Points Uniquely Associated with a Point on a Curved Surface.

by

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I. Introduction.

In the study of a curved surface in space, by projective differential methods, there are defined many unique points connected with every point on the surface. The purpose of this thesis is to collect the information with regard to these points, some of which has appeared elsewhere in miscellaneous papers, into a single paper, where it will be more available for use. In connection with these points, a geometrical interpretation will be given.
II. Representation of the Surface.

We assume that the homogeneous coordinates of a point \( P_y \) in space is given as an analytic function of the two variables:

\[
y^{(x)} = f^{(x)}(u, v), \quad x = 1, 2, 3, 4.
\]

It has been shown that a curved surface may be represented by two second order differential equations of the form:*  

\[
\begin{align*}
    y_{uu} + 2a y_u + 2b y_v + c y &= 0, \\
    y_{vv} + 2a' y_u + 2b' y_v + c' y &= 0,
\end{align*}
\]

where two directrix curves are uniquely determined on each point \( P_y \) on the surface. For the form of differential equations (1) the parametric curves on the surface are asymptotic curves. \( C_u \) is generated by allowing the variable \( u \) to change and keeping \( v \) equal to a constant. The other parametric curve, \( C_v \), is generated by allowing \( v \) to vary and keeping \( u \) equal to a constant.

III. Transformations of the Variables.

A set of surfaces, all projectively equivalent, was defined by (1). Wilczynski* has shown that the most general transformations which leave the form of the differential equations in (1) unchanged are of the form:

\[(2) \quad \bar{u} = \Phi(u), \quad \bar{v} = \Psi(v),\]

for the independent variable, and

\[(3) \quad \bar{y} = \lambda(u, v) \bar{y},\]

for the dependent variable. Both of these transformations given by (2) and (3) leave the surface, as well as the form of the equations in (1), unchanged.

From transformation (2) we have:

\[
\begin{align*}
\bar{u} &= \Phi(u) \bar{u}, \quad \bar{v} = \Phi(v) \bar{v}, \\
\bar{y} &= \Phi^2(\bar{u}, \bar{v}) + \Phi(u) \bar{y}, \\
\bar{y} &= \Phi^2(\bar{y}) + \Phi(v) \bar{y}, \\
\bar{y} &= \Phi(u) \Phi(v) \bar{y},
\end{align*}
\]

When the values of these derivatives are substituted in (1) we have a new set of equations of the form:

\[
\begin{align*}
\bar{y} &= \left( \frac{\partial^2 \bar{y}}{\partial \bar{u} \partial \bar{v}} \right) + 2A \bar{y} + 2B \bar{y} + C \bar{y} = 0, \\
\bar{y} &= \left( \frac{\partial^2 \bar{y}}{\partial \bar{u} \partial \bar{v}} \right) + 2A' \bar{y} + 2B' \bar{y} + C' \bar{y} = 0,
\end{align*}
\]

---

where:

\[ R = \frac{1}{\Phi_u} (a + \frac{1}{2} \psi) \]

\[ R' = \frac{\psi_u}{\psi} \psi , \]

\[ B = \frac{\psi_u}{\Phi_u} \psi , \]

\[ B' = \frac{1}{\psi} (a' + \frac{1}{2} \psi) , \]

\[ C = \frac{1}{\Phi_u} c , \]

\[ C' = \frac{1}{\psi} c , \]

and

\[ \gamma = \frac{\psi_u}{\psi} ; \quad f = \frac{\psi_u}{\psi} . \]

Some of the derivatives of the above coefficients will be needed later in this paper. These are:

\[ B'_u = \frac{1}{\Phi_u} \psi_v \psi'_u \]

\[ B'_v = \frac{\Phi_u}{\psi_v^3} (a'_u - 2 \theta \psi) , \]

\[ \gamma'_u = \frac{1}{\psi_v^2} (a'_u + \theta \psi) ; \quad \gamma_v = \frac{1}{\psi_v} \psi_v a_v , \]

\[ B_u = \frac{\psi_v}{\Phi_u^3} (\nu_u - 2 \nu \theta) , \]

\[ B'_v = \frac{1}{\Phi_u^2} (\nu_v + \nu \theta) . \]

Now let us make the dependent variable transformation on equation (5). If we differentiate, we have:

\[ \bar{y} = \lambda \bar{y} , \]

\[ \bar{y}_u = \lambda \bar{y}_u + \lambda \bar{y}_u , \]

\[ \bar{y}_v = \lambda \bar{y}_v + \lambda \bar{y}_v , \]

\[ \bar{y}_{uu} = \lambda \bar{y}_{uu} + 2 \lambda \bar{y}_u \bar{y}_u + \lambda \bar{y}_u \bar{y}_u , \]

\[ \bar{y}_{uv} = \lambda \bar{y}_{uv} + \lambda \bar{y}_u \bar{y}_v + \lambda \bar{y}_u \bar{y}_v + \lambda \bar{y}_v \bar{y}_u , \]

\[ \bar{y}_{vv} = \lambda \bar{y}_{vv} + 2 \lambda \bar{y}_v \bar{y}_v + \lambda \bar{y}_v \bar{y}_v . \]
If these are substituted in (5) we have a set of equations of the form

\begin{align*}
\dot{y} + 2\tilde{A} \tilde{y} + 2\tilde{B} \tilde{y} + \tilde{C} \tilde{y} &= 0, \\
\ddot{y} + 2\tilde{A}' \dot{y} + 2\tilde{B}' \dot{y} + \tilde{C}' \dot{y} &= 0,
\end{align*}

where the coefficients are expressed in terms of the original coefficients and functions of the transformations, as follows

\begin{align*}
\tilde{A} &= A + \frac{\lambda u}{\lambda}, \\
\tilde{B} &= B, \\
\tilde{A}' &= A', \\
\tilde{B}' &= B + \frac{\lambda v}{\lambda}, \\
\tilde{C} &= C + \frac{\lambda u + 2\alpha \lambda u + 2\lambda^2}{\lambda}, \\
\tilde{C}' &= C' + \frac{\lambda v + 2\alpha' \lambda u + 2\beta' \lambda v}{\lambda},
\end{align*}
IV. Invariants and Covariants.

A function of the coefficients and their derivatives which remains unchanged under both the dependent and the independent variable transformation is called an Invariant. If this function is unchanged except for a factor it is called a relative invariant.

A function of the coefficients, their derivatives, the dependent variable, and its derivatives which is unchanged under both the dependent and independent variable transformations is called a Covariant. If this function is unchanged except for a factor it is called a relative Covariant.

If transformations (2) and (3) are made on (1), we arrive at a system of equations given by (10). If the coefficients in this system of equations are fixed to such an extent that they can only be changed by a factor, then we say that this set of equations is in the Canonical form.

Stouffer has shown that by means of the two transformations (2) and (3), the following coefficients and derivatives can be made to vanish at a point where \( \mu = 0, \nu = 0 \):

\[
\vec{a} = \vec{b} = \vec{d} \big|_{\mu = 0, \nu = 0} = \vec{e} \big|_{u = 0, \nu = 0} = 0
\]
Under these conditions the transformations are fixed to such an extent that the coefficients, the variables, and the derivatives of each, are completely determined except for a factor. We have therefore arrived at a Canonical form. In the form (10) after the conditions given by (11) have been imposed, each of the coefficients leads, by direct substitution back, to an Invariant in terms of the original coefficients. The variables and their derivatives in (10) lead by direct substitution back, to Covariants in terms of the original variables and coefficients.

It is easily seen how the transformations that impose conditions (11) are found. The independent variable transformation is fixed by making:

\[ R' = \frac{1}{\psi'} \left( a' + \gamma a' \right) = 0; \quad \gamma = -\frac{a' \omega}{\omega'(\omega,0)} \]

\[ B' = \frac{1}{\psi'} \left( \lambda \psi + \gamma \beta \right) = 0; \quad \lambda = -\frac{\psi(\psi,0)}{\psi'(\psi,0)} \]

The conditions above are maintained by (2) if and only if:

\[ \gamma \equiv \frac{\psi \phi}{\phi} = 0 \quad \text{or} \quad \phi = a \text{const} \]

\[ \lambda \equiv \frac{\psi \psi'}{\psi'} = 0 \quad \text{or} \quad \psi = a \text{const} \]

In order to impose the other conditions in (11) we use the dependent variable transformation. This transformation is determined by:

\[ \tilde{R} = R + \frac{\lambda u}{\lambda} = 0; \quad \frac{\lambda u}{\lambda} = -R, \]

\[ \tilde{B}' = B' + \frac{\lambda \psi}{\lambda} = 0; \quad \frac{\lambda \psi}{\lambda} = -B'. \]
It is not in general possible to put two conditions, such as those in (14) on the transformation in (3). It is, however, possible to do it in this case because of the integrability conditions that had to be put on the coefficients in the original equations. The conditions in (14) are maintained if and only if $\lambda = \cosh$.

Now by direct substitution back, we shall calculate the covariants of (10) under conditions (11) in terms of the original variables and coefficients. From (9) we have

$$
\bar{y} = \lambda \tilde{y},
$$

$$
\bar{y}_u = \lambda \tilde{y}_u + \lambda \tilde{y}_u,
$$

with

$$
\frac{\lambda}{\lambda} = -\alpha', \quad \frac{\lambda}{\lambda} = -\beta',
$$

From (2), (4), and (8) we have further

$$
\mathcal{A} = \frac{1}{\rho_u} (\alpha + 2\tilde{\alpha}), \quad \bar{y}_u = \frac{1}{\rho_u} y_u,
$$

with

$$
\mathcal{A} = -\frac{\partial \tilde{\alpha}}{\partial \rho_u},
$$

which shows that,

$$(15) \quad \bar{y}_u = \frac{1}{\lambda \rho_u} \left[ y_u + \beta \frac{\partial \phi_v}{\partial \phi_v} \right].$$

This holds for $v=0$ and $v=\infty$, but if we go back and look at the quantities used in calculating this expression we see that it did not matter whether $v=0$ or not. This will hold at any point in general; therefore, we have an covariant in (15). By methods similar to those above it is easily verified that

$$(15) \quad \bar{y}_u = \frac{1}{\lambda \rho_u} \left[ y_u + \beta \frac{\partial \phi_v}{\partial \phi_v} \right],$$

$$(17) \quad \bar{y}_u = \frac{1}{\lambda \rho_u} \left[ y_u + \beta \frac{\partial \phi_v}{\partial \phi_v} \right].$$
In (15), (16), and (17) we have three relative covariant expressions representing three covariant points of the coordinate system in (1). If the factor multiplying each of these relative covariant expressions be disregarded, these three points, with the point \( P_y \), may be used as the vertices of a tetrahedron of reference for the study of the surface.

It has been found that the conditions in (11) give rise to the same covariant points that Wilczynski used for his canonical tetrahedron. He defined (15) and (16), geometrically, by use of the osculating ruled surface. By an osculating ruled surface we mean the surface formed by all the tangents to one set of asymptotic curves along another asymptotic curve. All the tangents to the \( C_u \) curves along one \( C_v \) curve forms the osculating ruled surface which contains the line joining \( P_y \) to \( Y_u \). Since this is a line of the ruled surface, there will be two flecnodes points on it. Wilczynski defined the vertex of his tetrahedron on this line as the harmonic conjugate of \( P_y \) with respect to these two flecnodes points.

Other sets of conditions different from (11) may be put on (2) and (3), and we obtain another set of covariant points. Next we shall show that we can make
(18) \( \bar{\alpha} = \bar{\beta}' = \bar{\alpha}'(\varphi = 0, \varphi = \varphi') = 1 \bar{\alpha}(\varphi = 0, \varphi = 0) = 0, \)

at the point \( \alpha = 0, \varphi = 0. \) From (8) we see that we can determine (2) by making \( \alpha' \) and \( \beta' \) vanish. We cannot make these vanish identically but only at a point. Thus we have

\[
\bar{\alpha}' = \frac{\psi'}{\psi^2}(\alpha' - 2 \psi') = 0,
\]

which gives

\[
\beta = \frac{\alpha'}{\psi^2},
\]

also

\[
\bar{\beta} = \frac{\psi'}{\psi^2}(\alpha' - 2 \psi') = 0,
\]

from which it follows that

\[
\gamma = \frac{\alpha''(\psi')}{2},
\]

\( \bar{\alpha}' \) and \( \bar{\beta} \) are maintained equal to zero at the point \( \alpha = 0, \varphi = 0 \) if and only if \( \beta \) and \( \gamma \) are both equal to zero, or what is equivalent, if \( \psi \) and \( \psi' \) are kept constant.

Transformation (3) is determined precisely as it was under conditions (11) for we have exactly the same conditions to impose again.

The two transformations are determined and if they are made on (1) under the conditions in (18), an equation of the form (10) will result. The variables in (11) will, by direct substitution back, give rise to relative covariant expressions in terms of the original coefficients and variables. It is very easy to verify that the variables in (10), under the conditions in (18) give rise to the relative covariants given by
These covariant points are the ones that Green used for his tetrahedron of reference but he defined them geometrically. If we call the covariant point, on the asymptotic tangent to $C_U$, that Green used, $C_1$ and the covariant point that he used on the asymptotic tangent to $C_V$ as $C_2$, then he defined these points as follows: $C_1$ is the pole of the line $P_yY_v$ with respect to the osculating conic to $C_U$. $C_2$ is the pole of the line $P_yY_u$ with respect to the osculating conic to $C_V$.

To simplify our notation we shall now define what we are going to mean by corresponding points on the two asymptotic tangents. If the two transformations in (2) and (3) are made on (1) with the conditions on the transformations that
\[
\begin{align*}
\vec{a} &= 0; \quad \vec{b}' = 0; \\
\ell \frac{\vec{p}_u'}{\vec{a}} + m \frac{\vec{p}_v}{\vec{b}} &= 0; \quad \ell \frac{\vec{p}_v'}{\vec{b}} + m \frac{\vec{p}_u}{\vec{a}} &= 0, \quad \alpha + (u = v = 0),
\end{align*}
\]
there shall result one covariant point on each asymptotic tangent. These two covariant points are called corresponding points on the two lines. We see that (11) and (18) are special cases of (20) and, therefore, $D_1$ corresponds to $D_2$, and $C_1$ to $C_2$. 

(19) $\vec{G}_u = \frac{1}{\lambda} \phi_u \left[ y_u + (\alpha + \frac{\ell u}{4}) y \right] = C_u$,

(20) $\vec{G}_v = \frac{1}{\lambda} \phi_v \left[ y_v + (\alpha' + \frac{\ell v}{4}) y \right] = C_v$. 

\[
\begin{align*}
\vec{a} &= 0; \quad \vec{b}' = 0; \\
\ell \frac{\vec{p}_u'}{\vec{a}} + m \frac{\vec{p}_v}{\vec{b}} &= 0; \quad \ell \frac{\vec{p}_v'}{\vec{b}} + m \frac{\vec{p}_u}{\vec{a}} &= 0, \quad \alpha + (u = v = 0),
\end{align*}
\]
If in (20) \( l = m \) we have two other corresponding points determined analytically. These we shall call \( F_1 \) and \( F_2 \). Fubini used these points as two vertices of his canonical tetrahedron, but he defined them geometrically instead of analytically. These points expressed in terms of the original variables and coefficients are

\[
F_1 = \frac{\ell}{\lambda_{\nu} u} = \frac{1}{\lambda_{\nu} u} \left[ y_u + \left( a + \frac{1}{2} \left\{ \frac{a^2 + \frac{a'}{a}}{2} \right\} \right) \right],
\]

(21)

\[
F_2 = \frac{\ell}{\lambda_{\nu} v} = \frac{1}{\lambda_{\nu} v} \left[ y_v + \left( a' + \frac{1}{2} \left\{ \frac{a^2 + \frac{a'}{a}}{2} \right\} \right) \right].
\]

It is very easy to verify analytically that the four points \( P_y, D, C, F \) on the asymptotic tangent to \( C_u \) form a harmonic set. Also the four corresponding points on the tangent to \( C_v \) form a harmonic set. We shall now show that the four points \( P_y, D, C, F \) do form a harmonic set on the tangent to \( C_u \). Any point on this tangent may be represented as a linear combination of \( P_y \) and \( y^* \) of the form \( y_u + \lambda y \). The coordinates of the four particular points we are studying are

\[
\begin{align*}
P_y &= (\infty, 1, 0, 0), \\
D &= (a - \frac{1}{2} \frac{a^2}{a'}, 1, 0, 0), \\
C &= (a + \frac{1}{2} \frac{a^2}{a'}, 1, 0, 0), \\
F &= (a + \frac{1}{2} \left\{ \frac{a^2 + \frac{a'}{a}}{2} \right\}, 0, 0).
\end{align*}
\]

If we form the double ratio of these four points

\* The point \( Y \) is the point whose coordinates are \( y^* \).
\[ (P_4, C, D, F_t) = \frac{a_m + b_n}{2a^' + 4b^'} = -1, \]

which shows that \( F_1 \) is the harmonic conjugate of \( P_4 \) with respect to \( P_y \) and \( C_1 \).

We shall show that all the lines joining corresponding points on the asymptotic tangents go through the same point. We shall find the coordinates of the point of intersection of the lines \( D_1D_2 \) and \( C_1C_2 \) and show that the line joining any two points in general go through this point. If \( D \) is any point on the line joining \( D_1 \) and \( D_2 \) then

\[ D = \left[ y_u + (a - \frac{a_m}{2a^'}) y + \kappa_1 \left( y_v + (l^' - \frac{b_n}{2b^'}) y \right) \right], \]

and the coordinates of \( D \) are

\[ (23) \left( a - \frac{a_m}{2a^'}, \frac{b_n}{2b^'}, \kappa_1, \kappa_2, 0 \right), \]

Let \( C \) be any point on the line \( C_1C_2 \), then

\[ C = \left[ (a + \frac{b_n}{2b^'}) y + \kappa_1 \left( y_v + (l^' + \frac{a_m}{4a^'}) y \right) \right], \]

and the coordinates of \( C \) are

\[ \left( a + \frac{b_n}{2b^'}, \kappa_1 \left( l^' + \frac{a_m}{4a^'}, 1, \kappa_2, 0 \right). \]

These two lines will intersect where the coordinates of \( D \) are equal to the coordinates of \( C \). In examining the coordinates of \( D \) and \( C \) we see that the \( \frac{2b_n}{4b^'} \) and \( \frac{a_m}{4a^'} \) are identical. We can make the \( y_v \) coordinate the same by making \( \kappa_1 = \kappa_2 \). If we set the two \( Y \) coordinates equal.
We can solve for the value of $k_1$ which when substituted in (23) will give the coordinates of the point of intersection of $D_1 D_2$ and $C_1 C_2$. If we let $L$ be any point on the line joining $I_1$ on the asymptotic tangent to $C_u$ to the corresponding point $I_4$ on the asymptotic tangent to $C_v$, then

$$L = \left[ y_u + (a + \frac{\ell c_u}{\alpha} + m \frac{c_u}{\alpha}) y + k_3 \left( \ell_1 + \frac{\ell c_v + m c_v}{\alpha} \right) \right],$$

and the coordinates of $L$ in terms of the original coordinate system are

$$ (24) \left( a + \frac{\ell c_u}{\alpha} + m \frac{c_u}{\alpha} \right) + k_3 \left( \ell_1 + \frac{\ell c_v + m c_v}{\alpha} \right), 1, k_3, 0$$

The value of $k_1$ which gave the point of intersection of the two lines $D_1 D_2$ and $C_1 C_2$ was

$$k_1 = - \frac{\ell c_u}{\alpha} + \frac{\ell c_v}{\alpha}.$$ 

If $k_3$ in (24) is made equal to $k_1$ then it is easily verified that (24) and (25) are the same point, which shows that all of these lines joining corresponding points on the asymptotic tangents go through a point. This point is called the canonical point and has the coordinates

$$ (a + \frac{\ell c_u}{\alpha} - \left( \frac{\ell c_u}{\alpha} + \frac{\ell c_v}{\alpha} \right) (\ell_1 + \frac{\ell c_v}{\alpha}, 1) - \left( \frac{\ell c_u}{\alpha} + \frac{\ell c_v}{\alpha} \right), 0).$$

If $\ell = 2m$ in (20) we have a special case in which the covariant point on each asymptotic tangent has moved to the point $P_y$. We cannot put the conditions in (20) on (2) and (3) for the case for which $\ell = 2m$. 

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V. Osculating Quadrics.

An osculating quadric to a surface at a point on the surface is the quadric which has second order contact with the surface at that point. We shall first derive an osculating quadric known as the quadric of Lie.

If we let the surface be represented by the set of equations in (1) we can develop the equation of an osculating ruled surface along the asymptotic tangent to the \( C_u \) curve. That is, it will have \( C_v \) as one directrix curve. If we let \( \rho = y_u \), then by differentiation and from (1) we have:

\[
\rho_{uv} - 4a'x y_v + (c' - 2d'c) y + (2a'c + c') \rho = 0. \tag{25}
\]

(25) with the second equation in (1) is the equation of the osculating ruled surface. These are two equations of the form

\[
\begin{align*}
\rho_{uv} + 2f_{uv} y_v + 2f_{v2} \rho_v + g_{11} y_v + g_{12} \rho = 0, \\
\rho_{uv} + 2f_{v1} y_v + 2f_{22} \rho_v + g_{21} y_v + g_{22} \rho = 0,
\end{align*}
\tag{26}
\]

where

\[
\begin{align*}
f_{11} &= 0; & g_{11} &= c', \\
f_{12} &= 0; & g_{12} &= 2a', \\
f_{21} &= -2a'b; & g_{21} &= c' - 2a'c, \\
f_{22} &= 0; & g_{22} &= 2a'c + c'.
\end{align*}
\]
In the study of a ruled surface it is found that there is a point, on the tangent to each asymptotic curve, defined as follows

\[ \mathbf{r} = \mathbf{y} + \mathbf{f}_1 y + \mathbf{f}_2 \rho = \mathbf{y}, \]
\[ \mathbf{v} = \rho v + \mathbf{f}_1 y + \mathbf{f}_2 \rho = \rho v + 2a_1 b_1 y. \]

Lie's quadric is the quadric which contains all the asymptotic tangents to the ruled surface, in (26) along the line \( P_y P \). Any point on the quadric may be represented in the form

\[ \lambda (\mathbf{y} + \alpha \rho) + \beta (\mathbf{y} + \alpha (\rho v + 2a_1 b_1 y)). \]

This same point may be represented in terms of the original coordinate system as

\[ x, y + x_2 \rho + x_3 y + x_4 \beta. \]

If we equate coefficients in (28) and (29) we have

\[ x_1 = (\lambda - 2a_1 b_1 \beta), \]
\[ x_2 = \lambda \alpha, \]
\[ x_3 = \beta, \]
\[ x_4 = \alpha \beta, \]

If from these we eliminate \( \alpha \beta \), and \( \lambda \) we have

\[ x_1 x_4 - x_2 x_3 + 2a_1 b_1 x_4^2 = 0, \]

as the equation of the quadric of Lie. The quadric to the other osculating ruled surface, is, from symmetry, identical with (30).
Next we shall derive the equation of Fubini's quadric in terms of the original coordinate system. The equation of his quadric when referred to his canonical tetrahedron is

\[(31) \quad \lambda f_1 x y - \lambda f_3 x_3 = 0,\]

He set up his tetrahedron of reference by means of transformation (3) under the conditions

\[(32) \quad \frac{\lambda f_1}{\lambda} = -\alpha; \quad \frac{\lambda f_2}{\lambda} = -\beta,\]

where

\[\alpha = a + \frac{1}{2} \left( \frac{f_{11}}{\lambda} + \frac{f_{12}}{\lambda} \right); \quad \beta = \frac{f_{22}}{2} \left( \frac{f_{12}}{\lambda} + \frac{f_{22}}{\lambda} \right).\]

The covariants of his system of equations, when expressed in the original coordinate system, were

\[(33) \quad \bar{y}_{\tilde{u} \tilde{v}} = \frac{1}{\lambda} \left[ y_{\tilde{u} \tilde{v}} + \alpha y_{\tilde{v}} \right] \equiv F_1; \]
\[\bar{y}_{\tilde{v} \tilde{v}} = \frac{1}{\lambda} \left[ y_{\tilde{v} \tilde{v}} + \beta y_{\tilde{v}} \right] \equiv F_2.\]

The covariant point $\bar{y}_{\tilde{u} \tilde{v}}$ may be found by means of differentiation, and one of the integrating conditions which must be placed on the original coefficients. If we differentiate either of the covariant expressions in (33) we have

\[(34) \quad \bar{y}_{\tilde{u} \tilde{v}} = \frac{1}{\lambda} \left[ y_{\tilde{u} \tilde{v}} + \alpha y_{\tilde{v}} + \beta y_{\tilde{v}} + (\alpha + \beta) y_{\tilde{v}} \right].\]

Any point in the original coordinate system may be represented in the form

\[(35)\]

\[x, y + x_2 y_2 + x_3 y_3 + x_4 y_4,\]

and the same point in the transformed coordinate system can be expressed as

\[(36)\]

\[\bar{x}_1 \bar{y}_1 + \bar{x}_2 \bar{y}_2 + \bar{x}_3 \bar{y}_3 + \bar{x}_4 \bar{y}_4.\]

If we substitute (33) and (34) in (36) we can equate coefficients and obtain an expression for the transformation from one coordinate system to the other. Thus we have

\[
\begin{align*}
\bar{x}_1 y + x_2 y_2 + x_3 y_3 + x_4 y_4 &\equiv \frac{1}{\lambda} \left[ \bar{x}_1 y + \bar{x}_2 (y_2 + \alpha y_4) + \bar{x}_3 (y_3 + \beta y_4) + \bar{x}_4 (y_4 + \gamma y_4) \right], \\
\end{align*}
\]

from which it follows that

\[
\begin{align*}
\bar{x}_1 &= \lambda (x_1 - \alpha x_2), \\
\bar{x}_2 &= \lambda (x_2 - \beta x_3), \\
\bar{x}_3 &= \lambda (x_3 - \gamma x_4), \\
\end{align*}
\]

\[(37)\]

Where

\[
\begin{align*}
\bar{x}_1 &= \lambda \left[ x_1 - \alpha x_2 - \beta x_3 + (\alpha \beta - \alpha \gamma) x_4 \right],
\end{align*}
\]

When (37) is substituted in (31) we have

\[(38)\]

\[
x_1 \bar{x}_4 - x_2 x_3 - \alpha \sqrt{x_4^2} = 0 \text{ where } \alpha = a + \frac{1}{2} \left( \frac{\alpha^2}{a^2} + \frac{\mu}{\lambda} \right)
\]

as the equation of Fubini's Canonical quadric.

The equation of Green's canonical quadric referred to his tetrahedron of reference is*

\[(39)\]

\[\bar{x}_1 \bar{x}_4 - \bar{x}_2 \bar{x}_3 = 0.\]

---

It is easily determined by the same method as above that the relation between the coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ and the original coordinates $x_1, x_2, x_3, x_4$ is

\begin{align*}
\bar{x}_4 &= \lambda \psi \omega x_4, \\
\bar{x}_3 &= \psi \omega [x_3 - x_4(a + \frac{1}{2})], \quad \gamma = \frac{\lambda a}{2}, \\
\bar{x}_2 &= \lambda \psi [x_2 - x_4(a + \frac{1}{2})], \quad \rho = \frac{\lambda a}{2}, \\
x_1 &= \lambda [x_1 - x_4(a + \frac{1}{2}) - x_3(a + \frac{1}{2}) - x_4\{a_0 - (a + \frac{1}{2})\{a + \frac{1}{2}\}}].
\end{align*}

Substitution of (40) into (39) gives

\begin{equation}
(41) \quad x_1 x_4 - x_2 x_3 - a_0 x_4^2 = 0,
\end{equation}

as the equation of Green's quadric in the original coordinate system.
VI. Polar Reciprocals.

We shall prove that the polar plane at any point on the asymptotic tangent to \( C_u \) is the same with respect to any quadric of the form

\[
(42) \quad x_u x_v - x_1 x_2 - \kappa x^2_3 = 0.
\]

The coordinates of any point on this tangent are \((t, 1, 0, 0)\). By the use of partial derivatives it is easily found that the equation of the polar plane at \((t, 1, 0, 0)\) is

\[
(43) \quad x_2 - t x^2_3 = 0,
\]

which is independent of \( K \).

From symmetry we have that the polar plane at the corresponding point \((s, 0, 1, 0)\) on the asymptotic tangent to \( C_v \) is

\[
(44) \quad x_2 - 5 x^2_3 = 0.
\]

The line of intersection of (43) and (44) is defined as the polar reciprocal of the line joining the two corresponding points with respect to the quadric in (42). \( P_y \) is one point on this polar reciprocal and another point on both polar planes, and hence is on their point of intersection, is the point given by \((0, s, t, 1)\). Any point on the polar reciprocal may be expressed as

\[
(45) \quad k_y + s y_u + t y_v + y_w,
\]

and the coordinates of this point are
Thus far we have three unique polar reciprocal lines determined. These are the polar reciprocals of the lines $D_1D_2, C_1C_2$, and $F_1F_2$. Each of these polar lines will meet all three of the quadrics given in (30), (39) and (41), thus determining nine unique points in the polar plane.

The three points on (30) are

$$P_1: [{\alpha}, \beta, \alpha' \beta', \beta, \alpha', 1], \text{ with } \alpha_1 = \alpha - \frac{\alpha' \beta'}{2 \alpha'},$$

$$P_2: [{\alpha}, \beta, \alpha' \beta', \beta_2, \alpha, 1], \quad \beta_2 = \beta - \frac{\beta_2}{2 \beta}.$$;

$$P_3: [{\alpha}, \beta, \alpha_2 \beta_2, \beta_2, \alpha, 1], \quad \alpha_2 = \alpha + \frac{\alpha_2}{4 \alpha}.$$;

The three on (39) are

$$P'_1: [{\alpha}, \beta, \alpha' \beta', \beta, \alpha', 1], \quad \beta = \beta' + \frac{\alpha' \beta'}{4 \alpha'},$$

$$P'_2: [{\alpha}, \beta, \alpha' \beta', \beta_2, \alpha, 1], \quad \alpha = \alpha + \frac{\alpha_2}{\beta_2 + \frac{\alpha_2}{\beta}}.$$;

$$P'_3: [{\alpha}, \beta, \alpha_2 \beta_2, \beta_2, \alpha, 1], \quad \beta = \beta + \frac{\alpha_2}{\beta_2 + \frac{\alpha_2}{\beta}}.$$;

and finally the three points of intersection on (41) are

$$P''_1: [{\alpha}, \beta, \alpha' \beta', \beta, \alpha', 1],$$

$$P''_2: [{\alpha}, \beta, \alpha' \beta', \beta, \alpha, 1],$$

$$P''_3: [{\alpha}, \beta, \alpha_2 \beta_2, \beta, \alpha, 1].$$
VII. Focal Points.

The double infinity of lines formed by joining the two corresponding covariant points, one on each asymptotic tangent, forms a congruence of lines. Each line of this congruence is a line of two developable surfaces.* The two points on each line in which the line meets the edges of regression of the developables are unique points, and are called the focal points of the line.

If we let $\rho$ and $\sigma$ be the two corresponding points on the asymptotic tangents to $C_u$ and $C_v$, respectively, we shall now find the focal points on the line $\rho \sigma$. Any point on this line may be expressed as

$$\Phi = \rho + \lambda \sigma$$

where $\lambda$ is a parameter. To determine $\lambda$ to make $\Phi$ a focal point we have to impose the condition that the points determined by $\Phi$, $\rho$, $\sigma$ shall all lie in a plane.*

From definition we have

$$\begin{align*}
\rho &= y_u + \alpha \frac{\partial}{\partial \Phi}, \\
\sigma &= y_v + \beta \frac{\partial}{\partial \Phi}.
\end{align*}$$

By differentiation we have

\[ (52) \begin{align*}
\Phi_u - \lambda \Phi &= \lambda y u + (\alpha - 2a + \beta y) y - (\alpha u - c + \beta u) y, \\
\Phi_v - \lambda \Phi &= y v - 2a' y + (\alpha - 2l') y + (\alpha v - c + \beta v) y.
\end{align*} \]

Making the points represented by (51) and (52) lie in a plane is equivalent to making the points \( \Phi_u, \Phi_v, \rho, \sigma \) lie in a plane. To make the points in (51) and (52) lie in a plane we must have

\[
\begin{vmatrix}
0 & 0 & 1 & \alpha \\
0 & 1 & 0 & \beta \\
\lambda & -2\lambda & \alpha - 2a + \beta \lambda & \alpha u - c + \lambda \beta u \\
1 & \alpha - 2l' + \beta \lambda & -2a' & \alpha v - c' + \lambda \beta v
\end{vmatrix} = 0.
\]

Expanding the determinant above we have

\[
(54) \lambda(c' + \beta v - 2l' + \beta) + \lambda(\alpha - 2a' - \alpha v + \beta u) + \alpha u - c + 2l' \beta = 0,
\]

a quadratic equation in \( \lambda \) the solution of which will give the two values of \( \lambda \) that will make (50) the focal points of the line \( \rho \sigma \). If we call the two solutions of (54) \( \lambda_1 \) and \( \lambda_2 \), we have as the two focal points

\[
\begin{align*}
\Phi_1 &= \rho + \lambda_1 \sigma, \\
\Phi_2 &= \rho + \lambda_2 \sigma.
\end{align*}
\]

Another uniquely determined point with simpler coordinates is given by taking the harmonic conjugate of \( \rho \) with respect to the two points in (55).
If we take the polar reciprocal of each of the lines joining the points \( P \) and \( \sigma \), as the point \( P_y \) moves over the surface, we have another congruence of lines known as the polar reciprocal congruence. Each line of this congruence will also have two focal points. If \( \Sigma = y_{uv} + y_{vy} \), then any point on a line of the polar reciprocal congruence may be expressed as

\[
\begin{align*}
T &= \Sigma + \lambda y, \\
\lambda &\text{ is a parameter, the value of which we are going to find so that (56) will represent the focal points on the line joining } P_y \text{ to } Z. \\
\end{align*}
\]

By differentiation in (56), we have

\[
\begin{align*}
T_u - 2a y &= (\alpha - 2a) y_{uv} + 4a' e' - 2b' c + \alpha u - 2a' b) y_v \\
T_v - 2a y' &= (\beta - 2b') y_{uv} + (\mu + 4a' e - 2b' c + \alpha u - 2a' b) y_v \\
&+ (4a' e - 2b' c') y_{uv} + (2a' e - 2a' b - b' c) y_v + (2a' e - 2a' b - b' c) y_v. \\
\end{align*}
\]

Now the condition that the points \( T_u, T_v, y, \Sigma \) all lie in a plane is given by making

\[
\begin{vmatrix}
\alpha - 2a & 4a' e - 2b' c + \alpha u - 2a' b & 2a' e - 2a' b - b' c \\
\beta - 2b' & 4a' e - 2b' c + \alpha u - 2a' b & 2a' e - 2a' b - b' c \\
0 & \alpha & \beta \\
0 & 0 & 0 \\
\end{vmatrix} = 0.
\]

If (58) is expanded we have

\[
(\mu - \mu) (\mu - \mu) = 0,
\]

a quadratic in the two solutions of which, when substituted in (56) will give the two focal points of the line. If we call \( \mu, \) and \( \mu, \) the two solutions of (59)
we have

\[ z + x' y = \tau_1, \]
\[ z + x' y = \tau_2, \]

as the two focal points of the line \( P_y Z \). Again we see that the expressions representing these focal points are very complicated. If we determine the harmonic conjugate of \( P_y \) with respect to the two points in (60) we have another uniquely determined point the coordinates of which are not as complicated as the coordinates of the two points in (60). If we call this point \( \tau_c \), then we have

\[ \tau_c = z + (2ax + 3a - 2b + -a^2 - 6a - x + 4a + y + 2ab') y. \]
VIII. Points Defined by Unique Conjugate Curves.

Any double infinity of curves on a surface such that two and only two curves pass through each point, is defined as a Net of Curves on a surface. The parametric curves satisfy this definition, and therefore represent a net of curves on the surface. Let us look at the differential equation

\[(62) \frac{P}{\partial u^2 + Q \partial u \partial v + R \partial v^2} = 0.\]

The solution of this quadratic for the ratio of \(\partial u\) to \(\partial v\) determines two directions at every point on the surface. If \(Q\) vanishes in (62), then this equation represents a conjugate net of curves. By a conjugate net of curves we mean a set of curves such that at each point the tangents to these curves separate the asymptotic tangents harmonically.

Green has shown that a pair of differential equations of the form

\[(63) \begin{align*}
\frac{\partial u}{\partial v} &= f^2 y_{uu} + f^1 y_{uv} + g_{uv} + g_{t2} y, \\
\frac{\partial u}{\partial v} &= f_{uv} y_{uu} + g_{uv} y_{uv} + g_{uv} y,
\end{align*}\]

are associated with a surface for which the parametric curves are conjugate curves. The most general trans-


# This is true only if the directrix curves are the asymptotic curves on the surface.
formations which can be made on (63) so that the form of the equations and the surface will both remain unchanged are given by (2) and (3). There are also some conditions of integrability which must be maintained on the original conditions. The only one that we shall have occasion to use is

\[(64) \quad (\ell + 2c') v = (2c' - \frac{c + a}{a}) u^2.\]

Two covariant points of (63) under (2) and (3) are given by\(*\)

\[(65) \quad \rho_i = y_u - y_v y_i, \quad \sigma_i = y_u - y_v y_v.\]

If we think of the same surface represented by both (1) and (63), we see that there should be a transformation of the form

\[(66) \quad \tilde{y} = y(x, y), \quad \tilde{v} = y(x, y),\]

which would transform (1) into (63). From (66) we have

\[(67) \quad \begin{aligned}
    y_u &= \varphi_u \tilde{y} + \psi_u \tilde{v}, \\
    y_v &= \varphi_v \tilde{y} + \psi_v \tilde{v}, \\
    y_{uu} &= \varphi_{uu} \tilde{y} + 2 \varphi_u \psi_u \tilde{v} + \psi_u \tilde{v} + \varphi_u \tilde{y} + \psi_u \tilde{v}, \\
    y_{uv} &= \varphi_{uv} \tilde{y} + (\varphi_u \psi_v + \psi_v \psi_u) \tilde{v} + \varphi_u \tilde{y} + \psi_u \tilde{v}, \\
    y_{vv} &= \varphi_{vv} \tilde{y} + 2 \varphi_v \psi_v \tilde{v} + \psi_v \tilde{v} + \varphi_v \tilde{y} + \psi_v \tilde{v},
\end{aligned}\]

* * *

By substitution in (11) we obtain

\[(68) \ \phi_u^2 \hat{y}_{uv} + 2 \phi_u \phi_v \hat{y}_{uv} + \phi_u \phi_v \hat{y}_{uv} + \phi_u \phi_v \hat{y}_{uv} + \psi_u \hat{y}_{uv} + \psi_v \hat{y}_{uv} = \partial \phi_u \hat{y}_{uv} + \partial \phi_v \hat{y}_{uv} + \partial \psi_u \hat{y}_{uv} + \partial \psi_v \hat{y}_{uv} + \partial \hat{y}_{uv}.
\]

and

\[(69) \ \phi_v^2 \hat{y}_{uv} + 2 \partial \phi_u \hat{y}_{uv} + \phi_v \phi_u \hat{y}_{uv} + \phi_v \phi_u \hat{y}_{uv} + \psi_v \hat{y}_{uv} + \psi_u \hat{y}_{uv} = \partial \phi_u \hat{y}_{uv} + \partial \phi_v \hat{y}_{uv} + \partial \psi_u \hat{y}_{uv} + \partial \psi_v \hat{y}_{uv} + \partial \hat{y}_{uv}.
\]

If (68) and (69) are solved for \(\hat{y}_{uv}\) and \(\hat{y}_{uv}\), we have two equations of the form

\[(70) \ \begin{align*}
\hat{y}_{uv} &= \hat{f}_1 \hat{y}_{uv} + \hat{f}_2 \hat{y}_{uv} + \hat{f}_3 \hat{y}_{uv} + \hat{f}_4 \hat{y}_{uv} + \hat{f}_5 \hat{y}_{uv}, \\
\hat{y}_{uv} &= \hat{g}_1 \hat{y}_{uv} + \hat{g}_2 \hat{y}_{uv} + \hat{g}_3 \hat{y}_{uv} + \hat{g}_4 \hat{y}_{uv} + \hat{g}_5 \hat{y}_{uv},
\end{align*}
\]

where the coefficients are expressions in terms of the coefficients in (1) and functions of the transformations. Thus we have

\[(71) \ \begin{align*}
\hat{g}_1 &= \frac{\partial \phi_u \psi_v - \partial \phi_v \psi_u}{2 \phi_u \partial \phi_v - \partial \psi_u \partial \psi_v}, \\
\hat{g}_2 &= \frac{\partial \phi_u \psi_v - \partial \phi_v \psi_u}{\phi_u \partial \phi_v - \partial \psi_u \partial \psi_v} - \phi_u \phi_v \psi_v, \\
\hat{g}_3 &= \frac{\partial \phi_u \psi_v - \partial \phi_v \psi_u}{\phi_u \partial \phi_v - \partial \psi_u \partial \psi_v} - \phi_u \phi_v \psi_u.
\end{align*}
\]

The condition that (71) represent a surface on which the parametric curves are conjugate is that \(q_1\) vanishes. The vanishing of \(q_1\) puts the condition on the transformation that

\[(72) \ \frac{\partial \psi_v}{\partial \phi_u} = \frac{\psi_v}{\psi_u}.
\]
The positive sign in (72) cannot be used for in that case the Jacobian of the transformation would vanish and there would be no transformation. Therefore, the condition on (66) that transforms (1) into (63) is

\[
\frac{\psi_u}{\psi_v} = -\frac{\psi_u}{\psi_v}.
\]

The curves in which the developables of the projective normal congruence cut the surface are a pair of unique curves on the surface. These two conjugate curves have tangents on each of which there is a unique point. The unique points are given by (65). We are now going to find the expressions for the points in (65) when referred to the coordinate system in (1).

The differential equation of the unique pair of conjugate curves defined above is*

\[
Q dv - Q' du = 0,
\]

where

\[
Q = [\Theta uu - 2\Theta uv + (f - 2k)\theta],
Q' = [\Theta vv - 2\Theta iv + (g - 2k')\theta],
\]

\[
\theta = \frac{1}{\sqrt{a''}},
\]

\[
f = c - a u - a v - 2 kl',
\]

\[
g = cl' - l' v - l'' v - 2aa',
\]

From (66) and (73) we have
\[ du = \frac{\psi \nu t \nu - \psi \nu u}{2 \varphi \nu \nu} , \]
(75) \[ dv = \frac{\psi \nu t \nu - \psi \nu u}{2 \varphi \nu \nu} , \]

If (75) are substituted in (74) we have
\[ (\varphi \psi \nu t - \varphi \psi \nu u)^2 du^2 + 2(\varphi \psi \nu u \psi \nu - \varphi \psi \nu \nu \nu) dv dt \]
(76) \[ + (\varphi \psi \nu t - \varphi \psi \nu u) dv^2 = 0 . \]

If the \( u = \text{const} \) and the \( v = \text{const} \) curves are to be the directrix curves on the surface, the coefficients of \( du^2 \) and \( dv^2 \) must vanish. To make these coefficients vanish we must have
\[ \frac{\psi \nu}{\psi \nu} = \pm \sqrt{\frac{\varphi'}{\varphi}} ; \quad \frac{\psi \nu}{\psi \nu} = \mp \sqrt{\frac{\varphi}{\varphi'}} , \]
(77)

If the transformation in (66) is made on (63), under the conditions in (77), we can calculate the covariant point on each of the tangents to the conjugate, giving these points in (65) expressed in terms of the variables and coordinates in (1) as
\[ \rho_i = \frac{1}{2 \psi \nu} \left[ \psi (\frac{\varphi}{\varphi'}) u + \psi \nu + \frac{\varphi}{\varphi'} \left\{ 2 \varphi + \varphi \sqrt{\frac{\varphi}{\varphi'}} + (\frac{\varphi}{\varphi'})^2 \right\} \right] y_i \]
(78)
\[ \sigma_i = \frac{1}{2 \psi \nu} \left[ \psi \nu - \sqrt{\frac{\varphi}{\varphi'}} y_n + \frac{\varphi}{\varphi'} \left\{ 2 \varphi + \varphi \sqrt{\frac{\varphi}{\varphi'}} - (\frac{\varphi}{\varphi'})^2 \right\} \right] y_i \]
If the two points in (78) are joined by a line this line will intersect each of the asymptotic tangents since all of these lines are in the tangent plane to the surface. If we call the point of intersection of the line $\beta_1 \sigma_1$ with the asymptotic tangent to $C_u M_1$ and $N_2$ as the point of intersection with the tangent to $C_v$, then we can join each of these points to the canonical point and determine two other unique points. The coordinates of each of these points is easily found from (78). For example to find the coordinates of $M_1$ it is necessary to take a linear combination of $\beta$ and $\sigma$ so that the $y$ coordinate will be zero, etc.
IX. Darboux and Segre Directions.

Darboux found that there were three uniquely determined directions at every point on a surface. These three directions are the ones along which the osculating quadric has third order contact with the surface. The differential equation which gives the Darboux directions is*

\[ b^3 du^3 + b' dv^3 = 0. \]

The conjugate directions of the Darboux directions are called the Segre directions and are given by the differential equation**

\[ b' du^3 - b dv^3 = 0. \]

Throughout the discussion which follows we shall use \( \epsilon \) to represent any one of the three cube roots of unity.

The differential equations which represent a pair of these conjugate curves on the surface is given by

\[ \sqrt{3} b' du + \sqrt{2} b dv = 0, \]

\[ \sqrt{3} b' du - \epsilon \sqrt{2} b dv = 0. \]

If the transformation in (66), under the condition in (73), is made on (81) we arrive at two new equations given by

\[ \begin{align*}
    d\bar{u} + \bar{x} \, d\bar{v} &= 0, \\
    d\bar{u} + \bar{x}' \, d\bar{v} &= 0,
\end{align*} \tag{82} \]

where

\[ \begin{align*}
    \chi &= \frac{\varepsilon \sqrt{a'} \phi_u - \varepsilon \sqrt{a'} \phi_v}{\sqrt{a'} \psi_u - \varepsilon \sqrt{a'} \psi_v}, \\
    \chi' &= \frac{\varepsilon \sqrt{a'} \phi_u + \varepsilon \sqrt{a'} \phi_v}{\omega_u \sqrt{a'} + \varepsilon \psi_u \sqrt{a'}},
\end{align*} \tag{83} \]

If (83) is to represent a pair of conjugate curves we must have \( \chi = -\chi' \). Putting this condition on the transformation is equivalent to making

\[ \begin{align*}
    \frac{\phi_u}{\phi_v} &= \pm \frac{\sqrt{a'}}{\varepsilon}, \quad \frac{\psi_u}{\psi_v} = \pm \frac{\sqrt{a'}}{\varepsilon}, \\
    \tau &= \sqrt{-1}.
\end{align*} \tag{84} \]

By the same method that was used in the preceding section we can find the covariant point on each of the tangents to the conjugate curves. If we call these two points \( \rho_d \) and \( \zeta \) we have

\[ \begin{align*}
    \rho_d &= \frac{1}{\varepsilon a} \left[ 2\frac{\phi'}{a} y' + 2 y' + \frac{\phi'}{a} \left[ \frac{\partial}{\partial u} - \frac{\partial}{\partial u} \right] \right], \\
    \zeta &= \frac{\sqrt{a'}}{\varepsilon} \left( \frac{\phi'}{a} \right) y' - \frac{\sqrt{a'}}{\varepsilon} \left[ \frac{\partial}{\partial u} - \frac{\partial}{\partial u} \right] y'.
\end{align*} \tag{850} \]
and

$$\tau d = \frac{1}{\eta_0} \left[ z y_{\nu} - 2 \frac{z}{y_{\nu}} \sqrt{a^2} + \frac{z}{y_{\nu}} \left\{ \frac{1}{y_{\nu}} \right\} \right] + \frac{z}{y_{\nu}} \sqrt{a^2} - \left( \frac{3}{y_{\nu}} \right) \nu,$$

Again we can join the two covariant points, this time the ones given in (86) and (90). This line will intersect each of the asymptotic tangents in a unique point. These points on the asymptotic tangents may be joined to the canonical point by a line and thus determine other points on the tangents to the asymptotic curves. We still have the possibility of using the harmonic conjugate of any point with any other points already found and so there are a great number of points which are uniquely associated with any point $P_y$ on a curved surface in space.