Points Uniquely Associated with a Point on a Curved Surface.

by

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I. Introduction.

In the study of a curved surface in space, by projective differential methods, there are defined many unique points connected with every point on the surface. The purpose of this thesis is to collect the information with regard to these points, some of which has appeared elsewhere in miscellaneous papers, into a single paper, where it will be more available for use. In connection with these points, a geometrical interpretation will be given.

II. Representation of the Surface.

We assume that the homogeneous coordinates of a point P_y in space is given as an analytic function of the two variables:

$$y^{(k)} = f^{(k)}(u,v) \qquad k = 1, 2, 3, 4.$$

It has been shown that a curved surface may be represented by two second order differential equations of the form:*

$$(1) \quad \begin{aligned} & y_{uu} + 2a y_u + 2b y_v + c y = 0, \\ & y_{vv} + 2a' y_u + 2b' y_v + c' y = 0, \end{aligned}$$

where two directrix curves are uniquely determined on each point P_y on the surface. For the form of differential equations (1) the parametric curves on the surface are asymptotic curves. C_u is generated by allowing the variable u to change and keeping v equal to a constant. The other parametric curve, C_v, is generated by allowing v to vary and kepping u equal to a constant.

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* E. J. Wilczynski: Differential Geometry of Curved Surfaces Trans. American Math. Society. p.244. Volume 8.

III. Transformations of the Variables.

A set of surfaces, all projectively equivalent, was defined by (1). Wilczynski* has shown that the most general transformations which leave the form of the differential equations in (1) unchanged are of the form:

(2) $\bar{u} = \varphi(u)\ , \bar{v} = \psi(v),$

for the independent variable, and

(3) $y = \lambda(u,v)\ \bar{y},$

for the dependent variable. Both of these transformations given by (2) and (3) leave the surface, as well as the form of the equations in (1), unchanged.

From transformation (2) we have:

$$
(4) \quad
\begin{aligned}
y_u &= \varphi_u \bar{y}_{\bar{u}}, \qquad y_v = \psi_v \bar{y}_{\bar{v}}, \\
y_{uu} &= \varphi_u^2 \bar{y}_{\bar{u}\bar{u}} + \varphi_{uu} \bar{y}_{\bar{u}}, \\
y_{vv} &= \psi_v^2 \bar{y}_{\bar{v}\bar{v}} + \psi_{vv} \bar{y}_{\bar{v}}, \\
y_{uv} &= \varphi_u \psi_v \bar{y}_{\bar{u}\bar{v}},
\end{aligned}
$$

When the values of these derivatives are substituted in (1) we have a new set of equations of the form:

$$
(5) \quad
\begin{aligned}
&\bar{y}_{\bar{u}\bar{u}} + 2A\bar{y}_{\bar{u}} + 2B\bar{y}_{\bar{v}} + C\bar{y} = 0, \\
&\bar{y}_{\bar{v}\bar{v}} + 2A'\bar{y}_{\bar{u}} + 2B'\bar{y}_{\bar{v}} + C'\bar{y} = 0,
\end{aligned}
$$

E. J. Wilczynski: Differential Geometry of Curved Surfaces. Transactions of the American Math. Society. p.244. V8.

where:

$$\mathcal{A} = \frac{1}{\varphi_u}\left(a + \frac{1}{2}\eta\right) \;; \qquad \mathcal{A}' = \frac{\varphi_u}{\psi_v^2} a',$$

(6)
$$\mathcal{B} = \frac{\psi_v}{\varphi_u^2} b, \qquad \mathcal{B}' = \frac{1}{\psi_v}\left(b' + \frac{1}{2}\rho\right),$$

$$\mathcal{C} = \frac{1}{\varphi_u^2} c, \qquad \mathcal{C}' = \frac{1}{\psi_v^2} c',$$

and (7) $\eta = \dfrac{\varphi_{uu}}{\varphi_u} \;; \; \rho = \dfrac{\psi_{vv}}{\psi_v}$,

Some of the derivatives of the above coefficients will be needed later in this paper. These are:

(8)
$$\mathcal{B}'_u = \frac{1}{\varphi_u \psi_v} b'_u \;; \qquad \mathcal{A}'_v = \frac{\varphi_u}{\psi_v^3}\left(a'_v - 2\rho a'\right),$$

$$\mathcal{A}'_u = \frac{1}{\psi_v^2}\left(a'_u + \eta a'\right) ; \; \mathcal{A}_v = \frac{1}{\varphi_u \psi_v} a_v,$$

$$\mathcal{B}_u = \frac{\psi_v}{\varphi_u^3}\left(b_u - 2\eta b\right),$$

$$\mathcal{B}_v = \frac{1}{\varphi_u^2}\left(b_v + \rho b\right).$$

Now let us make the dependent variable transformation on equation (5). If we differentiate,we have:

(9)
$$\bar{y} = \lambda \bar{\bar{y}},$$
$$\bar{y}_{\bar{u}} = \lambda_{\bar{u}} \bar{\bar{y}} + \lambda \bar{\bar{y}}_{\bar{u}},$$
$$\bar{y}_{\bar{v}} = \lambda_{\bar{v}} \bar{\bar{y}} + \lambda \bar{\bar{y}}_{\bar{v}},$$

$$\bar{y}_{\bar{u}\bar{u}} = \lambda \bar{\bar{y}}_{\bar{u}\bar{u}} + 2\lambda_{\bar{u}} \bar{\bar{y}}_{\bar{u}} + \lambda_{\bar{u}\bar{u}} \bar{\bar{y}},$$

$$\bar{y}_{\bar{u}\bar{v}} = \lambda \bar{\bar{y}}_{\bar{u}\bar{v}} + \lambda_{\bar{v}} \bar{\bar{y}}_{\bar{u}} + \lambda_{\bar{u}} \bar{\bar{y}}_{\bar{v}} + \lambda_{\bar{u}\bar{v}} \bar{\bar{y}},$$

$$\bar{y}_{\bar{v}\bar{v}} = \lambda \bar{\bar{y}}_{\bar{v}\bar{v}} + 2\lambda_{\bar{v}} \bar{\bar{y}}_{\bar{v}} + \lambda_{\bar{v}\bar{v}} \bar{\bar{y}}.$$

If these are substituted in (5) we have a set of equations of the form

$$
(10) \qquad \begin{aligned} &\bar{\bar{y}}_{\bar{u}\bar{u}} + 2\bar{A}\bar{\bar{y}}_{\bar{u}} + 2\bar{B}\bar{\bar{y}}_{\bar{v}} + \bar{C}\bar{\bar{y}} = 0, \\ &\bar{\bar{y}}_{\bar{v}\bar{v}} + 2\bar{A}'\bar{\bar{y}}_{\bar{u}} + 2\bar{B}'\bar{\bar{y}}_{\bar{v}} + \bar{C}'\bar{\bar{y}} = 0, \end{aligned}
$$

where the coefficients are expressed in terms of the original coefficients, and functions of the transformations, as follows

$$
(11) \qquad \begin{aligned} &\bar{A} = A + \frac{\lambda_u}{\lambda}, \\ &\bar{B} = B, \\ &\bar{A}' = A', \\ &\bar{B}' = B + \frac{\lambda_v}{\lambda}, \\ &\bar{C} = C + \frac{\lambda_{uu} + 2a\lambda_u + 2b\lambda_v}{\lambda}, \\ &\bar{C}' = C' + \frac{\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v}{\lambda}. \end{aligned}
$$

IV. Invariants and Covariants.

A function of the coefficients and their derivatives which remains unchanged under both the dependent and the independent variable transformation is called an Invariant. If this function is unchanged except for a factor it is called a relative invariant.

A function of the coefficients, their drivatives, the dependent variable, and its derivatives which is unchanged under both the dependent and independent variable transformations is called a Covariant. If this function is unchanged except for a factor it is called a relative Covariant.

If transformations (2) and (3) are made on (1), we arrive at a system of equations given by (10). If the coefficients in this system of equations are fixed to such an extent that they can only be changed by a factor, then we say that this set of equations is in a ~~the~~ Canonical form.

Stouffer has shown, that by means of the two transformations (2) and (3), the following coefficients and derivatives can be made to vanish at a point where $u=0, v=0$:

(11) $$\bar{A} = \bar{B}' = \bar{A}'_u(u=u, v=0) = \bar{B}_v(u=0, v=v) = 0$$

Under these conditions the transformations are fixed to such an extent that the coefficients, the varaibles, and the derivatives of each, are completely determined except for a factor. We have therefore arrived at a Canonical form. In the form (10) after the conditions given by (11) have been imposed, each of the coefficients leads, by direct subtitution back, to an Invariant in terms of the original coefficients. The variables and their derivatives in (10) lead by direct substitution back, to Covariants in terms of the original variables and coefficients.

It is easilly seen how the transformations that impose conditions (11) are found. The independent variable transformation is fixed by making:

$$(12) \quad \begin{aligned} A'_u &= \frac{1}{\psi_v^2}(a'_u + \eta a') = 0 \ ; \quad \eta = -\frac{a'_u(u,0)}{a'(u,0)} \\ B_v &= \frac{1}{\varphi_u^2}(b_v + \rho b) = 0 \ ; \quad \rho = -\frac{b_v(0,v)}{b(0,v)} \end{aligned}$$

The conditions above are maintained by (2) if and only if:

$$(13) \quad \begin{aligned} \eta &\equiv \frac{\varphi_{uu}}{\varphi_u} = 0 \qquad \text{or } \varphi_u = \text{a const} \\ \rho &\equiv \frac{\psi_{vv}}{\psi_v} = 0 \qquad \text{or } \psi_v = \text{a const} \end{aligned}$$

In order to impose the other conditions in (11) we use the dependent variable transformation. This transformation is determined by:

$$(14) \quad \begin{aligned} \bar{A} &= A + \frac{\lambda_u}{\lambda} = 0; \quad \frac{\lambda_u}{\lambda} = -A, \\ \bar{B}' &= B' + \frac{\lambda_v}{\lambda} = 0; \quad \frac{\lambda_v}{\lambda} = -B'. \end{aligned}$$

It is not in general possible to put two conditions, such as those in (14) on the transformation in (3). It is, however, possible to do it in this case because of the integrability conditions that had to be put on the coefficients in the original equations. The conditions in (14) are maintained if and only if $\lambda = const.$

Now by direct substitution back, we shall calculate the covariants of (10) under conditions (11) in terms of the original variables and coefficients. From (9) we have

$$\bar{y} = \lambda \bar{\bar{y}},$$
$$\bar{y}_{\bar{u}} = \lambda_{\bar{u}} \bar{\bar{y}} + \lambda \bar{\bar{y}}_{u},$$

with $$\frac{\lambda_u}{\lambda} = -A, \quad \frac{\lambda_v}{\lambda} = -B',$$

From (2), (4), and (8) we have further

$$A = \frac{1}{\varphi_u}\left(a + \tfrac{1}{2}q\right), \quad \bar{y}_{\bar{u}} = \frac{1}{\varphi_u} y_u$$

with $$q = -\frac{a'_u(u,0)}{a'(u,0)},$$

which shows that,

(15) $$\bar{\bar{y}}_{\bar{u}} = \frac{1}{\lambda\varphi_u}\left[y_u + \left(a - \frac{a'_u}{2a'}\right) y\right]$$

This holds for $u = u$ and $v = 0$, but if we go back and look at the quantities used in calculating this expression we see that it did not matter whether $v = 0$ or not. This will hold at any point in general; therefore, we have a covariant in (15). By methods similar to those above it is easilly verified that

(16) $$\bar{\bar{y}}_{\bar{v}} = \frac{1}{\lambda\psi_v}\left[y_v + \left(b' - \frac{b_u}{2b}\right) y\right],$$

(17) $$\bar{\bar{y}}_{\bar{u}\bar{v}} = \frac{1}{\lambda\varphi_u\psi_v}\left[y_u + \left(a - \frac{a'_u}{2a'}\right) y_v + \left(b' - \frac{b_u}{2b}\right) y_u + \left(a_v - a b'\right) y\right].$$

In (15),(16),and (17) we have three relative covariant expressions representing three covariant points of the coordinate system in (1). If the factor multiplying each of these relative covariant expressions be disregarded, these three points, with the point P_y, may be used as the vertices of a tetrahedron of reference for the study of the surface.

It has been found that the conditions in (11) give rise to the same covariant points that Wilczynski used for his canonical tetrahedron. He defined (15) and (16), geometrically, by use of the osculating ruled surface. By an osculating ruled surface we mean the surface formed by all the tangents to one set of asymptotic curves along another asymptotic curve. All the tangents to the C_u curves along one C_v curve forms the osculating ruled surface which contains the line joining P_y to Y_u^*. Since this is a line of the ruled surface there will be two flecnode points on it. Wilczynski defined the vertex of his tetrahedron on this line as the harmonic conjugate of P_y with respect to these two flecnode points.

Other sets of conditions different from (11) may be put on (2) and (3), and we obtain other sets of covariant points. Next we shall show that we can make

(18) $\bar{A} = \bar{B}' = \bar{A}'_v(u=0, v=v) = \bar{B}_u(u=u, v=0) = 0,$

at the point $u=0, v=0$. From (8) we see that we can determine (2) by making $\bar{A}'_v$ and $\bar{B}_u$ vanish? We cannot make these vanish identically but only at a point. Thus we have

$$\bar{A}'_v = \frac{\varphi_u}{\psi_v^3}(a'_v - 2\rho a') = 0,$$

which gives

$$\rho = \frac{a'_v(0,v)}{2a(0,v)},$$

also
$$\bar{B}_u = \frac{\psi_v}{\varphi_u^3}(b_u - 2\eta b) = 0,$$

from which it follows that

$$\eta = \frac{b_u(u,0)}{2b(u,0)},$$

$\bar{A}'_v$ and $\bar{B}_u$ are maintained equal to zero at the point $u=0, v=0$ if and only if ρ and η are both equal to zero, or what is equivalent, if ψ_v and φ_u are kept constant.

Transformation (3) is determined precisely as it was under conditions (11) for we have exactly the same conditions to impose again.

The two tansformations are determined and if they are made on (1) under the conditions in (18), an equation of the form (10) will result. The variables in (11) will, by direct substitution back, give rise to relative covariant expressions in terms of the original coefficients and variables. It is very easy to verify that the variables in (10), under the conditions in (18) give rise to the relative covariants given by

$$(19)\qquad \bar{\bar{y}}_{\bar u} = \frac{1}{\lambda_1 \varphi_u}\left[y_u + \left(a + \frac{b_u}{4b}\right) y\right] \equiv C_1 ,$$

$$\bar{\bar{y}}_{\bar v} = \frac{1}{\lambda_1 \psi_v}\left[y_v + \left(b' + \frac{a'_v}{4a'}\right) y\right] \equiv C_2 .$$

These covariant points are the ones that Green used for his tetrahedron of reference but he defined them geometrically. If we call the covariant point, on the asymptotic tangent to C_u, that Green used, C_1 and the covariant point that he used on the asymptotic tangent to C_v as C_2, then he defined these points as follows: C_1 is the pole of the line P_yY_v with respect to the osculating conic to C_u. C_2 is the pole of the line P_yY_u with respect to the osculating conic to C_v.

To simplify our notation we shall now define what we are going to mean by corresponding points on the two asymptotic tangents. If the two transformations in (2) and (3) are made on (1) with the conditions on the transformations that

$$(20)\qquad \bar{A} = 0 \; ; \; \bar{B}' = 0 ,$$

$$l\,\frac{\bar{A}'_u}{\bar{A}'} + m\,\frac{\bar{B}_u}{\bar{B}} = 0 ; \; l\,\frac{\bar{B}_v}{\bar{B}} + \frac{\bar{A}'_v}{\bar{A}} = 0 \;, \text{ at } (u = 0, v = 0),$$

there shall result one covariant point on each asymototic tangent. These two covariant points are called corresponding points on the two lines. We see that (11) and (18) are special cases of (20) and, therefore, D_1 corresponds to D_2, and C_1 to C_2.

If in (20) $l=m$ we have two other corresponding points determined analytically. These we shall call F_1 and F_2. Fubini used these points as two vertices of his canonical tetrahedron, but he defined them geometrically instead of analytically. These points expressed in terms of the original variables and coefficients are

$$(21)\qquad \begin{aligned} F_1 &\equiv \bar{\bar{y}}_{2u} = \frac{1}{\lambda_2 \varphi_{2u}}\left[y_u + \left(a + \frac{1}{2}\left\{\frac{a'_u}{a'} + \frac{b_u}{b}\right\}\right)y\right], \\ F_2 &\equiv \bar{\bar{y}}_{2v} = \frac{1}{\lambda_2 \psi_{2v}}\left[y_v + \left(b' + \frac{1}{2}\left\{\frac{b_v}{b} + \frac{a'_v}{a'}\right\}\right)y\right]. \end{aligned}$$

It is very easy to verify analytically that the four points $P_y D_1 C_1 F_1$ on the asymptotic tangent to C_u form a harmonic set. Also the four corresponding points on the tangent to C_v form a harmonic set. We shall now show that the four points $P_y D_1 C_1 F_1$ do form a harmonic set on the tangent to C_u. Any point on this tangent may be represented as a linear combiantion of P_y and Y^*_u of the form $y_u + \lambda y$. The coordinates of the four particlar points we are studying are

$$(22)\qquad \begin{aligned} P_y &\equiv (\infty,\ 1,\ 0,\ 0), \\ D_1 &\equiv \left(a - \tfrac{1}{2}\tfrac{a'_u}{a'},\ 1, 0, 0\right), \\ C_1 &\equiv \left(a + \tfrac{b_u}{4b},\ 1, 0, 0\right), \\ F_1 &\equiv \left(a + \tfrac{1}{2}\left\{\tfrac{a'_u}{a'} + \tfrac{b_u}{b}\right\}, 1, 0, 0\right) \end{aligned}$$

If we form the double ratio of these four points we have

* The point Y_u is the point whose coordinates are y_{iu} i = 1?2,3,4.,ect.

$$(P_y C_1 D_1 F_1) = \frac{\frac{a'_u}{2a'} + \frac{b_u}{4b}}{-\frac{a'_u}{2a'} - \frac{b_u}{4b}} = -1,$$

which shows that F_1 is the harmonic conjugate of P_y with respect to P_y and C_1.

We shall show that all the lines joining corresponding points on the asymptotic tangents go through the same point. We shall find the coordinates of the point of intersection of the lines D_1D_2, and C_1C_2 and show that the line joining any two points in general go through this point. If D is any point on the line joining D_1 and D_2 then

$$D \equiv \left[y_u + \left(a - \frac{a'_u}{2a'}\right)y + \kappa_1\left\{y_v + \left(b' - \frac{b_v}{2b}\right)y\right\}\right],$$

and the coordinates of D are

$$(23)\quad \left(a - \frac{a'_u}{2a'} + \kappa_1\left\{b' - \frac{b_v}{2b}\right\},\ 1,\ \kappa_1,\ 0\right),$$

Let C be any point on the line C_1C_2, then

$$C \equiv \left[\left(a + \frac{b_u}{4b}\right)y + y_u + \kappa_2\left\{y_v + \left(b' + \frac{a'_v}{4a'}\right)y\right\}\right],$$

and the coordinates of C are

$$\left(a + \frac{b_u}{4b} + \kappa_2\left\{b' + \frac{a'_v}{4a'}\right\},\ 1,\ \kappa_2,\ 0\right).$$

These two lines will intersect where the coordinates of D are equal to the coordinates of C. In examining the coordinates of D and C we see that the 2nd and 4th are identical. We can make the y_v coordinate the same by making $\kappa_1 = \kappa_2$. If we set the two Y coordinates equal

We can solve for the value of k_1 which when substituted in (23) will give the coordinates of the point of intersection of D_1D_2 and C_1C_2. If we let L be any point on the line joining L_i on the asymptotic tangent to C_u to the corresponding point L_j on the asymptotic tangent to C_v, then

$$L = \left[y_u + \left(a + \frac{l\frac{a'_u}{a'} + m\frac{b_u}{b}}{2(2m-l)}\right)y + k_3\left\{(y_v) + \left(b' + \frac{l\frac{b_v}{b} + m\frac{a'_v}{a'}}{2(2m-l)}\right)y\right\}\right],$$

and the coordinates of L in terms of the original coordinate system are

$$(24)\quad \left(a + \frac{l\frac{a'_u}{a'} + m\frac{b_u}{b}}{2(2m-l)} + k_3\left\{b' + \frac{l\frac{b_v}{b} + m\frac{a'_v}{a'}}{2(2m-l)}\right\},\ 1,\ k_3,\ 0\right)$$

The value of k_1 which gave the point of intersection of the two lines D_1D_2 and C_1C_2 was

$$k_1 = -\frac{\frac{b_u}{2b} + \frac{a'_u}{a'}}{\frac{b_v}{b} + \frac{a'_v}{2a'}},$$

If k_3 in (24) is made equal to k_1 then it is easilly verified that (24) and (23) are the same point, which shows that all of these lines joining corresponding points on the asymptotic tangents go through a point. This point is called the canonical point and has the coordinates

$$\left(a + \frac{b_u}{4b} - \left\{\frac{\frac{b_u}{2b} + \frac{a'_u}{a'}}{\frac{b_v}{b} + \frac{a'_v}{2a'}}\right\}\left(b' + \frac{a'_v}{4a'}\right),\ 1,\ -\left\{\frac{\frac{b_u}{2b} + \frac{a'_u}{a'}}{\frac{b_v}{b} + \frac{a'_v}{2a'}}\right\},\ 0\right),$$

If $l = 2m$ in (20) we have a special case in which the covariant point on each asymptotic tangent has moved to the point P_y. We cannot put the conditions in (20) on (2) and (3) for the case for which $l = 2m$.

V. Osculating Quadrics.

An osculating quadric to a surface at a point on the surface is the quadric which has second order contact with the surface at that point. We shall first derive an osculating quadric known as the Quadric of Lie.

If we let the surface be represented by the set of equations in (1) we can develop the equation of an osculating ruled surface along the asymptotic tangent to the C_u curve. That is, it will have C_v as one directrix curve. If we let $\rho = y_u$, then by differentiation and from (1) we have

$$(25) \qquad \rho_{vv} - 4a'b\, y_v + (c_u - 2a'c)\, y + (2a'_u + c')\rho = 0.$$

(25) with the second equation in (1) is the equation of the osculating ruled surface. These are two equations of the form

$$(26) \qquad \begin{aligned} y_{vv} + 2p_{11} y_v + 2p_{21}\rho_v + q_{11} y + q_{12}\rho &= 0, \\ \rho_{vv} + 2p_{21} y_v + 2p_{22}\rho_v + q_{21} y + q_{22}\rho &= 0, \end{aligned}$$

where

$$\begin{aligned} p_{11} &= 0 \quad ; & q_{11} &= c', \\ p_{12} &= 0 \quad ; & q_{12} &= 2a', \\ p_{21} &= -2a'b; & q_{21} &= c'_u - 2a'c, \\ p_{22} &= 0 \quad ; & q_{22} &= 2a'_u + c', \end{aligned}$$

In the study of a ruled surface it is found that there is a point, on the tangent to each ~~directrix~~ asymptotic curve, defined as follows

$$(27)\qquad r = y_v + p_{11}y + p_{12}\rho \equiv y_v,$$
$$\sigma = \rho_v + p_{21}y + p_{22}\rho = \rho_v + 2a'b\,y.$$

Lie's quadric is the quadric which contains all the asymptotic tangents to the ruled surface, in (26) along the line $P_y\rho$. Any point on the quadric may be represented in the form

$$(28)\qquad \lambda(y + \alpha\rho) + \beta(y_v + \alpha\{\rho_v + 2a'b\,y\}).$$

This same point may be represented in terms of the original coordinate system as

$$(29)\qquad x_1 y + x_2\rho + x_3 y_v + x_4\rho_v.$$

If we equate coefficients in (28) and (29) we have

$$x_1 = (\lambda - 2a'b\,\alpha\beta),$$
$$x_2 = \lambda\alpha,$$
$$x_3 = \beta,$$
$$x_4 = \alpha\beta,$$

If from these we eliminate α, β, and λ we have

$$(30)\qquad x_1x_4 - x_2x_3 + 2a'b\,x_4^2 = 0,$$

as the equation of the quadric of Lie. The quadric to the other osculating ruled surface, is, from symmetry, identical with (30).

Next we shall derive the equation of Fubini's Quadric in terms of the original coordinate system. The equation of his quadric when referred to his canonical tetrahedron is*

(31) $\bar{X}_1\bar{X}_4 - \bar{X}_2\bar{X}_3 = 0,$

He set up his tetrahedron of reference by means of transformation (3) under the conditions

(32) $\frac{\lambda_u}{\lambda} = -\alpha \quad ; \quad \frac{\lambda_v}{\lambda} = -\beta,$

where

$$\alpha = a + \frac{1}{2}\left(\frac{a'_u}{a'} + \frac{b_u}{b}\right); \beta = b' + \frac{1}{2}\left(\frac{b_v}{b} + \frac{a'_v}{a'}\right).$$

The covariants of his system of equations, when expressed in the original coordinate system, were

(33) $$\bar{\bar{y}}_{\bar{u}} = \frac{1}{\lambda}\left[y_u + \alpha y\right] \equiv F_1,$$
$$\bar{\bar{y}}_{\bar{v}} = \frac{1}{\lambda}\left[y_v + \beta y\right] \equiv F_2.$$

The covariant point $\bar{\bar{y}}_{\bar{u}\bar{v}}$ may be found by means of, differentiation, and one of the integrating conditions which must be placed on the original coefficients. If we differentiate either of the covariant expressions in (33) we have

(34) $$\bar{\bar{y}}_{\bar{u}\bar{v}} = \frac{1}{\lambda}\left[y_{uv} + \alpha y_v + \beta y_u + (\alpha_v + \alpha\beta) y\right].$$

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* Wilczynski; Diff. Projective Geometry of Curved Surfaces Trans. Am. Math. Society. Vol. 9. p. 112

Any point in the original coordinate system may be represented in the form

(35) $x_1 y + x_2 y_u + x_3 y_v + x_4 y_{uv}$,

and the same point in the transformed coordinate system can be expressed as

(36) $\bar{x}_1 \bar{\bar{y}} + \bar{x}_2 \bar{\bar{y}}_{\bar{u}} + \bar{x}_3 \bar{\bar{y}}_{\bar{v}} + \bar{x}_4 \bar{\bar{y}}_{\bar{u}\bar{v}}$,

If we substitute (33) and (34) in (36) we can equate coefficients and obtain an expression for the transformation from one coordinate system to the other. Thus we have

$$x_1 y + x_2 y_u + x_3 y_v + x_4 y_{uv} \equiv \frac{1}{\lambda}\left[\bar{x}_1 y + \bar{x}_2(y_u + \alpha y) + \bar{x}_3(y_v + \beta y) + \bar{x}_4(y_{uv} + \alpha y_v + \beta y_u + (\alpha\beta + \alpha_v) y)\right],$$

from which it follows that

$$\begin{aligned} \bar{x}_4 &= \lambda x_4, \\ \bar{x}_3 &= \lambda(x_3 - \alpha x_4), \\ \bar{x}_2 &= \lambda(x_2 - \beta x_4), \\ \bar{x}_1 &= \lambda[x_1 - \alpha x_2 - \beta x_3 + (\alpha\beta - \alpha_v) x_4], \end{aligned} \tag{37}$$

When (37) is substituted in (31) we have

(38) $x_1 x_4 - x_2 x_3 - \alpha_v x_4^2 = 0$ where $\alpha = a + \frac{1}{2}\left(\frac{a'_u}{a'} + \frac{b_u}{b}\right)$

as the equation of Fubini's Canonical quadric.

The equation of Green's canonical quadric referred to his tetrahedron of reference is*

(39) $\bar{x}_1 \bar{x}_4 - \bar{x}_2 \bar{x}_3 = 0$.

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* Green: Memoir on the General Theory of Surfaces. Trans. Am. Math. Society. Vol. 20 p. 98

It is easily determined by the same method as above that the relation between the coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ and the original coordinates $x_1\, x_2\, x_3\, x_4$ is

$$
(40) \qquad
\begin{aligned}
\bar{x}_4 &= \lambda\, \varphi_u \psi_v x_4, \\
\bar{x}_3 &= \lambda\, \psi_v \left[x_3 - x_4\left(a + \tfrac{1}{2}\eta\right)\right], \qquad ; \ \eta = \frac{b_u}{2a}, \\
\bar{x}_2 &= \lambda\, \varphi_u \left[x_2 - x_4\left(b' + \tfrac{1}{2}\rho\right)\right] \qquad ; \ \rho = \frac{a'_v}{2b}, \\
x_1 &= \lambda\left[x_1 - x_2\left(a + \tfrac{1}{2}\eta\right) - x_3\left(b' + \tfrac{1}{2}\rho\right) - x_4\left\{a_v - \left(a + \tfrac{1}{2}\eta\right)\left(b' + \tfrac{1}{2}\rho\right)\right\}\right].
\end{aligned}
$$

Substitution of (40) into (39) gives

$$
(41) \qquad x_1 x_4 - x_2 x_3 - a_v x_4^2 = 0,
$$

as the equation of Green's quadric in the original coordinate system.

VI. Polar Reciprocals.

We shall prove that the polar plane at any point on the asymototic tangent to C_u is the same with respect to any quadric of the form

(42) $x_1 x_4 - x_2 x_3 - K x_4^2 = 0.$

The coordinates of any point on this tangent are $(t,1,0,0)$. By the use of partial derivatives it is easily found that the equation of the polar plane at $(t,1,0,0)$ is

(43) $x_3 - t x_4^2 = 0,$

which is independent of K.

From symmetry we have that the polar plane at the corresponding point $(s,0,1,0)$ on the asymototic tangent to C_v is

(44) $x_2 - s x_4^2 = 0.$

The line of intersection of (43) and (44) is defined as the polar reciprocal of the line joining the two corresponding points with respect to the quadric in (42). P_y is one point on this polar reciprocal and another point on both polar planes, and hence is on their point of intersection, is the point given by $(0,s,t,1)$. Any point on the polar reciprocal may be expressed as

(45) $ky + s y_u + t y_v + y_{uv},$

and the coordinates of this point are

(46) $(h, s, t, 1)$.

Thus far we have three unique polar reciprocal lines determined. These are the polar reciprocals of the lines D_1D_2, C_1C_2, and F_1F_2. Each of these polar lines will meet all three of the quadrics given in (30), (39) and (41), thus determining nine unique points in the polar plane.

The three points on (30) are

(47)
$$P_1 : [\alpha_1\beta_1 - 4a'b, \beta_1, \alpha_1, 1],$$
$$P_2 : [\alpha_2\beta_2 - 4a'b, \beta_2, \alpha_2, 1],$$
$$P_3 : [\alpha\beta - 4a'b, \beta, \alpha, 1],$$

with:
$$\alpha_1 = a - \frac{a'u}{2a'},$$
$$\beta_1 = b' - \frac{bu}{2b},$$
$$\alpha_2 = a + \frac{bu}{4b},$$
$$\beta_2 = b' + \frac{a'v}{4a'},$$
$$\alpha = a + \frac{1}{2}\left(\frac{a'u}{a'} + \frac{bu}{b}\right),$$
$$\beta = b' + \frac{1}{2}\left(\frac{a'v}{a'} + \frac{bv}{b}\right),$$

The three on (39) are

(48)
$$P_1' : [av + \alpha_1\beta_1, \beta_1, \alpha_1, 1],$$
$$P_2' : [av + \alpha_2\beta_2, \beta_2, \alpha_2, 1],$$
$$P_3' : [av + \alpha\beta, \beta, \alpha, 1],$$

and finally the three points of intersection on (41) are

(49)
$$P_1'' : [\alpha v + \alpha_1\beta_1, \beta_1, \alpha_1, 1],$$
$$P_2'' : [\alpha v + \alpha_2\beta_2, \beta_2, \alpha_2, 1],$$
$$P_3'' : [\alpha v + \alpha\beta, \beta, \alpha, 1],$$

VII. Focal Points.

The double infinity of lines formed by joining the two corresponding covariant points, one on each asymototic tangent, forms a congruence of lines. Each line of this congruence is a line of two developable surfaces.* The two points on each line in which the line meets the edges of regression of the developables are unique points, and are called the focal points of the line.

If we let ρ and σ be the two corresponding points on the asymototic tangents to C_u and C_v, respectively, we shall now find the focal points on the line $\rho\sigma$. Any point on this line may be expressed as

$$\Phi = \rho + \lambda\sigma \tag{50}$$

where λ is a parameter. To determine λ to make Φ a focal point we have to impose the condition that the points determined by Φ_u, Φ_v, ρ and σ shall all lie in a plane.*

From definition we have

$$\begin{aligned} \rho &= y_u + \alpha y, \\ \sigma &= y_v + \beta y. \end{aligned} \tag{51}$$

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* Gabriel M. Green: Memoir on the General Theory of Surfaces. Trans. Am. Math. Society. p.90. vol 20.

By differentiation we have

$$(52)\quad \begin{aligned} \Phi_u - \lambda_u \sigma &= \lambda y_{uv} + (\alpha - 2a + \beta\lambda) y_u - 2b y_v + (\alpha_u - c + \beta_u \lambda) y, \\ \Phi_v - \lambda_v \sigma &= y_{uv} - 2a' y_u + (\alpha - 2b'\lambda + \beta\lambda) y_v + (\alpha_v - c'\lambda + \lambda\beta_v) y, \end{aligned}$$

Making the points represented by (51) and (52) lie in a plane is equivalent to making the points $\Phi_u, \Phi_v, \rho, \sigma$ lie in a plane. To make the points in (51) and (52) lie in a plane we must have

$$(53)\quad \begin{vmatrix} 0 & 0 & 1 & \alpha \\ 0 & 1 & 0 & \beta \\ \lambda & -2b & \alpha - 2a + \beta\lambda & \alpha_u - c + \lambda\beta_u \\ 1 & \alpha - 2b'\lambda + \beta\lambda & -2a' & \alpha_v - c'\lambda + \lambda\beta_v \end{vmatrix} = 0,$$

Expanding the determinant above we have

$$(54)\quad \lambda^2(c' + \beta_v - 2b' + \beta^2) + \lambda(\alpha - 2a'\alpha - \alpha_v + \beta_u) + \alpha_u - c + 2b\beta = 0,$$

a quadratic equation in λ, the solution of which will give the two values of λ that will make (50) the focal points of the line $\rho\sigma$. If we call the two solutions of (54) λ_1 and λ_2 we ahve as the two focal points

$$(55)\quad \begin{aligned} \Phi_1 &= \rho + \lambda_1 \sigma, \\ \Phi_2 &= \rho + \lambda_2 \sigma, \end{aligned}$$

Another uniquely determined point with simpler coordinates is given by taking the harmonic conjugate of ρ with respect to the two points in (55).

If we take the polar reciprocal of each of the lines joining the points ρ and σ, as the point P_y moves over the surface, we have another congruence of lines known as the polar reciprocal congruence. Each line of this congruenc will also have two focal points. If $z = y_{uv} + \alpha y_v + \beta y_u$ then any point on a line of the polar reciprocal congruence may be expressed as

$$(56) \qquad \tau = z + \mu y,$$

μ is a parameter, the value of which we are going to find so that (56) will represent the focal points on the line joining P_y to Z. By differentiation in (56) we have

$$(57) \qquad \begin{aligned} \tau_u - \mu_u y &= (\alpha - 2a) y_{uv} + (4bb' - 2b_v - c + \alpha_u - 2b\beta) y_v \\ &\quad + (4a'b - 2a_v + \beta_u - 2a\beta + \mu) y_u + (2bc' - c_v - c\beta) y, \\ \tau_v - \mu_v y &= (\beta - 2b') y_{uv} + (\mu + 4a'b - 2b'_u + \alpha_v - 2\alpha b') y_v \\ &\quad + (4aa' - 2a'_u - c' - 2\alpha a' + \beta_v) y_u + (2a'c - c'_u - c'\alpha) y, \end{aligned}$$

Now the condition that the points τ_u, τ_v, y, z all lie in a plane is given by making

$$(58) \qquad \begin{vmatrix} \alpha - 2a & \begin{matrix} 4bb' - 2b_v - c \\ + \alpha_u - 2b\beta \end{matrix} & \begin{matrix} 4a'b - 2a_v + \beta_u \\ - 2\alpha\beta + \mu \end{matrix} & 2bc' - c_v - c\beta \\ \beta - 2b' & \begin{matrix} 4a'b - 2b'_u + \alpha_v \\ - 2\alpha b' + \mu \end{matrix} & \begin{matrix} 4aa' - 2a'_u - c' \\ - 2\alpha a' - \beta_v \end{matrix} & 2a'c - c'_u - c'\alpha \\ 1 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0,$$

If (58) is expanded we have

$$(59) \qquad (\mu - \mu_1)(\mu - \mu_2) = 0.$$

a quadratic in the two solutions of which, when substituted in (56) will give the two focal points of the line. If we call μ_1 and μ_2 the two solutions of (59) we have

$$
(60)\qquad
\begin{aligned}
z + \mu_1 y &\equiv \tau_1, \\
z + \mu_2 y &\equiv \tau_2,
\end{aligned}
$$

as the two focal points of the line $P_y Z$. Again we see that the expressions representing these focal points are very complicated. If we determine the harmonic conjugate of P_y with respect to the two points in (60) we have another uniquely determined point the coordinates of which are not as complicated as the coordinates of the two points in (60). If we call this point τ_c, then we have

$$
(61)\qquad \tau_c = z + (2a\alpha + \beta\alpha - 2b'\alpha - \alpha^2 - 8a'b + 4av - \beta u + 2a\beta - \alpha v + 2\alpha b')y.
$$

VIII. Points Defined by Unique Conjugate Curves.

Any double infinity of curves on a surface such that two and only two curves pass through each point, is defined as a Net of Curves on a surface. The parametric curves satisfy this definition, and therefore represent a net of curves on the surface. Let us look at the differential equation

$$(62) \quad P\,du^2 + Q\,du\,dv + R\,dv^2 = 0.$$

The solution of this quadratic for the ratio of du to dv determines two directions at every point on the surface. If Q vanishes in (62), then this equation represents a conjugate net of curves.# By a conjugate net of curves we mean a set of curves such that at each point the tangents to these curves separate the asymototic tangents harmonically.

Green* has shown that a pair of differential equations of the form

$$(63) \quad \begin{aligned} y_{uu} &= p_{11} y_{vv} + p_{12} y_u + q_{11} y_v + q_{12} y, \\ y_{uv} &= \phantom{p_{11} y_{vv} + {}} p_{22} y_u + q_{21} y_v + q_{22} y, \end{aligned}$$

are associated with a surface for which the parametric curves are conjugate curves. The most general trans-

* Green: Nets on a Curved Surface. American Journal of Mathematics. Vol. 37. p.221.

\# This is true only if the directrix curves are the asymptotic curves on the surface.

formations which can be made on (63) so that the form of the equations and the surface will both remain un- changed are given by (2) and (3). There are also some conditions of integrability which must be maintained on the original conditions. The only one that we shall have ocassion to use is

$$(64) \quad (b + 2c')_v = \left(2b' - \frac{c+a}{a}\right)_u.$$

Two covariant points of (63) under (2) and (3) are given by*

$$(65) \quad \begin{aligned} \rho_1 &= y_u - g_{21} y, \\ \sigma_1 &= y_v - f_{22} y. \end{aligned}$$

If we think of the same surface represented by both (1) and (63), we see that there should be a trans- formation of the form

$$(66) \quad \bar{u} = \varphi(u,v), \quad \bar{v} = \psi(u,v),$$

which would transform (1) into (63). From (66) we have

$$(67) \quad \begin{aligned} y_u &= \varphi_u \bar{y}_{\bar{u}} + \psi_u \bar{y}_{\bar{v}}, \\ y_v &= \varphi_v \bar{y}_{\bar{u}} + \psi_v \bar{y}_{\bar{v}}, \\ y_{uu} &= \varphi_u^2 \bar{y}_{\bar{u}\bar{u}} + 2\varphi_u \psi_u \bar{y}_{\bar{u}\bar{v}} + \psi_u^2 \bar{y}_{\bar{v}\bar{v}} + \varphi_{uu} \bar{y}_{\bar{u}} + \psi_{uu} \bar{y}_{\bar{v}}, \\ y_{uv} &= \varphi_u \varphi_v \bar{y}_{\bar{u}\bar{u}} + (\varphi_u \psi_v + \varphi_v \psi_u) \bar{y}_{\bar{u}\bar{v}} + \psi_u \psi_v \bar{y}_{\bar{v}\bar{v}} \\ &\quad + \varphi_{uv} \bar{y}_{\bar{u}} + \psi_{uv} \bar{y}_{\bar{v}}, \\ y_{vv} &= \varphi_v^2 \bar{y}_{\bar{u}\bar{u}} + 2\varphi_v \psi_v \bar{y}_{\bar{u}\bar{v}} + \psi_v^2 \bar{y}_{\bar{v}\bar{v}} + \varphi_{vv} \bar{y}_{\bar{u}} + \psi_{vv} \bar{y}_{\bar{v}}, \end{aligned}$$

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Green: Nets on a Curved Surface.
Americal Journal of Math. Vol. 37. p. 225.

By substitution in (1) we obtain

$$(68)\quad \varphi_u^2\bar{y}_{\bar{u}\bar{u}} + 2\varphi_u\psi_u\bar{y}_{\bar{u}\bar{v}} + \psi_u^2\bar{y}_{\bar{v}\bar{v}} + \varphi_{uu}\bar{y}_{\bar{u}} + \psi_{uu}\bar{y}_{\bar{v}}$$
$$= \bar{a}\varphi_u\bar{y}_{\bar{u}} + \bar{a}\psi_u\bar{y}_{\bar{u}} + \bar{b}\varphi_v\bar{y}_{\bar{u}} + \bar{b}\psi_v\bar{y}_{\bar{v}} + \bar{c}\bar{y},$$
$$\bar{a} = -2a;\ \bar{b} = -2b;\ \bar{c} = -c,$$

and

$$(69)\quad \varphi_v^2\bar{y}_{\bar{u}\bar{u}} + 2\varphi_v\psi_v\bar{y}_{\bar{u}\bar{v}} + \psi_v^2\bar{y}_{\bar{v}\bar{v}} + \varphi_{vv}\bar{y}_{\bar{u}} + \psi_{vv}\bar{y}_{\bar{v}}$$
$$= \bar{a}'\varphi_u\bar{y}_{\bar{u}} + \bar{a}'\psi_u\bar{y}_{\bar{v}} + \bar{b}'\varphi_v\bar{y}_{\bar{u}} + \bar{b}'\psi_v\bar{y}_{\bar{v}} + \bar{c}'\bar{y},$$
$$\bar{a}' = -2a';\ \bar{b}' = -2b';\ \bar{c}' = -c',$$

If (68) and (69) are solved for $\bar{y}_{\bar{u}\bar{u}}$ and $\bar{y}_{\bar{u}\bar{v}}$ we have two equations of the form

$$(70)\quad \begin{aligned} \bar{y}_{\bar{u}\bar{u}} &= f_1\bar{y}_{\bar{v}\bar{v}} + f_2\bar{y}_{\bar{u}} + f_3\bar{y}_{\bar{v}} + f_4\bar{y}, \\ \bar{y}_{\bar{u}\bar{v}} &= q_1\bar{y}_{\bar{v}\bar{v}} + q_2\bar{y}_{\bar{u}} + q_3\bar{y}_{\bar{v}} + q_4\bar{y}, \end{aligned}$$

where the coefficients are expressions in terms of the coefficients in (1), and functions of the transformations. Thus we have

$$q_1 = \frac{\varphi_v^2\psi_u^2 - \varphi_u^2\psi_v^2}{2\varphi_u^2\varphi_v\psi_v - 2\varphi_v^2\psi_u\varphi_u},$$

$$(71)\quad -q_2 = \frac{(\bar{a}\varphi_u + \bar{b}\psi_v - \varphi_{uu})\varphi_v^2 + (\varphi_{vv} - \bar{a}'\varphi_u - \bar{b}'\varphi_v)\varphi_u^2}{4\varphi_u^2\varphi_v\psi_v},$$

$$-q_3 = \frac{(\bar{a}\psi_u + \bar{b}\psi_v - \psi_{uu})\varphi_v^2 + (\psi_{vv} - \bar{b}'\psi_v - \bar{a}'\psi_u)\varphi_u^2}{4\varphi_u^2\varphi_v\psi_v}.$$

The conditon that (71) represent a surface on which the parametric curves are conjugate is that q_1 vanishes. The vanishing of q_1 puts the condition on the transformation that

$$(72)\quad \frac{\varphi_u}{\varphi_v} = \pm\frac{\psi_u}{\psi_v}$$

The positive sign in (72) cannot be used for in that case the Jacobian of the transformation would vanish and there would be no transformation. Therefore, the condition on (66) that transforms (1) into (63) is

$$(73) \qquad \frac{\varphi_u}{\varphi_v} = -\frac{\psi_u}{\psi_v}.$$

The curves in which the developables of the projective normal congruence cut the surface are a pair of unique curves on the surface. These two conjugate curves have tangents on each of which there is a unique point. The unique points are given by (65). We are now going to find the expressions for the points in (65) when referred to the coordinate system in (1). The differential equation of the unique pair of conjugate curves defined above is*

$$(74) \qquad Q\,du^2 - Q'\,dv^2 = 0,$$

where

$$Q = [\theta_{uu} - 2b\theta_u + (f - 2b_v)\theta],$$

$$Q' = [\theta_{vv} - 2a'\theta_v + (g - 2a'_u)\theta],$$

$$\theta = \frac{1}{\sqrt{a'b}},$$

$$f = c - a_u - a^2 - 2bb',$$

$$g = c' - b'_v - b'^2 - 2aa',$$

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* Gabriel M. Green: Memoir on the General Theory of Surfaces. Trans. Am. Math. Society. Vol. 20. p 145.

From (66) and (73) we have

$$(75)\qquad du = \frac{\varphi_u d\bar{v} - \psi_u d\bar{u}}{2\varphi_u\psi_v}, \qquad dv = \frac{\psi_v d\bar{u} - \varphi_v d\bar{v}}{2\varphi_u\psi_v},$$

If (75) are substituted in (74) we have

$$(76)\qquad (Q\psi_u^2 - Q'\psi_v^2)\,d\bar{u}^2 + 2(Q'\varphi_v\psi_v - Q\varphi_u\psi_u)\,d\bar{u}\,d\bar{v} + (Q\varphi_u^2 - Q'\varphi_v^2)\,d\bar{v}^2 = 0.$$

If the $u = \text{const.}$ and the $v = \text{const.}$ curves are to be the directix curves on the surface, the coefficients of $d\bar{u}^2$ and $d\bar{v}^2$ must vanish. To make these coefficients vanish we must have

$$(77)\qquad \frac{\varphi_u}{\varphi_v} = \pm\sqrt{\frac{Q'}{Q}}\,; \quad \frac{\psi_u}{\psi_v} = \mp\sqrt{\frac{Q'}{Q}},$$

If the transformation in (66) is made on (63), under the conditions in (77), we can calculate the covariant point on each of the tangents to the conjugate, giving these points in (65) expressed in terms of the variables and coordinates in (1) as

$$(78)\qquad \rho_1 = \frac{1}{2\varphi_v}\Big[\sqrt{\frac{Q}{Q'}}\,y_u + y_v + \frac{Q}{2Q'}\Big\{\bar{b} + \bar{a}\sqrt{\frac{Q'}{Q}} + \Big(\sqrt{\frac{Q'}{Q}}\Big)_u - \sqrt{\frac{Q'}{Q}}\Big(\sqrt{\frac{Q'}{Q}}\Big)_v - \bar{b}'\frac{Q'}{Q} - \bar{a}'\Big(\frac{Q'}{Q}\Big)^{3/2}\Big\}y\Big],$$

$$\sigma_1 = \frac{1}{2\psi_v}\Big[y_v - \sqrt{\frac{Q'}{Q}}\,y_u + \frac{Q}{2Q'}\Big\{\bar{b} + \bar{a}\sqrt{\frac{Q'}{Q}} - \Big(\sqrt{\frac{Q'}{Q}}\Big)_u - \sqrt{\frac{Q'}{Q}}\Big(\sqrt{\frac{Q'}{Q}}\Big)_v - \bar{b}'\Big(\frac{Q'}{Q}\Big) - \bar{a}'\Big(\frac{Q'}{Q}\Big)^{3/2}\Big\}y\Big].$$

If the two points in (78) are joined by a line this line will intersect each of the asymototic tangents since all of these lines are in the tangent plane to the surface. If we call the point of intersection of the line $\rho_1\sigma_1$ with the asymototic tangent to C_u, M_1, and, N_2 as the point of intersection with the tangent to C_v, then we can join each of these points to the canonical point and determine two other unique points. The coordinates of each of these points is easily found from (78). For example to find the coordinates of M_1 it is necessary to take a linear combination of ρ_1 and σ_1 so that the y_1 coordinate will be zero, ect.

IX. Darboux and Segre Directions.

Darboux found that there were three uniquely determined directions at every point on a surface. These three directions are the ones along which the osculating quadric has third order contact with the surface. The differential equation which gives the Darboux directions is*

(79) $$b' du^3 + a' dv^3 = 0,$$

The conjugate directions of the Darboux directions are called the Segre directions and are given by the differential equation**

(80) $$b' du^3 - a' dv^3 = 0.$$

Throughout the discussion which follows we shall use ϵ to represent any one of the three cube roots of unity.

The differential equations which represent a pair of these conjugate curves on the surface is given by

(81) $$\sqrt[3]{b'}\, du + \epsilon \sqrt[3]{a'}\, dv = 0,$$
$$\sqrt[3]{b'}\, du - \epsilon \sqrt[3]{a'}\, dv = 0.$$

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* Green: Memoir on the General Theory of Surfaces. Trans. Am. Math. Society. Vol. 20. p.142.

If the transformation in (66), under the condition in (73), is made on (81) we arrive at two new equations given by

$$(82)\qquad \begin{aligned} d\bar{u} + \chi\, d\bar{v} &= 0, \\ d\bar{u} + \chi' d\bar{v} &= 0, \end{aligned}$$

where

$$(83)\qquad \begin{aligned} \chi &= \frac{\varepsilon\sqrt[3]{b}\,\varphi_u - \sqrt[3]{a'}\,\varphi_v}{\sqrt[3]{a'}\,\psi_v - \varepsilon\sqrt[3]{b}\,\psi_u}, \\ \chi' &= \frac{\sqrt[3]{a'}\,\varphi_v + \varepsilon\,\varphi_u\sqrt[3]{b}}{\psi_v\sqrt[3]{a'} + \varepsilon\,\psi_u\sqrt[3]{b}}, \end{aligned}$$

If (83) is to represent a pair of conjugate curves we must have $\chi = -\chi'$. Putting this condition on the transformation is equivalent to making

$$(84)\qquad \frac{\varphi_u}{\varphi_v} = \pm\frac{i}{\varepsilon}\sqrt[3]{\frac{a'}{b}}\,; \quad \frac{\psi_u}{\psi_v} = \mp\frac{i}{\varepsilon}\sqrt[3]{\frac{a'}{b}},$$

$$i \equiv \sqrt{-1}.$$

By the same method that was used in the preceeding section we can find the covariant point on each of the tangents to the conjugate curves. If we call these two points ρ_d and σ_d we have

$$(85)\qquad \begin{aligned} \rho_d = \frac{1}{4a}\Big[& 2\frac{\varepsilon}{i}\sqrt[3]{\frac{b}{a'}}\,y_u + 2y_v + \frac{\varepsilon}{i}\sqrt[3]{\frac{b}{a'}}\Big\{ b - a\frac{i}{\varepsilon}\sqrt[3]{\frac{a'}{b}} + \Big(\frac{i}{\varepsilon}\sqrt[3]{\frac{a'}{b}}\Big)_u, \\ & - \frac{i}{\varepsilon}\sqrt[3]{\frac{a'}{b}}\Big(\frac{i}{\varepsilon}\sqrt[3]{\frac{a'}{b}}\Big)_v + a'\frac{i}{\varepsilon}\sqrt[3]{\frac{a'}{b}} - b'\sqrt[3]{\frac{a'^2}{b^2}}\Big\}\, y\Big]. \end{aligned}$$

and

$$
\sigma_d = \frac{1}{4\psi_v}\Big[2y_v - 2\frac{1}{\varepsilon}y_u\sqrt[3]{\frac{a'}{b}} + \frac{\varepsilon}{i}\Big\{\sqrt[3]{\frac{b}{a'}}\Big\}\Big\{b + b\frac{1}{\varepsilon}\sqrt[3]{\frac{a'}{b}} - \Big(\frac{1}{\varepsilon}\sqrt[3]{\frac{a'}{b}}\Big)_u, \tag{86}
$$
$$
- \frac{1}{\varepsilon}\sqrt[3]{\frac{a'}{b}}\Big(\frac{1}{\varepsilon}\sqrt[3]{\frac{a'}{b}}\Big)_v - b'\frac{1}{\varepsilon}\sqrt[3]{\frac{a'^2}{b^2}} - a\sqrt{\frac{a'}{a}}\Big\}y\Big].
$$

Again we can join the two covariant points,this time the ones given in (850 and (86),and this line will intersect each of the asymptotic tangents in a unique points. These points on the asymptotic tangents may be joined to the canonical point by a line ad thus determine other points on the tangents to the asymotic curves. We still have the possibility of using the harmonic conjugate of any point with any other points already found and so there are a great number of points which are uniquely associated with any point P_y on a curved surface in space.