

Three Dimensional Jacobian Derivations And Divisor Class Groups

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Abstract

In this thesis, we use P. Samuel's purely inseparable descent methods to investigate the divisor class groups of the intersections of pairs of hypersurfaces of the form $w_1^p = f$, $w_2^p = g$ in affine 5-space with f, g in $A = k[x, y, z]$; k is an algebraically closed field of characteristic $p > 0$. This corresponds to studying the divisor class group of the kernels of three dimensional Jacobian derivations on A that are regular in codimension one. Our computations focus primarily on pairs where f, g are quadratic forms. We find results concerning the order and the type of these groups. We show that the divisor class group is a direct sum of up to three copies of \mathbb{Z}_p , is never trivial, and is generated by those hyperplane sections whose forms are factors of linear combinations of f and g .

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To my parents, wife and precious two little daughters, Mariam and Fatimah

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Introduction

Let k an algebraically closed field of characteristic $p > 0$, $R = k[x_1, \dots, x_n]$, and for each f_1, \dots, f_n in R , let $J(f_1, \dots, f_n)$ denote the determinant of the $n \times n$ matrix, $[\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)}]$. A map $D : R \rightarrow R$ is a n -dimensional Jacobian derivation if there exists $f_1, \dots, f_{n-1} \in R$ such that for all $h \in R$, $D(h) = J(h, f_1, \dots, f_{n-1})$.

P. Samuel in his 1964 Tata notes studied the divisor class group of the kernels of two dimensional Jacobian derivations. Normal surfaces in \mathbb{A}_k^5 that are defined by an equation of the form $w^p = f(x, y)$, where $f \in k[x, y]$, are called Zariski surfaces and a lot of effort has focused on studying their divisor class groups. If $D(f) = J(f, -) = f_y \frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y}$ is such a derivation on $k[x, y]$ and f_x and f_y are relatively prime, then, in the case where f is a homogeneous polynomial of degree 2 in $k[x, y]$, Samuel shows that the kernel of $D(f)$ is isomorphic to the coordinate ring of the Zariski surface $w^p = f(x, y)$ and that the divisor class group of the kernel of $D(f)$ is isomorphic to one copy of \mathbb{Z}_p [1, pages 65, 66].

J. Lang studied the divisor class group of $w^p = f(x, y)$ for a more general $f(x, y) \in k[x, y]$ [7], and he described an algorithm that determine the divisor class group of $D(f)$ up to isomorphism [12]. Moreover, most of the efforts have been on exploring three features of the divisor class group of f of various degrees and forms: their properties for a generic f , all groups that are obtainable as divisor class groups of the kernels of $D(f)$, and explicit sets of independent generators of the class groups.

This thesis represents an initial step in extending these explorations to the divisor class group of

the kernels of three dimensional Jacobian derivations that are regular in codimension one. For the most part we concentrate on the case where the defining polynomials, f and g , are homogeneous polynomials of degree 2. As mentioned above, in the case of a two dimensional Jacobian derivation defined by a quartic form, it was relatively easy to prove that the divisor class group of the kernel is isomorphic to \mathbb{Z}_p . However, the three dimensional case is more complex. We will show that the divisor class group of the kernel when f and g are homogeneous of degree 2 is isomorphic to a direct sum of up to three copies of \mathbb{Z}_p , is always non trivial, and is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ generically. Through a complete analysis of the quadratic case, we also prove that for all f and g , the divisor class group is generated by the hyperplane sections whose forms are factors of linear combinations of f and g .

In chapter one, section one, we give a brief definition of the divisor class group of a Krull domain A . In section two, we present a brief discussion of some properties of the divisor class group and, specifically, its relation with the unique factorization domains. Section three is a brief summary of Galois descent techniques. The background and preliminaries given in chapter one can be found in a detailed fashion in the work of P. Samuel's 1964 Tata notes and R. Fossum's book, titled, The Divisor Class Group of Krull Domains.

In chapter two, section one, we show that the kernels of the three dimensional Jacobian derivation defined by $f, g \in k[x, y, z]$ is isomorphic to the coordinate ring of the intersections of the two hypersurfaces, $w_1^p = f, w_2^p = g$ in \mathbb{A}_k^5 and that the divisor class group is isomorphic to the additive group of polynomial logarithmic derivatives of the derivation. This section includes several computational results as well. Section two provides several key examples that will be used in the chapter three.

In chapter three, section one, we discuss the divisor class group when f and g are forms both of degree one. Section two discusses the divisor class group in the case where f and g are homogeneous of degrees two and one, respectively. In section three, we present a detailed and complete analysis of the case where f, g are quadratic forms.

In chapter four, section one, we provide a case of homogeneous polynomials f and g of distinct degrees greater than two, for which their corresponding divisor class group is trivial generically. In section two, we will conclude with several key conjectures concerning Jacobian derivations in general.

Notations

Notation 1. k -an algebraically closed field of characteristic $p \neq 0$.

Notation 2. k^* denote all the nonzero elements of k .

Notation 3. k^n -set of all n -tuples of elements in k .

Notation 4. $k^{n \times n}$ denote all the n by n matrices whose entries are elements in k .

Notation 5. \mathbb{A}_k^n -affine n -space over k .

Notation 6. \mathbb{F}_p denote the elementary or prime subfield of k .

Notation 7. If A is a Krull ring we denote by $Cl(A)$ the divisor class group of A .

Notation 8. Given $f \in k[x, y, z]$, $g \in k[x, y, z]$, and $p > 0$ a prime number, let $X_{f,g} \subseteq \mathbb{A}_k^3$ be the hypersurface defined by the intersection of the two hypersurfaces $w_1^p = f$ and $w_2^p = g$. Also, let $Cl(X_{f,g})$ denote the divisor class group of the coordinate ring of $X_{f,g}$.

Notation 9. For $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ we denote by:

- $\deg(f)$ - the total degree of f .
- $\deg_{x_i}(f)$ - the degree of f in the variable x_i .

Notation 10. Given $f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$, $J(f_1, \dots, f_n)$ denote the determinant of the following Jacobian matrix :

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \dots & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \dots & \dots & \dots & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \dots & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Chapter 1

Background and Preliminaries

This chapter contains some definitions and results obtained from P. Samuel's 1964 Tata notes [1] and R. Fossum's The divisor class group of a Krull domain [2]. These results form the framework and the foundation for this thesis. In section one below, we give a brief definition of the divisor class group of a Krull domain A . In section two, we discuss some properties of such a divisor class group. In section three, we discuss how Samuel developed the technique of Galois descent to study the divisor class group of the kernel of a derivation acting on a Krull domain.

1.1 The Divisor Class Group $Cl(A)$

Definition 1.1.1. Let A be an integral domain and let K be its field of fractions or its quotient field. A **fractional ideal** \mathcal{I} of A is an A sub-module of K with the property that there exists a non-zero element $d \in A$ such that $d\mathcal{I} \subset A$. Moreover, we say that \mathcal{I} is **principal** if there exists $x \in K$ such that $\mathcal{I} = (x)$; i.e. \mathcal{I} is generated by a single element. \mathcal{I} is said to be **integral** if $\mathcal{I} \subset A$. We say that a non-zero fractional ideal \mathcal{I} is **divisorial** if it is an intersection of principal ideals.

Definition 1.1.2. For a fractional ideal \mathcal{I} of A , let us write

$$\mathcal{I}^{-1} = \{y \in K \mid y\mathcal{I} \subset A\} = \bigcap_{I \in \mathcal{I}} (I^{-1})$$

By definition, \mathcal{I}^{-1} is a divisorial fractional ideal. Hence, $(\mathcal{I}^{-1})^{-1}$ is again a divisorial ideal with the property that $\mathcal{I} \subset (\mathcal{I}^{-1})^{-1}$. Observe that $(\mathcal{I}^{-1})^{-1}$ is contained in every principle fractional ideal containing \mathcal{I} and hence it is the smallest divisorial ideal containing \mathcal{I} . Let us denote $(\mathcal{I}^{-1})^{-1}$ by $\overline{\mathcal{I}}$.

Definition 1.1.3. Let $I(A) = \{\mathcal{I} \subset K \mid \mathcal{I} \neq 0, \mathcal{I} \text{ is fractional ideal of } A\}$. So, $I(A)$ denote the set of non-zero fractional ideals of A . In $I(A)$, let us define an equivalence relation " \sim " -named **Artin equivalence**- as follows:

We say that two fractional ideals \mathcal{I} and \mathcal{V} are equivalent (written $\mathcal{I} \sim \mathcal{V}$) $\iff \overline{\mathcal{I}} \sim \overline{\mathcal{V}}$.

The quotient set of $I(A)$ defined by the Artin equivalence relation is called the set of divisors of A and denoted by $D(A)$. So, for each $\mathcal{I} \in I(A)$, $\overline{\mathcal{I}}$ denote the equivalence class of \mathcal{I} in $D(A)$.

Lemma 1.1.1. *If \mathcal{I}, \mathcal{V} and \mathcal{C} are fractional ideals in $I(A)$ then:*

$$\text{If } \mathcal{I} \sim \mathcal{V} \text{ then } \mathcal{I}\mathcal{C} \sim \mathcal{V}\mathcal{C}.$$

Proof. See [3, page 26] □

Definition 1.1.4. Let A be a domain. A is a Krull ring if there exists a family of discrete valuations $(v_i)_{i \in I}$ of K such that:

1. $A = \bigcap_i R_{v_i}$, where R_{v_i} denotes the ring of (v_i) .
2. For every non-zero $x \in A$, $v_i(x) = 0$ for almost $i \in I$.

Theorem 1.1.1. *A Noetherian integrally closed domain is a Krull ring.*

Proof. For the full proof of this theorem see [1, page 5] . □

Definition 1.1.5. Let A be a Krull ring. It follows from Lemma 1.1.1 that the multiplication of fractional ideals passes down to a well defined composition on the set of divisors $D(A)$ which can be written additively: $\overline{\mathcal{I}} + \overline{\mathcal{V}} = \overline{\mathcal{I}\mathcal{V}}$. Thus, $D(A)$ acquires the structure of an abelian group with identity element \overline{A} .

Let $F(A)$ be the subgroup of $D(A)$ generated by the principle divisors (equivalence classes of principle ideals). The quotient group $D(A)/F(A)$ is called the **divisor class group** of A and we denote it by $Cl(A)$.

1.2 Properties of $Cl(A)$

Lemma 1.2.1. *Every non-zero prime ideal contains a non zero prime ideal which is divisorial.*

Proof. See [1, page 7] □

It follows from Lemma 1.2.1 that there are no strict containments between divisorial prime ideals. Hence, we have the following important result.

Theorem 1.2.1. *$Cl(A)$ is generated by the classes of the height one prime ideals of A .*

Proof. See [1, page 7] □

The following theorem is a good measure of the extent to which the Krull ring A fails to be a unique factorization domain.

Theorem 1.2.2. *The following are equivalent for a domain A :*

1. *A is Krull ring and every prime ideal of height one is principle .*
2. *A is UFD .*
3. *$Cl(A) = 0$.*

Definition 1.2.1. A is said to be factorial if $Cl(A) = 0$.

Let A and B be Krull rings such that $B \subset A$. Let $Q \subset B$ and $q \subset A$ be a prime ideals. We say that q lies over Q if $q \cap B = Q$ and we write it as $q|Q$.

The ramification index of Q in q is the unique integer $e(q, Q)$ such that $QA_q = q^e A_q$ (A_q is the localization of the ring A at q).

Theorem 1.2.3. *Let $B \subset A$ be Krull domains such that for every prime ideal q of A , $\text{height}(q \cap B) \leq 1$. Then there is a well defined group homomorphism $\phi : Cl(B) \longrightarrow Cl(A)$.*

P. Samuel [1, page 20] proved that the condition in Theorem 1.2.3 is satisfied if A has either one of the following properties:

- A is integral over B or
- A is a flat B algebra.

Remark 1.2.1. If A is a factorial ring, then $Cl(A) = 0$. Hence, the kernel of the map ϕ in Theorem 1.2.3 is equal to $Cl(B)$. Theorem 1.3.1 in the next section provides a nice characterization of $\ker(\phi)$.

Remark 1.2.2. By Theorem 1.2.1, the homomorphism $\phi : Cl(B) \longrightarrow Cl(A)$ - described in Theorem 1.2.3 - can be determined by its action on height one prime ideals. So, if Q is a height one prime ideal of B , then the map ϕ can be defined by the following assignment:

$$Q \longrightarrow \sum_{q|Q} e(q, Q)Q$$

The sum is taken over all height one prime ideals q in A lying over Q and the sum has to be finite since A is a Krull ring. And then ϕ can be extended by linearity.

The following theorem was developed by M. Nagata [1, page 21] and it is very useful tool for characterizing the factoriality of Krull rings.

Theorem 1.2.4. *Let A be a Krull ring and S a multiplicatively closed subset in A . Then $S^{-1}A$ is a flat Krull ring and:*

1. $\phi(Cl(A)) \longrightarrow Cl(S^{-1}A)$ is surjective
2. If S is generated by prime elements then ϕ is surjective.

Proof. See [1, page 21]

□

Gauss had used Theorem 1.2.4 to establish and prove the following theorem:

Theorem 1.2.5. *Let A be a Krull ring. Then $A[x]$ is a Krull ring and $\phi(Cl(A)) \longrightarrow Cl(A[x])$ is bijective.*

Proof. See [1, page 22] □

1.3 Galois Descent Techniques

P. Samuel [1] has studied the kernel of a derivation acting on a Noetherian integrally closed domain (i.e Krull domain) with a characteristic $p \neq 0$. He has used this technique to study the relationship between the divisor class groups and the group of logarithmic derivatives. He was able to come up with a large class of examples of factorial rings -as well as non factorial rings- with certain characteristics.

Throughout the rest of this paper, all rings studied are assumed to be Noetherian integrally closed domains and are thus Krull rings. Also, we will focus on the case where the quotient field of A over the quotient field of B is purely inseparable extension.

Let A be a Krull ring of characteristic $p > 0$ and let K be the quotient field of A . Let D be the derivation of K such that $D(A) \subset A$. Let L be the kernel of D and $B = A \cap L$. Then B is a Krull ring and in particular, A is integral over B .

By Theorem 1.2.3, we have a well defined homomorphism $\phi(Cl(B)) \longrightarrow Cl(A)$.

Set $\mathcal{L} = \{t^{-1}D(t) \mid t \in K \text{ and } t^{-1}D(t) \in A\}$ and $\mathcal{L}' = \{u^{-1}D(u) : u \text{ is a unit in } A\}$.

Lemma 1.3.1. 1. \mathcal{L} is an additive subgroup of A and

2. \mathcal{L}' is a subgroup of \mathcal{L} .

Proof. 1. Follows directly by product rule.

2. Obvious. □

\mathcal{L} is called the *group of logarithmic derivatives* of D .

Theorem 1.3.1. 1. *There exists a canonical monomorphism $\bar{\phi} : \ker(\phi) \longrightarrow \mathcal{L}/\mathcal{L}'$.*

2. *If $[K : L] = p$ and $D(A)$ is not contained in any height one prime of A , then $\bar{\phi}$ is an isomorphism.*

Proof. See [1, page 62] □

Remark 1.3.1. If $I \in \ker(\phi) \subset Cl(B)$, then $\phi(I) = aA$ for some $a \in K$. Samuel proves that $a^{-1}D(a) \in A$ and then he defines $\bar{\phi}(I) = a^{-1}D(a)$. Since $\mathcal{L} \subset A$ and the characteristic of K is p , each nonzero element of $\ker(\phi)$ has order p .

Remark 1.3.2. In Theorem 1.3.1(2), each height one prime Q of B has a unique ideal q in A lying over it (specifically, $q = \{b \in A : b^p \in Q\}$) and $e(q : Q) = 1$ or p . If $Q \in \ker(\phi)$ with $e(q : Q) = p$ and q is principle, generated by some $b \in A$, then $Q = b^pB$ and, hence, is trivial in $Cl(B)$. In particular, if A is a unique factorization domain, then $Cl(B)$ is generated by the non-principle unramified height one primes of B .

The next theorem is very helpful in calculating the divisor class group of some examples in this paper.

Theorem 1.3.2. *If $[K : L] = p$, then*

1. *There exists $a \in B$ such that $D^p = aD$.*
2. *An element $t \in K$ is equal $v^{-1}D(v)$ for some $v \in K$ if and only if $D^{p-1}(t) - at = -t^p$.*

Proof. See [1, pages 63, 64] □

The following two results are more like observations, but they will be stated as lemmas since they appear in many of the computations of the divisor class group.

Lemma 1.3.2. *If A is factorial ring and $u, v \in A$ are relatively prime in A such that $uv \in B$, then $u^{-1}D(u) \in A$.*

Proof. Since $uv \in B$, then $D(uv) = 0$. By product rule, $\frac{D(uv)}{uv} = \frac{D(u)}{u} + \frac{D(v)}{v} = 0 \implies vD(u) = uD(v)$.
Since u, v are relatively prime in A , then $u^{-1}D(u) \in A$. □

Lemma 1.3.3. 1. If $t \in \mathcal{L}$, then $t = h^{-1}D(h)$ for some $h \in A$.

2. If A is factorial, then \mathcal{L} is generated by $\{h^{-1}D(h) : h \in A \text{ and } h \text{ is irreducible}\}$.

Proof. Let $t \in \mathcal{L}$. Then, there exists some $g \in K$ such that $t = g^{-1}D(g)$, where $g = \frac{a}{b}$ and $a, b \in A$.
Choose $h = b^{p-1}a$, then $t = h^{-1}D(h)$. Part (2) is a consequence of Lemma 1.3.2. □

Chapter 2

Calculating the Group of Logarithmic Derivatives

Throughout the rest of this thesis, we will assume that k is algebraically closed field of characteristic $p \geq 3$, f and g belong to $A = k[x, y, z]$ where A is the polynomial ring in 3 variables x, y and z over k . Let $B = k[x^p, y^p, z^p, f, g]$ and let $K = k(x, y, z)$ and $L = k(x^p, y^p, z^p, f, g)$ be the quotient fields of A and B respectively.

Let $h_1, h_2, h_3 \in A$ and let $J(h_1, h_2, h_3)$ denote the determinant of the 3×3 Jacobian matrix $[\frac{\partial(h_1, h_2, h_3)}{\partial(x, y, z)}]$. Let $D(h) : K \rightarrow K$ -where $h \in K$ - be the k -derivation defined by $D(h) = J(h, f, g)$. Then, $D(A) \subset A$, $L \subset \ker D$ and $k \subset \ker D$.

Let $\mathbf{X}_{f,g}$ denote the intersection of the two hypersurfaces $w_1^p = f$ and $w_2^p = g$ in \mathbb{A}_k^5 . In section 2.1, we will prove that the divisor class group of $X_{f,g}$ is isomorphic to the additive group of logarithmic derivatives \mathcal{L} of D in A . In section 2.2, we will provide six key examples which we will use in chapter three in our analysis of $Cl(X_{f,g})$ when f and g are general quadratic forms.

Notation 11. If the 2×2 minors of the Jacobian matrix, $[\frac{\partial(f,g)}{\partial(x,y,z)}]$, have greatest common divisor 1 in A , then we refer to this assumption as **the gcd condition**.

Observe that part 2 of Theorem 1.3.1 is partially not applicable if $D(A)$ is contained in any height one prime ideal of A . Hence, throughout this paper, we will assume that both f and g satisfy **the**

gcd condition.

2.1 Calculating the Group of Logarithmic Derivatives

Remark 2.1.1. Since both f and g satisfy the gcd condition, then as a consequence of that the following relations must hold:

- $f, g \notin A^{(p)} = k[x^p, y^p, z^p]$.
- $g \notin A^{(p)}[f] = k[x^p, y^p, z^p, f]$ and $f \notin A^{(p)}[g] = k[x^p, y^p, z^p, g]$.
- $f, g \notin K^{(p)} = k(x^p, y^p, z^p)$.
- $f \notin K^{(p)}(g) = k(x^p, y^p, z^p, g)$ and $g \notin K^{(p)}(f) = k(x^p, y^p, z^p, f)$.

Lemma 2.1.1. *The coordinate ring of $X_{f,g}$ is isomorphic to $B = k[x^p, y^p, z^p, f, g]$.*

Proof. The coordinate ring of $X_{f,g}$ is $R = A[w_1, w_2]/I$, where I is the ideal in $A[w_1, w_2]$ generated by $(w_1^p - f, w_2^p - g)$. Let $\phi : A[w_1, w_2] \rightarrow B$ be the mapping that sends $\alpha \in k$ to α^p , x to x^p , y to y^p , z to z^p , w_1 to f , and w_2 to g . This is a surjective homomorphism since k is perfect. We have $\dim B = 3$. Thus by Theorem 1.8 in [4, page 6], $\ker(\phi)$ is a height 2 prime ideal in $A[w_1, w_2]$. Since $w_1^p - f$ and $w_2^p - g$ belong to $\ker(\phi)$, then $I \subseteq \ker(\phi)$. By Remark 2.1.1, we have $f \notin A^{(p)}$ and $g \notin A^{(p)}[f]$. Hence, $w_1^p - f$ generates a prime ideal in $A[w_1]$ and the image of $w_2^p - g$ in $A[w_1, w_2]/(w_1^p - f)$ does so as well. Also, since I is generated by two elements and $(0) \subsetneq (w_1^p - f) \subsetneq (w_1^p - f, w_2^p - g) = I$, then its height must be equal to 2. So, it follows that I is a height 2 prime ideal in $A[w_1, w_2]$. But since I is a prime ideal of height 2 contained in $\ker \phi$, then we must have $\ker(\phi) = I$. Therefore, R is isomorphic to B . \square

Lemma 2.1.2. $D^{-1}(0) \cap A = B$.

Proof. By Remark 2.1.1, we have

$$K^{(p)} \subsetneq K^{(p)}(f) \subsetneq K^{(p)}(f, g) \subseteq D^{-1}(0) \subsetneq K$$

Now, since $[K : K^{(p)}] = p^3$, $[K^{(p)}(f, g) : K^{(p)}(f)] = p$, and $[K^{(p)}(f) : K^{(p)}] = p$, then we must have the field of quotients $K^{(p)}(f, g) = D^{-1}(0)$, which means that B and $D^{-1}(0) \cap A$ have the same quotient field.

Also, observe that the gcd condition implies that R is regular in codimension 1. Hence, by Lemma 2.1.1, B is normal and, Thus, is integrally closed.

Since $D^{-1}(0) \cap A$ is integral over B and they have the same quotient field, the two must be equal; i.e. $D^{-1}(0) \cap A = B$. □

The next lemma shows that as long as the gcd condition on f and g holds, $Cl(X_{f,g})$ can be computed by calculating \mathcal{L} .

Lemma 2.1.3. *Let $\mathcal{L} = \{h^{-1}D(h) : h \in K \text{ and } h^{-1}D(h) \in A\}$. Then $Cl(X_{f,g}) \approx \mathcal{L}$.*

Proof. By Theorem 1.3.1(1), $Cl(X_{f,g})$ injects into $\frac{\mathcal{L}}{\mathcal{L}}$, where $\mathcal{L}' = \{u^{-1}D(u) : u \text{ is a unit in } A\}$. Since the units of A are exactly the units of k , $\mathcal{L}' = 0$.

By Lemma 2.1.2, we have $[K : D^{-1}(0)] = p$ and $D^{-1}(0) = K^{(p)}(f, g) = L$. Since f and g satisfy the gcd condition, then the image $D(A)$ is not contained in any hight one prime ideal of A . Thus, by Theorem 1.3.1(2), $Cl(X_{f,g})$ is isomorphic to \mathcal{L} . □

Note that we have proved Lemmas 2.1.1, 2.1.2 and 2.1.3 for any f and g . But since the primary focus of this paper is on f and g that are homogeneous of degree 2, then from now on(unless specified otherwise) we will assume that f and g are quadratic forms.

Remark 2.1.2. *If the characteristic of $k = 2$* , then the gcd condition does not hold for f and g since they are both homogeneous of degree 2. This explains the assumption in this chapter and the next that the characteristic of k is at least 3.

Definition 2.1.1. If D is a derivation on a field K and $t \in K$, we will say that t is a **logarithmic derivative of D if there exists $h \in K$ such that $t = h^{-1}D(h)$.**

Lemma 2.1.4. *Let $t \in A$ be a nonzero logarithmic derivative of D . Then $deg t = 1$.*

Proof. It is easily checked that if h is a homogeneous polynomial of degree w , then $D(h)$ is a homogeneous polynomial of degree $w + 1$ and therefore $D^p(h)$ is a homogeneous of degree $p + w$. By Theorem 1.3.2(1), we have $D^p(h) = aD(h)$ with $a \in B$. Comparing the degrees we see that a is homogeneous of degree $p - 1$.

Let t be a polynomial in A which is a logarithmic derivative of D . Then we have that $t = h^{-1}D(h)$ for some $h \in K$.

Write $t = t_\alpha + \dots + t_\beta$ and $h = h_d + \dots + h_s$ where the t_j and h_i are all none zero homogeneous elements, where t_α, h_d are the components of smallest degrees α and d , and t_β, h_s are the components of largest degrees β and s . Then, we get

$$D(h_d) + \dots + D(h_s) = (h_d + \dots + h_s)(t_\alpha + \dots + t_\beta)$$

The right hand side of the above equation has $h_d t_\alpha$ as its term of lowest degree and $h_s t_\beta$ as its term of largest degree. The degree of the first nonzero term on the left hand side is at least $d + 1$, while the degree of the last nonzero term is at most $s + 1$. Hence, we get two inequalities

$$d + 1 \leq d + \alpha$$

and

$$s + 1 \geq s + \beta$$

Thus, we have

$$1 \leq \alpha \leq \beta \leq 1$$

So, we conclude That is $\alpha = \beta = 1$. □

Proposition 2.1.1. $Cl(X_{f,g})$ is a p -group of type (p, p, p) of order p^c , where $c \leq 3$.

Proof. By Lemma 2.1.2, we have that K is purely inseparable extension of L of degree p . By Proposition 1.3.2, there exists $a \in D^{-1}(0) \cap A$ such that $D^p = aD$ and t is logarithmic derivative if

and only if

$$D^{p-1}(t) - at = -t^p \quad (2.1)$$

By Lemma 2.1.4, we write $t = a_1x + a_2y + a_3z$, where $a_i \in k$. Substituting this expression for t in equation (2.1), we get on the left side of this equation a polynomial in x, y and z whose coefficients are linear expressions in the a_i with coefficients in k . Thus,

$$\begin{aligned} D^{p-1}(a_1x + a_2y + a_3z) - a(a_1x + a_2y + a_3z) &= -(a_1x + a_2y + a_3z)^p \\ D^{p-1}(a_1x + a_2y + a_3z) - a(a_1x + a_2y + a_3z) &= -(a_1x)^p - (a_2y)^p - (a_3z)^p \end{aligned} \quad (2.2)$$

By comparing coefficients on both side of equation (2.2), we get the equations

$$L(a_1, a_2, a_3) = a_i^p$$

and

$$L'(a_1, a_2, a_3) = 0$$

Thus, t is a logarithmic derivative if and only if $L(a_1, a_2, a_3) = a_i^p$ and $L'(a_1, a_2, a_3) = 0$, for $1 \leq i \leq 3$. The ring $C = k[a_1, a_2, a_3]$ with the property $L(a_1, a_2, a_3) = a_i^p$ is a finite dimensional vector space over k . Moreover, the spanning set of the vector space is the set:

$$\{(a_1^\delta a_2^\gamma a_3^\theta \mid \delta, \gamma, \theta < p)\}$$

Since C satisfies the minimal condition, then C is an Artinian ring. Thus, by Proposition 8.3 [4, page 89], C has a finite number of maximal ideals.

Henceforth, the 3-equations system $L(a_1, a_2, a_3) = a_i^p$ have a finite number of solutions in k , by Bezout's theorem [5, page 198], there are at most p^3 solutions in k . Since \mathcal{L} is an additive group, then it must be an elementary p -group of order at most p^3 .

□

Proposition 2.1.2. *Let $t_1, \dots, t_s \in A$ be logarithmic derivatives of D ; i.e. $t_1, \dots, t_s \in \mathcal{L}$.*

The set $\{t_1, \dots, t_s\}$ is Linearly independent over k if and only if it is linearly independent over \mathbb{F}_p .

Proof. Assume that the set $\{t_1, \dots, t_s\}$ is linear independent over \mathbb{F}_p but not over k . Let t_i be the first element such that $t_i = a_1 t_1 + a_2 t_2 + \dots + a_{i-1} t_{i-1}$, where $a_i \in k$, not all zeroes, and $1 \leq i \leq s$.

By Theorem 1.3.2, there exists $a \in B$ such that $D^p = aD$ and t_i is logarithmic derivative if and only if $D^{p-1}(t_i) - a_i = -t_i^p$. Thus,

$$\begin{aligned}
 D^{p-1}\left(\sum_{j=1}^{i-1} a_j t_j\right) - a\left(\sum_{j=1}^{i-1} a_j t_j\right) &= -\left(\sum_{j=1}^{i-1} a_j t_j\right)^p \\
 \sum_{j=1}^{i-1} a_j (D^{p-1}(t_j) - a t_j) &= -\left(\sum_{j=1}^{i-1} a_j t_j\right)^p \\
 \sum_{j=1}^{i-1} a_j (-t_j^p) &= -\sum_{j=1}^{i-1} a_j^p t_j^p
 \end{aligned} \tag{2.3}$$

By comparing coefficients on both sides of equation(2.3), we get

$$a_j^p = a_j \quad \forall j = 1, 2, \dots, i-1$$

Hence, $a_j \in \mathbb{F}_p \quad \forall j = 1, 2, \dots, i-1$ which contradicts the assumption that $\{t_1, \dots, t_s\}$ is linear independent over \mathbb{F}_p .

The proof of reverse direction is obvious. □

Lemma 2.1.5. *1. If for some $h \in A^{(p)}[g]$, $f + h = uv$ with $u, v \in A$, then $u^{-1}D(u) = J(u, v, g)$; i.e. $u^{-1}D(u) \in \mathcal{L}$.*

2. If for some $h \in A^{(p)}[f]$, $g + h = uv$ with $u, v \in A$, then $u^{-1}D(u) = J(u, f, v)$; i.e. $u^{-1}D(u) \in \mathcal{L}$.

Proof. Suppose $h \in A^{(p)}[g]$, $f + h = uv$. Since $h \in A^{(p)}[g]$, then $J(u, h, g) = 0$. Hence,

$$D(u) = J(u, f, g) = J(u, f, g) + J(u, h, g) = J(u, f + h, g) = J(u, uv, g)$$

Now, we have

$$\begin{aligned}
D(u) &= \begin{vmatrix} u_x & u_y & u_z \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ u_x v + uv_x & u_y v + uv_y & u_z v + uv_z \\ g_x & g_y & g_z \end{vmatrix} \\
&= u \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ g_x & g_y & g_z \end{vmatrix} + v \begin{vmatrix} u_x & u_y & u_z \\ u_x & u_y & u_z \\ g_x & g_y & g_z \end{vmatrix} \\
&= u \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ g_x & g_y & g_z \end{vmatrix} = uJ(u, v, g)
\end{aligned}$$

We conclude from the previous argument that $u^{-1}D(u) = J(u, v, g) \in \mathcal{L}$.

The proof of part (2) is similar. □

Lemma 2.1.6. *If $\deg_z D(x) = 0$, $\deg_z D(y) = 0$, $\deg_z D(z) \leq 1$ and $t \in \mathcal{L}$, then $t \in k[x, y]$.*

Proof. By Lemma 1.3.3(1), $t = h^{-1}D(h)$ for some $h \in A$. Also, since we have $D(h) = h_x D(x) + h_y D(y) + h_z D(z)$, then $\deg_z D(h) \leq \deg_z h$. It follows that $\deg_z h^{-1}D(h) = \deg_z t = 0$ which means that t is a polynomial in two variables; i.e. $t \in k[x, y]$. □

Proposition 2.1.3. *Let $t \in A$ be a logarithmic derivative of D at h . Let $\phi : A \rightarrow A$ be the k -automorphism induced by the homogeneous change of coordinates map that is defined by*

$\phi(x, y, z) = (x, y, z) M$, *where $M \in k^{3 \times 3}$ and the determinant of $M \neq 0$. Let $D_\phi(h)$ be the determinant of the Jacobian matrix $[\frac{\partial(h, \phi(f), \phi(g))}{\partial(x, y, z)}]$. Then $D_\phi(h)$ is a derivation on K and*

$$\phi(t) = \frac{1}{\det M} \frac{D_\phi(\phi(h))}{\phi(h)}$$

Proof. Let $M = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$, where $\alpha_{ij} \in k$ for each $1 \leq i, j \leq 3$.

Observe that if $h \in A$, then $\phi(h(x, y, z)) = h(\phi(x), \phi(y), \phi(z))$. Hence,

$$\phi(h) = h(\alpha_{11}x + \alpha_{21}y + \alpha_{31}z, \alpha_{12}x + \alpha_{22}y + \alpha_{32}z, \alpha_{13}x + \alpha_{23}y + \alpha_{33}z).$$

By taking the partial derivative of $\phi(h)$ with respect to x , y and z respectively, we get

$$(\phi(h))_x = \alpha_{11}h_x(\phi(x), \phi(y), \phi(z)) + \alpha_{12}h_y(\phi(x), \phi(y), \phi(z)) + \alpha_{13}h_z(\phi(x), \phi(y), \phi(z))$$

$$(\phi(h))_y = \alpha_{21}h_x(\phi(x), \phi(y), \phi(z)) + \alpha_{22}h_y(\phi(x), \phi(y), \phi(z)) + \alpha_{23}h_z(\phi(x), \phi(y), \phi(z))$$

$$(\phi(h))_z = \alpha_{31}h_x(\phi(x), \phi(y), \phi(z)) + \alpha_{32}h_y(\phi(x), \phi(y), \phi(z)) + \alpha_{33}h_z(\phi(x), \phi(y), \phi(z))$$

By repeating the same process for f and g as well, we get the following system of equations:

$$\begin{pmatrix} \phi(h)_x & \phi(f)_x & \phi(g)_x \\ \phi(h)_y & \phi(f)_y & \phi(g)_y \\ \phi(h)_z & \phi(f)_z & \phi(g)_z \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} h_x(\phi) & f_x(\phi) & g_x(\phi) \\ h_y(\phi) & f_y(\phi) & g_y(\phi) \\ h_z(\phi) & f_z(\phi) & g_z(\phi) \end{pmatrix}$$

From the left hand side of the above equation we have

$$\det\left[\frac{\partial(\phi(h), \phi(f), \phi(g))}{\partial(x, y, z)}\right] = J(\phi(h), \phi(f), \phi(g)) = D_\phi(\phi(h)) =$$

While from the right hand side we have

$$\det\left[M \phi\left(\frac{\partial(h, f, g)}{\partial(x, y, z)}\right)\right] = \det M \phi(J(h, f, g)) = \det M \phi(D(h))$$

Thus;

$$D_\phi(\phi(h)) = \det M \phi(D(h)) \implies \phi(D(h)) = \frac{1}{\det M} D_\phi(\phi(h)) \implies \phi\left(\frac{D(h)}{h}\right) = \frac{1}{\det M} \frac{D_\phi(\phi(h))}{\phi(h)} \implies \phi(t) = \frac{1}{\det M} \frac{D_\phi(\phi(h))}{\phi(h)}.$$

So, we have proved that if $t \in A$ is a logarithmic derivative of D at h , then $\det M \phi(t)$ is the logarithmic derivative of D_ϕ at $\phi(h)$. □

2.2 Examples

Example 2.2.1. Let $f = x^2 - z^2$ and $g = y^2 - z^2$. Then, it is clear that f and g satisfy the gcd condition. By Lemma 2.1.5, we have

- $t_1 = \frac{D(y+z)}{y+z} = J(y+z, f, y-z) = 4x \in \mathcal{L}$ since $g = (y-z)(y+z)$.
- $t_2 = \frac{D(x+z)}{x+z} = J(x+z, x-z, g) = -4y \in \mathcal{L}$ since $f = (x-z)(x+z)$.
- $t_3 = \frac{D(x+y)}{x+y} = J(x+y, x-y, g) = 4z \in \mathcal{L}$ since $f - g = (x+y)(x-y)$.

Note that the t_i are the logarithmic derivatives of D at factors of linear combinations of f and g . By Proposition 2.1.1, the size of \mathcal{L} is at most p^3 . By Proposition 2.1.2, since t_i are linear independent over \mathbb{F}_p , they are linear independent over k . Henceforth, \mathcal{L} and $Cl(X_{f,g})$ are isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$; i.e.

$$Cl(X_{f,g}) \approx \mathcal{L} \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Example 2.2.2. Let $f = x^2 + y^2 + z^2$ and $g = xy$. Then f and g satisfy the gcd condition and by Lemma 2.1.5, we have

- $t_1 = \frac{D(x)}{x} = J(x, f, y) = -2z \in \mathcal{L}$ since $g = xy$.
- $t_2 = \frac{D(x-y+z)}{x-y+z} = J(x-y+z, x-y+z, g) = 2(x+y) \in \mathcal{L}$ since
 $f - 2g = (x-y+z)(x-y-z)$.
- $t_3 = \frac{D(x+y+z)}{x+y+z} = J(x+y+z, x+y-z, g) = 2(x-y) \in \mathcal{L}$ since
 $f + 2g = (x+y+z)(x+y-z)$.

Note again that the t_i are the logarithmic derivatives of D at factors of linear combinations of f and g .

Following the same argument demonstrated in Example 2.2.1, we conclude that

$$Cl(X_{f,g}) \approx \mathcal{L} \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Example 2.2.3. Let $f = xy$, $g = x^2 + z^2$. Observe that f and g satisfy the gcd condition.

Let $h \in A$. Then,

$$D(h) = 2xz h_x - 2yz h_y - 2x^2 h_z$$

Note that $\deg_y D(h) \leq \deg_y h$. Hence,

$$\deg_y \frac{D(h)}{h} = 0$$

So, if t is a logarithmic derivative of D ; i.e. $t \in \mathcal{L}$ then, by Lemma 2.1.4, t is linear and by Lemma 2.1.6, $\deg_y t = 0$. Hence, t is a logarithmic derivative of D if and only if $t = ax + bz \in k[x, z]$, where $a, b \in k$. Moreover, by Lemma 2.1.5, we have

- $t_1 = \frac{D(x)}{x} = J(x, y, g) = 2z \in \mathcal{L}$ since $f = xy$.
- $t_2 = \frac{D(x + \sqrt{-1}z)}{x + \sqrt{-1}z} = J(x + \sqrt{-1}z, f, x - \sqrt{-1}z) = 2\sqrt{-1}x \in \mathcal{L}$ since $g = (x + \sqrt{-1}z)(x - \sqrt{-1}z)$. $\sqrt{-1}$ is a fixed root in k of $X^2 + 1$.

Since t_1 and t_2 are linearly independent over \mathbb{F}_p , then by Proposition 2.1.2, the set $\{t_1, t_2\}$ generates the divisor class group of $X_{f,g}$. Note that the generators $\{t_1, t_2\}$ of \mathcal{L} are the logarithmic derivatives of D at factors of linear combinations of f and g .

Thus, by Proposition 2.1.1 and Lemma 2.1.3, the divisor class group $X_{f,g}$ is isomorphic to \mathcal{L} which is exactly a direct sum of two copies of \mathbb{Z}_p , i.e

$$Cl(X_{f,g}) \approx \mathcal{L} \approx \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Example 2.2.4. Let $f = x^2 - xy$ and $g = y^2 + xz$. Note that f and g satisfy the gcd condition and we have the following observations:

- $D(x) = -x^2 \implies D^2(x) = 2x^3 \implies D^3(x) = -6x^4 \implies D^4(x) = 24x^5 \implies$
..... $\implies D^n(x) = (-1)^n n! x^{n+1}$. In particular, when $n = p$ and $n = p - 1$ then we have
 $D^p(x) = -p! x^{p+1} = 0$ and by Wilson theorem [6, page 22], we have $D^{p-1}(x) = (p-1)! x^p = -x^p$.
- $D(y) = xy - 2x^2 \implies D^2(y) = 2x^3 = D^2(x) \implies D^n(y) = D^n(x) = (-1)^n n! x^{n+1}$.
- $D(z) = 4xy - 2y^2 + xz$.

If $t \in \mathcal{L}$ then by Lemma 2.1.4, $t \in k[x, y]$ since $\deg_z t = 0$. Hence $t = ax + by$, for some $a, b \in k$. By Theorem 1.3.2 we have t is a logarithmic derivative of D if and only if $t = ax + by$ if and only if $D^{p-1}(t) - at = -t^p$ where $D^p = aD$ and $a \in B$. But since $D^p(x) = 0$ and $D(x) \neq 0$, then we must have $a = 0$. Henceforth,

$$t \text{ is a logarithmic derivative of } D \iff t = ax + by \iff D^{p-1}(t) = -t^p \iff D^{p-1}(ax + by) = -(ax + by)^p \iff aD^{p-1}(x) + bD^{p-1}(y) = -a^p x^p - b^p y^p \iff aD^{p-1}(x) + bD^{p-1}(x) = -a^p x^p - b^p y^p \iff -(a+b)x^p = -a^p x^p - b^p y^p \iff a = a^p \text{ and } b = 0.$$

So, $t = ax$ for some $a \in \mathbb{F}_p$. We conclude that \mathcal{L} is generated by the logarithmic derivative of D at x , which of course is a factor of f , and hence

$$Cl(X_{f,g}) \approx \mathcal{L} \approx \mathbb{Z}_p$$

Example 2.2.5. Let $f = xy + z^2$, $g = yz$. Both f and g satisfy the gcd condition. We have the following observations:

- $D(y) = -y^2 \implies D^2(y) = 2y^3 \implies D^3(y) = -6y^4 \implies D^4(y) = 24y^5 \implies$
..... $\implies D^n(y) = (-1)^n n! y^{n+1}$. If $n = p$ then $D^p(y) = -p! y^{p+1} = 0$ and $D^{p-1}(y) = (p-1)! y^p = -y^p$. Also, $t = \frac{D(y)}{y} = J(y, f, z) = -y \in \mathcal{L}$ since $g = yz$.
- $D(z) = yz \implies D^2(z) = D(yz) = D(g) = 0 \implies D^{p-1}(z) = 0$.

- Since $D(x) = xy - 2z^2$ and $\deg_x x^{-1}D(x) = 0$ then by Lemma 2.1.6 and Lemma 2.1.4, we have $t = by + cz$ for some $b, c \in k$.

By Theorem 1.3.2, there exists $a \in B$ such that $D^p(y) = aD(y)$.

Since $D^p(y) = 0$ and $D(y) \neq 0$ we must have $a = 0$. Hence, t is a logarithmic derivative of D if and only if $D^{p-1}(t) = -t^p$ if and only if $D^{p-1}(by + cz) = -(by + cz)^p$ if and only if $bD^{p-1}(y) + cD^{p-1}(z) = -b^p y^p - c^p z^p$ if and only if $-by^p = -b^p y^p - c^p z^p$.

Comparing coefficients on both sides of the last equation, we get $b = b^p$ and $c = 0$, which means that $b \in \mathbb{F}_p$. Therefore, \mathcal{L} is generated by the logarithmic derivative of D at y , which is a factor of g , and hence

$$Cl(X_{f,g}) \approx \mathcal{L} \approx \mathbb{Z}_p$$

Example 2.2.6. Let $f = yz + x^2$, $g = xz + y^2$. f and g satisfy the gcd condition. We will break this example into two cases:

Case (I) : Assume $p = 3$. Then

- $D(x) = xz - 2y^2 = g \implies D^2(x) = D^3(x) = 0$.
- $D(y) = yz - 2x^2 = f \implies D^2(y) = D^3(y) = 0$.
- $D(z) = xy + 2z^2 \implies D^2(z) = x^3 + y^3 - z^3$.

By Theorem 1.3.2, for $t \in A$, we have $t \in \mathcal{L}$ if and only if $D^2(t) - at = -t^3$, where $D^3 = aD$. Since $D^3(x) = 0$ then we must have $a = 0$. By Lemma 2.1.4, if $t \in \mathcal{L}$, then $t = ax + by + cz$ for some $a, b, c \in k$. Hence,

$t \in \mathcal{L} \iff D^2(t) + t^3 = 0 \iff c(x^3 + y^3 - z^3) + (ax + by + cz)^3 = 0 \iff t = c(-x - y + z)$ for some $c \in \mathbb{F}_3$.

Therefore, \mathcal{L} is generated by $-x - y + z$. Observe that $\frac{D(x-y)}{x-y} = -x - y + z$ and $x - y$ is a factor of $f - g$. Hence,

$$Cl(X_{f,g}) \approx \mathcal{L} \approx \mathbb{Z}_3$$

Case (II) : Assume $p \geq 5$.

This example is an interesting one. As we shown previously when $p = 3$, the divisor class group is isomorphic to one copy of \mathbb{Z}_p and we may think that the same fact will hold for every prime greater than or equal to 3 . But it turns out that this is not the case for $p > 3$. For each $\lambda \in k^*$, we have $f - \lambda g = x^2 - \lambda y^2 + (y - \lambda x)z$. Hence,

$$f - \lambda g \text{ is reducible} \iff x^2 - \lambda y^2 + y - \lambda x \text{ is reducible} \iff \left(x + \frac{\lambda}{2}\right)^2 - \lambda \left(y - \frac{1}{2\lambda}\right)^2 - \frac{1}{4\lambda}(\lambda^3 - 1) \text{ is reducible}$$

$$\iff \lambda^3 = 1 .$$

On the other hand observe that

$$D(y - \lambda x) = \begin{vmatrix} -\lambda & 1 & 0 \\ 2x & z & y \\ z & 2y & x \end{vmatrix} = (2\lambda^2 x + 2\lambda y + z)(y - \lambda x)$$

$$\implies \frac{D(y - \lambda x)}{y - \lambda x} = 2\lambda^2 x + 2\lambda y + z$$

So, for each such λ , $t_\lambda = 2\lambda^2 x + 2\lambda y + z$ is a logarithmic derivative of D ; i.e. $t_\lambda \in \mathcal{L}$.

Since $p \neq 3$, then $\lambda^3 - 1$ has 3 distinct solutions say λ_1, λ_2 , and λ_3 .

The matrix of coefficients of the t_{λ_i} is

$$M = \begin{pmatrix} 2\lambda_1^2 & 2\lambda_1 & 1 \\ 2\lambda_2^2 & 2\lambda_2 & 1 \\ 2\lambda_3^2 & 2\lambda_3 & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda_1^2 & \lambda_1 & 1 \\ \lambda_2^2 & \lambda_2 & 1 \\ \lambda_3^2 & \lambda_3 & 1 \end{pmatrix}$$

Since M is column equivalent to the Vandermonde matrix for the λ_i ; M is nonsingular, which shows that the t_{λ_i} are independent. Thus, we have \mathcal{L} is generated by the the logarithmic derivatives of D at factors of linear combination of f and g .

By Proposition 2.1.1 the order of \mathcal{L} is at most p^3 , hence,

$$Cl(X_{f,g}) \approx \mathcal{L} \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Chapter 3

Analysis of the Quadratic Case

In this chapter we prove three major facts regarding $Cl(X_{f,g})$ for f and g quadratic forms satisfying the gcd condition. Throughout section 3.3 we will show that $X_{f,g}$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ for a generic f and g , that is $Cl(X_{f,g})$ cannot be zero, and that \mathcal{L} is generated by the set of $h^{-1}D(h)$ such that h is a factor of some linear combination of f and g . Before taking up this case, it is appropriate to first discuss the cases where f and g are forms of lower degrees, which we do in section 3.1 and 3.2.

3.1 $\{\deg(f) = \deg(g) = 1\}$

Lemma 3.1.1. $Cl(X_{f,g}) = \{0\}$

Proof. Without a loss of generality we may assume after a change of coordinates that $f(x, y, z) = x$ and $g(x, y, z) = y$. Then f and g satisfy the gcd condition. By Lemma 2.1.1, we have

$$A^{(p)}[f, g] = A^{(p)}[x, y] = k[x, y, z^p] \approx \frac{A[w_1, w_2]}{(w_1^p - x, w_2^p - y)}$$

Since $k[x, y, z^p]$ is a unique factorization domain then by Definition 1.2.2, we get that $Cl(X_{f,g}) = \{0\}$. □

3.2 $\{\deg(f) = 2, \deg(g) = 1\}$

Without loss of generality we may assume after a change of coordinates that $g(x, y, z) = x$ and $f(x, y, z) = \alpha_{11}xy + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$, where $\alpha_{ij} \in k$.

Lemma 3.2.1. *Let $t \in A$ be a logarithmic derivative of D . Then $t \in k$.*

Proof. Let $h \in A$ be a homogeneous polynomial of degree w . Since $\deg(f) = 2$ and $\deg(g) = 1$, then $D(h)$ is a homogeneous polynomial of degree w and therefore D_h^p is a homogeneous of degree w as well. Now, by Comparing the degrees in the formula $D^p(h) = aD(h)$, we see that a is homogeneous of degree 0. Thus, $a \in k$.

The rest of the proof is similar to the techniques used to prove Lemma 2.1.4. □

Lemma 3.2.2. *$Cl(X_{f,g}) \approx \{0\}$ or $Cl(X_{f,g}) \approx \mathbb{Z}_p$*

Proof. If t is a logarithmic derivative of D ; i.e. $t \in \mathcal{L}$ then By Lemma 3.2.1, we have $t \in k$. Hence, $D^p(t) = D^{p-1}(t) = \dots = D(t) = 0$ and by Theorem 1.3.2, $t \in \mathcal{L}$ if and only if $at = t^p$ for $a \in k$. If $a \neq 0$, then $at = t^p \iff t \in \mathbb{F}_p$. Hence,

$$Cl(X_{f,g}) \approx \mathbb{Z}_p$$

If $a = 0$ then the logarithmic derivative $t = 0$ and hence

$$Cl(X_{f,g}) \approx \{0\}$$

□

Actually it turns out that in the case $Cl(X_{f,g}) \approx \mathbb{Z}_p$, we can calculate an exact formula for a and the logarithmic derivative t .

Claim 1.

$$a = (f_{yz}^2 - f_{yy}f_{zz})^{\frac{p-1}{2}}$$

and

$$t = \sqrt{f_{yz}^2 - f_{yy}f_{zz}}$$

Proof of the claim:

$$D(y) = \alpha_{10}x + \alpha_{01}y + 2\alpha_{00}z = f_z, \text{ and by the gcd condition } f_z \neq 0.$$

$$D^2(y) = \alpha_{01}f_z - 2\alpha_{00}f_y$$

$$D^3(y) = (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})f_z$$

$$D^4(y) = \alpha_{01}(\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})f_z - 2\alpha_{00}(\alpha_{01}^2f_z - 4\alpha_{02}\alpha_{00})f_y = (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})D^2(y)$$

$$D^5(y) = (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})^2f_z$$

$$D^6(y) = (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})^2D^2(y)$$

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If n is an even number, then

$$D^n(y) = (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})^{\frac{n-2}{2}} D^2(y)$$

If n is an odd number, then

$$D^n(y) = (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})^{\frac{n-1}{2}} f_z$$

In particular, if $n = p$ then

$$D^p(y) = (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})^{\frac{p-1}{2}} f_z$$

By Theorem 1.3.2, we have $D^p(y) = aD(y)$. Hence,

$$D^p(y) = aD(y) \implies (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})^{\frac{p-1}{2}} f_z = a f_z \implies a = (\alpha_{01}^2 - 4\alpha_{02}\alpha_{00})^{\frac{p-1}{2}} = (f_{yz}^2 - f_{yy}f_{zz})^{\frac{p-1}{2}}.$$

By solving for the logarithmic derivative t in the formula $at = t^p$, we get

$$t = \sqrt{\alpha_{01}^2 - 4\alpha_{02}\alpha_{00}} = \sqrt{f_{yz}^2 - f_{yy}f_{zz}}, \text{ which generates } \mathcal{L} \text{ and hence completes the proof of the claim.}$$

Remark 3.2.1. Let α be a none zero fixed root of $X^2 - f_{yz}^2 + f_{yy}f_{zz}$. To obtain a polynomial whose logarithmic derivative equals α , let $h = (f_{yz} + \alpha)f_y - f_{yy}f_z$. By the gcd condition, $h \neq 0$, and we have $D(h) = (f_{yz} + \alpha)(f_{yz}f_y - f_{yy}f_z) + f_{yy}(f_{yz}f_z - f_{zz}f_y) = (\alpha^2 + \alpha f_{yz})f_y - \alpha f_{yy}f_z = \alpha h$. Thus, $h^{-1}D(h) = \alpha$. By a similar calculation, if $\hat{h} = (f_{yz} - \alpha)f_y - f_{yy}f_z$, then $\hat{h}^{-1}D(\hat{h}) = -\alpha$, from which it follows that $D(h\hat{h}) = 0$; i.e. $h\hat{h} \in B$. Therefore we conclude that $h\hat{h}$ is a linear combination of f and g^2 ; i.e. h is a factor of linear combination of f and g^2 .

Remark 3.2.2. Lemmas 3.2.1 and 3.2.2 still hold for none homogeneous polynomials f and g .

From Lemmas 3.2.1, 3.2.2, Remark 3.2.1, Theorem 1.3.1, Remark 1.3.1, and Proposition 2.1.3, we have the following theorem :

Theorem 3.2.1. *Let f, g be homogeneous polynomials in A with $\deg(f) = 2$ and $\deg(g) = 1$ respectively, then $Cl(X_{f,g})$ is isomorphic to \mathbb{Z}_p for a generic f and g and \mathcal{L} is generated by $h^{-1}D(h)$, where h is a factor of $af + bg^2$ for some $a, b \in k$.*

3.3 Analysis of the Quadratic Case

In this section we assume that f and g are homogeneous polynomials of degree 2. We want to prove the following three assertions:

A₁) For a generic pair f and g , $Cl(X_{f,g})$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$.

A₂) For all f and g , $Cl(X_{f,g}) \neq 0$.

A₃) For all f and g , the group of logarithmic derivatives \mathcal{L} is generated by

$$\{h^{-1}D(h) : h \text{ divides } af + bg \text{ for some } a, b \in k\}.$$

Notation 12. In general, when a pair of polynomials of the same degree satisfy the property mentioned and stated in A_3 , we will say that f and g satisfy the **logarithmic derivative factor property**.

From Theorem 1.3.1 and the remarks in section 1.3, the logarithmic derivative property means that the divisor class group of $w_1^p - f = w_2^p - g = 0$ is generated by hyperplane sections whose defining forms are factors of linear combinations of f and g .

To prove the three assertions, we will repeatedly employ two reduction steps in the cases considered below.

The first reduction step is to replace the pair f, g by the pair $f, g + af$ (or by the pair $f + ag, g$) for some $a \in k$. This operation is valid since B, D and \mathcal{L} are unchanged by it and the logarithmic derivative factor property and the gcd condition are preserved under it.

The second reduction step is to replace f and g by $\theta(f)$ and $\theta(g)$ where $\theta : A \rightarrow A$ is a k -automorphism induced by a homogeneous change of coordinates. This operation is valid by Proposition 2.1.3 and by the fact that the order of $Cl(X_{f,g})$ remains unchanged after a change of coordinates. Also, it is important to note that the gcd condition still holds under this operation.

Below we will start our investigations of assertions A_1, A_2 , and A_3 by considering the case where $f = x^2 + \alpha_1 yz + \alpha_2 z^2$ and $g = y^2 + \beta_1 xz + \beta_2 z^2$, with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in k$.

Proposition 3.3.1. *Let $f = x^2 + \alpha_1 yz + \alpha_2 z^2, g = y^2 + \beta_1 xz + \beta_2 z^2 \in A$. Then f and g satisfy the logarithmic derivative factor property and the divisor class group $Cl(X_{f,g})$ is not trivial (the order of $Cl(X_{f,g})$ is at least p).*

Proof.

For each $\lambda \in k^*$, we have $f - \lambda g$ is reducible

$$\begin{aligned}
&\iff x^2 + \alpha_1 yz + \alpha_2 z^2 - \lambda(y^2 + \beta_1 xz + \beta_2 z^2) \text{ is reducible} \\
&\iff x^2 + \alpha_1 yz + \alpha_2 z^2 - \lambda(y^2 + \beta_1 xz + \beta_2 z^2) \text{ is reducible} \\
&\iff x^2 - \lambda\beta_1 xz - \lambda y^2 + \alpha_1 yz + (\alpha_2 - \lambda\beta_2)z^2 \text{ is reducible} \\
&\iff \left(x - \frac{\lambda\beta_1}{2}z\right)^2 - \lambda\left(y - \frac{\alpha_1}{2\lambda}z\right)^2 + \left(\alpha_2 - \lambda\beta_2 - \frac{\beta_1^2\lambda^2}{4} + \frac{\alpha_1^2}{4\lambda}\right)z^2 \text{ is reducible} \\
&\iff \left(x - \frac{\lambda\beta_1}{2}z\right)^2 - \lambda\left(y - \frac{\alpha_1}{2\lambda}z\right)^2 - \frac{1}{4\lambda}(\beta_1^2\lambda^3 + 4\beta_2\lambda^2 - 4\alpha_2\lambda - \alpha_1^2)z^2 \text{ is reducible} \\
&\iff \left(x - \frac{\lambda\beta_1}{2}z\right)^2 - \lambda\left(y - \frac{\alpha_1}{2\lambda}z\right)^2 - \frac{1}{4\lambda}C(\lambda)z^2 \text{ is reducible, where} \tag{3.1}
\end{aligned}$$

$$C(\lambda) = \beta_1^2\lambda^3 + 4\beta_2\lambda^2 - 4\alpha_2\lambda - \alpha_1^2. \tag{3.2}$$

By replacing x by $x + \frac{\lambda\beta_1}{2}z$ and replacing y by $y + \frac{\alpha_1}{2\lambda}z$ in equation (3.1), then we get

$$f - \lambda g \text{ is reducible} \iff x^2 - \lambda y^2 - \frac{1}{4\lambda}C(\lambda)z^2 \text{ is reducible} \iff C(\lambda) = 0.$$

Since k is algebraically closed field then $C(\lambda)$ must have at least one root.

Hence, if $C(\lambda) = 0$, then $f - \lambda g = uv$, where $u = x - \frac{\lambda\beta_1}{2}z + \sqrt{\lambda}\left(y - \frac{\alpha_1}{2\lambda}z\right)$,

$v = x - \frac{\lambda\beta_1}{2}z - \sqrt{\lambda}\left(y - \frac{\alpha_1}{2\lambda}z\right)$, and $\sqrt{\lambda}$ is a fixed root in k of $X^2 - \lambda$.

By Lemma 1.3.2, we have for each $\lambda \in k^*$ such that $C(\lambda) = 0$, $t_\lambda \in \mathcal{L}$, where

$$t_\lambda = u^{-1}D(u) = 2\sqrt{\lambda}\beta_1x + \frac{2}{\sqrt{\lambda}}\alpha_1y + (\beta_1^2\lambda\sqrt{\lambda} + 4\beta_2\sqrt{\lambda})z.$$

Since both f and g satisfy the gcd condition, then $t_\lambda \neq 0$ and, thus, $\mathcal{L} \neq 0$. Hence, by Lemma 2.1.3 we have $Cl(X_{f,g}) \neq \{0\}$. Also, t_λ is a logarithmic derivative of D at a factor of a linear combination of f and g . □

Actually it turns out that the number of distinct roots of $C(\lambda)$ in the previous proposition plays a major role in identifying the nature and the order of the Divisor class group of f and g . The following three Corollaries address this relation.

Corollary 3.3.1. *Let $f = x^2 + \alpha_1 yz + \alpha_2 z^2$, $g = y^2 + \beta_1 xz + \beta_2 z^2 \in A$ and $C(\lambda) = \beta_1^2\lambda^3 + 4\beta_2\lambda^2 - 4\alpha_2\lambda - \alpha_1^2$. If $C(\lambda)$ has three distinct roots then f and g satisfy the logarithmic derivative factor*

property and

$$Cl(X_{f,g}) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Proof. Assume that $C(\lambda)$ has the following three distinct roots λ_1 , λ_2 , and λ_3 . We will prove this corollary by breaking it into 4 cases. The first case is a generic assumption on f and g .

Case (I) : If $\alpha_1 \neq 0$ and $\beta_1 \neq 0$.

As in the proof of Proposition 3.3.1, for each λ_j , we have

$$t_{\lambda_j} = 2\sqrt{\lambda_j}\beta_1x + \frac{2}{\sqrt{\lambda_j}}\alpha_1y + (\beta_1^2\lambda_j\sqrt{\lambda_j} + 4\beta_2\sqrt{\lambda_j})z, j = 1, 2, 3$$

t_{λ_j} are independent over k if and only if the following matrix of coefficients of the t_{λ_j} is nonsingular,

$$M_1 = \begin{pmatrix} 2\sqrt{\lambda_1}\beta_1 & \frac{2}{\sqrt{\lambda_1}}\alpha_1 & \beta_1^2\lambda_1\sqrt{\lambda_1} + 4\beta_2\sqrt{\lambda_1} \\ 2\sqrt{\lambda_2}\beta_1 & \frac{2}{\sqrt{\lambda_1}}\alpha_1 & \beta_1^2\lambda_2\sqrt{\lambda_2} + 4\beta_2\sqrt{\lambda_2} \\ 2\sqrt{\lambda_3}\beta_1 & \frac{2}{\sqrt{\lambda_1}}\alpha_1 & \beta_1^2\lambda_3\sqrt{\lambda_3} + 4\beta_2\sqrt{\lambda_3} \end{pmatrix}$$

Using column operations we get, $\det M_1 = -4\alpha_1\beta_1^3 \det M_2$, where

$$M_2 = \begin{pmatrix} \sqrt{\lambda_1} & \frac{1}{\sqrt{\lambda_1}} & \lambda_1\sqrt{\lambda_1} \\ \sqrt{\lambda_2} & \frac{1}{\sqrt{\lambda_2}} & \lambda_2\sqrt{\lambda_2} \\ \sqrt{\lambda_3} & \frac{1}{\sqrt{\lambda_3}} & \lambda_3\sqrt{\lambda_3} \end{pmatrix}$$

Note that $\det M_2 = \frac{-1}{\sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3}} \det V$, where

$$V = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}$$

V is the Vandermonde matrix of $\lambda_1, \lambda_2, \lambda_3$ which has nonzero determinant since λ_j 's are distinct.

Therefore, we conclude that t_{λ_j} are linearly independent over k which means that \mathcal{L} is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$. By Lemma 2.1.3, we conclude that $Cl(X_{f,g})$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ for a generic choice of f and g .

Case (II): If $\alpha_1 = \beta_1 = 0$.

Then, we have

$$f = x^2 + \alpha_2 z^2, \quad g = y^2 + \beta_2 z^2$$

And $C(\lambda)$ becomes

$$C(\lambda) = 4\beta_2 \lambda^2 - 4\alpha_2 \lambda$$

Since f and g satisfy the gcd condition then we must have $\alpha_2 \beta_2 \neq 0$, which means that $C(\lambda)$ has no multiple roots.

Replacing f by $\frac{f}{\alpha_2}$ and g by $\frac{g}{\beta_2}$, we get

$$f = \frac{x^2}{\alpha_2} + z^2, \quad g = \frac{y^2}{\beta_2} + z^2$$

Let $\sqrt{-\alpha_2}$, $\sqrt{-\beta_2}$ be fixed roots of $X^2 + \alpha_2$, $X^2 + \beta_2$, respectively. Replacing x by $x\sqrt{-\alpha_2}$ and replacing y by $y\sqrt{-\beta_2}$, we obtain

$$f = x^2 - z^2, \quad g = y^2 - z^2$$

Hence, $Cl(X_{f,g})$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ and f and g satisfy the logarithmic derivative factor property by Example 2.2.1.

Case (III): If $\alpha_1 = 0$ and $\beta_1 \neq 0$.

Then f , g and $C(\lambda)$ become:

$$f = x^2 + \alpha_2 z^2, \quad g = y^2 + \beta_1 xz + \beta_2 z^2, \quad C(\lambda) = \beta_1^2 \lambda^3 + 4\beta_2 \lambda^2 - 4\alpha_2 \lambda$$

Since $\alpha_1 = 0$, then by the gcd condition we must have $\alpha_2 \neq 0$.

For each $\lambda \in k$, we have $f - \lambda g = x^2 + \alpha_2 z^2 - \lambda(y^2 + \beta_1 xz + \beta_2 z^2) = (x - \frac{\lambda\beta_1}{2}z)^2 - \lambda y^2 + (\alpha_2 - \beta_2\lambda - \frac{\beta_1^2}{4}\lambda^2)z^2 = (x - \frac{\lambda\beta_1}{2}z)^2 - \lambda y^2 + C_0(\lambda)z^2$, where $C_0(\lambda) = \alpha_2 - \beta_2\lambda - \frac{\beta_1^2}{4}\lambda^2$. Observe that since $\alpha_1 = 0$, then we have the relation $C(\lambda) = -4\lambda C_0(\lambda)$. Since $C(\lambda)$ has no repeated roots then $C_0(\lambda)$ must have a none zero simple roots. Let λ_1, λ_2 be the two distinct nonzero roots of $C_0(\lambda)$ and let $\sqrt{\lambda_1}, \sqrt{\lambda_2}$ be the fixed roots in k of $X^2 - \lambda_1, X^2 - \lambda_2$, respectively.

Let $u_i = x + \sqrt{\lambda_i}y - \frac{\lambda_i\beta_1}{2}z$ and $v_i = x - \sqrt{\lambda_i}y - \frac{\lambda_i\beta_1}{2}z$ for $i = 1, 2$. Then, for each i , $u_i v_i = f - \lambda_i g$, and by Lemma 2.1.5 we have

$$t_i = u_i^{-1}D(u_i) = J(u_i, v_i, g) = -\sqrt{\lambda_i}(2\beta_1 x + (4\beta_2 + \beta_1^2\lambda_i)z) \in \mathcal{L} \text{ for } i = 1, 2$$

On the other hand, since $f = x^2 + \alpha_2 z^2 = (x + \sqrt{-\alpha_2}z)(x - \sqrt{-\alpha_2}z)$, where $\sqrt{-\alpha_2}$ is a fixed root of $X^2 + \alpha_2$, then by Lemma 2.1.5 we also have

$$t_3 = \frac{D(x + \sqrt{-\alpha_2}z)}{x + \sqrt{-\alpha_2}z} = J(x + \sqrt{-\alpha_2}z, x - \sqrt{-\alpha_2}z, g) = 4\sqrt{-\alpha_2}y \in \mathcal{L}$$

Since $\alpha_2 \neq 0, \beta_1 \neq 0$ and $\lambda_1 \neq \lambda_2$, then $\det H = -8\sqrt{-\alpha_2}\beta_1^3\sqrt{\lambda_1}\sqrt{\lambda_2}(\lambda_2 - \lambda_1) \neq 0$, where H is the following matrix of coefficients of t_i ,

$$H = \begin{pmatrix} -2\beta_1\sqrt{\lambda_1} & 0 & -\sqrt{\lambda_1}(4\beta_2 + \beta_1^2\lambda_1) \\ -2\beta_1\sqrt{\lambda_2} & 0 & -\sqrt{\lambda_2}(4\beta_2 + \beta_1^2\lambda_2) \\ 0 & 4\sqrt{-\alpha_2} & 0 \end{pmatrix}$$

Since $\det H \neq 0$, then it follows that the three t_i are clearly linear independent over k . Hence, each t_i is a logarithmic derivative of D at a factor of a linear combination of f and g and by Lemma 2.1.3 and Proposition 2.1.1 we conclude that

$$Cl(X_{f,g}) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Case (IV): If $\alpha_1 \neq 0$ and $\beta_1 = 0$.

Note that this case is essentially same as Case (III), since one is obtained from the other by trans-

posing x and y . □

Corollary 3.3.2. *Let $f = x^2 + \alpha_1 yz + \alpha_2 z^2$, $g = y^2 + \beta_1 xz + \beta_2 z^2 \in A$ and $C(\lambda) = \beta_1^2 \lambda^3 + 4\beta_2 \lambda^2 - 4\alpha_2 \lambda - \alpha_1^2$. If $C(\lambda)$ has two distinct roots then f and g satisfy the logarithmic derivative factor property and*

$$Cl(X_{f,g}) \approx \mathbb{Z}_p \bigoplus \mathbb{Z}_p$$

Proof. Assume that $C(\lambda)$ has two distinct roots λ_1 and λ_2 in k . Assume that λ_1 is the double root of $C(\lambda)$.

We will prove this corollary by breaking it into 4 cases.

Case (I) : If $\alpha_1 \neq 0$ and $\beta_1 \neq 0$.

By the same argument as used in the proof of Proposition 3.3.1, we obtain two linearly independent elements of \mathcal{L} ,

$$t_{\lambda_j} = 2\sqrt{\lambda_j} \beta_1 x + \frac{2}{\sqrt{\lambda_j}} \alpha_1 y + (\beta_1^2 \lambda_j \sqrt{\lambda_j} + 4\beta_2 \sqrt{\lambda_j}) z, \quad j = 1, 2$$

Note that both of t_{λ_j} are logarithmic derivatives of D at factors of linear combinations of f and g .

Hence, \mathcal{L} will have an order of at least p^2 . Since λ_1 is a double root, then we have

$$C(\lambda_1) = \beta_1^2 \lambda_1^3 + 4\beta_2 \lambda_1^2 - 4\alpha_2 \lambda_1 - \alpha_1^2 = 0$$

$$C'(\lambda_1) = 3\beta_1^2 \lambda_1^2 + 8\beta_2 \lambda_1 - 4\alpha_2 = 0$$

$$C''(\lambda_1) = 6\beta_1^2 \lambda_1 + 8\beta_2 \neq 0$$

By the gcd condition we must have $\lambda_1 \neq 0$.

By replacing f by $f - \lambda_1 g$ and then replacing x by $x + \frac{1}{2} \lambda_1 \beta_1 z$ and y by $y + \frac{1}{2\lambda_1} \alpha_1 z$, we then have:

$$f = x^2 - \lambda_1 y^2 - \frac{1}{4\lambda_1} C(\lambda_1) z^2 = x^2 - \lambda_1 y^2 \quad (\text{since } C(\lambda_1) = 0) \text{ and}$$

$$g = (y + \frac{1}{2\lambda_1} \alpha_1 z)^2 + \beta_1 (x + \frac{1}{2} \lambda_1 \beta_1 z) z + \beta_2 z^2 = y^2 + \frac{\alpha_1}{\lambda_1} yz + \beta_1 xz + \frac{1}{4\lambda_1^2} (\alpha_1^2 + 4\beta_2 \lambda_1^2 + 2\beta_1^2 \lambda_1^3) z^2$$

Since $C'(\lambda_1) - C(\lambda_1) = \alpha_1^2 + 4\beta_2\lambda_1^2 + 2\beta_1^2\lambda_1^3 = 0$, then we can reduce f and g to where

$$f = x^2 - \lambda_1 y^2 = (x - \sqrt{\lambda_1}y)(x + \sqrt{\lambda_1}y) \text{ and } g = y^2 + \frac{\alpha_1}{\lambda_1}yz + \beta_1xz.$$

Note that $D(x) = -2\lambda_1y(\frac{\alpha_1}{\lambda_1}y + \beta_1x)$, $D(y) = -2x(\frac{\alpha_1}{\lambda_1}y + \beta_1x)$, and $D(z) = 2x(2y + \frac{\alpha_1}{\lambda_1}z) + 2\lambda_1\beta_1yz$.

By Lemma 2.1.6 if $t \in \mathcal{L}$, then $\deg_z(t) = 0$. Hence, $t = ax + by$ for some $a, b \in k$ and, thus, by Proposition 2.1.2 the order of \mathcal{L} is at most p^2 . So we conclude that $p^2 \leq |\mathcal{L}| \leq p^2$ which means that the order of \mathcal{L} is exactly p^2 . Note again f and g satisfy the logarithmic derivative factor property. By Lemma 2.1.3 and Proposition 2.1.1, we obtain the desired result, that is,

$$Cl(X_{f,g}) \approx \mathcal{L} \approx \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Case (II) : If $\alpha_1 = \beta_1 = 0$.

This case can not happen because it violates the gcd condition.

Case (III) : If $\alpha_1 = 0$ and $\beta_1 \neq 0$.

By the same argument used in the beginning of the proof of the third case in Proposition 3.3.1, we have $\alpha_2 \neq 0$, $f = x^2 + \alpha_2z^2$, $g = y^2 + \beta_1xz + \beta_2z^2$, $C(\lambda) = \beta_1^2\lambda^3 + 4\beta_2\lambda^2 - 4\alpha_2\lambda$ and for each $\lambda \in k$ we have $f - \lambda g = x^2 + \alpha_2z^2 - \lambda(y^2 + \beta_1xz + \beta_2z^2) = (x - \frac{\lambda\beta_1}{2}z)^2 - \lambda y^2 + (\alpha_2 - \beta_2\lambda - \frac{\beta_1^2}{4}\lambda^2)z^2 = (x - \frac{\lambda\beta_1}{2}z)^2 - \lambda y^2 + C_0(\lambda)z^2$, where $C_0(\lambda) = \alpha_2 - \beta_2\lambda - \frac{\beta_1^2}{4}\lambda^2$ and $C(\lambda) = -4\lambda C_0(\lambda)$. Since $\alpha_2 \neq 0$ then the roots of $C_0(\lambda)$ must be different from zero and $C_0(\lambda)$ has a double root if and only if $C(\lambda)$ has a double root. Hence, $\lambda_1 \neq 0$ is a double root for $C_0(\lambda)$.

Replacing g by $f - \lambda_1g$, we obtain, $f = x^2 + \alpha_2z^2$ and $g = (x - \frac{\lambda_1\beta_1}{2}z)^2 - \lambda_1y^2$.

Replacing x by $x + \frac{\lambda_1\beta_1}{2}z$, we obtain, $g = x^2 - \lambda_1y^2$ and $f = x^2 + \lambda_1\beta_1xz + (\frac{\beta_1^2}{4}\lambda_1^2 + \alpha_2)z^2$. But since $C_0(\lambda_1) - \lambda_1C_0'(\lambda_1) = \frac{\beta_1^2}{4}\lambda_1^2 + \alpha_2 = 0$, then we obtain $g = x^2 - \lambda_1y^2$ and $f = x^2 + \lambda_1\beta_1xz$.

Replacing z by $\frac{1}{\lambda_1\beta_1}(z - x)$, we obtain, $f = xz$ and $g = x^2 - \lambda_1y^2$. Then, we have $D(x) = -2\lambda_1xy$, $D(y) = 2x^2$, $D(z) = -2\lambda_1yz$. So, if $t \in \mathcal{L}$, then by Lemma 2.1.6 we have $\deg_z(t) = 0$; i.e. $t = ax + by$ for some $a, b \in k$. Hence, by Proposition 2.1.2, the dimension of \mathcal{L} over \mathbb{F}_p is at most two; i.e. the order of \mathcal{L} is at most p^2 . Since both of $t_1 = \frac{D(x)}{x} = -2\lambda_1y$ and $t_2 = \frac{D(x - \sqrt{\lambda_1}y)}{x - \sqrt{\lambda_1}y} = -2\lambda_1x$ belong to \mathcal{L} and are linear independent over k , then

$$Cl(X_{f,g}) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p$$

and f and g satisfy the logarithmic derivative factor property.

Case (IV) : If $\alpha_1 \neq 0$ and $\beta_1 = 0$.

This case is similar to the previous case since one is obtained from the other by interchanging x and y . □

Corollary 3.3.3. *Let $f = x^2 + \alpha_1 yz + \alpha_2 z^2$, $g = y^2 + \beta_1 xz + \beta_2 z^2 \in A$ and $C(\lambda) = \beta_1^2 \lambda^3 + 4\beta_2 \lambda^2 - 4\alpha_2 \lambda - \alpha_1^2$. If $C(\lambda)$ has a triple root then f and g satisfy the logarithmic derivative factor property and*

$$Cl(X_{f,g}) \approx \mathbb{Z}_p$$

Proof. Let $\lambda_1 \in k$ be a triple root of $C(\lambda)$. Then, we have

$$C(\lambda_1) = \beta_1^2 \lambda_1^3 + 4\beta_2 \lambda_1^2 - 4\alpha_2 \lambda_1 - \alpha_1^2 = 0 \quad (3.3)$$

$$C'(\lambda_1) = 3\beta_1^2 \lambda_1^2 + 8\beta_2 \lambda_1 - 4\alpha_2 = 0 \quad (3.4)$$

$$C''(\lambda_1) = 6\beta_1^2 \lambda_1 + 8\beta_2 = 0 \quad (3.5)$$

If $\lambda_1 = 0$, then it follows from equations 3.3, 3.4, and 3.5 that $\alpha_1 = \alpha_2 = \beta_2 = 0$. Hence, the gcd condition does not hold. So, we must have $\lambda_1 \in k^*$.

To prove this corollary, we will discuss two cases:

Case (I) : If $p = 3$.

Then equation 3.4 and 3.5 become $C'(\lambda_1) = 8\beta_2 \lambda_1 - 4\alpha_2$ and $C''(\lambda_1) = 8\beta_2$, respectively. But since $C'(\lambda_1) = C''(\lambda_1) = 0$, then we obtain $\alpha_2 = \beta_2 = 0$. Thus, we have $f = x^2 + \alpha_1 yz$ and $g = y^2 + \beta_1 xz$. By the gcd condition, we have $\alpha_1 \beta_1 \neq 0$. Let a_0 and b_0 be fixed roots in k of $X^3 - \alpha_1$ and $X^3 - \beta_1$.

Replacing x by $a_0^2 b_0 x$, and then y by $a_0 b_0^2 y$, we obtain, $f = a_0^4 b_0^2 x^2 + a_0^4 b_0^2 yz$ and $g = a_0^2 b_0^4 y^2 +$

$a_0^2 b_0^4 xz$.

Now we replace f by $\frac{f}{a_0^4 b_0^2}$ and replace g by $\frac{g}{a_0^2 b_0^4}$, then we obtain $f = x^2 + yz$ and $g = y^2 + xz$.

Hence, By Example 2.2.6, f and g satisfy the logarithmic derivative factor property and

$$Cl(X_{f,g}) \approx \mathbb{Z}_p$$

Case (II) : If $p > 3$.

As in the proof of Proposition 3.3.1, By replacing f by $f - \lambda_1 g$ and then replacing x by $x + \frac{1}{2} \lambda_1 \beta_1 z$, y by $y + \frac{1}{2\lambda_1} \alpha_1 z$, f and g can be reduced to the case

$$f = x^2 - \lambda_1 y^2 - \frac{1}{4\lambda_1} C(\lambda_1) z^2 = x^2 - \lambda_1 y^2 \text{ (since } C(\lambda_1) = 0\text{),}$$

$$g = y^2 + \frac{\alpha_1}{\lambda_1} yz + \beta_1 xz + \frac{1}{4\lambda_1^2} (\alpha_1^2 + 4\beta_2 \lambda_1^2 + 2\beta_1^2 \lambda_1^3) z^2.$$

By multiplying equation 3.4 by λ_1 and then subtracting equation 3.5 we get

$$\lambda_1 C'(\lambda_1) - C(\lambda_1) = 2\beta_1^2 \lambda_1^3 + 4\beta_2 \lambda_1^2 + \alpha_1^2 = 0$$

We thus have reduced to the case :

$$f = x^2 - \lambda_1 y^2, \quad g = y^2 + \frac{\alpha_1}{\lambda_1} yz + \beta_1 xz$$

By the gcd condition we must have $\alpha_1 \beta_1 \neq 0$.

Replacing x by $\frac{x}{\beta_1}$, we obtain $f = \frac{1}{\beta_1^2} x^2 - \lambda_1 y^2$, $g = y^2 + (\frac{\alpha_1}{\lambda_1} y + x)z$.

Replacing x by $\frac{\alpha_1}{\lambda_1} y + x$, then f and g are reduced to :

$$f = \frac{1}{\beta_1^2} x^2 - 2\frac{\alpha_1}{\beta_1^2} xy + \lambda_1^2 \beta_1^2 (\alpha_1^2 - \lambda_1^3 \beta_1^2) y^2, \quad g = y^2 + xz$$

Moreover, it follows from equations 3.3, 3.4, and 3.5 that

$$\lambda_1^2 C''(\lambda_1) - (\lambda_1 C'(\lambda_1) - C(\lambda_1)) = \alpha_1^2 - \beta_1^2 \lambda_1^3 = 0$$

We thus can reduce f and g to the case

$$f = \frac{1}{\beta_1^2}x^2 - 2\frac{\alpha_1}{\beta_1^2}xy, \quad g = y^2 + xz$$

Replace f by $\beta_1^2 f$ and then replace y by $\frac{y}{2\alpha_1}$ we obtain, $f = x^2 - xy$, $g = \frac{y^2}{4\alpha_1^2} + xz$.

If we now Replace g by $4\alpha_1^2 g$ and then replace z by $\frac{z}{4\alpha_1^2}$, we obtain $f = x^2 - xy$, $g = y^2 + xz$.

By Example 2.2.4, f and g satisfy the logarithmic derivative factor property and

$$Cl(X_{f,g}) \approx \mathbb{Z}_p$$

□

Note that if we refer to the multiplicities of the roots of $C(\lambda)$ instead of the number of its distinct roots, then we can combine the results of Proposition 3.3.1, Corollary 3.3.1, Corollary 3.3.2, and Corollary 3.3.3 into one statement regarding the order of $Cl(X_{f,g})$ as in the next theorem.

Theorem 3.3.1. *Let $f = x^2 + \alpha_1 yz + \alpha_2 z^2$, $g = y^2 + \beta_1 xz + \beta_2 z^2 \in A$ and $C(\lambda) = \beta_1^2 \lambda^3 + 4\beta_2 \lambda^2 - 4\alpha_2 \lambda - \alpha_1^2$. Then f and g satisfy the logarithmic derivative factor property. The order of $Cl(X_{f,g})$ is equal to p^3 if $C(\lambda)$ has no repeated roots, The order of $Cl(X_{f,g})$ is equal to p^2 if $C(\lambda)$ has a double root, and The order of $Cl(X_{f,g})$ is equal to p if $C(\lambda)$ has a triple root.*

Now, we turn our focus on the investigation of assertions A_2 and A_3 by considering a more general case where $f = \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$ and $g = \beta_{20}x^2 + \beta_{02}y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2$, with $\alpha_{20}, \alpha_{02}, \alpha_{10}, \alpha_{01}, \alpha_{00}, \beta_{20}, \beta_{02}, \beta_{10}, \beta_{01}, \beta_{00} \in k$.

Proposition 3.3.2. *Let $f = \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$, $g = \beta_{20}x^2 + \beta_{02}y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2 \in A$. If $\alpha_{20}\beta_{02} - \alpha_{02}\beta_{20} \neq 0$, which is a generic assumption on f and g , then f and g satisfy the logarithmic derivative factor property and*

$$Cl(X_{f,g}) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Proof. Replacing f and g by linear independent combinations of them, we may assume that

$f = x^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$ and $g = y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2$. Hence, we have

$$f = x^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2 = \left(x + \frac{\alpha_{10}}{2}z\right)^2 + \alpha_{01}yz + \left(\alpha_{00} - \frac{1}{4}\alpha_{10}^2\right)z^2, \text{ and}$$

$$g = y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2 = \left(y + \frac{1}{2}\beta_{01}z\right)^2 + \beta_{10}xz + \left(\beta_{00} - \frac{1}{4}\beta_{01}^2\right)z^2.$$

After Replacing x and y by $x + \frac{\alpha_{10}}{2}z$, $y + \frac{1}{2}\beta_{01}z$, respectively; we have $f = x^2 + \alpha_1yz + \alpha_2z^2$ and

$g = y^2 + \beta_1xz + \beta_2z^2$, with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in k$. In the proof of Corollary 3.3.1, we showed that f

and g satisfy the logarithmic derivative factor property. Also, the proof shows that

$$Cl(X_{f,g}) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$$

□

Remark 3.3.1. With Proposition 3.3.2, we have completed our analysis of $Cl(X_{f,g})$ in the case where $f = \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$, $g = \beta_{20}x^2 + \beta_{02}y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2$, and $\alpha_{20}\beta_{02} - \alpha_{02}\beta_{20} \neq 0$. Therefore, we continue our investigation of assertions A_2 and A_3 by turning our attention to the case where $\alpha_{20}\beta_{02} - \alpha_{02}\beta_{20} = 0$. If we replace g by a combination of f and g , then we can cancel out the terms that contain x^2 and y^2 in g and, hence, we may assume

$$f = \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2, g = \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2 = (\beta_{10}x + \beta_{01}y + \beta_{00}z)z.$$

If $\beta_{10} = \beta_{01} = 0$, then the gcd condition doesn't hold. So we have either $\beta_{10} \neq 0$ or $\beta_{01} \neq 0$.

Without loss of generality we may assume that $\beta_{10} \neq 0$. By replacing x by $\frac{1}{\beta_{10}}(x - \beta_{01}y - \beta_{00}z)$,

we obtain $g = xz$ and after another combination of f and g , we may assume that $f = \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{11}xy + \alpha_{01}yz + \alpha_{00}z^2$ and $g = xz$. In the next two propositions, we divide our analysis of

this pair of polynomials into two cases, first we will consider the case where $\alpha_{02} \neq 0$ and then we

consider the case where $\alpha_{02} = 0$. Observe that if $\alpha_{02} \neq 0$, then after replacing f by $\frac{1}{\alpha_{02}}f$, we can

assume that $\alpha_{02} = 1$, which we do in the following proposition:

Proposition 3.3.3. *Let $f = ax^2 + bxy + y^2 + cyz + dz^2 \in A$ and $g = xz$. Then f and g satisfy the logarithmic factor property. Moreover, if $(a - \frac{b^2}{4})(d - \frac{c^2}{4}) \neq 0$, then $Cl(X_{f,g})$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$, and if $(a - \frac{b^2}{4})(d - \frac{c^2}{4}) = 0$, then $Cl(X_{f,g})$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$.*

Proof. If we replace y by $y - \frac{b}{2}x$ and consequently replace f by $f + \frac{bc}{2}g$, we get,

$$f = (a - \frac{b^2}{4})x^2 + y^2 + cyz + dz^2 \text{ and } g = xz.$$

If we now replace y by $y - \frac{c}{2}z$, then we obtain

$$f = (a - \frac{b^2}{4})x^2 + y^2 + (d - \frac{c^2}{4})z \text{ and } g = xz.$$

Now, we will prove the proposition by discussing the following four cases:

Case (I): If $(a - \frac{b^2}{4})(d - \frac{c^2}{4}) \neq 0$.

Let r_1 and r_2 be fixed roots in k of $X^2 - a + \frac{b^2}{4}$ and $X^2 - d + \frac{c^2}{4}$, respectively. If we replace x by $\frac{1}{r_1}x$, z by $\frac{1}{r_2}z$, and g by r_1r_2g , then we get $f = x^2 + y^2 + z^2$ and $g = xz$. Then by Example 2.2.1, f and g satisfy the logarithmic derivative factor property and

$$Cl(X_{f,g}) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Case (II) : If $(a - \frac{b^2}{4}) = (d - \frac{c^2}{4}) = 0$.

Such a case can not happen since the gcd condition would be violated.

Case (III) : If $(a - \frac{b^2}{4}) \neq 0, (d - \frac{c^2}{4}) = 0$.

Then $f = (a - \frac{b^2}{4})x^2 + y^2$ and $g = xz$. Replacing x by $\frac{1}{r_1}x$, and then replacing g by r_1g , we obtain, $f = x^2 + y^2$ and $g = xz$. By Example 2.2.3, we have f and g satisfy the logarithmic derivative factor property and

$$Cl(X_{f,g}) \approx \mathbb{Z}_p \oplus \mathbb{Z}_p$$

Case (IV) : If $(a - \frac{b^2}{4}) = 0, (d - \frac{c^2}{4}) \neq 0$.

This case is exactly similar to the third Case above. □

Proposition 3.3.4. *Let $f = ax^2 + bxy + cyz + dz^2 \in A$ and $g = xz$. Then f and g satisfy the logarithmic factor property. Moreover, if $bc \neq 0$, then $Cl(X_{f,g})$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$, and if $bc = 0$, then $Cl(X_{f,g})$ is isomorphic to \mathbb{Z}_p .*

Proof. **Case (I) :** If $bc \neq 0$.

If we replace y by $\frac{1}{b}(y - ax)$ and then f by $f = \frac{ac}{b}g$, we obtain, $f = xy + \frac{c}{b}yz + dz^2$ and $g = xz$. Now replace y by $\frac{b}{c}(y - dz)$ and then f by $f + \frac{bd}{c}g$, we obtain, $f = xy + yz$ and $g = xz$. Also, we have $D(x) = x(x+z)$, $D(y) = y(z-x)$, and $D(z) = -z(x+z)$. Hence, by Lemma 2.1.6, we have that if $t \in \mathcal{L}$, then $\deg_y t = 0$; i.e. $t = \alpha x + \beta y$ for some $\alpha, \beta \in k$. Hence, the dimension of \mathcal{L} over k is at most two, which by Proposition 2.1.2 implies the order of \mathcal{L} is at most p^2 . Since $t_1 = \frac{D(x)}{x} = x+z \in \mathcal{L}$ and $t_2 = \frac{D(x+z)}{x+z} = x-z \in \mathcal{L}$, t_1 and t_2 are a basis of \mathcal{L} over \mathbb{F}_p . Therefore, f and g satisfy the logarithmic derivative factor property and we have

$$Cl(X_{f,g})\mathbb{Z}_p \oplus \mathbb{Z}_p$$

Case (II) : If $bc = 0$.

Since f and g satisfy the gcd condition then we can not have the case $b = c = 0$. Without loss of generality, we may assume $b \neq 0$ and $c = 0$. If we replace y by $\frac{1}{b}(y - ax)$, then we can reduce to the case $f = xy + dz^2$ and $g = xz$. Because of the gcd condition we can not have $d = 0$. If we replace z by $\frac{1}{\sqrt{d}}z$ and then replace g by $\sqrt{d}g$, then we obtain, $f = xy + z^2$ and $g = xz$. By Example 2.2.5, we have that f and g satisfy the logarithmic derivative factor property. Also by the same example we have

$$Cl(X_{f,g}) \approx \mathbb{Z}_p$$

□

So far we have completed our analysis of $Cl(X_{f,g})$ when f and g are quadratic forms such that $f = \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$, $g = \beta_{20}x^2 + \beta_{02}y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2 \in A$. In all cases, we found that $Cl(X_{f,g}) \neq \{0\}$ and that the logarithmic derivative factor property holds. We are left with the most general form of f and g . In the next proposition, we will prove that if $f = \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$, $g = \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2 \in A$, then f and g can be reduced to the case where $f = \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$, $g = \beta_{20}x^2 + \beta_{02}y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2$.

Proposition 3.3.5. Let $f = \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 + \alpha_{10}xz + \alpha_{01}yz + \alpha_{00}z^2$, and

$g = \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2 + \beta_{10}xz + \beta_{01}yz + \beta_{00}z^2$ be two homogeneous polynomials in A , each of degree 2. Then there exists a k -automorphism $\theta : A \rightarrow A$ induced by a homogeneous change of coordinates such that $(\theta(f))_{xy} = (\theta(g))_{xy} = 0$.

Proof. Let $\phi : F[x, y, z] \rightarrow F[x, y, z]$ be the F -automorphism induced by the change of coordinates that is defined by:

$$\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ where } M = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, x_{ij} \text{ are distinct variable entries, and } F \text{ is the field extension of } k \text{ generated by the } x_{ij}.$$

Since $\phi(f(x, y, z)) = f(\phi(x), \phi(y), \phi(z))$, then

$$\begin{aligned} \phi(f) &= \alpha_{20}(x_{11}x + x_{12}y + x_{13}z)^2 + \alpha_{11}(x_{11}x + x_{12}y + x_{13}z)(x_{21}x + x_{22}y + x_{23}z) + \alpha_{02}(x_{21}x + x_{22}y + x_{23}z)^2 \\ &+ \alpha_{10}(x_{11}x + x_{12}y + x_{13}z)(x_{31}x + x_{32}y + x_{33}z) + \alpha_{01}(x_{21}x + x_{22}y + x_{23}z)(x_{31}x + x_{32}y + x_{33}z) + \alpha_{00}(x_{31}x + x_{32}y + x_{33}z)^2 \\ &= \dots + (2\alpha_{20}x_{11}x_{12} + \alpha_{11}(x_{11}x_{22} + x_{12}x_{21}) + 2\alpha_{02}x_{21}x_{22} + \alpha_{10}(x_{11}x_{32} + x_{12}x_{31}) + \alpha_{01}(x_{21}x_{32} + x_{22}x_{31}) + 2\alpha_{00}x_{31}x_{32})xy + \dots \end{aligned}$$

Hence, by taking the partial with respect to xy we get

$$(\phi(f))_{xy} = 2\alpha_{20}x_{11}x_{12} + \alpha_{11}(x_{11}x_{22} + x_{12}x_{21}) + 2\alpha_{02}x_{21}x_{22} + \alpha_{10}(x_{11}x_{32} + x_{12}x_{31}) + \alpha_{01}(x_{21}x_{32} + x_{22}x_{31}) + 2\alpha_{00}x_{31}x_{32}.$$

Similarly,

$$(\phi(g))_{xy} = 2\beta_{20}x_{11}x_{12} + \beta_{11}(x_{11}x_{22} + x_{12}x_{21}) + 2\beta_{02}x_{21}x_{22} + \beta_{10}(x_{11}x_{32} + x_{12}x_{31}) + \beta_{01}(x_{21}x_{32} + x_{22}x_{31}) + 2\beta_{00}x_{31}x_{32}.$$

Let $\alpha = (\phi(f))_{xy}$, $\beta = (\phi(g))_{xy}$ and let $\delta = \det M$.

Now, Let us rewrite α , β and δ in the following forms:

$$\begin{aligned} \alpha &= (2\alpha_{20}x_{12} + \alpha_{11}x_{22} + \alpha_{10}x_{32})x_{11} + (\alpha_{11}x_{12} + 2\alpha_{02}x_{22} + \alpha_{01}x_{32})x_{21} + (\alpha_{10}x_{12} + \alpha_{01}x_{22} + 2\alpha_{00}x_{32})x_{31} \\ &= f_x(x_{12}, x_{22}, x_{32})x_{11} + f_y(x_{12}, x_{22}, x_{32})x_{21} + f_z(x_{12}, x_{22}, x_{32})x_{31}. \end{aligned}$$

$$\beta = (2\beta_{20}x_{12} + \beta_{11}x_{22} + \beta_{10}x_{32})x_{11} + (\beta_{11}x_{12} + 2\beta_{02}x_{22} + \beta_{01}x_{32})x_{21} + (\beta_{10}x_{12} + \beta_{01}x_{22} + 2\beta_{00}x_{32})x_{31}$$

$$= g_x(x_{12}, x_{22}, x_{32})x_{11} + g_y(x_{12}, x_{22}, x_{32})x_{21} + g_z(x_{12}, x_{22}, x_{32})x_{31}.$$

$$\delta = (x_{22}x_{33} - x_{23}x_{32})x_{11} + (x_{13}x_{32} - x_{12}x_{33})x_{21} + (x_{12}x_{23} - x_{13}x_{22})x_{31}.$$

Thus, we have

$$\begin{bmatrix} \alpha \\ \beta \\ \delta \end{bmatrix} = \mathcal{C} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}, \text{ where}$$

$$\mathcal{C} = \begin{bmatrix} 2\alpha_{20}x_{12} + \alpha_{11}x_{22} + \alpha_{10}x_{32} & \alpha_{11}x_{12} + 2\alpha_{02}x_{22} + \alpha_{01}x_{32} & \alpha_{10}x_{12} + \alpha_{01}x_{22} + 2\alpha_{00}x_{32} \\ 2\beta_{20}x_{12} + \beta_{11}x_{22} + \beta_{10}x_{32} & \beta_{11}x_{12} + 2\beta_{02}x_{22} + \beta_{01}x_{32} & \beta_{10}x_{12} + \beta_{01}x_{22} + 2\beta_{00}x_{32} \\ x_{22}x_{33} - x_{23}x_{32} & x_{13}x_{32} - x_{12}x_{33} & x_{12}x_{23} - x_{13}x_{22} \end{bmatrix}$$

$$= \begin{bmatrix} f_x(x_{12}, x_{22}, x_{32}) & f_y(x_{12}, x_{22}, x_{32}) & f_z(x_{12}, x_{22}, x_{32}) \\ g_x(x_{12}, x_{22}, x_{32}) & g_y(x_{12}, x_{22}, x_{32}) & g_z(x_{12}, x_{22}, x_{32}) \\ x_{22}x_{33} - x_{23}x_{32} & x_{13}x_{32} - x_{12}x_{33} & x_{12}x_{23} - x_{13}x_{22} \end{bmatrix}$$

Note that α and β are linear and homogeneous in $L[x_{11}, x_{21}, x_{31}]$ with coefficients from $L = k(x_{12}, x_{22}, x_{32})$. By the gcd condition, α and β must be distinct. Hence, (α, β) -the ideal generated by α and β -must be prime in $k[x_{12}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}]$. Since x_{13}, x_{23}, x_{33} are linear independent of the choices of $x_{12}, x_{21}, x_{32}, x_{11}, x_{12}$, and x_{13} , then (α, β) is prime ideal in

$$k[x_{12}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}, x_{13}, x_{23}, x_{33}].$$

To show that there exists a $\theta : A \rightarrow A$ as described in the statement of the proposition, it is enough to show that there exists $Q \in k^3$ such that $\alpha(Q) = \beta(Q) = 0$ and $\delta(Q) \neq 0$, which is equivalent to showing that $\delta \notin \text{Rad}((\alpha, \beta))$; i.e. δ does not belong to the radical of the ideal (α, β) in $k[x_{12}, x_{22}, x_{32}, x_{11}, x_{21}, x_{31}, x_{13}, x_{23}, x_{33}]$. But since (α, β) is prime ideal, as mentioned above, then it is sufficiently enough to show that $\delta \notin (\alpha, \beta)$; i.e. δ is not a linear combination of α and β . That is equivalent to showing that the matrix of coefficients \mathcal{C} of α , β , and δ has rank 3. Hence, it is enough to verify that for some specialization of \mathcal{C} .

Since f and g satisfy the gcd condition, we may assume, without loss of generality, that $c(x, y, z) =$

$\det\left[\frac{\partial(f,g)}{\partial(x,y)}\right] \neq 0$. In particular, let $c_0 = c(x_{12}, x_{22}, x_{32}) \neq 0$, then at least one of $c(x_{12}, x_{22}, 0)$, $c(x_{12}, 0, x_{32})$, $c(0, x_{22}, x_{32})$ is nonzero. However,

- If $c(x_{12}, 0, x_{32}) \neq 0$ and we set $x_{13} = x_{23} = x_{22} = 0$, then $\delta \neq 0$ and

$$\begin{aligned} \det \mathcal{C} &= -x_{12}x_{33} \det \begin{pmatrix} 2\alpha_{20}x_{12} + \alpha_{10}x_{32} & \alpha_{10}x_{12} + 2\alpha_{00}x_{32} \\ 2\beta_{20}x_{12} + \beta_{10}x_{32} & \beta_{10}x_{12} + 2\beta_{00}x_{32} \end{pmatrix} \\ &= -x_{12}x_{33} c(x_{12}, 0, x_{32}) \neq 0. \end{aligned}$$

- If $c(x_{12}, x_{22}, 0) \neq 0$ and we set $x_{32} = x_{23} = x_{33} = 0$, then $\delta \neq 0$ and

$$\begin{aligned} \det \mathcal{C} &= x_{13}x_{22} \det \begin{pmatrix} 2\alpha_{20}x_{12} + \alpha_{11}x_{22} & \alpha_{11}x_{12} + 2\alpha_{02}x_{22} \\ 2\beta_{20}x_{12} + \beta_{11}x_{22} & \beta_{11}x_{12} + 2\beta_{02}x_{22} \end{pmatrix} \\ &= x_{13}x_{22} c(x_{12}, x_{22}, 0) \neq 0. \end{aligned}$$

- If $c(0, x_{22}, x_{32}) \neq 0$, and we let $x_{12} = x_{13} = x_{32} = 0$, then $\delta \neq 0$ and

$$\begin{aligned} \det \mathcal{C} &= x_{13}x_{22} \det \begin{pmatrix} 2\alpha_{02}x_{22} + \alpha_{01}x_{32} & \alpha_{01}x_{22} + 2\alpha_{00}x_{32} \\ 2\beta_{02}x_{22} + \beta_{01}x_{32} & \beta_{01}x_{22} + 2\beta_{00}x_{32} \end{pmatrix} \\ &= x_{13}x_{22} c(0, x_{22}, x_{32}) \neq 0. \end{aligned}$$

Note that in all cases above we always have $\delta \neq 0$ and we found a specialization of \mathcal{C} that is nonsingular. Hence, we have proved by the method of specialization that the rank of the coefficient matrix \mathcal{C} is 3. \square

Proposition 3.3.5 completes our analysis of $Cl(X_{f,g})$ when f and g are homogeneous polynomials in A of degree 2. In all cases, we found that $Cl(X_{f,g}) \neq 0$ and that the logarithmic derivative factor property holds.

The next theorem summarizes our findings:

Theorem 3.3.2. *Let f and $g \in k[x, y, z]$ be homogeneous polynomials of degree two and assume f and g satisfy the gcd condition. Then $Cl(X_{f,g}) \neq 0$ and the logarithmic derivative factor property holds. Also, for a generic f and g , $Cl(X_{f,g})$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$.*

Chapter 4

Closing Remarks

In chapter three, for a homogeneous pair of f and g , we discussed the cases where the degrees of f and g are 2 or lower and we were able to come up with a full characterization of $Cl(X_{f,g})$. Unlike the quadratic case, We found that $Cl(X_{f,g})$ could be trivial group in the case where $\deg f = 2$ and $\deg g = 1$. In section one below, we provide a case of homogeneous polynomials f and g of distinct degrees greater than 2, for which $Cl(X_{f,g}) = 0$. In section two, we will conclude with some conjectures .

4.1 $\{\deg(f) = m, \deg(g) = n\}$

Observe that, as we noted earlier in chapter three, that Lemmas 2.1.1, 2.1.2, and 2.1.3 still hold for f and g of various degrees and forms in $k[x,y,z]$ as long as the gcd condition holds. In Proposition 4.1.1 below, we will combine Proposition 2.2.1 and Lemma 2.1.4 and prove a similar result for the more general statement when $\deg(f) = m$ and $\deg(g) = n$.

Proposition 4.1.1. *If $\deg(f) = m$ and $\deg(g) = n$, then $Cl(X_{f,g})$ is a p -group of type (p, \dots, p) of order p^r with $r \leq \binom{m+n}{3}$. If, in addition, f and g are homogeneous, then $r \leq \binom{m+n-1}{2}$.*

Proof. Let $t \in \mathcal{L}$. Then, by Lemma 1.3.3, $t \in A$ and $t = \frac{D(h)}{h}$ for some $h \in A$. Since $\deg(D(h)) \leq$

$\deg(h) + m + n - 3$, we have $\deg(t) \leq m + n - 3$. Thus,

$$t = \sum_{i+j+l=0}^{m+n-3} \alpha_{ij} x^i y^j z^l, \text{ with } \alpha_{ij} \in k. \quad (4.1)$$

By theorem 1.3.2 we have, $D^{p-1}(t) - at = -t^p$, where $a \in B$ is such that $D^p = aD$.

If we substitute t into both sides of equation (4.1) and compare coefficients, we see that the coefficients of t satisfy linear homogeneous equations in the α 's with coefficients in k . Also for each triple (i, j, l) with $i + j + l \leq m + n - 3$, the coefficients of t also satisfy a linearized equation of the form $L_{ijl} = \alpha_{ijl}^p$, where L_{ijl} is a linear expression in the α 's with coefficients in k . By Bezout's theorem, the linearized system has at most p^N solutions, where N is the number of monomials of degree less than or equal to $m + n - 3$. Since N is the $m + n - 2$ -tetrahedral number, we have,

$$N = \binom{m+n-1}{3}.$$

Suppose now that f and g are homogeneous of degrees m, n , respectively, and $t \in \mathcal{L}$ with $t \neq 0$. First note that if $w \in A$ is homogeneous and for some r , $D^r(w) \neq 0$, then $D^r(w)$ is homogeneous and $\deg(D^r(w)) = \deg(w) + r(m + n - 3)$. From this it follows that if $a \in B$ is such that $D^p = aD$, then either $a = 0$ or a is a homogeneous of degree $(p - 1)(m + n - 3)$.

Again, by Theorem 1.3.2, we have, $D^{p-1}(t) - at = -t^p$. Thus, if r is the degree of the leading homogeneous form of t , then the preceding paragraph together with a comparison of both sides of this equation yields that $pr \geq r + (p - 1)(m + n - 3)$. Hence, $r \geq m + n - 3$. Since we have already seen that the degree of t is at most $m + n - 3$, we obtain that t is homogeneous of degree $m + n - 3$. Since a homogeneous polynomial of this degree has at most $\binom{m+n-1}{2}$ nonzero coefficients, the same argument as used above shows that the order of \mathcal{L} is at most p^r . \square

Corollary 4.1.1. *Suppose $\deg(f) = m$ and $\deg(g) = n$. If $t \in \mathcal{L}$, then $\deg(t) \leq m + n - 3$. If f and g are homogeneous and $t \neq 0$, then t is homogeneous of degree $m + n - 3$ and $t = h^{-1}D(h)$ for some homogeneous $h \in A$.*

Proof. Each statement is from the proof of Proposition 4.1.1, except the very last one. Let $t \in \mathcal{L}$ be homogeneous of degree $n + m - 3$. Then there exists $h \in A$ such that $t = h^{-1}D(h)$ by Lemma 1.3.3. If $\deg(h) = r$, then $h = h_0 + h_1 + \dots + h_r$, where each h_i equals zero or is nonzero homogeneous in A of degree i . Then $\sum h_i t = \sum D(h_i)$ and each $D(h_i)$ that is not zero is homogeneous of degree $i + m + n - 3$. Hence, for each i such that $h_i \neq 0$, $D(h_i) = h_i t$, which proves the last assertion. \square

The following result is proved in Proposition 2.12 in [7]. It states that if $n \geq 4$ is an integer, then for a generic $g \in k[x, y]$ of degree n , the divisor class group of the surface $z^p = g$ is trivial. The proof uses Galois descent as described in section 1.3 and reduces to showing that for a generic g , the derivation D_g on $k(x, y)$ defined by $D_g(h) = g_y h_x - g_x h_y$ has no nonzero logarithmic derivatives in $k[x, y]$.

Proposition 4.1.2. *Assume $p \geq 3$. For a generic $g \in k[x, y]$ of degree $n \geq 4$, if $h \in k(x, y)$ and $h^{-1}(h_x g_y - h_y g_x) \in k[x, y]$, then $h_x g_y - h_y g_x = 0$.*

Proof. See [8, page 358]. \square

Proposition 4.1.3. *Assume $p \geq 5$. Let f and g be homogeneous polynomials in $k[x, y, z]$ with $\deg(f) = m$ and $\deg(g) = n$. Assume that $p \nmid m$ and $p \mid n$. Then for a generic g , $Cl(X_{f,g})$ is isomorphic to a direct sum of $s - 1$ copies of \mathbb{Z}_p , where s is the number of distinct irreducible factors of f . In particular, $Cl(X_{f,g}) = \{0\}$ for a generic pair f, g .*

Proof. By Proposition 4.12, we can safely assume that $g(x, y, 1)$, $g(x, 1, z)$, and $g(1, y, z)$ have no nonzero polynomial logarithmic derivatives since this is the case for a generic g . Let $f = f_1 f_2 \dots f_s$. If $f_i = f_j$ for $i \neq j$ and $1 \leq i, j \leq s$, then the gcd condition does not hold. Hence, we may assume that f_1, \dots, f_s are distinct, irreducible and homogeneous in A .

Our goal here is to show that $\{f_i^{-1}D(f) : 1 \leq i \leq s - 1\}$ is a basis of \mathcal{L} over \mathbb{F}_p . By Lemma 1.3.3 and corollary 4.1.1, we conclude that \mathcal{L} is generated by $\{h^{-1}D(h) : \text{where } h^{-1}D(h) \in A, h \in A \text{ is homogeneous and irreducible}\}$.

Now, assume that $t \in \mathcal{L}$, $t \neq 0$, and $t = h^{-1}D(h)$, for some $h \in A$, where h is irreducible and $\deg(h) = r$.

By Euler's formula we have,

$$D(h) = J(h, f, g) = \frac{1}{x} \begin{vmatrix} rh & h_y & h_z \\ mf & f_y & f_z \\ 0 & g_y & g_z \end{vmatrix} = \frac{1}{y} \begin{vmatrix} h_x & rh & h_z \\ h_y & mf & f_z \\ h_z & 0 & g_z \end{vmatrix} = \frac{1}{z} \begin{vmatrix} h_x & h_y & rh \\ h_y & f_y & mf \\ h_z & g_y & 0 \end{vmatrix}$$

Thus, h divides $mf(h_x g_y - h_y g_x)$, $mf(h_x g_z - h_z g_x)$, and $mf(h_y g_z - h_z g_y)$ in A .

Now, we consider the following cases:

- If h divides $h_x g_y - h_y g_x$, then $h(x, y, 1)$ divides the Jacobian $J(h(x, y, 1), g(x, y, 1))$ in $k[x, y]$. Hence, By our assumptions on g and by Proposition 4.1.2, we obtain, $h_x g_y - h_y g_x = 0$.
- If h divides $h_x g_z - h_z g_x$, then $h(x, 1, z)$ divides the Jacobian $J(h(x, 1, z), g(x, 1, z))$ in $k[x, z]$ and, hence, by Proposition 4.1.2, we conclude that $h_x g_z - h_z g_x = 0$.
- If h divides $h_y g_z - h_z g_y$, then $h(1, y, z)$ divides the Jacobian $J(h(1, y, z), g(1, y, z))$ in $k[y, z]$ and, thus, by Proposition 4.1.2, we get $h_y g_z - h_z g_y = 0$.

Based on the above cases, if h does not divide f , then $D(h)$ must equal to zero, which implies $t = 0$.

This conclusion contradicts our assumption that $t \neq 0$. Hence, we must have h divides f . Since the factors of f are irreducible, then $h = \alpha f_j$ for some $\alpha \in k^*$ and some $j = 1, 2, \dots, s$. Therefore, $t = h^{-1}D(h) = f_j^{-1}D(f_j)$.

By Lemma 2.1.5, we have $f_i^{-1}D(f_i) \in \mathcal{L}$, for each $i = 1, 2, \dots, s$. By Lemma 1.3.2, we have $\sum_{i=1}^s f_i^{-1}D(f_i) = f^{-1}D(f) = 0$. So, we have so far proved that the set $\{f_i^{-1}D(f_i) : 1 \leq i \leq s-1\}$ generates the group of logarithmic derivatives \mathcal{L} .

To show that $\{f_i^{-1}D(f_i) : 1 \leq i \leq s-1\}$ is independent, assume that $\sum_{i=1}^s \beta_i f_i^{-1}D(f_i) = 0$, for some $\beta_1, \dots, \beta_{s-1}$ in \mathbb{F}_p . Then, we have $D(f_1^{\beta_1} \dots f_{s-1}^{\beta_{s-1}}) = 0$ which by Proposition 2.1.2 implies $f_1^{\beta_1} \dots f_{s-1}^{\beta_{s-1}} = \sum_{i=0}^{p-1} w_i f^i$, where each $w_i \in A^p[g]$ and is homogeneous.

Since each w_i is necessarily of degree congruent to $0 \pmod{p}$ and by our assumption $m \neq 0 \pmod{p}$, it must be that there exists a j such that $f_1^{\beta_1} \dots f_{s-1}^{\beta_{s-1}} = w_j f^j$.

If $j \neq 0$, then f_s divides $f_1^{\beta_1} \dots f_{s-1}^{\beta_{s-1}}$, which contradicts the fact that all f_i 's are distinct and irreducible.

On the other hand, if $j = 0$, then $f_1^{\beta_1} \dots f_{s-1}^{\beta_{s-1}} \in A^p[g]$. But by Lemma 1.3.2, our opening assumption on g , and the gcd condition, this can only occur if each $\beta_i = 0$ for each $i = 1, 2, \dots, s-1$. Hence, the set $\{f_i^{-1}D(f) : 1 \leq i \leq s\}$ forms a basis for \mathcal{L} over \mathbb{F}_p . Therefore, \mathcal{L} and $Cl(X_{f,g})$ are isomorphic to $s-1$ copies of \mathbb{Z}_p .

In regard to the last statement in the proposition:

Since f is generically irreducible then based on all of the previous argument, we conclude that $Cl(X_{f,g}) = \{0\}$. □

4.2 Future Work

Throughout our analysis of the quadratic case in section 3.3, we have found that $Cl(X_{f,g}) \neq \{0\}$ for a generic f and g . From this fact, a natural analogue emerges about homogeneous polynomials f and g in $k[x, y, z]$ of the same degrees. We state the analogue in the following conjecture:

Conjecture 1. *Assume $f, g \in A$ are homogeneous such that the gcd condition holds and such that $\deg(f) = \deg(g) \geq 3$. Then $Cl(X_{f,g}) \neq \{0\}$ for a generic f and g .*

We generally tend to believe that conjecture one is true.

Also, throughout our investigation of the quadratic case, we have found that the generators of $Cl(X_{f,g})$ are a linear combinations of f and g ; i.e. f, g satisfy the logarithmic derivative factor property. Hence, based on this result, we formulate the following conjecture in regard to the more general case:

Conjecture 2. *The logarithmic derivative factor property holds for all homogeneous pairs $f, g \in A$ of the same degree that satisfies the gcd condition.*

The following conjecture is motivated by Theorem 3.2.1 in section 3.2 :

Conjecture 3. *Let $f, g \in A$ be homogeneous polynomials in A such that the gcd condition holds. If $\deg(g) = 1$ and $t \in \mathcal{L}$, Then there exists $a, b \in k$ and a factor h of $af + bg^{\deg(f)}$ such that $t = h^{-1}D(h)$.*

Let $R = k[x_1, \dots, x_n]$ and let $f_1, f_2, \dots, f_{n-1} \in R$ be homogeneous polynomials of degree 2. Let $D : k(x_1, \dots, x_n) \rightarrow k(x_1, \dots, x_n)$ be the n -th dimensional k -derivation defined by :

$$D(h) = \det\left[\frac{\partial(h, f_1, \dots, f_{n-1})}{\partial(x_1, \dots, x_n)}\right] = J(h, f_1, \dots, f_{n-1}) \text{ for all } h \in k(x_1, \dots, x_n).$$

Furthermore, assume that the $(n-1) \times (n-1)$ maximal minors of the Jacobian matrix $\left[\frac{\partial(f_1, \dots, f_{n-1})}{\partial(x_1, \dots, x_n)}\right]$ have the greatest common divisor equals 1 in R . Let $R^p = k[x_1^p, \dots, x_n^p]$, then we have the following conjecture:

Conjecture 4. 1. $Cl(R^p[f_1, \dots, f_{n-1}]) \neq 0$.

2. For each $r = 1, \dots, n$, there exists $f_1, \dots, f_{n-1} \in R$ such that the order of $Cl(R^p[f_1, \dots, f_{n-1}])$ equals to p^r .

3. The order of $Cl(R^p[f_1, \dots, f_{n-1}])$ equals to p^n for a generic $f_1, \dots, f_{n-1} \in R$.

4. For all f_1, \dots, f_{n-1} , the group of logarithmic derivatives \mathcal{L} is generated by :

$\{h^{-1}D(h) : h \text{ divides } \alpha_1 f_1 + \dots + \alpha_{n-1} f_{n-1} \text{ for some } \alpha_1, \dots, \alpha_{n-1} \in k\}$. Hence, the natural analogue of the logarithmic derivative factor property holds for $f_1, \dots, f_{n-1} \in R$.

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