ENVELOPES

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SECTION I.

Definitions of an Envelope of a Family of Curves Involving a Single Parameter

A. Let us consider a family of curves

$$F(x\ y\ a) = 0,$$

a being the variable parameter, that is, a being allowed to take on various values. If the curves of the family are drawn which correspond to a system of infinitesimally close values of the parameter, arranged in order of magnitude, the locus of the intersection of consecutive curves of the family

$$F(x\ y\ a) = 0$$

is defined as the envelope of the family of curves.

Having defined an envelope of the family of curves,

$$F(x, y, a) = 0$$

let us find the equation of that envelope, provided one exists. Assuming, for the time, that an envelope does exist, let a and $a + \delta a$ be two values of the parameter, where δ a is an infinitesimally small quantity. The equation of two such curves we know to be

$$F(x, y, a) = 0 \qquad (1)$$

and

$$F(x, y, a + \delta a) = 0 \qquad (2)$$

Since the limiting position of the points of intersection of two curves of the family whose parameters differ by an infinitesimally small quantity, as $\delta a \longrightarrow 0$, is, by definition, points of the envelope, we have that the equation

$$\frac{F(x, y, a + \delta a) - F(x, y, a)}{a} = 0 \qquad (3)$$

represents a curve which passes through all the intersections of the two curves (1) and (2).

Since a is constant for any one of the curves of the family, we have by the definition of a partial derivative that

$$\lim_{\delta a \to 0} \frac{F(x, y, a + \delta a) - F(x\ y\ a)}{\delta a} =$$

$$F'a = 0 \qquad (4)$$

Hence

$$F(x, y, a) = 0$$

and

$$F'a = 0$$

intersect in those points of the envelope which lie upon

$$F(x, y, a) = 0.$$

Therefore, to obtain the equation of the envelope, we have only to eliminate a from equations (1) and (4).

B. Given the family of curves

$$F (x, y, a) = 0 \qquad (1)$$

where as is the variable parameter. If each of the curves of the family, as a takes on all possible values, is tangent to a curve E, we say that E is the envelope of the family of curves (1)

Let us now find the equation of this envelope, provided one exists. From the definition given above the curves of the family (1) are tangent to the envelope, and the point of tangency moves along the envelope as a varies. The equation of the envelope may, therefore, be written.

$$x = \phi (a) \quad , y = \psi (a), \text{ with}$$

$$F (\phi, \psi, a) = 0 \qquad (2)$$

where the first two equations show that the points on the envelope are functions of the parameter a, and the last one shows that each point of the envelope lies on some curve of the family

$$F (x,y,a) = 0.$$

Differentiating (2) with respect to a, we have

$$F'_x \phi' (a) + F'_y \psi'(a) + F'_a = 0 \qquad (3)$$

From our definition of an envelope, we know that the tangents to the curves E and F (x,y,a) = 0 coincide for all values of a. Let δx and δy be two quantities proportional to the direction cosines of the tangent to the curve

$$F(x,y,a) = 0$$

and let $\frac{dx}{da}$ and $\frac{dy}{da}$ be the derivatives of the functions $x = \phi(a)$ and $y = \psi(a)$ respectively. The necessary condition for tangency is

$$\frac{\frac{dx}{da}}{\delta x} = \frac{\frac{dy}{da}}{\delta y}$$

But since a, for any one curve of the family has a constant value, we have

$$F'_x \; \delta x + F'_y \; \delta y = 0 \qquad (4)$$

which determines the tangent to the curve

$$F(x,y,a) = 0$$

By combining (3) and (4) we have

$$F'_a(x,y,a) = 0 \qquad (5)$$

If we eliminate a in equations (1) and (5) we have the equation of the envelope.

C. Let us consider the envelope to a family of curves whose equation is of the form

$$A a^2 + 2 B a + C = 0$$

where A, B and C are any functions of x and y, and a is the variable parameter.

$$F(xya) = Aa^2 + 2\ Ba + C = 0 \qquad (1)$$

$$F(xya) = 2Aa + 2B = 0$$

Eliminating a in equations (1) and (2) we see that the envelope to the family of curves (1) is given by the equation

$$B^2 = AC$$

D. The discriminant of

$$F(x,y,a) = Ao\ a^n + A_1\ a^{n-1} \text{ -------- } A_n = 0, \qquad (1)$$

where A_i ($i = 0$ -----n) is any function of x and y, and a is the variable parameter, is that integral function of the coefficients of

$$F(x,y,a) = 0, \qquad (2)$$

that is, a function of x and y, whose fanishing is a necessary and sufficient condition that $F(a) = 0$ has multiple (equal) roots.

When this discriminant is expressed in its simplest rational integral form, it is called the a- discriminant of (2), and may be denoted by

$$\Delta a\ (F) = 0$$

We know from our work in Theory of Equations, that the discriminant of (1) is found by eliminating a between

$$F(x,y,a) = 0$$

and

$$F'a = 0 \qquad (3)$$

Hence, since from previous definitions of the envelope, we know that its equation is obtained by eliminating a in (2) and (3), the equation of the discriminant is the equation of the envelope or contains curves which are the envelope of the family of curves.

E. Given the family of curves

$$F(x,y,a) = 0, \tag{1}$$

and suppose that as a varies, we can find a curve to which each of the curves of the family (1) is tangent, that is, that there is an envelope for this family of curves.

Now let us consider equation (1) as that of a surface. If we section this surface by a series of planes parallel to the x,y plane, that is, if we let a assume various values, and project the various curves formed by the intersections of the surface

$$F(x,y,a) = 0$$

and the planes

$$a = k,$$

where k takes on all possible values, onto the x,y plane, these projections will be the family of given curves (1) in the x,y plane, the two curves with the same value of a being coincident.

If now, we project into space the envelope of the given family of curves in the x,y plane by erecting perpendiculars at all points of the envelope, this projection will be a curve in space which lies on the surface

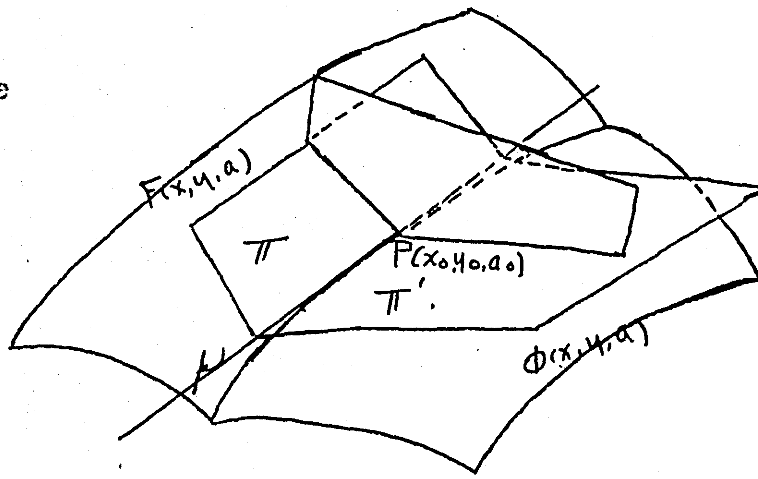
$$F(x,y,a) = 0$$

If through this curve we pass another surface,

$$\phi(x,y,a) = 0, \qquad (2)$$

distinct from $F(x,y,a) = 0$, and let it be required to find the tangent line to this curve in space, at the point $P(x_o, y_o, a_o)$

The tangent line p through $P(x_o, y_o, z_o)$ is the intersection of the tangent planes π and π' to the two surfaces $F(x,y,a) = 0$ and $\phi(x,y,a) = 0$ respectively, for p is tangent to both surfaces, and hence must lie in both tangent planes.



The equations of the tangent planes π and π' at $P(x_o, y_o, a_o)$ are

$$\frac{\partial F_o}{\partial x_o}(x-x_o) + \frac{\partial F_o}{\partial y_o}(y-y_o) + \frac{\partial F_o}{\partial a_o}(a-a_o) = 0 \qquad (3)$$

and

$$\frac{\partial \phi_0}{\partial x_0}(x-x_0) + \frac{\partial \phi_0}{\partial y_0}(y-y_0) + \frac{\partial \phi_0}{\partial a_0}(a-a_0) = 0 \qquad (4)$$

respectively. Reducing these equations (3) and (4) to the determinant form we have

$$\frac{x - x_0}{\begin{vmatrix} \frac{\partial \phi_0}{\partial y_0} & \frac{\partial F_0}{\partial y_0} \\ \frac{\partial \phi_0}{\partial a_0} & \frac{\partial F_0}{\partial a_0} \end{vmatrix}} = \frac{y - y}{\begin{vmatrix} \frac{\partial \phi_0}{\partial a_0} & \frac{\partial F_0}{\partial a_0} \\ \frac{\partial \phi_0}{\partial x_0} & \frac{\partial F_0}{\partial x_0} \end{vmatrix}} = \frac{a - a}{\begin{vmatrix} \frac{\partial \phi_0}{\partial x_0} & \frac{\partial F_0}{\partial x_0} \\ \frac{\partial \phi_0}{\partial y_0} & \frac{\partial F_0}{\partial y_0} \end{vmatrix}} \qquad (5)$$

$$\frac{x - x_0}{\begin{vmatrix} \frac{\partial \phi_0}{\partial y_0} & \frac{\partial F_0}{\partial y_0} \\ \frac{\partial \phi_0}{\partial a_0} & \frac{\partial F_0}{\partial a_0} \end{vmatrix}} = \frac{y - y_0}{\begin{vmatrix} \frac{\partial \phi_0}{\partial a_0} & \frac{\partial F_0}{\partial a_0} \\ \frac{\partial \phi_0}{\partial x_0} & \frac{\partial F_0}{\partial x_0} \end{vmatrix}} \qquad (6)$$

is the projection of the tangent line to $F(x,y,a) = 0$ and $\phi\ (x,y,a) = 0$ at the point $P\ (x_0, y_0, a_0)$ onto the x,y plane.

Let it be required to find the tangent to the curve

$$F\ (x,y,\ a_0) = 0 \qquad (7)$$

at the point (x_0, y_0). This equation is

$$\frac{\partial F_0}{\partial x_0}(x - x_0) + \frac{\partial F_0}{\partial y_0}\, y - y_0) = 0,$$

$$\frac{\partial F_0}{\partial x_0}(x - x_0) = -\frac{\partial F_0}{\partial y_0}(y - y_0) = 0,$$

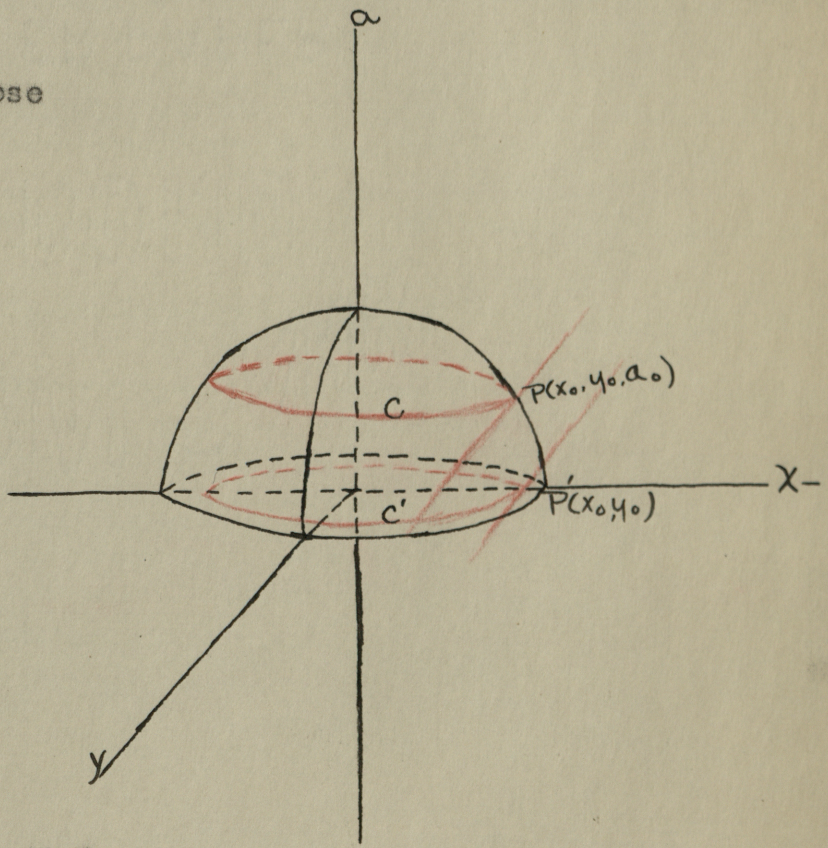
or

$$\frac{x - x_0}{\frac{\partial F_0}{\partial y_0}} = \frac{y - y_0}{-\frac{\partial F_0}{\partial x_0}} \tag{8}$$

The projection of a tangent to a curve C in space at the point P (x_0, y_0, a_0) is tangent at $P'(x_0, y_0)$, the projection of P (x_0, y_0, a_0), to the curve C', the projection of C onto the x, y plane.

Let Q be a point on the curve C, whose projection on the x,y plane is Q', a point on the curve C' different from P'.

As the chord PQ of C becomes the tangent at P to C by moving Q up to P, the chord $P'Q'$ of C' becomes the tangent at P' to C' by moving Q' up to P'.



Hence we have that

$$\frac{x - x_0}{\begin{vmatrix} \frac{\partial \phi_0}{\partial y_0} & \frac{\partial F_0}{\partial y_0} \\[2ex] \frac{\partial \phi}{\partial a_0} & \frac{\partial F_0}{\partial a_0} \end{vmatrix}} = \frac{y - y_0}{\begin{vmatrix} \frac{\partial \phi_0}{\partial a_0} & \frac{\partial F_0}{\partial a_0} \\[2ex] \frac{\partial \phi_0}{\partial x_0} & \frac{\partial F_0}{\partial x_0} \end{vmatrix}}$$

and

$$\frac{x - x_0}{\frac{\partial F_0}{\partial y_0}} = -\frac{y - y_0}{\frac{\partial F_0}{\partial x_0}}$$

represent the same line in the x, y plane, that is

$$\frac{\begin{vmatrix} \frac{\partial \phi_0}{\partial y_0} & \frac{\partial F_0}{\partial y_0} \\[2ex] \frac{\partial \phi_0}{\partial a_0} & \frac{\partial F_0}{\partial a_0} \end{vmatrix}}{\begin{vmatrix} \frac{\partial \phi_0}{\partial a_0} & \frac{\partial F_0}{\partial a_0} \\[2ex] \frac{\partial \phi_0}{\partial x_0} & \frac{\partial F_0}{\partial x_0} \end{vmatrix}} = \frac{\frac{\partial F_0}{\partial y_0}}{-\frac{\partial F_0}{\partial x_0}} \qquad (9)$$

$$\frac{\dfrac{\partial \phi_0}{\partial y_0}\dfrac{\partial F_0}{\partial a_0} - \dfrac{\partial F_0}{\partial y_0}\dfrac{\partial \phi_0}{\partial a_0}}{\dfrac{\partial \phi_0}{\partial a_0}\dfrac{\partial F_0}{\partial x_0} - \dfrac{\partial F_0}{\partial a_0}\dfrac{\partial \phi_0}{\partial x_0}} \qquad \frac{\dfrac{\partial F_0}{\partial y_0}}{-\dfrac{\partial F_0}{\partial x_0}}$$

$$-\frac{\partial \phi_0}{\partial y_0}\frac{\partial F_0}{\partial a_0}\frac{\partial F_0}{\partial x_0} + \frac{\partial F_0}{\partial y_0}\frac{\partial \phi_0}{\partial a_0}\frac{\partial F_0}{\partial x_0} = \frac{\partial \phi_0}{\partial a_0}\frac{\partial F_0}{\partial x_0}\frac{\partial F_0}{\partial y_0} - \frac{\partial F_0}{\partial a_0}\frac{\partial \phi_0}{\partial x_0}\frac{\partial F_0}{\partial y_0}$$

$$\frac{\partial F_0}{\partial a_0}\frac{\partial \phi_0}{\partial x_0}\frac{\partial F_0}{\partial y_0} - \frac{\partial \phi_0}{\partial y_0}\frac{\partial F_0}{\partial a_0}\cdot\frac{\partial F_0}{\partial x_0}$$

$$\frac{\partial F_0}{\partial a_0}\left[\frac{\partial \phi_0}{\partial x_0}\frac{\partial F_0}{\partial y_0} - \frac{\partial \phi}{\partial y_0}\frac{\partial F_0}{\partial x_0}\right] = 0$$

$$\frac{\partial F_0}{\partial a_0}\begin{vmatrix} \dfrac{\partial \phi_0}{\partial x_0} & \dfrac{\partial F_0}{\partial x_0} \\[2ex] \dfrac{\partial \phi}{\partial y} & \dfrac{\partial F_0}{\partial y_0} \end{vmatrix} = 0 \tag{10}$$

If the determinant (10) does not vanish, we know that

$$\frac{\partial F_0}{\partial a_0} = 0$$

We shall now prove that the determinant

$$\begin{vmatrix} \dfrac{\partial \phi_0}{\partial x_0} & \dfrac{\partial F_0}{\partial x_0} \\ \\ \dfrac{\partial \phi_0}{\partial y_0} & \dfrac{\partial F_0}{\partial y_0} \end{vmatrix} = \Delta \qquad (11)$$

cannot be zero.

In order that the curve (7) may have a tangent at (x_0, y_0) it is sufficient that $\dfrac{\partial F_0}{\partial x_0}$ and $\dfrac{\partial F_0}{\partial y_0}$ should be continuous in the neighborhood of (x_0, y_0) and that not both $\dfrac{\partial F_0}{\partial x_0}$ and $\dfrac{\partial F_0}{\partial y_0}$ be zero. If $\dfrac{\partial F_0}{\partial x_0} = 0$ and $\dfrac{\partial F_0}{\partial y_0} = 0$, there would be no tangent, for equation (8) would vanish.

Likewise in order that the curve

$$\phi(x, y, a_0) = 0 \qquad (12)$$

may have a tangent at (x_0, y_0) it is sufficient that $\dfrac{\partial \phi_0}{\partial x_0}$ and $\dfrac{\partial \phi_0}{\partial y_0}$ be continuous in the neighborhood of (x_0, y_0) and that notboth $\dfrac{\partial \phi_0}{\partial x_0}$ and $\dfrac{\partial \phi_0}{\partial y_0}$ be zero.

Hence the determinant Δ cannot reduce to zero by having all the elements of either column be zero.

The two foregoing theorems do not take care of the case where $\frac{\partial F_0}{\partial x_0} = 0$ and $\frac{\partial F_0}{\partial y_0} \neq 0$ and $\frac{\partial \phi_0}{\partial x_0} = 0$ and $\frac{\partial \phi_0}{\partial y_0} \neq 0$. Should this be the case we would have

$$\begin{vmatrix} 0 & 0 \\ \frac{\partial \phi_0}{\partial y_0} & \frac{\partial F_0}{\partial y_0} \end{vmatrix} = 0$$

It is easily seen that this condition cannot exist, for, if $\frac{\partial F_0}{\partial x_0}$ and $\frac{\partial \phi_0}{\partial x_0}$ are both zero, equation (9), the equation upon which this entire discussion is based, would vanish. Likewise, $\frac{\partial F_0}{\partial y_0}$ and $\frac{\partial \phi_0}{\partial y_0}$ cannot both be zero.

Hence the determinant Δ cannot reduce to zero by having all the elements of either row be zero.

These two special cases are, in fact, taken care of by the following general discussion.

To prove that if $\Delta = 0$, even though the elements of any row or column are not all zero, that is, to prove that if the determinant vanishes at the point (x_0, y_0); the two functions (7) and (12) are not independent functions.

Let

$$u = F(x, y\ a_0) \qquad (13)$$

and

$$v = \phi\ (x, y, a_0), \qquad (14)$$

where u and v are supposed not to reduce to constants, and hence we may assume that at least one of the partial derivatives, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, does not reduce to zero. Let $\frac{\partial F}{\partial y}$ be the derivative which does not vanish at the point (x_0, y_0). Then equation (13) may be solved for y, hence we have

$$y = f\ (x, u, a_0)$$

Substituting in (14), we have

$$v = g\ [x,\ f\ (x, u a_0)\ a_0]$$

$$v = g\ (x, u, a_0) \qquad (15)$$

Differentiating and remembering that x and y are independent variables, we have

$$\frac{\partial v}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\,\frac{\partial u}{\partial x} \qquad (16)$$

$$\frac{\partial v}{\partial y} \qquad\qquad \frac{\partial g}{\partial u}\,\frac{\partial u}{\partial y}$$

But we have assumed that

$$\begin{vmatrix} \dfrac{\partial \phi_0}{\partial x_0} & \dfrac{\partial F_0}{\partial x_0} \\[2ex] \dfrac{\partial \phi_0}{\partial y_0} & \dfrac{\partial F_0}{\partial y_0} \end{vmatrix} = 0$$

Since

$$\begin{vmatrix} \dfrac{\partial \phi}{\partial x} & \dfrac{\partial F}{\partial x} \\[2ex] \dfrac{\partial \phi}{\partial y} & \dfrac{\partial F}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \\[2ex] \dfrac{\partial v}{\partial y} & \dfrac{\partial u}{\partial y} \end{vmatrix} = 0$$

equations (16) hold only if $\dfrac{\partial g}{\partial x} = 0$, that is if

g has no x - terms, and

$$v = g\,(u,a)$$

This is contrary to our assumption that

$$F(x, y, a_0) = 0$$

and $$\phi(x,y,a_0) = 0$$

are independent functions.

Hence,

$$F_a'(x,y, a) = 0 \qquad (17)$$

By eliminating a from equations (1) and (17) if such elimination be possible, we obtain the equation

$$\psi(x,y) = 0$$

which is the equation of the envelope to the family of curves

$$F(x,y,a) = 0$$

should such an envelope exist.

F. Let us consider a family of curves depending on one parameter, that is, one represented by the equation

$$F(x,y,a) = 0 \qquad (1)$$

whose partial derivatives, at least of the first two orders, are continuoüs. We may assume further that if there are any singular points, that is, points which satisfy the equations

$$F(x,y,a) = F_x'(x,y,a) = F_y'(x,y,a) = 0, \qquad (2)$$

they are isolated points.

Let $M(x,y)$ be an ordinary point on the curve $F(x,y,a)$, so that at least one of the partial derivatives, $F'x$ or $F'y$, is not zero. Assuming that there are a finite number of singular points in any region we wish to consider, a region of values of x,y, and a, including the point $M(x,y)$, can be determined which contains no singular points of any curve of the family of curves (1).

The equation

$$F\,(x,y,a+\delta a\,) = 0 \qquad (3)$$

of the curve of the family which differs from $F(x,y,a)=0$ by an infinitesimally small quantity, may be written in the form

$$F\,(x,y,a) + F(x,y,a+\delta a\,) - F\,(x,y,a) = 0 \qquad (4)$$

But, $F(x,y,a+\delta a\,) - F\,(x,y\,a)$, by the Law of the Mean, is equal to

$$\delta a\,\left[F'a\,(x,y,a) + \theta\,\delta a\right]$$

Substituting in equation (3), we have as the equation of the curve $F(x,y,a+\delta a\,) = 0$,

$$F(x\;y\;a) + \delta a\,F'a\,(x,y\;a) + \theta\,(\delta a)^2$$
$$= 0 \qquad (5)$$

Let us assume without proof that the shortest distance σ of M from the curve (4) satisfies the relation

$$\sigma = A\ F\ (x, y, a + \delta a)\ (1 + \epsilon\ )$$

or

$$= A \left\{F\ (x,y,a) + \delta a\ F'_a(x,y,a) + \theta(\delta a)^2\right\}$$
$$\left\{1 + \epsilon\right\} \qquad (6)$$

where $A \neq 0$, (x,y) are the coordinates of M, and $\epsilon \to 0$ when $\sigma \to 0$

Since M (x,y) is an ordinary point, we have that $\sigma \to 0$ when $\delta a \to 0$, so that $\epsilon \to 0$ when $\delta a \to 0$. But $AF(x,y,a) = 0$, hence equation (5), which may be written in the form

$$\sigma = \left[A\ F(x,y,a) + AF'_a\ (x,y,a)\ \delta a + A\theta(\ \delta a)^2\right]$$
$$\left[1 + \epsilon\right]$$

reduces to

$$\sigma = \left[AF'_\alpha(x,y,a)\ \delta a + A\ \theta(\delta\ a)^2\right]\left[1 + \epsilon\right]$$

Therefore, as $\delta a \to 0$

$$\sigma = A\ F'_\alpha(x,y,a)\ \delta a + A\theta(\ \delta a)^2]\ \left[1+0\right].$$

This distance will be of the second, or higher order of smallness, if and only if

$$F'_a\ \ (x,y,a) = 0.$$

An ordinary point on the curve (1), whose distance from the curve (3) is of second order of smallness at least, is called a characteristic point of the curve (1). These points are ordinary points at which $F'_a(x,y,a) = 0$. It may happen that the curve (1) is entirely composed of characteristic points, but in general these characteristic points are isolated.

Definition:

The envelope of a family of curves (1) is the locus of its isolated characteristic points.

If there is an envelope, its points satisfy the equations

$$F(x,y,a) = 0$$

and

$$F'_a(x,y,a) = 0, \tag{7}$$

and the equation of the envelope is obtained by eliminating a between these two equations.

In all the definitions of an envelope given in this paper, we have assumed that an envelope exists, and that its equation is obtained by eliminating the parameter a between the equations (7).

The complete result of such an elimination we know from definition (D) of the envelope of a family of curves involving a single parameter to be the a -

discriminant of

$$F(x,y,a) = 0.$$

We cannot, however, be certain that any curve contained in the a - discriminant is the envelope or any part of the envelope. For instance it may be possible to satisfy the two equations (7), by values of a independent of x and y, and then the curve (1), where a has taken on a particular value, will be composed entirely of characteristic points. Such a curve will be included in the a-discriminant, but will not be a part of the envelope. Again, suppose that the family (1) contains a locus of singular points, that is points of (1) at which both $F'x$ and $F'y$ are zero. The coordinates of these singular points satisfy the equations

$$F(x,y,a) = 0$$

$$F'x = 0$$

$$F'y = 0,$$

and also the equation obtained by differentiating $F(x,y,a) = 0$ with respect to a, that is

$$F'x \frac{dx}{da} + F'y \frac{dy}{da} + F'a = 0,$$

where $\frac{dx}{da}$ and $\frac{dy}{da}$ are determined from the equation of the locus of singular points. Thus we see

$$F_a(x,y,a) = 0$$

is also satisfied by the coordinates (x,y) of these singular points. Hence any locus of singular points of the curves of the family (1) will be part of the a-discriminant, and must be distinguished from the envelope. These loci of singular points may be a nodal locus, a cusp locus, or a tac locus.

In general, no such locus exists for, in general, singular points occur only for isolated value of x and y for any given value of a, that is, they occur only for isolated members of the family, and hence may or may not lie on the a - discriminat.

If we are to find the equation of the true envelope to a family of curves (1), the equation found by eliminating a between equations (7) must be tested for curves composed entirely of characteristic points, also for loci of singular points, in case the singular points lie on the a - discriminant. When the equations of such curves have been discarded, any remaining curves constitute the envelope of the family of curves.

The envelope of a family of curves may be one single continuous curve, or it may be composed of two or more curves; and it may happen that either the whole curve or one or more of the component curves may coincide with a particular curve or curves of the system

$$F (x, y, a) = 0.$$

SECTION II

Properties of Envelopes of Curves Involving a Single Parameter.

The envelope is tangent to each of the curves of the family.

Every point of the envelope is a point on some curve of the family, so that what we have to prove is that at such a point the two curves have the same slope.

The slope of the family of curves

$$F\ (x,y,a) = 0 \qquad (1)$$

at the point (x,y) is given by the equation,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \quad \frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Since the equation of the envelope is found by eliminating a from the equations $F\ (x,y,a) = 0$ and $F'a\ (x,y,a) = 0$ we may solve $F'a$ for a and substitute this value in equation (1). Hence we have that the equation of the envelope may be given in the form

$$F(x,y,a) = 0,$$

where a is no longer a constant but a function of x and y. Differentiating with respect to x, since a is a function of x and y, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial a}\frac{da}{dx} = 0 \qquad (2)$$

But from the equation of the envelope we know that

$$F_a'(x,y,a) = \frac{\partial F}{\partial a} = 0$$

Solving for $\frac{dy}{dx}$ in (2), we find that the slope of the envelope is

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Thus we have proved that the slope of the curve (1) and the slope of the envelope to the family of curves (1) are identical.

SECTION III

Definitions of an Envelope of a Family of Curves Involving Two Parameters.

Having considered only those curves involving a single parameter, let us now consider the family of curves

$$F\,(x,y,a,\,b) = 0,$$

involving the two parameters a and b.

A. Given the family of curves

$$F\,(x,y,a,b) = 0, \qquad (1)$$

where a and b, the two parameters, are connected by some given equation as

$$\phi(a,b) = 0 \qquad (2)$$

it is possible, by solving the given equation (2) to find one of the parameters in terms of the other. Substituting this value in the original equation, we have reduced our problem to that of finding the envelope to a family of curves involving only one parameter.

It is sometimes difficult, however, to eliminate one parameter as described above. In such a case, it is better to differentiate the two given equations (1) and (2), using one parameter as the independent variable and the other parameter as the dependent variable. If we take a as the independent variable, we have the four equations,

$$F\,(x, y, a\ b) = 0$$

$$\phi\,(a, b) = 0$$

$$F'a + F'b\,\frac{db}{da} = 0 \qquad (3)$$

$$\phi'a + \phi'b\,\frac{db}{da} = 0 \qquad (4)$$

From these four equations we may eliminate the three terms a, b, and $\frac{db}{da}$, and the resulting equation

$$\psi\,(x, y) = 0$$

is the equation of the envelope to the family of curves (1).

B. Given the equation of a family of curves,

$$x = f\,(u, k) \qquad (1)$$

and

$$y = g\,(u, k), \qquad (2)$$

where u and k are variable parameters, to find the envelope of the family of curves (1) and (2). This may be done in two ways.

First, we may eliminate one of the parameters, say k, from equations (1) and (2), and thus reduce the problem to that of finding the envelope of a family of curves

$$F(x, y, u) = 0$$

which can be done by eliminating u from equations

$$F\ (x,y,u\ ) = 0$$

and

$$F'u\ (x,y,u\ ) = 0.$$

As a second method, we may consider one of the parameters, say u, as the independent variable and k as the dependent variable. Differentiating (1) and (2), we obtain the equations

$$\frac{\partial f}{\partial u} + \frac{\partial f}{\partial k}\ \frac{dk}{du} = 0 \qquad (3)$$

$$\frac{\partial g}{\partial u} + \frac{\partial g}{\partial k}\ \frac{dk}{du} = 0 \qquad (4)$$

If we eliminate $\frac{dk}{du}$ from equations (3) and (4), we have

$$\frac{\partial f}{\partial u}\ \frac{\partial g}{\partial k} - \frac{\partial g}{\partial u}\ \frac{\partial f}{\partial k} = 0.$$

But this equation may be written in the form

$$\begin{vmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial k} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial k} \end{vmatrix} = 0 \qquad (5)$$

By eliminating u and k in equations (1), (2) and (5), we obtain the equation of the envelope to the family of curves, (1) and (2).

SECTION IV.

Definition of Envelopes of Surfaces.

Let us consider a surface S of the family of surfaces

$$f(x,y,z,a) = 0$$

where a is the variable parameter. If the surface S and S' of the family, which correspond to a system of infinitesimally close values of the parameter, arranged in order of magnitude, are drawn, the limiting position of the curve of intersection as S' approaches S as a limit, is known as the characteristic curve C. From the discussion of envelopes of curves, we know that this curve is represented by the equations

$$f\,(\,x\;y\;z\;a\,) = 0 \qquad (1)$$

and

$$f'_{a}\,(\,x\;y\;z\;a\,) = 0 \qquad (2)$$

If there exists a surface E which is tangent to each of the surfaces S along this characteristic curve C, E is called the envelope of the family of surfaces (1). The curve C is the curve of tangency of the two surfaces C and E. Furthermore, the curve C lies on the envelope E, that is, we may define the envelope as the surface generated by C as a varies.

Let (x_o y_o z_o) be the coordinate of a point P on the curve C. If P (x_o y_o z_o) is not a singular point of S, the equation of the plane tangent to S at P(x_o y_o z_o) is

$$\frac{\partial f_o}{\partial x_o}(x - x_o) + \frac{\partial f_o}{\partial y_o}(y-y_o) + \frac{\partial f_o}{\partial z_o}(z - z_o)$$

$$= 0 \qquad (3)$$

Since the curve C on S lies on E also, we know that equation (1) must be satisfied by the points of the envelope. This being true, we know from our work in elementary calculus on total differentials, that

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial a}da = 0 \qquad (4)$$

But, since the tangent plane to S at P (x_o, y_o, z_o) is represented by equation (3) in order that E whose points must satisfy equation (4), should coincide with the tangent plane, that is for E to be tangent to the surface, its points must also satisfy equation (3). Hence from (3) and (4) we have that

$$\frac{\partial f}{\partial a} = f'(a) = 0$$

As in the case of envelopes of curves, the equation

$$\psi(x\ y\ z) = 0 \qquad (6)$$

of the envelope, is obtained by eliminating a from equations (1) and (5). Equation (6) may, however, represent the loci of singular points as well as the true envelope. In each case, as was true of the envelope of a family of curves, equation (6) must be examined for the true envelope.

Since many of the surfaces to be studied involve two variable parameters, we must consider the case where S is a surface of the family of surfaces

$$f\ (x, y, z, a, b) = 0, \qquad (7)$$

a and b being related by some equation say

$$\phi(a, b) = 0 \qquad (8).$$

It is easily seen that in general, there does not exist any one surface which is tangent to each member of this family along a characteristic curve. We can, however, as in the case of the family of curves, solve for b in terms of a in equation (7), that is

$$b = \phi_1\ (a) \qquad (9)$$

If we substitute this value in (7) we have reduced our problem to that of finding the envelope to a family of surfaces involving one parameter,

$$f\ (x\ y\ z\ a\phi_1(a)) = 0 \qquad (10)$$

Then equations (7) and (9) and the equation

$$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\,\phi_1'(a) = 0 \qquad (11)$$

represents the envelope of this one - parameter family or, for any fixed value of a, they represent the characteristic curve on the corresponding surface S. Hence for some constant value of a, we have a characteristic curve on the surface S along which the envelope is tangent to S.

In case it is difficult to solve for one variable parameter in terms of the other, we may consider one parameter, say a, as the independent variable and the other as the dependent variable, and differentiate (7) and (8) with respect to a. We will then have the four equations

$$f\ (x\ y, z\ a\ b) = 0$$

$$\phi\ (a, b) = 0$$

$$f'a + f'b\,\frac{db}{da} = 0 \qquad (12)$$

$$\phi'a + \phi'b\,\frac{db}{d\ a} = 0 \qquad (13)$$

From these equations we may eliminate the three terms a, b, and $\frac{db}{da}$ and hence obtain the equation of the envelope to the surfaces (7).

Let us now consider the family of surfaces

$$f\ (x,y,z,\ a,b\ ) = 0 \qquad (14)$$

where a and b are independent variable parameters. If a, as well as b, is allowed to take on all possible values, the characteristic curves do not, in general, form a surface. We shall now try to find whether there is a surface E which touches each of the surfaces of the family (14) in one or more points, not along a curve. If such a surface exists, the coordinates (x y z) of the point of tangency, which is a point on E, as in the case of a family of surfaces involving one parameter, must satisfy the equation

$$f\ (x,y,z,a,b) = 0$$

and their total differentials with respect to the variable parameters, a and b, must satisfy the equation

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial a}\,da + \frac{\partial f}{\partial b}\,db = 0 \qquad (15)$$

Moreover, in order that E be tangent to S, it is necessary that we should have

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 0 \qquad (16)$$

Hence from (15) and (16) we have that

$$\frac{\partial f}{\partial a}\ da + \frac{\partial f}{\partial b}\, db = 0$$

But, since a and b are independent variables, we know that the equations $\dfrac{\partial f}{b\ a} = 0$ and $\dfrac{\partial f}{\partial b} = 0$ (17)

must be satisfied simultaneously by the coordinates (xyz) of the point of tangency. Hence, we shall obtain the equation

$$g\ (x,y,z) = 0 \tag{18}$$

of the envelope to the family of surfaces (14) by eliminating a and b between the equations (14) and (17).

The surface (18) thus obtained will be tangent to S at the point P (xyz) unless the equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

are satisfied simultaneously by the values (x y z) and then this surface is the locus of singular points.

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