SOME CONFIGURATIONS AND PROJECTIVE PROPERTIES
OF A COMPLETE FIVE-POINT.

by

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PART I

SECTION I. INTRODUCTION

The study of geometry led to an early recognition of regular polygons, including among them the regular pentagon. The Pythagoreans knew how to construct a regular pentagon by the use of isosceles triangles. This led them to the discovery of the star pentagon which they regarded as a symbol of health. Menelaus used the complete 4-line in the first century A.D. Carnot was the first to use the term "complete four-side" but Girard considered the complete n-point in the plane for $n < 7$ in his studies of the triangle and the diagonals of the simple n-points. Lachlan in 1893 used the terms polystigma and polygram to denote a complete n-line in the plane. In 1910, Veblen and Young published the definition of a complete n-line in a 3-space. Euclid gives the four problems of inscribing and circumscribing regular pentagons to circles and of inscribing and circumscribing circles to regular pentagons.

It is the purpose of this paper to collect known properties of a complete five-point and a complete five-line and to make a further study of the properties of these figures.

T. L. Heath, Euclid's Elements, vol. II pp. 77-100
Georges Dostor, Journal de Mathematique, Series 3, Vol. VI; p. 343
Tropfke, Vol. II. pp. 52-53
Lachlan, p. 83
Heath, Euclid's Elements, Vol. II, p. 100
SECTION II  PRELIMINARY DEFINITIONS

1. A simple $n$-point, in the plane, is a set of $n$ points, no three collinear together with the $n$ lines joining them in a definite order.

2. A complete $n$-point, in the plane is a set of $n$ points, no three collinear, together with the $\frac{n(n-1)}{2}$ lines joining them.

3. A diagonal point of an $n$-point is the point of intersection of any two lines of the $n$-point which do not intersect in one of the given $n$ points.

4. A plane configuration is a set of points and lines so arranged that each point contains the same number of lines and each line contains the same number of points. (The number of points on a line need not be the same as the number of lines on a point.)

1. Veblen and Young, Projective Geometry, Vol. I.
SECTION III KNOWN PROPERTIES

One of the first properties discovered in connection with five points or five lines is their determination of conic. This was known by Apollonius who seems to have been familiar with the construction of a conic when given either five points or five lines. Because of this well known property, the five-point figure has been studied most extensively in connection with conics.

An attempt has been made to assemble these known properties. These will be given largely in chronological order except in cases of closely connected articles.

THEOREM 1. IF CIRCLES ARE CIRCUMSCRIBED TO THE FOUR TRIANGLES FORMED BY A COMPLETE FOUR-LINE, THE FOUR CIRCLES INTERSECT IN A POINT WHICH IS THE FOCUS OF THE PARABOLA TANGENT TO THE FOUR LINES.

This theorem and the one following are given first by A. Miquel and the point of intersection is usually referred to as the Miquel point. This second theorem has been stated and proved in a variety of ways by different authorities but may be summarized as follows:

\[ \text{T.L. Heath, Apollonius of Perga, pp. cxxx and cli.} \]
\[ \text{Auguste Miquel, Journal de Mathematique III, pp. 480-487; IX, pp. 20-21.} \]
THEOREM II. GIVEN A COMPLETE FIVE-LINE, THE FIVE
MIQUEL POINTS DETERMINED BY THE FIVE LINES TAKEN FOUR AT
A TIME, LIE ON A CIRCLE.

This theorem has been generalized for $2N$ and $2N-1$
lines by Clifford, Fuortes and Kantor. Each figure of $2N$
lines has associated with it a unique point corresponding
to the Miquel point and it is called the Clifford point.
The circle uniquely determined by a figure of $2N+1$ lines
is the Clifford circle. The complete network is the
Clifford chain. The conditions for the failure of this
chain are enumerated by F. Morley and Walter B. Carver.

S. Kantor discussed the Geometry of the complete five-
line in two articles in 1878. These discussions are based
on the relations of the Miquel Circle and the Steiner
Mittelpunktskreise. The first paper treats of the conic
determined by the intersection of the five sides with the
corresponding normals to the lines from a Miquel point.

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'Miquel, Ibid.
E. Catalan, Theoremes et Problemes de Geometrie Elementaire
P. 50.
W.K. Clifford, Collected Works, pp. 38-54; Messenger of
Mathematics, V, pp. 124-141; Jahrbuch der
Fortschritte der Mathematik, II, p. 394, hereafter
referred to as Jahrbuch.
T. Fuortes, Educational Times, X, p. 46; Jahrbuch, III, p. 260;
Giornale di Mathematica, IX, pp. 55-57; Jahrbuch,
X, p. 384.
Jacob Steiner, Synthetische Geometrie, p. 124.
Clifford, Collected Works, pp. 38-54.
Fuortes, Giornale di Mathematica, IX, pp. 55-57.
S. Kantor, Jahrbuch, IX, p. 424.

F. Morley, Transactions of American Mathematical Society,
I, pp. 97-115.
W.B. Carver, American Journal of Mathematics, XLII, p. 137.
This conic also passes through the Miquel point. The Miquel circle is the locus of the foci of all curves of third class which are tangent to the five lines and have the line at infinity as a double tangent. The point of intersection of the Steiner circles treated in this manner gives an equilateral hyperbola.

This subject was not treated systematically from this point of view again until 1900 when F. Morley took it up analytically by the use of circular coordinates, reaching the same conclusions as the earlier writers.

In the same year Frank H. Loud published an article in which he treated the subject of directed lines. He gave the two following theorems relative to the subject:

**THEOREM III.** LET THERE BE GIVEN FIVE STRAIGHT LINES, EACH OF SPECIFIED DIRECTION. ANY FOUR OF THESE DETERMINE A QUADRANGLE (VIZ. THE VERTICES OF THE LATTER ARE POINTS IN WHICH THE EQUIDISTANT AXES OF PAIRS OF LINES TAKEN FROM THE GIVEN FOUR CONVERGE BY THREES.) WHICH QUADRANGLE IS INSCRIPTIBLE IN A CIRCLE $C_4$ AND THE FIVE CIRCUMCIRCLES $C_i$ HAVE THEIR CENTERS ON A NEW CIRCLE $C_5$.

**THEOREM IV.** GIVEN FIVE DIRECTED STRAIGHT LINES, THE FIVE CIRCLES $C_i$ (OF THEOREM III) MEET IN A POINT $N_5$.

A. Clebsch discussed the five-line figure in connection

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with the equation of fifth degree and also the simple five-point figure with its diagonals, in reality making a complete five-point figure. In the second of these articles he considered a simple five-point having each interior angle less than a straight angle. By drawing the diagonals of such a figure, he obtained a second five-point which is projective with the first and corresponds to it in a certain collineation. If this process of drawing diagonals is continued, the limit of the five-point figure is a definite point of the plane. This point is a vertex of the fundamental triangle of the collineation and is a root of the cubic equation of the collineation.

G. Kohn arrived at this same conclusion by considering the simple five-point or five-line as a Poncelet Polygon (a polygon inscribed in one conic and circumscribed to a second).

The theory of five arbitrary points in the plane has received considerable attention from F. Rathke in his Dissertation which is based on a theorem, stated without proof by Jacob Steiner. As with a large number of the other papers here referred to, the complete work of Rathke has been inaccessible to the writer and the synopsis given in "Jahrbuch der Fortschritte der Mathematik" has furnished

1A. Clebsch, Mathematische Annalen, IV, p. 284.
2Ibld., p. 476.
3G. Kohn, Jahrbuch, XXIII, p. 672.
all of the information regarding this Dissertation. Since these theorems concerning the five-point are unestablished in the synopsis, the writer has supplied proofs of them in so far as possible. A review of Steiner’s and Rathke’s papers follows with the references to these proofs which are given in Part II of this paper.

Five arbitrary points A, B, C, D, E in a plane determine ten lines \( \mathcal{Q} \) which intersect in fifteen new points \( \mathcal{R} \) called diagonal points of the complete five-point. (Theorem II, Part II). These are located in sixes upon ten conics \( \mathcal{K} \). These fifteen points \( \mathcal{R} \) are again joined in pairs by seventy-five lines (Theorem IX, Part II) of two types \( \mathcal{L} \) and \( \mathcal{H} \). Fifteen of these, \( \mathcal{L} \), are sides of the five diagonal triangles of the five four-points formed by taking the original points four at a time. (Theorem II' and VI, Part II). These lines \( \mathcal{L} \) are tangent in sixes to ten conics \( \mathcal{K} \) which are related in a definite manner to the ten conics \( \mathcal{K} \). These fifteen lines \( \mathcal{L} \) intersect by threes in ten points 1, 2, 3 - - - - 10 (Theorem V, Part II) which are collinear in fours with one of the original points (Theorem I, Part II). These five lines are tangent to the conic determined by the five original points. (Theorems I' and V, Part II). The other sixty lines \( \mathcal{H} \) depend upon all of the original points (Theorem X, Part II). Each point \( \mathcal{R} \) is the intersection of eight lines \( \mathcal{H} \) and two lines \( \mathcal{L} \). (Theorems VI, IX, and X, Part II). Taken in a fixed order in pairs, these lines \( \mathcal{H} \) intersect in thirty new points \( \mathcal{S} \) which are
collinear in sixes with the five original points in the
lines a, b, c, d, e (Theorem X, Part II) and these six
points form an involution. (Theorem XI, Part II). These
thirty points lie in threes on the first ten lines $g$.
(Theorem XIII, Part II). The five lines a, b, c, d, e again
determine ten points t. (Theorem I', Part II). This gives
$15r + 30s + 10t$ or fifty-five points all of which are
dependent on all of the original points and are related to
the conics $K$ while another group of points dependent upon
the original points four at a time are connected with the
conics $K$.

Various discussions of the construction of a conic
when given five points or five lines have been given. The
most common method is the one based on the special case of
Pascal's theorem in which two of the points coincide (Theorem
VIII, Part II)

Obviously, the type of conic determined by five points
of the plane is dependent upon their location with respect
to one another. A method for the determination of the
type is given by G. Stiner. He showed that given five
arbitrary points $A_1, A_2, A_3, A_4, A_5$ determining a conic $K$, the
circle through $A_1, A_2, A_3$ cuts the line $A_3, A_5$ in a second
point $A'_3$; and the conic $K$ is then a (1) hyperbola,
(2) parabola, or (3) ellipse on the condition that the

T.L. Heath, Apollonius of Perse
J. Thomas, Jahrbuch, XXV, p. 931
O. Schlomilch, Jahrbuch, XXV, p. 991
G. Stiner, Jahrbuch, XXVI, p. 611
circle $A_1^1 A_2^1 A_3$ intersects the line $A_1 A_2$ (1) in two real distinct points, (2) two real coincident points or (3) two imaginary points.

Since five points always determine a conic and since any conic may be projected into a circle, any five points of the plane may be projected into five points of a circle. Studies of this were made by S. Kantor and F. Schur.

Kantor has also made a study of the configurations formed by $(n - 1)$ perspective $n$-points and by $n$ perspective $n$-points. He showed that if $(n - 1)$ $n$-points are perspective from a point, they determine $(n - 1)$ perspective $(n - 1)$-points for each of the $n$ arrangements of the $n$ points taken $(n - 1)$ at a time. Each of these arrangements determines a point $T_{n-1}$ and these $n$ points $T_{n-1}$ are collinear in a line $s_n$. This affords a configuration of $\binom{2n - 1}{n - 1}$ points each of which contains $n$ lines and $\binom{2n - 1}{n - 1}$ lines each containing $n$ points of 
\[ \left[ \binom{2n - 1}{n}, \binom{2n - 1}{n - 1} \right]. \]

If $n$ complete $n$-points are perspective from a point then the $n$ lines $s_n$, which are determined by the $n$ arrangements of the $n$ $n$-points $(n - 1)$ at a time, will meet in a point $T_n$. This affords a configuration of $\binom{2n}{n}$ points each of which contains $n$ lines and $\binom{2n}{n}$ lines each of which contains $(n + 1)$ points of 
\[ \left[ \binom{n - 1}{n_{n+1}}, \binom{2n}{n - 1} \right]. \]

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1. S. Kantor, Jahrbuch, X, p. 386; XI, p. 412
2. F. Schur, Jahrbuch, XXV, p. 991
3. S. Kantor, Jahrbuch, XI, p. 411
Jan De Vries has also discussed these configurations based on p perspective complete g-points. He based his work on that of Kantor and also showed that this group belongs to the general class known as polyhedral configurations.

In considering the simple figure, composed of five points and five lines, Reye has shown that the configurations is \((5_g, 5_l)\) or is made up of five points two on each line and five lines two on each point. Jan De Vries treated a group of configurations of the form \(\binom{2n}{n}\). The beginning of this series is the configuration formed by a complete five line and is \((10_g, 5_l)\).

Arthur Cayley considered the simple pentagons in a Desargues Configuration consisting of ten points and ten lines and reached the following conclusions:

The Desargues Configuration may be considered in the form of two pentagons each inscribed to and circumscribed about the other. (This may be done in six different ways.)

Having given any pentagon, a second may be found such that the two are inscribed and circumscribed to each other.

Miss S.F. Richardson has written an article on systems

Jan De Vries, *Jahrbuch*, XXIII, p. 550
in- and circumscribed polygons in which she finds:

If $A B C D E$ be any pentagon and if the points
$(AC, BD), (BD, CE), (DA, EC), (BE, DA), (AC, EB)$ be
named $e, a, b, c, d$ respectively, then if $ABCD$ and
$ABDE$ are regarded as two simple pentagons, there will be
a poristic system of pentagons if $ab$ be projected on $bc,$
$bc$ on $cd, cd$ on $de,$ $de$ on $ea,$ $ea$ on $ab$ from the vertices
$B, C, D, E, A$ respectively. She proved this by a process
of testing for closure when the vertices of $ABCD$ and $A-
B D E$ are the points projected.

Catalan has extended the theorem of Menelaus to any
polygon and gives the following:

If the sides of a polygon are cut by a transversal,
the product of the non-consecutive segments is equal to
the product of the others.

He also gives a theorem on the regular polygon which
is inscriptible in a circle. In a regular convex pentagon,
the diagonals cut mutually in extreme and mean ratio.

The study of the regular pentagon has been continued
since early times. E. Sachse treated the relation of the
sides of a regular pentagon to the radius of the circum-
scribing circle.

H. Schröter's method of construction of a regular
pentagon is reported by A. Cayley.

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1E. Catalan, Théorèmes et Problèmes de Géométrie
   Elémentaire, p. 97.

2Ibid. p. 261.

3E. Sachse, Jahrbuch, I, p. 264.

4H. Schröter, Messenger of Mathematics, XII, p. 177.
SECTION IV ALIGNMENTS AND CONFIGURATIONS

The notation used in this paper is as follows:

1. \( P_i (i=1,2,3,4,5) \), the five original points.

2. \( l_{ij} (i,j=1,2,3,4,5) \), the ten lines joining the points \( P_i \).

3. \( D_{ijkl} (i \neq j \neq k \neq l) \), the diagonal points formed by the intersection of \( l_{ij} \) and \( l_{kl} \).

4. \( K \), the conic determined (uniquely) by the five points \( P_i \).

5. \( t_i (i=1,2,3,4,5) \), the five tangents to \( K \) at the points \( P_i \).

6. \( T_{ij} (i \neq j) \), the points of intersection of \( t_i \) and \( t_j \).

7. \( a_{ijk} (i \neq j \neq k \neq l) \), the diagonal lines determined by \( T_{ij} \) and \( T_{kl} \).

8. \( T_{ijk} (i \neq j \neq k) \), the points of intersection of \( t_i \) and \( l_{jk} \).

9. \( l_{ijk} (i \neq j \neq k) \), the joins of \( P_i \) and \( T_{jk} \).

10. \( a_{ijk} (i \neq j \neq k) \), the Pascal lines of the triangles \( P_i \), \( P_j \), and \( P_k \).

11. \( A_{ijk} (i \neq j \neq k) \), the Brianchen points of the triangles \( t_i \), \( t_j \), and \( t_k \).
THEOREM I. FIVE ARBITRARY POINTS \( P_1 \) IN THE PLANE AND THE TEN LINES \( l_{ij} \) JOINING THEM FORM A CONFIGURATION OF \((5, 10)\).

PROOF: The two theorems, being plane duals, a proof is given for the one on the left and the second follows from it. Each of the five points \( P_i \) may be joined to four others making four lines on each point. The total number of lines is \( C_2^5 \) or ten lines and each line contains only two points.

THEOREM II. THE TEN LINES INTERSECT IN FIFTEEN DIAGONAL POINTS \( D_{ijkl} \) THREE ON EACH LINE.

PROOF: Any line as \( l_{ij} \) meets all of the other nine lines: \( l_{13}, l_{14}, l_{15} \) in point \( P_1 \); \( l_{23}, l_{24}, l_{25} \) in \( P_2 \); \( l_{34} \) in \( D_{1234} \); \( l_{25} \) in \( D_{1235} \); and \( l_{45} \) in \( D_{1245} \). Hence there are three points \( D_{ijkl} \) on each line \( l_{ij} \).

There are two lines \( l_{ij} \) on each point \( D_{ijkl} \) and we have \( \frac{10 \times 3}{2} \) or fifteen such points.

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2. Jan De Vries, Jahrbuch, XXIII, p. 560.
3. This custom will be followed in proving dual theorems.
Theorem III. The fifteen diagonal points $d_{jkl}$ and the ten lines $l_{ij}$ of a complete five-point form a configuration cf. $(15, 10)$. Theorem III' follows immediately from the proof of theorem II.

Theorem IV. The five points $P_{i}$ taken four at a time form five complete four-points whose diagonal points are diagonal points $d_{ijl}$ of the complete five-point.

Proof: Take any complete four-point as $P_{i}P_{j}P_{k}P_{l}$ and the six lines joining these are lines of the complete five-point. The diagonal points of the four-point are the intersections of opposite sides but since the diagonal points of the complete five-point include all of the possible intersections of these lines, the diagonal points of the four-point coincide with three of the diagonal points of the five-point. The number of complete four-points thus formed is $C_{4}^{5}$ or five.
THEOREM V. THE SIDES OF THE DIAGONAL TRIANGLES OF THESE COMPLETE FOUR-POINTS ARE CONCURRENT IN THREE ON THE TEN POINTS T_{ij} OF THE COMPLETE TEN LINES l_{ij} OF THE COMPLETE FIVE-LINE FORMED BY THE FIVE-POINT FORMED BY THE POINTS LINES t_k.

PROOF: Take any quadrangle as P_1 P_2 P_3 P_4. Then D_{1234} T_{23} and T_{14} are collinear for: "If the vertices of a complete quadrangle are points of a point conic, the tangents at a pair of vertices meet in a point of the line joining the diagonal points of the quadrangle which are not on the side joining the two vertices."

In quadrangle P_1 P_2 P_3 P_5, T_{14}, T_{25}, D_{1245} and D_{1354} are collinear on d_{1425}.

In quadrangle P_1 P_2 P_3 P_5, T_{14}, T_{35}, D_{1245} and D_{1345} are collinear on d_{1435}.

Hence d_{1425}, d_{1425}, and d_{1435} are concurrent in T_{14}, the intersection of T_{14} and t_{14}.


Veblen and Young, Projective Geometry, I, Theorem 8, p. 115. Hereafter referred to as Veblen and Young.
PROOF: In any quadrangle $P_i P_j P_k P_l$, the two diagonal points $D_{i k l}$ and $D_{j k l}$ and the points $T_{i k}$ and $T_{j l}$ are collinear by Theorem V. $T_{i k}$ and $T_{j l}$ determine a diagonal line $d_{i k l}$ of the complete five-line and $D_{i k l}$ and $D_{j k l}$ determine a side of the diagonal triangle of $P_i P_j P_k P_l$. Hence the line $D_{i k l}$ and $D_{j k l}$ is identical with the line $d_{i k l}$.

THEOREM VII. THE FIGURE FORMED BY THE FIFTEEN DIAGONAL POINTS $D_{i k l}$ AND THE FIFTEEN DIAGONAL LINES $d_{i k l}$ IS SELF-DUAL AND FORMS A CONFIGURATION of $(15, 15)$. This theorem is an immediate consequence of Theorem V.

THEOREM VIII. IF A SIMPLE FIVE-POINT $P_1 P_2 P_3 P_4 P_5$ BE INSCRIBED IN A CONIC, THE PAIRS OF LINES $l_{12}, l_{45}; l_{15}, l_{23}; l_{34} t$, MEET IN THREE POINTS OF A LINE.

This Theorem is a corollary of Pascal's Theorem.

THEOREM VIII'. IF A SIMPLE FIVE-LINE $t_1 t_2 t_3 t_4 t_5$ BE CIRCUMSCRIBED TO A CONIC, THE JUNCTIONS OF THE THREE PAIRS OF POINTS $t_{12} t_{45}; t_{15} t_{34}; T_3$, $T_{34} P_1$ ARE CONCURRENT.

This theorem is a corollary of Brianchon's Theorem.

Winger, Projective Geometry p. 119
Ibid. p. 122.
In a complete $N$-point, there are $\frac{(n - 1)!}{2}$ distinct simple $n$-points, hence in a complete five-point there are twelve simple five-points. Each simple five-point with a tangent at any point determines a Pascal line and dually, every simple five-line formed by the tangents to any point of tangency $P_\omega$ determines a Brianchon point. Therefore for a complete five-point and a complete five-line formed by the tangents to $K$ at the points $P_\omega$, there exist sixty Pascal lines and sixty Brianchon points.

**Theorem IX. The Sixty Pascal Lines of the Complete Five-Point and the Fifteen Diagonal Lines of the Complete Five-Line Are the Only Possible Joins of the Fifteen Points.** $D_{ijkl}$, and form a configuration $cf(15_\omega, 75_\omega)$.

**Theorem IX'. The Sixty Brianchon Points of the Complete Five-Line and the Fifteen Diagonal Points of the Complete Five-Point Are the Only Possible Intersections of the Fifteen Lines $d_{ijkl}$ and form a configuration $cf(75_\omega, 15_\omega)$.

**Proof:** In choosing any point $D_{ijkl}$ there are fifteen choices but for the second, there are only ten choices since five of the points are on the two lines $l_j$ which determine $D_{ijkl}$. Since either order of choice gives the same line there are seventy-five lines, ten on each point, as the total number of possibilities of joining the points $D_{ijkl}$.
THEOREM X. THE SIXTY

PASCAL LINES AND THE FIFTEEN

DIAGONAL POINTS DETERMINE A

CONFIGURATION. cf(15, 60). A CONFIGURATION cf(60, 15)

PROOF: Every Pascal line contains two and only two points D_{ijkl}. Every diagonal point D_{ijkl} may be connected to any one of ten others but two of these lines are diagonal lines of the complete five-line. Therefore there are only eight Pascal lines on each point D_{ijkl}.

THEOREM XI. THE PASCAL

LINES OF A COMPLETE FIVE-

POINT ARE CONCURRENT IN TWO'S

WITH A TANGENT \(t_i\) AND A LINE

\(l_{ij}\) IN THIRTY POINTS WHICH ARE

COLLINEAR IN SIXES ON THE

FIVE TANGENTS \(t_i\) AT THE POINTS \(P_i\).

PROOF: The Pascal line of the simple five-point \(P_1, P_2, P_3, P_4, P_5\) contains \(D_{1245}, D_{1323}\) and \(T_{1234}\) (the intersection of \(t_i\) and \(l_{134}\)). By Theorem VIII.

The Pascal line of \(P_1, P_5, P_3, P_4, P_6\) contains \(D_{1524}, D_{1235}\) and \(T_{134}\) by Theorem VIII. Hence, \(T_{134}\) is the intersection of two Pascal lines, a tangent \(t_5\) and a line \(l_{ij}\).
There are ten lines $l_{ij}$. A tangent $t_i$ cuts four of them in a point $P_i$ and the other six in points $T_{ijk}$, hence there are six such points on each tangent and a total of thirty.

**THEOREM XII.** THE SIX POINTS $T_{ijk}$ COLLINEAR WITH $P_i$

ON THE TANGENT $t_i$ ARE PAIRS OF POINTS OF AN INVOLUTION OF WHICH $P_i$ IS A DOUBLE POINT.

**PROOF:** The six lines $l_{jk}$ cutting the tangent $t_i$ in the six points $T_{ijk}$ are the sides of the complete four-point formed by the four points exclusive of the point of tangency of $t_i$.

These points form an involution for, "If the vertices of a complete quadrangle are on a conic, the pairs of opposite sides meet the tangent at any other point in pairs of an involution of which the point of contact of the tangent is a double point."

**THEOREM XIII.** THERE ARE THREE POINTS $T_{ijk}$ ON EACH OF THE LINES $l_{ij}$ OF A COMPLETE FIVE POINT FORMED BY THE POINTS $P_i$.

**PROOF:** Take any line $l_{ij}$.

It intersects $t_i$ in $P_j$; $t_j$ in $P_{ij}$; $t_k$ in $T_{ij}k$; $t_l$ in $T_{lj}$.

Veblen and Young, I, p. 127.
and \( t_m \) in \( T_{m-j} \).

Hence, there are three such points on each line \( l_{ij} \).

**THEOREM XIV.** **THE THIRTY POINTS** \( T_{ijk} \)** **ARE** **COLLINEAR** **IN** **LINES** \( l_{ijk} \)** **ARE** **CONCURRENT** **THREES** **ON** **TEN** **OTHER** **LINES** \( a_{ijk} \)** **IN** **THREES** **IN** **TEN** **NEW** **POINTS** \( A_{ijk} \)**

**PROOF:** Take any triangle \( P_{i}P_{j}P_{k} \). The points \( T_{jik} \) and \( T_{kij} \) are collinear on a line \( a_{ijk} \) for, we have as a special case of Pascal's theorem:

"If a triangle \( P_{i}P_{j}P_{k} \) be inscribed in a conic, the tangents \( t_i, t_j, t_k \) at the vertices meet the opposite sides in three points of a line."

The number of triangles formed by the five points \( P_i \) is \( C_5^3 \) or ten.

Hence, there are ten lines \( a_{ijk} \).

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Winger, Projective Geometry, p. 124.
The theorems so far proved have established the existence of the following configurations:

- $\text{cf}(5, 10)$ Theorem I
- $\text{cf}(15, 10)$ Theorem III
- $\text{cf}(15, 15)$ Theorem VII
- $\text{cf}(15, 75)$ Theorem IX
- $\text{cf}(15, 60)$ Theorem X

Twelve of $\frac{(n - 1)}{2}$ (formed by the distinct simple five-points) distinct simple five-lines)
SECTION V  CHARACTERISTIC DOUBLE RATIOS

Since any four independent points may be projected into any four independent points and double ratios are preserved under projection most of the foregoing results were also found analytically.

Choose the five points $P_i$ as follows

\[ \begin{align*}
P_1 : (1, 1, 1) & \quad P_2 : (1, -1, 1) \\
P_3 : (1, m, n) & \quad P_5 : (1, -1, -1)
\end{align*} \]

Then writing the equations of the conic $K$ through the five points we have

\[ (m^2-n^2)x_i^2 + (n^2-1^2)x_i^3 + (1^2-m^2)x_i^4 = 0. \]

The equations and coordinates of the ten lines $l_{ij}$ are

\[ \begin{align*}
l_{12} : & \quad x_i - x_j = 0 \quad \begin{bmatrix} 1, 1, 1 \end{bmatrix} \\
l_{13} : & \quad (n-m)x_i - (n-1)x_j + (m-1)x_j = 0 \quad \begin{bmatrix} n-m, n, (n-1), (m-1) \end{bmatrix} \\
l_{14} : & \quad x_i - x_j = 0 \quad \begin{bmatrix} 1, 1, 1 \end{bmatrix} \\
l_{15} : & \quad x_i - x_j = 0 \quad \begin{bmatrix} 1, 1, 1 \end{bmatrix} \\
l_{23} : & \quad (n+m)x_i + (n-1)x_j - (m+1)x_j = 0 \quad \begin{bmatrix} n+m, (n-1), (m+1) \end{bmatrix} \\
l_{24} : & \quad x_i + x_j = 0 \quad \begin{bmatrix} 0, 1, 1 \end{bmatrix} \\
l_{25} : & \quad x_i + x_j = 0 \quad \begin{bmatrix} 0, 1, 1 \end{bmatrix} \\
l_{34} : & \quad (n+m)x_i - (n+1)x_j + (m-1)x_j = 0 \quad \begin{bmatrix} n+m, - (n+1), (m-1) \end{bmatrix} \\
l_{35} : & \quad (m-n)x_i - (n+1)x_j + (m+1)x_j = 0 \quad \begin{bmatrix} m-n, (n+1), (m+1) \end{bmatrix} \\
l_{45} : & \quad x_i + x_j = 0 \quad \begin{bmatrix} 1, 1, 1 \end{bmatrix}
\end{align*} \]

The equations and coordinates of the fifteen diagonal points $D_{ijkl}$ are

\[ \begin{align*}
D_{1234} : & \quad (n+1)u_i - (2m-1+n)u_j + (n+1)u_j = 0 \quad \begin{bmatrix} f, 2m-1-n, 1 \end{bmatrix} \\
\end{align*} \]

Four coplaner points are independent if no three are collinear.
By finding the double ratios of these points in various arrangements we find the following theorem:

THEOREM XV. IN A COMPLETE FIVE-POINT THE DOUBLE RATIO OF THE FOUR LINES $l_{ij}$ OF A PENCIL ON ONE OF THE POINTS $P_i$ IS THE SAME AS THE DOUBLE RATIO DETERMINED BY THE CENTER OF THE PENCIL $P_i$ AND ANY THREE DIAGONAL POINTS $D_{ijkh}$ COLLINEAR WITH THE CENTER.

PROOF: The characteristic double ratios as found analytically by our notation are as follows:

$D_{1235} : (n+1)u_1 + (2m+1-n)u_2 + (n+1)u_3 = 0 \quad \frac{1}{\frac{n+1}{m+1}}, \frac{2m+1-n}{1} \quad (0, 0, 1)

D_{1245} : u_2 = 0

D_{1324} : (m+u-2l)u_1 + (n-m)u_2 - (n-m)u_3 = 0 \quad \frac{m+n-2l}{n-m}, 1, -l

D_{1325} : (m-1)u_1 - (m-1)u_2 + (1+m-2n)u_3 = 0 \quad \frac{1}{\frac{n}{m-1}}, 1, -l

D_{1345} : (n-1)u_1 - (n-2m)u_2 - (n-1)u_3 = 0 \quad \frac{1}{\frac{n-1}{m-1}}, 1, -l

D_{1423} : (m+1)u_1 + (m+1)u_2 + (2n-m-1)u_3 = 0 \quad \frac{1}{\frac{m+1}{n+m}}, 1, 2n+m-1

D_{1425} : u_3 = 0

D_{1435} : (m+1)u_1 + (m+1)u_2 - (2n-m+1)u_3 = 0 \quad \frac{1}{\frac{m+1}{n+m}}, 1, 2n-m+1

D_{1523} : (m+2l-n)u_1 + (u+m)u_2 + (n+m)u_3 = 0 \quad \frac{m+2l-n}{n+m}, 1, l

D_{1524} : u_1 = 0

D_{1534} : (m-2l-n)u_1 - (n+m)u_2 - (n+m)u_3 = 0 \quad \frac{m-2l-n}{n+m}, -1

D_{1534} : - (n-1)u_1 - (2m+1+n)u_2 + (n-1)u_3 = 0 \quad \frac{-1}{\frac{2m+1+n}{n-1}}, 1

D_{2435} : (m+n+2l)u_1 + (m-n)u_2 - (m-n)u_3 = 0 \quad \frac{m+n+2l}{m-n}, 1, -l

D_{2534} : (m-1)u_1 - (n-1)u_2 - (2n+1+m)u_3 = 0 \quad \frac{-1}{\frac{m-1}{n-1}}, 1, 2n+1+m
\[ R_i (x, x_2, x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)} = r \]

**Lines on \( P_1 \):**

| Points on \( l_{12} \): | \( D_{1234} \) | \( P_1 \) | \( D_{1245} \) |
| Points on \( l_{13} \): | \( D_{1345} \) | \( P_1 \) | \( D_{1324} \) |
| Points on \( l_{14} \): | \( D_{1423} \) | \( P_1 \) | \( D_{1425} \) |
| Points on \( l_{15} \): | \( D_{1524} \) | \( P_1 \) | \( D_{1523} \) |

**Lines on \( P_2 \):**

| Points on \( l_{12} \): | \( D_{1235} \) | \( P_2 \) | \( D_{1234} \) |
| Points on \( l_{13} \): | \( D_{1345} \) | \( P_2 \) | \( D_{1324} \) |
| Points on \( l_{14} \): | \( D_{1423} \) | \( P_2 \) | \( D_{1425} \) |
| Points on \( l_{15} \): | \( D_{1524} \) | \( P_2 \) | \( D_{1523} \) |

**Lines on \( P_3 \):**

| Points on \( l_{13} \): | \( D_{1345} \) | \( P_3 \) | \( D_{1324} \) |
| Points on \( l_{14} \): | \( D_{1423} \) | \( P_3 \) | \( D_{1425} \) |
| Points on \( l_{15} \): | \( D_{1524} \) | \( P_3 \) | \( D_{1523} \) |

**Lines on \( P_4 \):**

| Points on \( l_{12} \): | \( D_{1235} \) | \( P_4 \) | \( D_{1245} \) |
| Points on \( l_{13} \): | \( D_{1345} \) | \( P_4 \) | \( D_{1324} \) |
| Points on \( l_{14} \): | \( D_{1423} \) | \( P_4 \) | \( D_{1425} \) |
| Points on \( l_{15} \): | \( D_{1524} \) | \( P_4 \) | \( D_{1523} \) |

**Lines on \( P_5 \):**

| Points on \( l_{12} \): | \( D_{1235} \) | \( P_5 \) | \( D_{1245} \) |
| Points on \( l_{13} \): | \( D_{1345} \) | \( P_5 \) | \( D_{1324} \) |
| Points on \( l_{14} \): | \( D_{1423} \) | \( P_5 \) | \( D_{1425} \) |
| Points on \( l_{15} \): | \( D_{1524} \) | \( P_5 \) | \( D_{1523} \) |
By changing the order of the points \( x, x', x'' \) we obtain six values for the double ratios. If \( R(x, x', x'', x_3) = r \), the other five values are \( \frac{1}{r}, 1-r, \frac{1}{1-r}, \frac{r-1}{r} \) and \( \frac{r}{r-1} \).

If \( r_1 \) is the double ratio of the four lines \( l_1 \) on \( P_1 \) we find the following relations between the values of \( r_1 \):

\[
\begin{align*}
r_2 r_3 &= \frac{r_1}{r_1 - 1} & r_2 r_4 &= \frac{r_1}{r_1 - 1} & r_2 r_5 &= \frac{r_1}{r_1 - 1} \\

\end{align*}
\]

Hence we conclude that only two of these are independent.

The equation of a tangent to the conic \( K \) at any point \( (x', x'', x_3') \) is

\[(m^2-n^2)x'x_3 + (n^2-1^2)x'x_3 + (1^2-m^2)x'x_3 = 0\]

Then at the points \( P_i \):

\[
\begin{align*}
t_1 : (m^2-n^2)x_1 + (n^2-1^2)x_2 + (1^2-m^2)x_3 &= 0 \\
t_2 : (m^2-n^2)x_1 - (n^2-1^2)x_2 + (1^2-m^2)x_3 &= 0 \\
t_3 : (m^2-n^2)x_1 + (n^2-1^2)x_2 + (1^2-m^2)x_3 &= 0 \\
t_4 : (m^2-n^2)x_1 + (n^2-1^2)x_2 - (1^2-m^2)x_3 &= 0 \\
t_5 : (m^2-n^2)x_1 - (n^2-1^2)x_2 - (1^2-m^2)x_3 &= 0 \\

\end{align*}
\]

The points, \( T_{ij} \), of intersection of the tangents are:

\[
\begin{align*}
T_{i2} : (1^2-m^2)u_1 - (m^2-n^2)u_3 &= 0 & \begin{pmatrix} m^2-1^2, 0, 1 \end{pmatrix} \\
T_{i3} : (n-m)(n^2-1^2)(1^2-m^2)u_1 - (n-1)(m^2-n^2)(1^2-m^2)u_3 + (m-1) \\
& (m^2-n^2)(n^2-1^2)u_3 &= 0 & \begin{pmatrix} m+1, l+m, 1 \end{pmatrix} \\
T_{i4} : (n^2-1^2)u_1 - (n^2-m^2)u_3 &= 0 & \begin{pmatrix} n^2-1^2, 1, -1 \end{pmatrix} \\
T_{i5} : (1^2-m^2)u_1 + (1^2-n^2)u_3 &= 0 & \begin{pmatrix} 0, l^2-m^2, 1 \end{pmatrix} \\
T_{i3} : (m-1)(1+n)u_1 - (m-n)(1-m)u_3 + (m-n)(1+n)u_3 &= 0 & \begin{pmatrix} m-1, -l-m, 1 \end{pmatrix} \\
\end{align*}
\]

\[\text{Veblen and Young, I, p. 161.}\]
\[
T_{24} : (1^2 - m^2)u_2 - (1 - n^2)u_3 = 0 \\
T_{25} : (n^2 - 1^2)u_1 + (u^2 - m^2)u_2 = 0 \\
T_{34} : (m+1)(1-n)u_1 - (1+m)(m-n)u_2 - (m-n(1-n)u_3 = 0 \\
T_{35} : -(m-1)(1-n)u_1 + (m+n)(1-m)u_2 + (m+n)(1-n)u_3 = 0 \\
T_{45} : (m^2 - 1^2)u_1 - (m^2 - n^2)u_3 = 0
\]

Through the double ratios of the four points \( T_{ij} \) on any tangent \( t \), we have the following theorem:

**THEOREM XVI.** IF FIVE POINTS \( P_1, P_2, P_3, P_4, P_5 \) ARE ON A CONIC \( K \), THE DOUBLE RATIO OF THE PENCIL ON \( P_1 \) FORMED BY \( l_{12}, l_{13}, l_{14}, l_{15} \) IS THE SAME AS THE DOUBLE RATIO OF THE FOUR POINTS \( T_{12}, T_{13}, T_{14}, T_{15} \), THE INTERSECTIONS OF \( t_1 \) WITH THE OTHER TANGENTS.

**PROOF:**

\[
\begin{align*}
(T_{12}, T_{13}, T_{14}, T_{15}) &= \frac{m-n}{m-1} \\
(T_{12}, T_{25}, T_{24}, T_{23}) &= \frac{1+m}{1-n} \\
(T_{13}, T_{23}, T_{34}, T_{25}) &= \frac{m^2 - 1^2}{m^2 - n^2} \\
(T_{14}, T_{24}, T_{34}, T_{14}) &= \frac{1-m}{1+n} \\
(T_{15}, T_{25}, T_{25}, T_{45}) &= \frac{m-n}{m+1}
\end{align*}
\]

Finding the points \( T_{ij} \) or the intersection of \( t_1 \) with the lines of the five points.

\[
T_{12} : (1+m)(21-m+n)u_1 + (m+1)(m+n)u_2 - (m+n)(m-2n-1)u_3 = 0 \\
T_{12} : \left( \frac{(1+m)(21-m+n)}{m+n(m-2n-1)} \right)^* = \left( \frac{m+1}{m-2n-1} \right) \\
T_{12} : (m^2 - 21 + n^2)u_1 - (m^2 - n^2)u_2 + (m^2 - n^2)u_3 = 0
\]
\[
\left( \frac{m^2 - 2l^2 + t^2}{m^2 - n^2}, -1, 1 \right) \\
T_{125} : \quad -(1^2 - m^2)u_x + (1^2 - m^2)u_x + (m^2 - 2n^2 + 1^2)u_3 = 0 \\
\left( -1, 1, \frac{m^2 - 2n^2 + 1^2}{1^2 - n^2} \right)
\]
\[
T_{134} : \quad (n+1)(n-2l-m)u_x - (m+n)(2m-n+1)u_x - (n+1)(m+n)u_3 = 0 \\
\left( \frac{n-2l-m-(2m-n+1)}{m+n}, -1 \right)
\]
\[
T_{135} : \quad (m+1)(n+1)u_x + (m+n)(2m-1+n)u_x + (n+1)(m+2n-1)u_3 = 0 \\
\left( \frac{m+1}{m+2n-1}, \frac{(1+m)(2m-1+n)}{(1+n)(m+2n-1)}, 1 \right)
\]
\[
T_{145} : \quad (n^2 - l^2)u_x - (2m^2 - n^2 - 1^2)u_x - (n^2 - 1^2)u_3 = 0 \\
\left( -1, \frac{2m^2 - n^2 - 1^2}{n^2 - 1^2}, 1 \right)
\]

The quadrangular set \( \{T_{123}, T_{124}, T_{125}, T_{145}, T_{135}, T_{134}\} \)

are in involution of which the point \( P_i \) is a double point.

From Theorem XII

\[
(P_i, T_{123}, T_{124}, T_{135}) = \frac{m^2 - l^2}{m^2 - n^2} \quad \text{Pencil } P_3 \text{ (page 24)}
\]

\[
(P_i, T_{145}, T_{125}, T_{134}) = \frac{m^2 - l^2}{m^2 - n^2} \quad \text{Pencil } P_2
\]

\[
(P_i, T_{125}, T_{124}, T_{123}) = \frac{1+m}{1+n} \quad \text{Pencil } P_4
\]

\[
(P_i, T_{134}, T_{135}, T_{145}) = \frac{1+m}{1+n} \quad \text{Pencil } P_5
\]

THEOREM XVII: IF A TRIANGLE \( P_i, P_j, P_k \) IS INSCRIBED IN A CONIC, THE TANGENT AT ANY VERTEX \( P_i \) MEETS THE OPPOSITE SIDE \( l_{jk} \) AND THE TANGENTS \( t_j \) AND \( t_k \) IN THREE POINTS WHICH WITH \( P_i \) FORM A HARMONIC SET.
PROOF:

\[(P_1 T_{12} T_{13} T_{14}) = -1\]
\[(P_1 T_{13} T_{14} T_{15}) = -1\]
\[(P_1 T_{12} T_{13} T_{15}) = -1\]
\[(P_1 T_{13} T_{12} T_{15}) = -1\]
\[(P_1 T_{14} T_{13} T_{15}) = -1\]
\[(P_1 T_{14} T_{12} T_{15}) = -1\]

OR

The two sides of a complete quadrangle which meet in a diagonal point are harmonic conjugates with respect to the two sides and of the diagonal triangle which meet in this point. Using the diagonal point \(D_{1524}\), we have the lines \(l_{15}\), \(l_{12}\), \(l_{425}\), and \(l_{1425}\) which cut the tangent \(T_1\) in \(P_1\), \(T_{12}\), \(T_{13}\), and \(T_{14}\) and \((P_1 T_{12} T_{13} T_{14}) = -1\)

Similarly use \(D_{1435}\), \(D_{1423}\), \(D_{1425}\), \(D_{1534}\) and \(D_{1345}\) we get the six harmonic sets on the tangent \(T_1\).

THEOREM XVIII: THE POINT OF TANGENCY \(P_1\) OF ANY TANGENT \(T_1\) AND ITS INTERSECTION WITH THE TANGENTS AT THREE OTHER POINTS \(P_2\) GIVES THE SAME DOUBLE RATIO AS THAT OF A PENCIL FORMED BY CONNECTING A FIFTY POINT OF THE CONIC TO EACH OF THE FOUR POINTS \(P_2\).

PROOF:

\[(P_1 T_{12} T_{13} T_{14}) = \frac{m-n}{m+n+1} \text{ Pencil } P_5 \text{ (page 24)}\]
\[(P_1 T_{15} T_{13} T_{12}) = \frac{1-m}{1+n} \text{ Pencil } P_4\]
\[(P_1 T_{15} T_{14} T_{13}) = \frac{1+m}{1-n} \text{ Pencil } P_2\]
\[(P_1 T_{14} T_{15} T_{12}) = \frac{m^2-x^2}{m^2-x^2} \text{ Pencil } P_3\]

Veblen and Young, I, Theorem VI, p. 82
If an involution has one double element it has a second distinct from the first.

If an involution has double points, they are harmonic conjugates with respect to every pair of the involution.

In the involution formed by \( q(T_{123} T_{124} T_{125}, T_{145} T_{135} T_{134}) \) and having the double point \( P \), we may solve for the second double point.

\[
\begin{align*}
(P, & \frac{X_1}{X_3}, T_{123} T_{124} ...) = -1, \\
(P, & \frac{X_1}{X_3}, T_{124} T_{125} ...) = -1, \\
(P, & \frac{X_1}{X_3}, T_{125} T_{134} ...) = -1, \\
(P, & \frac{X_1}{X_3}, T T ...) = -1,
\end{align*}
\]

\[
\frac{X_1}{X_3} = \frac{\delta(1+m(2m-1+n) + (m+n)(m-2n-1))}{\delta(m+n)(m-2n-1) + (1+m)(2m+n)},
\]

\[
3l^2 + 1m + ln - m^2 + mn - n^2,
\]

\[
3n^2 + 1m + ln - m^2 + mn - l^2,
\]

And in like manner

\[
\frac{X_2}{X_3} = \frac{\delta(1+m)(2m-1+n) - (1+n)(m+2n-1))}{\delta(1+m)(m+2n-1) - (1+m)(2m-1+n)},
\]

\[
3m^2 + 1m - l^2 + ln + mn - n^2,
\]

\[
3n^2 + 1m - l^2 + ln + mn - m^2.
\]

\[\text{Veblen and Young, I, Corollary IV, p. 103.}\]

\[\text{Ibid, Corollary I, p. 102.}\]
SECTION VI: POLARITIES WITH RESPECT TO THE CONIC $K$.  

By definition any tangent $t_i$ is the polar of its point of contact $P_i$ with respect to $K$.

THEOREM XIX. ANY DIAGONAL LINE $d_{ijkl}$ OF THE COMPLETE FIVE LINE FORMED BY THE TANGENTS $t_i$ TO $K$ IS THE POLAR OF THE DIAGONAL POINT $D_{ijkl}$ OF THE COMPLETE FIVE POINT FORMED BY THE POINTS OF TANGENCY $P_i$.

PROOF: The diagonal lines $d_{ijkl}$ are the sides of the diagonal triangles of the complete four-points formed by using the points $P_i$ in fours (Theorem VI) and

"The line joining two diagonal points of any complete quadrangle whose vertices are points of a conic is the polar of the other diagonal point with respect to the conic."

THEOREM XX. ANY LINE $l_{ij}$ IS THE POLAR OF THE CORRESPONDING POINT $T_{ij}$ WITH RESPECT TO $K$.

PROOF: The point $T_{ij}$ is the intersection of the tangents $t_i$ and $t_j$ at the points $P_i$ and $P_j$ and the join of the points of tangency of two tangents is the polar of the point of intersection of the tangents.

THEOREM XXI: THE POLAR OF ANY BRIANCHON POINT IS A PASCAL LINE CONTAINING THE TWO DIAGONAL POINTS $D_{ijkl}$ WHICH ARE THE POLES OF THE LINES $d_{ijkl}$ ON THE BRIANCHON POINT.

PROOF: The polar of the intersection of two lines is the join of the poles of the two lines.

\[1\text{Veblen and Young, I, p. 121}\]
\[2\text{Ibid.}\]
THEOREM XXII. ANY LINE $l_{ij}h$ IS THE POLAR OF A POINT $T_{ijk}$

PROOF:

$l_{ij}h$ is the join of $P_i$ and $T_{ijk}$

$P_i$ is the pole of $t_i$

$T_{ijk}$ is the pole of $l_{ij}k$

Hence, the pole of $l_{ij}h$ is $T_{ijk}$, the intersection of $t_i$ and $l_{ij}h$.

THEOREM XXIII. ANY LINE $a_{ij}k$ IS THE POLAR OF THE CORRESPONDING POINT $A_{ijk}$

PROOF:

$a_{ij}k$ contains three of the points $T_{ijk}$ which are the intersections of two Pascal lines. Then the pole of $a_{ij}k$ is a point which is the intersection of three lines $l_{ijk}$ which are joins of two Brianchon points.

Hence, $A_{ijk}$ is the pole of $a_{ij}k$.

THEOREM XXIV. THE PENCIL OF LINES ON $P_i$ FORMED BY THE FOUR LINES $l_{ij}^*$, ONE TANGENT $t_i$, AND THE SIX LINES $l_{ijk}$ IS PROJECTIVE WITH THE RANGE OF POINTS $t_i$ FORMED BY THE FOUR POINTS $T_{ij}$, ONE POINT $P_i$ AND THE SIX POINTS $T_{ijk}$

PROOF: The polar lines of the points of a range constitute a pencil projective with the range and dually.

THEOREM XXV. THE PENCIL OF LINES ON $T_{ij}$ FORMED BY THE TWO TANGENTS $t_i$, THE THREE DIAGONAL LINES $d_{ijk}$ AND THE THREE LINES $l_{ijk}$ IS PROJECTIVE WITH THE RANGE OF POINTS ON $l_{ij}k$ FORMED BY THE TWO POINTS $P_i$, THE THREE DIAGONAL POINTS $d_{ijk}$ AND THE THREE POINTS $T_{ijk}$.
**THEOREM XXVI.** THE PENCIL OF LINES ON $D_{ijkl}$ FORMED BY THE TWO LINES $l_{ij}$, THE TWO LINES $a_{ijkl}$ AND THE EIGHT PASCAL LINES IS PROJECTIVE WITH THE RANGE OF POINTS ON $a_{ijkl}$ BY THE TWO POINTS $T_{ij}$, THE TWO DIAGONAL POINTS $D_{ijkl}$ AND THE EIGHT BRIANCHON POINTS.

**THEOREM XXVII.** THE PENCIL OF LINES ON $T_{ijk}$ FORMED BY THE LINE $l_{ij}$, THE TANGENT $t_{i}$, THE LINE $a_{ijk}$ AND THE TWO PASCAL LINES IS PROJECTIVE WITH THE RANGE OF POINTS ON $l_{ijk}$ FORMED BY THE POINT $T_{ij}$, THE POINT $P_{i}$, THE POINT $A_{ijk}$ AND THE TWO BRIANCHON POINTS.
SECTION VII. LINEAR SECTIONS OF A COMPLETE FIVE-POINT.

THEOREM XXVIII. THE SECTION OF A COMPLETE FIVE-POINT DETERMINED BY A LINE \( l \), NOT ON A POINT \( P_L \) OR A DIAGONAL POINT, CONSISTS OF TEN POINTS WHICH DETERMINE FIVE QUADRANGULAR SETS.

PROOF: Given the complete five-point \( P_1, P_2, P_3, P_4, P_5 \) and the transversal \( l \) cutting the lines \( l_i \) in points \( P_{ij} \).

In quadrangles
\[
P_1 P_2 P_3 P_4 \quad \text{we have} \quad Q(P_{12}, P_{13}, P_{14}, P_{24}, P_{34}, P_{43})
\]
\[
P_1 P_2 P_4 P_5 \quad \text{"} \quad Q(P_{12}, P_{13}, P_{15}, P_{25}, P_{35}, P_{34})
\]
\[
P_1 P_2 P_5 P_4 \quad \text{"} \quad Q(P_{12}, P_{14}, P_{15}, P_{25}, P_{35}, P_{34})
\]
\[
P_1 P_3 P_4 P_5 \quad \text{"} \quad Q(P_{13}, P_{14}, P_{15}, P_{25}, P_{35}, P_{34})
\]
\[
P_2 P_3 P_4 P_5 \quad \text{"} \quad Q(P_{23}, P_{24}, P_{25}, P_{35}, P_{34})
\]

Hence, there are five quadrangular sets.

The four points omitted in each set are the section of the pencil which is independent of the determining four-point.
COROLLARY I. The section of a complete five-point determined by a line \( l \) on one and only one point \( P_x \) of the five-point determines one quadrangular set.

PROOF. A line \( l \) through any point as \( P_x \) cuts the six lines of the complete five-point not on \( P_x \) in six points which form a quadrangular set for they are the six sides of the complete four-point \( P_x P_y P_z P_s \).

COROLLARY II. The section of a complete five-point determined by a tangent \( t \) to the conic \( K \) determines an involution of which the point of contact \( P_x \) is a double point.

COROLLARY III. Any line \( l_{ij} \) considered as a section of the complete five-point \( P_x P_y P_z P_s \) contains two points \( P_i, P_j \) and three points \( D_{ijkl} \) and

The range of points made up of \( P_x \) and any three collinear points \( D_{ijkl} \) have the same double ratio as that of the other four points \( P_j \) on the Conic.

(Sect. II, Part II # XXVII).

THEOREM XXIX. IF TWO COMPLETE FIVE-POINTS CORRESPOND IN SUCH A WAY THAT EIGHT PAIRS OF HOMOLOGOUS SIDES, MEET IN POINTS OF ALINE \( l \), THE OTHER TWO PAIRS OF HOMOLOGOUS SIDES MEET IN POINTS OF THE SAME LINE.

---

Veblen and Young, Projective Geometry Vol. 1, Cor. of Th. XIX. p. 127.
GIVEN: the two complete five points $P, P_2, P_3, P_4, P_5$ and $P', P'_2, P'_3, P'_4, P'_5$ with

- $l_{12}$ meeting $l'_{12}$ in $P_{12}$
- $l_{13}$ meeting $l'_{13}$ in $P_{13}$
- $l_{14}$ meeting $l'_{14}$ in $P_{14}$
- $l_{34}$ meeting $l'_{34}$ in $P_{34}$
- $l_{25}$ meeting $l'_{25}$ in $P_{25}$

TO PROVE: $l_{35}$ meets $l'_{25}$ in $P_{35}$ a pt. of $l$.
and $l_{45}$ meets $l'_{45}$ in $P_{45}$ a pt. of $l$.

PROOF. In quadrangles

- $P, P_2, P_4, P_5$ we have $Q(P_{12}, P_{14}, P_{45}, P_{15}, P_{25}, P_{14})$
- $P', P'_2, P'_4, P'_5$ we have $Q(P'_{12}, P'_{14}, P'_{45}, P'_{15}, P'_{25}, P'_{14})$
Hence \( l_{45} \) meets \( l_{15} \) in \( P_{45} \) a pt. of \( l \).

In quadrangles

\[
P_1 P_2 P_3 P_5 \quad \text{we have} \quad Q(P_{12}, P_{13}, P_{15}, P_{25}, P_{23})
\]

\[
P'_1 P'_2 P'_3 P'_5 \quad \text{we have} \quad Q(P'_{12}, P'_{13}, P'_{15}, P'_{25}, P'_{23})
\]

Hence \( l_{35} \) meets \( l'_{35} \) in \( P_{35} \) a point of \( l \).

In a similar way, if any other eight pairs of homologous sides meet in points of \( l \), corresponding quadrangles can be so chosen as to establish the meeting on \( l \) of the other two pairs of homologous sides.

**COROLLARY I.** If two complete five points correspond in such a way that any two homologous simple five points are perspective from a line \( l \) and two other pairs of homologous sides meet in points of \( l \), the other three pairs of sides meet in points of the same line.

**GIVEN**/ two complete five-points \( P_1 P_2 P_3 P_4 P_5 \) and \( P'_1 P'_2 P'_3 P'_4 P'_5 \) corresponding in such a way that

\[
1_{12} \text{ meets } l'_{12} \text{ in } P_{12}
\]
\[
1_{13} \text{ " } l'_{13} \text{ " } P_{13}
\]
\[
1_{14} \text{ " } l'_{14} \text{ " } P_{14}
\]
\[
1_{15} \text{ " } l'_{15} \text{ " } P_{15} \quad \text{points of } l
\]
\[
1_{51} \text{ " } l'_{51} \text{ " } P_{51}
\]
\[
1_{13} \text{ " } l'_{13} \text{ " } P_{13}
\]
\[
1_{25} \text{ " } l'_{25} \text{ " } P_{25}
\]

**TO PROVE** \( l'_{14} \) meets \( l_{14} \) in \( P_{14} \) points of \( l \)

\[
1_{24} \text{ " } l'_{24} \text{ " } P_{24}
\]
\[
1_{15} \text{ " } l'_{25} \text{ " } P_{25}
\]
PROOF: In quadrangle

\[ P_1 P_3 P_4 P_5 \] we have \[ C(P_{12} P_{14} P_{15} P_{45} P_{35} P_{34}) \]

\[ P_1' P_3' P_4' P_5' \] we have \[ C(P_{12} P_{14} P_{15} P_{45} P_{35} P_{34}) \]

Hence \( l_{14} \) meets \( l_{14}' \) in \( P_{14} \) a point of \( l \)

In quadrangle

\[ P_1 P_2 P_3 P_4 \] we have \[ C(P_{12} P_{13} P_{14} P_{34} P_{32} P_{31}) \]

\[ P_1' P_2' P_3' P_4' \] we have \[ C(P_{12} P_{13} P_{14} P_{34} P_{32} P_{31}) \]

Hence \( l_{24} \) meets \( l_{24}' \) in \( P_{24} \) a pt. of \( l \).

In quadrangle

\[ P_1 P_2 P_3 P_5 \] we have \[ C(P_{12} P_{13} P_{15} P_{25} P_{23} P_{32}) \]

\[ P_1' P_2' P_3' P_5' \] we have \[ C(P_{12} P_{13} P_{15} P_{25} P_{23} P_{32}) \]

Hence \( l_{25} \) meets \( l_{25}' \) in \( P_{25} \) a pt. of \( l \).

COROLLARY II If two complete five-points \( P_1 P_2 P_3 P_4 P_5 \)
and \( P_1' P_2' P_3' P_4' P_5' \) correspond in such a way that \( D_{12} \) coincides with \( D_{12}' \) and any other six pairs of homologous sides meet in a point of a line, \( l \), through \( D_{12}' \), the other two pairs meet in points of \( l \).

GIVEN two complete five points \( P_1 P_2 P_3 P_4 P_5 \) and \( P_1' P_2' P_3' P_4' P_5' \)
with \( D_{1234} \equiv D_{1'2'3'4'} \) and

\[ l_{14} \text{ meeting } l_{14}' \text{ in } P_{14} \]

\[ l_{12} \quad " \quad l_{12}' \quad " \quad P_{12} \]

\[ l_{15} \quad " \quad l_{15}' \quad " \quad P_{15} \]

\[ l_{25} \quad " \quad l_{25}' \quad " \quad P_{25} \]

\[ l_{35} \quad " \quad l_{35}' \quad " \quad P_{35} \]

\[ l_{45} \quad " \quad l_{45}' \quad " \quad P_{45} \]

Where \( P_{ij} \) are points of a line \( \ell \), through \( D_{1234} \) .
TO PROVE \( l_{13} \) meets \( l_{13}' \) in \( P_{13} \)
\( l_{14} \) meets \( l_{14}' \) in \( P_{14} \)

PROOF. In quadrangles \( P_1 P_2 P_3 P_5 \) and \( P'_1 P'_2 P'_3 P'_5 \) we have on \( l \)
\[
Q(P_{12} P_{13} P_{15} P_{25} P_{23})
\]
and \[
Q'(P_{12} P_{13} P_{15} P_{25} P_{23})
\]
Hence \( l_{13} \) meets \( l_{13}' \) in \( P_{13} \) a point of \( l \).

In quadrangles \( P_1 P_2 P_3 P_7 \) and \( P'_1 P'_2 P'_3 P'_7 \) we have on \( l \)
\[
Q(D_{1234} P_{13} P_{14} D_{1234} P_{24} P_{23})
\]
and \[
Q'(D_{1234} P_{13} P_{14} D_{1234} P_{24} P_{23})
\]
Hence \( l_{14} \) meets \( l_{14}' \) in \( P_{14} \) a point of \( l \), for Veblen and Young have shown that the theorem on quadrangualr sets remains valid when the line, \( l \), contains a diagonal point of the quadrangles.

COROLLARY III. If two complete five-points \( P_1 P_2 P_3 P_4 P_5 \) and \( P'_1 P'_2 P'_3 P'_4 P'_5 \) correspond in such a way that two points \( D_{ij} \) coincide with the two corresponding points \( D'_{ij} \) and any other four pairs of homologous sides meet in points of the line, \( l \), joining the two diagonal points, the other two pairs of homologous sides meet in points of the same line \( l \).

Case 1. If \( l \) is a Pascal line.

Given the two complete five-points \( P_1 P_2 P_3 P_4 P_5 \) and \( P'_1 P'_2 P'_3 P'_4 P'_5 \) with \( D_{1234} = D'_{1234} \) and \( D_{2345} = D'_{2345} \).

'Veblen and Young, I p. 49.
and $l_{15}$ meeting $l'_{15}$ in $P_{15}$ on $D_{1234} D_{2345}$

$l_{25}$ " $l'_{25}$ " $P_{25}$ " " 

$l_{35}$ " $l'_{35}$ " $P_{35}$ " " 

$l_{13}$ " $l'_{13}$ " $P_{13}$ " "

TO PROVE $l_{24}$ meets $l'_{24}$ in $P_{24}$ " " 

$l_{14}$ " $l'_{14}$ " $P_{14}$ " "

In quadrangles $P_1 P_2 P_3 P_4$ and $P'_1 P'_2 P'_3 P'_4$

We have on $D_{1234} D_{2345}$

$Q(D_{2345} P_{34} P_{25} D_{2345} P_{35} P_{24})$

and $Q'(D_{2345} P_{34} P_{25} D_{2345} P_{35} P_{24})$

Hence $l_{24}$ meets $l'_{24}$ in $P_{24}$ a point of $D_{1234} D_{2345}$

In quadrangles $P_1 P_2 P_3 P_4$ and $P'_1 P'_2 P'_3 P'_4$

we have on $D_{1234} D_{2345}$

$Q(D_{1234} P_{13} P_{14} D_{1234} P_{24} P_{23})$

and $Q'(D_{1234} P_{13} P_{14} D_{1234} P_{24} P_{23})$

Hence $l_{14}$ meets $l'_{14}$ in $P_{14}$ a point of $D_{1234} D_{2345}$

Case 2. $l$ is a diagonal line $d_{ij} k l$.

GIVEN the two complete five-point $P_1 P_2 P_3 P_4 P_5$ and $P'_1 P'_2 P'_3$

$P'_4 P'_5$ with $D_{1234} \equiv D'_{1234}$

and $D_{1324} \equiv D'_{1324}$

$l_{15}$ meeting $l'_{15}$ in $P_{15}$

$l_{25}$ meeting $l'_{25}$ in $P_{25}$ points of $d_{1423}$

$l_{35}$ meeting $l'_{35}$ in $P_{35}$

$l_{45}$ meeting $l'_{45}$ in $P_{45}$

TO PROVE $l_{14}$ meets $l'_{14}$ in $P_{14}$ points of $d_{1423}$

$l_{23}$ " $l'_{23}$ " $P_{23}$
In the quadrangles $P_1 P_2 P_4 P_5$ and $P'_1 P'_2 P'_4 P'_5$
we have $Q(P_{14} P_{15} P_{45} P_{24})$
and $Q(P_{13} P_{15} P_{45} P_{24})$
Hence $l_{14}$ meets $l'_{14}$ in $P_{14}$ a point of $d_{1423}$ and similarly
$l_{23}$ meets $l'_{23}$ in $P_{23}$ a point of $d_{1423}$.

COROLLARY IV. If two complete five-points $P_1 P_2 P_3 P_4 P_5$
and $P'_1 P'_2 P'_3 P'_4 P'_5$ have the same diagonal line $d_{ijkl}$ and
any seven of the eight points of intersection of the lines
$l_{ij}$ with $d_{ijkl}$ coincide with the seven corresponding points
of intersection of the lines $l'_{ij}$ with $d_{ijkl}$, the eighth
points coincide

Fig. 5.
GIVEN the complete five points \( P_i \), \( P_{i+1} \), \( P_{i+2} \), \( P_{i+3} \), \( P_{i+4} \) and \( P'_i \), \( P'_{i+1} \), \( P'_{i+2} \), \( P'_{i+3} \), \( P'_{i+4} \) with the common diagonal line \( d_{1425} \) with:

- \( l_{13} \) meeting \( l'_{13} \) in \( P_{13} \)
- \( l_{14} \) meeting \( l'_{14} \) in \( P_{14} \)
- \( l_{23} \) meeting \( l'_{23} \) in \( P_{23} \)
- \( l_{25} \) meeting \( l'_{25} \) in \( P_{25} \)
- \( l_{34} \) meeting \( l'_{34} \) in \( P_{34} \)
- \( l_{35} \) meeting \( l'_{35} \) in \( P_{35} \)

and \( D_{1424} = D'_{1524} \)

TO PROVE \( D_{1245} = D'_{1245} \)

PROOF Case I

In the quadrangles \( P_i, P_{i+1}, P_{i+2}, P_{i+3}, P_{i+4}, P'_{i}, P'_{i+1}, P'_{i+2}, P'_{i+3}, P'_{i+4} \) we have:

- \( Q( P_{13}, P_{14}, P_{15}, D_{1245}, P_{35}, P_{34} ) \)
- \( Q'( P_{13}, P_{14}, P_{15}, P_{35}, P_{34} ) \)

Hence \( l_{15} \) meets \( l'_{15} \) in \( D_{1245} \), a point of \( d_{1425} \).

and \( D_{145} = D'_{1245} \)

Case II.

If given both diagonal points and five other points the case is trivial by Cor. III.
COROLLARY V. If two complete five-points \( P_1, P_2, P_3, P_4, P_5 \) and \( P'_1, P'_2, P'_3, P'_4, P'_5 \) correspond in such a way that \( D_{1234} = D'_{1234} \) and \( l_{24} \) meets \( l'_{24} \), \( l_{25} \) meets \( l'_{25} \), \( l_{45} \) meets \( l'_{45} \) in points of the diagonal line \( d_{134} \), then the other three pairs of homologous sides meet in points of \( d_{134} \).

[Diagram of five points and lines corresponding with their primes, showing intersections and line relationships.]
In the quadrangle

\( P_1 P_2 P_3 P_4 \)

we have

\( H(D_{1234} D_{1423} l_{124} l_{13}) \)

\( P'_1 P'_2 P'_3 P'_4 \)

we have

\( H(D_{1234} D_{1423} l_{124} ? ) \)

Hence \( l_{13} \) meets \( l'_{13} \) in \( P_{13} \) of \( d_{1324} \)

\( P_1 P_2 P_4 P_5 \)

we have

\( Q(D_{1234} D_{1423} P_{15} , P_{45} P_{25} P_{24}) \)

\( P'_1 P'_2 P'_4 P'_5 \) " "

\( Q(D_{1234} D_{1423} ? , P_{45} P_{25} P_{24}) \)

Hence \( l_{15} \) meets \( l'_{15} \) in \( P_{15} \) of \( d_{1324} \)

\( P_2 P_3 P_4 P_5 \)

we have

\( Q(D_{1423} P_{24} P_{25} , P_{45} P_{35} D_{134}) \)

\( P'_2 P'_3 P'_4 P'_5 \) " "

\( Q(D_{1423} P_{24} P_{25} , P_{45} ? D_{134}) \)

Hence \( l_{35} \) meets \( l'_{35} \) in \( P_{35} \) of \( d_{1324} \).
COROLLARY VI. The range of points determined on a diagonal line $d_{ijkl}$ of a complete five-point by the two diagonal points $D_{ijkl}$ and the four points of intersection of the diagonal line with the lines $l_{ij}$ of the pencil on the point which is independent of the diagonal points forms a quadrangular set.

Refering to Section II Part II on analytic work we have

$$\begin{align*}
(D_{1524}, D_{4216}, P_{12}, P_{45}) &= 1, \\
Q(D_{1524}, P_{23}, P_{44}, D_{4215}, P_{23}, P_{12}, P_{25}) &= A, B, C, D, \\
Q(D_{1524}, P_{36}, P_{13}, D_{4215}, P_{34}, P_{25}) &= A, B, C, D.
\end{align*}$$
\[
\begin{align*}
(D_{1524} & D_{1425}) P_{23} P_{34} = \frac{1-m}{1+m} \\
(D_{1524} & D_{1425}) P_{35} P_{13} = \frac{1-m}{1+m}
\end{align*}
\]

Hence \(D_{1524} D_{1425} P_{23} P_{34} \not\sim D_{1524} D_{1425} P_{35} P_{13}\)
and this determines \(Q(D_{1524} P_{23} P_{34}, D_{1425} P_{13} P_{25})\)
for "The necessary and sufficient condition for the pro-
jectivity on a line \(MNAB \not\sim MMA'B'\) (\(M \neq N\)) is
\(Q(MAB, NDA').\)

Similarly
\[
\begin{align*}
(D_{1524} & D_{1425} P_{35} P_{13}) = \frac{n-m}{n+m} \\
(D_{1524} & D_{1425} P_{13} P_{34}) = \frac{n-m}{n+m}
\end{align*}
\]

Hence \(D_{1524} D_{1425} P_{23} P_{34} \not\sim D_{1524} D_{1425} P_{13} P_{34}\) and we
have \(Q(D_{1524} P_{35} P_{23}, D_{1425} P_{34} P_{13})\)

**Theorem XXX.** The four points on a Pascal line of a
complete quadrangle form a harmonic set.

Considering the quadrangle \(P_1 P_3 P_4 P_5\) within the complete
five-point \(P_1 P_2 P_3 P_4 P_5\) and the special Pascal line of the
quadrangle \(d_{1524}\) we find
\[
(D_{1345} D_{1435} T_{24} T_{15}) = -1
\]

Similarly on any line \(d_{ijkl}\) we have
\[
(D_{ijkl} D_{ikjl} T_{ik} T_{jk}) = -1
\]

\[\text{Veblen and Young, I, p. 100}\]
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