack{\vec{l}=(l_1,\ldots,l_p): \\ 1 \le l_1 < \cdots < l_p = r.}} \Xi_r(\vec{l}, \vec{\tau}).$$

This completes the proof.

Take $\zeta \in \Upsilon_p$. We denote by $\mathscr{E}_{s,t}^{\zeta}$ the multiple integral

$$\mathscr{E}_{s,t}^{\zeta} = \int_s^t \int_s^{t_r} \cdots \int_s^{t_2} d\mathscr{E}_{s,t_1}^{\zeta_1} \cdots d\mathscr{E}_{s,t_{r-1}}^{\zeta_{p-1}} d\mathscr{E}_{s,t_r}^{\zeta_p}, \tag{3.7}$$

where

$$\mathscr{E}_{s,t}^{i}(y) = \sum_{r=1}^{N} \sum_{\gamma \in \Gamma_r} \mathscr{V}_\gamma I_i(y) x_{s,t}^{\gamma}\,, \quad y \in \mathbb{R}^d. \tag{3.8}$$

According to Proposition 4.2.2, the multiple integral $\mathscr{E}_{s,t}^{\zeta}$ can be expressed as a linear combination of elements in $\{x_{s,t}^{\alpha}, \alpha \in \Gamma\}$. Recall that $\Gamma = \cup_{r=1}^{\infty} \Gamma_r$ is the collection of multi-indices with elements in $\{1, \ldots, m\}$. The following lemma provides an explicit formula for this linear combination. Recall that for a permutation ρ on $\{1, \ldots, r\}$, we denote $\alpha \circ \rho = (\alpha_{\rho(1)}, \ldots, \alpha_{\rho(r)})$.

Lemma 4.3.2. *Take $p \in \mathbb{N}$ and $\zeta \in \Upsilon_p$. Assume that $V \in C^{N-1}$. Then for each $s,t \in [0,T]$ we have*

$$\mathscr{E}^{\zeta}_{s,t} = \sum_{r=p}^{N} \sum_{\substack{\vec{\tau}=(\tau_1,\dots,\tau_p):\\ 1\le\tau_1<\dots<\tau_p=r}} \sum_{\alpha\in\Gamma_r} \sum_{\rho\in\Xi_r(\vec{\tau})} H(\alpha\circ\rho,\zeta,\vec{\tau})x^{\alpha}_{s,t} + Q(N,p,\zeta)_{s,t}, \quad (3.9)$$

where

$$Q(N,p,\zeta)_{s,t} = \sum_{\substack{\gamma^1,\dots,\gamma^p\in\Gamma:\\ |\gamma^1|+\dots+|\gamma^p|>N\\ |\gamma^1|,\dots,|\gamma^p|\le N}} \sum_{\rho\in\Xi_{|\gamma|}(\vec{\tau}(\gamma))} H(\gamma,\zeta,\vec{\tau}(\gamma))x^{\gamma\circ\rho^{-1}}_{s,t},$$

$\gamma = (\gamma^1,\dots,\gamma^p)$, $\vec{\tau}(\gamma) = (\tau_1(\gamma),\dots,\tau_p(\gamma))$ *and* $\tau_i(\gamma) = |\gamma^1|+\dots+|\gamma^i|$, $i = 1,\dots,p$.

Proof It follows from (3.7) and (3.8) that

$$\mathscr{E}^{\zeta}_{s,t}(y) = \sum_{\substack{\gamma^1,\dots,\gamma^p\in\Gamma:\\ 1\le|\gamma^1|,\dots,|\gamma^p|\le N}} \mathscr{V}_{\gamma^1}I_{\zeta_1}(y)\cdots\mathscr{V}_{\gamma^p}I_{\zeta_p}(y)\int_s^t\int_s^{t_p}\cdots\int_s^{t_2} dx^{\gamma^1}_{t_1}\cdots dx^{\gamma^{p-1}}_{t_{p-1}}dx^{\gamma^p}_{t_p}.$$

We recall the notation $\gamma = (\gamma^1,\dots,\gamma^p)$, and $\tau_i(\gamma) = |\gamma^1|+\dots+|\gamma^i|$, $i = 1,\dots,p$. It follows from Proposition 4.2.2 and the definition of the function H in (3.5) that

$$\mathscr{E}^{\zeta}_{s,t}(y) = \sum_{\substack{\gamma^1,\dots,\gamma^p\in\Gamma:\\ 1\le|\gamma^1|,\dots,|\gamma^p|\le N}} H(\gamma,\zeta,\vec{\tau}(\gamma))(y) \sum_{\rho\in\Xi_{|\gamma|}(\vec{\tau}(\gamma))} x^{\gamma\circ\rho^{-1}}_{s,t},$$

or

$$\mathscr{E}^{\zeta}_{s,t}(y) = \left(\sum_{r=p}^{N}\sum_{|\gamma^1|+\dots+|\gamma^p|=r} + \sum_{\substack{|\gamma^1|+\dots+|\gamma^p|>N\\ |\gamma^1|,\dots,|\gamma^p|\le N}}\right) \sum_{\rho\in\Xi_{|\gamma|}(\vec{\tau}(\gamma))} H(\gamma,\zeta,\vec{\tau}(\gamma))(y)x^{\gamma\circ\rho^{-1}}_{s,t} \quad (3.10)$$

The second term in the above summation is exactly $Q(N,p,\zeta)_{s,t}(y)$. On the other hand, since

$$\{(\gamma^1,\dots,\gamma^p):\sum_{i=1}^p|\gamma^i|=r\} \quad = \bigcup_{\substack{\tau_1,\dots,\tau_p:\\ 1\le\tau_1<\cdots<\tau_p=r}} \{(\gamma^1,\dots,\gamma^p):|\gamma^i|=\tau_i-\tau_{i-1}, i=1,\dots,p\},$$

where $\tau_0=0$, we have

$$\begin{aligned}&\sum_{|\gamma^1|+\cdots+|\gamma^p|=r}\ \sum_{\rho\in\Xi_r(\vec{\tau}(\gamma))} H(\gamma,\zeta,\vec{\tau}(\gamma))x_{s,t}^{\gamma\circ\rho^{-1}}\\ &= \sum_{\substack{\vec{\tau}=(\tau_1,\dots,\tau_p):\\ 1\le\tau_1<\cdots<\tau_p=r}}\ \sum_{\substack{\gamma^i:|\gamma^i|=\tau_i-\tau_{i-1}\\ i=1,\dots,p}}\ \sum_{\rho\in\Xi_r(\vec{\tau})} H(\gamma,\zeta,\vec{\tau})x_{s,t}^{\gamma\circ\rho^{-1}}.\end{aligned} \tag{3.11}$$

Notice that for fixed $\vec{\tau}=(\tau_1,\dots,\tau_p)$ such that $1\le\tau_1<\cdots<\tau_p=r$ and $\rho\in\Xi_r(\vec{\tau})$, we have

$$\sum_{\substack{\gamma^i:|\gamma^i|=\tau_i-\tau_{i-1}\\ i=1,\dots,p}} H(\gamma,\zeta,\vec{\tau})x_{s,t}^{\gamma\circ\rho^{-1}} \quad = \quad \sum_{\gamma\in\Gamma_r} H(\gamma,\zeta,\vec{\tau})x_{s,t}^{\gamma\circ\rho^{-1}},$$

so the quantity in (3.11) is equal to

$$\sum_{\substack{\tau_1,\dots,\tau_p:\\ 1\le\tau_1<\cdots<\tau_p=r}}\ \sum_{\rho\in\Xi_r(\vec{\tau})}\ \sum_{\gamma:|\gamma|=r} H(\gamma,\zeta,\vec{\tau})x_{s,t}^{\gamma\circ\rho^{-1}}.$$

Since ρ is a bijection on $\{1,\dots,r\}$, by replacing $\gamma\circ\rho^{-1}$ by α, the above expression becomes

$$\sum_{\substack{\tau_1,\dots,\tau_p:\\ 1\le\tau_1<\cdots<\tau_p=r}}\ \sum_{\rho\in\Xi_r(\vec{\tau})}\ \sum_{\alpha\circ\rho:|\alpha|=r} H(\alpha\circ\rho,\zeta,\vec{\tau})x_{s,t}^{\alpha}.$$

Substituting the above expression into (3.10), we obtain identity (3.9).

Let $f \in C^N(\mathbb{R})$ and $s,t \in [0.T]$. It follows from (3.6) in Lemma 4.3.1 that

$$\begin{aligned}\sum_{r=1}^{N}\sum_{\alpha\in\Gamma_r} x_{s,t}^{\alpha}\mathscr{V}_\alpha f &= \sum_{r=1}^{N}\sum_{\alpha\in\Gamma_r} x_{s,t}^{\alpha}\sum_{p=1}^{r}\sum_{\zeta\in\Upsilon_p}\sum_{1\le\tau_1<\cdots<\tau_p=r}\sum_{\rho\in\Xi_r(\vec{\tau})} H(\alpha\circ\rho,\zeta,\vec{\tau})\partial_\zeta f \\ &= \sum_{1\le p\le r\le N}\sum_{\zeta\in\Upsilon_p}\sum_{\alpha\in\Gamma_r}\sum_{1\le\tau_1<\cdots<\tau_p=r}\sum_{\rho\in\Xi_r(\vec{\tau})} x_{s,t}^{\alpha}H(\alpha\circ\rho,\zeta,\vec{\tau})\partial_\zeta f \quad (3.12)\end{aligned}$$

On the other hand, it follows from (3.9) in Lemma 4.3.2 that

$$\begin{aligned}&\sum_{p=1}^{N}\sum_{\zeta\in\Upsilon_p}\left(\mathscr{E}_{s,t}^{\zeta} - Q(N,|\zeta|,\zeta)_{s,t}\right)\partial_\zeta f \\ &= \sum_{p=1}^{N}\sum_{\zeta\in\Upsilon_p}\partial_\zeta f\sum_{r=p}^{N}\sum_{\substack{\tau_1,\dots,\tau_p:\\ 1\le\tau_1<\cdots<\tau_p=r}}\sum_{\alpha\in\Gamma_r}\sum_{\rho\in\Xi_r(\vec{\tau})} H(\alpha\circ\rho,\zeta,\vec{\tau})x_{s,t}^{\alpha},\end{aligned}$$

which is equal to the right-hand side of (3.12). Therefore, we obtain the following identity

$$\sum_{\alpha\in\Gamma:1\le|\alpha|\le N} x_{s,t}^{\alpha}\mathscr{V}_\alpha f = \sum_{\zeta\in\Upsilon:1\le|\zeta|\le N}\mathscr{E}_{s,t}^{\zeta}\partial_\zeta f - \sum_{\zeta\in\Upsilon:1\le|\zeta|\le N} Q(N,|\zeta|,\zeta)_{s,t}\partial_\zeta f \quad (3.13)$$

Notice that

$$\begin{aligned}\mathscr{E}_{s,t}^{(N)}(y) &= \sum_{j=1}^{m}\int_s^t \sum_{\alpha\in\Gamma:0\le|\alpha|\le N-1} x_{s,u}^{\alpha}\mathscr{V}_\alpha V_j(y)dx_u^j \\ &= \sum_{j=1}^{m}\int_s^t\left(V_j(y) + \sum_{\alpha\in\Gamma:1\le|\alpha|\le N-1} x_{s,u}^{\alpha}\mathscr{V}_\alpha V_j(y)\right)dx_u^j. \quad (3.14)\end{aligned}$$

Then, applying (3.13) to the second term on the right-hand side of (3.14) with $f = V_j$, we obtain the following result.

Proposition 4.3.1. *Let $\mathscr{E}^{(N)}_{s,t}$ be the order-N Taylor expansion defined in (3.3). Assume that $V \in C^N$. Then the following equation holds true,*

$$\int_s^t \left(V(y) + \sum_{\zeta \in \Upsilon : 1 \le |\zeta| \le N} \mathscr{E}^{\zeta}_{s,u}(y) \partial_\zeta V(y) \right) dx_u - \mathscr{E}^{(N)}_{s,t}(y) \quad = \quad R^1_{s,t}(y), \qquad (3.15)$$

where

$$\begin{aligned} R^1_{s,t}(y) \quad &= \quad \sum_{\zeta \in \Upsilon_N} \int_s^t \mathscr{E}^{\zeta}_{s,u}(y) \partial_\zeta V(y) dx_u \\ &+ \sum_{\zeta \in \Upsilon : 1 \le |\zeta| \le N-1} \int_s^t Q(N-1, |\zeta|, \zeta)_{s,u}(y) \partial_\zeta V(y) dx_u \,. \qquad (3.16) \end{aligned}$$

Applying the chain rule repeatedly we obtain

$$V(y + \mathscr{E}^{(N)}_{s,t}(y)) = V(y) + \sum_{\zeta \in \Upsilon : 1 \le |\zeta| \le N} \mathscr{E}^{\zeta}_{s,t}(y) \partial_\zeta V(y) + \sum_{\zeta \in \Upsilon_{N+1}} \mathscr{E}^{\zeta}_{s,t} \left(\partial_\zeta V(y + \mathscr{E}^{(N)}_{s,\cdot}(y)) \right)(y), \qquad (3.17)$$

where for $\zeta \in \Upsilon_p$, we denote

$$\mathscr{E}^{\zeta}_{s,t}(f) \quad = \quad \int_s^t \int_s^{t_r} \cdots \int_s^{t_2} f_{t_1} d\mathscr{E}^{\zeta_1}_{s,t_1} \cdots d\mathscr{E}^{\zeta_{p-1}}_{s,t_{r-1}} d\mathscr{E}^{\zeta_p}_{s,t_r}. \qquad (3.18)$$

Note that the first two terms on the right-hand side of (3.17) are the integrands of (3.15), so Proposition 4.3.1 implies

$$\int_s^t V\left(y + \mathscr{E}^{(N)}_{s,u}(y)\right) dx_u - \mathscr{E}^{(N)}_{s,t}(y) \quad = \quad \sum_{\zeta \in \Upsilon_{N+1}} \int_s^t \mathscr{E}^{\zeta}_{s,u} \left(\partial_\zeta V \left(y + \mathscr{E}^{(N)}_{s,\cdot}(y) \right) \right)(y) dx_u + R^1_{s,t}(y).$$

In particular, the difference $\int_s^t V\left(y+\mathscr{E}_{s,u}^{(N)}(y)\right)dx_u - \mathscr{E}_{s,t}^{(N)}(y)$ is equal to the summation of multiple integrals of order higher than N. Our next result is a generalization of this property. We first introduce a modification of the order-N Taylor expansion.

Definition 4.3.1. *Let $\widetilde{\Gamma}$ be a finite subset of Γ (collection of multi-indices with elements in $\{1,\ldots,m\}$) and denote $N=\max\left\{|\alpha|,\alpha\in\widetilde{\Gamma}\right\}$. We define the* incomplete Taylor expansion

$$\widetilde{\mathscr{E}}_{s,t}^{(N)}(y)=\sum_{\gamma\in\widetilde{\Gamma}}\mathscr{V}_\gamma I(y)x_{s,t}^\gamma. \tag{3.19}$$

If $\mathscr{E}_{s,t}^{(N)}(y)$ is defined as in (3.3) and if $\widetilde{\Gamma}=\{\gamma\in\Gamma:|\gamma|\le N\}$ we have $\widetilde{\mathscr{E}}_{s,t}^{(N)}=\mathscr{E}_{s,t}^{(N)}$. In the following, $\widetilde{\mathscr{E}}_{s,t}^{\zeta}(f)$ is the multiple integral defined as in (3.18) by replacing $\mathscr{E}_{s,t}^{\zeta_j}$ by $\widetilde{\mathscr{E}}_{s,t}^{\zeta_j}$ in (3.18).

Proposition 4.3.2. *Let $\widetilde{\mathscr{E}}_{s,t}^{(N)}$ and $\mathscr{E}_{s,t}^{(N)}$, $t\in[0,T]$ be the incomplete Taylor expansion and the order-N Taylor expansion in Definition 4.3.1 with $N=\max_{\gamma\in\widetilde{\Gamma}}|\gamma|$. Assume that $V\in C^{N+1}$. Then the following equation holds true,*

$$\int_s^t V(y+\widetilde{\mathscr{E}}_{s,u}^{(N)}(y))dx_u-\widetilde{\mathscr{E}}_{s,t}^{(N)}(y)=\sum_{e=1}^4 R_{s,t}^e(y), \tag{3.20}$$

where $R_{s,t}^1(y)$ is the same as in (3.16), and

$$R_{s,t}^2(y) = \sum_{\zeta\in\Upsilon_{N+1}}\int_s^t \widetilde{\mathscr{E}}_{s,u}^{\zeta}\left(\partial_\zeta V(y+\widetilde{\mathscr{E}}_{s,\cdot}^{(N)}(y))\right)(y)dx_u, \tag{3.21}$$

$$R_{s,t}^3(y) = \sum_{\zeta\in\Upsilon:1\le|\zeta|\le N}\int_s^t\left(\widetilde{\mathscr{E}}_{s,u}^{\zeta}(y)-\mathscr{E}_{s,u}^{\zeta}(y)\right)\partial_\zeta V(y)dx_u, \tag{3.22}$$

$$R_{s,t}^4(y) = \mathscr{E}_{s,t}^{(N)}(y)-\widetilde{\mathscr{E}}_{s,t}^{(N)}(y). \tag{3.23}$$

Proof As in (3.17), applying the chain rule several times we obtain

$$V(y+\widetilde{\mathscr{E}}^{(N)}_{s,t}(y)) = V(y) + \sum_{\zeta\in\Upsilon:1\le|\zeta|\le N} \widetilde{\mathscr{E}}^{\zeta}_{s,t}(y)\partial_\zeta V(y) + \sum_{\zeta\in\Upsilon:|\zeta|=N+1} \widetilde{\mathscr{E}}^{\zeta}_{s,t}\left(\partial_\zeta V(y+\widetilde{\mathscr{E}}^{(N)}_{s,\cdot}(y))\right)(y). \tag{3.24}$$

Integrating both sides of the above equation with respect to dx_u over $[s,t]$ and then subtracting $\widetilde{\mathscr{E}}^{(N)}_{s,t}(y)$, we obtain

$$\begin{aligned}&\int_s^t V(y+\widetilde{\mathscr{E}}^{(N)}_{s,u}(y))dx_u - \widetilde{\mathscr{E}}^{(N)}_{s,t}(y)\\ &= \int_s^t V(y)dx + \int_s^t \left(\sum_{\zeta\in\Upsilon:1\le|\zeta|\le N} \mathscr{E}^{\zeta}_{s,u}\partial_\zeta V(y)\right)dx_u - \mathscr{E}^{(N)}_{s,t}(y) + \sum_{e=2}^{4} R^e_{s,t}(y).\end{aligned}$$

Applying Proposition 4.3.1 to the above equation we obtain equation (3.20).

Definition 4.3.2. *Take $\alpha,\alpha'\in\Gamma$ such that $|\alpha|=r$ and $|\alpha'|=r+1$. We say that α is contained in α', denoted by $\alpha\Subset\alpha'$, if there is an injection ρ from $\{1,\dots,r\}$ to $\{1,\dots,r+1\}$ such that $\alpha(i)=\alpha'(\rho(i))$, $i=1,\dots,r$.*

Definition 4.3.3. *We say that $\widetilde{\Gamma}\subset\Gamma$ has a hierarchical structure if for any $\alpha\in\Gamma\setminus\widetilde{\Gamma}$ and $\alpha\Subset\alpha'$, we have $\alpha'\in\Gamma\setminus\widetilde{\Gamma}$.*

The following result shows that the difference $\int_s^t V(y+\widetilde{\mathscr{E}}^{(N)}_{s,u}(y))dx_u - \widetilde{\mathscr{E}}^{(N)}_{s,t}(y)$ is equal to the summation of multiple integrals of order "higher" than those in $\{x^\alpha : \alpha\in\widetilde{\Gamma}\}$.

Proposition 4.3.3. *Let the assumptions be as in Proposition 4.3.2. Then the following statements hold true:*

(i) *$R^1_{s,t}(y)$ is a linear combination of the multiple integrals in $\left\{x^\alpha_{s,t} : \alpha\in\Gamma, |\alpha|\ge N+1\right\}$, and the coefficients of this combination are the products of $\mathscr{V}_\gamma I_i(y)$ and $\partial_\zeta V_j(y)$ for $\gamma\in\Gamma$ such that $|\gamma|\le N$ and $\zeta\in\Upsilon$ such that $|\zeta|\le N$, $i=1,\dots,d,\ j=1,\dots,m$.*

(ii) $R^4_{s,t}(y)$ *is a linear combination of the multiple integrals in* $\left\{x^{\alpha}_{s,t} : \alpha \in \Gamma \setminus \widetilde{\Gamma}\right\}$, *and the coefficients are* $\mathscr{V}_{\gamma} I(y)$ *for* $\gamma \in \Gamma \setminus \widetilde{\Gamma}$ *such that* $|\gamma| \leq N$.

(iii) *Assume that* $\widetilde{\Gamma}$ *has the hierarchical structure introduced in Definition 4.3.3. Then* $R^3_{s,t}(y)$ *is a linear combination of the multiple integrals in* $\left\{x^{\alpha}_{s,t} : \alpha \in \Gamma \setminus \widetilde{\Gamma}\right\}$, *and the coefficients are products of* $\mathscr{V}_{\gamma} I_i(y)$ *and* $\partial_{\zeta} V_j(y)$ *for* $\gamma \in \Gamma$ *such that* $|\gamma| \leq N$ *and* $\zeta \in \Upsilon$ *such that* $|\zeta| \leq N$, $i = 1, \ldots, d$, $j = 1, \ldots, m$.

With the help of Proposition 4.3.2 we can now derive an equation for the global error function of the incomplete Taylor scheme (1.9) associated with the incomplete Taylor expansion $\widetilde{\mathscr{E}}^{(N)}_{s,t}(y)$ in (1.8) or (3.19); that is, for the global error function of the numerical scheme

$$y^n_t \quad = \quad y^n_{t_k} + \widetilde{\mathscr{E}}^{(N)}_{t_k,t}(y^n_{t_k}), \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, \ldots, n-1. \tag{3.25}$$

Recall that $t_k = kT/n$, $k = 0, 1, \ldots, n-1$, $\lfloor a \rfloor$ is the integer part of a and $a \wedge b$ is the small of a and b for $a, b \in \mathbb{R}$.

Proposition 4.3.4. *Let* y *and* y^n *be the solutions of equation (3.1) and (3.25), respectively. Let* $R^e_{s,t}(y)$, $e = 1, 2, 3, 4$, *be the functions defined in (3.16), (3.21), (3.22) and (3.23). Assume that* $V \in C^{N+1}$. *Then the error function* $y - y^n$ *satisfies the equation*

$$y_t - y^n_t = \int_0^t (V(y_s) - V(y^n_s))\, dx_s + \sum_{e=1}^{4} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} R^e_{t_k, t_{k+1} \wedge t}(y^n_{t_k}), \tag{3.26}$$

for $t \in [0, T]$.

Remark 4.3.1. *Denote* $\varepsilon := y - y^n$, *and set*

$$\dot{V}_j(\xi, \xi') := \int_0^1 \partial V_j(\theta \xi + (1-\theta)\xi') d\theta, \tag{3.27}$$

for $\xi, \xi' \in \mathbb{R}^d$, $j = 1, \ldots, m$. The following linear differential equation for ε can be easily derived from (3.26),

$$\varepsilon_t = \sum_{j=1}^{m} \int_0^t \dot{V}_j(y_u, y_u^n)\varepsilon_u dx_u^j + \sum_{e=1}^{4} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} R^e_{t_k, t_{k+1}\wedge t}(y^n_{t_k})\,. \tag{3.28}$$

Proof of Proposition 4.3.4: By taking $s = t_k$, $t \in [t_k, t_{k+1}]$ and $y = y^n_{t_k}$ in (3.20) we obtain

$$\int_{t_k}^t V(y_u^n)dx_u - \widetilde{\mathscr{E}}^{(N)}_{t_k,t}(y^n_{t_k}) = \sum_{e=1}^{4} R^e_{t_k,t}(y^n_{t_k})\,.$$

This implies that

$$\sum_{i=0}^{k} \int_{t_i}^{t_{i+1}\wedge t} V(y_u^n)dx_u - \sum_{i=0}^{k} \widetilde{\mathscr{E}}^{(N)}_{t_i,t_{i+1}\wedge t}(y^n_{t_i}) = \sum_{i=0}^{k} \sum_{e=1}^{4} R^e_{t_i,t_{i+1}\wedge t}(y^n_{t_i}), \;\; t \in [t_k, t_{k+1}],$$

or

$$\int_0^t V(y_u^n)dx_u - y_t^n = \sum_{i=1}^{\lfloor \frac{nt}{T} \rfloor} \sum_{e=1}^{4} R^e_{t_i,t_{i+1}\wedge t}(y^n_{t_i}).$$

Equation (3.26) then follows by noticing equation (3.4).

4.3.2 The explicit expression for the error function

In this subsection we derive an explicit expression of the error function $y - y^n$, where y and y^n are solutions of equation (3.1) and (3.25), respectively, with $\widetilde{\mathscr{E}}^{(N)}_{s,t}(y)$ being the incomplete Taylor expansion (3.19).

We define the *fundamental equation* of (3.28),

$$\Phi_t = I + \sum_{j=1}^{m} \int_0^t \dot{V}_j(y_s, y_s^n) \Phi_s dx_s^j, \tag{3.29}$$

and its inverse,

$$\Psi_t = I - \sum_{j=1}^{m} \int_0^t \Psi_s \dot{V}_j(y_s, y_s^n) dx_s^j, \tag{3.30}$$

for $t \in [0, T]$. Recall that $\dot{V}$ is defined in (3.27) and I is the $d \times d$ identity matrix. The fact that Ψ is the inverse of the function Φ, i.e. $\Psi\Phi \equiv I$, can be shown by applying the product rule to $\Psi\Phi$ and taking into account the identity

$$\int_0^t \Psi_s \cdot d\Phi_s + \int_0^t d\Psi_s \cdot \Phi_s = 0.$$

The following result provides an explicit expression for the error function $y - y^n$ under the above assumptions. We denote $\eta(t) = t_k$ for $t \in [t_k, t_{k+1})$.

Theorem 4.3.1. *Let assumptions be as in Proposition 4.3.4. The following expression of $y - y^n$ holds true for $t \in [0, T]$,*

$$\begin{aligned} y_t - y_t^n &= \sum_{e=1}^{4} \Phi_t \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \int_{t_k}^{t_{k+1} \wedge t} \Psi_s dR_{t_k,s}^e(y_{t_k}^n) \\ &= \sum_{e=1}^{4} \Phi_t \int_0^t \Psi_s dR_{\eta(s),s}^e(y_{\eta(s)}^n) . \end{aligned} \tag{3.31}$$

Proof By applying the product rule to the quantity on the right-hand side of equation (3.31) and taking into account identities (3.29) and $\Phi\Psi \equiv I$, we can show that this quantity satisfies equation (3.28), and by the uniqueness of the solution of equation (3.28), we conclude that it is equal to $y_t - y_t^n$.

4.4 The incomplete Taylor scheme

Let y be the solution of the differential equation (3.1) and let x^j be Hölder continuous of order $\beta_j > 1/2$. Given any finite set $\widetilde{\Gamma}$ of Γ (collection of multi-indices with elements in $\{1,\dots,m\}$) let y^n be the approximation solution defined by (3.25), where $\widetilde{\mathscr{E}}_{s,t}^{(N)}(y)$ is the incomplete Taylor expansion in Definition 4.3.1. In this section, we study the convergence rate of y^n to y. Denote $\beta := \min_j \beta_j$.

Let $a, b \in [0,T]$ with $a < b$ and $\delta \in (0,1)$. For a function $z : [0,T] \to \mathbb{R}$, $\|z\|_{a,b,\delta}$ denotes the δ-Hölder seminorm of z on $[a,b]$, that is,

$$\|z\|_{a,b,\delta} = \sup\left\{\frac{|z_u - z_v|}{(v-u)^\delta} : a \le u < v \le b\right\}.$$

We will denote the uniform norm of z on the interval $[a,b]$ by $\|z\|_{a,b,\infty}$. When $a = 0$ and $b = T$, we will simply write $\|z\|_\infty$ for $\|z\|_{0,T,\infty}$ and $\|z\|_\delta$ for $\|z\|_{0,T,\delta}$.

The following lemma provides some upper bounds of y^n, Φ and Ψ.

Lemma 4.4.1. *Let $\widetilde{\Gamma}$ be a finite subset of Γ and assume $V \in C_b^{N+1}$. Let y^n be the solution of (3.25). We have the following estimate*

$$\|y^n\|_\beta \le C\|x\|_\beta \vee \|x\|_\beta^{1/\beta+N-1}, \tag{4.1}$$

where C is a constant independent of n. Furthermore, the following estimate holds true for the functions Φ and Ψ defined in (3.29) and (3.30),

$$\|\Phi\|_\beta \vee \|\Phi\|_\infty \vee \|\Psi\|_\beta \vee \|\Psi\|_\infty \le C\exp(C\|x\|_\beta^{1/\beta^2+(N-1)/\beta}).$$

Proof The upper bound estimate for y^n follows immediately from Lemma 4.8.4. We turn to the function Φ. Consider the following system of equations

$$\begin{aligned} y^n_t &= y_0 + \int_0^t dy^n_s\,, \\ y_t &= y_0 + \int_0^t V(y_s)dx_s\,, \\ \Phi_t &= I + \sum_{j=1}^m \int_0^t \dot{V}_j(y_u, y^n_u)\Phi_u dx^j_u\,. \end{aligned}$$

Applying Lemma 3.1 in [13] to the above system we obtain

$$\|\Phi\|_\beta \;\leq\; C\exp(C\|y^n\|_\beta^{1/\beta} + C\|x\|_\beta^{1/\beta}).$$

Applying the estimate (4.1) to the right-hand side of the above inequality, we obtain the upper bound for $\|\Phi\|_\beta$. The upper bound for $\|\Phi\|_\infty$ follows from the estimate of $\|\Phi\|_\beta$. The upper bounds for Ψ can be shown similarly.

We also need the following upper bound on $\widetilde{\mathscr{E}}^{(N)}_{s,\cdot}(y)$.

Lemma 4.4.2. *Take $s, s' \in [0,T]$ such that $s < s'$ and $y \in \mathbb{R}^d$. Assume that $V \in C_b^{N-1}$. Then for the incomplete Taylor expansion $\widetilde{\mathscr{E}}^{(N)}_{s,t}(y)$, $t \in [s,s']$ we have*

$$\|\widetilde{\mathscr{E}}^{(N)}_{s,\cdot}(y)\|_{s,s',\beta} \leq K\exp\left(K\sum_{j=1}^m \|x^j\|_{s,s',\beta_j}\right),$$

for some constant C independent of n.

Proof By Lemma 4.8.1, we have, for any $\alpha \in \Gamma_r$

$$\|x^\alpha_{s,\cdot}\|_{s,s',\beta} \;\leq\; \prod_{i=1}^r \|x^{\alpha_i}\|_{s,s',\beta_{\alpha_i}} \leq \exp\left(K\sum_{j=1}^m \|x^j\|_{s,s',\beta_j}\right).$$

The desired estimate follows immediately by noticing that $\widetilde{\mathscr{E}}_{s,t}^{(N)}(y)$ is a linear combination of multiple integrals in $\{x_{s,t}^{\alpha} : \alpha \in \widetilde{\Gamma}\}$.

For a given finite subset $\widetilde{\Gamma}$ of Γ, we define

$$\theta = \theta_{\widetilde{\Gamma}} := \min\left\{\beta_{\alpha(1)} + \cdots + \beta_{\alpha(|\alpha|)} - 1 : \alpha \in \Gamma \setminus \widetilde{\Gamma}\right\}. \tag{4.2}$$

Lemma 4.4.3. *Assume that α belongs to $\Gamma \setminus \widetilde{\Gamma}$. Assume that f is a Hölder continuous function of order β. Then there exists a constant K such that for $t, t' \in [t_k, t_{k+1}]$, $k = 0, 1, \ldots, n-1$, we have*

$$\left|\int_t^{t'} f_u dx_{t_k,u}^{\alpha}\right| \leq K(\|f\|_\beta + \|f\|_\infty) \exp\left(K \sum_{j=1}^{m} \|x^j\|_{\beta_j}\right) n^{-\theta-1}.$$

Proof Take $\alpha \in \Gamma$ such that $|\alpha| = r$. Applying Lemma 4.8.2 to the integral $\int_t^{t'} f_u dx_{t_k,u}^{\alpha}$, we obtain

$$\begin{aligned}\left|\int_t^{t'} f_u dx_{t_k,u}^{\alpha}\right| &\leq K(\|f\|_\beta + \|f\|_\infty) \left(\prod_{j=1}^{r} \|x^{\alpha_j}\|_{\beta_{\alpha_j}}\right) n^{-\sum_{j=1}^{r} \beta_{\alpha_j}} \\ &\leq K(\|f\|_\beta + \|f\|_\infty) \exp\left(K \sum_{j=1}^{m} \|x^j\|_{\beta_j}\right) n^{-\sum_{j=1}^{r} \beta_{\alpha_j}}.\end{aligned}$$

Since $\alpha \notin \widetilde{\Gamma}$, we see that $\sum_{j=1}^{r} \beta_{\alpha_j} \geq \theta + 1$, proving the desired estimate.

In the following, we consider the incomplete Taylor scheme (3.25) defined by any finite set $\widetilde{\Gamma}$.

Lemma 4.4.4. *Assume that $\widetilde{\Gamma}$ has the hierarchical structure introduced in Definition 4.3.3. Assume that f is a Hölder continuous function of order β, and $R_{s,t}^{e}(y)$, $e = 1, 2, 3, 4$, are the functions defined by (3.16), (3.21), (3.22) and (3.23). Assume that*

$V \in C_b^{N+1}$. *Then there exists a constant* K *such that for* $t, t' \in [t_k, t_{k+1}] \subset [0, T]$, *we have*

$$\left| \int_t^{t'} f_u dR^2_{t_k,u}(y) \right| \leq C\left(\|f\|_\beta + \|f\|_\infty\right) \exp\left(K \sum_{j=1}^m \|x^j\|_{\beta_j}\right) n^{-(N+2)\beta}, \tag{4.3}$$

and

$$\left| \int_t^{t'} f_u dR^e_{t_k,u}(y) \right| \leq C\left(\|f\|_\beta + \|f\|_\infty\right) \exp\left(K \sum_{j=1}^m \|x^j\|_{\beta_j}\right) n^{-\theta-1}, \quad e = 1, 2, 3, 4. \tag{4.4}$$

Proof According to Proposition 4.3.3, for $e = 1, 3, 4$, the integral $\int_t^{t'} f_u dR^e_{t_k,u}(y)$ is a linear combination of integrals of the form

$$\int_t^{t'} f_u dx^\alpha_{t_k,u}, \quad \alpha \in \Gamma \setminus \widetilde{\Gamma}.$$

So inequality (4.4) for $e = 1, 3, 4$ follows from Lemma 4.4.3.

Inequality (4.3) can be shown by applying Lemma 4.8.2 to the integral

$$\int_t^{t'} f_u dR^2_{t_k,u}(y) = \sum_{\zeta \in \Upsilon : |\zeta| = N+1} \int_t^{t'} \widetilde{\mathscr{E}}^\zeta_{s,u} \left(\partial_\zeta V(y + \widetilde{\mathscr{E}}^{(N)}_{s,\cdot}(y))\right)(y) f_u dx_u$$

and taking into account the estimates in Lemma 4.4.2. Finally, inequality (4.4) holds for $e = 2$ because it is easy to verify from the definition of θ that $(N+1)\beta \geq \theta + 1$.

The following theorem is the main result in this section.

Theorem 4.4.1. *Let* $\widetilde{\Gamma}$ *be a finite subset of* Γ *and let* θ *be defined by (4.2). Assume that* $\widetilde{\Gamma}$ *has the hierarchical structure introduced in Definition 4.3.3. Let* y *be the solution of equation (3.1) and let* y^n *be the solution to (3.25). Assume that* $V \in C_b^{N+1}$. *Then*

$$\sup_{t \in [0,T]} |y_t - y^n_t| \leq G n^{-\theta},$$

where

$$G = C\exp\left(C\|x\|_{\beta}^{1/\beta^2+(N-1)/\beta} + C\sum_{j=1}^{m}\|x^j\|_{\beta_j}\right).$$

Proof Because of identity (3.31) and the estimate of $\|\Phi\|_\infty$ in Lemma 4.4.1, we only need to show that the quantity

$$\sum_{e=1}^{4}\sum_{k=0}^{\lfloor \frac{nt}{T}\rfloor}\left|\int_{t_k}^{t_{k+1}\wedge t}\Psi_s dR^e_{t_k,s}(y^n_{t_k})\right|$$

is bounded by $Gn^{-\theta}$ for $t\in[0,T]$. Inequality (4.4) in Lemma 4.4.4 shows that the above quantity is bounded by

$$C\sum_{k=0}^{\lfloor \frac{nt}{T}\rfloor}\left(\|\Psi\|_\beta+\|\Psi\|_\infty\right)\exp\left(K\sum_{j=1}^{m}\|x^j\|_{\beta_j}\right)n^{-\theta-1},$$

which is less than

$$C\left(\|\Psi\|_\beta+\|\Psi\|_\infty\right)\exp\left(K\sum_{j=1}^{m}\|x^j\|_{\beta_j}\right)n^{-\theta}.$$

Applying the estimates on Ψ given in Lemma 4.4.1 to the above expression, we obtain the desired estimate.

Now we can apply this theorem to obtain the best Taylor scheme. It is clear that the possible rates of convergence are of the form $n^{-\theta}$, where θ is a nonnegative integer linear combination of β_i, $i=1,\dots,m$ subtracting one:

$$\theta = \sum_{j=1}^{m}k_j\beta_j - 1,\ k_j = 0,1,2,\dots \text{ and } j=1,\dots,m. \tag{4.5}$$

Given a rate of the above form we define

$$\Gamma(\theta) := \{\alpha \in \Gamma : \beta_{\alpha(1)} + \cdots + \beta_{\alpha(|\alpha|)} - 1 < \theta\}. \tag{4.6}$$

Lemma 4.4.5. *If θ has the form (4.5), then $\theta_{\Gamma(\theta)} = \theta$, where $\theta_{\Gamma(\theta)}$ is defined by (4.2).*

Proof Let $\theta = \sum_{j=1}^{m} k_j \beta_j - 1$ for some k_j, $j = 1, \cdots, m$. Consider

$$\alpha = (\overbrace{1, \ldots, 1}^{k_1}, \ldots, \overbrace{m, \ldots, m}^{k_m}).$$

Then $\beta_{\alpha(1)} + \cdots + \beta_{\alpha(|\alpha|)} - 1 = \theta$ and hence $\alpha \notin \Gamma(\theta)$. This shows that $\theta_{\Gamma(\theta)} \le \theta$. If $\theta_{\Gamma(\theta)} < \theta$, then, by the definition of $\theta_{\Gamma(\theta)}$, there is an $\alpha = (\alpha(1), \ldots, \alpha(r)) \in \Gamma \setminus \Gamma(\theta)$ such that $\beta_{\alpha(1)} + \cdots + \beta_{\alpha(|\alpha|)} - 1 < \theta$. On the other hand, by our definition of $\Gamma(\theta)$, $\alpha \in \Gamma(\theta)$. This is a contradiction. Thus $\theta_{\Gamma(\theta)} = \theta$.

Remark 4.4.1. *(i) From Lemma 4.4.5 and from Theorem 4.4.1, we see that a possible rate has the form $n^{-\theta}$, where θ is of the form (4.5), and for a rate of this form, the best choice of the incomplete Taylor scheme (3.25) is $\widetilde{\Gamma} = \Gamma(\theta)$.*

(ii) When $\beta_i = \beta$, $i = 1, \ldots, m$ for $\beta > 1/2$, $\theta = (N+1)\beta - 1$ and $\Gamma(\theta)$ becomes

$$\Gamma(\theta) \quad = \quad \{\alpha \in \Gamma : |\alpha| \le N\}.$$

So in this case, the best Taylor scheme is the complete Taylor scheme:

$$y_t^n \quad = \quad y_{t_k}^n + \mathscr{E}_{t_k,t}^{(N)}(y_{t_k}^n), \quad y_0^n = y_0,$$

for $t \in [t_k, t_{k+1}]$, $k = 0, 1, \ldots, n-1$. According to Theorem 4.4.1, its convergence rate is $n^{1-(N+1)\beta}$, which coincides with the result obtained in [8].

4.5 L_p-estimates of weighted random sums and multiple integrals

In the first subsection, we recall some definitions on fractional integrals and derivatives to fix the notation we are going to use. In the subsequent three subsections, we derive some L_p-estimates of weighted random sums and multiple integrals, which are needed to obtain the rate of convergence for the modified Taylor scheme (1.12).

4.5.1 Elements of fractional calculus

Take $f \in L_1([0,T])$ and $\delta > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order δ are defined, for almost all $t \in (a,b)$, by

$$I_{a+}^{\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} f(s) ds$$

and

$$I_{b-}^{\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_t^b (s-t)^{\delta-1} f(s) ds,$$

respectively, where $\Gamma(\delta) = \int_0^\infty r^{\delta-1} e^{-r} dr$ is the Gamma function. For $p \geq 1$, let $I_{a+}^{\delta}(L_p([0,T]))$ (respectively $I_{b-}^{\delta}(L_p([0,T]))$) be the class of functions f which may be represented as an I_{a+}^{δ}- (I_{b-}^{δ}-) integral of some L_p-function φ. If $f \in I_{a+}^{\delta}(L_p([0,T]))$ (respectively $f \in I_{b-}^{\delta}(L_p([0,T]))$) and $0 < \delta < 1$ then the fractional Weyl derivative is defined as

$$D_{a+}^{\delta} f(t) = \frac{1}{\Gamma(1-\delta)} \left(\frac{f(t)}{(t-a)^{\delta}} + \delta \int_a^t \frac{f(t)-f(s)}{(t-s)^{\delta+1}} ds \right) \mathbf{1}_{(a,b)}(t)$$

$$\left(\text{resp. } D^{\delta}_{b-}f(t) \;=\; \frac{1}{\Gamma(1-\delta)}\left(\frac{f(t)}{(b-t)^{\delta}}+\delta\int_t^b \frac{f(t)-f(s)}{(s-t)^{\delta+1}}ds\right)\mathbf{1}_{(a,b)}(t)\right),$$

where $a < t < b$.

4.5.2 L_p-estimate of weighted random sums

Let $\zeta = \{\zeta_{k,n}, n \in \mathbb{N}, k = 0,1,\ldots,n\}$ be a double sequence of random variables. The aim of this subsection is to provide an L_p-estimate of the weighted summations of this sequence, which we need for the rate of convergence of the modified Taylor scheme. We first introduce the space of Hölder continuous functions in L_p.

Definition 4.5.1. *Let f be a stochastic process on $[0,T]$ such that $f(t) \in L_p$ for each t. We say that f is* Hölder continuous of order β in L_p *if*

$$\|f(t)-f(s)\|_p \;\le\; K|t-s|^{\beta}, \quad s,t \in [0,T]$$

for $\beta > 0$. We define the seminorm

$$\|f\|_{\beta,p} \;=\; \sup\left\{\frac{\|f(t)-f(s)\|_p}{(t-s)^{\beta}} : 0 \le s < t \le T\right\}.$$

In the following, we denote $t_k = kT/n$, $k = 0,1,\ldots,n$ and $\eta(t) = t_k$ for $t \in [t_k, t_{k+1})$.

Proposition 4.5.1. *Let $p \ge 1$, $q, q' > p$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{q'}$ and let β, β' be in $(0,1)$ such that $\beta + \beta' > 1$. Let $\zeta = \{\zeta_{k,n}, n \in \mathbb{N}, k = 0,1,\ldots,n\}$ satisfy*

$$\mathbb{E}\left(\left|\sum_{k=j+1}^{i}\zeta_{k,n}\right|^{q}\right) \;\le\; L\left(\frac{i-j}{n}\right)^{\beta' q} \tag{5.1}$$

for all $i, j = 0,1,\ldots,n$, $i > j$ and for some constant $L > 0$. Let f be a continuous process and assume that f is Hölder continuous of order β in $L_{q'}$. Then for $i, j = 0,1,\ldots,n-1$,

$i > j$,

$$\left\| \sum_{k=j+1}^{i} f(t_k)\zeta_{k,n} \right\|_p \leq cL\|f\|_{\beta,q'} \left(\frac{i-j}{n}\right)^{\beta+\beta'} + cL\|f(t_j)\|_{q'} \left(\frac{i-j}{n}\right)^{\beta'},$$

where c is a constant depending on T and the parameters p, q, q', β, β'.

Proof For each $t \in [0,T]$ we denote

$$g_n(t) := \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \zeta_{k,n}.$$

Then we can write

$$\sum_{k=j+1}^{i} f(t_k)\zeta_{k,n} = \int_{(t_j,t_{i+1})} f(t)dg_n(t).$$

We shall use the following fractional integration by parts formula to deal with the above integral (see Theorem 2.4 in [49]).

$$\int_{(a,b)} f dg_n = \int_a^b D_{a+}^{\delta} f_a(t) D_{b-}^{1-\delta} g_{n,b}(t)dt + f(a)(g_n(b-) - g_n(a+)), \tag{5.2}$$

where we denote $f_a(t) = \mathbf{1}_{(a,b)}(t)(f(t) - f(a))$, $g_{n,b}(t) = \mathbf{1}_{(a,b)}(t)(g_n(t) - g_n(b-))$ and $\delta \in [0,1]$.

We denote $a := t_j$ and $b := t_{i+1}$. Let δ be such that $1 - \beta' < \delta < \beta$. By the definition of the fractional derivative it is easy to show that

$$\left\| D_{a+}^{\delta} f_{a+}(t) \right\|_{q'} \leq \frac{\|f\|_{\beta,q'}}{\Gamma(1-\delta)} \frac{\beta}{\beta - \delta} (t-a)^{\beta-\delta}. \tag{5.3}$$

On the other hand,

$$\left\|D_{b-}^{1-\delta} g_{n,b}(t)\right\|_q \leq \frac{1}{\Gamma(\delta)}\left(\left\|\frac{g_n(t)-g_n(b-)}{(b-t)^{1-\delta}}\right\|_q+(1-\delta)\left\|\int_t^b \frac{g_n(t)-g_n(s)}{(s-t)^{2-\delta}} ds\right\|_q\right) \quad (5.4)$$

We first consider the second term on the right-hand side of the above inequality. When $t \geq b-\frac{T}{n}$, we have $g_n(t)-g_n(s)=0$ and thus the second term is equal to zero. When $t<b-\frac{T}{n}$, we have

$$\begin{aligned}
&\left\|\int_t^b \frac{g_n(t)-g_n(s)}{(s-t)^{2-\delta}} ds\right\|_q \\
&=\left\|\int_{\eta(t)+\frac{T}{n}}^b \frac{g_n(t)-g_n(s)}{(s-t)^{2-\delta}} ds\right\|_q \leq L \int_{\eta(t)+\frac{T}{n}}^b \frac{[\eta(s)-\eta(t)]^{\beta'}}{(s-t)^{2-\delta}} ds \\
&\leq 2^{\beta'} L \int_{\eta(t)+2\frac{T}{n}}^b (s-t)^{\beta'+\delta-2} ds+L \int_{\eta(t)+\frac{T}{n}}^{\eta(t)+2\frac{T}{n}} \frac{[\eta(s)-\eta(t)]^{\beta'}}{(s-t)^{2-\delta}} ds \\
&\leq \frac{2^{\beta'} L}{\beta'+\delta-1}(b-t)^{\beta'+\delta-1}+\frac{L T^{\beta'}}{n^{\beta'}(\delta-1)}\left[\left(\eta(t)+2\frac{T}{n}-t\right)^{\delta-1}-\left(\eta(t)+\frac{T}{n}-t\right)^{\delta-1}\right] \\
&\leq \frac{2^{\beta'} L}{\beta'+\delta-1}(b-t)^{\beta'+\delta-1}+\frac{L T^{\beta'}}{n^{\beta'}(1-\delta)}\left(\eta(t)+\frac{T}{n}-t\right)^{\delta-1}, \qquad (5.5)
\end{aligned}$$

where in the first inequality we used the assumption (5.1). For the first term in (5.4), since

$$\|g_n(t)-g_n(b-)\|_q \leq L\left|b-\frac{T}{n}-\eta(t)\right|^{\beta'} \leq L|b-t|^{\beta'}$$

for $t \in (a,b)$, we have

$$\left\|\frac{g_n(t)-g_n(b-)}{(b-t)^{1-\delta}}\right\|_q \leq L(b-t)^{\beta'+\delta-1}.$$

Substituting the above inequality and (5.5) into (5.4) we obtain

$$\left\| D_{b-}^{1-\delta} g_{n,b}(t) \right\|_q \leq cL(b-t)^{\beta'+\delta-1} + cLn^{-\beta'}(\eta(t)+\frac{T}{n}-t)^{\delta-1}. \qquad (5.6)$$

By the fractional integration by parts formula (5.2) and by (5.3) and (5.6) we obtain

$$\begin{aligned} \left\| \int_a^b f(t) dg_n(t) \right\|_p &\leq cL\|f\|_{\beta,q'} \int_a^b (t-a)^{\beta-\delta} \left[(b-t)^{\beta'+\delta-1} + n^{-\beta'}(\eta(t)+\frac{T}{n}-t)^{\delta-1} \right] dt \\ &\quad + \|f(a)\|_{q'}(b-a)^{\beta'} \\ &\leq cL\|f\|_{\beta,q'}(b-a)^{\beta+\beta'} + cL\|f\|_{\beta,q'} \int_a^b (t-a)^{\beta-\delta} n^{-\beta'}(\eta(t)+\frac{T}{n}-t)^{\delta-1} dt \\ &\quad + \|f(a)\|_{q'}(b-a)^{\beta'}. \end{aligned}$$

The lemma then follows from the following calculation,

$$\begin{aligned} &\int_a^b (t-a)^{\beta-\delta} n^{-\beta'}(\eta(t)+\frac{T}{n}-t)^{\delta-1} dt \\ &\leq n^{-\beta'} \sum_{k=j}^{i-1} (t_{k+1}-a)^{\beta-\delta} \int_{t_k}^{t_{k+1}} (\eta(t)+\frac{T}{n}-t)^{\delta-1} dt \\ &\leq n^{-\beta'} \sum_{k=0}^{i-j} \left(\frac{k+1}{n}\right)^{\beta-\delta} \frac{1}{\delta} \left(\frac{T}{n}\right)^{\delta} = cn^{-\beta-\beta'} \sum_{k=0}^{i-j} (k+1)^{\beta-\delta} \\ &\leq cn^{-\beta-\beta'}(i-j)^{1+\beta-\delta} \leq c\left(\frac{i-j}{n}\right)^{\beta+\beta'}. \end{aligned}$$

□

4.5.3 Monotonicity in the L_2-norm of multiple integrals

In this subsection we derive a monotonicity result on the L_2-norm of multiple integrals with respect to the fractional Brownian motion. We recall that a standard one-dimensional fractional Brownian motion (fBm) is a centered Gaussian process

$B = \{B_t, t \geq 0\}$ with covariance given by

$$\mathbb{E}[B_t B_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

where $H \in (0,1)$ is the Hurst parameter. Our proof is based on the approximation of multiple integrals by sums of products of fBm increments. Throughout this subsection, we assume that, for $j = 1, \ldots, N$, B^j is either a fBm with Hurst parameter $H_j > 1/2$ or the identity function. Assume in addition that for $j, j' = 1, \ldots, N$, the two processes B^j and $B^{j'}$ are either mutually independent or equal.

We first recall a formula for the expectation of a product of increments. Take an even number r. There are $(r-1)!!$ ways to arrange the elements of $\{1, \ldots, r\}$ into pairs. We denote by $\mathscr{R}_r$ the collection of these ways. Assume that each $\tau \in \mathscr{R}_r$ is of the form $\tau = \{(\tau_1, \tau_2), \ldots, (\tau_{r-1}, \tau_r)\}$, where $\tau_i \in \{1, \ldots, r\}$. For a subinterval $I = (s,t)$ of $[0,T]$, we denote $B^j_I = B^j_{s,t} = B^j_t - B^j_s$. The following is a consequence of the Feynman diagram formula (see [21, page 16]).

Lemma 4.5.1. *Let I_i, $i = 1, \ldots, r$, be subintervals of $[0,T]$.*
(i) *If B^j, $j = 1, \ldots, r$ are fBms, then the following identity holds true,*

$$\mathbb{E}[B^1_{I_1} \cdots B^r_{I_r}] = \begin{cases} \sum_{\tau \in \mathscr{R}_r} \mathbb{E}[B^{\tau_1}_{I_{\tau_1}} B^{\tau_2}_{I_{\tau_2}}] \cdots \mathbb{E}[B^{\tau_{r-1}}_{I_{\tau_{r-1}}} B^{\tau_r}_{I_{\tau_r}}], & r \text{ is even}. \\ 0, & r \text{ is odd}. \end{cases}$$

(ii) *The following inequality holds true,*

$$\mathbb{E}[B^1_{I_1} \cdots B^r_{I_r}] \geq 0.$$

Proof The first result is the Feynman diagram formula for products of Gaussian random variables. The second result follows from (i) and the fact that the increments of a fBm with Hurst parameter $H > 1/2$ have positive correlation.

We recall an L_p-convergence result in Proposition 2.2 [13].

Proposition 4.5.2. *Assume that f and g are stochastic processes which are Hölder continuous of orders μ and λ in L_p (see Definition 4.5.1) for any $p \geq 1$, respectively, such that $\lambda + \mu > 1$. Assume that $f_0 \in L_p$. Then, the integral $\int_0^T f dg$ exists as a Riemann-Stieltjes integral of L_p-valued functions, and, as a consequence, we have the following convergence in L_p:*

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f_{t_k} g_{t_k,t_{k+1}} = \int_0^T f dg, \quad \text{where } t_k = kT/n,\ k = 0,1,\ldots,n.$$

We are ready to prove the main result of this subsection. We will make use of the notation

$$J_r(\mathscr{A}) \ := \ \int_0^T \cdots \int_0^T \mathbf{1}_{\mathscr{A}} dB^1_{s_1} \cdots dB^r_{s_r}$$

for any Borel subset $\mathscr{A} \subset [0,T]^r$ such that the above multiple integral exists as an iterated Riemann-Stieltjes integral defined using L_p-convergence. For any $0 \leq s < t \leq T$, we define

$$[s,t]^r_< = \{(s_1,\ldots,s_r) \in [0,T]^r : s \leq s_1 \leq \cdots \leq s_r \leq t\}.$$

Proposition 4.5.3. *Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be a partition of $[0,T]$. Take*

$$\mathscr{A} = \bigcup_{k=0}^{n-1} [t_k, t_{k+1}]^r_<, \tag{5.7}$$

and

$$\mathscr{A}' = \bigcup_{k=0}^{n-1} [t_k, t_{k+1}]^r.$$

Then $J_r(\mathscr{A})$ and $J_r(\mathscr{A}')$ are well defined as iterated Riemann-Stieltjes integrals, and we have

$$\mathbb{E}\left(|J_r(\mathscr{A})|^2\right) \leq \mathbb{E}\left(\left|J_r(\mathscr{A}')\right|^2\right). \tag{5.8}$$

Proof We denote $t_k^l = Tk/l$ and $I_k^l = [t_k^l, t_{k+1}^l]$ for $0 \leq k \leq l$. By Proposition 4.5.2, it is easy to see that

$$\mathbb{E}\left(|J_r(\mathscr{A})|^2\right) = \lim_{n_1 \to \infty} \mathbb{E}\left(\left|\sum_{k=0}^{n_1-1} B^r_{I_k^{n_1}} \int_0^T \cdots \int_0^T \mathbf{1}_{\mathscr{A}}(s_1, \ldots, s_{r-1}, t_k^{n_1}) dB^1_{s_1} \cdots dB^{r-1}_{s_{r-1}}\right|^2\right)$$

Applying Proposition 4.5.2 several times we obtain

$$\lim_{n_r \to \infty} \cdots \lim_{n_1 \to \infty} \mathbb{E}\left(\left|\sum_{k_r=0}^{n_r-1} \cdots \sum_{k_1=0}^{n_1-1} B^r_{I_{k_r}^{n_r}} \cdots B^1_{I_{k_1}^{n_1}} \mathbf{1}_{\mathscr{A}}(t_{k_1}^{n_1}, \ldots, t_{k_r}^{n_r})\right|^2\right) = \mathbb{E}\left(|J_r(\mathscr{A})|^2\right) \tag{5.9}$$

It is clear that this identity still holds when we replace $\mathscr{A}$ by $\mathscr{A}'$. On the other hand, since $\mathscr{A} \subset \mathscr{A}'$, it follows from Lemma 4.5.1(ii) that

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{k_r=0}^{n_r-1} \cdots \sum_{k_1=0}^{n_1-1} B^r_{I_{k_r}^{n_r}} \cdots B^1_{I_{k_1}^{n_1}} \mathbf{1}_{\mathscr{A}}(t_{k_1}^{n_1}, \ldots, t_{k_r}^{n_r})\right|^2\right) \\
&\leq \mathbb{E}\left(\left|\sum_{k_r=0}^{n_r-1} \cdots \sum_{k_1=0}^{n_1-1} B^r_{I_{k_r}^{n_r}} \cdots B^1_{I_{k_1}^{n_1}} \mathbf{1}_{\mathscr{A}'}(t_{k_1}^{n_1}, \ldots, t_{k_r}^{n_r})\right|^2\right).
\end{aligned}$$

By taking limits in both sides of the above inequality and taking into account (5.9) we obtain inequality (5.8).

Remark 4.5.1. *The monotonicity property in Proposition 4.5.3 can be generalized to any $\mathscr{A},\mathscr{A}' \subset [0,T]^r$ such that $\mathscr{A} \subset \mathscr{A}'$ as long as the multiple integrals $J_r(\mathscr{A})$ and $J_r(\mathscr{A}')$ are well defined as iterated Riemann-Stieltjes integrals. The same generalization holds true for the monotonicity property established in Proposition 4.5.4 below.*

Remark 4.5.2. *In the same way as in the proof of Proposition 4.5.3, we can show that*

$$\mathbb{E}\left(J_r(\mathscr{A})\right) = \lim_{n_r\to\infty}\cdots\lim_{n_1\to\infty}\mathbb{E}\left(\sum_{k_r=0}^{n_r-1}\cdots\sum_{k_1=0}^{n_1-1}B^r_{I^{n_r}_{k_r}}\cdots B^1_{I^{n_1}_{k_1}}\mathbf{1}_{\mathscr{A}}(t^{n_1}_{k_1},\ldots,t^{n_r}_{k_r})\right). \quad (5.10)$$

So the expectation of the multiple integral $J_r(\mathscr{A})$ is always zero when the number of fBms in $\{B^1,\ldots,B^r\}$ is odd.

Lemma 4.5.2. *Let I_i, $i=1,\ldots,2r$, be subintervals of $[0,T]$, where r is an even number. If B^j, $j=1,\ldots,2r$ are fBms, then the following identity holds true,*

$$\mathrm{Cov}(B^1_{I_1}\cdots B^r_{I_r}, B^{r+1}_{I_{r+1}}\cdots B^{2r}_{I_{2r}}) = \sum_{\tau\in\mathscr{R}'_{2r}}\mathbb{E}[B^{\tau_1}_{I_{\tau_1}}B^{\tau_2}_{I_{\tau_2}}]\cdots\mathbb{E}[B^{\tau_{2r-1}}_{I_{\tau_{2r-1}}}B^{\tau_{2r}}_{I_{\tau_{2r}}}],$$

where $\mathscr{R}'_{2r}$ is a subset of $\mathscr{R}_{2r}$ such that for $\tau\in\mathscr{R}_{2r}\setminus\mathscr{R}'_{2r}$, either $\tau_i,\tau_{i+1}\in\{1,\ldots,r\}$ or $\tau_i,\tau_{i+1}\in\{r+1,\ldots,2r\}$, $i=1,3,\ldots,2r-1$. In particular, the covariance of $B^1_{I_1}\cdots B^r_{I_r}$ and $B^{r+1}_{I_{r+1}}\cdots B^{2r}_{I_{2r}}$ is nonnegative.

Proof Note that

$$\mathrm{Cov}(B^1_{I_1}\cdots B^r_{I_r}, B^{r+1}_{I_{r+1}}\cdots B^{2r}_{I_{2r}}) = \mathbb{E}[B^1_{I_1}\cdots B^{2r}_{I_{2r}}] - \mathbb{E}[B^1_{I_1}\cdots B^r_{I_r}]\mathbb{E}[B^{r+1}_{I_{r+1}}\cdots B^{2r}_{I_{2r}}].$$

The lemma then follows immediately from Lemma 4.5.1 (i).

Recall that we define the centered multiple integral as

$$\widetilde{J}_r(\mathscr{A}) \quad := \quad J_r(\mathscr{A}) - \mathbb{E}\left[J_r(\mathscr{A})\right].$$

Following is the monotonicity result on the L_2-norm of this multiple integral.

Proposition 4.5.4. *Let $\mathscr{A}$ and $\mathscr{A}'$ be as in Proposition 4.5.3. Then we have*

$$\mathbb{E}\left(\left|\widetilde{J}_r(\mathscr{A})\right|^2\right) \quad \leq \quad \mathbb{E}\left(\left|\widetilde{J}_r(\mathscr{A}')\right|^2\right). \tag{5.11}$$

Proof We first notice that

$$\mathbb{E}\left(\left|\widetilde{J}_r(\mathscr{A})\right|^2\right) \quad = \quad \mathbb{E}\left(|J_r(\mathscr{A})|^2\right) - \mathbb{E}\left(J_r(\mathscr{A})\right)^2.$$

By applying (5.9) and (5.10) to the above equation, we have

$$\begin{aligned}
\mathbb{E}\left(\left|\widetilde{J}_r(\mathscr{A})\right|^2\right) \quad &= \quad \lim_{n_r\to\infty}\cdots\lim_{n_1\to\infty}\left\{\mathbb{E}\left(\left|\sum_{k_r=0}^{n_r-1}\cdots\sum_{k_1=0}^{n_1-1} B^r_{I^{n_r}_{k_r}}\cdots B^1_{I^{n_1}_{k_1}}\mathbf{1}_{\mathscr{A}}(t_{k_1},\ldots,t_{k_r})\right|^2\right)\right. \\
&\qquad \left. -\mathbb{E}\left(\sum_{k_r=0}^{n_r-1}\cdots\sum_{k_1=0}^{n_1-1} B^r_{I^{n_r}_{k_r}}\cdots B^1_{I^{n_1}_{k_1}}\mathbf{1}_{\mathscr{A}}(t_{k_1},\ldots,t_{k_r})\right)^2\right\} \\
&= \quad \lim_{n_r\to\infty}\cdots\lim_{n_1\to\infty}\sum_{k_r,k'_r=0}^{n_r-1}\cdots\sum_{k_1,k'_1=0}^{n_1-1}\mathrm{Cov}[B^r_{I^{n_r}_{k_r}}\cdots B^1_{I^{n_1}_{k_1}}, B^r_{I^{n_r}_{k'_r}}\cdots B^1_{I^{n_1}_{k'_1}}] \\
&\qquad \times\mathbf{1}_{\mathscr{A}}(t_{k_1},\ldots,t_{k_r})\mathbf{1}_{\mathscr{A}}(t_{k'_1},\ldots,t_{k'_r}).
\end{aligned} \tag{5.12}$$

Since $\mathscr{A}\subset\mathscr{A}'$ and $\mathrm{Cov}[B^r_{I^{n_r}_{k_r}}\cdots B^1_{I^{n_1}_{k_1}}, B^r_{I^{n_r}_{k'_r}}\cdots B^1_{I^{n_1}_{k'_1}}]\geq 0$ by Lemma 4.5.2, we have

$$\begin{aligned}
&\mathrm{Cov}[B^r_{I^{n_r}_{k_r}}\cdots B^1_{I^{n_1}_{k_1}}, B^r_{I^{n_r}_{k'_r}}\cdots B^1_{I^{n_1}_{k'_1}}]\mathbf{1}_{\mathscr{A}}(t_{k_1},\ldots,t_{k_r})\mathbf{1}_{\mathscr{A}}(t_{k'_1},\ldots,t_{k'_r}) \\
&\leq \mathrm{Cov}[B^r_{I^{n_r}_{k_r}}\cdots B^1_{I^{n_1}_{k_1}}, B^r_{I^{n_r}_{k'_r}}\cdots B^1_{I^{n_1}_{k'_1}}]\mathbf{1}_{\mathscr{A}'}(t_{k_1},\ldots,t_{k_r})\mathbf{1}_{\mathscr{A}'}(t_{k'_1},\ldots,t_{k'_r}).
\end{aligned}$$

By summing over $k_1, k_1', \ldots, k_r, k_r'$ and taking limits on both sides of the above inequality, and taking into account the identity (5.12), we obtain the inequality (5.11).

4.5.4 L_p-estimates of multiple integrals

In this subsection, we assume that B^1 is the identity function and denote $H_1 = 1$, and B^j is a fBm of Hurst parameter $H_j > 1/2$, $j = 2, \ldots, m$. We assume in addition that $B^2, \ldots, B^m$ are mutually independent. For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \Gamma_r$ (the collection of multi-indices of length r with elements in $\{1, \ldots, m\}$), and $\mathscr{A} \subset [0,T]^r$, we write

$$J_r^\alpha(\mathscr{A}) := \int_0^T \cdots \int_0^T \mathbf{1}_{\mathscr{A}}(s_1, \ldots, s_r) dB_{s_1}^{\alpha_1} \cdots dB_{s_r}^{\alpha_r},$$

provided that this multiple integral exists, as an integrated Riemann-Stieltjes integral in L_p, for all $p \geq 1$. Recall that $D = \{t_k = kT/n, k = 0, 1, \ldots, n\}$.

Proposition 4.5.5. *Denote (see (5.7)) $\mathscr{A}_k = [t_k, t_{k+1}]_<^r$, $k = 0, 1, \ldots, n-1$. Take $s, t \in D$, $\alpha \in \Gamma_r$, and denote $r' = \#\{i : \alpha_i \neq 1\}$. Then we have*

$$\left\| \sum_{k=ns/T}^{nt/T-1} J_r^\alpha(\mathscr{A}_k) \right\|_2 \leq C\nu_n(\alpha)(t-s)^{1/2}, \tag{5.13}$$

where C is a constant that depends on T, m and α, and

$$\nu_n(\alpha) = \begin{cases} n^{1-H_\alpha} & \text{if } \quad r' \text{ is even}, \\ n^{H-H_\alpha} & \text{if } \quad r' \text{ is odd}. \end{cases} \tag{5.14}$$

Here $H_\alpha := H_{\alpha_1} + \cdots + H_{\alpha_r}$ and $H = \max_{i:\alpha_i \neq 1} H_{\alpha_i}$.

Proof According to Proposition 4.5.3, it suffices to show that the L_p-estimate holds for $\mathscr{A}_k = [t_k, t_{k+1}]^r$. In this case, we have

$$J_r^\alpha(\mathscr{A}_k) = \prod_{i=1}^r B_{t_k,t_{k+1}}^{\alpha_i} = \prod_{j=1}^m (B_{t_k,t_{k+1}}^j)^{r_j}, \tag{5.15}$$

where $r_j = \#\{i : \alpha_i = j\}$, $j = 1, \ldots, m$ and $r_1 + \cdots + r_m = r$.

We first consider the case when $r_1 = 0$. Denote by μ_q the qth moment of a standard Gaussian random variable $G \sim \mathscr{N}(0,1)$, that is, $\mu_q = (q-1)!!$ when q is even and $\mu_q = 0$ when q is odd. It is well-known that we have the following Hermite decomposition:

$$x^q = \sum_{p=0}^q \frac{q!}{(q-p)!} \mu_{q-p} H_p(x), \quad x \in \mathbb{R},$$

where

$$H_q(x) = \frac{(-1)^q}{q!} e^{\frac{x^2}{2}} \frac{d^q}{dx^q} \left(e^{-\frac{x^2}{2}} \right).$$

If we denote $G_k^j = (\frac{n}{T})^{H_j} B_{t_k,t_{k+1}}^j$, then, applying the above decomposition to (5.15), we have

$$\begin{aligned} J_r^\alpha(\mathscr{A}_k) &= (\frac{n}{T})^{-\sum_{j=2}^m r_j H_j} \prod_{j=2}^m (G_k^j)^{r_j} \\ &= (\frac{n}{T})^{-\sum_{j=2}^m r_j H_j} \prod_{j=2}^m \sum_{p_j=0}^{r_j} \frac{r_j!}{(r_j - p_j)!} \mu_{r_j - p_j} H_{p_j}(G_k^j). \end{aligned} \tag{5.16}$$

Therefore, for any $0 \leq k_1 \leq k_2 \leq n-1$,

$$\begin{aligned}\left\|\sum_{k=k_1}^{k_2} J_r^{\alpha}(\mathscr{A}_k)\right\|_2^2 &= \left\|\sum_{k=k_1}^{k_2} (\frac{n}{T})^{-\sum_{j=2}^m r_j H_j} \prod_{j=2}^m \sum_{p_j=0}^{r_j} \frac{r_j!}{(r_j-p_j)!} \mu_{r_j-p_j} H_{p_j}(G_k^j)\right\|_2^2 \\ &\leq Cn^{-2\sum_{j=2}^m r_j H_j} \sum_{p_2=0}^{r_1} \cdots \sum_{p_m=0}^{r_m} \left\|\sum_{k=k_1}^{k_2} \prod_{j=2}^m \mu_{r_j-p_j} H_{p_j}(G_k^j)\right\|_2^2. \quad (5.17)\end{aligned}$$

Expanding the right-hand side of the above inequality, we have

$$\begin{aligned}&\left\|\sum_{k=k_1}^{k_2} \prod_{j=2}^m \mu_{r_j-p_j} H_{p_j}(G_k^j)\right\|_2^2 \\ &= \sum_{k,k'=k_1}^{k_2} \mathbb{E}\left[\prod_{j=2}^m \mu_{r_j-p_j} H_{p_j}(G_k^j) \prod_{j'=2}^m \mu_{r_{j'}-p_{j'}} H_{p_{j'}}(G_{k'}^{j'})\right] \\ &= \sum_{k,k'=k_1}^{k_2} \prod_{j=2}^m \mu_{r_j-p_j}^2 \mathbb{E}\left[H_{p_j}(G_k^j) H_{p_j}(G_{k'}^j)\right] \\ &= \left(\sum_{|k-k'|\leq 2} +2 \sum_{k>k'+2}\right) \prod_{j=2}^m \mu_{r_j-p_j}^2 \mathbb{E}\left[H_{p_j}(G_k^j) H_{p_j}(G_{k'}^j)\right] \\ &:= E_1 + E_2. \quad (5.18)\end{aligned}$$

We take $k_1 = \frac{ns}{T}$ and $k_2 = \frac{nt}{T} - 1$. Since

$$\begin{aligned}\mathbb{E}\left[H_{p_j}(G_k^j) H_{p_j}(G_{k'}^j)\right] &= \left(\mathbb{E}[G_k^j G_{k'}^j]\right)^{p_j} \\ &= \left(\alpha_{H_j} \int_k^{k+1} \int_{k'}^{k'+1} |u-v|^{2H_j-2} du dv\right)^{p_j}, \quad (5.19)\end{aligned}$$

where $\alpha_{H_j} = H_j(2H_j - 1)$, it is easy to see that

$$E_1 \leq Cn(t-s). \quad (5.20)$$

On the other hand, from (5.19) we have

$$\mathbb{E}\left[H_{p_j}(G_k^j)H_{p_j}(G_{k'}^j)\right] \leq C(k-k')^{(2H_j-2)p_j}, \quad k,k' : |k-k'| \geq 2,$$

and thus

$$\begin{aligned} E_2 &\leq C \sum_{k>k'+2} \prod_{j=2}^{m} \mu_{r_j-p_j}^2 (k-k')^{(2H_j-2)p_j} \\ &= C \sum_{k>k'+2} (k-k')^{\sum_{j=2}^{m}(2H_j-2)p_j} \prod_{j=2}^{m} \mu_{r_j-p_j}^2. \end{aligned} \tag{5.21}$$

Since $2H_j - 2 < 0$, the quantity $(k-k')^{(2H_j-2)\sum_{j=2}^{m} p_j}$ reaches maximum when $p_2 = \cdots = p_m = 0$. So

$$E_2 \leq C \sum_{k>k'+2} 1 \leq Cn^2(t-s)^2. \tag{5.22}$$

Substituting (5.20) and (5.22) into to (5.18) and taking into account (5.17) and the identity

$$\sum_{j=2}^{m} r_j H_j = H_{\alpha_1} + \cdots + H_{\alpha_r},$$

we obtain inequality (5.13) for the case when r is even.

We turn to the case when $r_1 = 0$ and $r = r'$ is odd. Note that

$$\prod_{j=2}^{m} \mu_{r_j-p_j} \neq 0$$

only if $r_2 - p_2, \ldots, r_m - p_m$ are all even, so for p_j, $j = 2, \ldots, m$ such that $E_2 \neq 0$,

$$\sum_{j=2}^{m} r_j - \sum_{j=2}^{m} p_j = r - \sum_{j=2}^{m} p_j$$

must be even, and so $\sum_{j=2}^{m} p_j$ must be odd. Therefore, for (5.21) we have

$$\begin{aligned} E_2 &\le C \sum_{k>k'+2} (k-k')^{(2H-2)} \\ &\le Cn^{2H}(t-s). \end{aligned} \tag{5.23}$$

Substituting the estimates (5.20) and (5.23) into (5.18) and taking into account (5.17), we obtain the estimate (5.13) when r is odd.

In the case $r_1 > 0$, it follows from the identity (5.15) that

$$\left\| \sum_{k=ns/T}^{nt/T-1} J_r^{\alpha}(\mathscr{A}_k) \right\|_2 \le \left(\frac{T}{n}\right)^{r_1} \left\| \sum_{k=ns/T}^{nt/T-1} \prod_{j=2}^{m} (B^j_{t_k,t_{k+1}})^{r_j} \right\|_2 .$$

Now we can apply the inequality (5.13) for the case $r_1 = 0$ to the right-hand side of the above equation to obtain the inequality (5.13) in the general case.

Remark 4.5.3. *By the monotonicity property in Proposition 4.5.3 we can also show that the rate $1-H_\alpha$ or $H-H_\alpha$ obtained in Proposition 4.5.5 is optimal; that is, there exists a constant C such that the right-hand side of inequality (5.13) is the lower bound for the quantity $\left\|\sum_{k=ns/T}^{nt/T-1} J_r^{\alpha}(\mathscr{A}_k)\right\|_2$ with $\mathscr{A}_k = [t_k, t_{k+1}]^r_<$. This follows by taking $\widetilde{\mathscr{A}_k}$ of the following form:*

$$\widetilde{\mathscr{A}_k} := [t_k + \frac{h}{2^r}, t_k + \frac{h}{2^{r-1}}] \times \cdots [t_k + \frac{h}{4}, t_k + \frac{h}{2}] \times [t_k + \frac{h}{2}, t_k + h],$$

where $h = \frac{T}{n}$ and $k = 0, 1, \dots, n-1$. Since $\widetilde{\mathscr{A}_k} \subset \mathscr{A}_k$, by the monotonicity property, it suffices to find a the lower bound for the quantity

$$\left\| \sum_{k=ns/T}^{nt/T-1} J_r^{\alpha}(\widetilde{\mathscr{A}_k}) \right\|_2 ,$$

which can be done in a similar way as in the proof of Proposition 4.5.5.

Remark 4.5.4. *Denote $r_j = \#\{i : \alpha_i = j\}$, $j = 1, \ldots, m$. From Remark 4.5.2 we see that $\mathbb{E}[J_r^\alpha(\mathscr{A})] = 0$ if $(r - r_1)$ is odd. In fact, in a similar way, we can show that $\mathbb{E}[J_r^\alpha(\mathscr{A})] \neq 0$ iff r_j, $j = 2, \ldots, m$ are all even numbers.*

We turn to the L_2-estimate of centered multiple integral $\widetilde{J}_r^\alpha(\mathscr{A}) := J_r^\alpha(\mathscr{A}) - \mathbb{E}[J_r^\alpha(\mathscr{A})]$. We shall prove that the rate in (5.13) will be improved and this is the basis for the introduction of the modified Taylor scheme. Recall that $H = \max_{i:\alpha_i \neq 1} H_{\alpha_i}$ and $H_\alpha = H_{\alpha_1} + \cdots + H_{\alpha_r}$.

Proposition 4.5.6. *Let $\mathscr{A}_k$, $k = 0, 1, \ldots, n-1$ be as in Proposition 4.5.5. Take $s, t \in D$ and $\alpha \in \Gamma_r$, $r \geq 2$. Denote $r_j = \#\{i : \alpha_i = j\}$, $j = 1, \ldots, m$. If r_j, $j = 2, \ldots, m$ are even, then we have*

$$\left\| \sum_{k=ns/T}^{nt/T-1} \widetilde{J}_r^\alpha(\mathscr{A}_k) \right\|_p \leq C(t-s)^{1/2} \omega_n(\alpha) \tag{5.24}$$

where

$$\omega_n(\alpha) = \begin{cases} n^{1/2 - H_\alpha} & \text{if } \frac{1}{2} < H < \frac{3}{4}, \\ n^{1/2 - H_\alpha} \sqrt{\log n} & \text{if } H = \frac{3}{4}, \\ n^{2H - 1 - H_\alpha} & \text{if } \frac{3}{4} < H < 1. \end{cases}$$

Proof According to Proposition 4.5.4, it suffices to consider the case when $\mathscr{A}_k = [t_k, t_{k+1}]^r$, $k = 0, 1, \ldots, n-1$. We first assume that $r_1 = 0$. By (5.16), we have

$$\mathbb{E}[J_r^\alpha(\mathscr{A}_k)] = \left(\frac{n}{T}\right)^{-\sum_{j=2}^m r_j H_j} \prod_{j=2}^m \mu_{r_j}.$$

So by denoting $\mathscr{P} = \{(p_2,\dots,p_m) : p_j = 0,1,\dots,r_j,\ j = 2,\dots,m\}$ and $\mathbf{0} = (0,\dots,0)$ we can write

$$\widetilde{J}_r^{\alpha}(\mathscr{A}_k) = \left(\frac{n}{T}\right)^{-\sum_{j=2}^m r_j H_j} \sum_{(p_2,\dots,p_m)\in\mathscr{P}\setminus\mathbf{0}} \prod_{j=2}^m \frac{r_j!}{(r_j-p_j)!}\mu_{r_j-p_j} H_{p_j}(G_k^j).$$

As in (5.18), we can write

$$\begin{aligned}\left\|\sum_{k=k_1}^{k_2} \widetilde{J}_r^{\alpha}(\mathscr{A}_k)\right\|_2^2 &\le Cn^{-2\sum_{j=2}^m r_j H_j} \sum_{(p_2,\dots,p_m)\in\mathscr{P}\setminus\mathbf{0}} \left\|\sum_{k=k_1}^{k_2}\prod_{j=2}^m \mu_{r_j-p_j} H_{p_j}(G_k^j)\right\|_2^2 \\ &\le Cn^{-2\sum_{j=2}^m r_j H_j} \sum_{(p_2,\dots,p_m)\in\mathscr{P}\setminus\mathbf{0}} (E_1+E_2). \qquad (5.25)\end{aligned}$$

Since r is even, by the same argument as in the proof of the Proposition 4.5.5, we see that for p_j, $j = 2,\dots,m$ such that $E_2 \neq 0$, $\sum_{j=2}^m p_j$ must be even. So, for $(p_2,\dots,p_m) \in \mathscr{P}\setminus\mathbf{0}$, we have $\sum_{j=2}^m p_j \ge 2$. This implies

$$E_2 \le C\sum_{k>k'+2}(k-k')^{2(2H-2)} \le \begin{cases} Cn(t-s) & \text{if } \frac{1}{2} < H < \frac{3}{4}, \\ Cn\log n(t-s) & \text{if } H = \frac{3}{4}, \\ Cn^{4H-2}(t-s) & \text{if } \frac{3}{4} < H < 1. \end{cases}$$

Applying (5.20) and the above estimate to (5.25), we obtain the upper bound estimate in (5.24).

Finally, the estimate (5.24) for the case $r_1 > 0$ follows immediately from the estimate in the case $r_1 = 0$. □

Remark 4.5.5. *As in Remark 4.5.3, we can show that the upper bound in Proposition 4.5.6 is optimal.*

Remark 4.5.6. *In terms of ν_n defined in (5.14), inequality (5.24) becomes*

$$\left\|\sum_{k=ns/T}^{nt/T-1}\widetilde{J}_r^{\alpha}(\mathscr{A}_k)\right\|_p \quad \le \quad C(t-s)^{1/2}\nu_n(\alpha)\sigma_n,$$

where

$$\sigma_n \quad = \quad \begin{cases} n^{-1/2} & \text{if} \quad \frac{1}{2}<H<\frac{3}{4}\,, \\ n^{-1/2}\sqrt{\log n} & \text{if} \quad H=\frac{3}{4}\,, \\ n^{2H-2} & \text{if} \quad \frac{3}{4}<H<1\,. \end{cases} \tag{5.26}$$

Remark 4.5.7. *For $t\in[0,T]$ and $\gamma\in\Gamma_r$, we define the function*

$$D_\gamma(t) := \mathbb{E}\left[B_{0,t}^{\gamma}\right] = \mathbb{E}[J_r^{\gamma}([0,t]_<^r)].$$

From Remark 4.5.4, $D_\gamma(t)=0$ if some $r_j=\#\{i:\gamma_i=j\}$, $j=2,\dots,m$ is odd. In the following we derive an explicit formula for $D_\gamma(t)$ when all r_j, $j=2,\dots,m$ are even.

Recall that when r is an even number, the set $\mathscr{R}_r$ is defined in Section 4.5.3. When r is an odd number, we define $\mathscr{R}_r$ to be the collection of ways to arrange $(1,\dots,r)$ into $(\frac{r-1}{2})$ pairs and one element and we write $\tau=\{(\tau_1,\tau_2),\dots,(\tau_{r-2},\tau_{r-1}),\tau_r\}$, where τ_i, $i=1,\dots,r$, are elements in $\{1,\dots,r\}$.

For $r\in\mathbb{N}$, we denote by $\mathscr{R}(\gamma)$ the subset of $\mathscr{R}_r$ such that for $\tau\in\mathscr{R}(\gamma)$ we have $\gamma_{\tau_i}=\gamma_{\tau_{i+1}}$ for $i=1,3,5,\dots$ and $i<r$, and when r is odd it satisfies an additional condition that $\tau_r=1$. We denote $\tau^*=\{i=1,3,5,\dots:\tau_i\neq 1, i<r\}$.

We denote by $\dot{B}_t^j$, $t\in[0,T]$ the fractional white noise associated with the fBm B^j (see [18]), $j=1,\dots,m$. Since

$$\mathbb{E}[\dot{B}_t^i\dot{B}_s^j] = \alpha_{H_j}|t-s|^{2H_j-2}\delta_{i,j},$$

where $\alpha_{H_j} = H_j(2H_j - 1)$, *we have*

$$\begin{aligned}\mathbb{E}\left[B^{\gamma}_{0,t}\right] &= \mathbb{E}\left[\int_0^t \cdots \int_0^{t_2} dB^{\gamma_1}_{t_1} \cdots dB^{\gamma_r}_{t_r}\right] \\ &= \mathbb{E}\left[\int_0^t \cdots \int_0^{t_2} \dot{B}^{\gamma_1}_{t_1} \cdots \dot{B}^{\gamma_r}_{t_r} dt_1 \cdots dt_r\right] \\ &= \sum_{\tau \in \mathscr{R}(\gamma)} \int_0^t \cdots \int_0^{t_2} \prod_{i \in \tau^*} \alpha_{H_{\tau_i}} |t_{\tau_i} - t_{\tau_{i+1}}|^{2H-2} dt_1 \cdots dt_r.\end{aligned}$$

4.6 L_p-rates for incomplete and modified Taylor schemes

In this section, we consider the numerical approximation of the solution for the SDE

$$dy_t = V(y_t)dB_t, \quad y_0 \in \mathbb{R}^d, \quad t \in [0,T], \tag{6.1}$$

where $B = (B^1, \dots, B^m)$ and B^j is a standard fractional Brownian motion (fBm) with Hurst parameter $H_j > 1/2$ for $j = 2, \dots, m$ and B^1 is the identity function. Assume in addition that $B^2, \dots, B^m$ are mutually independent.

4.6.1 The incomplete Taylor scheme for SDE driven by fBm

In this subsection, we consider the incomplete Taylor scheme (3.25) of the following form

$$\begin{aligned}y^n_t &= y^n_{t_k} + \widetilde{\mathscr{E}}^{(N)}_{t_k,t}(y^n_{t_k}), \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, \dots, n-1, \quad y^n_0 = y_0, \\ \widetilde{\mathscr{E}}^{(N)}_{s,t}(y) &= \sum_{\gamma \in \widetilde{\Gamma}} \mathscr{V}_\gamma I(y) B^{\gamma}_{s,t}\,.\end{aligned} \tag{6.2}$$

where $\widetilde{\Gamma}$ is a finite subset of Γ. Recall that $\Gamma = \cup_{r=1}^{\infty} \Gamma_r$, and Γ_r is the collection of multi-indices of length r with elements in $\{1, \dots, m\}$. We denote $N = \max_{\gamma \in \widetilde{\Gamma}} |\gamma|$.

Take $\alpha \in \Gamma$. Denote $r'(\alpha) = \#\{i : \alpha_i \neq 1\}$ and

$$\vartheta(\alpha) = \begin{cases} 1 & \text{when } r'(\alpha) \text{ is even} \\ \max\limits_{i:\alpha_i \neq 1} H_{\alpha_i} & \text{when } r'(\alpha) \text{ is odd} \end{cases} .$$

We define

$$\rho = \rho_{\widetilde{\Gamma}} = \min\left\{ H_\alpha - \vartheta(\alpha),\ \alpha \in \Gamma \setminus \widetilde{\Gamma} \right\}, \tag{6.3}$$

where recall that $H_\alpha = H_{\alpha_1} + \cdots + H_{\alpha_r}$ for $\alpha \in \Gamma_r$.

We first derive two auxiliary results. We take β such that $1/2 < \beta < \min_j H_j$.

Lemma 4.6.1. *Let $R^e_{s,t}$, $e = 1,3,4$ be defined by (3.16), (3.22) and (3.23). Assume that $\widetilde{\Gamma}$ has the hierarchical structure introduced in Definition 4.3.3. Assume that $V \in C_b^{N+2}$. Then the following estimate holds for any $i > j$*

$$\left\| \sum_{k=j}^{i-1} R^e_{t_k,t_{k+1}}(y^n_{t_k}) \right\|_p \leq Cn^{-\rho} \left(\frac{i-j}{n} \right)^{1/2} \qquad e = 1,3,4. \tag{6.4}$$

Proof According to Proposition 4.3.3, $R^e_{t_k,t_{k+1}}(y^n_{t_k})$, $e = 1,3,4$, are the summations of quantities of the form $U(y^n_{t_k})B^\alpha_{t_k,t_{k+1}}$ for $\alpha \in \Gamma \setminus \widetilde{\Gamma}$. Since $V \in C_b^{N+2}$, it is easy to see from Proposition 4.3.3 that $U \in C_b^1$. To prove the lemma, it suffices to show that the L_p-estimate (6.4) holds true for

$$\sum_{k=j}^{i-1} U(y^n_{t_k})B^\alpha_{t_k,t_{k+1}}, \quad \alpha \in \Gamma \setminus \widetilde{\Gamma}. \tag{6.5}$$

By Proposition 4.5.5 and the definition of ρ we have

$$\left\| \sum_{k=j}^{i-1} B^\alpha_{t_k,t_{k+1}} \right\|_p \leq Cn^{-\rho} \left(\frac{i-j}{n} \right)^{1/2}.$$

On the other hand, Lemma 4.4.1 implies that $\|y^n\|_{\beta,p} \le C$. So we can apply Proposition 4.5.1 to the quantity (6.5) to obtain the estimate

$$\left\| \sum_{k=j}^{i-1} U(y_{t_k}^n) B_{t_k,t_{k+1}}^{\alpha} \right\|_p \le Cn^{-\rho} \left(\frac{i-j}{n} \right)^{1/2}.$$

This completes the proof.

Lemma 4.6.2. *Let f be a stochastic process on $[0,T]$, such that $\mathbb{E}(\|f\|_\beta^p)$ and $\mathbb{E}(\|f\|_\infty^p)$ are finite for all $p \ge 1$. Let $\widetilde{\Gamma}$, $R_{s,t}^e(y)$, $e = 1,3,4$ and V be as in Lemma 4.6.1 and let $R_{s,t}^e(y)$ be defined in (3.21). Then the following estimate is true for $e = 1,2,3,4$ and $l = 0,1,\dots,n-1$,*

$$\left\| \sum_{k=0}^{l} \int_{t_k}^{t_{k+1}} f_u dR_{t_k,u}^e(y_{t_k}^n) \right\|_p \le Kn^{-\rho}, \tag{6.6}$$

where K depends on $\mathbb{E}(\|f\|_\beta^p)$, $\mathbb{E}(\|f\|_\infty^p)$ and the vector fields V_j.

Proof: According to the estimate (4.3) in Lemma 4.4.4 and taking into account the assumption that $\mathbb{E}(\|f\|_\beta^p)$ and $\mathbb{E}(\|f\|_\infty^p)$ are finite, we have

$$\left\| \int_{t_k}^{t_{k+1}} f_u dR_{t_k,u}^2(y_{t_k}^n) \right\|_p \le Kn^{-(N+2)\beta},$$

where we recall that $N = \max_{\gamma \in \widetilde{\Gamma}} |\gamma|$. Therefore,

$$\begin{aligned} \left\| \sum_{k=0}^{l} \int_{t_k}^{t_{k+1}} f_u dR_{t_k,u}^2(y_{t_k}^n) \right\|_p &\le \sum_{k=0}^{n-1} \left\| \int_{t_k}^{t_{k+1}} f_u dR_{t_k,u}^2(y_{t_k}^n) \right\|_p \\ &\le Kn^{1-(N+2)\beta} \\ &\le Kn^{-\rho}, \end{aligned}$$

where the last inequality follows by taking β sufficiently close to $\min_j H_j$ and the fact that we can find $\alpha \in \Gamma_{N+1}$ such that $H_\alpha - \vartheta(\alpha) < (N+2)\beta - 1$.

We turn to the case $e = 1,3,4$. We write

$$\begin{aligned}\int_{t_k}^{t_{k+1}} f_u dR^e_{t_k,u}(y^n_{t_k}) &= \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} df_v dR^e_{t_k,u}(y^n_{t_k}) + f_{t_k} R^e_{t_k,t_{k+1}}(y^n_{t_k}) \\ &=: R^{e,1}_k + R^{e,2}_k.\end{aligned}$$

Applying Proposition 4.5.1 to $\sum_{k=0}^{l} R^{e,2}_k$ and taking into account the estimate in Lemma 4.6.1 and the assumption that $\|f\|_{\beta,p}$ is finite, we obtain the inequality

$$\left\| \sum_{k=0}^{l} R^{e,2}_k \right\|_p \leq Kn^{-\rho}. \tag{6.7}$$

According to Proposition 4.3.3, the quantity $R^{e,1}_k$ is a linear combination of the terms

$$\int_{t_k}^{t_{k+1}} \int_{t_k}^{u} df_v dB^{\alpha}_{t_k,u}, \quad \alpha \notin \widetilde{\Gamma}.$$

Take $\beta_j < H_j$ for $j = 1,\dots,m$. Then by Lemma 4.8.2 we have

$$\left\| \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} df_v dB^{\alpha}_{t_k,u} \right\|_p \leq Kn^{-\beta_{\alpha_1}-\cdots-\beta_{\alpha_p}-\beta} \leq Kn^{-\rho-1},$$

where the last inequality follows by taking β_j, $j = 1,\dots,m$ sufficiently close to H_j and β to $\min_j H_j$. Therefore,

$$\left\| \sum_{k=0}^{l} R^{e,1}_k \right\|_p \leq \sum_{k=0}^{n-1} \left\| R^{e,1}_k \right\|_p \leq \sum_{k=0}^{n-1} Kn^{-\rho-1} = Kn^{-\rho}. \tag{6.8}$$

Combining (6.7) and (6.8), we obtain the inequality (6.6) for $e = 1,3,4$.

Theorem 4.6.1. *Let $\widetilde{\Gamma}$ be a finite subset of Γ. Assume that $\widetilde{\Gamma}$ satisfies the hierarchical structure defined in Definition 4.3.3. Let y be the solution of equation (6.1) and y^n be the solution of numerical equation (6.2) with $\widetilde{\mathscr{E}}^{(N)}_{s,t}$ defined in (6.2). Take $M > 0$. Assume that $V \in C_b^{N+2}$. Then*

$$\sup_{t\in[0,T]} \left(\mathbb{E}|\mathbf{1}_{\{\|B\|_\beta<M\}}(y_t - y_t^n)|^p\right)^{1/p} \quad \leq \quad C_M n^{-\rho}, \tag{6.9}$$

where C_M is a constant depending on M.

Proof By Theorem 4.3.1, we have

$$\begin{aligned}
&\mathbf{1}_{\{\|B\|_\beta<M\}}(y_t - y_t^n) \\
&= \sum_{e=1}^{4} \left[\mathbf{1}_{\{\|B\|_\beta<M\}}\Phi_t\right] \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \int_{t_k}^{t_{k+1}\wedge t} \left[\mathbf{1}_{\{\|B\|_\beta<M\}}\Psi_s\right] dR^e_{t_k,s}(y^n_{t_k}),
\end{aligned}$$

where Φ and Ψ are solutions of equations (3.29) and (3.30). According to the estimate of $\|\Phi\|_\infty$ in Lemma 4.4.1, the L_p-norm of the quantity $\mathbf{1}_{\{\|B\|_\beta<M\}}\Phi_t$ is less than a constant C_M is independent of n. So to prove the theorem, it suffices to show that the L_p-norm of

$$\sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} \int_{t_k}^{t_{k+1}\wedge t} \left[\mathbf{1}_{\{\|B\|_\beta<M\}}\Psi_s\right] dR^e_{t_k,s}(y^n_{t_k}) \tag{6.10}$$

is less than $C_M n^{-\rho}$. We take $f_s = \mathbf{1}_{\{\|B\|_\beta<M\}}\Psi_s$, then it follows again from Lemma 4.4.1 that $\mathbb{E}[\|f\|_\beta^p]$ and $\mathbb{E}[\|f\|_\infty^p]$ are bounded by a constant independent of n. So applying Lemma 4.6.2 to (6.10) we obtain the upper bound $C_M n^{-\rho}$. This completes the proof.

Remark 4.6.1. *With more careful estimates in Lemmas 4.6.1 and 4.6.2 and with the help of Remark 4.5.3, we can show that the convergence rate of the incomplete Taylor* *scheme obtained in Theorem 4.6.1 is optimal, that is, we can find a constant C such that the left-hand side of (6.9) is greater than its right-hand side.*

The following result follows immediately from Theorem 4.6.1.

Corollary 4.6.1. *Let the assumption be as in Theorem 4.6.1. The scaled error $n^{\rho}(y_t - y_t^n)$, $t \in [0,T]$ of the numerical scheme is bounded in probability (or tight), that is, for every $\varepsilon > 0$, there exists $C > 0$ such that*

$$P(n^{\rho}|y_t - y_t^n| > C) \leq \varepsilon \quad \textit{for all } n.$$

To obtain the best choice of $\widetilde{\Gamma}$ by Theorem 4.6.1, we will follow the same ideas as in (4.5) and (4.6). Take nonnegative integers $r_1, \dots, r_m$ and denote $r' = \sum_{j=2}^m r_j$. First, we see that a possible L_p-rate has the form

$$\rho = \begin{cases} \sum_{j=1}^m r_j H_j - 1 & \textit{if } r' \text{ is even} \\ \sum_{j=1}^m r_j H_j - \max\limits_{j>1: r_j>0} H_j & \textit{if } r' \text{ is odd}. \end{cases} \tag{6.11}$$

Given a ρ of the above form we define

$$\begin{aligned} \widehat{\Gamma}(\rho) &= \{\alpha \in \Gamma : H_\alpha - 1 < \rho,\ r'(\alpha) \text{ is even}\} \\ &\quad \bigcup \left\{\alpha \in \Gamma : H_\alpha - \max_{j:\alpha_j \neq 1} H_{\alpha_j} < \rho,\ r'(\alpha) \text{ is odd}\right\}. \end{aligned} \tag{6.12}$$

Lemma 4.6.3. *Given a ρ of the form (6.11), we define $\widehat{\Gamma}(\rho)$ by (6.12). Then*

$$\rho_{\widehat{\Gamma}(\rho)} = \rho\,.$$

Proof The proof of this lemma is similar to that of Lemma 4.4.5.

Remark 4.6.2. *From this lemma and Theorem 4.6.1, we see that all possible rates for the L_p-convergence has the form (6.11). Given a rate of the form (6.11) the best choice of $\widetilde{\Gamma}$ in (6.2) for the L_p-convergence is (6.12).*

Remark 4.6.3. *We compare Theorem 4.6.1 to the strong convergence results in [11, 22]. We consider the d-dimensional Stratonovich SDE:*

$$y_t = y_0 + \int_0^t V(y_s)dW_s,$$

where $W = (W^1, \dots, W^m)$, W^j is a standard Brownian motion for $j = 2, \dots, m$, $W_t^1 \equiv t$ and W^j, $j = 2, \dots, m$ are mutually independent, and the integral on the right-hand side is a Stratonovich integral. The numerical scheme studied in [22] coincides with the incomplete Taylor scheme (6.2) constructed here. Indeed, by taking $H_j = 1/2$, $j = 2, \dots, m$ and $H_1 = 1$ in (6.11) we see that the possible rates of convergence are $\{1, 2, 3, \dots\}$, and for $\rho = 1, 2, \dots$, the set $\widehat{\Gamma}(\rho)$ becomes

$$\widehat{\Gamma}(\rho) \quad = \quad \{\alpha \in \Gamma : r'(\alpha) + 2r_1(\alpha) < 2\rho + 1\},$$

or

$$\widehat{\Gamma}(\rho) \quad = \quad \{\alpha \in \Gamma : |\alpha| + r_1(\alpha) \leq 2\rho\}, \tag{6.13}$$

where $r_1(\alpha) = \#\{i : \alpha_i = 1\}$, $r'(\alpha) = \#\{i : \alpha_i \neq 1\}$ and $|\alpha|$ is the length of α. By taking

$$\widetilde{\mathscr{E}}_{s,t}^{(N)}(y) \quad = \quad \sum_{\gamma \in \widehat{\Gamma}(\rho)} \mathscr{V}_\gamma I(y) W_{s,t}^\gamma,$$

with $\widehat{\Gamma}(\rho)$ defined in (6.13), we obtain the numerical scheme considered in [22]:

$$y_t^n \;=\; y_{t_k}^n + \widetilde{\mathscr{E}}_{t_k,t}^{(N)}(y_{t_k}^n), \quad y_0^n = y_0,$$

for $t \in [t_k, t_{k+1}]$, $k = 0, 1, \ldots, n-1$. The following strong convergence result is obtained in [22]:

$$\mathbb{E}(|y_t - y_t^n|^2)^{1/2} \;\leq\; Cn^{-\rho}.$$

In particular, the convergence rate $n^{-\rho}$ of the incomplete Taylor scheme in the Brownian case coincides with the convergence rate in the fBm case. So we can consider Theorem 4.6.1 as a generalization of [22] to the fBm case.

Example 4.6.2. *We consider the scalar SDE*

$$y_t \;=\; y_0 + \int_0^t V(y_s)dB_s, \quad y_0 \in \mathbb{R}, \tag{6.14}$$

where B is a one-dimensional fBm with Hurst parameter $H > 1/2$. Take $N \in \mathbb{N}$. The order-N Taylor expansion in this case is

$$\mathscr{E}_{s,t}^{(N)}(y) \;=\; \frac{1}{r!}\sum_{r=1}^{N} \mathscr{V}_r I(y)(B_t - B_s)^r,$$

where $\mathscr{V}_1 = \mathscr{V}$, and $\mathscr{V}_{r+1} = \mathscr{V}_r\mathscr{V}$, r=1,..., N. So the global numerical scheme associated with this Taylor expansion is

$$y_t^n \;=\; y_{t_k}^n + \frac{1}{r!}\sum_{r=1}^{N} \mathscr{V}_r I(y_{t_k}^n)(B_t - B_{t_k})^r, \quad t \in [t_k, t_{k+1}]. \tag{6.15}$$

This is the numerical scheme studied in [10]. By taking $m = d = 1$, we recover from Theorem 4.6.1 the strong convergence result of (6.15) obtained in [10]: the numerical scheme y^n defined in (6.15) converges to the solution y of (6.14) with rate $n^{1-(N+1)H}$ when N is odd and with rate n^{-NH} when N is even.

4.6.2 Modified Taylor scheme

In this subsection, we briefly explain how to improve the convergence rate of the numerical scheme studied in Section 4.6.1 by a slight modification of the scheme. The proof of the main result in this subsection is similar to the previous subsection, and the proof is omitted.

By comparing Proposition 4.5.5 with Proposition 4.5.6, we see that the centered multiple integral

$$\widetilde{J}_r(\mathscr{A}) \quad = \quad J_r(\mathscr{A}) - \mathbb{E}\left[J_r(\mathscr{A})\right],$$

usually will have a smaller L_2-norm, where $\mathscr{A}$ is a subset of $[0,T]^r$ such that the multiple integral $J_r(\mathscr{A})$ is well defined. This leads us to consider the following modification of the Taylor expansion.

Take nonnegative integers $r_1, \ldots, r_m$ and denote $r' = \sum_{j=2}^m r_j$. Let ρ be of the form

$$\rho = \begin{cases} \sum_{j=1}^m r_j H_j - 1 & if\ r' \text{ is even} \\ \sum_{j=1}^m r_j H_j - \max_{j>1: r_j>0} H_j & if\ r' \text{ is odd} \end{cases}, \tag{6.16}$$

and define

$$\widehat{\Gamma}(\rho) \;=\; \{\alpha\in\Gamma : H_\alpha - 1 < \rho,\; r'(\alpha) \text{ is even}\}$$
$$\bigcup\{\alpha\in\Gamma : H_\alpha - \max_{j:\,\alpha_j\neq 1} H_{\alpha_j} < \rho,\; r'(\alpha) \text{ is odd}\},$$

where recall that $H_\alpha = H_{\alpha_1}+\cdots+H_{\alpha_r}$ and $r'(\alpha) = \#\{i : \alpha_i \neq 1\}$.

We denote $\rho' = \min\{\delta : \delta \text{ is of the form (6.16) and } \delta > \rho\}$, and define $\widehat{\Gamma}(\rho)' := \widehat{\Gamma}(\rho')\setminus\widehat{\Gamma}(\rho)$. Recall that we define the function $D_\gamma(t) := \mathbb{E}\left[B^\gamma_{0,t}\right]$; see Remark 4.5.7 for an explicit expression.

Definition 4.6.1. *Let ρ be of the form (6.16). We call*

$$\widehat{\mathscr{E}}_{s,t}(y) = \sum_{\gamma\in\widehat{\Gamma}(\rho)} \mathscr{V}_\gamma I(y) B^\gamma_{s,t} + \sum_{\gamma\in\widehat{\Gamma}(\rho)'} \mathscr{V}_\gamma I(y) D_\gamma(t-s)$$

the modified Taylor expansion.

Remark 4.6.4. *In Proposition 4.3.2 we have shown that the identity (3.20) holds for the incomplete Taylor expansion $\widetilde{\mathscr{E}}^{(N)}_{s,t}$. In fact, (3.20) also holds true when $\widetilde{\mathscr{E}}^{(N)}_{s,t}$ is replaced by the Taylor expansion $\widehat{\mathscr{E}}_{s,t}$. In fact, the only properties of the incomplete Taylor expansion $\widetilde{\mathscr{E}}^{(N)}_{s,t}$ we used in the proof of the proposition are:*

1. *The multiple integrals appearing in the proof are well defined;*
2. *the chain rule used in (3.24) holds true.*

We consider the following modified Taylor scheme:

$$y^n_t \;=\; y^n_{t_k} + \widehat{\mathscr{E}}_{t_k,t}(y^n_{t_k}), \quad t\in[t_k,t_{k+1}], \quad k = 0,1,\dots,n-1. \tag{6.17}$$

As in Remark 4.6.4, it is easy to show that if R^e, $e = 1,2,3,4$ are defined in (3.16), (3.21), (3.22) and (3.23) with $\widetilde{\mathscr{E}}_{s,t}^{(N)}$ replaced by $\widehat{\mathscr{E}}_{s,t}$ and y^n is the modified Taylor scheme (6.17), then identity (3.31) still holds true.

Based on estimate (5.24) and identity (3.31), we can prove the following stronger convergence result. Recall that for $\alpha \in \Gamma$, we denote $r_j(\alpha) = \#\{i : \alpha_i = j\}$, $j = 1, \dots, m$.

Theorem 4.6.3. *Assume that $V \in C_b^{N+2}$. Let y be the solution of equation (6.1) and y^n be the numerical scheme (6.17). Take $M > 0$. If $r_j(\alpha)$, $j = 2, \dots, m$ is even for each $\alpha \in \widehat{\Gamma}(\rho)'$, then*

$$\sup_{t\in[0,T]} \left(\mathbb{E} |\mathbf{1}_{\{\|B\|_\beta < M\}} (y_t - y_t^n)|^p \right)^{1/p} \quad \leq \quad C_M n^{-\rho} \sigma_n,$$

where σ_n is defined in (5.26) and C_M is a constant depending on M. In particular, the scaled error $\sigma_n^{-1} n^\rho (y_t - y_t^n)$ is bounded in probability for each $t \in [0,T]$.

Remark 4.6.5. *As in Remark 4.6.1, we can show that the convergence rate obtained in Theorem 4.6.3 is optimal.*

Following are two applications of the modified Taylor scheme. For simplicity, we assume from now on that $B = (B^1, \dots, B^m)$ is a m-dimensional standard fBm with Hurst parameter $H > 1/2$.

Example 4.6.4. *Take $\rho = 2H - 1$. Then $\widehat{\Gamma}(\rho) = \{1, \dots, m\}$ and $N = 1$. Take $\gamma \in \Gamma$ such that $|\gamma| = N + 1 = 2$, and denote $\gamma = (j, j')$. We calculate $D_\gamma(t)$,*

$$\begin{aligned} D_\gamma(t) = \mathbb{E}[B_{0,t}^\gamma] \quad &= \quad \mathbb{E}\left[\int_0^t \int_0^u dB_{u'}^j dB_u^{j'} \right] \\ &= \quad \frac{1}{2} t^{2H} \delta_{jj'}, \end{aligned}$$

where $\delta_{jj'}$ is the Kronecker function such that $\delta_{jj'} = 1$ when $j = j'$ and $\delta_{jj'} = 0$ other wise. Then the modified order-2 Taylor expansion is

$$\widehat{\mathscr{E}}(y) \quad = \quad V(y)B_{s,t} + \frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{d} V_j^i \partial_i V_j(y)(t-s)^{2H}.$$

The modified Taylor scheme associated with this scheme is

$$y_{t_{k+1}}^n \quad = \quad y_{t_k}^n + V(y_{t_k}^n)B_{t_k,t_{k+1}} + \frac{1}{2}\sum_{j=1}^{m}(\partial V_j V_j)(y_{t_k}^n)(T/n)^{2H},$$

for $k = 0, \ldots, n-1$. This is the modified Euler scheme introduced in [13]. By taking $\rho = 2H - 1$ we recover from Theorem 4.6.3 the convergence result (1.11).

Example 4.6.5. *Let N be an odd integer. We consider the model in Example 4.6.2. The modified Taylor expansion for this model becomes*

$$\widehat{\mathscr{E}}_{s,t}(y) \quad = \quad \sum_{r=1}^{N}\frac{1}{r!}\mathscr{V}_r I(y)(B_t - B_s)^r + \frac{1}{(N+1)!!}\mathscr{V}_{N+1}I(y)(t-s)^{(N+1)H}.$$

The modified Taylor scheme associated with this scheme is

$$\begin{aligned} y_{t_{k+1}}^n \quad = \quad & y_{t_k}^n + \sum_{r=1}^{N}\frac{1}{r!}\mathscr{V}_r I(y_{t_k}^n)(B_{t_{k+1}} - B_{t_k})^r \\ & + \frac{1}{(N+1)!!}\mathscr{V}_{N+1}I(y_{t_k}^n)(T/n)^{(N+1)H}. \end{aligned}$$

According to Theorem 4.6.3, the convergence rate of this scheme is $n^{1/2-H(N+1)}$ for $1/2 < H < 3/4$; $n^{1/2-3(N+1)/4}\sqrt{\log n}$ for $H = 3/4$ and $n^{-1-H(N-1)}$ for $H > 3/4$, which improves the numerical scheme (6.15).

4.7 Numerical approximation in the rough paths case

In this section, we consider the numerical approximation for the d-dimensional rough differential equation:

$$dy_t = V(y_t)dx_t \tag{7.1}$$

on $[0,T]$, where the control function $x \in C([0,T],\mathbb{R}^m)$ is not differentiable, but is enriched with a proper algebraic structure. The theory of rough paths analysis has been developed from the seminal paper by Lyons [25]. Our settings in this section will follow closely [8].

As in (3.2), we can define the Taylor scheme for the solution of (7.1) based on the Taylor expansion:

$$\mathscr{E}_{s,t}^{(N)}(y) = \sum_{\gamma\in\Gamma,|\gamma|\leq N} \mathscr{V}_\gamma I(y) x_{s,t}^{\gamma}, \quad y \in \mathbb{R}^d,$$

where $x_{s,t}^{\gamma}$ is a multiple rough integral that we will define later. Our aim in this section is to show that the expression for $y - y^n$ derived in (3.31), still holds true in the rough paths case. Notice that the results in Section 4.3.1 are only based on the algebraic properties of the differential equation (3.1), so to show (3.31), it suffices to derive a rough paths version of Proposition 4.2.2.

In the first subsection, we briefly review some concepts and results from the rough paths theory. In the second subsection, we generalize Proposition 4.2.2 to the rough paths case.

4.7.1 Elements of the rough paths theory

Denote by $C^{p\text{-var}}([s,t])$ the collection of continuous functions on $[s,t]$ with bounded p-variation. We first define the step-N signature.

Definition 4.7.1. *The step-N signature of $\gamma \in C^{1\text{-}var}([s,t];\mathbb{R}^m)$ is given by*

$$\begin{aligned} S_N(\gamma)_{s,t} &\equiv \left(1, \int_{s<u<t} d\gamma_u, \dots, \int_{s<u_1<\cdots<u_N<t} d\gamma_{u_1} \otimes \cdots \otimes d\gamma_{u_N}\right) \\ &\in \oplus_{k=0}^N (\mathbb{R}^m)^{\otimes k}. \end{aligned}$$

We denote by $G^N(\mathbb{R}^m)$ the so-called *free nilpotent group of step N* over $\mathbb{R}^m$, that is,

$$G^N(\mathbb{R}^m) := \left\{ S_N(\gamma)_{0,1} : \gamma \in C^{1\text{-var}}([0,1];\mathbb{R}^m) \right\}.$$

It is well-known that $G^N(\mathbb{R}^m)$ is a Lie group with respect to the tensor multiplication $\otimes$, and for every $g \in G^N(\mathbb{R}^m)$, the "Carnot-Carathéodory norm"

$$\|g\| := \inf \left\{ \int_0^1 |d\gamma| : \gamma \in C^{1\text{-var}}([0,1];\mathbb{R}^m) \text{ and } S_N(\gamma)_{0,1} = g \right\}$$

is finite and achieved at some minimizing path γ^*, i.e.

$$\|g\| = \int_0^1 |d\gamma^*| \text{ and } S_N(\gamma^*)_{0,1} = g.$$

The norm $\|\cdot\|$ leads to a metric $d(g,h) := \|g^{-1} \otimes h\|$ on $G^N(\mathbb{R}^m)$, called the *Carnot-Carathéodory metric*.

Consider a $G^N(\mathbb{R}^m)$-valued path $\mathbf{x}$ on the time interval $[0,T]$. For any $p \geq 1$, we denote by

$$\|\mathbf{x}\|_{p\text{-var};[s,t]} = \sup_{(t_i)\subset[s,t]} \left(\sum_i d(\mathbf{x}_{t_i}, \mathbf{x}_{t_{i+1}})^p \right)^{1/p}$$

the p-variation norm of $\mathbf{x}$, and when $s=0, t=T$, we simply write $\|\mathbf{x}\|_{p\text{-var}} = \|\mathbf{x}\|_{p\text{-var};[0,T]}$. The following proposition shows that an abstract path $\mathbf{x} : [0,T] \to G^N(\mathbb{R}^m)$ of bounded p-variation can be approximated by a sequence of step-N signatures; see [8]. We denote by d_∞ the infinity distance, that is,

$$d_\infty(\mathbf{x}, \mathbf{x}') := \sup_{t\in[0,T]} d(\mathbf{x}_t, \mathbf{x}'_t),$$

and $C^{p\text{-var}}([0,T]; G^N(\mathbb{R}^m))$ stands for

$$\left\{ \mathbf{x} \in C([0,T]; G^N(\mathbb{R}^m)) : \|\mathbf{x}\|_{p\text{-var}} < \infty \right\}.$$

Proposition 4.7.1. *Let $\mathbf{x} \in C^{p\text{-}var}([0,T]; G^N(\mathbb{R}^m))$, $p \geq 1$, with $\mathbf{x}_0 = (1,0,\ldots,0)$. Then there exists $(x^n)_n \subset C^1([0,T];\mathbb{R}^m)$, such that*

$$d_\infty(\mathbf{x}, S_N(x^n)) \to 0 \text{ as } n \to \infty, \quad \text{and} \quad \sup_n \|S_N(x^n)\|_{p\text{-}var} < \infty.$$

Consider the *rough differential equation* (RDE)

$$dy_t = V(y_t)d\mathbf{x}_t, \quad y_0 \in \mathbb{R}^d, \tag{7.2}$$

where $\mathbf{x} : [0,T] \to G^{\lfloor p \rfloor}(\mathbb{R}^m)$ is a *weak geometric p-rough path*, i.e. an element in $C^{p\text{-var}}([0,T]; G^{\lfloor p \rfloor}(\mathbb{R}^m))$.

Definition 4.7.2. *We take* $\mathbf{x} \in C^{p\text{-}var}([0,T];G^{\lfloor p \rfloor}(\mathbb{R}^m))$*. We say that* $y \in C([0,T];G^{\lfloor p \rfloor}(\mathbb{R}^d))$ *is a solution to equation (7.2) if for any sequence* $(x^n)_n$ *in* $C^{1\text{-}var}([0,T];\mathbb{R}^m)$ *such that*

$$d_\infty(\mathbf{x}, S_{\lfloor p \rfloor}(x^n)) \to 0 \text{ as } n \to \infty, \text{ and } \sup_n \|S_{\lfloor p \rfloor}(x^n)\|_{p\text{-}var} < \infty, \tag{7.3}$$

with y^n *the solution of the equation* $dy^n = V(y^n)dx^n$*, there exists a subsequence of* (x^n, y^n) *(which we still denote by* (x^n, y^n)*) such that* y^n *converges uniformly to* y *when* $n \to \infty$*.*

Theorem 4.7.1. *Assume that* $V = (V_j)_{1 \le j \le m}$ *is a collection of* $C_b^{\lfloor p \rfloor + 1}$*-vector fields on* $\mathbb{R}^d$*. Then, there exists a unique RDE solution to the equation (7.2). The conclusion still holds when* $V = (V_j)_{1 \le j \le m}$ *is a collection of linear vector fields.*

To define the rough path integral $\int_0^\cdot V(x_t)d\mathbf{x}_t$, we consider the following RDE

$$\begin{aligned} dz_t &= d\mathbf{x}_t, \\ dy_t &= V(z_t)d\mathbf{x}_t, \\ (z_0, y_0) &= (0,0). \end{aligned}$$

It follows from Theorem 4.7.1 that if $V \in C_b^{\lfloor p \rfloor + 1}(\mathbb{R}^d)$, then the above equation has a unique solution (z, y). We call y the *rough integral*, denoted as $\int_0^\cdot V(x_t)d\mathbf{x}_t$.

4.7.2 Multiple rough integrals

In this subsection, we consider some multiple rough integrals. We denote by $S_N(\mathbf{x})$, $N \ge \lfloor p \rfloor$, the so-called *Lyons lift* of $\mathbf{x}$, which satisfies $\pi_i(S_N(\mathbf{x})) = \pi_i(\mathbf{x})$ for $i = 1, \ldots, \lfloor p \rfloor$ and $S_N(\mathbf{x}) \in C^{p\text{-var}}([0,T];G^N(\mathbb{R}^m))$. We refer to Section 9.1.2 in [8] for the proof of the existence and uniqueness for the *Lyons lift* of a weak geometric p-rough path. Following is a basic fact on weak geometric rough paths. It shows that if $\mathbf{x}$ is a weak geometric p-rough path, $p \geq 1$, then the multiple integral

$$\int_s^t \cdots \int_s^{u_2} d\mathbf{x}_{u_1} \otimes \cdots \otimes d\mathbf{x}_{u_i}, \quad s,t \in [0,T],$$

coincides with the ith tensor level of the order-N *Lyons's lift* of $\mathbf{x}$, where $N \geq i$.

Lemma 4.7.1. *Let* $\mathbf{x}$ *be a weak geometric* p*-rough path and* $N \in \mathbb{N}$*. Then for each* $i = 1, \ldots, N$*, we have*

$$\pi_i(S_N(\mathbf{x})_{s,t}) = \int_s^t \int_s^{t_i} \cdots \int_s^{t_2} d\mathbf{x}_{t_1} \otimes \cdots \otimes d\mathbf{x}_{t_{i-1}} \otimes d\mathbf{x}_{t_i}.$$

Proof Without loss of generality, we assume $s = 0$. We consider the following linear system of equations

$$\begin{aligned} dz_t^1 &= d\mathbf{x}_t \\ dz_t^2 &= z_t^1 \otimes d\mathbf{x}_t \\ &\cdots \\ dz_t^N &= z_t^{N-1} \otimes d\mathbf{x}_t, \end{aligned}$$

with the initial value $(\mathbf{z}_0^1, \ldots, \mathbf{z}_0^N) = 0$. For convenience, we denote the equation system by

$$dz_t = V(z_t)d\mathbf{x}_t, \;\; z_0 = 0, \tag{7.4}$$

where $V = (V_1, \ldots, V_m)$ and $V_j(z) = A^j z + b^j$. It is easy to see that A^j and b^j are $(\sum_{i=1}^N m^i) \times (\sum_{i=1}^N m^i)$ and $(\sum_{i=1}^N m^i) \times 1$ matrices, respectively, whose entries take values 0 and 1. According to Theorem 4.7.1, the above equation system has a unique solution. In fact, it can be verified directly that

$$z_t^i = \int_0^t \int_0^{t_i} \cdots \int_0^{t_2} d\mathbf{x}_{t_1} \otimes \cdots \otimes d\mathbf{x}_{t_{i-1}} \otimes d\mathbf{x}_{t_i}, \quad i = 1, \ldots, N.$$

According to the definition of solution of RDE, for any sequence $(x^n)_n$ in $C^{1\text{-var}}([0,T];\mathbb{R}^m)$ satisfying (7.3), there exists a subsequence of $(x^n)_n$ (which we still denote by $(x^n)_n$) such that $S_N(x^n)$ converges to the solution $(1, z^1, \ldots, z^N)$ of (7.4) uniformly. On the other hand, according to Proposition 4.7.1, we can choose the sequence $(x^n)_n$ such that $S_N(x^n)$ converges to $S_N(\mathbf{x})$ uniformly. Therefore, we must have $S_N(\mathbf{x}) = (1, z^1, \ldots, z^N)$. This completes the proof.

Following is our main result in this subsection, which can be shown by approximation and with the help of Proposition 4.2.2 and Lemma 4.7.1. For $\alpha \in \Gamma$ such that $|\alpha| = r$, we denote

$$x_{s,t}^{\alpha} \quad := \quad \pi_r(S_r(\mathbf{x}))_{s,t}^{\alpha} = \int_s^t \int_s^{t_i} \cdots \int_s^{t_2} d\mathbf{x}_{t_1}^{\alpha_1} \cdots d\mathbf{x}_{t_{i-1}}^{\alpha_{r-1}} d\mathbf{x}_{t_i}^{\alpha_r},$$

where the second equality holds because of Lemma 4.7.1.

Proposition 4.7.2. *We take $\mathbf{x} \in C^{p\text{-}var}([0,T]; G^{\lfloor p \rfloor}(\mathbb{R}^m))$. Let γ^1, ..., γ^p be multi-indices in Γ. We denote $r = |\gamma^1| + \cdots + |\gamma^p|$ and $\vec{\tau} = (\tau_1, \ldots, \tau_p)$ such that $\tau_i = |\gamma^1| + \cdots + |\gamma^i|$, $i = 1, \ldots, p$. Denote $\gamma = (\gamma^1, \ldots, \gamma^p) \in \Gamma$. Then*

$$\int_s^t \int_s^{t_p} \cdots \int_s^{t_2} d\mathbf{x}_{s,t_1}^{\gamma^1} \cdots d\mathbf{x}_{s,t_{p-1}}^{\gamma^{p-1}} d\mathbf{x}_{s,t_p}^{\gamma^p} = \sum_{\rho \in \Xi_r(\vec{\tau})} x_{s,t}^{\gamma \circ \rho^{-1}}.$$

4.8 Appendix

4.8.1 Estimates of some multiple integrals

In this subsection we provide some estimates on multiple Riemann-Stieltjes integrals needed in this chapter. We also refer to [12] for more studies.

We take $r \in \mathbb{N}$ and $\beta_j \in (\frac{1}{2}, 1]$, $j = 1, \ldots, r$, and $s, s' \in [0, T]$. Let g^j be a Hölder continuous function of order β_j on $[s, s']$ for $j = 1, \ldots, m$. In this subsection, we consider the following multiple integral,

$$g^{\alpha}_{s,t} := \int_s^t \int_s^{s_1} \cdots \int_s^{s_{r-1}} dg^1_{s_r} \cdots dg^{r-1}_{s_2} dg^r_{s_1}, \tag{8.1}$$

where $\alpha = (1, \ldots, r)$.

Recall that for $f \in C^{\beta'}$ and $h \in C^{\beta}$ such that $\beta + \beta' > 1$, we have the following estimate (see, for instance, [13]):

$$\left| \int_s^t f dh \right| \leq K \left[\|f\|_{s,t,\infty} + \|f\|_{s,t,\beta'} (t-s)^{\beta'} \right] \|h\|_{\beta} (t-s)^{\beta}. \tag{8.2}$$

The following lemma provides an estimate for (8.1), which is obtained applying repeatedly (8.2).

Lemma 4.8.1. *There exists a constant $K > 0$ such that for $t, t' \in [s, s']$, we have*

$$\left| g^{\alpha}_{s,t'} - g^{\alpha}_{s,t} \right| \leq K \left(\prod_{j=1}^{r} \|g^j\|_{s,s',\beta_j} \right) (s-s')^{\sum_{j=1}^{r-1} \beta_j} (t'-t)^{\beta_r}. \tag{8.3}$$

Proof: We prove the lemma by induction. The inequality is clear when $r=1$. Suppose the lemma is true for $1,\dots,r-1$. In the case $\beta_r=1$, we have

$$\left|g^{\alpha}_{s,t'}-g^{\alpha}_{s,t}\right| = \left|\int_t^{t'} g^{\alpha-}_{s,u}\,dg^r_u\right| \le \|g^r\|_{s,s',\beta_r} \sup_{[s,s']}\left|g^{\alpha-}_{s,\cdot}\right|(t'-t). \tag{8.4}$$

By induction assumption we have

$$\left|g^{\alpha-}_{s,t}\right| \le K\left(\prod_{j=1}^{r-1}\|g^j\|_{s,s',\beta_j}\right)(s'-s)^{\sum_{j=1}^{r-1}\beta_j},\ \ s_1\in[s,s'].$$

Substituting the above inequality into (8.4) we obtain the estimate (8.3).

In the case $\beta_r\in(\frac{1}{2},1)$, it follows from inequality (8.2) that

$$\begin{aligned}\left|g^{\alpha}_{s,t'}-g^{\alpha}_{s,t}\right| &= \left|\int_t^{t'} g^{\alpha-}_{s,u}\,dg^r_u\right| \\ &\le K\left(\sup_{[s,s']}\left|g^{\alpha-}_{s,\cdot}\right| + \|g^{\alpha-}_{s,\cdot}\|_{s,s',\beta_{r-1}}(s'-s)^{\beta_{r-1}}\right)\|g^r\|_{s,s',\beta_r}(t'-t)^{\beta_r}\end{aligned} \tag{8.5}$$

By induction assumption we have

$$\left\|g^{\alpha-}_{s,\cdot}\right\|_{s,s',\infty} \le K\left(\prod_{j=1}^{r-1}\|g^j\|_{s,s',\beta_j}\right)(s'-s)^{\sum_{j=1}^{r-1}\beta_j},$$

and

$$\left\|g^{\alpha-}_{s,\cdot}\right\|_{s,s',\beta_{r-1}} \le K\left(\prod_{j=1}^{r-1}\|g^j\|_{s,s',\beta_j}\right)(s'-s)^{\sum_{j=1}^{r-2}\beta_j}.$$

Substituting the above two inequalities into (8.5) we have

$$\left|g^{\alpha}_{s,t'}-g^{\alpha}_{s,t}\right| \le K\left(\prod_{j=1}^{r-1}\|g^j\|_{s,s',\beta_j}\right)(s'-s)^{\sum_{j=1}^{r-1}\beta_j}\|g^r\|_{s,s',\beta_r}(t'-t)^{\beta_r}.$$

The proof is now complete.

Let $f_t = (f_t^1, \ldots, f_t^r)$, $t \in [0,T]$, be a function in $C^\beta(\mathbb{R}^r)$, where $\beta \in (1/2, 1]$. We denote

$$g_{s,t}^j(f^j) = \int_s^t f^j dg^j,$$

and

$$g_{s,t}^\alpha(f^\alpha) = \int_s^t \cdots \int_s^{s_2} f_{s_1}^1 dg_{s_1}^1 \cdots f_{s_r}^r dg_{s_r}^r.$$

Lemma 4.8.2. *There exists a constant $K > 0$ such that for $t, t' \in [s, s']$, we have*

$$\left| g_{s,t'}^\alpha(f^\alpha) - g_{s,t}^\alpha(f^\alpha) \right| \leq K(s-s')^{\sum_{j=1}^{r-1} \beta_j} (t'-t)^{\beta_r} \left(\prod_{j=1}^r \|g^j\|_{s,s',\beta_j} (\|f^j\|_{s,s',\infty} + \|f^j\|_{s,s',\beta}) \right).$$

Proof Applying Lemma 4.8.1 yields

$$\left| g_{s,t'}^\alpha(f^\alpha) - g_{s,t}^\alpha(f^\alpha) \right| \leq K \left(\prod_{j=1}^r \|g_{s,\cdot}^j(f^j)\|_{s,s',\beta_j} \right) (s-s')^{\sum_{j=1}^{r-1} \beta_j} (t'-t)^{\beta_r}. \tag{8.6}$$

In the case $\beta_j \in (\frac{1}{2}, 1)$, it follows from inequality (8.2) that

$$\left\| g_{s,\cdot}^j(f^j) \right\|_{s,s',\beta_j} \leq C(\|f^j\|_{s,s',\infty} + \|f^j\|_{s,s',\beta_j}) \|g^j\|_{s,s',\beta_j}. \tag{8.7}$$

In the case $\beta_j = 1$, we have

$$\left\| g_{s,\cdot}^j(f^j) \right\|_{s,s',\beta_j} \leq \|f^j\|_{s,s',\infty} \|g^j\|_{s,s',\beta_j}. \tag{8.8}$$

The lemma then follows by substituting (8.7) and (8.8) into (8.6).

4.8.2 Estimates of numerical solutions

In this subsection, we derive upper bound estimates for the numerical solutions. We follow the approaches of [13]. We first state an auxiliary result that provides estimates on integrals whose integrands are step functions. We define the seminorm,

$$\|x\|_{a,b,\beta,n} = \sup\left\{\frac{|x_u - x_v|}{|v-u|^\beta}; \quad u,v \in D\right\}.$$

Recall that $D = \{kT/n : k = 0,1,\dots,n\}$ is a partition of $[0,T]$. When $a = 0$ and $b = T$, we simply write $\|x\|_{\beta,n} = \|x\|_{a,b,\beta,n}$. We will denote $\eta(t) = t_k$ for $t \in [t_k, t_{k+1})$.

Lemma 4.8.3. *Let $y = \{y_t, t \in [0,T]\}$ be a function with values in $\mathbb{R}^m$ such that $\|y\|_{\beta,n} < \infty$, $n \geq 1$. Take $V \in C_b^1(\mathbb{R}^m)$, and $x \in C^{\beta'}([0,T])$ such that $\beta + \beta' > 1$. Then for $s,t \in D$ such that $s < t$ we have*

$$\left|\int_s^t V(y_{\eta(r)})dx_r\right| \;\; \leq \;\; K\left[1 + \|y\|_{s,t,\beta,n}(t-s)^\beta\right]\|x\|_{\beta'}(t-s)^{\beta'},$$

where the K is a constant depending on β, β', $\|V\|_\infty$ and $\|\partial V\|_\infty$.

Proof See [13].

Assume that $g = (g^1,\dots,g^m)$ and $g^i \in C^\beta([0,T])$, $i = 1,\dots,m$ for $\beta > \frac{1}{2}$. We fix $n \in \mathbb{N}$ and the partition of $[0,T]$ given by $t_i = i\frac{T}{n}$, $i = 0,1,\dots,n$. Consider the following differential equation,

$$y_t = y_0 + \sum_{l=1}^N \sum_{\alpha\in\Gamma_l} \int_0^t \varphi_\alpha(y_{\eta(s)})dg^\alpha_{\eta(s),s},\; t \in [0,T], \tag{8.9}$$

where $N \in \mathbb{N}$ is some constant, and φ_α, $\alpha \in \cup_{l=1}^N \Gamma_l$ are functions with values in $\mathbb{R}^{d\times m}$. Recall that Γ_l is the collection of multi-indices of length l with elements in $\{1,\dots,m\}$.

We shall derive some estimates for the Hölder seminorm and supremum norm of the solution of this equation.

The constants appearing in the following results depend on β, T, $\|\varphi_\alpha\|_\infty$ and $\|\partial\varphi_\alpha\|_\infty$ for $\alpha \in \Gamma$ of length less or equal to N.

Lemma 4.8.4. *Let y be the solution of equation (8.9). Assume that $\varphi_\alpha \in C_b^1$ for $\alpha \in \cup_{l=1}^N \Gamma_l$. Then there exists a positive constant C such that*

$$\|y\|_\beta \;\leq\; C\|g\|_\beta \vee \|g\|_\beta^{1/\beta+N-1}, \tag{8.10}$$

and

$$\|y\|_\infty \;\leq\; |y_0| + C\|g\|_\beta \vee \|g\|_\beta^{1/\beta+N-1}. \tag{8.11}$$

Furthermore, there exists $K_0 > 0$ such that for $s,t \in [0,T]$ and $\|g\|_\beta |t-s|^\beta \leq K_0$, we have

$$\|y\|_{s,t,\beta} \;\leq\; K\|g\|_\beta \vee \|g\|_\beta^N. \tag{8.12}$$

Proof Let $s,t \in [0,T]$ be such that $s < t$. It follows from (8.9) that

$$\begin{aligned} |y_t - y_s| \;\leq\; & \sum_{j=1}^m \left| \int_s^t \varphi_j(y_{\eta(u)}) dg_u^j \right| \\ & + \sum_{l=2}^N \sum_{\alpha\in\Gamma_l} \left| \int_s^t \varphi_\alpha(y_{\eta(u)}) dg^\alpha_{\eta(u),u} \right|. \end{aligned} \tag{8.13}$$

We first derive an estimate for $\|y\|_{\beta,n}$. Assume that $s,t \in D$, that is, $s = \eta(s)$ and $t = \eta(t)$. Applying Lemma 4.8.3 to the first term on the right-hand side of the above inequality yields

$$\sum_{j=1}^{m}\left|\int_s^t \varphi_j(y_{\eta(s)})dg_s^j\right| \le C(1+\|y\|_{s,t,\beta,n}(t-s)^\beta)\|g\|_\beta(t-s)^\beta. \tag{8.14}$$

On the other hand, for $\alpha \in \Gamma_l$ with $l = 1,\ldots,N$, we have

$$\begin{aligned}\left|\int_s^t \varphi_\alpha(y_{\eta(u)})dg^\alpha_{\eta(u),u}\right| &\le \sum_{k=ns/T}^{nt/T-1}\left|\int_{t_k}^{t_{k+1}} \varphi_\alpha(y_{t_k})dg^\alpha_{\eta(u),u}\right| \\ &\le C\sum_{k=ns/T}^{nt/T-1}|g^\alpha_{t_k,t_{k+1}}| \le Cn(t-s)(n^{-\beta}\|g\|_\beta)^l = Cn^{1-\beta l}(t-s)\|g\|_\beta^l, \end{aligned} \tag{8.15}$$

where the third inequality follows from Lemma 4.8.1. So, the second term on the right-hand side of (8.13) is bounded by

$$\sum_{l=2}^{N}\sum_{\alpha\in\Gamma_l}\left|\int_s^t \varphi_\alpha(y_{\eta(u)})dg^\alpha_{\eta(u),u}\right| \le C(t-s)\sum_{l=2}^{N}\|g\|_\beta^l. \tag{8.16}$$

Substituting (8.14) and (8.16) into (8.13), we obtain

$$|y_t - y_s| \le C\|y\|_{s,t,\beta,n}\|g\|_\beta(t-s)^{2\beta} + C(t-s)^\beta\sum_{l=1}^{N}\|g\|_\beta^l.$$

Dividing both sides of the above inequality by $(t-s)^\beta$, and then taking the seminorm $\|\cdot\|_{s,t,\beta,n}$ on the left-hand side we obtain

$$\|y\|_{s,t,\beta,n} \le C\|y\|_{s,t,\beta,n}\|g\|_\beta(t-s)^\beta + C\sum_{l=1}^{N}\|g\|_\beta^l. \tag{8.17}$$

If we assume that $T/n \le \frac{1}{2}(2C\|g\|_\beta)^{-1/\beta}$, then we can find an integer k_0 such that

$$\frac{1}{2}(2C\|g\|_\beta)^{-1/\beta} \le k_0T/n \le (2C\|g\|_\beta)^{-1/\beta}. \tag{8.18}$$

Denote $\Delta := k_0 T/n$ and take u, v such that $u - v = \Delta$, then from the second inequality in (8.18) we have

$$C\|g\|_\beta (u-v)^\beta \quad \leq \quad \frac{1}{2}.$$

Applying this inequality to (8.17) we obtain

$$\|y\|_{v,u,\beta,n} \leq \frac{1}{2}\|y\|_{v,u,\beta,n} + C\sum_{l=1}^{N} \|g\|_\beta^l ,$$

or

$$\|y\|_{v,u,\beta,n} \leq 2C\sum_{l=1}^{N} \|g\|_\beta^l . \tag{8.19}$$

This inequality provides the upper bound for $\|y\|_{v,u,\beta,n}$ for $u, v \in D : v - u = \Delta$.

For any $s, t \in D$ such that $t - s > \Delta$.

$$\begin{aligned} \frac{|y_t - y_s|}{(t-s)^\beta} \quad &\leq \quad \frac{|y_{s+\Delta} - y_s|}{(t-s)^\beta} + \frac{|y_{s+2\Delta} - y_{s+\Delta}|}{(t-s)^\beta} + \cdots + \frac{|y_t - y_{s+\lfloor \frac{t-s}{\Delta} \rfloor \Delta}|}{(t-s)^\beta} \\ &\leq \quad (\lfloor \frac{t-s}{\Delta} \rfloor + 1) \sup_{v \in [0, T-\Delta]} \|y\|_{v,v+\Delta,\beta,n} \frac{\Delta^\beta}{(t-s)^\beta} . \end{aligned}$$

Taking the supremum over $s, t \in D$ on both sides of the above inequality and taking into account (8.19), we obtain

$$\|y\|_{\beta,n} \leq C(\frac{T}{\Delta} + 1)^{1-\beta} (2C\sum_{l=1}^{N} \|g\|_\beta^l) .$$

From the first inequality in (8.18) we have

$$(\frac{T}{\Delta} + 1)^{1-\beta} \leq C(\|g\|_\beta^{1/\beta - 1} + 1).$$

Therefore,

$$\begin{aligned}\|y\|_{\beta,n} &\leq C(\|g\|_\beta^{1/\beta-1}+1)(\sum_{l=1}^N \|g\|_\beta^l) \\ &\leq C\|g\|_\beta \vee \|g\|_\beta^{1/\beta+N-1}. \end{aligned} \tag{8.20}$$

If we assume that $T/n \geq \frac{1}{2}(2C\|g\|_\beta)^{-1/\beta}$ or $n \leq 2T(2C\|g\|_\beta)^{1/\beta}$. Applying (8.15) to (8.13) we obtain

$$|y_t - y_s| \leq \sum_{l=1}^N Cn^{1-\beta l}(t-s)\|g\|_\beta^l.$$

Dividing both sides of the above inequality by $(t-s)^\beta$, and then taking the supremum over all $s,t \in D$, we obtain

$$\|y\|_{\beta,n} \leq \sum_{l=1}^N Cn^{1-\beta l}\|g\|_\beta^l.$$

Since $n \leq 2T(2C\|g\|_\beta)^{1/\beta}$, we have

$$\|y\|_{\beta,n} \leq C\|g\|_\beta^{1/\beta} + \sum_{l=2}^N C\|g\|_\beta^l. \tag{8.21}$$

Combining (8.20) and (8.21) we obtain the estimate

$$\|y\|_{\beta,n} \leq C\|g\|_\beta \vee \|g\|_\beta^{1/\beta+N-1}. \tag{8.22}$$

It follows from inequality (8.13) that for $s,t \in [t_k, t_{k+1}]$, $k = 0,1,\ldots,n-1$,

$$\|y\|_{s,t,\beta} \leq \sum_{l=1}^N \|g\|_\beta. \tag{8.23}$$

For any $s,t \in [0,T]$, we have

$$\frac{|y_t - y_s|}{|t-s|^\beta} \le \frac{|y_{\eta(s)+\frac{T}{n}} - y_s|}{|\eta(s)+\frac{T}{n} - s|^\beta} + \frac{|y_{\eta(t)} - y_{\eta(s)-\frac{T}{n}}|}{|\eta(t)-\eta(s)+\frac{T}{n}|^\beta} + \frac{|y_t - y_{\eta(t)}|}{|t-\eta(t)|^\beta}.$$

We apply (8.23) to the first and third term on the right-hand side of the above inequality and apply (8.22) to the second term to obtain

$$\frac{|y_t - y_s|}{|t-s|^\beta} \le C\|g\|_\beta \vee \|g\|_\beta^{1/\beta+N-1}.$$

The estimate (8.10) then follows by taking the supremum over $s,t \in [0,T]$ on the above left-hand side.

The estimate of $\|y\|_\infty$ follows immediately from (8.10). Indeed, by the definition of $\|\cdot\|_\beta$ we have

$$|y_t| \le |y_0| + T^\beta (C\|g\|_\beta \vee \|g\|_\beta^{1/\beta+N-1}).$$

Taking the supremum of $|y_t|$ over $t \in [0,T]$ we obtain (8.11).

Finally, it is easy to derive inequality (8.12) from (8.17).

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