

THE ANALYSIS OF ECLIPSING BINARY SYSTEMS

By

Robert L. Talley
A.B., University of Kansas, 1955

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Redacted Signature

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Instructor in charge~~

Redacted Signature

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For the department~~

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TABLE OF CONTENTS

Preface

I. Introduction	1
1. Emission of Radiation	5
II. Preliminary Elements	7
1. Spherical Model	8
A. Circular Orbits	19
a. Orbital Formalism	19
b. Complete Eclipses	22
c. Partial Eclipses	29
B. Eccentric Orbits	34
a. Orbital Formalism	34
b. Complete Eclipses	44
c. Partial Eclipses	49
2. Ellipsoidal Model	57
III. Distorted Stars	59
1. Figure and Density Distribution	61
A. Hydrodynamical Formulation	61
B. Rotational and Tidal Potential	63
C. Potential Theory of Heterogeneous Spheroids.	65
D. Dependence of Shape on Density Distribution.	70
E. Surface Gravity of Distorted Stars	76
2. Apical Motion	78
A. Formulation of Disturbing Function	78
B. Secular Motion of the Apse	80

III. Distorted Stars (Continued)

3. Gravity Darkening	88
4. Ellipticity and Reflection Effects.	94
A. Influence of Tidal Distortion	97
B. Influence of Rotational Distortion	101
C. Various Laws for the Reflection Effect	106
D. Luminous Efficiency.	114

IV. Rectification 118

1. Russell's Theorem	120
2. Rectification for Ellipticity	126
A. Exact Rectification	126
B. Approximate Rectification	130
3. Rectification for Reflection	133
A. By Subtraction	133
B. By Addition	134
4. Rectification for Asymmetries	138
5. Combined Standard Rectification Procedure	140
A. By Subtraction	140
B. By Addition	141

V. Unsolved Problems 144

Bibliography

PREFACE

The inception of the present investigation began when the writer was briefly introduced into the subject of eclipsing binaries in a course given by Dr. Norman W. Storer. The apprehension that intricate relationships exist between the multifarious parameters necessary to describe eclipsing systems, even in the elementary stages of analysis, stimulated the writer to strive toward their comprehension, with the intent of elucidating the rudimentary aspects of the subject to the best of his ability. Consequently, the subsequent discussion embodies, predominantly, a presentation and interpretation of many of the theoretical concepts and practical procedures requisite for determining the preliminary elements for eclipsing binary systems. In the discussion an attempt has been made to avoid the terse and succinct qualities so characteristic of current research publications. Originality has not been possible, of course, except in a few special cases. Some of the derivations and methods have been formulated independently, but are not original in the true sense. The only new technique is that presented for partial eclipsing systems which move in eccentric orbits, although the formulation of much of the section on eccentric orbits was independent. The procedure for rectification by subtraction was also an independent formulation. The writer accepts full responsibility for the interpretations presented, except in cases where explicit references are stated. If errors should exist in the writer's interpretations he can only beg condonation from the masters of the subject whose adroitness may be jeopardized by such statements.

The subject of eclipsing binaries encompasses such a vast realm that no single work should be regarded as the ultimate. Therefore, this treatment of the subject should merely complement others existing at present and serve as a guide to the original investigations. Two publications which are virtually indispensable for the serious practical investigator of eclipsing binary systems are "The Determination of Elements of Eclipsing Binaries", Contr. Princeton U. Obs., No. 26, 1952, written by H. N. Russell and J. E. Merrill, and "The Computation of Elements of Eclipsing Binary Systems", Harvard Obs. Mon., No. 8, 1950, written by Z. Kopal. The former publication is concerned primarily with the early stages of analysis and contains many valuable precepts for the computer. The latter treats the more advanced procedures for computing orbits of eclipsing binaries. Some notable omissions of salient methods are made. One of these which merits special mention is J. E. Merrill's method of nomographic solutions. This is admirably treated in Contr. Princeton U. Obs., No. 24, 1953.

The selection of topics in the present work is not completely arbitrary, but depends essentially on material which was deemed necessary to comprehend and vindicate certain simplifying assumptions underlying the current methods of analysis. The arrangement of topics was chosen purposely to indicate that complex dynamical considerations definitely have a place in elementary discussions. Originally, the writer hoped to abstain from discussing such topics, but shortly discovered that a rather thorough presentation of them was the only solution. Subsequently, the writer feels that this is perhaps the most prudent course to follow anyway, since it offers

one an opportunity to develop a sound physical basis for the majority of the considerations and leads naturally to more advanced stages of analysis.

Unfortunately, the vexing problem of attaining a consistent notation has not been resolved. It was found advantageous to depart from the wanton notation adopted by some investigators and to adhere to that of others. The superscripts "oc" and "tr" refer to an occultation and a transit, respectively. The letter "u" is used to represent the limb-darkening coefficient rather than "x", which is often used, to avoid confusing it with x when used as a variable. Similarly, τ is employed to denote the coefficient of gravity-darkening rather than "y".

Precise photometric quantities have been introduced with the purpose of eliminating the vague and equivocal terminology so often employed in the literature. Only two of the customary vague terms are retained, but they are given particular significance. The term "light" refers specifically to the flux density of radiation, and "brightness" denotes the specific intensity of radiation.

The writer is indebted to Dr. Henry G. Horak for his advice and encouragement throughout the course of investigation. Also he wishes to thank Dr. Norman W. Storer for pointing out the need of a complete treatment of the rudimentary aspects of eclipsing binaries and reading the entire manuscript; Dr. R. Stanley Alexander for several helpful and stimulating discussions; and Dr. Max Dresden for kindly consenting to read sections of the thesis. Finally, he would like to express his appreciation to Mrs. Kenneth C. Talley for typing a large portion of the material.

INTRODUCTION

Any description of an eclipsing binary system is complete only if the methods of reducing the photometric data are such as to educe all information which the limits of observational error will permit. At its best the description can only be tentative, because as observational accuracy increases additional parameters will be required to describe the system. The determination of these parameters, or elements as they are usually called, requires a different approach than an analogous problem in the computation of the orbit of an asteroid or comet. In the former instance the low accuracy of the observations makes it necessary to use many observations which are well distributed over the period. In the latter case, however, the observational data is quite accurate and the problem is to fit three or four observations exactly. The analysis of an eclipsing binary system is considerably more complex than the problem in classical celestial mechanics, because more quantities are necessary to describe the system completely and because the relationships between the parameters themselves are more complicated.

The seven parameters needed to define the relative orbit are:

- P, the period;
- t_0 , the time of primary minimum;
- i, the inclination of the orbital plane to the plane of the sky;
- a, the semi major axis of the relative orbit, which is taken as the unit of length;
- Ω , the position angle of the node;
- e, the eccentricity of the relative orbit; and

ω , the longitude of periastron, measured from the ascending node in the direction of motion of the primary component and in the plane of the orbit. The ascending node is that node at which the primary component is moving away from the observer.

Six more quantities are required to describe the eclipses, which are:

r_s, r_g , the radii of the smaller and larger (greater) star;
 L_s, L_g , the luminosities of the smaller and larger star; and
 u_s, u_g , the coefficients of limb-darkening of the smaller and larger star.

For close eclipsing binary systems the following six quantities are also needed;

g_s, g_g , the coefficients of gravity-darkening of the smaller and larger star;
 e_e , the coefficient of equatorial ellipticity of the components;
 e_m , the coefficient of meridional ellipticity of the components;
 $I_s(\phi), I_g(\phi)$, the light visible from the smaller and larger star at any phase due to the reflection effect.

The parameters for luminosity, reflection, limb-darkening, and gravity-darkening of each component will vary with wave length.

No general, direct solution for these nineteen unknowns has been found. Hence, the procedure adopted herein is to consider restricted models which can be extended to the actual physical model by means of successive approximations. At present there is no eclipsing binary

known whose components can be separated, either by telescope or interferometer. This means that the longitude of the node, Ω , must remain unknown. Unless the parallax of the system in question is known, the luminosities can not be determined absolutely. However, the fractional luminosities of the components can be found by expressing the total luminosity as unity. The actual length of the semi-major axis, a , can only be found from radial velocity measures. Thus it is usually adopted as unity and other distances expressed in terms of a . This reduces the total number of unknowns to sixteen at a maximum.

The following terminology will be used:

- primary minimum; the deeper minimum of the light curve,
- secondary minimum; the shallower minimum of the light curve,
- primary component; the component of greater surface brightness,
- secondary component; the component of lesser surface brightness,
- occultation; an eclipse of the smaller star by the larger one,
- transit; an eclipse of the larger star by the smaller one,
- total eclipse; the geometrical configuration at which time the light from one star is completely obscured from the observer by the interposition of the other star.
- annular eclipse; the geometrical configuration at which time the entire smaller star is seen projected on the larger star.
- complete eclipse; the geometrical orientation which permits total and annular eclipses to occur alternately.

partial eclipse; the geometrical orientation which permits one star to be only partially obscured from the observer by the other star at the minimum of light.

Emission Of Radiation

Radiation is said to come from, impinge on, or pass through a surface element, but not a point, of an emitting body. Each surface element of a star is pierced by rays propagated outward in all directions. Consider two surface elements $d\sigma_1$ and $d\sigma_2$ separated by a distance Δ . Let $d\sigma_1$ be an element on an extended source and $d\sigma_2$ be the receiving element. The amount of energy radiated per unit time in the frequency interval $\nu, \nu+d\nu$ through $d\sigma_1$ in a direction making an angle γ_1 with its outward normal and confined to an element of solid angle $d\omega_2 = \frac{\cos \gamma_2 d\sigma_2}{\Delta^2}$ is called the flux $dF_{12}(\nu)$. This flux is given by

$$dF_{12}(\nu) \propto d\sigma_1 \cdot \cos \gamma_1 \cdot d\omega_2 \cdot d\nu = d\sigma_1 \cdot \frac{\cos \gamma_1 d\sigma_2 \cos \gamma_2 d\nu}{\Delta^2}$$

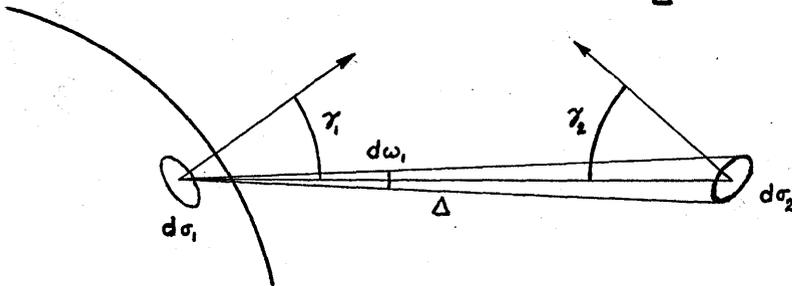


Figure 1

Introducing the proportionality constant I_ν , this becomes

$$(1) \quad dF_{12}(\nu) = I_\nu \frac{d\sigma_1 \cos \gamma_1 d\sigma_2 \cos \gamma_2 d\nu}{\Delta^2}$$

where I_ν is called the specific intensity of the emitting surface.

In the following pages the subscripts of and integration over ν will be dropped and specific note will be made as to whether the quantities are monochromatic or integrated.

Thus (1) becomes

$$dF_{12} = I \frac{d\sigma_1 \cos \gamma_1 d\sigma_2 \cos \gamma_2}{\Delta^2}$$

The flux density J —either monochromatic or integrated — is defined as the energy crossing a unit area normal to the incident beam in a unit of time. Therefore the flux density dJ at $d\sigma_2$ due to the element $d\sigma_1$ is given by

$$dJ = \frac{dF_{12}}{d\sigma_2} = \frac{I \, d\sigma_1 \, \cos \gamma_1 \, \cos \gamma_2}{\Delta^2} .$$

The total flux density received from the body is

$$J = \int_A \frac{I \, \cos \gamma_1 \, \cos \gamma_2 \, d\sigma_1}{\Delta^2} ,$$

where the integral is taken over the visible surface A of the source. Let R be the distance of $d\sigma_1$ from some arbitrary origin within a star. Then if the observational equipment is located so that $\Delta \gg R$ and it is arranged so that

$$\gamma_2 = 0 ,$$

$$J = \frac{1}{\Delta^2} \int_A I \, \cos \gamma_1 \, d\sigma_1 .$$

If the main interest is only in the relative flux density of the star taken on some arbitrary scale, and the distance Δ remains sensibly constant, the above equation reduces to

$$J = \int_A I \, \cos \gamma \, d\sigma ,$$

where the subscripts "1" may be dropped without confusion. This is the type of expression which is used in discussing the light received from eclipsing binary systems.

PRELIMINARY ELEMENTS

At this stage of analysis the objective is to determine the type of eclipse, i.e., an occultation or a transit, which produces the observed primary and secondary minimum, to decide whether the eclipse is total, annular or partial, and to derive a set of elements which satisfy reasonably well the photometric observations.

The solution of a light curve for these preliminary elements may be based on either of two idealized models; a spherical model or an ellipsoidal model. The choice of which model to use rests solely with the light variations outside eclipse. In order to be able to base a solution on either model the following must ensue:

1. The observations must cover the entire light curve and be at least of moderate accuracy.
2. The period must be accurately determined and, in the first instance, must be constant over the interval covered by the observations.¹

The spherical model may be used if the light between minima is essentially constant; the ellipsoidal model, if the light between minima varies appreciably.

¹ Variations in period seem to be frequent. In N. L. Pierce's Finding List, Contr. Princeton U. Obs., No. 22, 1947, about one-fifth of the tabulated stars have variable periods. The cause of most observed changes in period is unknown, but a few such changes have been explained as being due to the varying light time across the orbit of the eclipsing pair about a third body.

Spherical Model

The spherical model consists of two widely separated spherical components revolving about a common center of mass in a relative orbit which may be circular or eccentric. For this system the effects of reflection, ellipticity and gravity-darkening are negligible.

Before considering the details of computing an orbit, it is informative to develop formulae which express the light received from a spherical model at any phase of eclipse.¹ By means of these formulae, tables may be constructed. Such tables² facilitate making solutions to the extent that they should be considered as a necessary tool to have at hand before beginning eclipsing binary work.

If the apparent distance between the centers of the projected components becomes less than the sum of their radii, eclipses will occur. The loss of light during eclipses may be expressed by

$$(2) \quad \delta J = \int_A I(\gamma) \cos \gamma d\sigma$$

subject to the aforesaid restrictions of this formula, where $I(\gamma)$ denotes the specific intensity at any point of the star undergoing eclipse, γ is the angle between the normal to the surface element $d\sigma$

1 These formulae will also hold for the ellipsoidal model if certain coordinate transformations are made and certain restrictions placed on the gravity-darkening.

2 The following two tables are probably the best in existence at the present time.

a) J. E. Merrill, Contr. Princeton U. Obs., 23, 1950.

b) V. Zessivitch, Bull. Astr. Inst. of the U.S.S.R.

Acad. Sci., No. 45, 1939 and No. 50, 1940.

and the line joining $d\sigma$ with the observer, and A is the range of integration taken over the eclipsed area. This integral equation (2.) has not yet been solved directly for the case when both $I(\gamma)$ and the geometrical elements are treated as unknowns. However, the difficulty may be circumvented by employing the following law of limb-darkening

$$(3) \quad I(\gamma) = I_0(1 - u + u \cos \gamma),$$

where I_0 is the intensity at the center of the disc, u is the coefficient of limb-darkening and γ is defined above.

Although this law is only approximate it is found to agree with the specific intensity distribution of the solar disc. In the following analysis, this law will be assumed to hold for stars of other spectral types as well. More complex laws would only further complicate the solution of a light curve and would probably not augment the value of such a solution.

In order to obtain an explicit expression for the loss of light in terms of the geometry of the eclipses and the assumed law of limb-darkening consider an eclipse of a star (a) by another star (b). The loss of light becomes

$$(4) \quad \delta J = I_0 \left\{ (1 - u) \int_A \cos \gamma \, d\sigma + u \int_A \cos^2 \gamma \, d\sigma \right\}$$

where $I_0 = I(0)$. The total light of star (a) is (see figure 2)

$$(5) \quad J_a = \int_0^{2\pi} \int_0^{r_a} I_0(1 - u + u \cos \gamma) r \, dr \, d\xi$$

$$(6) \quad J_a = 2\pi I_0 \int_0^{r_a} \left(1 - u + u \sqrt{1 - \left(\frac{r}{r_a}\right)^2}\right) r \, dr = \pi r_a^2 I_0 \left(1 - \frac{u}{3}\right)$$

The fraction f of the total light from the eclipsed star which is lost is given by $\delta J / J_a$ or

$$(7) \quad \pi r_a^2 f = \frac{3(1-u)}{3-u} \int_A \cos \gamma d\sigma + \frac{3u}{3-u} \int_A \cos \gamma d\sigma.$$

In order to integrate these expressions, transform to rectangular co-ordinates xy in a plane perpendicular to the line of sight. Take the origin at the center of star (a) being eclipsed, and the positive x -axis in the direction of the projected center of the eclipsing star (b). (see figure 3).

The equation of the boundary of disc (a) is given by

$$(8) \quad r_a^2 = x^2 + y^2,$$

and that of the boundary of disc (b)

$$(9) \quad r_b^2 = (\delta - x)^2 + y^2,$$

where $\delta = \overline{OO'}$ is the apparent distance between centers of the two stars, which for computational purposes is always positive.

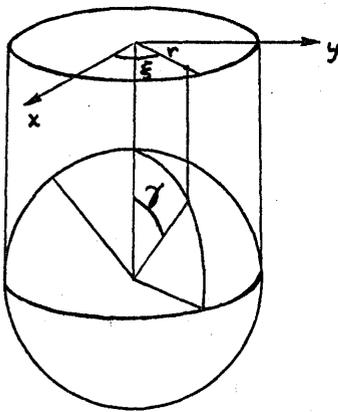


Figure 2

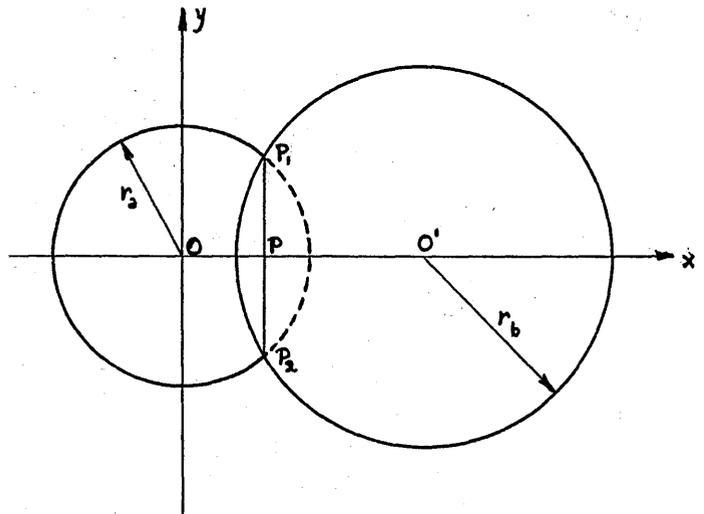


Figure 3

If $\delta \leq r_a + r_b$, the two circles defined by (8) and (9) intersect at the points $P_1(s, +\sqrt{r_a^2 - s^2})$, $P(s, -\sqrt{r_a^2 - s^2})$, where $s = \overline{OP}$ and

$$(10) \quad s = \frac{r_a^2 - r_b^2 + \delta^2}{2\delta}.$$

From equation (7)

$$(11) \quad \pi r_a^2 f = \frac{3(1-u)}{3-u} \int_A dx dy + \frac{3u}{3-u} \int_A \frac{\sqrt{r_a^2 - x^2 - y^2}}{r_a} dx dy.$$

If f represents the fractional loss of light during partial phase or partial eclipse of discs uniformly illuminated ($u = 0$), equation (11) gives

$$(12) \quad \pi r_a^2 f = \int_s^{r_a + \sqrt{r_a^2 - x^2}} \int_{-\sqrt{r_a^2 - x^2}}^{+\sqrt{r_b^2 - (\delta - x)^2}} dy dx + \int_{\delta - r_b}^s \int_{-\sqrt{r_b^2 - (\delta - x)^2}}^{+\sqrt{r_b^2 - (\delta - x)^2}} dy dx,$$

when $\delta \leq r_a + r_b$ and the eclipsed star (a) may be the larger or the smaller of the pair. This equation also applies to total eclipse when star (a) is the smaller and $\delta \leq r_b - r_a$. For discs completely darkened at the limb ($u = 1$) equation (11) yields for partial phase or partial eclipse

$$(13) \quad \frac{2}{3} \pi r_a^3 f = \int_s^{r_a + \sqrt{r_a^2 - x^2}} \int_{-\sqrt{r_a^2 - x^2}}^{+\sqrt{r_a^2 - x^2 - y^2}} dy dx + \int_{\delta - r_b}^s \int_{-\sqrt{r_b^2 - (\delta - x)^2}}^{+\sqrt{r_b^2 - (\delta - x)^2 - y^2}} dy dx,$$

where f is the fractional loss of light. This holds when $\delta \leq r_a + r_b$ and when the eclipsed star is the smaller or larger of the pair.

When the eclipsed star (a) is the larger and $\delta < r_a - r_b$ the eclipse becomes annular and equations (12) and (13) must be replaced by

$$(14) \quad \pi r_a^2 f = \int_{\delta - r_b}^{\delta + r_b} \int_{-\sqrt{r_b^2 - (\delta - x)^2}}^{+\sqrt{r_b^2 - (\delta - x)^2}} dy dx$$

and

$$(15) \quad \frac{2}{3} \pi r_a^3 f = \int_{\delta-r_b}^{\delta+r_b} \int_{-\sqrt{r_b^2-(\delta-x)^2}}^{\sqrt{r_b^2-(\delta-x)^2}} \sqrt{r_a^2-x^2-y^2} \, dy \, dx,$$

respectively.

For a disc with limb-darkening u the fractional loss f of light during any phase of the eclipse is

$$(16) \quad f = \frac{3(1-u)}{3-u} f + \frac{2u}{3-u} f.$$

Expressions (12) and (14) are simple to integrate. If the eclipse is partial, equation (12) gives

$$(17) \quad \pi r_a^2 f = \frac{r_a^2}{2} [\varphi_1 - \sin \varphi_1] + \frac{r_b^2}{2} [\varphi_2 - \sin \varphi_2],$$

where

$$\varphi_1 = 2 \cos^{-1} \frac{s}{r_a},$$

$$\varphi_2 = 2 \cos^{-1} \frac{\delta-s}{r_b}.$$

If the eclipse is annular, equation (14) gives

$$(18) \quad \pi r_a^2 f = \pi r_b^2,$$

which is the result one expects. Kopal¹ has shown that for a disc completely darkened at the limb during partial eclipse

$$(19) \quad \pi r_a^3 f = \pi r_b^3 - 2r_b^3 [(E-F)F(\phi, \kappa') - FE(\phi, \kappa')] \\ + \frac{r_b}{3} \sqrt{\frac{\delta}{r_b}} [7r_b^2 - 4r_a^2 + \delta^2] (2E-F) \\ - \frac{1}{3\delta} \sqrt{\frac{\delta}{r_b}} [5\delta^2 r_b^2 + 3(r_a^2 - r_b^2)^2 - 3\delta r_a^3] F,$$

where $F = F(\frac{\pi}{2}, \kappa)$ and $E = E(\frac{\pi}{2}, \kappa)$ represent the Legendre complete elliptic integrals of the first and second kind, with the modulus

$$(20) \quad \kappa^2 = \frac{1}{2} \left\{ 1 - \frac{\delta-s}{r_b} \right\},$$

¹ Z. Kopal, Harvard Obs. Mon., 6, 27, 1946.

and where $F(\phi, \kappa)$ and $E(\phi, \kappa)$ denote the Legendre incomplete elliptic integrals of the first and second kind with a complementary modulus

$$(21) \quad \kappa' = (1 - \kappa^2)^{\frac{1}{2}}$$

and amplitude

$$(22) \quad \phi = \sin^{-1} \sqrt{\frac{2\delta}{r_a + r_b + \delta}} .$$

When such an eclipse is annular equation (15) gives

$$(23) \quad \pi r_a^3 f = 2r_a^3 \left[(E - F)F(\phi, \kappa') - FE(\phi, \kappa') \right] \\ + \frac{1}{3} \left[7r_b^2 - 4r_a^2 + \delta^2 \right] \sqrt{2\delta(r_b - \delta + s)} E \\ + \frac{1}{3} \left[(\delta^2 - r_b^2)^2 - r_a^2(2r_a^2 - r_b^2) + 6\delta r_a^3 \right] \frac{F}{\sqrt{2\delta(r_b - \delta + s)}} .$$

Where the modulus of the complete elliptic integrals is $1/\kappa$ and the amplitude of the incomplete integrals is

$$(24) \quad \phi = \sin^{-1} \sqrt{\frac{r_a + r_b - \delta}{r_a + r_b + \delta}} .$$

In order to be able to apply the f -functions to the solution of problems it is desirable to tabulate them in terms of certain parameters. The functions $^o f$ and $^i f$ can be made to depend on δ/r_g and r_s/r_g , itself, depends on the ratio of the radii, k , of the two stars, where the dimensionless quantity k is defined by

$$(25) \quad k = r_s/r_g$$

and is restricted to the values $k \leq 1$. Because of expressions which will arise later it is convenient to define another dimensionless parameter, p , the geometrical depth of the eclipse, by the relation

1 The slight change in notation at this point is quite straightforward, because in the previous sections either r_a or r_b could be the larger or smaller depending on the circumstances of eclipse.

$$(26) \quad p = \frac{\delta - r_g}{r_s},$$

and let f be expressed in terms of u, k , and p as $f(u, k, p)$. This definition of p holds whether the eclipse is an occultation or a transit. An eclipse will be complete, partial or absent according as $p < -1, 1 > p > -1, p > 1$. Regardless of u , $p = +1$ and $f(u, k, 1) = 0$ at the time of first contact. At internal tangency $p = -1$ and two possibilities arise for f : for an occultation $f_s(u_s, k, -1) = 1$; and for a transit, when annular phase begins,¹

$$(27) \quad f_s(u_s, k, -1) = \frac{3k^2}{3 - u_s} \left\{ 1 - u_s + u_s \varphi(k) \right\} = \beta^{tr}(u_s, k)$$

where

$$(28) \quad \varphi(k) = \frac{4}{3\pi k^2} \left\{ \sin^{-1} \sqrt{k} + \frac{1}{3} (4k - 3)(2k + 1) \sqrt{k(1-k)} \right\}.$$

The functions $f(u, k, p)$ are more suitable for tabulation if they are expressed in terms of functions which vary from zero at first contact to unity at internal tangency. Therefore set²

$$(29) \quad f(u, k, p) = \alpha(u, k, p) \beta(u, k)$$

where α is the ratio of light lost at any phase of the eclipse to that lost at internal tangency, and β is the ratio of light lost at internal

1 Cf. Z. Kopal, Harvard Obs. Mon., 6, 29, 1946.

2 Both relations β and $\varphi(k)$ are introduced not because they are unique but because both appear in the literature and knowing their interconnection permits an investigator to use tables of β or $\varphi(k)$ whichever is at hand irrespective of the notation employed during the remainder of analysis. The development of formulae could proceed if only one of the two quantities were introduced. V. Zessevitch tabulates $\varphi(k)$ to six significant figures from $k = 0.20$ to 1.00 at intervals of 0.01 , in Bull. Astr. Inst. of the U.S.S.R. Acad. Sci., No. 50, 1940.

K. Ferrari tabulates $\beta(k)$ to four decimals for arguments in k of 0.05 , in Sitzungsb. d. Wien Akad. d. Wiss., Abt. IIa, 148, 221, 1939.

tangency to the total light of the star in question. From the preceding it is evident that for an occultation

$$(30) \quad \beta^{oc}(u_s, k) = f_s(u_s, k, -1) = 1,$$

and for a transit $\beta^{tr} < 1$ (unless $k = 1$) and $f_g = \alpha^{tr} \beta^{tr}$. In general one finds for β^{tr} the expression

$$(31) \quad \beta^{tr}(u_g, k) = (1 - z^{tr}) \circ \beta^{tr}(k) + z^{tr} {}^1 \beta^{tr}(k)$$

where $z^{tr} = \frac{2u_g}{3 - u_g}$. At the time of internal tangency

$$(32) \quad \circ \beta^{tr}(k) = k^2$$

$${}^1 \beta^{tr}(k) = \frac{3}{2} k^2 \varphi(k)$$

Thus, explicitly, the normalized fractional loss of light α^{oc} of the smaller star is

$$(33) \quad \alpha^{oc}(u_s, k, p) = \frac{f_s(u_s, k, p)}{f_s(u_s, k, -1)}$$

if the eclipse is an occultation, and

$$(34) \quad \alpha^{oc}(u_s, k, p) = (1 - z^{oc}) \circ \alpha^{oc}(k, p) + z^{oc} {}^1 \alpha^{oc}(k, p)$$

where $z^{oc} = \frac{2u_s}{3 - u_s}$.

Similarly,

$$(35) \quad \alpha^{tr}(u_g, k, p) = \frac{f_g(u_g, k, p)}{f_g(u_g, k, -1)}$$

if the eclipse is a transit, and

$$(36) \quad \alpha^{tr}(u_g, k, p) = (1 - z^{tr}) \circ \alpha^{tr}(k, p) + z^{tr} {}^1 \alpha^{tr}(k, p)$$

where $z^{tr} = \frac{u_g \varphi(k)}{1 - u_g + u_g \varphi(k)}$.

These new α -functions will vary from 0 to 1 as p varies from +1 to -1, independent of k or u . In the preceding equations

$\circ \alpha^{oc}(k, p) = \circ \alpha^{tr}(k, p)$, so that a single table for $\circ \alpha$ may be used.

No such simple relation exists between $\alpha^{oc}(k,p)$ and $\alpha^{tr}(k,p)$ (unless $k = 1$), and separate tables are required for occultations and transits.

Written in terms of the α -functions the fractional loss of light of the smaller star during occultation is

$$(37) \quad f_s = (1 - z^{oc}) \alpha^{oc}(k,p) + z^{oc} \alpha^{oc}(k,p),$$

and the loss of the larger star during transit becomes

$$(38) \quad f_g = (1 - z^{tr}) \alpha^{tr}(k,p) \beta^{tr}(k) + z^{tr} \alpha^{tr}(k,p) \beta^{tr}(k)$$

where again $z^{oc} = \frac{2u_s}{3 - u_s}$ and $z^{tr} = \frac{2u_g}{3 - u_g}$.

Before the fractional loss of light f can have practical meaning, a unit of total light of each star must be adopted. The light between minima is taken as the unit and the fractional lights L_s and L_g of the smaller and larger components are restricted by

$$(39) \quad L_s + L_g = 1.$$

During an occultation the fraction, ℓ , of light of the system received by the observer is

$$(40) \quad \ell^{oc} = (1 - f_s)L_s + L_g = 1 - f_s L_s,$$

and during a transit

$$(41) \quad \ell^{tr} = (1 - f_g)L_g + L_s = 1 - f_g L_g.$$

These expressions may be written in terms of the α -functions giving

$$(42) \quad \ell^{oc} = 1 - L_s \alpha^{oc}(u_s, k, p),$$

$$(43) \quad \ell^{tr} = 1 - L_g \alpha^{tr}(u_g, k, p) \beta^{tr}(u_g, k).$$

Add equations (40) and (41) to give

$$(44) \quad f_s(u_s, k, p) = \alpha^{oc}(u_s, k, p) = 1 - \ell^{oc} + (1 - \ell^{tr}) \frac{f_s}{f_g}.$$

Now from equations (37) and (38) a new function q is defined which is particularly useful in treating partial eclipses,

$$(45) \quad q(u_s, u_g, k, p) = \frac{f_g}{f_s} = \frac{(1 - z^{tr})^0 \alpha^{tr}(k, p) \beta^{tr}(k) + z^{tr} \alpha^{tr}(k, p) \beta^{tr}(k)}{(1 - z^{oc})^0 \alpha^{oc}(k, p) + z^{oc} \alpha^{oc}(k, p)} \\ = \frac{\alpha^{tr}(u_g, k, p) \beta^{tr}(u_g, k)}{\alpha^{oc}(u_s, k, p)}.$$

The function $q(u_s, u_g, k, p)$ is the ratio of the fractional light loss of the large star at any phase of the transit to the fractional light loss of the smaller star at the same geometrical phase during occultation. The definition applies to complete as well as partial eclipses.

Therefore, at any phase of eclipse

$$(46) \quad l^{oc} = 1 - L_g \alpha^{oc},$$

$$(47) \quad l^{tr} = 1 - L_g \alpha^{oc} q$$

and

$$(48) \quad \alpha^{oc} = 1 - l^{tr} \frac{(1 - l^{tr})}{q}.$$

For mid-eclipse, where p_0 is the maximum geometrical depth of the eclipse and $\lambda^{oc}, \lambda^{tr}$ are the values of l , (48) becomes

$$(49) \quad \alpha_0^{oc} = \alpha^{oc}(u_s, k, p_0) = 1 - \lambda^{oc} + \frac{(1 - \lambda^{tr})}{q_0},$$

where α_0^{oc} is the maximum obscuration attained during occultation.

If the eclipse is complete λ^{oc} and λ^{tr} denote the fractional intensities during total and annular eclipses, respectively. Then (49) yields

$$(50) \quad q_0 = q(u_s, u_g, k, p_0) = \frac{1 - \lambda^{tr}}{\lambda^{oc}} = (1 - z^{tr})^0 \beta^{tr}(k) + z^{tr} \alpha^{tr}(k, p_0) \beta^{tr}(k)$$

for $p_0 \leq -1$. If the larger star is uniformly illuminated

$$(51) \quad k^2 = \frac{1 - \lambda^{tr}}{\lambda^{oc}}.$$

The ratio of the normalized light losses at different phases of the same eclipse can also be found directly from the observations.

Form the ratios, using equations (42) and (43),

$$(52) \quad n^{tr} = \frac{\alpha^{tr}}{\alpha_0^{tr}} = \frac{\alpha^{tr}(u_s, k, p)}{\alpha^{tr}(u_s, k, p_0)} = \frac{1 - \ell^{tr}}{1 - \lambda^{tr}}, \quad n^{oc} = \frac{\alpha^{oc}}{\alpha_0^{oc}} = \frac{\alpha^{oc}(u_s, k, p)}{\alpha^{oc}(u_s, k, p_0)} = \frac{1 - \ell^{oc}}{1 - \lambda^{oc}},$$

or dropping the superscripts,

$$(53) \quad \frac{\alpha}{\alpha_0} = \frac{1 - \ell}{1 - \lambda} = n$$

where λ is the light observed at maximum obscuration α_0 . When an eclipse is complete $\alpha_0 = 1$ during occultation. Furthermore, when the larger star is uniformly illuminated $\alpha_0 = 1$ during annular eclipse.

From the equations (46) and (47) the ratio of the light of the two stars is

$$(54) \quad \frac{L_e}{L_s} = \frac{1(1 - \ell^{tr})}{q(1 - \ell^{oc})} = \frac{1(1 - \lambda^{tr})}{q_0(1 - \lambda^{oc})}.$$

Define the ratio of the mean surface brightness by

$$(55) \quad \frac{L_e}{L_s} = k^2 \frac{I_s}{I_e}.$$

Then

$$(56) \quad \frac{I_e}{I_s} = \frac{k^2(1 - \ell^{tr})}{q(1 - \ell^{oc})} \\ = \frac{k^2(1 - \lambda^{tr})}{q_0(1 - \lambda^{oc})}.$$

Finally the ratio of central intensities is given by

$$(57) \quad \left(\frac{I_e}{I_s} \right)_0 = \frac{(3 - u_s) I_e}{(3 - u_e) I_s}.$$

Circular Orbit

The following additional requirement must be satisfied if the spherical model is to be used to represent a system with a circular orbit. The primary and secondary minima must be symmetric about their mid-points, must be of equal duration, and must be separated by half the period. Assuming that this condition is satisfied and that tables calculated using the previous formula are at hand, their use in determining a set of elements will now be discussed.

Plot the light curve and decide which of the following cases apply:

a) The light shows a constant phase in one or both minima; such an eclipse is complete. When the larger star is darkened at the limb, the light will vary during annular phase making this minimum easy to distinguish. Should the larger component be of uniform surface brightness both minima will exhibit a constant phase.

b) The light shows no constant phase in either minimum; such eclipses are partial, except for the special case $k \approx 1$, $i \approx 90^\circ$.

c) Only one minimum is observed, and it shows a constant phase; this eclipse may be complete.¹ Whether the minimum corresponds to a total or annular eclipse must be determined from the results of analysis.²

¹ In this case one must decide whether the spherical model may hold from symmetry of the minimum alone. If the minimum is asymmetric the spherical model will not hold, but the converse does not necessarily mean that this model must apply, because the secondary minimum may even be partial if the orbit is eccentric.

² A useful computing trick is given by R. G. Aitken, The Binary Stars, 2nd ed. (New York, 1935), p. 193.

d) Only one minimum is observed, and it shows no constant phase; the solution for this situation is generally indeterminate by any method at the present time¹.

Since the period P and the time of primary minimum t_0 can be determined from the observations quite independent of the other elements, the remaining unknowns for the spherical model are r_s, r_g, i, L_s (or L_g), u_s and u_g . The coefficients of limb-darkening for each component must be assumed from the spectral class, surface brightness, or intuition at this point of analysis. J. E. Merrill² suggests that the choice of limb-darkening coefficients for components whose spectral types are known should be approximately as follows for observations with effective wave lengths of 4500 Å to 5000 Å:³

Spectral Type	B0 - A1	A2 - F4	F5 - G2	G3 -
u	0.6	0.4	0.6	0.8

When only the spectral type of a single component is known he suggests choosing its appropriate value of u , and a coefficient of $u = 0.0$ or one of magnitude 0.2 greater than u , for the second star, depending on differences in eclipse depths. If neither spectral type is known, an assumed value $u = 0.6$ should suffice for a first approximation. In some cases, if the accuracy of observations permits, the correctness of the assumed values of limb-darkening can be tested in subsequent stages.

¹ The restriction is placed by limits of observational accuracy rather than by undeveloped methods of analysis, as is shown by Z. Kopal, *Harvard Obs. Mon.*, 8, 55, 1950.

² J. E. Merrill, *Contr. Princeton U. Obs.*, 23, 1953.

³ In a later publication, *Contr. Princeton U. Obs.*, 23, 1953. Table x=1.0 notes that "darkening coefficients appropriate to the range in wave length of 1P21 photocells used without filters, seem to be turning out slightly smaller than suggested ... [above] ... particularly for the hotter stars."

Take the center of the component being eclipsed as the origin of co-ordinates and let θ be the mean anomaly of the second component, θ being measured in the plane of the orbit from inferior conjunction to the second star. The apparent distance δ between centers of the two stars at any time t is (refer to figure 4 on page 35)

$$(58) \quad \delta^2 = R^2(\cos^2 i + \sin^2 i \sin^2 \theta),$$

where

$$(59) \quad \theta = \frac{2\pi}{P} (t - t'),$$

i is the inclination of the orbital plane to the plane of the sky, R is the actual radius vector of the relative orbit, and t' is some arbitrary epoch. In a circular orbit R is constant, and is taken as unity in the case considered here, but in an eccentric orbit R depends on the true anomaly. The combination of equations (25), (26) and (58) gives the first fundamental equation,

$$(60) \quad r_g^2 (1 + kp)^2 = \cos^2 i + \sin^2 i \sin^2 \theta.$$

In this equation $p = p(u, k, \alpha)$ is obtained by numerical inversion of the function $\alpha = \alpha(u, k, p)$, and the values of θ are found from the photometric observations.

Complete Eclipses

Assume that situation a) holds for the light curve. Analysis will first be made of the minimum produced by total eclipse¹, if that is distinct, or of the primary minimum where $t' = t_0$ if two constant minima are manifest. The unknown parameters in this case are r_s, r_g , and i . Three equations of type (60) are theoretically sufficient to determine these unknowns, but since the number of observations must compensate for the low observational accuracy, the practical determination of r_s, r_g , and i must depend on points distributed over the whole light curve.

Russell originally eliminated r_g and i in order to solve for k in the following way. For three values of θ_j and α_j (from which p_j follow) where $j = 1, 2, 3$, equation (60) gives

$$(61) \quad \sin^2 \theta_1 - \sin^2 \theta_2 = r_g^2 \csc^2 i \left[(1 + kp_1)^2 - (1 + kp_2)^2 \right]$$

and then

$$\frac{\sin^2 \theta_1 - \sin^2 \theta_2}{\sin^2 \theta_2 - \sin^2 \theta_3} = \frac{(1 + kp_1) - (1 + kp_2)}{(1 + kp_2) - (1 + kp_3)} = \psi(u, k, \alpha_1, \alpha_2, \alpha_3).$$

Two points (θ_2, α_2) and (θ_3, α_3) may be assigned and (θ_1, α_1) left arbitrary. The definition of the new function ψ then becomes

$$(62) \quad \frac{\sin^2 \theta - \sin^2 \theta_2}{\sin^2 \theta_2 - \sin^2 \theta_3} = \psi(u, k, \alpha).$$

¹ Russell and Merrill give analytical procedures based on the minimum produced by annular eclipses in Contr. Princeton U. Obs., 26, 13, 1952, but certain difficulties and disadvantages tend to make this method inferior to one based on the above minima. In certain cases when the total eclipse is poorly observed and annular eclipse of a limb-darkened star gives rise to the primary minimum, such an analysis as given by Russell and Merrill is necessary to obtain a solution. This method is discussed on page 28.

The values of ψ are different for occultations and transits,¹ but the definition applies in each case. Now put

$$A = \sin^2 \theta_1 ,$$

$$B = \sin^2 \theta_2 - \sin^2 \theta_3$$

to obtain

$$(63) \quad \sin^2 \theta = A + B \psi(u, k, \alpha) .$$

Here A and B each have different values for a transit and an occultation for the same pair of stars unless $u = 0$. Then at external and internal tangency, denoted by superscripts e and i respectively,

$$\sin^2 \theta^e = A^{oc} + B^{\alpha} \psi^{\alpha}(u, k, 0) = A^{tr} + B^{tr} \psi^{tr}(u, k, 0)$$

$$\sin^2 \theta^i = A^{oc} + B^{\alpha} \psi^{\alpha}(u, k, 1) = A^{tr} + B^{tr} \psi^{tr}(u, k, 1) .$$

From equation (60)

$$r_g^2 (1+k)^2 \csc^2 i = \cot^2 i + A + B \psi(u, k, 0)$$

$$r_g^2 (1-k)^2 \csc^2 i = \cot^2 i + A + B \psi(u, k, 1) .$$

Therefore,

$$(64) \quad \cot^2 i = -A + \frac{B}{\Phi_2(u, k)} ,$$

$$(65) \quad r_g^2 = \frac{B \sin^2 i}{\Phi_1(u, k)} ,$$

where $\Phi_2(u, k)$ and $\Phi_1(u, k)$ are given by

$$(66) \quad \Phi_2(u, k) = \frac{4k}{(1-k)^2 \psi(u, k, 0) - (1+k)^2 \psi(u, k, 1)} .$$

$$(67) \quad \Phi_1(u, k) = \frac{4k}{\psi(u, k, 0) - \psi(u, k, 1)} .$$

¹ Extensive tables have been computed adopting $\alpha_2 = 0.6$ and $\alpha_3 = 0.9$. Contr. Princeton U. Obs., 23, 1953. Tables of ψ^{α} are given from $\alpha^{oc} = 0$ to 1, while those for ψ^{tr} include values of α_g greater than 1.

The Φ - functions have been tabulated by Merrill for both occultations and transits, with various degrees of darkening as well as for uniform discs.

Summary of the Procedure for obtaining a Solution Using Russell's Method.

1) Determine the period P and the time of primary minimum t_0 from the observations.

2) Plot the observations on a suitable scale with intensity ℓ vs. phase θ . Draw a smooth free hand curve through the points. Determine the intensities of light λ_1, λ_2 at mid-primary and secondary minimum. Let $\lambda_1 = L_g$, and then $L_s = 1 - L_g$. Such an assumption of a total eclipse at primary minimum always leads to a physically possible solution.¹

3) Calculate ℓ° for selected values of α° , such as 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.8, 0.9, 0.95, 0.98, from $\ell^{\circ} = 1 - L_s \alpha^{\circ}$. Or choose certain values of ℓ° and calculate α° from $\alpha^{\circ} = \frac{1 - \ell^{\circ}}{1 - \lambda_1}$.

4) Read values of θ , corresponding to the calculated or chosen ℓ , from the free hand curve.

5) Compute A, B from, $A^{\circ} = \sin^2 \theta_2$, $\alpha_2 = 0.6$

$$B^{\circ} = A - \sin^2 \theta_3, \alpha_3 = 0.9.$$

These values are the same regardless of the coefficient u_s .

6) Compute $\psi^{\circ}(u_s, k, \alpha)$ for each θ from

$$\psi^{\circ}(u_s, k, \alpha) = \frac{\sin^2 \theta - A^{\circ}}{B^{\circ}}.$$

¹ On the other hand, an assumption of an annular eclipse at primary minimum gives an acceptable solution only if $1 - \lambda_1 < k^2$; otherwise the calculated light from the smaller star would be negative.

7) Knowing α° , ψ° and assuming u_s , interpolate in the appropriate table to find k . If the free hand and theoretical curve are approximately the same shape, these values of k will be nearly equal. If not, vary A and B until the mean k derived from parts of the curve for which $\theta < 0.6$, $0.6 < \theta < 0.9$, $0.9 < \theta$ are the same. In extreme cases change u_s and/or give double weight to values of k derived from α between 0.2 and 0.4 and between 0.95 and 0.99. Thus find a mean value of k .

8) Using the mean k and α , take ψ from the appropriate table and compute θ and ℓ from

$$\sin^2 \theta = A + B\psi(u_s, k, \alpha) \quad \text{and} \quad \ell = 1 - L_s \alpha.$$

9) Plot ℓ against θ in order to obtain the theoretical light curve. If this curve agrees with the observations adopt the value of k . Before calculating i and r_g apply the Φ -function criterion. Since $\cot^2 i$ cannot be negative for a real value of k , this implies that for a real i , $\Phi_1 \leq B/A$.

10) Use equations (64) and (65) to compute i and r_g . If $\cot^2 i$ turns out to be negative obtain the best possible fit for $\cos^2 i = 0$. See Ap.J., 36, 252-254, 1912.

11) Determine whether the minimum analyzed was an occultation or a transit. Difficulty arises only when the larger star is uniformly illuminated and $k^2 = (1 - \lambda^{tr}) / \lambda^{\circ}$. This expression gives two values for k depending on whether $\lambda^{\circ} = \lambda_1$, or λ_2 . One such k should be close to the value obtained by analysis, and the corresponding type of eclipse is immediately apparent. However, when $\lambda_1 = \lambda_2$ this method fails and one must resort to spectroscopic analysis in order to extract the type of eclipse.

The selection of the two fixed points (α_2, α_3) is of no concern theoretically. For practical reasons, though, Russell adopted two points where the fractional loss of light α reaches 0.6 and 0.9. The light curve is relatively steep for these and other large values of α , which facilitates reading θ from the calculated ℓ .

Russell's method has two main disadvantages:

a) Values of A and B read from the free-hand light curve are subject to appreciable error. Small changes in A or B radically affect the determined value of k. Thus the magnitude of k calculated from values of A and B which are subject to error may itself be in considerable error.

b) Values of k determined from the free hand curve vary greatly. Since the derivative $\frac{\partial \psi(u, k, \alpha)}{\partial \alpha}$ is variable, the problem arises as how best to find a weighted mean of the k's. The method suggested by Russell and summarized in step 7, is often unconsoling to a computer.

Kopal¹ suggested an elegant graphical method for obtaining k which eliminates these difficulties and simultaneously determines $\cot^2 i$ and $(\cos^2 i)/r_g^2$. The essentials of his procedure are as follows: In equation (60) set²

$$Y = \sin^2 \theta \quad \text{and} \quad X(u, k, \alpha) = (1 + kp)^2$$

This gives

$$(68) \quad Y = r_g^2 \csc^2 i \cdot X(u, k, \alpha) - \cot^2 i.$$

1 Z Kopal, Ap.J., 94, 145, 1941.

2 A similar procedure holds if $Y = \sin^2 \theta$ and $Z = \psi(u, k, \alpha)$ in Russell's original notation, although it is probably less convenient than Kopal's treatment.

As a result of r_g and i being constant, equation (68) should represent a straight line in the XY plane. However, X and Y are not independent but are related by

$$\sin^2 \theta = h(\alpha) = j(k, p),$$

and points given by (X, Y) must lie on the light curve. The problem becomes that of finding the value of k which makes equation (68) a straight line. The intercepts of such a line on the X and Y axes are, respectively, $-\cot^2 i$ and $(\cos^2 i)/r_g^2$. In using this method take the steps noted below.

1) Find P, t_0 , and L_g as in Russell's analysis. Then determine θ and α for every individual observation or normal point, from

$$\theta = \frac{2\pi(t-t_0)}{P}, \quad \alpha = \frac{1-l}{1-\lambda}.$$

2) Find a provisional value of k from $k^2 = (1-\lambda_1)/\lambda_2$.

3) Use the approximate k and appropriate tables of $p(u, k, \alpha)$ to calculate the values of X and Y , where

$$X = (1 + kp)^2, \quad Y = \sin^2 \theta.$$

Plot the points (X, Y) . They will generally lie along a curve, whose curvature depends upon k . In order to save time use only tabular values of k . Only when plots show opposite curvature is interpolation for k and $p(u, k, \alpha)$ necessary.

4) Continue this method until all observations plotted lie as close as possible to a straight line. Obviously, for any real value of i the intercept of the Y axis must be negative. If this intercept is positive, probably the incorrect coefficient of limb-darkening, u_s , was used. From the intercepts, i and r_g may be found.

Kopal's method utilizes actual observations rather than a free hand curve. The error in k can be estimated by plotting additional points with slightly different values of k . The computation for this latter scheme is apt to be greater and it is doubtful if the increased reliability of the results would compensate for the extra effort.

In view of the possibility that the total eclipse may be poorly observed, or the annular eclipse of a limb-darkened star gives rise to the primary minimum, or the annular eclipse must be utilized to obtain certain necessary information as, for example, in dealing with eccentric orbits, the analytical procedure based on the minima produced by annular eclipses must be considered. The following procedure was developed by Russell and Merrill.¹

During an annular eclipse $\alpha^{tr} > 1$, and at the middle of annular eclipse $\alpha_0^{tr} > 1$. In order to make a solution one must determine α_0^{tr} . This is possible if an estimate of the light at internal tangency l_i^{tr} can be made. From equation (52)

$$\alpha_0^{tr} = \frac{1 - \lambda^{tr}}{1 - l_i^{tr}},$$

where $\alpha^{tr} = 1$ at internal tangency. Using equation (53) one then finds at any other phase

$$\alpha^{tr} = \frac{1 - l^{tr}}{1 - l_i^{tr}}.$$

Difficulty may arise in estimating l_i^{tr} , but usually this can be done with fair accuracy if u is not close to unity, and if the annular

¹ H. N. Russell and J. E. Merrill, Contr. Princeton U. Obs., 26 13-14, 1952.

phase is long. In unfavorable circumstances when u is close to unity and annular phase is short, the estimate will be weak and will become impossible if $u = 1$.¹ Nomographs² may help to get ℓ_i^{tr} and α^{tr} . After a reasonably accurate value of ℓ_i^{tr} is obtained, values of θ corresponding to assigned values of α^{tr} may be read from the light curve, and a solution made as before using.

Since the computed curve is adjusted to pass through three points of the observed light curve, an error in the estimate of ℓ_i^{tr} will not greatly affect the agreement of this part of the curves. The most sensitive region is during the annular phase. An important equation can easily be obtained from equation (63) by noting that at mid-eclipse $\theta=0$ and $\alpha=\alpha_0$. Therefore,

$$\psi(u, k, \alpha_0) = -\frac{A}{B}$$

Here B is always positive and A is negative if $\alpha_0 < 0.6$, (using Russell's definition of ψ). This equation should be applied early in the calculation. Obviously, comparison of the computed light curve with the observations is quite important here.

Partial Eclipses

Let the light curve now come under category b) on page 19 for partial eclipses. In this case the maximum obscuration α_0 of the eclipsed star is not unity and becomes an unknown along with r_s , r_g and i . A solution for these four parameters is usually possible using the depths

¹ H. N. Russell and H. Shapley, *Ap.J.*, 36, 385-386, 1912.

² The nomographs are graphs from which graphical solutions may be obtained quite readily. Their importance is not to be overlooked, particularly at this point. For a complete treatment of nomographs see H. N. Russell and J. E. Merrill, *Contr. Princeton U. Obs.*, 26, 19-35, 1952, and J. E. Merrill, *Contr. Princeton U. Obs.*, 24, 1953.

of the two minima and the shape of the deeper.

Let λ_1 and λ_2 be the values of light observed at mid-primary and mid-secondary minimum, respectively. Equation (53) with $\lambda = \lambda_1$, and the light curve define a relation between θ and n . Any value of $\theta(n)$ may be read directly from the light curve. At mid-primary minimum $\theta = 0$ and $n = 1$. Equation (60) gives

$$(69) \quad \cos^2 i + \sin^2 i \cdot \sin^2 \theta(n) = r_g^2 [1 + kp(u, k, n \alpha_0)],$$

$$(70) \quad \cos^2 i = r_g^2 [1 + kp(u, k, \alpha_0)],$$

from which

$$(71) \quad \sin^2 \theta(n) = r_g^2 \csc^2 i \left\{ [1 + kp(u, k, n \alpha_0)]^2 - [1 + kp(u, k, \alpha_0)]^2 \right\}.$$

Dividing equation (71) by a similar one expressed for a fixed value of n such as $n = n_1$, yields the definition of Russell's χ - function.

$$(72) \quad \chi(u, k, \alpha_0, n) = \frac{\sin^2 \theta(n)}{\sin^2 \theta(n_1)} = \frac{[1 + kp(u, k, n \alpha_0)]^2 - [1 + kp(u, k, \alpha_0)]^2}{[1 + kp(u, k, n_1 \alpha_0)]^2 - [1 + kp(u, k, \alpha_0)]^2}$$

or

$$(73) \quad \sin^2 \theta(n) = \sin^2 \theta(n_1) \chi(u, k, \alpha_0, n).$$

If values of u and n are given, χ may be tabulated in terms of k and α_0 .

Separate tables of χ are required for occultations and transits.

In order to define a relation between k and α_0 , assume a value of the limb-darkening coefficient u , choose a single value of n which is an argument in the χ tables, compute l from $n = (1 - l)/(1 - \lambda_1)$, and read the corresponding $\theta(n)$ from the light curve. Compute $\chi(u, k, \alpha_0, n)$ using equation (73). Enter the appropriate table with χ and interpolate to find the value of α_0 corresponding to tabular values

1 H. N. Russell, *Ap.J.* 35, 315, 1912.

2 Excellent tables of χ have been tabulated by J. E. Merrill in *Contr. Princeton U. Obs.*, 23, 1950, based on $n = 0.5$.

of k , or vice versa. Each interpolation gives a point (k, α_0) .

Plot these points on a suitable scale to obtain a shape curve. Since the investigator has no means of deciding beforehand whether an occultation or a transit gives rise to the primary minimum, both the χ^{oc} and χ^{tr} tables must be entered with χ . Therefore two shape curves will be obtained; one relating k and α_0^{oc} , the other relating k and α_0^{tr} . Inquiry should be made into both assumptions because both may lead to solutions for the same light curve.

It seems reasonable to suppose that for two different values of n the corresponding k - α_0 curves obtained from χ might intersect, giving unique values of k and α_0 . Russell¹ discovered, however, that such a solution was virtually indeterminate. All pairs (k, α_0) which give a fixed value of $\chi(u, k, \alpha_0, n_1)$ give rise to values of $\chi(u, k, \alpha_0, n_2)$, for any other n_2 , which are nearly the same. Therefore, the shape curve gives only one usable relation between k and α_0 .

A second correlation of k and α_0 may be obtained from the depth of the minima by means of the q -function defined by equation (45). For partial eclipses $p = p(u_s, k, \alpha_0^{oc})$; consequently, $q = q(u_s, u_g, k, \alpha_0^{oc})$. In practice the q -function is more useful when it is inverted to give

$$(74) \quad k = k(u_s, u_g, q, \alpha_0^{oc}).$$

This function is tabulated for various values of u_s and u_g , with q_0 and α_0^{oc} as arguments.

At the moment of mid-eclipse equation (49) applies

$$\alpha_0^{oc} = 1 - \lambda^{oc} + \frac{(1 - \lambda^{tr})}{q_0}.$$

Again denote the light at mid-primary and mid-secondary minimum by λ ,

1 H. N. Russell, Ap.J., 35, 327, 1912.

and λ_2 . Adopt tabular values of α_o^{oc} , compute q_0 from (49), and find k from tables constructed for (74). If an occultation gives rise to the primary minimum $\lambda^{oc} = \lambda_1$, and k, α_o^{oc} are found directly. These values are then plotted on the same diagram as the shape curve for an occultation. When a transit produces the primary minimum, $\lambda^{oc} = \lambda_2$, and k, α_o^{oc} are again found. In this instance α_o^{oc} refers to the secondary minimum and α_o^{tr} is needed before a plot can be carried out. Tables of $k(u_s, u_g, q_0, \alpha_o^{oc})$ constructed by Merrill¹ give tabular values of α_o^{tr} corresponding to each k . Thus when the table is entered with α_o^{oc} and q_0 values of α_o^{tr} and k may be extracted. Plot these values on the same diagram as the shape curve for a transit.

On each diagram the two curves will intersect and unique values of k and α_o are found. The remainder of the elements may then be found. Either the χ or the ψ -functions yield r_s, r_g and i . with the χ -functions use

$$(75) \quad \sin^2 \theta^e = \sin^2 \theta(0.0) = \sin^2 \theta(n_1) \chi(u, k, \alpha_o, 0.0),$$

$$(76) \quad r_g^2 = \frac{\sin^2 \theta^e}{(1+k)^2 - [1+kp(u, k, \alpha_o)]^2 \cos^2 \theta^e},$$

$$(77) \quad r_s = k r_g,$$

$$(78) \quad \cos i = [1+kp(u, k, \alpha_o)] r_g.$$

¹ Tables of $k(u_s, u_g, q_0, \alpha_o^{oc})$ are given for different degrees of limb-darkening of each star and are denoted, in the notation here, by $^{u_s, u_g} k_{\alpha}$. See J. E. Merrill's "Auxiliary Tables". Contr. Princeton U. Obs., 23, 1953.

With the ψ -functions use

$$\sin^2 \theta^c = \sin^2 \theta(n) \psi(u, k, \alpha_0, 0, 0) \quad ,$$

$$(79) \quad B = \frac{\sin^2 \theta^c}{\psi(u, k, 0) - \psi(u, k, \alpha_0)} \quad ,$$

$$(80) \quad A = -B \psi(u, k, \alpha_0) \quad ,$$

$$\cot^2 i = -A + \frac{B}{\Phi_2(u, k)} \quad ,$$

$$r_g^2 = \frac{B \sin^2 i}{\Phi_1(u, k)} \quad ,$$

$$r_s = k r_g \quad ,$$

$$p_s = p(u, k, \alpha_0) = \frac{\cos i - r_g}{r_s} \quad .$$

To compute a light curve when the known data is $k, r_g, i, \alpha_0^c, \alpha_0^{tr}, \lambda^c$, and λ^{tr} , assume values of α^c and α^{tr} and compute l from

$$l^c = 1 - \frac{(1 - \lambda^c) \alpha^c}{\alpha_0^c} \quad \text{and} \quad l^{tr} = 1 - \frac{(1 - \lambda^{tr}) \alpha^{tr}}{\alpha_0^{tr}} \quad .$$

To find the corresponding mean anomaly θ calculate A and B for each minimum from

$$B^c = r_g^2 \csc^2 i \Phi_1(u_s, k) \quad , \quad B^{tr} = r_g^2 \csc^2 i \Phi_1(u_g, k) \quad ,$$

$$A^c = \frac{B^c}{\Phi_2(u_s, k)} - \cot^2 i \quad , \quad A^{tr} = \frac{B^{tr}}{\Phi_2(u_g, k)} - \cot^2 i \quad .$$

Then for the assumed α^c and α^{tr}

$$\sin^2 \theta^c = A^c + B^c \psi^c(u_s, k, \alpha^c) \quad , \quad \text{and} \quad \sin^2 \theta^{tr} = A^{tr} + B^{tr} \psi^{tr}(u_g, k, \alpha^{tr}) \quad .$$

The light of each star, L_s and L_g , becomes

$$L_s = \frac{1 - \lambda^c}{\alpha_0^c} \quad \text{and} \quad L_g = 1 - L_s \quad .$$

Eccentric Orbit

An eclipsing system whose light curve satisfies the conditions stated may be represented by a spherical model with an eccentric orbit if the primary and secondary minima are symmetric about their mid-points, and are of unequal duration or are of unequal spacing, or both. It will facilitate comparing photometric and spectrographic results if the primary component is considered to move about the secondary. Let e represent the eccentricity of the relative orbit, and ω the longitude of periastron measured from the ascending node in the direction of motion. As in spectroscopic binaries, the ascending node is taken as the node at which the star is receding from the observer. Let v denote the true anomaly measured from periastron in the direction of motion. Let ϕ represent the true anomaly measured from superior conjunction in the direction of motion to the primary component. As before let θ denote the mean anomaly measured from mid-eclipse. The actual distance R between the centers of the two stars is given by

$$(81) \quad R = \frac{a(1 - e^2)}{1 + e \cos v},$$

where a is the semi-major axis of the relative orbit. It is customary for the discussion of eccentric orbits to express all distances in terms of a .

The general treatment of eccentric orbits is considerably more complex than that of circular orbits because the minima fail to coincide with the conjunctions. As a result the areas of each component eclipsed at the respective minima will generally be unequal. For

spherical stars the minima will occur when the apparent distance δ between star centers is a minimum. The expression for δ can be found by considering the figure below:

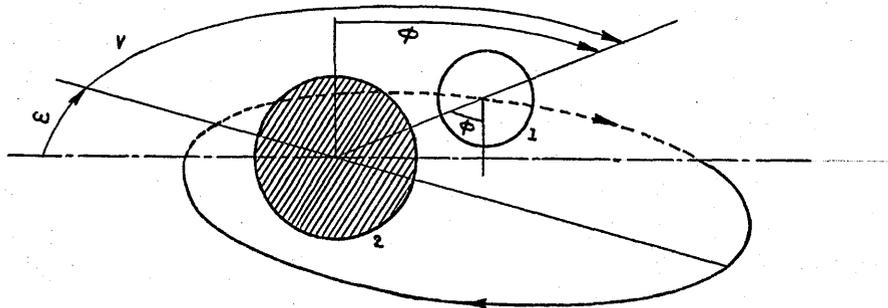


Figure 4

$$(82) \quad \delta^2 = R^2 (\cos^2 i + \sin^2 i \cos^2 \phi) = \frac{(1 - e^2)^2 (1 - \sin^2 i \cos^2 \phi)}{(1 + e \cos v)^2}.$$

Substitute $v = 90^\circ + \phi - \omega$ to obtain

$$(83) \quad \delta^2 = \frac{(1 - e^2)^2 (1 - \sin^2 i \cos^2 \phi)}{[1 - e \sin(\phi - \omega)]^2} = r_g^2 \{1 + kp(u, k, \alpha)\}^2.$$

In order to obtain the maxima and minima for δ , set $\frac{d\delta}{d\phi} = 0$; thus

$$1 - e \sin(\phi - \omega) = 0,$$

$$(84) \quad [1 - e \sin(\phi - \omega)] \sin^2 i \sin 2\phi + 2e \cos(\phi - \omega) [1 - \sin^2 i \cos^2 \phi] = 0.$$

The first equation gives a singular solution and is of no practical importance, since it gives a real solution only if $e \geq 1$ or if $\omega = 0^\circ$ or 180° . When $i = 90^\circ$ is introduced into the second equation, the roots giving minima are found to be 0° and 180° . For inclined orbits where $i \neq 90^\circ$, the roots of equation (84) near 0° and 180° must be investigated. Equation (84) reduces to the exact fourth order equations which follow:

$$(85) \quad (\sin^4 i) x^4 + (2e \sin \omega \sin^2 i) x^3 + [e^2 (\cos^2 \omega \cos^4 i + \sin^2 \omega) - \sin^4 i] x^2 - (2e \sin \omega \sin^2 i) x - e^2 \sin^2 \omega = 0,$$

where $x = \cos \phi$, or

$$(86) \quad (\sin^4 i) y^4 + (2 e \cos \omega \sin^2 i \cos^2 i) y^3 \\ + [e^2 (\sin^2 \omega \cos^2 2i + \cos^2 \omega \cos^4 i) - \sin^4 i] y^2 \\ - (2 e \cos \omega \sin^2 i \cos^2 i) y - e^2 \cos^4 i \cos^2 \omega = 0,$$

where $y = \sin \phi$.

The values of $\sin \phi$ will be very small so an approximate solution may be obtained by setting $\sin \phi = \phi$ and $\cos \phi = 1 - \frac{\phi^2}{2}$, and neglecting third and higher powers of e and ϕ . The roots of interest then become

$$(87) \quad \phi_1 = -e \cos \omega \cot^2 i \cdot (1 - e \sin \omega \csc^2 i) + \dots,$$

$$(88) \quad \phi_2 = \pi + e \cos \omega \cot^2 i (1 - e \sin \omega \csc^2 i) + \dots$$

For most cases encountered in practice these approximation formulas are sufficient to give answers to the number of significant figures justified by the values of i, e and ω . However, if greater accuracy seems warranted the approximate values found may be corrected using the exact expressions.

It is apparent that the true anomaly reckoned from periastron passage at the moment t , is given by

$$(89) \quad v = 90^\circ - \omega + \phi.$$

Since the angular momentum, and hence the areal velocity, is constant for a two body central force orbit, the time interval $t_2 - t_1$, between primary and secondary minima, expressed in terms of the period P is

$$\int_{t_1}^{t_2} \frac{1}{P} dt = \frac{1}{2\pi a^2 \sqrt{1-e^2}} \int_{v_1}^{v_2} R^2 dv,$$

or

$$(90) \quad \frac{2\pi}{P} (t_2 - t_1) = (1 - e^2)^{3/2} \int_{v_1}^{v_2} \frac{dv}{(1 + e \cos v)^2}$$

The series expansion for this integral is to be avoided if one seeks a solution in closed form. A transformation of variable

$$(91) \quad \tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{v}{2},$$

where E is called the eccentric anomaly, permits one to perform the integration easily. Note the following relations for an ellipse:

$$(92) \quad dv = \frac{1 - e^2}{1 - e \cos E} dE$$

and

$$(93) \quad \frac{1}{1 + e \cos v} = \frac{1 - e \cos E}{(1 - e^2)}$$

Therefore the displacement becomes

$$\begin{aligned} n(t_2 - t_1) &= \int_{E_1}^{E_2} (1 - e \cos E) dE = E - e \sin E \Big|_{E_1}^{E_2} \\ &= 2 \tan^{-1} \sqrt{\frac{1-e}{1+e}} \tan \frac{v}{2} - \frac{e \sqrt{1-e^2} \sin v}{1 + e \cos v} \Big|_{v_1 = 90^\circ + (\phi_1 - \omega)}^{v_2 = 90^\circ + (\phi_2 - \omega)}, \end{aligned}$$

where $n = 2\pi/P$. Or

$$\begin{aligned} (94) \quad n(t_2 - t_1) &= 2 \tan^{-1} \sqrt{\frac{1-e}{1+e}} \left(\frac{1 + \sin(\phi_2 - \omega)}{\cos(\phi_2 - \omega)} \right) \\ &\quad - 2 \tan^{-1} \sqrt{\frac{1-e}{1+e}} \left(\frac{1 + \sin(\phi_1 - \omega)}{\cos(\phi_1 - \omega)} \right) \\ &\quad + e \sqrt{1 - e^2} \left[\frac{\cos(\phi_1 - \omega)}{1 - e \sin(\phi_1 - \omega)} - \frac{\cos(\phi_2 - \omega)}{1 - e \sin(\phi_2 - \omega)} \right]. \end{aligned}$$

This is the exact solution.

In order to complete the analysis, the duration of the ascending and descending branches must be investigated; for this purpose introduce the quantities

$t'_{1,2}$, the time of first contact of the primary (subscript 1) and secondary (subscript 2) eclipse.

$t''_{1,2}$, the time of last contact of the primary and secondary eclipse.

$\psi'_{1,2}, \psi''_{1,2}$ the true anomalies at first and last contact measured irrespective of sign from the respective conjunction which occurs at time $t_{1,2}$.

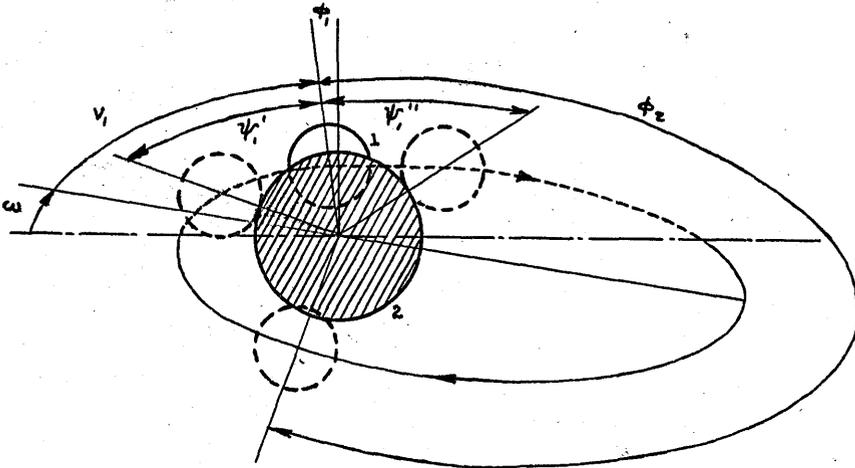


Figure 5

The durations of the ascending and descending branches of both minima are given by

$$(95) \quad \tau''_{1,2} = t''_{1,2} - t_{1,2} ,$$

$$(96) \quad \tau'_{1,2} = t_{1,2} - t'_{1,2} ,$$

respectively. Using the law of areas one finds

$$(97) \quad n \tau'_{1,2} = (1 - e^2)^{3/2} \int_{\psi'_{1,2}}^{\psi''_{1,2}} \frac{dv}{(1 + e \cos v)^2}$$

and

$$(98) \quad n \tau''_{1,2} = (1 - e^2)^{3/2} \int_{\psi''_{1,2}}^{\psi'_{1,2}} \frac{dv}{(1 + e \cos v)^2} ,$$

where R is given by equation (81), and $\psi'_{1,2}$ and $\psi''_{1,2}$ by

$$(99) \quad \sqrt{1 - \sin^2 i \cos^2 \psi'_{1,2}} = h [1 \pm e \sin(\omega + \psi'_{1,2})],$$

$$(100) \quad \sqrt{1 - \sin^2 i \cos^2 \psi''_{1,2}} = h [1 \pm e \sin(\omega - \psi''_{1,2})],$$

utilizing equation (81), where $h = \frac{r_s + r_g}{1 - e^2}$, and r_s, r_g are the radii of the components expressed in units of the semi-major axis. Exact expressions from which solutions may be made are

$$(101) \quad \begin{aligned} & \{[\sin^2 i - h^2 e^2 \cos 2\omega]^2 + 4h^4 e^4 \cos^2 \omega \sin^2 \omega\} x^4 \\ & + \{4h^2 e \sin \omega [\sin^2 i - h^2 e^2 \cos 2\omega] + 8h^4 e^3 \sin \omega \cos^2 \omega\} x^3 \\ & + \{4h^4 e^2 + 2[\sin^2 i - h^2 e^2 \cos 2\omega] \cdot [h^2(1 + e^2 \cos^2 \omega) - 1] - 4h^4 e^4 \sin^2 \omega \cos^2 \omega\} x^2 \\ & + \{4h^4 e \sin \omega [h^2(1 + e^2 \cos^2 \omega) - 1] - 8h^4 e^3 \sin \omega \cos^2 \omega\} x \\ & + \{[h^2(1 + e^2 \cos^2 \omega) - 1]^2 - 4h^4 e^2 \cos^2 \omega\} = 0, \end{aligned}$$

where $x = \cos \psi'_{1,2}$. Also one finds

$$(102) \quad \begin{aligned} & \{[\sin^2 i - h^2 e^2 \cos 2\omega]^2 + 4h^4 e^4 \cos^2 \omega \sin^2 \omega\} y^4 \\ & - \{4h^2 e \sin \omega [\sin^2 i - h^2 e^2 \cos 2\omega] + 8h^4 e^3 \sin \omega \cos^2 \omega\} y^3 \\ & + \{4h^4 e^2 + 2[\sin^2 i - h^2 e^2 \cos 2\omega] \cdot [h^2(1 + e^2 \cos^2 \omega) - 1] - 4h^4 e^4 \sin^2 \omega \cos^2 \omega\} y^2 \\ & - \{4h^2 e \sin \omega [h^2(1 + e^2 \cos^2 \omega) - 1] - 8h^4 e^3 \sin \omega \cos^2 \omega\} y \\ & + \{[h^2(1 + e^2 \cos^2 \omega) - 1]^2 - 4h^4 e^2 \cos^2 \omega\} = 0, \end{aligned}$$

where $y = \cos \psi''_{1,2}$. From the form of these equations it is apparent that the roots of one equation are the negative roots of the other, so that only one equation need be solved to obtain roots for both.

As far as the inverse problem of obtaining the elements from the light curve is concerned, equations (101) and (102) are virtually worthless, because the coefficients of x and y therein are also unknown. However, one can utilize these equations in determining the magnitude of the asymmetries of the minima. Exact expressions for the durations of the descending and ascending branches of the

minima can be obtained from equations (97) and (98)

$$(103) \quad n\tau'_{1,2} = 2 \tan^{-1} \frac{\sqrt{1-e} \left(\frac{1 - \sin(\omega - \phi_{1,2})}{\cos(\omega - \phi_{1,2})} \right) - 2 \tan^{-1} \frac{\sqrt{1-e} \left(\frac{1 - \sin(\omega + \psi'_{1,2})}{\cos(\omega + \psi'_{1,2})} \right)}{\sqrt{1+e} \left(\frac{1 - \sin(\omega - \phi_{1,2})}{\cos(\omega - \phi_{1,2})} \right) - 2 \tan^{-1} \frac{\sqrt{1-e} \left(\frac{1 - \sin(\omega + \psi'_{1,2})}{\cos(\omega + \psi'_{1,2})} \right)}{\sqrt{1+e} \left(\frac{1 - \sin(\omega - \phi_{1,2})}{\cos(\omega - \phi_{1,2})} \right)}}{-e\sqrt{1-e^2} \left[\frac{\cos(\omega - \phi_{1,2})}{1 + e \sin(\omega - \phi_{1,2})} - \frac{\cos(\omega + \psi'_{1,2})}{1 + e \sin(\omega + \psi'_{1,2})} \right]},$$

$$n\tau''_{1,2} = 2 \tan^{-1} \frac{\sqrt{1-e} \left(\frac{1 - \sin(\omega - \psi''_{1,2})}{\cos(\omega - \psi''_{1,2})} \right) - 2 \tan^{-1} \frac{\sqrt{1-e} \left(\frac{1 - \sin(\omega - \phi_{1,2})}{\cos(\omega - \phi_{1,2})} \right)}{\sqrt{1+e} \left(\frac{1 - \sin(\omega - \psi''_{1,2})}{\cos(\omega - \psi''_{1,2})} \right) - 2 \tan^{-1} \frac{\sqrt{1-e} \left(\frac{1 - \sin(\omega - \phi_{1,2})}{\cos(\omega - \phi_{1,2})} \right)}{\sqrt{1+e} \left(\frac{1 - \sin(\omega - \psi''_{1,2})}{\cos(\omega - \psi''_{1,2})} \right)}}{-e\sqrt{1-e^2} \left[\frac{\cos(\omega - \psi''_{1,2})}{1 + e \sin(\omega - \psi''_{1,2})} - \frac{\cos(\omega - \phi_{1,2})}{1 + e \sin(\omega - \phi_{1,2})} \right]}.$$

Then the durations of both minima are given by the sums

$$\tau'_1 + \tau''_1 \quad \text{and} \quad \tau'_2 + \tau''_2,$$

and the asymmetries by the differences

$$\tau'_1 - \tau''_1 \quad \text{and} \quad \tau'_2 - \tau''_2.$$

The preceding equations should reveal that the durations are equal when $\omega = 0^\circ$ or 180° and the asymmetries vanish only if $\omega = 90^\circ$ or 270° .

For values ordinarily encountered in practice the asymmetry, when at its largest value, is found to be but a few per cent of the displacement. This means, that in most practical cases the asymmetries will be too small to be detected by observation.¹ From the stand point of the inverse problem, this information is of utmost importance, because it opens up a possibility of determining the true elements of eclipsing systems with eccentric orbits rather easily in certain cases.

Before outlining the computational procedure it is advantageous to note the explicit forms which some of the previous formulae assume

¹ Z. Kopal, Harvard Obs. Mon., 6, 103, 1946.

for the special cases $i = 90^\circ$. As was mentioned before, primary and secondary minima occur at

$$\phi_1 = 0^\circ, \quad v_1 = 90^\circ - \omega,$$

$$\phi_2 = 180^\circ, \quad v_2 = 270^\circ - \omega,$$

respectively. The displacement of the secondary minimum is given by

$$(104) \quad n(t_2 - t_1) = -2 \tan^{-1} \frac{\sqrt{1-e^2}}{e \cos \omega} + \frac{2e\sqrt{1-e^2} \cos \omega}{1 - e^2 \sin^2 \omega},$$

or

$$= \pi + 2 \tan^{-1} \frac{e \cos \omega}{\sqrt{1-e^2}} + \frac{2e\sqrt{1-e^2} \cos \omega}{1 - e^2 \sin^2 \omega}.$$

Now put

$$D = \pi + 2 \tan^{-1} \frac{e \cos \omega}{\sqrt{1-e^2}},$$

and obtain

$$(105) \quad n(t_2 - t_1) = D - \sin D.$$

Equation () is a particular case of Kepler's equation, which may be solved in a variety of different ways.¹ One can use this equation to obtain the quantity $\frac{e \cos \omega}{\sqrt{1-e^2}}$, knowing the displacement of the minima. When this quantity is expanded in a series there results

$$(106) \quad \frac{e \cos \omega}{\sqrt{1-e^2}} = \left(e + \frac{1}{2} e^3 + \frac{1 \cdot 3}{2^2 \cdot 2!} e^5 + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!} e^7 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 \cdot 4!} e^9 + \dots \right) \cos \omega.$$

If terms of the order of e^3 and higher are neglected, this expression reduces to the component $e \cos \omega$. When e is large, greater accuracy will be attained if more terms are used.

The durations of the ascending and descending branches of both minima are still given by equations (95) and (96), respectively. Using the law of areas directly one finds

$$(107) \quad n \tau'_{1,2} = \frac{1}{a^2 \sqrt{1-e^2}} \int_{v_{1,2} - v'_{1,2}}^{v_{1,2}} R^2 dv,$$

¹ See e.g., W. M. Smart, Celestial Mechanics, (London, 1953), p. 19-20.

and

$$(108) \quad n\tau_{1,2}'' = \frac{1}{a^2\sqrt{1-e^2}} \int_{\psi_{1,2}}^{\psi_{1,2} + \psi_{1,2}''} R^2 dv,$$

where R is again given by equation (81), and $\psi_{1,2}'$ and $\psi_{1,2}''$ by

$$(109) \quad R \sin \psi_{1,2}' = r_s + r_g,$$

$$(110) \quad R \sin \psi_{1,2}'' = r_s + r_g,$$

where r_s and r_g are expressed in units of the semi-major axis. Then

$$(111) \quad \sin \psi_{1,2}' = h \{1 \pm e \sin(\omega + \psi_{1,2}')\},$$

$$(112) \quad \sin \psi_{1,2}'' = h \{1 \pm e \sin(\omega - \psi_{1,2}'')\},$$

where $h = \frac{r_s + r_g}{1 - e^2}$. The solutions to these equations are

$$(113) \quad \sin \psi_{1,2}' = \frac{h}{U^2} \left\{ 1 - he \cos \omega \pm e \sin \omega \sqrt{U^2 - h^2} \right\},$$

$$(114) \quad \sin \psi_{1,2}'' = \frac{h}{V^2} \left\{ 1 + he \cos \omega \pm e \sin \omega \sqrt{V^2 - h^2} \right\},$$

where

$$U^2 = (1 + h^2 e^2 - 2he \cos \omega),$$

$$V^2 = (1 + h^2 e^2 + 2he \cos \omega).$$

The durations of the descending and ascending branches of the minima become, respectively,

$$(115) \quad n\tau_{1,2}' = 2 \tan^{-1} \left\{ \frac{\sqrt{1-e^2} \tan \frac{\psi_{1,2}'}{2}}{1 \pm e(\sin \omega + \cos \omega \tan \frac{\psi_{1,2}'}{2})} \right\} \\ \pm \sqrt{1-e^2} \left\{ \frac{e \cos(\omega + \psi_{1,2}')}{1 \pm e \sin(\omega + \psi_{1,2}')} - \frac{e \cos \omega}{1 \pm e \sin \omega} \right\},$$

$$(116) \quad n\tau_{1,2}'' = 2 \tan^{-1} \left\{ \frac{\sqrt{1-e^2} \tan \frac{\psi_{1,2}''}{2}}{1 \pm e(\sin \omega - \cos \omega \tan \frac{\psi_{1,2}''}{2})} \right\} \\ \mp \sqrt{1-e^2} \left\{ \frac{e \cos(\omega - \psi_{1,2}'')}{1 \pm e \sin(\omega - \psi_{1,2}'')} - \frac{e \cos \omega}{1 \pm e \sin \omega} \right\}.$$

The durations of both minima are given by the same sums as before

$$\tau_1' + \tau_1'' \quad \text{and} \quad \tau_2' + \tau_2'',$$

and the asymmetries by the same differences

$$\tau_1' - \tau_1'' \quad \text{and} \quad \tau_2' - \tau_2''.$$

Equations (115) and (116) reveal that the durations are equal only when $\omega = 0^\circ$ or 180° , and reach a maximum difference at $\omega = 90^\circ$ or 270° . On the other hand, the asymmetries vanish only if $\omega = 90^\circ$ or 270° and reach a maximum when $\omega = 0^\circ$ or 180° . Hence, the differences in durations of the minima and their asymmetries can never vanish simultaneously.

The magnitudes of $\tau_{1,2}'$ and $\tau_{1,2}''$ are related as follows:

$$\begin{aligned} \tau_1' < \tau_1'' & \quad \text{and} \quad \tau_2' > \tau_2'' & \quad (-90^\circ < \omega < 90^\circ), \\ \tau_1' > \tau_1'' & \quad \text{and} \quad \tau_2' < \tau_2'' & \quad (90^\circ < \omega < 270^\circ). \end{aligned}$$

The branch of each minimum which lies nearest to the neighboring minimum is found to be the steeper of the two.

The solution of the inverse problem, that is of finding elements which, when substituted into equation (83), will reproduce the known light curve to within observational accuracy, is in general somewhat tedious. Usually such definitive elements can only be obtained after adjusting preliminary approximate values. Since the minima of the light curves of eccentric systems are very nearly symmetrical, preliminary "circular" elements may be found by treating each minimum separately as though the orbit were a circle of radius equal to the instantaneous radius vector at the time of each conjunction. From these circular elements the approximate true elements may be determined.

The formulae which relate the true to the circular elements should yield closely approximate values even for large eccentricity. However, the more accurate formulae are usually the more complex. As T. E. Sterne remarks;¹

¹ T. E. Sterne, Proc. Nat. Acad. Sci., 26, 37, 1940.

It is desirable for the formulae to be simple and easy of application ; and since the resulting approximate values of the true orbital elements should probably always be tested and if necessary improved by the exact equation, $[(e\omega)]$, it is permissible to sacrifice some accuracy in order to gain simplicity.

Russell¹ gives such equations which are practical to use only when the eccentricity does not exceed about 0.2 . More accurate formulae, which yield values approximate to within about one per cent of the true values for eccentricities up to about 0.5, are given by Sterne² and are presented here.³

Complete Eclipses

First consider complete eclipses, for which the elements may in principle be obtained from each minimum independently. Upon regarding the minima as symmetrical one actually assumes that the variation of R or $\frac{dv}{dt}$ within each minimum may be ignored. Let this assumption be made and consider the eclipse of the primary component by the secondary.

At superior conjunction $v = 90^\circ - \omega$ and

$$(117) \quad R = \frac{a(1 - e^2)}{1 + e \sin \omega} .$$

Then by the law of areas

$$R^2 \frac{dv}{dt} = \frac{2\pi a^2 \sqrt{1 - e^2}}{p} ,$$

$$\text{or } (118) \quad \frac{dv}{dt} = \frac{n(1 + e \sin \omega)^2}{(1 - e^2)^{3/2}} .$$

1 H. N. Russell, Ap. J., 36, 54, 1912

2 T. E. Sterne, Proc. Nat. Acad. Sci., 26, 36, 1940

3 Other treatments of eccentric systems for small eccentricities include

A. Pannekoek and E. Van Dien, B.A.N., 8, 141, 1937.

A. B. Wyse, Lick Bull., No. 496, 1939.

Denote the true anomaly measured from superior conjunction by

$$\phi = \nu + \omega - 90^\circ$$

and the mean anomaly from mid-primary minimum by θ' .

To a first approximation suppose that primary minimum and superior conjunction coincide. Then approximately,

$$(119) \quad \phi = \frac{\theta' (1 + e \sin \omega)^2}{(1 - e^2)^{3/2}} .$$

The eccentricity e is usually small unless the stars are widely separated, and then θ' is small during eclipse. Therefore replacing the angles by their sines one finds

$$(120) \quad \sin^2 \phi = \frac{(1 + e \sin \omega)^4}{(1 - e^2)^3} \sin^2 \theta' .$$

Substitution of this in equation (83) gives

$$(121) \quad \sin^2 i \left(\frac{(1 - e^2)^3}{(1 + e \sin \omega)^4} \cot^2 i + \sin^2 \theta' \right) = r_g^2 \frac{(1 - e^2)}{(1 + e \sin \omega)^2} (1 + kp)^2 .$$

If r_g' and i' are defined by the quantities

$$r_g' = \frac{\sqrt{1 - e^2}}{1 + e \sin \omega} r_g ,$$

and

$$\cot i' = \frac{(1 - e^2)^{3/2} \cot i}{(1 + e \sin \omega)^2} ,$$

then

$$\frac{\sin^2 i'}{\sin^2 i''} (\cos^2 i' + \sin^2 i' \sin^2 \theta') = r_g'^2 (1 + kp)^2 .$$

Approximately $\frac{\sin^2 i'}{\sin^2 i''} = 1$. The final equations relating the fictitious

circular elements and true elliptical elements at superior conjunction (primary minimum) are

$$(122) \quad \cos^2 i' + \sin^2 i' \sin^2 \theta' = r_g'^2 (1 + kp)^2 ,$$

$$(123) \quad r_g' = \frac{\sqrt{1 - e^2}}{1 + e \sin \omega} r_g ,$$

$$(124) \quad \cot i' = \frac{(1-e^2)^{3/2}}{(1+e \sin \omega)^2} \cot i \quad .$$

Circular elements at inferior conjunction (secondary minimum) are given by the above formulae when ω is increased by 180° . Hence,

$$(125) \quad \cos^2 i'' + \sin^2 i'' \sin^2 \theta' = r_g''^2 (1+kp)^2 \quad ,$$

$$(126) \quad r_g'' = \frac{\sqrt{1-e^2}}{(1-e \sin \omega)} r_g \quad ,$$

$$(127) \quad \cot i'' = \frac{(1-e^2)^{3/2}}{(1-e \sin \omega)^2} \cot i \quad .$$

Analyzing each minimum using the methods outlined in the earlier section on complete eclipses will yield a value of k , the ratio of the radii, accurate to the order of approximation made in assuming symmetric minima. Either minimum may be an occultation or a transit and both possibilities must be investigated. Superior conjunction will almost always occur at primary minimum, however. In this way, values of i' , r_g' and i'' , r_g'' will be obtained, from which the corresponding true values may be found.

If the eclipses are wide it is more accurate to use values of R and $\frac{dv}{dt}$ averaged over the eclipses. Equations based on such average values are the same as those given in this section except that $e \sin \omega$ in equations (122) - (127) should be replaced by $\eta e \sin \omega$. With sufficient approximation¹

$$(128) \quad \eta = \frac{2}{3} + \frac{1}{3} \cos \frac{1}{2}(\theta_i + \theta_e) \quad ,$$

where θ_i and θ_e are the mean anomalies at the beginning and end of the

¹ H. W. Russell, Ap.J., 36, 56, 1912.

respective eclipse. When e is large the eclipses will usually be narrow, and η will be close to unity regardless of asymmetry. In such circumstances the equations as given without η should be sufficiently accurate to give reasonably good values for the true orbital elements.

The true elements may now be found by a proper combination of the other known quantities. One obtains the so-called radial component of eccentricity from either of the ratios

$$\frac{r_1^i}{r_2^i} = \frac{(1 - e \sin \omega)}{(1 + e \sin \omega)}, \quad \frac{\cot i'}{\cot i''} = \frac{(1 - e \sin \omega)^2}{(1 + e \sin \omega)^2}.$$

Probably greater accuracy will be obtained using the latter ratio.

If $x = \frac{r_1^i}{r_2^i}$ or $\left(\frac{\cot i'}{\cot i''}\right)^{\frac{1}{2}}$, then for the corresponding formula

$$(129) \quad e \sin \omega = \frac{1 - x}{1 + x}.$$

For consistency one must have

$$(130) \quad \left(\frac{r_1^i}{r_2^i}\right)^2 = \left(\frac{r_1^i}{r_2^i}\right)^2 = \frac{\cot i'}{\cot i''}.$$

The tangential component of eccentricity, $e \cos \omega$, may be obtained from equation (105) as a first approximation. This relation

$$n(t_2 - t_1) = D - \sin D,$$

where $D = \frac{e \cos \omega}{\sqrt{1 - e^2}}$ is exact for $i = 90^\circ$, and is very nearly true for the inclinations encountered in practice. Using these two relations, determine e and ω . Next calculate the true i , r_g , and r_s from equations (123) - (127). Adopt as final values the means or weighted means of the true elements obtained from each minimum.

As a first check on the elements compute the displacement of the secondary minimum using equation (94) and ϕ_1 and ϕ_2 found using the

approximation formulas (87) and (88). If this is not the same as the observed displacement, make minor changes in e , ω , and i , while still maintaining the previous ratios, until they agree. For plots of light curves within minima, equation (119) is sufficient for finding the mean anomaly, provided ϕ is augmented by ϕ_1 or ϕ_2 .

It now becomes important to plot the complete light curve from the preliminary elements. This is the final test as to whether or not the elements of the adopted model represent the observed light curve. The final stages of analysis at the preliminary level lie in adjusting the elements and coercing them to represent the observations with at least moderate accuracy.

Assuming various values of the true anomaly ϕ , compute $p(u, k, \alpha)$ from (). Then enter the $\alpha(u, k, p)$ - tables¹ with p, k , and the assumed u , and extract α . Compute the mean anomaly from (94) with ϕ_2 taken as ϕ , and t_2 as t . Corresponding to each ϕ , plot the light ℓ , using (42) and (43), where $\beta^{tr}(u_g, k)$ must be found from tables², against the respective θ .

Some ambiguity may arise when the stars are darkened strongly at the limb, because it may not be evident whether the variable light within annular eclipse is due to limb-darkening or to a partial eclipse. In the first instance, if the type of eclipse is unknown, it is probably wise to consider the variable light as due to limb-darkening and follow the preceding outline; then, if values of the elements derived from the two minima vary widely indicating that a partial

¹ For example, the "Auxiliary Tables", Contr. Princeton U. Obs., 23, 1953.

² Again the "Auxiliary Tables", will be found to be useful.

eclipse is likely, treat the problem from the stand point of partial eclipses given next.

Partial Eclipses

Partial eclipses produced by two limb-darkened spherical stars moving in a relative orbit which is eccentric, offer one of the most complex problems in eclipsing binary work. Particularly this is true if the eccentricity is large and the inclination of the orbit is of such magnitude that while the minimum near periastron is visible the one near apastron is non-existent. As Kopal says,¹ "a solution for the elements in such extreme cases can evidently proceed only by successive approximations, based on the general properties of the problem of two bodies,...". The analysis to be developed here will apply only to cases where both minima are visible. Even in this case one obtains a solution only with the expenditure of a great amount of labor.

The method used previously for finding "circular" elements requires that each minimum be solved independently for the elements. This is usually possible, unless one of the eclipses is total or annular and the other is partial -- something heretofore not encountered! The extension of the method of "circular" elements to cases where the eclipses are partial is not immediately obvious, since in a solution for a circular orbit both minima were combined to determine the maximum normalized fractional loss of light α . The difficulty arises because for a relative orbit which is eccentric the maximum geometrical depth p_0 at the bottom of both minima is not the same, while in a

¹ Z.Kopal, Harvard Obs. Mon., 8, 115, 1950.

circular orbit it is.

First investigate the information which may be obtained from the shape of each minimum. Let the maximum normalized fractional loss of light at superior and inferior conjunctions be α_o^{oc} and α_o^{tr} , respectively, Equation (52) still defines the following

$$n^{tr} = \frac{\alpha_o^{tr}}{\alpha_o^{tr}} = \frac{1 - \ell^{tr}}{1 - \lambda^{tr}}, \quad n^{oc} = \frac{\alpha_o^{oc}}{\alpha_o^{oc}} = \frac{1 - \ell^{oc}}{1 - \lambda^{oc}},$$

where as before λ^{tr} and λ^{oc} are the observed values of light at mid-minimum corresponding to a transit and an occultation.

As in the case of partial eclipses for circular orbits the shape or χ - function may be defined by (72), but now is defined in terms of the fictitious inclination and radius of the larger star. From (122) and (125)

$$(131) \quad \cos^2 i^f + \sin^2 i^f \sin^2 \theta^m = r_g^{f^2} [1 + k p(u, k, n \alpha_o)],$$

where the superscript f denotes the fictitious elements, and m , denotes the true mean anomaly measured from the mid-point of the particular minimum in question. At mid-minimum $\theta = 0$ and $n = 1$, so that

$$(132) \quad \cos^2 i^f = r_g^{f^2} [1 + k p(u, k, \alpha_o)].$$

The χ -function now becomes

$$(133) \quad \chi(u, k, \alpha_o, n) = \frac{\sin^2 \theta^m(n)}{\sin^2 \theta^m(n_1)} = \frac{[1 + k p(u, k, n \alpha_o)]^2 - [1 + k p(u, k, \alpha_o)]^2}{[1 + k p(u, k, n, \alpha_o)]^2 - [1 + k p(u, k, \alpha_o)]^2}.$$

To meet the objective of using the χ -function to obtain a relation between k and α_o , assume a value of u , choose a single value of n which is an argument in the χ - tables, and compute ℓ from

$$\ell = 1 - n(1 - \lambda)$$

where $\lambda = \lambda_1$ for primary minimum and $\lambda = \lambda_2$ for secondary minimum.

Using this value of ℓ , read the corresponding θ^m from the light curve and compute $\chi(u, k, n, \alpha_0)$ from equation (133) for primary minimum.

Enter the appropriate χ^{oc} and χ^{tr} tables with χ and interpolate to find α_0 corresponding to the tabular values of k , or vice versa. If p_0^i and p_0^s are the values of p at inferior and superior conjunction, the α_0^s extracted for primary minimum will be $\alpha^{oc}(u_s, k, p_0^s)$ and $\alpha^{tr}(u_g, k, p_0^s)$. Plot these points to give two curves; one relating k and α_0^{oc} , and another relating k and α_0^{tr} .

Investigation of the depths of the two minima yields other useful relations, although these are not as simple as the corresponding ones for circular orbits. Light losses during an occultation and a transit must now be designated with individual values of the geometrical depth. Hence at mid-minima

$$(134) \quad 1 - \lambda^{oc} = L_s \alpha^{oc}(u_s, k, p_0^s),$$

$$(135) \quad 1 - \lambda^{tr} = L_g \alpha^{tr}(u_g, k, p_0^i) \beta^{tr}(u_g, k)$$

The q -function is still defined as the ratio of the fractional light loss f_g of the larger star at any phase of transit to the fractional light loss f_s of the smaller star at the same geometrical phase during occultation. For eccentric systems, however, this function is usable only in the approximate sense and only at mid-minima where the geometrical phase is known to be the same. Since there are two values of p , i.e., p_0^i and p_0^s , a varied mathematical definition of q must be used. If the eclipse which occurs at primary minimum is an occultation, then

$$(136) \quad q_0^{oc} = q_0^{oc}(u_s, u_g, k, p_0^s, p_0^i) = \frac{f_g}{f_s} = \frac{\alpha^{tr}(u_g, k, p_0^i) \beta^{tr}(u_g, k)}{\alpha^{oc}(u_s, k, p_0^s)}$$

$$\frac{(1 - z^{tr}) \alpha^{tr}(k, p_0^i) \beta^{tr}(k) + z^{tr} \alpha^{tr}(k, p_0^i) \beta^{tr}(k)}{(1 - z^{oc}) \alpha^{oc}(k, p_0^s) + z^{oc} \alpha^{oc}(k, p_0^s)}$$

where $z^{tr} = \frac{2u_s}{3-u_s}$ and $z^{oc} = \frac{2u_g}{3-u_g}$. If it is a transit

$$(137) \quad q_0^{tr} = q^{tr}(u_s, u_g, k, p_0^s, p_0^i) = \frac{f_g}{f_s} = \frac{\alpha^{tr}(u_g, k, p_0^s) \beta^{tr}(u_g, k)}{\alpha^{oc}(u_s, k, p_0^i)}$$

$$= \frac{(1-z^{tr})^0 \alpha^{tr}(k, p_0^s) \beta^{tr}(k) + z^{tr} \alpha^{tr}(k, p_0^s) \beta^{tr}(k)}{(1-z^{oc})^0 \alpha^{oc}(k, p_0^i) + z^{oc} \alpha^{oc}(k, p_0^i)}$$

the z 's being the same as above.

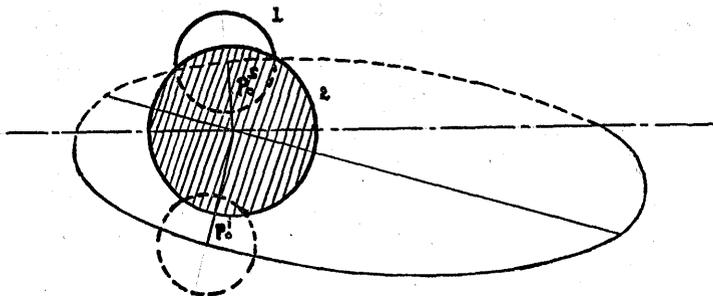


Figure 6. Primary minimum is an occultation.

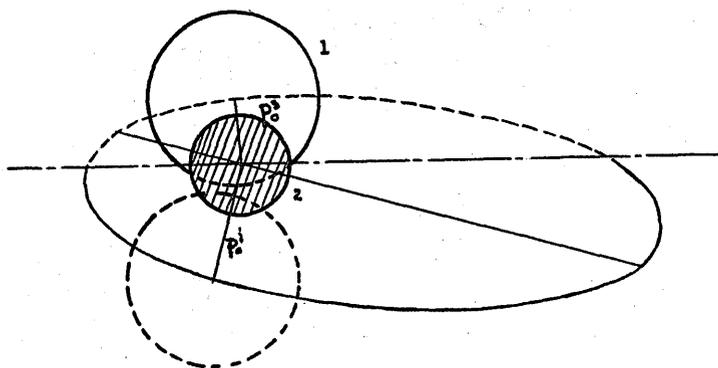


Figure 7. Primary minimum is a transit.

Assuming primary minimum to be an occultation one finds

$$1 - \lambda_1 = L_s \alpha^{oc}(u_s, k, p_0^s),$$

$$1 - \lambda_2 = L_g \alpha^{oc}(u_s, k, p_0^s) q^{oc}(u_s, u_g, k, p_0^s, p_0^i) = L_g \alpha^{tr}(u_g, k, p_0^i) \beta^{tr}(u_g, k),$$

and

$$(138) \quad \alpha^{oc}(u_s, k, p_0^s) = 1 - \lambda_1 + \frac{(1 - \lambda_2)}{q_0^{oc}}.$$

Then for secondary minimum $\alpha^{tr}(u_g, k, p_0^i) \beta^{tr}(u_g, k)$ becomes

$$(139) \quad \alpha^{tr}(u_g, k, p_0^i) \beta^{tr}(u_g, k) = q_0^{oc} (1 - \lambda_1) + (1 - \lambda_2).$$

Assuming primary minimum to be a transit one has

$$1 - \lambda_2 = L_s \alpha^{oc}(u_s, k, p_o^i),$$

$$1 - \lambda_1 = L_g \alpha^{oc}(u_s, k, p_o^i) q^{tr}(u_s, u_g, k, p_o^s, p_o^i) = L_g \alpha^{tr}(u_g, k, p_o^s) \beta^{tr}(u_g, k),$$

and

$$(140) \quad \alpha^{oc}(u_s, k, p_o^i) = 1 - \lambda_2 + \frac{(1 - \lambda_1)}{q_o^{tr}}.$$

For secondary minimum, $\alpha^{tr}(u_g, k, p_o^s) \beta^{tr}(u_g, k)$ becomes

$$(141) \quad \alpha^{tr}(u_g, k, p_o^s) \beta^{tr}(u_g, k) = q_o^{tr}(1 - \lambda_2) + (1 - \lambda_1).$$

One must analyze primary minimum using both assumptions, because a priori one does not know which minimum is an occultation and which is a transit.

If primary minimum is assumed to be an occultation, adopt tabular values of $\alpha^{oc}(u_s, k, p_o^s)$ and for a selected u_s compute q_o^{oc} from (138). Since q_o^{oc} is not a convenient quantity to tabulate, no tables of it are available. Hence, one must resort to using the tabulated q 's in a particular way. The q -function which is tabulated is defined by

$$(142) \quad q_o^i(u_s, u_g, k, p_o^i) = \frac{\alpha^{tr}(u_g, k, p_o^i) \beta^{tr}(u_g, k)}{\alpha^{oc}(u_s, k, p_o^i)}.$$

Using equations (136) and (142) one finds

$$(143) \quad q_o^i = q_o^{oc} \left[\frac{\alpha^{oc}(u_s, k, p_o^s)}{\alpha^{oc}(u_s, k, p_o^i)} \right].$$

Thus far only the depth and shape relations have been used. An additional relation is needed to determine the ratio $\frac{\alpha^{oc}(u_s, k, p_o^i)}{\alpha^{oc}(u_s, k, p_o^s)} = c$ but such a relation is not known at present.¹ Therefore, this ratio must be estimated, the values of k and the α_o 's found, and the light

¹ If the shape curves from both minima are available it may be possible to relax this statement.

curve plotted. If the computed light curve agrees closely with the observed curve, this indicates, though not necessarily, that the adopted ratio is the correct one. The estimation of this ratio serves the purpose of estimating the ratio of geometrical depths p_o^s/p_o^i .

Using the estimate, compute $\alpha^{oc}(u_s, k, p_o^i)$ and q_o^i for each assumed $\alpha^{oc}(u_s, k, p_o^s)$. With the various $\alpha^{oc}(u_s, k, p_o^i)$ and q_o^i , enter the tables of $k(u_s, u_g, q_o^i, \alpha^{oc})$ and extract k . Plot k against $\alpha^{oc}(u_s, k, p_o^i)$ on the corresponding shape curve to obtain a unique value of k and $\alpha^{oc}(u_s, k, p_o^i)$ at the intersection of the two curves. Now compute $\alpha^{tr}(u_s, k, p_o^i)$, q_o^{oc} and q_o^i for these unique values of k and $\alpha^{oc}(u_s, k, p_o^s)$. Then enter the $k(u_s, u_g, q_o^i, \alpha^{oc})$ tables with the particular q_o^i and $\alpha^{oc}(u_s, k, p_o^i)$, and find $\alpha^{tr}(u_g, k, p_o^i)$. Alternately, one could enter the tables¹ with $\alpha^{oc}(u_s, k, p_o^i)$ and k , and obtain $\alpha^{tr}(u_g, k, p_o^i)$, thus eliminating the necessity of calculating q_o^{oc} and q_o^i .

The light curve within minima must now be drawn to ascertain whether or not the assumed ratio is the correct one. From the observed light curve read $\theta(n = \frac{1}{2})$, where $n^{oc} = \frac{\alpha^{oc}}{\alpha_o^{oc}}$ and $n^{tr} = \frac{\alpha^{tr}}{\alpha_o^{tr}}$ for primary and secondary minimum, respectively. Compute the true mean anomaly θ^e at first or last contact measured from the corresponding mid-minimum using

$$(114) \quad \sin^2 \theta^{oc e} = \sin^2 \theta^{oc}(n = \frac{1}{2}) \chi^{oc}(u_s, k, \alpha_o^{oc}(p_o^s), n=0) \quad (\text{primary minimum}),$$

$$(115) \quad \sin^2 \theta^{tr e} = \sin^2 \theta^{tr}(n = \frac{1}{2}) \chi^{tr}(u_g, k, \alpha_o^{tr}(p_o^i), n=0) \quad (\text{secondary minimum}).$$

Then determine

$$(116) \quad B^{oc} = \frac{\sin^2 \theta^{oc e}}{\chi^{oc}(u_s, k, 0) - \chi^{oc}(u_s, k, \alpha_o^{oc}(p_o^s))},$$

$$(117) \quad A^{oc} = -B^{oc} \chi^{oc}(u_g, k, \alpha_o^{oc}(p_o^s)),$$

¹ This statement refers specifically to J. E. Merrill's tables.

$$\text{and (1148)} \quad B^{\text{tr}} = \frac{\sin^2 \theta^{\text{tr}} e}{\psi^{\text{tr}}(u_g, k, 0) - \psi^{\text{tr}}(u_g, k, \alpha_o^{\text{tr}}(p_o^i))},$$

$$(1149) \quad A^{\text{tr}} = -B^{\text{tr}} \psi^{\text{tr}}(u_g, k, \alpha_o^{\text{tr}}(p_o^i)).$$

In order to obtain the light ℓ and the true mean anomaly θ^m from mid-eclipse use the formulae

$$(150) \quad \sin^2 \theta^{\text{oc}^m} = A^{\text{oc}} + B^{\text{oc}} \psi^{\text{oc}}(u_s, k, \alpha^{\text{oc}}) \quad (\text{primary minimum}),$$

$$(151) \quad \sin^2 \theta^{\text{tr}^m} = A^{\text{tr}} + B^{\text{tr}} \psi^{\text{tr}}(u_g, k, \alpha^{\text{tr}}) \quad (\text{secondary minimum}),$$

$$\text{and (152)} \quad \ell^{\text{oc}} = 1 - \frac{(1-\lambda_1) \alpha^{\text{oc}}}{\alpha^{\text{oc}}(u_s, k, p_o^s)},$$

$$(153) \quad \ell^{\text{tr}} = 1 - \frac{(1-\lambda_2) \alpha^{\text{tr}}}{\alpha^{\text{tr}}(u_g, k, p_o^i)},$$

and choose different tabular values of the α 's. Plot the values (ℓ, θ) and obtain the light curve within minima. If this curve is not in agreement with the observations, an adjustment of the ratio c will be necessary.

Repeat the above procedure of successive approximations until a satisfactory fit is procured, if this is possible using the assumption that primary minimum is an occultation.

Alternately one may begin, assuming primary minimum to be a transit. A similar procedure is to be followed, except that one changes the order of operations slightly. The adopted values are $\alpha^{\text{oc}}(u_s, k, p_o^s)$ for an assumed u_s . q_o^{tr} is computed using (1140). The q -function to be used is

$$(154) \quad q_o^s(u_s, u_g, k, p_o^s) = q_o^{\text{tr}} \left[\frac{\alpha^{\text{oc}}(u_s, k, p_o^s)}{\alpha^{\text{oc}}(u_s, k, p_o^s)} \right]$$

The ratio $\frac{\alpha^{\text{oc}}(u_s, k, p_o^s)}{\alpha^{\text{oc}}(u_s, k, p_o^s)}$ must now be estimated. Compute $\alpha^{\text{oc}}(u_s, k, p_o^s)$ for each assumed $\alpha^{\text{oc}}(u_s, k, p_o^s)$. Enter the $k(u_s, u_g, q_o^s, \alpha_o^{\text{oc}})$ tables¹ with $\alpha^{\text{oc}}(u_s, k, p_o^s)$

¹ This can be done if Merrill's tables are used.

and q_0^S , and extract k and $\alpha^{tr}(u_g, k, p_0^S)$. Plot k against $\alpha^{tr}(u_g, k, p_0^S)$ on the corresponding shape curve to find unique values of k and $\alpha^{tr}(u_g, k, p_0^S)$. Enter the $k(u_s, u_g, q_0^S, \alpha^{oc})$ tables with these values and obtain $\alpha^{oc}(u_s, k, p_0^S)$. Then calculate $\alpha^{oc}(u_s, k, p_0^i)$. The remainder of the method is the same if the names "primary" and "secondary" in parentheses are interchanged, and the α dependence on p is interchanged, i.e., interchange p_0^S and p_0^i .

After a satisfactory representation of both minima is attained, fictitious elements may be calculated according to equations (75)-(78) using the fictitious r_g and i for each minimum, or with the equations utilizing the ψ -functions as listed below. Since A^m and B^m are known for the particular minimum in question, the formulae become

$$(155) \cot^2 i^f = - \frac{A^m + B^m}{\Phi_2(u, k)},$$

$$(156) \quad (r_g^f)^2 = \frac{B^m \sin i^f}{\Phi_1(u, k)},$$

$$(157) \quad r_s^f = k r_g^f,$$

$$(158) \quad L_s = \frac{1 - \lambda^{oc}}{\alpha_0^{oc}},$$

$$(159) \quad L_g = 1 - L_s.$$

Using these equations r_s^f, r_g^f, i^f and r_s^g, r_g^g, i^g are determined. Knowing these, compute the true elements in the same manner as for complete eclipses produced by spherical stars moving in eccentric orbits. The final stages of analysis here should also include a plot of the exact light curve as before, with subsequent modifications of the elements if necessary to attain a good representation of the observations.

Ellipsoidal Model

Often one must consider eclipsing systems for which the observed light between minima does not remain constant. The rigorous analysis of such systems, including all observable effects, is still one of the problems of contemporary astrophysical research, although much progress has been made on this problem in recent years. Eclipsing binaries of this type occur whenever the components are separated by less than about eight or ten radii. In such systems the stars are distorted by rotational and tidal effects, and exhibit variations in intensity over their surfaces as a result of differences in the value of local surface gravity and of the heating of the surface of each component by the radiation from the other. Exact representation of the entire light curve with any simple model is hopeless.

Russell¹ found, however, that light curves of these proximate eclipsing systems could be represented, with a sufficient degree of approximation to serve as a basis for a more refined solution, by a model, intermediate in complexity between the spherical and the actual physical model, consisting of a pair of similar prolate ellipsoids. This ellipsoidal model embodies the practical advantages of the spherical one, since it incorporates the use of the same functions. Furthermore, the procedure is largely the same after the actual light curve is "rectified" by transforming the observed light and phases into a curve which corresponds to that produced by a pair of spherical

¹ H. N. Russell, Ap.J., 36, 54, 1912.

Rectification for flux density was introduced early and was systematized by Russell² in 1912. Rectification for phase was developed much later by Merrill³. Upon utilizing the important contribution of phase rectification all the equations⁴ of the spherical model become applicable, while rectification for light alone gives transformed values to which most of the equations of the spherical model apply. Most of the constants necessary for the rectification are obtainable from the light variations outside minima. From the spherical elements derived from the rectified curve, those of the ellipsoidal model follow quite readily. For the purpose of simplicity, the combined rectification procedure will be developed here.

Before doing this it is quite beneficial to consider distorted stars and some of their properties, in order to grasp the complexity of this problem and, hopefully, an understanding of the theoretical ideas on which rectification is based. Such an excursion should give one fluency with the ideas concerning proximate systems to the extent that more advanced stages of analysis could be undertaken without undue effort. The writer found that familiarity with the material was also necessary before an estimate of the validity of the approximations involved could be made.

1 Z. Kopal has shown, in Proc. American Phil. Soc., 85, 399, 1942, that rectification can only be performed with limited accuracy within minima, because the terms neglected from the treatment of distorted systems in the process are appreciable in size compared to the terms which are retained.

2 H. N. Russell, Ap.J., 36, 64, 1912.

3 See H. N. Russell's paper; Ap.J., 108, 389, 1948.

4 This procedure including rectification for phase is presented in a form inapplicable to systems which show appreciable eccentricity. Most close systems have orbits that are nearly circular, which at least partially justifies this restriction. To deal with orbits with large eccentricity it appears necessary, at present, to use Russell's method which was developed in 1912 and noted above.

DISTORTED STARS

In view of the preceding idealized models which have been used to represent the components of eclipsing binary systems, one may legitimately ask "what physical models are used to represent eclipsing binary components, and how do they differ from the idealized models"? A complete answer to this question would lead beyond the intended scope of this investigation. However, the geometrical aspects of physical stellar models in proximate binary systems will be considered.

At the outset one would expect rotational and tidal forces that are present in binary systems to distort the components from spherical form, unless the stars were rigid. Since the stars are known to be gaseous, it is highly unlikely that they can be regarded as rigid bodies. The consideration of rotationally and tidally distorted components make it necessary to determine the geometrical aspects of each component as a function of its size, distribution of matter, rate of rotation, mass and separation from the other component. Therefore, a more general type of analysis than is applicable in cases of rigid bodies is required. The classical type of analysis¹ which will be followed was initiated by Clairaut and perfected by Legendre, Laplace and Radau. The procedure is quite general as long as the distortion is small, and it applies to all stellar models; for large distortions the analysis is not practical and should be superseded by a different method. In most eclipsing binary problems the method is quite adequate.

¹ For a general discussion see F. Tisserand, Traite de Mechanique Celeste (Paris, 1891), II, Ch. XVIII; H. Poincare, Lecons sur les Figures d'Equilibre (Paris, 1903), Ch. IV; or H. Jeffreys, The Earth (Cambridge, 1924), Ch. XIII.

The important recognition to be made is that observed ellipticities and apsidal motions of eclipsing binaries must depend on the density distribution of matter in the components. For any given density distribution it is desirable to be able to compute the corresponding ellipticities and apsidal motions in order to verify or disprove the distributions indicated by theories of stellar interiors. One aspect to be considered here is the theory of apsidal motion for binary stars composed of compressible fluids, in the limiting case where the stellar distortion is periodic in the orbital period, and is given at each instant by the equilibrium theory of the tides. The derived results should be applicable except in very close systems and in cases where the resonance of partial tides, which are generated by motion in an eccentric orbit, become salient. Apparently the latter is not likely to occur.¹ The procedure developed herein was initiated by Sterne² and is patterned after the earlier investigation of Russell.³ It surpasses the accuracy of Russell's, and later Cowling's⁴, treatment of the problem and is presented in a form applicable to all stellar models.

It is significant to note that a more general perturbation theory describing stellar configurations distorted by tidal and centrifugal forces has been advanced by Krogdahl.⁵ This method is a generalization, again applicable to any stellar model, of the ideas presented earlier

1 T. E. Sterne, M.N., 99, 452, 1939.

2 T. E. Sterne, M.N., 99, 451, 1939.

3 H. N. Russell, M.N., 88, 641, 1928.

4 T. G. Cowling, M.N., 98, 734, 1938.

5 W. Krogdahl, Ap.J., 96, 124, 1942.

on polytropes by Milne,¹ von Zeipel,² and Chandrasekhar.³ Certainly it does not transcend the difficulties encountered with partial tides, or achieve a greater accuracy than the classical analysis. Krogdahl's theory treats the equilibrium of rotationally and tidally distorted compressible fluid configurations characterized by an arbitrary pressure-density relation as a perturbation problem of an initially undistorted configuration, whereas the emphasis of the Clairaut theory is on the variations in the forms of the surfaces of constant pressure or density throughout the star. In practice the approach of Krogdahl yields information about static stars. On the other hand, the Clairaut approach ultimately gives information about the actual density distribution in the components of binary systems. It is in this sense, primarily, that one theory differs from the other. Chandrasekhar and Krogdahl⁴ have shown the formal equivalence of the two approaches.

Figure and Density Distribution

Stars are known to be gaseous with very low viscosity, and may therefore be regarded as almost perfectly fluid. A fundamental property of a perfect fluid is that it cannot be in a state of stress which is such that the mutual action between two adjacent parts is oblique to the common surface.⁵

1 E. A. Milne, M.N., 83, 118, 1923

2 H. von Zeipel, M.N., 84, 665, 684, 1924

3 S. Chandrasekhar, M.N., 93, 390, 449, 462, 539, 1933.

4 S. Chandrasekhar and W. Krogdahl, Ap.j., 96, 151, 1942.

5 One must keep in mind that hydrostatic derivations may not always be applicable in cases of astrophysical interest. See e.g. Z. Kopal, Proc. Nat. Acad. Sci., 27, 364, 1941.

Consider the fluid of volume v , within a closed surface S , moving with the fluid and containing the same particles, and thus the same mass. Let dv be a volume element of this enclosed region also moving with the fluid and containing the same particles and having a constant mass ρdv , where ρ is the fluid density. If \vec{v} is the velocity, and $\frac{d\vec{v}}{dt}$ the acceleration of the element, the rate of increase of momentum of the element is $\frac{d\vec{v}}{dt} \rho dv$. If \vec{F} is the impressed force per unit mass acting on the fluid at time t , the force on this element is $\vec{F} \rho dv$. The force on a surface element dA of surface S due to the pressure p of the surrounding fluid is $-p\hat{n}.dA$, where \hat{n} is the outward unit-normal at dA . Equating the rate of increase of momentum of the entire fluid to the total force acting on the fluid, one finds

$$\begin{aligned} \int \frac{d\vec{v}}{dt} \rho dv &= \int_v \rho \vec{F} dv - \int_S p \hat{n} dA \\ &= \int_v (\rho \vec{F} - \nabla p) dv. \end{aligned}$$

Since this equality must hold for the region bounded by any closed surface in the fluid, the integrands must be equal, or

$$(160) \quad \frac{1}{\rho} \nabla p = \vec{F} - \frac{d\vec{v}}{dt}.$$

This vector equation immediately gives Euler's three scalar equations of motion for the fluid.

If the impressed forces are conservative then \vec{F} may be written in terms of a potential function V , i.e., $\vec{F} = \nabla V$. Then

$$\text{or} \quad \frac{1}{\rho} \nabla p = \nabla V - \frac{\partial \vec{v}}{\partial t} - \vec{v} \cdot \nabla \vec{v},$$

$$(161) \quad \frac{1}{\rho} \nabla p = \nabla V - \frac{\partial \vec{v}}{\partial t} - \frac{1}{2} \nabla(\vec{v} \cdot \vec{v}) + \vec{v} \times \text{curl } \vec{v}.$$

The primary and secondary stars will be assumed to rotate with direct¹ constant angular velocities ω_1 and ω_2 about axes normal to the orbital plane. Because there is no certainty as to how actual stars rotate, these assumptions are probably justifiable on grounds of simplicity. The problem based on the assumption that ω does not vary with depth is the simplest, although the more general problem with varying ω has been tackled by Kopal.² Kopal concludes that the postulate of uniform ω is valid at least to a first approximation. Additional complications arising from inclined axes of rotation do not appear to be justifiable at present.³

Taking the z-axis as the axis of rotation of the primary one finds for the velocity of any fluid element

$$\vec{v} = \omega_1 (x\vec{j} - y\vec{i}),$$

and it readily follows that $\frac{\partial \vec{v}}{\partial t} = 0$, and $\text{curl } \vec{v} = 0$. Equation (160) becomes.

$$(162) \quad \frac{1}{\rho} \nabla p = \nabla V - \frac{1}{2} \nabla (\vec{v} \cdot \vec{v}) = \nabla (V + V_r),$$

where the disturbing potential V_r due to rotation is

$$(163) \quad V_r = \frac{\omega_1^2}{2} (x^2 + y^2).$$

Thus

$$(164) \quad dp = \rho d\psi,$$

where ψ is the total potential, gravitational, tidal and centrifugal.

1 Direct rotation has been observed spectroscopically.

2 Z. Kopal, Proc. Nat. Acad. Sci., 27, 359, 1941.

3 Krogdahl's investigation includes rotation about any axis, not necessarily one normal to the plane of motion; Ap.J., 96, 124, 1942.

From this equation it follows¹ that p is a function of Ψ , and that p is either the same everywhere or another function of Ψ . Over any surface of the star characterized by equal pressure and density

$$(165) \quad \Psi = \text{constant.}$$

Therefore, the immediate concern is to determine the total potential Ψ .

Let m_1 and m_2 denote the masses of the primary and secondary, respectively. Take the center of mass of the primary as the origin of coordinates with the x -axis in the direction of the secondary, and let R be the distance between centers of mass of the two components. The potential for the inverse square field of a particle, or a spherically symmetric configuration, of mass m_2 at a point P , in the neighborhood of the primary and at a distance r from O , is

$$V = \frac{Gm_2}{R[1 - 2\left(\frac{r}{R}\right)\sin\vartheta\cos\psi + \left(\frac{r}{R}\right)^2]} ,$$

where ϑ is the angle between the z -axis and line OP , and ψ is the azimuthal angle. Developing this in a series of Legendre functions P_j of the first kind gives

$$(166) \quad V = \frac{Gm_2}{R} \sum_{j=0}^{\infty} \left(\frac{r}{R}\right)^j P_j(\sin\vartheta\cos\psi) . . .$$

Since the secondary is not an arbitrary shaped body, but one which, like the primary, is in hydrostatic equilibrium, one might expect the approximation of replacing the secondary by a point mass to be useful for finding the distortion of the primary and the consequent disturbance of its external gravitational potential caused by this distortion. It has been shown^{2,3} that expression (166) is correct to terms of the order of $\frac{Gm_2}{R} \left(\frac{a_2}{R}\right)^5$, where a_2 is the mean radius of the secondary, when the dis-

1 H. Jeffreys, *The Earth* (Cambridge, 1924), p. 186.

2 S. Chandrasekhar, *M.N.*, 93, 449, 1933.

3 T. E. Sterne, *M.N.*, 99, 451, 1939.

tortion of the secondary is taken into account. Therefore to a reasonable degree of accuracy the above approximation is valid.

The first term in (166) is constant and does not give rise to forces; the second term causes an acceleration $\frac{Gm_2}{R^2}$ of all elements of the primary parallel to the line of centers and towards the secondary.¹ This acceleration is merely the entire relative orbital acceleration. The remaining terms contributing to the tidal potential, V_t , within the adopted degree of accuracy are

$$(167) \quad V_t = \frac{Gm_2}{R} \sum_{j=2}^4 \left(\frac{r}{R}\right)^j P_j(\sin \mathcal{J} \cos \psi),$$

where R is a function of time if the orbit is eccentric.

Now consider the potential at point $P(r, \mathcal{J}, \psi)$, which lies outside the primary due to a mass element dm' at an arbitrary point $Q(r', \mathcal{J}', \psi')$ of this star. The total potential at P will be

$$(168) \quad V_1 = G \int \frac{dm'}{R},$$

where G is the constant of gravitation, and the mass element dm' is

$$dm' = \rho r'^2 \sin \mathcal{J}' dr' d\mathcal{J}' d\psi',$$

where

$$R = (r^2 + r'^2 - 2rr' \cos \gamma)^{1/2},$$

and

$$\cos \gamma = \cos \mathcal{J} \cos \mathcal{J}' + \sin \mathcal{J} \sin \mathcal{J}' \cos(\psi - \psi'),$$

or

$$(169) \quad V_1 = G \int_0^{r_t} \int_0^\pi \int_0^{2\pi} \frac{\rho r'^2 \sin \mathcal{J}' dr' d\mathcal{J}' d\psi'}{R}.$$

Here r_t is the radius vector to the surface from the origin O at the center of mass of the primary. Expand $\frac{1}{R}$ in terms of Legendre polynomials $P_j(\cos \gamma)$ to obtain

¹ J. H. Jeans, Problems of Cosmogony and Stellar Dynamics, (Cambridge, 1918), p. 43.

$$V_{e1} = \frac{G}{r} \int_0^{r_t} \int_0^\pi \int_0^{2\pi} \rho r'^2 \sin \mathcal{J}' dr' d\mathcal{J}' d\psi' \\ + G \sum_{j=1}^{\infty} \frac{1}{r^{j+1}} \int_0^{r_t} \int_0^\pi \int_0^{2\pi} \rho P_j(\cos \gamma) r'^{j+2} \sin \mathcal{J}' dr' d\mathcal{J}' d\psi' ,$$

or

$$(170) \quad V_{e1} = \frac{G m_1}{r} + G \sum_{j=1}^{\infty} \frac{1}{r^{j+1}} V_{j1} ,$$

where

$$(171) \quad V_{j1} = \int_0^{r_t} \int_0^\pi \int_0^{2\pi} \rho P_j(\cos \gamma) r'^{j+2} \sin \mathcal{J}' dr' d\mathcal{J}' d\psi' .$$

Suppose the star is divided into strata separated by surfaces of equal density. Because of the completeness of the surface spherical harmonics S'_j the radius vector r' to the surface of equal density may be expanded as

$$(172) \quad r' = a' \left\{ 1 + \sum_{i=0}^{\infty} S'_i(a', \mathcal{J}', \psi') \right\} ,$$

a' being the parameter of surfaces of equal density. If the surfaces of equal density were all spherical and ρ a function of r' only, the evaluation of expression (171) would be trivial. In distorted bodies the density ρ will not remain constant within a spherical layer, but will in general depend on \mathcal{J}' and ψ' as well. If one considers a' to be the variable parameter instead of r' and lets a' vary from zero at the center to a , at the surface of the body, all the strata will be included. Let a' be chosen as the mean radius of a sphere having the same volume as that interior to a stratum of equal density corresponding to a' . Physically, a' may be regarded as the mean radius of a stratum of equal density. To the first degree the density ρ will be the same at all points lying within the layer defined by the two level surfaces with parameters a' and $a'+dd$. Denote the density within this layer by ρ' . Of course, ρ' will vary from one

stratum to another, indicating that it is a function of α' alone. Using this result the apparent integration difficulties disappear because

$$(173) \quad dr' = \frac{\partial r'}{\partial \alpha'} d\alpha',$$

and equation (171) becomes

$$(174) \quad V_{j_i} = \frac{1}{j+3} \int_0^{\alpha'} \rho' \frac{\partial}{\partial \alpha'} \left\{ \int_0^\pi \int_0^{2\pi} P_j(\cos \gamma) r'^{j+3} \sin \mathcal{J}' d\mathcal{J}' d\psi' \right\} d\alpha'.$$

Now substitute r' from (172) into (174) and neglect squares and cross products of the S_i' 's to obtain ¹

$$(175) \quad V_{j_i} = \frac{4\pi}{2j+1} \int_0^{\alpha'} \rho' \frac{\partial}{\partial \alpha'} \left\{ \alpha'^{j+3} S_j'(a', \mathcal{J}, \psi) \right\} d\alpha', \quad j=1, 2, \dots$$

$$(176) \quad V_{0_i} = \frac{4\pi}{3} \int_0^{\alpha'} \rho' \frac{\partial}{\partial \alpha'} \left\{ \alpha'^3 [1 + 3 S_0'(a', \mathcal{J}, \psi)] \right\} d\alpha' = m_i,$$

where the following relation has been used

$$\int_0^\pi \int_0^{2\pi} P_j(\cos \gamma) S_i'(a', \mathcal{J}, \psi) \sin \mathcal{J}' d\mathcal{J}' d\psi' = \begin{cases} 0, & i \neq j \\ \frac{4\pi}{2j+1} S_j'(a', \mathcal{J}, \psi), & i=j. \end{cases}$$

If v' is the volume interior to the arbitrary stratum which corresponds to the parameter α' , then

$$\begin{aligned} v' &= \int_0^{\alpha'} \int_0^\pi \int_0^{2\pi} r'^2 \sin \mathcal{J}' dr' d\mathcal{J}' d\psi' \\ &= \frac{1}{3} \int_0^{\alpha'} \frac{\partial}{\partial \alpha'} \left\{ \int_0^\pi \int_0^{2\pi} r'^3 \sin \mathcal{J}' d\mathcal{J}' d\psi' \right\} d\alpha'. \end{aligned}$$

Upon substituting for r' from equation (172) and again neglecting squares and cross products of S_i' one finds

$$v' = \frac{4\pi}{3} \int_0^{\alpha'} \frac{\partial}{\partial \alpha'} \left\{ \alpha'^3 [1 + 3 S_0'(a', \mathcal{J}, \psi)] \right\} d\alpha'.$$

¹ The adopted procedure is closely approximate if the surface differs but little from a sphere.

If, for the parameter a' , one takes the radius of the sphere having the same volume as that contained within this layer,

$$v' = \frac{4\pi a'^3}{3} = \frac{4\pi}{3} \int_0^{a'} \frac{\partial}{\partial a'} \{a'^3\} da'$$

and this requires

$$\int_0^{a'} \frac{\partial}{\partial a'} \{a'^3 S'_0(a', \mathcal{J}, \psi)\} da' = a'^3 S'_0(a', \mathcal{J}, \psi) = 0$$

for all a' . The result can only be true if $S'_0(a', \mathcal{J}, \psi) = 0$, which holds for arbitrary \mathcal{J} and ψ . Therefore equation (172) may be written

$$(177) \quad r' = a' \left\{ 1 + \sum_{i=1}^{\infty} S'_i(a', \mathcal{J}, \psi) \right\}.$$

Equation (176) now becomes

$$(178) \quad m_1 = 4\pi \int_0^{a_1} \rho' a'^2 da'.$$

Since the origin is taken at the center of mass of the primary star the term including $S'_1(a', \mathcal{J}, \psi)$ must vanish¹. Thus,

$$\int_0^{a_1} \rho' \frac{\partial}{\partial a'} \{a'^4 S'_1(a', \mathcal{J}, \psi)\} da' = 0,$$

and the potential finally becomes

$$(179) \quad V_{e1} = \frac{Gm_1}{r} + G \sum_{j=2}^{\infty} \frac{4\pi}{(2j+1)r^{j+1}} \int_0^{a_1} \rho' \frac{\partial}{\partial a'} \{a'^{j+3} S'_j(a', \mathcal{J}, \psi)\} da'.$$

Similarly, one may calculate the potential due to a stratum of the type considered previously, at a point interior to the layer. Equation (168) still applies, but if the layer extends from r_s to r_t where $r_t > r_s$ then equation (169) becomes

$$(180) \quad V_{ii} = G \int_{r_s}^{r_t} \int_0^{\pi} \int_0^{2\pi} \frac{\rho r'^2}{R} \sin \mathcal{J}' dr' d\mathcal{J}' d\psi'.$$

¹ See F. Tisserand, Traite de Mechanique Celeste (Paris, 1891), II, p. 295; or T. M. McRoberts, Spherical Harmonics (New York, 1948), p. 174.

Again $\frac{1}{R}$ may be expanded in terms of Legendre polynomials $P_j(\cos\gamma)$ to give

$$(181) \quad V_{ii} = G \sum_{j=0}^{\infty} r^j V'_{j,i},$$

where

$$(182) \quad V'_{j,i} = \int_{r_s}^{r_t} \int_0^{\pi} \int_0^{2\pi} \rho P_j(\cos\gamma) r'^{1-j} \sin\gamma' dr' d\gamma' d\psi'.$$

Upon introducing the variable a' for r' by equation(173), and letting the parameters a , and a_s , correspond to r_t and r_s , respectively, one finds

$$(183) \quad V'_{j,i} = \frac{1}{2-j} \int_{a_s}^{a_1} \rho' \frac{\partial}{\partial a'} \left\{ \int_0^{\pi} \int_0^{2\pi} P_j(\cos\gamma) r'^{2-j} \sin\gamma' d\gamma' d\psi' \right\} da', \text{ if } j \neq 2,$$

or

$$(184) \quad V'_{2,i} = \int_{a_s}^{a_1} \rho' \frac{\partial}{\partial a'} \left\{ \int_0^{\pi} \int_0^{2\pi} P_2(\cos\gamma) \ln r' \sin\gamma' d\gamma' d\psi' \right\} da', \text{ if } j=2.$$

Then expanding r' from equation (177) these give

$$(185) \quad V'_{j,i} = \frac{4\pi}{2j+1} \int_{a_s}^{a_1} \rho' \frac{\partial}{\partial a'} \left\{ a'^{2-j} S'_j(a', \gamma, \psi) \right\} da',$$

$$(186) \quad V'_{0,i} = 4\pi \int_{a_s}^{a_1} \rho' a' da',$$

$$V'_{2,i} = \frac{4\pi}{5} \int_{a_s}^{a_1} \rho' \frac{\partial}{\partial a'} \left\{ S'_2(a', \gamma, \psi) \right\} da'.$$

Hence, $V'_{2,i}$ also comes from equation (185) so the expression for this potential becomes

$$(187) \quad V_{ii} = 4\pi G \int_{a_s}^{a_1} \rho' a' da' + 4\pi G \sum_{j=1}^{\infty} \frac{r^j}{(2j+1)} \int_{a_s}^{a_1} \rho' \frac{\partial}{\partial a'} \left\{ a'^{2-j} S'_j(a', \gamma, \psi) \right\} da'.$$

The potential at a point Q, on the level surface corresponding to the parameter α , inside the spheroidal primary star of heterogeneous density may be found by considering the spheroid separated

into two parts by the level surface through Q. This potential is the sum of the potential V_{e_1} due to the interior spheroid and the potential V_{ii} due to the spheroidal stratum outside this level surface. The potential at Q becomes

$$(188) V_{ii} + V_{e_1} = \frac{4\pi G}{r} \int_0^a \rho' a'^2 da' + 4\pi G \sum_{j=2}^{\infty} \frac{1}{(2j+1)r^{j+1}} \int_0^a \rho' \frac{\partial}{\partial a'} \{a'^{j+3} S'_j(a', \vartheta, \psi)\} da' \\ + 4\pi G \int_a^{a_1} \rho' a' da' + 4\pi G \sum_{j=1}^{\infty} \frac{r^j}{(2j+1)} \int_a^{a_1} \rho' \frac{\partial}{\partial a'} \{a'^{2-j} S'_j(a', \vartheta, \psi)\} da' .$$

If one assumes that the primary star is in hydrostatic equilibrium throughout, then Ψ_1 must be constant over all surfaces of equal density

$$(189) \quad \Psi_1 = V_{ii} + V_{e_1} + V_t + V_r = \text{constant},$$

where now V_r from equation (163) is to be expressed in terms of spherical polar co-ordinates. Substituting in equation (189) the value of r' given by

$$(190) \quad r' = a' \left\{ 1 + \sum_{j=1}^{\infty} S'_j(a', \vartheta, \psi) \right\} ,$$

it is apparent that Ψ_1 reduces to a function of a' alone. In the scheme of approximation adopted here, whenever r , the distance to a layer under consideration, appears multiplied by a small quantity such as S'_j , ω^2 , or $1/R^3$, etc., it will be replaced by a_1 . The above substitution gives

$$\int_a^{a_1} \rho' a' da' + \sum_{j=1}^{\infty} \frac{a_1^j}{2j+1} \int_a^{a_1} \rho' \frac{\partial}{\partial a'} \{a'^{2-j} S'_j(a', \vartheta, \psi)\} da' \\ + \frac{1}{a} \left\{ 1 - \sum_{j=1}^{\infty} S'_j(a', \vartheta, \psi) \right\} \int_0^a \rho' a'^2 da' + \sum_{j=1}^{\infty} \frac{1}{(2j+1)a^{j+1}} \int_0^a \rho' \frac{\partial}{\partial a'} \{a'^{j+3} S'_j(a', \vartheta, \psi)\} da' \\ + \frac{m_2}{4\pi R} \sum_{j=2}^4 \left(\frac{a_1}{R}\right)^j P_j(\sin \vartheta \cos \psi) - \frac{1}{12\pi G} \omega_1^2 a_1^2 P_2(\cos \vartheta) = \text{constant} .$$

Now neglect constant terms which may give rise to an expansion of the configuration as a whole, but cannot distort the body from a spherical shape. Hence,

$$\begin{aligned} & \sum_{j=1}^{\infty} \left(-S'_j(a, \vartheta, \psi) \int_0^a \rho' a'^2 da' + \frac{1}{(2j+1)a^{j+1}} \int_0^a \rho' \frac{\partial}{\partial a'} \{ a'^{j+3} S'_j(a', \vartheta, \psi) \} da' \right. \\ & \quad \left. + \frac{a'^j}{2j+1} \int_a^{a_1} \rho' \frac{\partial}{\partial a'} \{ a'^{2-j} S'_j(a', \vartheta, \psi) \} da' \right) \\ & = \frac{-m_2}{4\pi R} \sum_{j=2}^4 \left(\frac{a}{R} \right)^j P_j(\sin \vartheta \cos \psi) + \frac{\omega^2 a^2 P_2(\cos \vartheta)}{12\pi G} = \frac{-1}{4\pi G} \sum_{j=1}^{\infty} a^j Y'_j(\vartheta, \psi), \end{aligned}$$

where in the last equality it is assumed that the disturbing terms may be developed as a series of spherical harmonics $r^j Y'_j(\vartheta, \psi)$ and $Y'_j(\vartheta, \psi)$ is a surface spherical harmonic of degree j .

There results for each j an equation

$$\begin{aligned} (191) \quad & \frac{S'_j(a, \vartheta, \psi)}{a} \int_0^a \rho' a'^2 da' + \frac{1}{(2j+1)a^{j+1}} \int_0^a \rho' \frac{\partial}{\partial a'} \{ a'^{j+3} S'_j(a', \vartheta, \psi) \} da' \\ & + \frac{a^j}{2j+1} \int_a^{a_1} \rho' \frac{\partial}{\partial a'} \{ a'^{2-j} S'_j(a', \vartheta, \psi) \} da' = \frac{a^j}{4\pi G} Y'_j(\vartheta, \psi), \end{aligned}$$

where

$$(192) \quad Y'_j = 0 \quad j \neq 2, 3, 4,$$

$$(193) \quad Y'_2 = \frac{Gm_2}{R^3} P_2(\sin \vartheta \cos \psi) - \frac{\omega^2}{3} P_2(\cos \vartheta),$$

$$(194) \quad Y'_3 = \frac{Gm_2}{R^4} P_3(\sin \vartheta \cos \psi),$$

$$(195) \quad Y'_4 = \frac{Gm_2}{R^5} P_4(\sin \vartheta \cos \psi).$$

Expression (191) gives a separate integral equation for each S_j' ($j \geq 1$).

However, an easier expression with which to work may be obtained by multiplying both sides of the equation by a^{j+1} , by differentiating with respect to a and finally by dividing by a^{2j} . Then

$$(196) \quad -\left\{ j \frac{S_j'(a, \mathcal{J}, \psi)}{a^{j+1}} + \frac{\partial S_j'(a, \mathcal{J}, \psi)}{a^j \partial a} \right\} \int_0^a \rho' a'^2 da' \\ + \int_a^{a_1} \rho' \frac{\partial}{\partial a'} \left\{ \frac{S_j'(a', \mathcal{J}, \psi)}{a'^{j-2}} \right\} da' = -\frac{(2j+1)}{4\pi G} Y_j'(\mathcal{J}, \psi).$$

Differentiate again with respect to a to obtain

$$(197) \quad \frac{\partial^2 S_j'(a, \mathcal{J}, \psi)}{\partial a^2} - j(j+1) \frac{S_j'(a, \mathcal{J}, \psi)}{a^2} + \frac{6\rho'}{F(a)} \left\{ a^2 \frac{\partial S_j'(a, \mathcal{J}, \psi)}{\partial a} + a S_j'(a, \mathcal{J}, \psi) \right\} = 0,$$

where by definition

$$(198) \quad F(a) = 3 \int_0^a \rho' a'^2 da'.$$

Now introduce the new variable

$$(199) \quad \eta_j = \frac{a}{S_j'(a, \mathcal{J}, \psi)} \frac{\partial S_j'(a, \mathcal{J}, \psi)}{\partial a}.$$

Equation (197) then becomes

$$(200) \quad a \frac{d\eta_j}{da} + \eta_j^2 - \eta_j + \frac{6\rho'}{\bar{\rho}} \{ \eta_j + 1 \} = j(j+1),$$

where ρ is the density at distance a from the origin and $\bar{\rho}$ is the mean density interior to a in a spherical distribution of matter, not necessarily in equilibrium, in which the density is the same function of a that ρ , in the actual distorted star, is of a .¹

At the center of the star $a = 0$ and $\eta_j = j - 2$. Poincaré² has shown that for any internal density-distribution for which $\frac{d\rho}{da}$ is negative

¹ T. E. Sterne, Proc. Nat. Acad. Sci., 27, 95, 1941.

² H. Poincaré, Leçons sur les Figures d'Equilibre (Paris, 1902), Ch. IV.

from the center to the surface, $j - 2 \leq \eta_j(a) \leq j + 1$. Callandreau¹ proved further that if both $\frac{d\rho}{da}$ and $\frac{d^2\rho}{da^2}$ are negative throughout the interior, $\eta_j(a)$ increases steadily from the center to the surface. If the entire mass is concentrated at the center of the star, giving the Roche model, then $\eta_j(a) = j - 1$ at all points where $a \neq 0$. On the other hand, if the entire mass is concentrated in a thin surface shell, the solution of equation(200) reduces to $\eta_j(a) = -1$, regardless of the value of j . Therefore, making no assumption about the internal density-distribution of a distorted star, one finds the absolute limits of η_j at the surface to be $-1 \leq \eta_j(a) \leq j + 1$.²

For any model, equation(200) may be solved, numerically or otherwise, in order to find the value $\eta_j(a)$ of η_j at the boundary $a = a$.³ The equations for the superficial distortion which have been derived may be used in the computation of the theoretical light curves within minima when the eclipsing systems are distorted. This problem has been investigated in detail by Kopal⁴ and will not be discussed here.

1 Callandreau, Comptes Rendus, 100, 1024, 1885.

2 Z. Kopal, Harvard Obs. Mon., 6, 117, 1946.

3 Since the purpose here is to develop the general theory of distorted stars, nothing specific can be concluded in regard to particular models. For applications of the theory one is referred to the following papers: L. Motz, "The Apsidal Motion In Stars Built on A Point Source Convective-Core Model With Varying Guillotine Factor", Ap.J., 94, 253, 1941; L. Motz, "The Apsidal Motion of Giant Binary Stars", Ap.J., 112, 434, 1950; and G. Keller, "On The Physical And Chemical Composition Of The Sun", Ap.J., 108, 347, 1948. The earlier paper by Motz is particularly significant because the equations of distortion and apsidal motion are put into a form more suitable for numerical integration.

4 Z. Kopal, Proc. American Phil. Soc., 85, 399, 1942.

When $a = a_1$, the mean radius of the star, the disturbing potential introduces the terms

$$(201) \quad S_j'(a_1, \vartheta, \psi) = \Delta_j \frac{a_1^{j+1}}{Gm_1} Y_j'(\vartheta, \psi),$$

where $\Delta_j = \frac{(2j+1)}{j + \eta_j(a_1)}$ and $\eta_j(a_1)$ is the value of η_j at the outer surface.

The external shape of the star then becomes

$$(202) \quad r = a_1 \left\{ 1 + \sum_{j=2}^4 S_j'(a_1, \vartheta, \psi) \right\}$$

In applications it is often desirable to separate the rotational and tidal terms. This is readily accomplished by re-defining the Y 's as follows,

$$(203) \quad Y_j = 0 \quad j \neq 2, 3, 4,$$

$$(204) \quad Y_2 = \frac{Gm_2}{R^3} P_2(\sin \vartheta \cos \psi), \quad Y_r = -\frac{\omega^2}{3} P_2(\cos \vartheta),$$

$$(205) \quad Y_3 = Y_3' = \frac{Gm_2}{R^4} P_3(\sin \vartheta \cos \psi),$$

$$(206) \quad Y_4 = Y_4' = \frac{Gm_2}{R^5} P_4(\sin \vartheta \cos \psi),$$

and introducing the additional term

$$(207) \quad S_r(a_1, \vartheta, \psi) = -\Delta_2 \frac{\omega^2 a_1^2}{3m_1 G} P_2(\cos \vartheta) = \frac{\Delta a_1^3}{m_1 G} Y_r'.$$

Then equation(202) becomes

$$(208) \quad r = a_1 \left\{ 1 + S_r(a_1, \vartheta, \psi) + \sum_{j=2}^4 S_j(a_1, \vartheta, \psi) \right\}.$$

The portion of the external potential which is contributed by the distortion of the primary is easily obtained. Set $a = a_1$ in equation (196) and utilize (201) and (179), the last of which may be written

$$(209) \quad V_{e1} = \frac{4\pi G}{r} \int_0^{a_1} \rho' a'^2 da' + \sum_{j=2}^4 U_{j1} + U_{r1},$$

the second term being due to the distortion. In this expression

$$(210) \quad U_{j1} = \frac{a_1^{2j+1}}{r^{j+1}} \frac{j+1-\eta_j(a_1)}{(j+\eta_j(a_1))} Y_j, \quad U_{r1} = \frac{a_1^5 (3-\eta_2(a_1))}{r^3 (2+\eta_2(a_1))} Y_r,$$

where the subscript j denotes the particular harmonic approximation and the second subscript, 1 , denotes the primary star.

The tidal and rotational distortions easily follow from the preceding results. The fractional elongation due to rotation is

$$(211) \quad \frac{r-a_1}{a_1} = S_r$$

and that due to tides is

$$(212) \quad \frac{r-a_1}{a_1} = S_2 + S_3 + S_4.$$

For a component of a double star the total fractional distortion is

$$(213) \quad \sigma(\mathcal{J}, \psi) = S_2 + S_3 + S_4 + S_r,$$

where, following Chandrasekhar¹, $\sigma(\mathcal{J}, \psi)$ is introduced to facilitate the computation of ellipticities.

In the equatorial plane (the xy plane), for the primary star,

$$(214) \quad \sigma_1\left(\frac{\pi}{2}, \psi\right) = \Delta_2 \frac{\omega^2 a_1^3}{6Gm_1} + \frac{m_2}{m_1} \sum_{j=2}^4 \Delta_j \left(\frac{a_1}{R}\right)^{j+1} P_j(\cos \psi);$$

¹ S. Chandrasekhar, M.N., 93, 468, 1933. Physically, $\sigma(\mathcal{J}, \psi)$ represents the fractional deviation from a spherical volume having the same volume as the distorted configuration. This is immediately obvious in the present analysis where the expansion of the rotating star as a whole has been accounted for.

in the principle meridian (the xz plane) through the center of mass of the secondary

$$(215) \quad \sigma_1(\mathcal{J}, 0) = -\Delta_2 \frac{\omega_1^2 a_1^3}{3Gm_1} P_2(\cos \mathcal{J}) + \frac{m_2}{m_1} \sum_{j=2}^4 \Delta_j \left(\frac{a_1}{R}\right)^{j+1} P_j(\sin \mathcal{J});$$

and in the diametral plane (the yz plane)¹

$$(216) \quad \sigma_1(\mathcal{J}, \frac{\pi}{2}) = -\Delta_2 \frac{\omega_1^2 a_1^3}{3Gm_1} P_2(\cos \mathcal{J}) + \frac{m_2}{m_1} \sum_{j=2}^4 \Delta_j \left(\frac{a_1}{R}\right)^{j+1} P_j(0).$$

The ellipticities² $\epsilon_e, \epsilon_m,$ and ϵ_d in the equatorial plane, in the principal meridian, and in the diametral plane, respectively, become

$$(217) \quad \epsilon_e = \sigma_1\left(\frac{\pi}{2}, 0\right) - \sigma_1\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{3}{2} \frac{m_2}{m_1} \left(\frac{a_1}{R}\right)^3 \Delta_2,$$

$$(218) \quad \epsilon_m = \sigma_1\left(\frac{\pi}{2}, 0\right) - \sigma_1(0, 0) = \frac{a_1^3}{m_1} \left(\frac{3}{2} \frac{m_2}{R^3} + \frac{\omega_1^2}{3G}\right) \Delta_2,$$

$$(219) \quad \epsilon_d = \sigma_1\left(\frac{\pi}{2}, \frac{\pi}{2}\right) - \sigma_1(0, 0) = \frac{\omega_1^2 a_1^3}{2Gm_1} \Delta_2.$$

where quantities of the order of $\left(\frac{a_1}{R}\right)^4$ and $\left(\frac{a_1}{R}\right)^5$ are neglected, so the configurations become ellipsoids.

In view of a succeeding discussion of gravity darkening it is important to realize how the surface gravity of each component depends on its size, rate of rotation, distribution of density, mass, and distance from the other component.

The total potential Ψ_1 , in the neighborhood of the primary star, 1, is

$$(220) \quad \Psi_1 = \frac{Gm_1}{r} + \sum_{j=2}^4 U_{j1} + \sum_{j=2}^4 r^j Y_j + U_{r1} + r^2 Y_r,$$

where the last terms are contributed by the rotational and tidal

1 It should be noticed that usually $\sigma\left(\frac{\pi}{2}, 0\right) > \sigma\left(\frac{\pi}{2}, \pi\right) > \sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) > \sigma(0, 0)$.

2 The term "ellipticity" as used in eclipsing binary work is analogous to the term "oblateness", but the former is used in a more general sense. Sometimes, however, ellipticity may refer to eccentricity. Each case must be examined individually.

potential. The surface gravity of the primary is

$$(221) \quad g = -\frac{d\psi}{dn},$$

evaluated at the surface, where dn is in the direction of the outward normal. Since ψ and hence the acceleration of gravity is continuous at the surface, only the value of ψ outside is needed. Sterne¹ remarks that the outward normal makes a small angle with the radius vector r which is of the order of magnitude of $S_2(a, \mathcal{J}, \psi)$ or smaller, thus permitting the cosine of this angle to differ at most from unity by a quantity of the order of magnitude of the square of $S_2(a, \mathcal{J}, \psi)$. To this degree of approximation

$$(222) \quad g = -\frac{d\psi}{dr},$$

again evaluated at the surface $r = a_1 \left\{ 1 + S_r + \sum_{j=2}^4 S_j \right\}$. Use expressions (201), (207) and (210) to obtain the following

$$(223) \quad \frac{g - g_0}{g_0} = -[1 + \eta_2(a_1)]S_r - \sum_{j=2}^4 [1 + \eta_j(a_1)]S_j,$$

where

$$(224) \quad g_0 = \frac{Gm_1}{a_1^2}.$$

¹ T. E. Sterne, Proc. Nat. Acad. Sci., 27, 93, 1941.

Apsidal Motion

The secular motion of the apse may now be investigated on the basis of the preceding development. Immediately, it is apparent that one must be cognizant of the particular form which the equations of motion obtained from general classical perturbation theory assume in order to apprehend a way to utilize the foregoing results. Convenience in derivation suggests the following approach.

Adopt as a fundamental hypothesis that an unaccelerated and non-rotating reference frame in Euclidean space can be chosen. Let the position vectors of the primary and secondary stars be X_{i1} and X_{i2} ($i = 1, 2, 3$), respectively, with respect to such a rectangular system which has its origin at 0. Represent the total disturbing potential due to the primary by $U_1(X_{i2} - X_{i1})$ and that due to the secondary by $U_2(X_{i2} - X_{i1})$, where $U_1 = \sum_{j=2}^4 U_{j1} + U_{r1}$ from equation (210), and from a similar equation for the secondary $U_2 = \sum_{j=2}^4 U_{j2} + U_{r2}$. The force function V for the system becomes

$$(225) \quad V = \frac{G m_1 m_2}{|X_{i2} - X_{i1}|} + m_2 U_1(X_{i2} - X_{i1}) + m_1 U_2(X_{i2} - X_{i1}).$$

The equations of motion become,

$$(226) \quad m_1 \ddot{X}_{i1} = \frac{\partial V}{\partial X_{i1}} = \frac{G m_1 m_2 [X_{i2} - X_{i1}]}{|X_{i2} - X_{i1}|^3} + m_2 \frac{\partial U_1(X_{i2} - X_{i1})}{\partial X_{i1}} + m_1 \frac{\partial U_2(X_{i2} - X_{i1})}{\partial X_{i1}},$$

$$(227) \quad m_2 \ddot{X}_{i2} = \frac{\partial V}{\partial X_{i2}} = \frac{-G m_1 m_2 [X_{i2} - X_{i1}]}{|X_{i2} - X_{i1}|^3} + m_2 \frac{\partial U_1(X_{i2} - X_{i1})}{\partial X_{i2}} + m_1 \frac{\partial U_2(X_{i2} - X_{i1})}{\partial X_{i2}},$$

where parentheses denote functional dependence and brackets denote factors.

Introduce center of mass coordinates to further simplify these expressions. Let x be the position vector from 0 to the center of mass of the components such that

$$(228) \quad x_i = \frac{m_1 X_{i1} + m_2 X_{i2}}{m_1 + m_2} \quad i = 1, 2, 3,$$

and let

$$(229) \quad X_i = X_{i2} - X_{i1} \quad i = 1, 2, 3,$$

be the position vector of the secondary with respect to the primary.

Then the distance R between the centers of mass of the primary and secondary components becomes $R^2 = \sum_i X_i^2 = \sum_i [X_{i2} - X_{i1}]^2$. Upon introducing these variables ∇ becomes a function of X_i only. The equations of relative motion take the form

$$\ddot{X}_i = \frac{1}{m_2} \frac{\partial \nabla}{\partial X_{i2}} - \frac{1}{m_1} \frac{\partial \nabla}{\partial X_{i1}},$$

or since $\frac{\partial \nabla}{\partial X_{i2}} = \frac{\partial \nabla}{\partial X_i}$ and $\frac{\partial \nabla}{\partial X_{i1}} = -\frac{\partial \nabla}{\partial X_i}$,
one has

$$(230) \quad \ddot{X}_i = \frac{m_1 + m_2}{m_1 m_2} \frac{\partial \nabla}{\partial X_i}.$$

Use equation (225) and (230) to get

$$\ddot{X}_i = \frac{m_1 + m_2}{m_1 m_2} \left\{ -\frac{G m_1 m_2}{R^3} X_i + \left[m_2 \frac{\partial U_1}{\partial X_i}(X_i) + m_1 \frac{\partial U_2}{\partial X_i}(X_i) \right] \right\},$$

or

$$(231) \quad \ddot{X}_i + \frac{\mu X_i}{R^3} = \frac{\partial}{\partial X_i} \left\{ \left[\frac{m_1 + m_2}{m_1} \right] U_1(X_i) + \left[\frac{m_1 + m_2}{m_2} \right] U_2(X_i) \right\},$$

where $\mu = G(m_1 + m_2)$ and the disturbing function \mathcal{Q} is

$$(232) \quad \mathcal{Q} = \left\{ \left[\frac{m_1 + m_2}{m_1} \right] U_1(X_i) + \left[\frac{m_1 + m_2}{m_2} \right] U_2(X_i) \right\}.$$

From the variation of the orbital elements the following expression emerges for the rate of apsidal motion¹

$$(233) \quad \dot{\omega} = \frac{\sqrt{1-e^2}}{e\sqrt{\mu A}} \frac{\partial \mathcal{R}}{\partial e} + \frac{\tan \frac{i}{2}}{\sqrt{(1-e^2)}\sqrt{\mu A}}$$

which for motion in a plane reduces to

$$(234) \quad \dot{\omega} = \frac{\sqrt{1-e^2}}{e\sqrt{\mu A}} \frac{\partial \mathcal{R}}{\partial e}$$

where A is the semi-major axis of the relative orbit. In general the expression for \mathcal{R} is a function of ϵ , the mean longitude at the epoch, of ω , the longitude of periastron, of A , and of e , as well as of time. Here, since only $\frac{\partial \mathcal{R}}{\partial e}$ is of concern, \mathcal{R} may be taken as a function of the mean anomaly M , of e and of A .²

The total disturbing potential $V_t - V_r$ which acts on the primary is

$$\left(\frac{Gm_2}{R^3} r^2 P_2(\sin \mathcal{J} \cos \psi) - \frac{\omega_1^2}{3} r^2 P_2(\cos \mathcal{J}) \right)_{j=2} + \left(\frac{Gm_2}{R^4} r^3 P_3(\cos \mathcal{J}) \right)_{j=3} + \left(\frac{Gm_2}{R^5} r^4 P_4(\cos \mathcal{J}) \right)_{j=4},$$

where the subscripts j indicate the j th order harmonic distortion.

From equation (210) the respective terms contributed to the external gravitational potential of the primary star are

$$(235) \quad U_1 = \left(2k_{2,1} \frac{a_1^5}{r^3} \left[\frac{Gm_2}{R^3} P_2(\sin \mathcal{J} \cos \psi) - \frac{\omega_1^2}{3} P_2(\cos \mathcal{J}) \right] \right)_{j=2} \\ + \left(2k_{3,1} \frac{a_1^7}{r^4} \left[\frac{Gm_2}{R^4} P_3(\sin \mathcal{J} \cos \psi) \right] \right)_{j=3} + \left(2k_{4,1} \frac{a_1^9}{r^5} \left[\frac{Gm_2}{R^5} P_4(\sin \mathcal{J} \cos \psi) \right] \right)_{j=4},$$

¹ Cf. H. C. Plummer, Dynamical Astronomy, (Cambridge, 1916), p. 147.

² T. E. Sterne, M.N., 99, 457, 1939.

where

$$(236) \quad k_{2,1} = \frac{3 - \eta_2(a_1)}{4 + 2\eta_2(a_1)},$$

$$(237) \quad k_{3,1} = \frac{4 - \eta_3(a_1)}{6 + 2\eta_3(a_1)},$$

$$(238) \quad k_{4,1} = \frac{5 - \eta_4(a_1)}{8 + 2\eta_4(a_1)}.$$

The first subscript on the k 's indicates the j th order harmonic and the second denotes the component in question; also the $\eta_j(a_1)$'s are the solutions of equation (200) when $j = 2, 3, 4$ and $a' = a_1$.

This potential acts on the secondary when $\mathcal{J} = \frac{\pi}{2}$, $\psi = 0$, and produces an acceleration of the secondary

$$(239) \quad \left(\frac{\partial U_1}{\partial r} \right)_{r=R} = - \left(6k_{2,1} \frac{a_1^5}{R^4} \left[\frac{Gm_2}{R^3} + \frac{\omega_1^2}{6} \right] \right)_{j=2} - \left(8k_{3,1} \frac{a_1^7}{R^5} \frac{Gm_2}{R^4} \right)_{j=3} - \left(10k_{4,1} \frac{a_1^9}{R^6} \frac{Gm_2}{R^5} \right)_{j=4},$$

where the Legendre coefficients have been evaluated using the expressions below.

$$P_2(x) = \frac{1}{2} (3x^2 - 1),$$

$$P_3(x) = \frac{1}{2} (5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3).$$

Since the acceleration is radial, the \mathcal{J} and ψ terms may be evaluated independent of r , and doing this fixes the direction under consideration. The above acceleration is directed toward the center of mass of the primary. Corresponding to this acceleration is the potential function

$$(240) \quad U_1 = \left(k_{2,1} a_1^5 \left[\frac{Gm_2}{R^6} + \frac{\omega_1^2}{3R^3} \right] \right)_{j=2} + \left(k_{3,1} a_1^7 \frac{Gm_2}{R^8} \right)_{j=3} + \left(k_{4,1} a_1^9 \frac{Gm_2}{R^{10}} \right)_{j=4}.$$

Similarly, the total distortion of the secondary star 2 due to its rotation with angular velocity ω_2 and to tidal effects of the primary gives rise to an acceleration of the primary

$$(241) \quad \left(\frac{\partial U_2}{\partial r} \right)_{r=R} = - \left(6k_{2,2} \frac{a_2^5}{R^4} \left[\frac{Gm_1}{R^3} + \frac{\omega_2^2}{6} \right]_{j=2} \right) - \left(8k_{3,2} \frac{a_2^7}{R^5} \frac{Gm_1}{R^4} \right)_{j=3} - \left(10k_{4,2} \frac{a_2^9}{R^6} \frac{Gm_1}{R^5} \right)_{j=4},$$

where a_2 is the mean radius of the secondary and

$$(242) \quad k_{2,2} = \frac{3 - \eta_2(a_2)}{4 + 2\eta_2(a_2)},$$

$$(243) \quad k_{3,2} = \frac{4 - \eta_3(a_2)}{6 + 2\eta_3(a_2)},$$

$$(244) \quad k_{4,2} = \frac{5 - \eta_4(a_2)}{8 + 2\eta_4(a_2)}.$$

The corresponding potential function is

$$(245) \quad U_2 = \left(k_{2,2} a_2^5 \left[\frac{Gm_1}{R^6} + \frac{\omega_2^2}{3R^3} \right] \right)_{j=2} + \left(k_{3,2} a_2^7 \frac{Gm_1}{R^8} \right)_{j=3} + \left(k_{4,2} a_2^9 \frac{Gm_1}{R^{10}} \right)_{j=4}.$$

The terms R^{-3} , R^{-6} , R^{-8} and R^{-10} appearing in equations (240) and (245) will also appear in \mathcal{R} , and must be expanded in terms of M , e , and A .

Upon expanding $\frac{A}{R}$ in a Fourier series one readily finds¹

$$(246) \quad \frac{A}{R} = 1 + 2 \sum_{l=1}^{\infty} J_l(le) \cos lM,$$

where $J_l(le)$ is the Bessel coefficient of order l and argument le .

In this expansion the relation below have been employed

$$\frac{A}{R} = (1 - e \cos E)^{-1},$$

$$E - e \sin E = M,$$

¹ H. C. Plummer, Dynamical Astronomy, (Cambridge, 1918), p. 35.

and

$$J_l(le) = \frac{1}{\pi} \int_0^{\pi} \cos(lE - le \sin E) dE,$$

E being the eccentric anomaly.

Raising this expansion to the j th power would give an infinite series in e^2 , and in addition a set of such series each multiplied by a cosine of a multiple of M . Were this entire series substituted in (234), the terms which were periodic would remain so and contribute only periodic terms to the apsidal motion. The consideration of such periodic contributions in the apsidal motion of eclipsing binary systems is not feasible at present, mainly because the accuracy of observations is not high enough to warrant it. However, the constant terms, namely, the power series in e^2 , cannot be neglected as they make the longitude of periastron become a linear function of time. The longitude of periastron will, therefore, increase indefinitely and, consequently, will cause an observable motion of the apse. Such motion is called secular motion.

The series in e^2 in question can most easily be obtained by making a binomial expansion of $\left(\frac{A}{R}\right)^j$, thus giving

$$(247) \quad \left(\frac{A}{R}\right)^j = (1 - e \cos E)^{-j} = \sum_{l=1}^{\infty} \frac{(j+l-1)!}{(j-1)! l!} e^l \cos^l E.$$

Expand $\cos^l E$ in a Fourier series in multiples of E and observe that terms appear which are equal to

$$\frac{l!}{2^l \left[\left(\frac{l}{2}\right)!\right]^2} + \left(\begin{array}{l} \text{terms involving even} \\ \text{multiples of } E \end{array} \right), \text{ for } n \text{ even,}$$

and

$$\frac{l! \cos E}{2^{l-1} \left(\frac{l-1}{2}\right)! \left(\frac{l+1}{2}\right)!} + \left(\begin{array}{l} \text{terms involving odd} \\ \text{multiples of } E \end{array} \right), \text{ for } n \text{ odd.}$$

If one develops $\cos mE$ in a series there results¹

$$\cos mE = -\frac{1}{2}e + m \sum_{k=1}^{\infty} \left\{ J_{k-m}(ke) - J_{k+m}(ke) \right\} \cos kM, (m > 0),$$

$$\cos mE = 1, \quad (m = 0).$$

The expression has as its only term independent of M , 1 , $-\frac{e}{2}$, or 0 according to whether $m = 0$, $m = 1$ or $m > 1$. Therefore the coefficient of e^{ℓ} in the non-periodic part of $\left(\frac{A}{R}\right)^j$ becomes

$$(248) \quad \left(\frac{A}{R}\right)^j = \sum_{\ell=0}^{\infty} \frac{(j+\ell-1)! e^{\ell}}{(j-1)! 2^{\ell} \left[\left(\frac{\ell}{2}\right)!\right]^2} - \sum_{\ell=1}^{\infty} \frac{(j+k-1)! e^{\ell+1}}{(j-1)! 2^{\ell} \left(\frac{\ell-1}{2}\right)! \left(\frac{\ell+1}{2}\right)!},$$

where each term of the first sum is non-zero only if ℓ is even, and each term of the second sum is non-zero only when ℓ is odd. This may be expanded in a more convenient form to give

$$(249) \quad \left(\frac{A}{R}\right)^j = \sum_{k=2}^{\infty} \frac{(j+k-2)! e^k}{(j-2)! 2^k \left[\left(\frac{k}{2}\right)!\right]^2},$$

where k only takes on even values. The coefficient of e^0 is unity.

Knowing the expression for $\left(\frac{A}{R}\right)^j$ which gives rise to the secular terms in the motion of the apse, one may substitute equation(240) and (245) into (232) and then (232) into (234) to give

$$(250) \quad \frac{\dot{\omega}}{n} = \frac{\sqrt{1-e^2}}{e} \left\{ \frac{k_{2,1} a_1^5}{A^5} \left[\frac{m_2}{m_1} \frac{\partial (A)^6}{\partial e (R)} + \frac{\omega_1^2}{3m_1 G} \frac{\partial (A)^3}{\partial e (R)} \right] + \frac{k_{2,2} a_2^5}{A^5} \left[\frac{m_1}{m_2} \frac{\partial (A)^6}{\partial e (R)} + \frac{\omega_2^2}{3m_2 G} \frac{\partial (A)^3}{\partial e (R)} \right] \right\}_{j=2}$$

$$+ \frac{\sqrt{1-e^2}}{e} \left\{ \frac{k_{3,1} a_1^7}{A^7} \frac{m_2}{m_1} \frac{\partial (A)^8}{\partial e (R)} + \frac{k_{3,2} a_2^7}{A^7} \frac{m_1}{m_2} \frac{\partial (A)^8}{\partial e (R)} \right\}_{j=3}$$

$$+ \frac{\sqrt{1-e^2}}{e} \left\{ \frac{k_{4,1} a_1^9}{A^9} \frac{m_2}{m_1} \frac{\partial (A)^{10}}{\partial e (R)} + \frac{k_{4,2} a_2^9}{A^9} \frac{m_1}{m_2} \frac{\partial (A)^{10}}{\partial e (R)} \right\}_{j=4},$$

upon making use of the relation $n^2 A^3 = G(m_1 + m_2) = \mu$, where n is the mean motion. The terms $\frac{\sqrt{1-e^2}}{e} \frac{\partial (A)^j}{\partial e (R)}$ for the corresponding j become

¹ Cf. W. M. Smart, Celestial Mechanics, (New York, 1953), P. 39.

$$(251) \quad \frac{\sqrt{1-e^2}}{e} \frac{\partial}{\partial e} \left(\frac{A}{R} \right)^3 = 3(1 + 2e^2 + 3e^4 + 4e^6 + \dots) = 3g_2(e),$$

$$\frac{\sqrt{1-e^2}}{e} \frac{\partial}{\partial e} \left(\frac{A}{R} \right)^6 = 15 \left(1 + \frac{13}{2} e^2 + \frac{181}{8} e^4 + \frac{465}{8} e^6 + \dots \right) = 15f_2(e)$$

$$\frac{\sqrt{1-e^2}}{e} \frac{\partial}{\partial e} \left(\frac{A}{R} \right)^8 = 28 \left(1 + \frac{43}{4} e^2 + \frac{449}{8} e^4 + \frac{12941}{64} e^6 + \dots \right) = 28f_3(e),$$

$$\frac{\sqrt{1-e^2}}{e} \frac{\partial}{\partial e} \left(\frac{A}{R} \right)^{10} = 45 \left(1 + 16e^2 + \frac{467}{4} e^4 + \frac{8975}{16} e^6 + \dots \right) = 45f_4(e),$$

where these expressions define the functions f and g . These expressions converge for all $e < 1$. Sterne¹ has given closed expressions for these series which are more convenient to use. They are

$$(252) \quad g_2(e) = (1-e^2)^{-2}$$

$$f_2(e) = (1-e^2)^{-5} \left(1 + \frac{3}{2} e^2 + \frac{1}{8} e^4 \right)$$

$$f_3(e) = (1-e^2)^{-7} \left(1 + \frac{15}{4} e^2 + \frac{15}{8} e^4 + \frac{5}{64} e^6 \right)$$

$$f_4(e) = (1-e^2)^{-9} \left(1 + 7e^2 + \frac{35}{4} e^4 + \frac{35}{16} e^6 + \frac{7}{128} e^8 \right).$$

These have the values unity when $e = 0$ and approach infinity, thus causing the secular motion to approach infinity, as e approaches unity.

Using these functions, the expansion for secular motion of the apse becomes

$$(253) \quad \frac{\dot{\omega}}{n} = \frac{\dot{\omega}_1}{n} + \frac{\dot{\omega}_2}{n} + \frac{\dot{\omega}_3}{n}$$

where

$$(254) \quad \frac{\dot{\omega}_1}{n} = \frac{k_{2,1} a_1^5}{A^5} \left(15 \frac{m_2}{m_1} f_2(e) + \frac{\omega_1^2}{m_1 G} g_2(e) \right) + \frac{k_{2,2} a_2^5}{A^5} \left(15 \frac{m_1}{m_2} f_2(e) + \frac{\omega_2^2}{m_2 G} g_2(e) \right),$$

$$\frac{\dot{\omega}_2}{n} = 28 \left[\frac{k_{3,1} a_1^7}{A^7} \frac{m_2}{m_1} + \frac{k_{3,2} a_2^7}{A^7} \frac{m_1}{m_2} \right] f_3(e),$$

$$\frac{\dot{\omega}_3}{n} = 45 \left[\frac{k_{4,1} a_1^9}{A^9} \frac{m_2}{m_1} + \frac{k_{4,2} a_2^9}{A^9} \frac{m_1}{m_2} \right] f_4(e).$$

1. T. E. Sterne, M.N., 99, 461, 1939.

But $\frac{\dot{\omega}}{n} = \frac{P}{P'}$, where P and P' are the orbital and apsidal periods, respectively. Therefore

$$(255) \quad \frac{P}{P'} = \frac{\dot{\omega}_1}{n} + \frac{\dot{\omega}_2}{n} + \frac{\dot{\omega}_3}{n} .$$

When the components are widely separated $\frac{\dot{\omega}_1}{n}$ should give a sufficiently accurate value of $\frac{P}{P'}$. To the adopted degree of accuracy the theory predicts a precession.

It is of significance to note that when the angular velocities of the stars are set equal to n , the mean motion, the following less general expression appears for $\frac{\dot{\omega}_1}{n}$

$$(256) \quad \frac{\dot{\omega}_1}{n} = \frac{k_{2,1} a_1^5}{A^5} \left[\frac{m_2}{m_1} (15 f_2(e) + g_2(e)) + g_2(e) \right] \\ + \frac{k_{2,2} a_2^5}{A^5} \left[\frac{m_1}{m_2} (15 f_2(e) + g_2(e)) + g_2(e) \right] .$$

In applying the theory of apsidal motion equation(200) can always be solved and the k_j 's determined. For any one-parameter family of models, e.g. the family of polytropes, the expected rate of apsidal motion may be computed for several values of the parameter, and by inverse interpolation the value of the parameter found which corresponds to the observed motion of the apse. In general for any two families of models at most only one k_j from each family will be equal to a k_j of the other family. Therefore, from the observed motion of the apse it is not possible to calculate k_2, k_3, k_4 , without selecting some stellar model. Sterne¹, Russell², and others³, who have investigated

1 T. E. Sterne, *M.N.*, 99, 662, 1939.

2 H. N. Russell, *Ap.J.*, 90, 641, 1939.

3 e.g. J. Ashbrook, *A.J.*, 55, 2, 1949.

the apsidal motion of several eclipsing binary stars adopt a one-parameter family of models and determine the value of the parameter which makes the computed and observed rates of apsidal motion identical. The family of polytropes is probably the most convenient single-parameter family to use. The parameter for this family is the polytropic index n .

An extensive table of values for $\eta_j(a)$ and k_j with argument n , based on the polytropic models, has been published¹ recently and is recommended for use in computing, since the earlier tables computed by Chandrasekhar² apparently contain errors.

Further, Sterne³ notes that the values of polytropic indices are not very sensitive to change in the velocity of rotation ω , the eccentricity e , or the observed ratio P/P' .

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- 1 R. A. Brooker and T. W. Ollie, M.N., 115, 105, 1955.
 - 2 S. Chandrasekhar, M.N., 93, 462, 1933.
 - 3 T. E. Sterne, M.N., 99, 663, 1939

Gravity Darkening

The concept of gravity darkening rests on a theorem first proved by von Zeipel¹ and later by Chandrasekhar.² It may be stated formally as follows: "The emergent net flux of total radiation over the surface of a rotationally or tidally distorted star in radiative equilibrium varies proportionally to the local gravity".

Von Zeipel has shown³ that the surfaces of constant potential are also surfaces of constant total pressure P , density ρ , and temperature T . In this proof he assumes that the nature of the gas is constant over an equipotential surface, but that it may change from one level surface to another. The equation of radiative equilibrium in the form of a differential equation is⁴

$$(257) \quad \operatorname{div} \left(\frac{1}{\kappa \rho} \nabla p_r \right) = - \frac{4\pi \epsilon \rho}{c},$$

p_r being the radiation pressure, ϵ the rate of liberation of energy per unit mass, and κ the opacity coefficient. Since P and p_r are constant over an equipotential surface, p_r may be considered as a function of P and the above equation may be written

$$(258) \quad \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(\frac{1}{\kappa} \frac{dp_r}{dP} \frac{1}{\rho} \frac{\partial P}{\partial x_i} \right) = - \frac{4\pi \epsilon \rho}{c},$$

or

$$(259) \quad \frac{d}{dP} \left(\frac{1}{\kappa} \frac{dp_r}{dP} \right) \cdot \frac{1}{\rho} \left(\frac{dP}{dn} \right)^2 + \frac{1}{\kappa} \frac{dp_r}{dP} \operatorname{div} \left(\frac{1}{\rho} \nabla P \right) = - \frac{4\pi \epsilon \rho}{c},$$

¹ H. von Zeipel, M.N., 84, 665, 1924.

² S. Chandrasekhar, M.N., 93, 549, 1933.

³ H. von Zeipel, M.N., 84, 665, 1924.

⁴ Cf., E. A. Milne, M.N., 83, 122, 1923; or H. von Zeipel, M.N., 84, 668, 1924.

upon using the relation

$$(260) \quad \left(\frac{dP}{dn}\right)^2 = \sum_{i=1}^3 \left(\frac{\partial P}{\partial x_i}\right)^2.$$

The equation of hydrostatic equilibrium given by (164) is

$$(261) \quad \frac{1}{\rho} \nabla P = \nabla \Psi = \nabla (V + V_t + V_r).$$

But V and V_t satisfy Poisson's and Laplace's equations, respectively

$$\operatorname{div} \nabla V = -4\pi G\rho$$

$$\operatorname{div} \nabla V_t = 0$$

and

$$V_r = \frac{1}{2} \omega^2 (x^2 + y^2).$$

Thus

$$(262) \quad \operatorname{div} \left(\frac{1}{\rho} \nabla P \right) = -4\pi G\rho + 2\omega^2.$$

Using this relation, equation (259) becomes

$$(263) \quad \frac{d}{dP} \left(\frac{1}{x} \frac{dP}{dP} \right) \cdot \frac{1}{\rho} \left(\frac{dP}{dn} \right)^2 = -\frac{4\pi G\rho}{c} + \frac{2}{x} \frac{dP}{dP} (2\pi G\rho - \omega^2).$$

The right side of this equation is constant over an equipotential surface. Consequently, the left side must be constant over a level surface or identically zero. If it were constant, the differential dn would necessarily be constant, requiring the equipotential surfaces to be parallel. Another proof by von Zeipel¹ shows that the level surfaces of a rotating liquid or gaseous configuration in hydrostatic equilibrium cannot be parallel. Hence,

$$(264) \quad \frac{d}{dP} \left(\frac{1}{x} \frac{dP}{dP} \right) = 0,$$

or

¹ H. von Zeipel, M.N., 84, 680, 1924.

$$(265) \quad \frac{1}{\kappa} \frac{dp_r}{d\rho} = \text{constant} = \gamma.$$

It readily follows from either the exact integral equation for radiative equilibrium, or from the differential equation, that the net flux across any elementary area of an equipotential surface per unit area is given by

$$(266) \quad \pi F_n = -\frac{c}{\kappa \rho} \frac{dp_r}{dn}.$$

Use equation (265) to obtain

$$(267) \quad \pi F_n = -\frac{c\gamma}{\rho} \frac{d\rho}{dn}.$$

From the differential equation (261) one finds

$$(268) \quad \pi F_n = -\frac{c\gamma}{dn} d\psi,$$

which is what was to be proved.

Explicit note should be made of the limitations of this theorem. If convection currents exist near the stellar surface, the equation of hydrostatic equilibrium is inapplicable and the theorem is not valid. The source of error arising from the non-hydrostatic state near a particular level surface is not as important, however, as the deviation caused by convective heat transport. The latter may render the flux over the surface no longer proportional to the temperature gradient and, hence, to the gravity. Even if exact proportionality holds for the total radiation, it is unlikely that it will hold for the net flux in a single wave length. Finally, one must realize that the formulae themselves are a good approximation for the interior regions of the gaseous mass, but not for regions too close to its outer surface. For strict application of the theorem the further

assumption must be made that the production of energy in the neighborhood of the surface is insignificant compared to the total radiation. The net flux just below the limiting surface is then very nearly equal to the net flux through the outer surface.

To predict theoretically the extent to which the gravity darkening theorem should apply in actual stars would be a prodigious task. Fortunately, observations of light between minima have been found to agree with the predicted variation based on the gravity darkening theorem. Therefore, it is probably safe to assume that the theorem will apply to the total integrated net flux in most cases where complications are not known to exist a priori.

On the basis of the theorem the total net flux πF emerging normally from the stellar atmosphere varies as

$$(269) \quad \frac{\pi F - \pi F_0}{\pi F_0} = \frac{g - g_0}{g_0} ,$$

where g and g_0 denote the local and mean surface gravity, respectively, and πF_0 is the value of πF at $g = g_0$. From this it is apparent that the normal specific intensity I' as influenced by the distortions S_j and S_r varies as

$$(270) \quad \frac{I' - I_0}{I_0} = \frac{g - g_0}{g_0} .$$

or

$$(271) \quad \frac{I}{I_0} = 1 + \left(\frac{g - g_0}{g_0} \right) = 1 - [1 + \eta_2(a) S_r] - \sum_{j=2}^4 [1 + \eta_j(a)] S_j ,$$

where I_0 is the average normal specific intensity, and $\eta_j(a)$ is a constant determined from equation (200).

¹ Cf. H. N. Russell, *Ap.J.*, 90, 641, 1939; or Z Kopal, *Annals New York Acad. Sci.*, 41, 13, 1941.

In order to account for deviations from the theorem a gravity darkening coefficient τ may be introduced in such a way that

$$(272) \quad \frac{I'}{I_0} = 1 + \tau \left(\frac{g - g_0}{g_0} \right),$$

where now I' and I_0 are for a given wave length. Here τ is the coefficient^{1,2} by which the full intensity variation predicted by the theorem must be multiplied to obtain the actual variation. If the theorem is strictly applicable $\tau = 1$, and if the intensity is independent of surface gravity $\tau = 0$. By using τ , no assumption of strict applicability of the theorem at individual wave lengths needs to be made.

The apparent intensity distribution I over the surface of distorted stars must still be given by

$$(273) \quad I = I' (1 - u + u \cos \gamma)$$

γ being the angle between the outward normal and the line of sight, and u , the coefficient of limb-darkening. Then

$$(274) \quad I = I_0 \left\{ 1 - \tau \left(1 - \frac{g}{g_0} \right) \right\} (1 - u + u \cos \gamma)$$

Physically this distribution of apparent brightness may be clarified by noting that the limb-darkening tends to make the parts of the surface nearest the observer the brightest, while the gravity darkening renders those parts closest to the star's center the

¹ The coefficient τ is identical with Russell's y . See Ap.J., 90, 641, 1939.

² If a star is considered to radiate like a black body, an explicit form of τ emerges and is given by $\tau = c_2 / 4\lambda T_0 (1 - e^{-c_2/\lambda T_0})$, where c_2 is a Planck radiation constant, λ is the wave length of radiation under consideration, and T_0 is the mean effective temperature averaged over the whole star. Cf. Z. Kopal, Harvard Obs. Mon., 6, 126, 1946.

brightest. The effect of gravity contributes two parts: the intensity at the poles of a star is increased due to rotation compared to the mean along the equator; and the intensity at points in line with the companion is diminished compared to the mean along the diametral section perpendicular to this line. If no limb-darkening were present the isophotes for each effect separately would be ellipses, to a first approximation, similar in the former case to the projection of the equator, and in the latter case to that of the diametral section, but generally not concentric with either. When both effects are present and when limb-darkening occurs, the isophotes become curves of higher degree.¹ If the combined effects are considered the point of a distorted star nearest the observer, and usually the brightest, will not in general be the geometric center of the apparent disc. Thus, as would be expected, the isophotes are not necessarily radially symmetric. Furthermore, the system of isophotes depend on the phase and vary as the stars revolve. Lastly, limb-brightening is physically possible if a distorted star is viewed along the direction of its longest axis.

¹ H. N. Russell, Ap.J., 108, 398, 1948.

Ellipticity and Reflection Effects

The light variations of close binary systems are not restricted to eclipses, but extend over the complete period. Two causes are responsible for the variations. First, the apparent surface area, and hence the observed light, of distorted stars change continuously during the course of a revolution. This is known as the "ellipticity effect". Second, a portion of the emitted radiation of each component strikes the surface of the other, and is absorbed and re-emitted, or is scattered, in all directions. The portion of this "reflected" light¹ which reaches the observer varies with phase and gives rise to what is known as the "reflection effect". These changes occur quite independent of the eclipses. The reflection effect should, and indeed does, occur in systems consisting of spherical as well as distorted components. The magnitude of the reflection effect for systems with sufficiently wide separation to be regarded as having spherical components is usually quite small. The ellipticity effect, on the other hand, occurs only in distorted systems.

Since for proximate stars the magnitude of the light variation between minima due to either effect is appreciable, solutions for preliminary elements from light curves corresponding to systems exhibiting such effects must currently be considered under the subject of rectification.

The light changes due to ellipticity and reflection should

¹ This includes secondary reflections, i.e., the light from one star which is reflected from the second back to the first and then again reflected into the line of sight. Reflections of higher order than the second are usually neglected.

disappear only if the plane of the orbit is perpendicular to the line of sight, and further only if the orbit is circular. If the orbit is eccentric, independent of the inclination, the separation changes and causes the distortion and surface brightness to change in the orbital period. As a consequence there is a difference in the maxima of light between minima. In addition, the changing distance between the components of eccentric systems also causes the reflection effect to vary, rendering it no longer symmetrical with respect to the conjunctions. Each of these effects contributes to the so-called "periastron effect".

Myers¹ and Roberts² first explained the continuous light variations between eclipses for special systems by the ellipticity of the binary components. Myers recognized that the ellipticity would be caused by tidal effects, and noted that the shape of the components of a close binary system which differ in density distribution also differ in form. Darwin³ had shown that the figure of such stars is probably nearly ellipsoidal, with the longest axis in the line joining the components, and the shortest perpendicular to the orbit plane. To avoid complications Myers and Roberts assumed the binary components to be uniformly illuminated similar prolate ellipsoids. Russell⁴, and Russell and Shapley⁵ investigated the photometric consequences of uniformly illuminated and completely darkened

1 G. W. Myers, Ap.J., 7, 8, 1898.

2 A. W. Roberts, Ap.J., 13, 177, 1901; M.N., 63, 527, 1903, M.N., 65, 706, 1905.

3 G. H. Darwin, Phil. Trans., 206, 160, 1906.

4 H. N. Russell, Ap.J., 36, 54, 1912.

5 H. N. Russell and H. Shapley, Ap.J., 36, 385, 1912.

similar ellipsoids. Walter¹ interpreted the light changes dynamically, and on the basis of the observed light variations between minima and concluded that irrespective of limb-darkening, the components would have to be nearly homogeneous. This contradicted conclusions based on the theory of internal constitution of the stars, because such configurations of low density concentration were believed to be unstable. The difficulty was resolved by Takeda,² who found that the photometric effects between minima for close binary systems are strengthened by gravity-darkening. Refinements in the previous investigations have been made by Russell,³ Sterne⁴ and Kopal.⁵ These form the basis for the present discussion, although Sterne's work is followed more closely than that of the others.

The object now is to investigate the theoretical light variations between minima as caused by ellipticity. The accuracy of the results obtained will exceed the exactness requisite for rectification procedures, but may be found useful in later stages of analysis—not treated here — when the exact light curve is represented.

The radius vector to any point of the surface of the primary star is given by

$$(275) \quad r = a_1 \left\{ 1 - v_{\eta_1} P_2(\cos \vartheta) + \sum_{j=2}^4 w_{j_1} P_j(\sin \vartheta \cos \psi) \right\},$$

where

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- 1 K. Walter, Konigsberg Veroff., 2, 1931.
 2 G. Takeda, Kyoto Mem., A, 17, 197, 1934.
 3 H. N. Russell, Ap.J., 90, 641, 1939.
 4 T. E. Sterne, Proc. Nat. Acad. Sci., 27, 99, 106, 1941.
 5 Z. Kopal, Annals New York Acad. Sci., 41, 13, 1941; and Harvard Obs. Mon., 6, 131, 1946.

$$(276) \quad v_{r,1} = \frac{\Delta_2 a_1^3 \omega_1^2}{3 m_1 G}$$

and

$$(277) \quad w_{j,1} = \frac{m_2}{m_1} \Delta_j \left(\frac{a_1}{R} \right)^{j+1}.$$

The rotational distortion will add no observable effects to the light between minima since the orientation of the axis of rotation with respect to the line of sight is fixed. However, this distortion must be included in the analysis to avoid making a systematic error in the derived luminosity, which is to be free from the effects of the unequal distribution of intensity over the surface of the distorted star. To be consistent with the degree of accuracy adopted heretofore the squares and cross products of the S's will be neglected, thus making the fractional effects of different distortions additive. Therefore the tidal and rotational distortions may be considered separately.

For the tidal distortion the radius vector to a point on the surface is

$$(278) \quad r = a_1 \left\{ 1 + \sum_{j=2}^4 w_{j,1} P_j(\sin \mathcal{J} \cos \psi) \right\}.$$

Now express the position of a point P on the unit sphere in terms of astrocentric spherical polar coordinates δ, μ , where δ is the angle between r and the line of centers, and μ is the azimuthal angle measured in the diametral plane from the plane of the orbit.

From figure (8) one readily finds

$$(279) \quad \cos \delta = \sin \mathcal{J} \cos \psi.$$

Then (278) becomes

$$(280) \quad r = a_1 \left\{ 1 + \sum_{j=2}^4 \bar{w}_{j,1} P_j(\cos \delta) \right\} .$$

The polar coordinates of the outward normal to the surface at the point corresponding to P may be denoted by δ', μ^1 . Realizing that

$$\delta = \delta' + \frac{dr}{r d\delta} ,$$

to the first order in small quantities, and noting that

$$\begin{aligned} \frac{dr}{r d\delta} = & -3 \frac{w_{2,1}}{2^{3,1}} \sin \delta \cos \delta - \frac{w_{3,1}}{2^{3,1}} (15 \sin \delta \cos \delta - 3 \sin \delta) \\ & - \frac{w_{4,1}}{2^{4,1}} (35 \sin \delta \cos^3 \delta - 15 \sin \delta \cos \delta) , \end{aligned}$$

one finds

$$(281) \quad \delta = \delta' - 3 \frac{w_{2,1}}{2^{3,1}} \sin \delta' \cos \delta' - \frac{w_{3,1}}{2^{3,1}} (15 \sin \delta' \cos \delta' - 3 \sin \delta') \\ - \frac{w_{4,1}}{2^{4,1}} (35 \sin \delta' \cos^3 \delta' - 15 \sin \delta' \cos \delta') .$$

The difference between the angles, being small, permits one to set the cosine of small terms equal to unity, thus giving

$$(282) \quad r = a_1 \left\{ 1 + \sum_{j=2}^4 \bar{w}_{j,1} P_j(\cos \delta') \right\} .$$

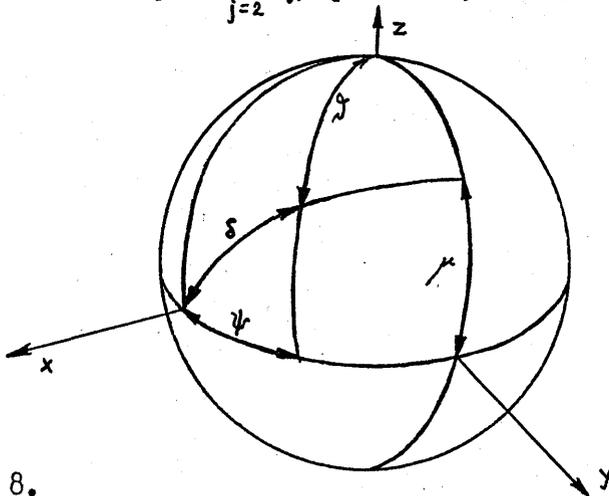


Figure 8.

The surface element is given by $dA_1 = r^2 \sin \delta d\delta d\mu$ and must be expressed in terms of the coordinates δ', μ to facilitate integration. Performing these substitutions there results

¹ Since a tidally distorted star is symmetric about the line of centers, μ does not change.

$$dA_1 = a_1^2 \{ 1 - 4w_{2,1} P_2(\cos \delta') - 10w_{3,1} P_3(\cos \delta') - 18w_{4,1} P_4(\cos \delta') \} \sin \delta' d\delta' d\mu,$$

or

$$(283) \quad dA_1 = a_1^2 \{ 1 - 4w_{2,1} P_2(\cos \delta') - 10w_{3,1} P_3(\cos \delta') - 18w_{4,1} P_4(\cos \delta') \} d\Omega,$$

where $d\Omega$ is an element of solid angle in the normal system. This element may, of course, be expressed in any convenient coordinates compatible with the normal system when the integration is performed, providing appropriate quantities are substituted for $\cos \delta'$ in equation (283). It is prudent to utilize any symmetry properties which exist when the coordinates are chosen.

In the above case adopt a rectangular reference frame $x'y'z'$ with origin at the center of the primary star, the z' - axis coincident with the line of sight, and the x' -axis in the plane of the sky towards the apparent center of the secondary component. Let r, γ, ζ be the spherical polar coordinates in this system such that γ , the polar angle, is equal to the angle of foreshortening and ζ is the azimuthal angle measured from the x' -axis.

The direction cosines of the line of centers in the primed system are $\sin \epsilon$, 0 , $\cos \epsilon$, where ϵ is the angle between the z' -axis and the x -axis. From figure (9) one obtains

$$(284) \quad \cos \epsilon = \sin i \cos \phi,$$

ϕ being the true anomaly of the secondary component from inferior conjunction. It is easily seen that ϕ is the same angle introduced earlier in connection with eccentric orbits.

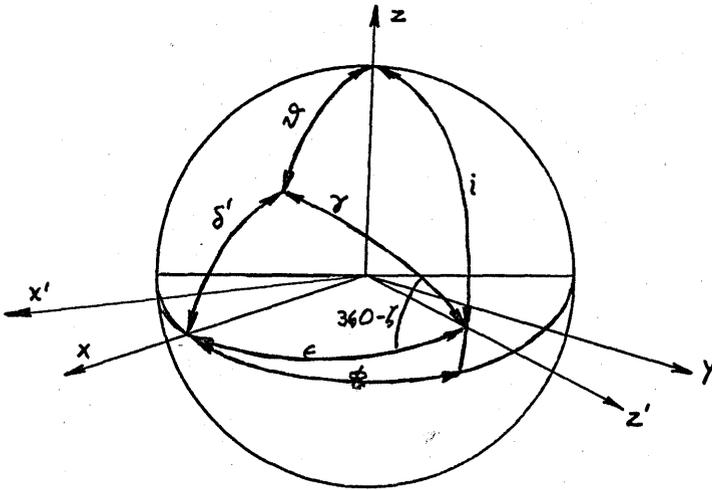


Figure 9.

Furthermore, one finds

$$(285) \quad \cos \delta' = \cos \epsilon \cos \gamma + \sin \epsilon \sin \gamma \cos \zeta.$$

In the $\gamma\zeta$ system

$$(286) \quad d\Omega = \sin \gamma d\gamma d\zeta.$$

Therefore, using equations (2) and (3), the relative flux density of monochromatic light of the primary component outside eclipse as seen by an observer is

$$(287) \quad J_{t_1} = a_1^2 I_0 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (1 - u_1 + u_1 \cos \gamma) \left\{ 1 - w_{2,1} (\Theta_{2,1} + 4) P_2(\cos \delta') \right. \\ \left. - w_{3,1} (\Theta_{3,1} + 10) P_3(\cos \delta') - w_{4,1} (\Theta_{4,1} + 18) P_4(\cos \delta') \right\} \sin \gamma \cos \gamma d\gamma d\zeta,$$

where $\Theta_{j,1} = \tau_j [1 + \eta_j(a)]$. Upon substituting equation (285) into (287) and integrating¹ one obtains

¹ All the integrals involved are elementary.

$$(288) \quad J_{t_1} = \pi a_1^2 L_b (1 - \frac{u_1}{3}) \left\{ 1 - \frac{w_{2,1} (\theta_{2,1} + 4)(15 + u_1) P_2(\cos \epsilon)}{20(3 - u_1)} \right. \\ \left. - \frac{w_{3,1} (\theta_{3,1} + 10) u_1 P_3(\cos \epsilon)}{4(3 - u_1)} + \frac{w_{4,1} (\theta_{4,1} + 18)(1 - u_1) P_4(\cos \epsilon)}{8(3 - u_1)} \right\}.$$

For rotational distortion

$$(289) \quad r = a_1 \left\{ 1 - v_{r,1} P_2(\cos \mathcal{J}') \right\}.$$

Because of rotational symmetry about the z-axis it is convenient to formulate the problem in a different coordinate system than was employed before. The spherical polar coordinates of the radius vector to the point P on the unit sphere in terms of the original x y z system are \mathcal{J}, ψ , and those of the outward normal to the surface at P are, say \mathcal{J}', ψ' . One easily obtains

$$\mathcal{J} = \mathcal{J}' + \frac{dr}{r d\mathcal{J}'},$$

to the first order in small quantities. Also

$$\frac{dr}{r d\mathcal{J}'} = 3v_{r,1} \cos \mathcal{J}' \sin \mathcal{J}'.$$

Thus

$$\mathcal{J} = \mathcal{J}' + 3v_{r,1} \cos \mathcal{J}' \sin \mathcal{J}'$$

and

$$r = a_1 \left\{ 1 - v_{r,1} P_2(\cos \mathcal{J}') \right\}.$$

The surface element of the primary as influenced by rotation only becomes

$$dA_1 = a_1^2 \left\{ 1 + 4v_{r,1} P_2(\cos \mathcal{J}') \right\} \sin \mathcal{J}' d\mathcal{J}' d\psi,$$

or

$$(290) \quad dA_1 = a_1^2 \left\{ 1 + 4v_{r,1} P_2(\cos \mathcal{J}') \right\} d\Omega,$$

where $d\Omega$ is again an element of solid angle in the normal system.

Adopt a new rectangular coordinate system $x''y''z''$ with origin at the center of the primary component, the z'' -axis coincident with the line of sight, and the x'' -axis in the plane of the sky along the projected axis of rotation. Let ν, γ, ξ be the spherical polar coordinates in this system, where γ is the polar angle — the angle between the surface normal and the line of sight — and ξ is the azimuthal angle measured from the x'' -axis. In this system

$$d\Omega = \sin \gamma d\gamma d\xi$$

and

$$(291) \quad \cos \mathcal{J}' = \cos i \cos \gamma + \sin i \sin \gamma \cos \xi,$$

(see figure 10.).

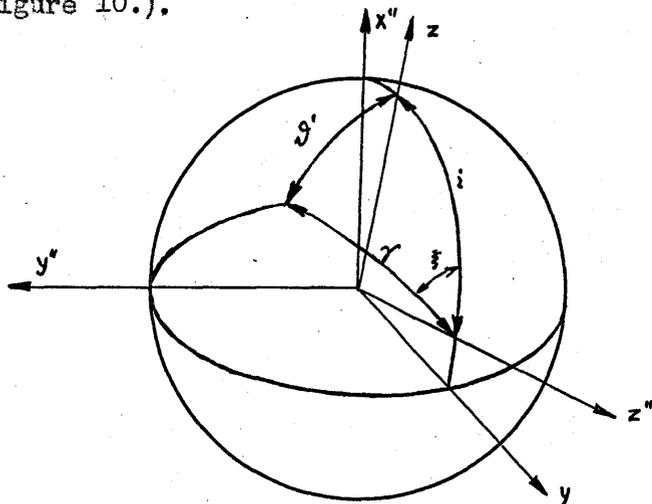


Figure 10.

From equations (2) and (3) the contribution to the relative flux density of the primary star outside eclipse as influenced by total distortion only is

$$(292) \quad J_{r_1} = a_1^2 I_0 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (1 - u_1 + u_1 \cos \gamma) \left\{ 1 + v_{r_1} (\Theta_{2,1} + 4) P_2(\cos \mathcal{J}') \right\} \sin \gamma \cos \gamma d\gamma d\xi,$$

where $\Theta_{2,1}$ is defined the same as before. Substituting (291) into (292) and integrating, the latter becomes

$$(293) \quad J_{r,1} = \pi a_1^2 I_0 (1 - \frac{u_1}{3}) \left\{ 1 + \frac{v_{r,1} (\Theta_{2,1} + 4)(15 + u_1)}{20(3 - u_1)} P_2(\cos i) \right\}.$$

Since this relation is independent of ϕ it is apparent that the light variations will not depend on this term.

Combining relations (288) and (293) the total flux density contributed by the primary star at any phase is

$$(294) \quad J_1 = \pi a_1^2 I_0 (1 - \frac{u_1}{3}) \left\{ 1 + \frac{v_{r,1} (\Theta_{2,1} + 4)(15 + u_1)}{20(3 - u_1)} P_2(\cos i) \right. \\ \left. - \frac{w_{2,1} (\Theta_{2,1} + 4)(15 + u_1)}{20(3 - u_1)} P_2(\sin i \cos \phi) - \frac{w_{3,1} (\Theta_{3,1} + 10) u_1}{4(3 - u_1)} P_3(\sin i \cos \phi) + \frac{w_{4,1} (\Theta_{4,1} + 18)(1 - u_1)}{8(3 - u_1)} P_4(\sin i \cos \phi) \right\}.$$

A similar relation holds for the light of the secondary. By introducing another constant the above becomes the following equality to the present order of accuracy,

$$(295) \quad J_1 = L_1 \left\{ 1 + \frac{v_{r,1} (\Theta_{2,1} + 4)(15 + u_1)}{20(3 - u_1)} P_2(\cos i) \right\} \left\{ 1 - \sum_{j=2}^4 z_{j,1} P_j(\sin i \cos \phi) \right\},$$

where

$$(296) \quad z_{2,1} = \frac{w_{2,1} (\Theta_{2,1} + 4)(15 + u_1)}{20(3 - u_1)}, \\ z_{3,1} = \frac{w_{3,1} (\Theta_{3,1} + 10) u_1}{4(3 - u_1)}, \\ z_{4,1} = -\frac{w_{4,1} (\Theta_{4,1} + 18)(1 - u_1)}{8(3 - u_1)},$$

and where L_1 is the light of the primary star free from effects of rotational and tidal distortion. When expression (295) is integrated over a sphere to find the total energy per unit time at a given frequency, the surface harmonics contribute nothing; L_1 depends only on the luminosity of the primary and its distance from the

observer, if reflection is neglected. It is of interest to note that a variation of light arises from the third order harmonic only if there is limb-darkening, and a change results from the fourth only if the limb-darkening is not complete. This property of harmonics higher than the second was first noticed by Russell.¹ Equation (295) applies to any stellar model for which the components move in a relative orbit which is circular.

For completeness one must consider the expressions arising when the orbit is eccentric. This introduces additional complications because the distortion and hence the distribution of intensity over the surface of each component will change. Let the semi-major axis of the relative orbit be A , and the distance between centers of mass of the two stars be R . Then

$$(297) \quad \frac{A}{R} \frac{(1 + e \cos v)}{1 - e^2} = \frac{1 + e \sin(\omega - \phi)}{1 - e^2},$$

where $\phi = \omega + v - 90^\circ$ as given by equation (89). In order to generalize equation (295), $w_{j,i}$ given by expression (277) must be replaced by²

$$(298) \quad w'_{j,i} = \frac{m_2}{m_1} \Delta_j \left(\frac{a_i}{A}\right)^{j+1} = \left(\frac{R}{A}\right)^{j+1} w_{j,i}.$$

Making this substitution, the light from the primary becomes

$$(299) \quad J_1 = L_1 \left\{ 1 + \frac{v_{r1} (\Theta_{z1} + 4) (15 + u_1) P_2(\cos i)}{20(3 - u_1)} \right\} \left\{ 1 - \sum_{j=2}^4 z'_{j1} P_j(\sin i \cos \phi) \right\},$$

¹ H. N. Russell, *Ap.J.*, 24, 1, 1906.

² A small systematic error is introduced here because the mean radius will also change with the distortion. This effect is undoubtedly negligible within the adopted degree of accuracy as it is a second order effect small compared to the change in R . Hence, w'_j and v'_j remain essentially constant over the entire period.

where

$$(300) \quad \begin{aligned} z'_{2,1} &= \frac{w'_{2,1} (\Theta_{2,1} + 4)(15 + u_1)}{20(3 - u_1)}, \\ z'_{3,1} &= \frac{w'_{3,1} (\Theta_{3,1} + 10)u_1}{4(3 - u_1)}, \\ z'_{4,1} &= -\frac{w'_{4,1} (\Theta_{4,1} + 18)(1 - u_1)}{8(3 - u_1)}. \end{aligned}$$

The corresponding expressions for the combined light of the primary and secondary stars moving in circular orbits become¹

$$(301) \quad \begin{aligned} \ell &= (\ell_1^m + \ell_2^m) - (z_{2,1}\ell_1^m + z_{3,2}\ell_2^m) P_2(\sin i \cos \phi) \\ &\quad - (z_{3,1}\ell_1^m - z_{3,2}\ell_2^m) P_3(\sin i \cos \phi) \\ &\quad - (z_{4,1}\ell_1^m + z_{4,2}\ell_2^m) P_4(\sin i \cos \phi), \end{aligned}$$

where

$$(302) \quad \ell_1^m = L_1 \left\{ 1 + \frac{v_{r,1} (\Theta_{2,1} + 4)(15 + u_1) P_2(\cos i)}{20(3 - u_1)} \right\}, \quad \ell_2^m = L_2 \left\{ 1 + \frac{v_{r,2} (\Theta_{3,2} + 10)u_2 P_2(\cos i)}{20(3 - u_2)} \right\},$$

and

$$(303) \quad \begin{aligned} z_{2,2} &= \frac{w_{2,2} (\Theta_{2,2} + 4)(15 + u_2)}{20(3 - u_2)}, \\ z_{3,2} &= \frac{w_{3,2} (\Theta_{3,2} + 10)u_2}{4(3 - u_2)}, \\ z_{4,2} &= -\frac{w_{4,2} (\Theta_{4,2} + 18)(1 - u_2)}{8(3 - u_2)}. \end{aligned}$$

If $\ell_1^m + \ell_2^m$ is set equal to ℓ^m and the coefficients of the Legendre polynomials, which are weighted means of the z 's, are replaced by a single letter z_j , where j denotes the order of the harmonic distortion, then the expression for the combined light becomes

$$(304) \quad \ell = \ell^m \left(1 - \sum_{j=2}^4 z_j P_j(\sin i \cos \phi) \right).$$

¹ In equation (295) one must remember that for the secondary component ϕ must be replaced by $(180 + \phi)$.

For the combined light in eccentric orbits one finds

$$(305) \quad l = (l_1^m + l_2^m) - (z'_{2,1} l_1^m + z'_{2,2} l_2^m) P_2(\sin i \cos \phi) \frac{[1 + e \sin(\omega - \phi)]^3}{(1 - e^2)^3} \\ - (z'_{3,1} l_1^m - z'_{3,2} l_2^m) P_3(\sin i \cos \phi) \frac{[1 + e \sin(\omega - \phi)]^4}{(1 - e^2)^4} \\ - (z'_{4,1} l_1^m + z'_{4,2} l_2^m) P_4(\sin i \cos \phi) \frac{[1 + e \sin(\omega - \phi)]^5}{(1 - e^2)^5},$$

or if $l_1^m + l_2^m = 1$ and the weighted means are denoted by z'_j ¹ then

$$(306) \quad l = l^m \left\{ 1 - \sum_{j=2}^4 z'_j P_j(\sin i \cos \phi) \left[\frac{1 + e \sin(\omega - \phi)}{1 - e^2} \right]^{j+1} \right\}.$$

A close inspection of equation () reveals that small differences in the combined light should occur at $\phi = 90^\circ$ and $\phi = 270^\circ$, i.e., roughly half way between minima, except when $\omega = 90$ or 270 .

The reflection effect was first noticed by Dugan² in 1908 during his study of RT Persei. Shortly afterward it was discovered independently by Stebbins³ and Nordmann⁴. All of these pioneers correctly explained the effect. Eddington⁵ first investigated the physical processes underlying the effect and pointed out that the heat albedo of stars in radiative equilibrium must be unity. He derived expressions for the reflected total radiation based on Lambert's law of reflection. Later, Milne⁶ formulated the equations of transfer of reflected radiation and obtained an approximate solution using the method of linear flow. Milne⁷ gave an alternate

1 The $z'_{j,1}$ and $z'_{j,2}$ are obtained from $z_{j,1}$ and $z_{j,2}$, respectively, given by (296) and (303) by replacing $w_{j,1}$ by $w'_{j,1}$ and $w_{j,2}$ by $w'_{j,2}$.

2 R.S. Dugan, Science, 28, 854, 1908; Contr. Princeton U. Obs., 1, 1910.

3 J. Stebbins, Ap.J., 32, 213, 1910.

4 C. Nordmann, B.A.N., 27, 145, 1910.

5 A. S. Eddington, M.N., 86, 320, 1926.

6 E. A. Milne, M.N., 87, 43, 1926.

7 In addition to E. A. Milne's 1926 paper, see his article in Handbuch der Astrophysik, 3, 134, 1930.

reasoning for the heat albedo requirement, which contains the following ideas.

The heat emitted from a star in the absence of incident radiation, is generated in the deep interior. The outer layers merely adjust themselves so as to let this heat pass through. This flow of energy will not be influenced by radiation incident on the outer layers, because the resulting increment in radiation pressure is quite small and is incapable of altering the rate of energy production in the interior. The incident radiation maintains the outer layers at a higher local temperature than they would otherwise possess. For a star in radiative equilibrium this temperature increase is just sufficient to enable the outer layers to radiate out into space an amount of energy equal to that incident on them, in addition to the energy which they would radiate if no external radiation were present. Otherwise the outer layers would be transmitting the missing heat to the interior and the net flow from the interior would be reduced—contrary to the assumption of radiative equilibrium. Thus all of the incident energy must be returned to space, which is equivalent to saying that the heat albedo is unity.

Before considering the method of transforming a heat curve into a light curve for a particular frequency range, it is necessary to note the way in which energy is redistributed by reflection.

The approximate distribution of energy re-emitted by a sphere exposed to a uniform beam of parallel radiation was determined by Milne, assuming an opacity coefficient independent of wave length. He determined the ratio, $s(\epsilon)$, of the reflected integrated flux

density at any phase angle to the ordinary integrated flux density at full phase. This ratio is given by

$$(307) \quad s(\epsilon) = \frac{\mathcal{L}_2}{\mathcal{L}_1} \left(\frac{r_1}{R} \right)^2 f(\epsilon) ,$$

if the secondary is the illuminating component and the primary is the reflecting component. In this expression \mathcal{L}_1 and \mathcal{L}_2 are the true luminosities, r_1 is the radius of the primary star, R is the distance between centers of the stars, and $f(\epsilon)$ is given by

$$(308) \quad f(\epsilon) = \frac{\sin \epsilon + (\pi - \epsilon) \cos \epsilon}{3\pi} + \frac{(1 - \cos \epsilon)(1 - 3 \cos \epsilon)}{32 \cos \frac{1}{2} \epsilon} \ln \cot \frac{\epsilon}{4} + \frac{1 + 5 \cos \epsilon}{16} ,$$

or, in terms of a numerical expression, by

$$(309) \quad f(\epsilon) = 0.278 + 0.347 \cos \epsilon + 0.073 \cos 2\epsilon + 0.005 \cos 3\epsilon + 0.002 \cos 4\epsilon .$$

It is apparent that such a ratio will also hold for the integrated flux density received by an observer. This is the type of expression of interest here. Therefore, at any phase the integrated flux density $\mathcal{L}'_1(\epsilon)$ reaching the observer which results from reflection of the secondary's light, \mathcal{L}_2^m , as increased by light from the primary, from the primary is

$$(310) \quad \mathcal{L}'_1(\epsilon) = \mathcal{L}_2^m \left(\frac{r_1}{R} \right)^2 f(\epsilon)$$

where $f(\epsilon)$ may have several forms depending on the accuracy and the method of derivation. The first of these forms, which was obtained by Milne, is given by equation (309).

The result found by Eddington using Lambert's law was

$$(311) \quad f(\epsilon) = 0.270 + 0.333 \cos \epsilon + 0.060 \cos 2\epsilon + 0.002 \cos 4\epsilon$$

Sahade and Gescó¹ calculated $f(\epsilon)$ using Chandrasekhar's² method

1 Quoted from Z. Kopal, Harvard Obs. Mon., 6, 154, 1946.

2 S. Chandrasekhar, Ap.J., 101, 348, 1945.

in the fourth approximation and obtained results in close agreement with Milne's values.

The close agreement obtained for $f(\epsilon)$ in the first order reflection effect using Lambert's law has prompted others to investigate the problem in more detail, assuming this law to hold for higher order reflection terms and taking into account the finite size of the reflecting and illuminating components. Such results are probably more exact than the above expressions, but no completely satisfactory theory of the reflection effect, applicable to any stellar model, has appeared, even for spherical stars. Departures from Lambert's law become important in refined work and must be taken into account in a definitive treatment.¹ At the present stage of knowledge these refinements would require detailed numerical integrations.

Takeda² initiated the refinements using Lambert's law by taking into account the convergence of the incident beam. He carried the solution to the fourth order in the fractional radii of the components, but made the mistake of extending the cone of incident light beyond the horizon of visibility of the illuminating point source, thus vitiating the fourth order term. His analysis was further limited to systems for which $i = 90^\circ$.

Sen³ corrected Takeda's results and found for the light reflected from the primary component, in the present notation

¹ A detailed investigation of the related problem of diffuse reflection from planetary atmospheres, reflecting with any arbitrary phase function, has been made by H. G. Horak, Ap.J., 112, 445, 1950.

² S. Takeda, Kyoto Mem., A, 17, 197, 1934.

³ H. K. Sen, Proc. Nat. Acad. Sci., 34, 311, 1948.

$$\begin{aligned}
 (312) \quad f(\epsilon) = & \left\{ 0.270 + 0.062 \left(\frac{r_1}{R} \right) + 0.068 \left(\frac{r_1}{R} \right)^2 \right\} \\
 & + \left\{ 0.333 + 0.250 \left(\frac{r_1}{R} \right) \right\} \cos \epsilon \\
 & + \left\{ 0.060 + 0.188 \left(\frac{r_1}{R} \right) + 0.028 \left(\frac{r_1}{R} \right)^2 \right\} \cos 2\epsilon \\
 & + \left\{ 0.002 - 0.050 \left(\frac{r_1}{R} \right)^2 \right\} \cos 4\epsilon .
 \end{aligned}$$

Clearly this method does not account for the influence of the finite size of the illuminating component on the reflected light.

Russell¹ first made a reconnaissance of the penumbral part² of the reflection effect by an approximate method which would be very difficult to generalize. Russell concluded that the contribution of the penumbral zone is probably less than one per cent of the total reflected light. He also found that the effect outside eclipse yields the terms

$$(313) \quad \Delta f(\epsilon) = 0.49 \left(\frac{r_2}{R} \right)^2 - 0.032 \left(\frac{r_2}{R} \right)^2 \cos 2\epsilon - 0.006 \left(\frac{r_2}{R} \right)^2 \cos 4\epsilon ,$$

which are to be added to expression (312).

Kopal³ has recently considered the mutual illumination of two finite spheres, again reflecting according to Lambert's law. Further, the finite size of the illuminating component and the contribution of light from the penumbral zone are included. Limitations of this treatment mainly involve the extent to which the true reflecting properties of the external layers of a star deviate from Lambert's law, the degree to which both components may be regarded as spherical, and the measure to which the line of sight can

1 H. N. Russell, Harvard Obs. Circ., No. 452, 1949.

2 The penumbral zone in the region on the reflecting star in which an observer would see the apparent disc of the illuminating star partly below the horizon.

3 Z. Kopal, M.N., 114, 101, 1954.

be considered to lie in the plane of the eclipsing orbit. Quantities to the fourth order in the fractional radii of the components were carried throughout this investigation. Kopal found $f(\epsilon)$, in the present notation, to be

$$\begin{aligned}
 (314) \quad f(\epsilon) = & \left\{ 0.270 + 0.062 \left(\frac{r_1}{R} \right) + 0.068 \left(\frac{r_1}{R} \right)^2 - 0.202 k \left(\frac{r_2}{R} \right)^2 \right\} \\
 & + \left\{ 0.333 + 0.250 \left(\frac{r_1}{R} \right) \right\} \cos \epsilon \\
 & + \left\{ 0.060 + 0.188 \left(\frac{r_1}{R} \right) + 0.027 \left(\frac{r_1}{R} \right)^2 + 0.135 k \left(\frac{r_2}{R} \right)^2 \right\} \cos 2\epsilon \\
 & + \left\{ 0.002 + 0.050 \left(\frac{r_1}{R} \right)^2 + 0.027 k \left(\frac{r_2}{R} \right) \right\} \cos 4\epsilon,
 \end{aligned}$$

where

$$(315) \quad k = 1 - \frac{12}{\pi} \frac{[5 + (\pi - 5)u_2]}{3 - u_2}.$$

His results indicate that the contribution of the penumbral zone is greater than Russell estimated. It should be noted that there are some difficulties in Kopal's analysis stemming from Sen's formulation¹ which may influence the magnitude of the penumbral zone effect, although the writer is not in a position to state their influence on the value of $f(\epsilon)$ given above, except to mention that probably they are small.

Cowling² pointed out that if Kopal's phase law were to be complete

¹ Kopal implies that his basic equation [37] is exact, while actually it is not. However, his equations and analysis are likely to be closely approximate for components which are separated widely enough to be considered spherical.

² This was noted by Kopal in an addendum to his paper M.N., 111, 101, 1954.

to quantities of the order of $\left(\frac{r_1 r_2}{R^2}\right)^2$, the effects of secondary reflection should be included. This is easily adjoined if ℓ_2^m denotes the intrinsic light of the secondary component, and ℓ_1^m , that of the primary. The actual light of the secondary will be increased to

$$(316) \quad \ell_2^m + \ell_1^m \left\{ \frac{2}{3} \frac{r_2^2}{R^2} + \frac{1}{2} \frac{r_2^3}{R^3} + \dots \right\}.$$

In final form Kopal's phase law for reflected light becomes

$$(317) \quad \ell_1'(\epsilon) = \ell_2^m \left(\frac{r_1}{R}\right)^2 f(\epsilon) + \frac{2}{3} \ell_1^m \left(\frac{r_1 r_2}{R^2}\right)^2 \left\{ \frac{(\pi - \epsilon) \cos \epsilon + \sin \epsilon}{\pi} \right\}.$$

The phase law, as expressed in the various forms, can be made to apply to the light reflected from the secondary by replacing r_1 by r_2 and ϵ by $\pi - \epsilon$.² The total amount of integrated light added by reflection to the intrinsic light of the system is given by

$$(318) \quad \ell'(\epsilon) = \ell_1'(\epsilon) + \ell_2'(\epsilon) = \ell_2^m \left(\frac{r_1}{R}\right)^2 f(\epsilon) + \ell_1^m \left(\frac{r_2}{R}\right)^2 f(\pi - \epsilon) + \frac{2}{3} \left(\frac{r_1 r_2}{R^2}\right)^2 \cos \epsilon \left[(\pi - \epsilon) \ell_1^m - \epsilon \ell_2^m \right],$$

where in general³ $\cos \epsilon = \cos \phi \sin i$, if ϕ denotes the true anomaly of the secondary component measured from inferior conjunction.

When $\phi = 0^\circ$ the hotter star, i.e., the primary, is behind the cooler.

As equation (318) stands it applies whether the relative orbit of the two components is circular or eccentric. If the orbit is circular, R is constant and ϵ is directly proportional to the time.⁴

1 The reason why Takeda, Sen and Kopal only considered terms up to and including the fourth order can be seen as follows. According to the previous results regarding distorted stars, the distortion will manifest itself in terms of the order of $(r_1/R)^3$. The reflection effect itself is proportional to the exposed area, a quantity of degree $(r_1/R)^2$. Thus the total reflected light should be affected by distortion in the terms $(R)^6$ and higher. Having considered spherical stars, these authors were restricted to quantities of the order $(R)^4$.

2 In Kopal's expression (314), u_2 of (315) must be replaced by u_1 .

3 Although none of the phase laws have been derived using an arbitrary inclination, this approximation is customarily used.

4 This is true only if $i = 90^\circ$.

If the orbit is an ellipse $\cos \epsilon$ is the same as noted above, and the separation R varies as

$$(319) \quad \frac{r}{R} = \frac{r [1 + e \sin (\omega - \phi)]}{A (1 - e^2)}.$$

When relation (319) is substituted into (318) the amount of reflected light will be observed to vary, and it will become asymmetric with respect to the conjunctions.

To the degree of accuracy necessary in a solution for the preliminary elements by the procedure of rectification, the expression¹

$$(320) \quad f(\epsilon) = 0.29 + 0.38 \cos \epsilon + 0.09 \cos 2\epsilon$$

may be adopted as a good general approximation for the effect of reflection from the primary. The combined effect of reflection then becomes

$$\begin{aligned} \ell'(\epsilon) &= \ell_2^m \left(\frac{r_1}{R}\right)^2 f(\epsilon) + \ell_1^m \left(\frac{r_2}{R}\right)^2 f(\pi - \epsilon) \\ &= \left[\ell_2^m \left(\frac{r_1}{R}\right)^2 + \ell_1^m \left(\frac{r_2}{R}\right)^2 \right] (0.29 + 0.09 \cos 2\epsilon) \\ &\quad + 0.38 \left[\ell_2^m \left(\frac{r_1}{R}\right)^2 - \ell_1^m \left(\frac{r_2}{R}\right)^2 \right] \cos \epsilon, \end{aligned}$$

or

$$(321) \quad \begin{aligned} \ell'(\epsilon) &= \left[\ell_2^m \left(\frac{r_1}{R}\right)^2 + \ell_1^m \left(\frac{r_2}{R}\right)^2 \right] (0.29 - 0.09 \cos^2 i + 0.09 \sin^2 i \cos 2\phi) \\ &\quad + 0.38 \left[\ell_2^m \left(\frac{r_1}{R}\right)^2 - \ell_1^m \left(\frac{r_2}{R}\right)^2 \right] \sin i \cos \phi. \end{aligned}$$

The term in $\cos 2\phi$ merges with the corresponding terms due to ellipticity and diminishes them numerically. The term in $\cos \phi$ led

¹ This expression is obtained by substituting into Kopal's equation (314) the values $\frac{r_1}{R} = \frac{r_2}{R} = 0.2$, and $u = 0.6$.

to the discovery of the effect, but it disappears for stars of equal surface brightness.

Equation (318) and (321) apply to the total integrated radiation, since it is only for this case that the heat albedo is known. If the theoretical results obtained are to be useful in comparison with observation carried out in any spectral range, it must be possible to translate the heat curves into light curves for any given wavelength. It is important to note that if the illuminating and reflecting component are of different spectral types, the incident light will be returned to the outside by gaseous matter whose temperature is different from that of the emitting source. If Thomson scattering predominates, as is likely in the atmospheres of early type stars, the spectral distribution of the returned light will remain practically unchanged. However, when the reflection effect is dominated by the process of absorption followed by re-emission, the spectral distribution of reflected light may differ greatly from that of the incident radiation. Unless the temperatures of both components of a binary system are the same — neglecting scattering by free electrons — the observed efficiency of the incident and reflected radiation will generally be different. Consequently, the observed monochromatic reflection may be greater or less than that appropriate for total radiation.

Let E_1 and E_2 represent the luminous efficiencies of the primary and secondary stars, respectively, and let J_1 and J_2 denote the respective monochromatic flux densities of either star at the other. The energy from the secondary incident on the primary is J_2/E_2 .

Suppose this is absorbed and re-emitted with luminous efficiency E_1 . The light stimulating ability of the secondary will be given by J_2/E_2 . This quantity may be substituted for Q_2^m in equation (321), if the corresponding light stimulating ability J_1/E_1 of the primary is substituted for Q_1^m . The term on the left will then be the contribution of monochromatic flux density at any phase by reflection. To obtain a more convenient notation set

$$(322) \quad G_1 = \frac{J_2 E_1}{E_2} \left(\frac{r_1}{R} \right)^2$$

and

$$(323) \quad G_2 = \frac{J_1 E_2}{E_1} \left(\frac{r_2}{R} \right)^2.$$

Then equation (321) becomes

$$(324) \quad \mathcal{L}'(\epsilon) = (G_2 + G_1)(0.29 - 0.07 \cos^2 i + 0.09 \sin^2 i \cos 2\phi) - 0.38 (G_2 - G_1) \sin i \cos \phi.$$

If I_1 and I_2 denote the mean surface brightness of the primary and secondary stars then $J_1 = I_1 \left(\frac{r_1}{R} \right)^2$, $J_2 = I_2 \left(\frac{r_2}{R} \right)^2$ and

$$(325) \quad \frac{G_2}{G_1} = \frac{I_1/E_1^2}{I_2/E_2^2}.$$

For the purpose of rectification it is sufficient to set I_1/I_2 equal to the ratio of the eclipse depths, rectified for ellipticity.¹

The values of E_1 and E_2 are more difficult to determine.

The obvious assumption that stars radiate like grey bodies is weak in this connection. Not only are the energy curves of normal stars distorted by variations of opacity with wave length, but also they are modified by molecular absorption, in the cool stars, and by the hydrogen absorption, in class A stars. In B type stars electron

¹ H. N. Russell, Ap.J., 104, 157, 1946.

scattering becomes important and light is reflected without appreciable change of wave length.

An investigation by Pismis¹ of the differences between the observed reflection effect and that computed for integrated light from the elements of several eclipsing systems showed a systematic trend with the ratio of luminous efficiencies of the two components. The systematic trend was largely removed by an application of the grey body values for the luminous efficiency.

Presently, grey body theory is probably the best guide available. The following table of values was given by Russell,² and is based on $C_2/\lambda T = 4$, for which E is very near its maximum value which is taken as unity. On the grey body assumption I/E^2 depends only on the temperature. Kuiper's³ temperatures corresponding to particular spectral types were used in the calculation of this table.

$$\lambda = 5290$$

Type	B0	B5	A0	A5	F0	F5	G0	dG5	dK0	dM0
log I	1.44	1.05	0.67	0.35	0.17	-0.08	-0.23	-0.46	-0.67	-1.54
log I/E	3.12	1.81	0.93	0.41	0.18	-0.07	-0.19	-0.35	-0.45	-0.66

$$\lambda = 4250$$

log I	1.24	0.85	0.38	0.02	-0.22	-0.54	-0.71	-0.98	-1.26	-2.35
log I/E	2.37	1.29	0.45	0.01	-0.18	-0.40	-0.48	-0.56	-0.62	-0.62

1 P. Pismis, Ap.J., 104, 161, 1946.

2 H. N. Russell, Ap.J., 104, 156, 1946.

3 G. P. Kuiper, Ap.J., 88, 429, 1938.

Russell and Merrill¹ have plotted these values, in a later publication, which makes the data readily accessible. They conclude that it is better to use the value of I_1/I_2 derived from the depths of minima even if the spectral type is known, because the spectral type of the bright side of the secondary may be greatly altered by ionization due to the radiation of the primary.

To use the table it is necessary to know the spectral type of the system. Usually the observed spectrum of the combined light will be that of the primary component. Thus the values of I_1 and I_1/E_1^2 can be determined. Then with sufficient accuracy one may assume the ratio I_1/I_2 to be equal to the ratio of the depths of minima, rectified for ellipticity. The value of I_2 may then be calculated and I_2/E_2^2 taken from the table.

¹ H. N. Russell and J. E. Merrill, Contr. Princeton U Obs., 26, 47, 1952.

RECTIFICATION

The manifest properties of eclipsing binary systems which require the use of the ellipsoidal model, and hence rectification, were discussed earlier. The process of rectification will now be investigated in detail.¹

Consider two stars of any equilibrium form and of any distribution of intensity over their surfaces. The variable light l_g from the larger star as influenced by its changing presentation but not by eclipse will be related to its contribution J_g to the observed light by

$$(326) \quad J_g = l_g(1 - F_g)$$

where F_g is the fraction of l_g lost by eclipse at any phase. A similar equation holds for the smaller star,

$$(327) \quad J_s = l_s(1 - F_s) .$$

The combined light of the system at any phase will be

$$(328) \quad l = l_g(1 - F_g) + l_s(1 - F_s) .$$

Then during occultation

$$(329) \quad F_s = \frac{l - l_s}{l_s} ,$$

and during transit

$$(330) \quad F_g = \frac{l^{tr} - l_g}{l_g} .$$

In order to express F_s and F_g conveniently in terms of the functions which apply to the spherical model, the geometrical and radiative properties of the idealized system must be related simply

¹ These opening arguments were advanced by H. N. Russell, Ap.J., 108, 390, 1948, and Ap.J., 104, 153, 1946.

and uniquely to the parameters u , k and p , which appear as arguments in tables of the spherical model functions α and β .

The ratio of the radii of the circular discs, k , in the spherical model is equivalent, in a given idealized system, to the condition that the apparent discs must at all phases be bounded by similar curves. This requires the surfaces of the components to be similar in form, the ratio of the dimensions being k .

The same significance for p may be retained if one assumes that the boundaries of the discs are centrally symmetrical, similar and similarly orientated. If δ is the apparent distance of the centers at any phase and a_s and a_g are the radii of the discs along the line joining the centers, then the relation

$$(331) \quad \delta = a_g + p a_s = a_g(1 + kp)$$

will make $p = -1$ at internal tangency, $p = 1$ at external tangency, and $p = 0$ when the center of the smaller disc lies on the boundary of the larger.

By requiring F to have the circular value, f , for two uniformly illuminated discs, one demands that these discs shall be reducible to circles by transformations in which the area of every element in the plane is altered in the same ratio. This and the restriction of central symmetry demands elliptical discs and ellipsoidal components.

Assuming that these restrictions have been made, it is now possible to write equation (328) as

$$(332) \quad l = l_g(1 - \alpha^{tr}\beta^{tr}) + l_s(1 - \alpha^{oc}) = l_g(1 - f_g) + l_s(1 - f_s),$$

where $\alpha^{oc} = 0$ except during occultation, and $\alpha^{tr} = 0$ except during

transit. Hence, outside eclipse $l = l_s + l_g$. During occultation

$$(333) \quad l^{oc} = l_s + l_g - \alpha^c l_s = l_s + l_g - f_s l_s$$

and during transit

$$(334) \quad l^{tr} = l_s + l_g - \alpha^{tr} \beta^{tr} l_g = l_s + l_g - f_g l_g.$$

In these relations l is the observed quantity and l_s and l_g are not known individually.

When the procedure of rectification was first introduced it was applied to uniformly illuminated discs only. In view of increased knowledge about variations in intensity over the surface of distorted components, the question arises as to whether this condition is a necessary one. Obviously it is not. The process of rectification will still apply exactly to limb- and gravity-darkened ellipsoids provided that

1) the isophotes on each apparent disc of the distorted components, at any time, are curves similar to and concentric with the limb, and

2) the apparent surface brightness along an isophote of dimensions $\sin \gamma$ times that of the limb is $I_0(1 - u + u \cos \gamma)$.

Now one may ask whether the isophotes on the apparent discs of limb- and gravity-darkened ellipsoids ever constitute such a family of curves.

Russell¹ investigated this problem in detail and found the above conditions to be satisfied in two cases: when the discs are uniformly illuminated; and when they are completely darkened and the gravity-darkening coefficient is given, in the present notation, by

¹ H. N. Russell, Ap.J., 95, 352, 1942, and Ap.J., 102, 11, 1945.

$$(335) \quad \tau = \frac{\Delta_2}{5 - \Delta_2},$$

where $\Delta_2 = \frac{5}{2 - \eta_2(a)}$, and $\eta_2(a)$ is the solution of equation (200) for $j = 2$.

These expressions for the laws of surface brightness, which together make the isophotes on the apparent discs similar to and concentric with the boundaries, may be derived as follows.

Let the ellipsoid E be given by

$$(336) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = a^2.$$

The transformation

$$(337) \quad \begin{aligned} x &= ax', \\ y &= by', \\ z &= cz', \end{aligned}$$

will convert it into the sphere S' defined by

$$(338) \quad x'^2 + y'^2 + z'^2 = a^2.$$

Any line — including the line of sight — with direction cosines l_0, m_0, n_0 transforms into a new line with direction cosines l'_0, m'_0, n'_0 given by

$$(339) \quad \begin{aligned} l'_0 &= \frac{l_0}{ha}, \\ m'_0 &= \frac{m_0}{hb}, \\ n'_0 &= \frac{n_0}{hc}, \end{aligned}$$

where

$$(340) \quad h^2 = \frac{l_0^2}{a^2} + \frac{m_0^2}{b^2} + \frac{n_0^2}{c^2}.$$

A plane tangent to the ellipsoid at any point of its surface is given by

$$(341) \quad lx + my + nz = p,$$

where l, m, n are the direction cosines of the outward surface normal at the point of tangency. This transforms into

$$(342) \quad l'x' + my' + nz' = p',$$

where

$$(343) \quad \begin{aligned} l' &= \frac{al}{H}, \\ m' &= \frac{bm}{H}, \\ n' &= \frac{cn}{H}, \\ p' &= a. \end{aligned}$$

The quantity H is expressed by the relation

$$(344) \quad H^2 = a^2l^2 + b^2m^2 + c^2n^2.$$

To the first order in a, b, c , the distance r to any point on the surface of the ellipsoid from its center is $r = aH$.

The angle γ between the normal to the plane (341) and the line of sight is given by

$$(345) \quad \cos \gamma = ll_0 + mm_0 + nn_0,$$

and the angle γ' between the transformed line of sight and the transformed plane becomes

$$(346) \quad \cos \gamma' = l'l'_0 + m'm'_0 + n'n'_0.$$

The substitution of expressions (339) and (343) into (346) reveals

$$(347) \quad \cos \gamma = hH \cos \gamma'$$

An element of surface $d\sigma$ on E with direction cosines l, m, n transforms into an element $d\sigma'$ on S' with direction cosines l', m', n' .

The ratios of their projections of the planes $x'=0$, $y'=0$ and $z'=0$ are, respectively,

$$(348) \quad \begin{aligned} \frac{\ell d\sigma}{\ell' d\sigma'} &= \frac{dy dz}{dy' dz'} = bc, \\ \frac{m d\sigma}{m' d\sigma'} &= \frac{dx dz}{dx' dz'} = ac, \\ \frac{n d\sigma}{n' d\sigma'} &= \frac{dx dy}{dx' dy'} = ab. \end{aligned}$$

Then

$$(349) \quad H d\sigma = abc d\sigma'$$

and

$$(350) \quad \cos \gamma d = abch \cos \gamma' d\sigma'.$$

Let I be the monochromatic specific intensity of any surface element $d\sigma$ on E as seen by an observer O at a great distance Δ .

The flux density received by the observer O will be

$$(351) \quad dJ = \frac{I \cos \gamma d\sigma}{\Delta^2}.$$

An observer O' at the same distance Δ from S' will receive the flux density

$$(352) \quad dJ' = \frac{I' \cos \gamma' d\sigma'}{\Delta^2},$$

where I' is the corresponding specific intensity at a given frequency of element $d\sigma'$ on S' . If the transformation is to be useful the observed flux densities from each element must be the same, requiring that

$$(353) \quad I' = abch I,$$

where h is independent of position on E or S' but depends on the direction from which E is observed.

The isophotes on the apparent discs of E and S' will pass through

corresponding points according to the transformation (337). If the isophotes on S' are concentric circles, those on E will be ellipses similar to and concentric with the limb, and vice versa,¹ Use the preceding results and the expression

$$(354) \quad I = I_0(1 - u - u \cos \gamma) \left\{ 1 - \tau \left(1 - \frac{g}{g_0} \right) \right\},$$

giving the distribution of intensity over the surface of a distorted star, to find the conditions for which the isophotes on S' are concentric circles.

Upon considering ellipsoids, only the second order harmonic terms in the expressions for distorted stars need to be considered. Realizing this the value of the relative surface gravity becomes, using equation (223),

$$(355) \quad 1 - \frac{g}{g_0} = [1 - \eta_2(a)] \cdot (S_r + S_2) = \left[\frac{5}{\Delta_2} - 1 \right] \cdot (S_r + S_2).$$

On comparing equation (208) with $r = aH$ one finds

$$(356) \quad H = 1 + S_r + S_2.$$

Substituting (347), (355) and (356) into (354) yields

$$(357) \quad I = I_0 \left\{ (1 - u)(1 - qs) + uh(1 + s - qs) \cos \gamma' \right\},$$

where

$$q = \tau \left[\frac{5}{\Delta_2} - 1 \right],$$

$$s = S_r + S_2.$$

Equation (357) becomes simple in only two cases, i.e. when $u = q = 0$, for which

$$(358) \quad I = I_0 \quad \text{and} \quad I' = abch I_0,$$

and when $u = q = 1$, for which

$$(359) \quad I = I_0 \frac{\cos \gamma'}{H} \quad \text{and} \quad I' = I_0 abch^2 \cos \gamma'.$$

The first case corresponds to uniformly illuminated discs with $\tau = 0$; the second corresponds to complete limb-darkening and the partial gravity-darkening $\tau = \frac{\Delta_2}{5 - \Delta_2}$. In either case the isophotes $I' = \text{constant}$ yield concentric circles on S' and concentric ellipses on E , irrespective of the phase. A similar theorem¹ holds for powers of $\cos \gamma$ and h , but the only cases of physical significance are those cited here. Kopal² remarks that if an extension to higher harmonics is made the isophotes can under no circumstances form a symmetrical family of curves, whatever the amount of limb or gravity darkening.

Now substitute relations (358) and (359) individually into (352) and integrate to obtain the respective total relative monochromatic flux densities of E toward O , or S' toward O' ,

$$(360) \quad J = \pi a^2 abch I_0 ,$$

$$(361) \quad J = \frac{2\pi a^2}{3} abch I_0 .$$

The direction cosines of the line of sight with respect to the x y z axes are

$$(362) \quad \begin{aligned} \ell_0 &= \sin i \cos \phi , \\ m_0 &= \sin i \sin \phi , \\ n_0 &= \cos i . \end{aligned}$$

The equations of this section are still quite general if the following substitutions are made

$$(363) \quad \begin{aligned} a^2 &= 1 , \\ \frac{b^2}{a^2} &= 1 - e_e^2 , \\ \frac{c^2}{a^2} &= 1 - e_m^2 , \end{aligned}$$

where e_e denotes the equatorial, and e_m the meridonal eccentricity.

¹ H. N. Russell, Ap.J., 102, 11, 1945.

² Z. Kopal, Harvard Obs. Mon., 6, 203, 1946.

When equations (362) and (363) are used in (340) there results

$$(364) \quad h^2 = h_0^2 \left\{ 1 - e_e^2 \sin^2 j \cos^2 \phi \right\},$$

where

$$(365) \quad h_0^2 = \frac{\sin^2 i}{1 - e_e^2} + \frac{\cos^2 i}{1 - e_m^2}$$

and

$$(366) \quad \frac{1 - e_e^2}{1 - e_m^2} = \frac{\tan^2 i}{\tan^2 j} = \frac{b^2}{c^2}.$$

It is customary to write (364) as

$$(367) \quad h^2 = h_0^2 \left\{ 1 - z \cos^2 \phi \right\}$$

with

$$(368) \quad z = e_e^2 \sin j,$$

z being called the "effective ellipticity".

The light received at a distant point by either component will be

$$(369) \quad \ell_{s,g} = \ell_{s,g}^m (1 - z \cos^2 \phi)^{\frac{1}{2}},$$

if $u = 0$ and $\tau = 0$, $\ell_{s,g}^m$ being the maximum light due to the particular component. Similarly,

$$(370) \quad \ell_{s,g} = \ell_{s,g}^m (1 - z \cos^2 \phi)$$

if $u = 1$ and τ is given by (335).

It is now instructive, although not very practical, to consider the process of rectification for the above two cases. For these cases rectification is exact. In practice such conditions are seldom, if ever, encountered. For the ellipsoidal model, with primary and secondary components denoted by "1" and "2" respectively, one finds using (369) and remembering that $I = \text{constant}$

$$(371) \quad \frac{\ell_1}{\ell_1^m} = \frac{\ell_2}{\ell_2^m} = (1 - z \cos^2 \phi)^{\frac{1}{2}}.$$

The total light then becomes

$$(372) \quad \ell = \ell^m (1 - z \cos^2 \phi)^{\frac{1}{2}}$$

when the effect of eclipses is not included. If equation (372) is divided by $(1 - z \cos^2 \phi)^{\frac{1}{2}}$ to obtain

$$(373) \quad \ell_r = \ell^m = \frac{\ell}{(1 - z \cos^2 \phi)^{\frac{1}{2}}},$$

the effect of ellipticity is eliminated and the rectified light ℓ_r is constant between eclipses just as for the spherical model. The rectified light at any phase is readily obtained by replacing ℓ^m by equation (332) written for the maximum light. This is permissible since the phase variation is included entirely in the $(1 - z \cos^2 \phi)^{\frac{1}{2}}$ term.

The total light corresponding to (370) is

$$(374) \quad \ell = \ell^m (1 - z \cos^2 \phi).$$

When this is divided by $(1 - z \cos^2 \phi)$ one finds

$$(375) \quad \ell_r = \ell^m = \frac{\ell}{(1 - z \cos^2 \phi)},$$

which is again completely rectified. An analogous statement applies to the light at any phase as is given above.

From equation (304) it is clear that only tidal distortion gives rise to light variations between minima, the rotational distortion merely having the effect of introducing a term, which is constant for any given model, into the apparent light of each star. The influence of the slight changes in intensity distribution produced by rotation on the light received from an eclipsing system will be apparent only in the shape of the minima. These changes may be regarded as refinements, which should be handled by perturbation methods. If all the

elements of two distorted stars are the same up through the second order harmonic terms, except the lengths of the polar axes, the light observed from them will be identical outside eclipse if the value of j given by (366) is the same.¹ This is the justification for considering the ellipsoidal model to consist of prolate ellipsoids.

Employ this model and take the smaller and larger stars to be similar prolate ellipsoids with semi-major axes a_s and a_g , and semi-minor axes b_s and b_g , respectively. Assume the eccentricity of the meridian section to be e_m . In projection on a plane perpendicular to the line of sight the two ellipsoids will appear as two elliptical discs whose major axes coincide with the line of centers. The semi-minor axes b'_s and b'_g of the apparent discs will always be b_s and b_g , but the projected semi-major axes a'_s and a'_g will usually be smaller than a_s or a_g . At any instant a'_g is given exactly by

$$(376) \quad a'_g{}^2 = a_g^2 (1 - z \cos^2 \phi),$$

where

$$(377) \quad z = e_m^2 \sin^2 j.$$

Since equation (331) holds for the apparent distance ξ between centers of the two components, the fundamental equation (60) now becomes

$$(378) \quad \cos^2 j + \sin^2 j \sin^2 \phi = a_g^2 (1 + kp)^2 (1 - z \cos^2 \phi).$$

Remember that for circular orbits $\phi = \theta$. This equation may be modified for phase by setting²

$$(379) \quad \sin^2 \theta = \frac{\sin^2 \phi}{1 - z \cos^2 \phi}.$$

¹ Theoretically this applies only to the two special cases of darkening discussed previously.

² It may be possible to rectify for phase in eccentric systems approximately by assuming the components to rotate uniformly rather than with the variable angular velocity as indicated in the equilibrium theory of tides.

Then

$$(380) \quad 1 - z \cos^2 \phi = \frac{1 - z}{1 - z \sin^2 \Theta} .$$

Placing (380) into (378) yields

$$(381) \quad \frac{\cos^2 j}{1 - z} + \frac{(\sin^2 j - z) \sin^2 \Theta}{1 - z} = a_g^2 (1 + kp)^2 .$$

Define a new fictitious inclination i_r by

$$(382) \quad \cos^2 i_r = \frac{\cos^2 j}{1 - z} .$$

Hence, equation (381) becomes

$$(383) \quad \cos^2 i_r + \sin^2 i_r \sin^2 \Theta = a_g^2 (1 + kp)^2 ,$$

which is identical in form with the equation for spherical stars moving in circular orbits.¹

In summary these results show that if the observed light ℓ , given by relation (372) or (374), is rectified by dividing by $(1 - z \cos^2 \phi)^{\frac{1}{2}}$ or $(1 - z \cos^2 \psi)$, respectively, and the observed mean anomaly is rectified by equation (379), the resulting light curve will be identical with that produced by a pair of spherical stars having radii a_g , ka_g and moving in a circular orbit with inclination i_r .

Having rectified the curve in the previous manner and solved for a_g , k and i_r by methods outlined in the section on circular orbits, the remaining elements may be determined. From equations (366) and (382) one finds²

$$(384) \quad \tan^2 i_r = (1 - z) \sec^2 j - 1 = (1 - e_m^2) \tan^2 j ,$$

and then from (368)

$$e_m^2 = z \csc^2 j .$$

¹ H. N. Russell, Ap.J., 108, 388, 1948.

² From equations (366) and (384) it follows that $a \tan i_r = b \tan j = c \tan i$.

Further

$$(385) \quad b_g^2 = a_g^2(1 - e_m^2)$$

and

$$(386) \quad a_s = ka_g, \quad \text{and} \quad b_s = kb_g,$$

The quantity z may be determined by plotting the observed values ρ^2 , given by (372), against $\cos^2\phi$, or ℓ , given by (374), against $\cos^2\phi$. The first factor of (372) and (374) will be constant outside minima. The plots near $\phi = 90^\circ$ will be straight lines with slopes z , which may be found quite readily.¹

The realization that virtually all eclipsing systems show intermediate degrees of darkening should also suggest that the foregoing rectification procedure will no longer be exact if applied to such systems. It is at this stage where the significance of the term "model" becomes salient.

The theoretical expression for the light at any phase between minima, neglecting reflection, has already been given by equation (304). If only first order terms are retained one obtains a formula useful for rectification purposes. The earlier conclusions concerning distorted stars indicate that the tidal distortion S_2 makes the surface of a star approximately a prolate ellipsoid with the greatest axis along the line of centers. Since only the tidal distortion influences the observed light between minima and the expression for the combined light involves weighted means of the z 's, one may assume the tidal oblateness ϵ_m in the principle meridian² to be the same for each component. The neglect of rotational distortion demands that

¹ This method for determining z was advanced by H. N. Russell, Ap.J., 36, 54, 1912.

² The expression to be used for ϵ_m is obtained from equation (218) by neglecting the rotational part $\frac{a^3\omega^2}{3mG}\Delta_2$.

$\sin i$ be replaced by $\sin j$ as indicated before. Thus equation (304) for the total light between eclipses of stars moving in circular orbits becomes

$$(387) \quad \ell = \ell^m \left(1 - \frac{3z_2 \sin^2 j}{4} - \frac{3z_2 \sin^2 j \cos 2\phi}{4} \right),$$

where the constant term of the second order Legendre polynomial is absorbed in ℓ^m .

This may be compared with the expression corresponding to uniformly illuminated discs, which in the present notation becomes

$$(388) \quad \ell = \ell^m \left(1 - \frac{3w_2 \sin^2 j}{4} - \frac{3w_2 \sin^2 j \cos 2\phi}{4} \right).$$

In the derivation of (387) and (388), only terms of the first order in the oblateness were retained. To this degree of accuracy the tidal oblateness ϵ_m and the eccentricity e_m of the principal meridian are related by

$$(389) \quad e_m^2 = 2\epsilon_m = 3w_2.$$

At the preliminary stages of analysis one can only make some very general assumptions about the system in order to perform the initial rectification, unless some spectroscopic data are available. Here rectification for ellipticity will be pursued when the only information at one's disposal is the photometric observations.

Russell¹ suggests, from a study of the apsidal motions, that actual stars are considerably concentrated; sufficiently concentrated for the values of $\eta_2(a) = 3$, derived for the Roche model, to be adopted, as a first approximation. Let this be used for each component, and set $m_1 = m_2$ to obtain

$$(390) \quad z_2 = \frac{e_m^2 (15 + u)(1 + \tau)}{3(15 - 5u)}.$$

¹ H. N. Russell, Ap.J., 90, 671, 1939.

The darkening coefficients may be assumed equal at this point.

Introducing the eccentricity into equations (387) and (388), respectively, there results

$$(391) \quad \ell = \ell^m \left(1 - \frac{zN}{4} - \frac{zN}{4} \cos 2\phi\right) = \ell^m \left(1 - \frac{zN}{2} \cos^2\phi\right)$$

where

$$(392) \quad N = \frac{(15 + u)(1 + \tau)}{15 - 5u},$$

and

$$(393) \quad \ell = \ell^m \left(1 - \frac{z}{4} - \frac{z}{4} \cos 2\phi\right) = \ell^m \left(1 - \frac{z}{2} \cos^2\phi\right).$$

To the first order, therefore, the combined effects of limb and gravity-darkening upon the light curve between minima may be represented by increasing the true ellipticities in the ratio \sqrt{N} , or the effective ellipticity in the ratio N . Russell¹ calls the latter increased value the "photometric ellipticity".

Let the observed light between eclipses, as influenced by ellipticity only, be represented by

$$(394) \quad \ell = A'_0 + A'_2 \cos 2\phi,$$

where A'_2 is negative. Then

$$(395) \quad \ell^m = A'_0 - A'_2$$

and

$$(396) \quad zN = \frac{-4A'_2}{A'_0 - A'_2}.$$

The rectified light ℓ_r is then found by division according to the relation

$$(397) \quad \ell_r = \frac{\ell(A'_0 - A'_2)}{A'_0 - A'_2 \cos 2\phi} = \ell^m.$$

¹ H. N. Russell, Ap.J., 90, 641, 1939.

The phase is again rectified by equation (379).

Obviously this rectification procedure requires a knowledge of u and τ . The correlation between u and τ is probably sufficient to permit N to be estimated from u . Russell and Merrill¹ suggest using the tentative values for N of 2.2, 2.6, or 3.2, corresponding to the respective values for u of 0.4, 0.6, or 0.8. The uncertainty in the adopted value of N will affect only the rectification of phase and the calculated eccentricity. Fortunately the effect on rectification will be small.

In order to perform a complete rectification it is necessary to account for the reflection effect. Because of the presence of terms in $\cos \phi$, with opposite signs, in the uneclipsed light reflected from the components, it is impossible to assume the ratio of either to their sum is constant. This immediately excludes rectification by division which was used for ellipticity. The terms in $\cos \phi$ must be removed before rectification by division can be attempted. Such a removal of terms may be accomplished in two ways; by subtracting from the observed light the quantity²

$$\begin{aligned}
 (398) \quad \ell' &= (G_2 + G_1)(0.29 - 0.09 \cos^2 i + 0.09 \sin^2 i \cos 2\phi) \\
 &\quad - 0.38 (G_2 - G_1) \sin i \cos \phi \\
 &= C_0' + G_1' \cos \phi + G_2' \cos 2\phi,
 \end{aligned}$$

or by adding to the observed light the quantity

¹ H. N. Russell and J. E. Merrill, Contr. Princeton U. Obs., 26, 42, 1952.

² Whether i , j , or i_r is used for the inclination is quite immaterial at this point because the reflection formulae were derived for an inclination of 90° , and thus are only approximate at the outset.

$$\begin{aligned}
 (399) \quad \ell' &= (G_2 + G_1)(0.29 - 0.09 \cos^2 i + 0.09 \sin^2 i \cos 2\phi) \\
 &\quad + 0.38 (G_2 - G_1) \sin i \cos \phi \\
 &= G_0 + G_1 \cos \phi + G_2 \cos 2\phi
 \end{aligned}$$

When rectification is performed by subtraction one immediately obtains the light curve as influenced by ellipticity only, except for an error within minima due to the subtracted light. At this stage there is no way to compensate for the difference between the true loss of light (including reflection) of the eclipsed star and the rectified light obtained. The result will be that the shape of the rectified minima may not be exactly representable by the ellipsoidal model. The total light of each star as calculated from either minimum of the rectified curve will be influenced by the errors in light at mid-minima. The subtraction of light will render the minima too deep, thereby making the computed light too small by $0.76 G_1$ for the primary star, and by $0.076 G_2$ for the secondary. In order to obtain the light of each star the following corrections must be applied.

$$\begin{aligned}
 (400) \quad \Delta L_1 &= + 0.76 L_2 \left(\frac{r_1}{R}\right)^2 \left(\frac{E_1}{E_2}\right) \\
 \Delta L_2 &= + 0.76 L_1 \left(\frac{r_2}{R}\right)^2 \left(\frac{E_2}{E_1}\right)
 \end{aligned}$$

When rectification is performed by addition, the symmetric part of the reflection effect is doubled. It may be removed by introducing appropriate terms in the process of rectification by division. From equation (399) it is evident that the addition of light diminishes the apparent effect of ellipticity. The addition of light will make the rectified minima too shallow, thus making the computed light too great by $0.76 G_1$ for the primary star, and by $0.076 G_2$ for the secondary. To obtain the light of each star the calculated values must be reduced by applying the corrections

1 The light as corrected will still be influenced by distortion.

$$(401) \quad \Delta L_1 = -0.76 L_2 \left(\frac{r_1}{R}\right)^2 \left(\frac{E_1}{E_2}\right)$$

$$\Delta L_2 = -0.76 L_1 \left(\frac{r_2}{R}\right)^2 \left(\frac{E_2}{E_1}\right).$$

If the observed light outside eclipse is represented by

$$(402) \quad \mathcal{L} = A_0 + A_1 \cos \phi + A_2 \cos 2\phi,$$

then A_1 will be influenced by reflection only, but A_2 by reflection and ellipticity. In this expression A_1 and A_2 are both negative if ϕ is measured from mid-primary minimum. Hence from expression (324) for the actual contribution of reflected light one finds

$$C'_1 = A_1,$$

$$(403) \quad C'_0 = - (0.76 - 0.24 \cos^2 i) \frac{(G_2 + G_1)}{(G_2 - G_1)} A_1 \csc i,$$

$$C'_2 = -0.24 \frac{(G_2 + G_1)}{(G_2 - G_1)} A_1 \sin i,$$

where C'_1 is negative and C'_0 and C'_2 are positive. If one compares (324) with the terms (399) to be added in rectification one finds

$$G_1 = -A_1,$$

$$(404) \quad G_0 = - (0.76 - 0.24 \cos^2 i) \frac{(G_2 + G_1)}{(G_2 - G_1)} A_1 \csc i,$$

$$G_2 = -0.24 \frac{(G_2 + G_1)}{(G_2 - G_1)} A_1 \sin i,$$

where all of the G 's are positive.

For practical application of either method the values of G_2 and G_1 must be known. The method of finding $\frac{I_2}{I_1}$ and $\frac{G_2}{G_1}$ has been discussed on page 115. However, if $\frac{G_2}{G_1}$ is greater than two, equations (403) or (404) may be used. When determining the elements for the first time one must set $\sin i = 1$, unless a better value is available.

When the depths of minima are nearly equal then the term A , is so small that it is useful only for removing the term $A \cos \phi$. The values of G_2 and G_1 become approximately the same and the values of the G 's and the C 's are poorly determined by (403) and (404), rendering the rectification procedure outlined above inaccurate. In such cases equations (398) or (399) must be employed. From equation (322) and (323) one finds

$$(405) \quad G_2 + G_1 = \frac{G_2 + G_1}{(G_1 G_2)^{\frac{1}{2}}} (J_1' J_2')^{\frac{1}{2}} \left(\frac{F_1 F_2}{R^2} \right),$$

where the first factor has a value close to two. The term $(J_1' J_2')^{\frac{1}{2}}$ is at most 0.50. If either eclipse shows a constant phase, the loss of light in it rectified for ellipticity is J_1'' or J_2'' and the second factor is obtained exactly. Should the eclipse be partial, J_1' and J_2' are greater than the losses in primary and secondary minima, respectively. Russell and Merrill¹ found that for a depth of $p = 0.20$, $(J_1' J_2')^{\frac{1}{2}} > 0.40$; hence, the error of assuming a mean value 0.45 would seldom exceed ten per cent. The value of the last factor can be estimated from the duration of eclipses. Writing equation (60) for the times of external

¹ H. N. Russell, and J. E. Merrill, Contr. Princeton U. Obs., 26, 49, 1952.

contact and conjunction when $\phi = \theta_e$ and $\varphi = 0$, respectively, one obtains

$$(406) \quad \frac{r_1 r_2}{R^2} = \frac{\sin^2 \theta_e \sin^2 i}{(1 - p_0)(2 + k + kp_0)} = F \sin^2 \theta_e \sin^2 i .$$

The following values of F are typical for certain eclipse types.¹

Type	p_0	$k = 1.0$	$k = 0.8$	$k = 0.6$	$k = 0.4$
Very shallow partial	-0.5	0.571	0.625	0.690	0.769
Shallow "	0.0	0.333	0.357	0.384	0.417
Deep "	-0.5	0.267	0.278	0.290	0.303
Grazing total	-1.0	0.250	0.250	0.250	0.250
Concentric total	-1/k	0.250	0.247	0.234	0.204
Depth for total ecl. ($u=0$)		0.50	0.39	0.26	0.14

Combining results for the three factors, relation (405) becomes approximately

$$(407) \quad G_2 + G_1 = 0.24 \sin^2 \theta_e$$

for total eclipses. This value is the same for grazing total eclipses, but it increases with increasing eclipse depths as the eclipses become partial. Deep partial eclipses are more likely to be observed than shallow ones, so a tentative value of 0.30 for the coefficient may be used when the minima are of nearly equal depths.

For the particular case under consideration, rectification for reflection can be made by subtracting from or by adding to the observed light the quantity

$$(408) \quad \mathcal{L}' = (0.070 + 0.022 \cos 2\phi) \sin^2 \theta_e$$

for complete eclipses, or

$$(409) \quad \ell = (0.087 + 0.027 \cos 2\phi) \sin^2 \theta_e$$

for partial eclipses. Also the term $A_1 \cos \phi$ derived empirically must be subtracted in both cases. In the above expressions θ_e is to be estimated from the light curve. If the ellipticity is large, a preliminary rectification for it may be necessary before the estimate can be made.

At the stage of analysis thus reached one might suspect that according to the model proposed, there would be a possibility for significant outstanding errors to remain when the observations are represented by equation (402), merely due to neglect of higher order terms in the distortion. Indeed, the entire run of observations outside eclipse may be represented by

$$(410) \quad \ell = A_0 + A_1 \cos \epsilon + A_2 \cos 2\epsilon + A_3 \cos 3\epsilon + A_4 \cos 4\epsilon,$$

to the accuracy which the terms have been retained in the theories of ellipticity and reflection. However, in practice one cannot determine, from the observations, all of the above coefficients simultaneously with sufficient accuracy to be of any value¹. Hence, one must use values of A_3 and A_4 computed from theory with the aid of preliminary elements. Their values are quite small in the majority of cases² and should not be considered as a part of the determination of preliminary elements.

Despite the fact that expression (402) should closely represent the light between minima, often it will not. Asymmetries of the

1 Z. Kopal, Harvard Obs. Mon., 8, 129, 1950.

2 H. N. Russell mentions that according to the actual physical model, equation (402) should represent the light of the system outside eclipse to within a few thousandths of a magnitude. Cf. Ap.J., 108, 405, 1948.

light curve both inside and outside minima are frequently observed. When these variations are periodic the non-eclipse variations may be represented by the first few terms of a general Fourier series

$$(411) \quad \ell = A_0 + A_1 \cos \phi + A_2 \cos 2\phi + B_1 \sin \phi + B_2 \sin 2\phi + \dots$$

No theories are currently available to explain the presence of the sine terms. At present only an empirical rectification method is used, which is without a sound theoretical basis.

The light loss due to eclipse can presumably be determined by extending the curve (411) across the minima. The actual light loss f at eclipse, which should be used in determining the elements, may be quite different. Furthermore, the processes which introduce the sine terms may also influence the cosine terms, thus affecting the values of ellipticity and reflection derived from observation.

At the outset it is unknown whether the complications arise in one component or both, or in what proportion. Russell¹ remarks;

If they affect the brightness of both in the same proportion but produce no asymmetry of shape or surface brightness, their effect can be completely removed by division by $A_0 + B_1 \sin \theta + B_2 \sin 2\theta$, leaving a symmetrical light-curve for discussion in the usual way. If they affect star 1 alone (with the same restrictions), the subtraction of $B_1 \sin \theta + B_2 \sin 2\theta$ from the observed light should give a symmetrical light curve for the eclipse of star 2 and outside eclipse. Correction during the other eclipse demands the subtraction of $(1 - f_1)(B_1 \sin \theta + B_2 \sin 2\theta)$ from the observed light, or the addition of $f_1(B_1 \sin \theta + B_2 \sin 2\theta)$ to the light corrected as above.

If ℓ_1 and ℓ_2 are known approximately, f_1 is given by a slight modification of (333) and the rectification for asymmetries may be made. The standard rectification is then carried out and the elements determined as before. The accuracy of the derived elements will be low, but worst of all, their precision is incapable of being estimated.

¹ H. N. Russell, Ap.J., 108, 405, 1948.

A brief summary of each of the standard rectification procedures is now in order. Either type of rectification is practical when the light between minima may be represented by

$$(412) \quad \ell = A_0 + A_1 \cos \phi + A_2 \cos 2\phi \quad ,$$

where A_1 and A_2 are negative if ϕ is measured from primary minimum.

In a circular orbit $\phi = \theta$.

Rectify the observations for reflection by subtraction according to

$$(413) \quad \ell_r = \ell_{\text{obs}} - C'_0 - C'_1 \cos \phi - C'_2 \cos 2\phi \quad ,$$

where ℓ_{obs} is the light observed at any phase. For the light outside eclipse

$$(414) \quad \ell_r = (A_0 - C'_0) + (A_2 - C'_2) \cos 2\phi \quad .$$

The constants C' are obtained from (403) by setting $\sin i = 1$. Thus

$$C'_1 = A_1 \quad ,$$

$$(415) \quad C'_0 = -0.6 \frac{(G_2 + G_1)}{(G_2 - G_1)} A_1 \quad ,$$

$$C'_2 = +0.3 C_0 \quad .$$

Normalize the curve to the light at $\phi = 90^\circ$ by noting that

$$(416) \quad \ell_r^m = A_0 - C'_0 - A_2 + C'_2 \quad .$$

Theoretically the light due to ellipticity should vary as

$$(417) \quad \ell_r = \ell_r^m \left(1 - \frac{zN}{4} - \frac{zN}{4} \cos 2\phi \right) \quad .$$

Comparing this with (414) one obtains

$$(418) \quad -\ell_r \frac{zN}{4} = A_2 - C'_2 \quad ,$$

or

$$(419) \quad -\frac{zN}{4} = \frac{A_2 - C'_2}{A_0 - C'_0 - A_2 + C'_2} \quad .$$

Then

$$(420) \quad \ell_r = \ell_r^m \left\{ 1 + \frac{A_2 - C_2'}{A_0 - C_0' - A_2 + C_2'} + \frac{(A_2 - C_2') \cos 2\phi}{A_0 - C_0' - A_2 + C_2'} \right\} \\ = \ell_r^m \frac{A_0 - C_0' + (A_2 - C_2') \cos 2\phi}{A_0 - C_0' - A_2 + C_2'}$$

Rectify for the actual ellipticity by

$$(421) \quad \ell_r' = \ell_r^m = \frac{\ell_r [A_0 - C_0' - A_2 + C_2']}{A_0 - C_0' + (A_2 - C_2') \cos 2\phi}$$

Rectification for reflection by addition is carried out according to

$$(422) \quad \ell_r = \ell_{\text{obs}} + C_0 + C_1 \cos \phi + C_2 \cos 2\phi,$$

where again ℓ_{obs} is the light observed at any phase. For the light outside eclipse

$$(423) \quad \ell_r = (A_0 + C_0) + (A_2 + C_2) \cos 2\phi.$$

The constants C are obtained from equation (404) by setting $\sin i = 1$.

Therefore,

$$C_1 = -A_1,$$

$$(424) \quad C_0 = -0.8 \frac{(G_2 + G_1) A_1}{(G_2 - G_1)},$$

$$C_2 = +0.3 C_0,$$

where all the C 's are positive. Normalize the curve to the light at

$\phi = 90^\circ$ by noting

$$(425) \quad \ell_r^m = A_0 + C_0 - (A_2 + C_2).$$

Theoretically the light ℓ_r to be treated for ellipticity should vary as

$$(426) \quad \ell_r = \ell_r^m \left(1 - \frac{zN}{4} - \frac{zN}{4} \cos 2\phi \right).$$

Comparing this with (423) one obtains

$$(427) \quad -\ell_r \frac{zN}{4} = A_2 + C_2,$$

or

$$(428) \quad \frac{-zN}{4} = \frac{A_2 + C_2}{A_0 + C_0 - (A_2 + C_2)}.$$

Then

$$(429) \quad l_r = l_r^m \left\{ 1 + \frac{(A_2 + C_2)}{A_0 + C_0 - (A_2 + C_2)} + \frac{(A_2 + C_2) \cos 2\phi}{A_0 + C_0 - (A_2 + C_2)} \right\} \\ = l_r^m \left\{ \frac{A_0 + C_0 + (A_2 + C_2) \cos 2\phi}{A_0 + C_0 - (A_2 + C_2)} \right\}.$$

Rectify for the apparent ellipticity by

$$(430) \quad l_r' = l_r^m = \frac{l_r [A_0 + C_0 - A_2 - C_2]}{A_0 + C_0 + (A_2 + C_2) \cos 2\phi}.$$

Rectification for phase, in circular orbits, is made by division by the equation

$$(431) \quad \sin \Theta = \frac{\sin \phi}{1 - z \cos^2 \phi},$$

where

$$(432) \quad zN = \frac{-4(A_2 - C_2')}{A_0 - C_0 - A_2 + C_2'},$$

or

$$(433) \quad zN = \frac{-4(A_2 - C_2)}{A_0 - C_0 - A_2 + C_2}$$

depending on which rectification procedure is used for flux density. Actually the expressions for zN are identical.

The constants A_0, A_1, A_2 are computed from the light between minima¹, and C_0, C_1, C_2 , and N from reasonable estimates using procedures outlined in the text. The eccentricity of the components may be found from $e_m^2 = z \csc^2 j$.

After the light curve is rectified, solve for the elements L_s , a_g, k, i_r using equation (383) just as indicated in the section on the spherical model. From these elements find a_g, a_s, b_g, b_s by equations (385) and (386), and i from (384) — if a reasonable estimate of b/c

¹ A very expedient method for finding the A 's was advanced by J. E. Merrill and is given in *Ap.J.*, 108, 388, 1948. This article also includes a discussion of the choice of observations which are most favorable to current techniques of analysis.

can be made from theory. The light of each component is obtained by applying to the values obtained from the rectified curve the corrections given by (400), if rectification by subtraction was used, or by (401) if rectification by addition was used.

UNSOLVED PROBLEMS

The field of eclipsing binaries is still confronted with some practical and theoretical difficulties to challenge the mental acuity of investigators. At present, the determination of physically sound expressions for the reflection effect, both inside and outside eclipse, and the explanation of the observed asymmetries and period changes remain as the foremost problems. Explanation of these observed effects will probably uncover new and more difficult problems, but with the consequence of helping to illuminate men's minds with the light of understanding that may assist posterity to discern the workings of nature.

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