THE CONTINUITY OF INTEGRAL TRANSFORMATIONS WITH POSITIVE KERNELS BETWEEN L^P SPACES WITH MIXED NORMS

by

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CHAPTER ONE

INTRODUCTION

Let (A,Ω,μ) and $(B,\mathcal{A},\nu)$ be two totally σ-finite measure spaces, M and N the linear spaces of real valued functions which are measurable and finite a.e. on A and B respectively, with the metric topology of convergence in measure on all subsets of finite measure, and $K(x,y)$ a real valued measurable function on the product space $A \times B$. We define the integral transformation with kernel $K(x,y)$ by

$$(1.1) \qquad Kf(y) = \int K(x,y)f(x)\,d\mu(x)$$

for $f \in M$ if the integral on the right exists and is finite for a.e. $y \in B$. The linear subspace of all such $f \in M$ is called the domain of K and denoted by $D(K)$. The linear transformation K^* is the integral transformation defined by

$$K^*(y,x) = \overline{K(x,y)}.$$

If $X \subset M$ and $Y \subset N$ we write $K:X \to Y$ whenever $X \subset D(K)$ and for all $f \in X$, $K(f) \in Y$. If $X \subset M$ and $Y \subset N$ are given Banach spaces with norms $\|\ \|_x$ and $\|\ \|_y$ respectively we can ask if K is continuous as a function defined on X with range in Y. In particular we say that "$K:X \to Y$ is bounded with $\|K\| \leq c$" if and only if

$$(1.2) \qquad \frac{\|Ku\|_y}{\|u\|_x} \leq c$$

for all $u \in X$ such that $\|u\|_x \neq 0$. The case of interest is when

X,Y are continuously contained in M and N. Then $K:X\to Y$ implies K is continous (See [2]). Thus, the question of continuity of K reduces to the following: Given X,Y, find conditions on K in order that $K:X \to Y$. Many authors have investigated the continuity properties of integral transforms. For example, see [2,3,4,7,8,9, 10 and 12]. One set of investigations led to the following two results for L^p spaces, which were discussed by Gagliardo [9].

Result 1: (N. Aronszajn) Let $1 < q \leq p < +\infty$, $0 < c < +\infty$, $(1/p') + (1/p) = (1/q') + (1/q) = 1$ and $K(x,y) \geq 0$ a.e. be measurable on $A \times B$. If for every real $\epsilon>0$ there exist measurable functions $\varphi(x) > 0$ and $\psi(y) > 0$ and finite a.e. such that

(i) $(K\varphi)(y) \leq (c+\epsilon)(\psi(y))^{q'/q}$

(ii) $(K^*\psi)(x) \leq (c+\epsilon)(\varphi(x))^{p/p'}$

(iii) $p = q$ or $\iint_{X\times Y} K(x,y)\varphi(x)\psi(y)\,dxdy \leq c+\epsilon$

then

$$K:L^p \to L^q$$

is bounded with

$$||K|| \leq c.$$

A converse of the above is given by the following.

Result 2: (E. Gagliardo) Let $1 < p < +\infty$, $1 < q < +\infty$, $(1/p')+(1/p) = (1/q')+(1/q) = 1$ and $K(x,y) \geq 0$ a.e. be measurable on $A \times B$. If $K:L^p \to L^q$ is bounded with $||K|| \leq c$, then for every real $\epsilon > 0$ there exist functions $\varphi \in L^p$, $\psi \in L^{q'}$, $\varphi,\psi > 0$ a.e. such that

(i) $(K\varphi)(y) \leq (c+\varepsilon)(\psi(y))^{q'/q}$

(ii) $(K^*\psi)(x) \leq (c+\varepsilon)(\varphi(x))^{p/p'}$

(iii) $||\varphi||_p \leq 1$ and $||\psi||_{q'} \leq 1$

Remark: Conditions (i), (ii) and (iii) in Result 2 imply condition (iii) of Result 1. Thus, for $q \leq p$, we have necessary and sufficent conditions for the boundedness of an integral transformation with a positive kernel.

In this paper, we will be concerned with extensions of Results 1 and 2 to the spaces L^P and L^Q of functions of several variables with mixed norms defined for the multi-indices $P = (p_1,\ldots,p_n)$ and $Q = (q_1,\ldots,q_m)$ by applying successively the p_i or q_j norm to the variable x_i or y_j. Corresponding to Results 1 and 2 we have Theorems 5 and 4 respectively. These theorems only form exact converses when $p_i \geq p_{i+1}$ for $i=1,\ldots,n$ and $q_j \leq q_{j+1}$ for $j=1,\ldots,m$. An open question, which is discussed toward the end of this paper, concerns the strengthening of Theorem 4 to give a converse of Theorem 5 without this restriction.

Of particular interest would be the generalization of these results to ordered Banach spaces or to Banach Function Spaces, as defined by Luxemburg and Zaanen [14], using duality mappings and stating the restriction $q \leq p$ in terms of these mappings. Theorem 1 indicates that Result 2 above can be generalized to these settings.

CHAPTER TWO

MAPPINGS BETWEEN ORDERED BANACH SPACES

Most of the terminology and basic results needed in this paper are available in [13, 14].

An ordered Banach space X is a Banach space with a linear partial ordering. If X is an ordered Banach space, then we define $X_+ = \{u \mid u \geq 0\}$ and postulate that for $u, v \in X_+$,

(2.1) $\qquad u \leq v$ implies $||u|| \leq ||v||$.

We will call an ordered Banach space $X \neq \{0\}$ an Mc ordered Banach space if X satisfies the following monotone convergence principle: For every increasing sequence $\{u_n\} \subset X_+$, such that $||u_n|| \leq \alpha$ for all n and some $\alpha > 0$ there exists $u \in X_+$ such that $u_n \to u$.

Let $(A, \mathcal{A}, \mu)$ be a totally σ-finite measure space and let M denote the linear space of all measurable functions on A and define an order relation $\leq$ as follows. For $f, g \in M$

(2.2) $\qquad f \leq g$ iff $f(x) \leq g(x)$ a.e.

and

$$M_+ = \{f \in M \mid f(x) \geq 0 \text{ a.e.}\}.$$

Define $R^e_+ = R^1_+ \cup \{+\infty\}$ to be the extended positive half axis. Following Luxemburg and Zaanen [14 § 2p. 138 and § 3p. 148] a function $\rho: M_+ \to R^e_+$ is called function norm if and only if

(i) $\rho(f) = 0$ if and only if $f = 0$ a.e.

(ii) $\rho(\alpha f) = \alpha\rho(f)$ for all $f \in M_+$ and $\alpha \in R^1_+$

(iii) $\rho(f+g) \leq \rho(f) + \rho(g)$ for all $f, g \in M_+$

(iv) if $f, g \in M_+$ and $f \leq g$ then $\rho(f) \leq \rho(g)$.

For $f \in M$ define $\rho(f) = \rho(|f|)$.

If for a function norm ρ, we let

(2.3) $$L_\rho = \{u \in M \mid \rho(u) < \infty\}$$

then L_ρ is a normed linear space with norm ρ and a linear partial ordering defined by (2.2). If $f_n, f \in M_+$, then we say $f_n \uparrow f$ if and only if $f_n(x) \uparrow f(x)$ a.e. A function norm ρ is said to have the Fatou property if for $f_n, f \in M_+$,

(2.4) $$f_n \uparrow f \text{ implies } \rho(f_n) \uparrow \rho(f).$$

It can be shown [14 § 3 p. 149] that if ρ satisfies the Fatou property, then L_ρ is complete and hence an ordered Banach space.

We say that a function norm ρ is smooth if and only if for $f_n, f \in L_\rho$

(2.5) $$f_n(x) \to f(x) \text{ a.e. and } \rho(f_n) \to \rho(f),$$

$$\text{implies } \rho(f_n - f) \to 0.$$

Let ρ be a smooth function norm with the Fatou property and $f_n \in M_+$ be such that $f_{n+1} \geq f_n$ and $\rho(f_n) \leq \alpha$. Then there exists $f \in M$ such that $f_n \uparrow f$ so by the Fatou property $\rho(f_n) \to \rho(f)$ and $\rho(f) \leq \alpha$. But ρ is smooth so $\rho(f_n - f) \to 0$ and hence $f_n \to f$. Thus, if ρ is a smooth function norm satisfying the Fatou property, then L_ρ is an Mc ordered Banach space.

Let X be an ordered Banach space; then for u, $v \in X$ let $w = u \vee v$ if and only if $u \leq w$, $v \leq w$ and $w \leq w_1$ whenever $u \leq w_1$ and $v \leq w_1$. A Banach lattice is an ordered Banach space X in which

$u \vee v$ exists and is unique for any $u, v \in X$. We are moreover assuming that if $|u| \leq |v|$, then $||u|| \leq ||v||$. In a Banach lattice we can define $u \wedge v = -[(-u) \vee (-v)]$, $u^+ = u \vee o$, $u^- = (-u) \vee o$, and $|u| = u^+ + u^-$ for any $u, v \in X$. We note that $u = u^+ - u^-$. Banach function spaces are examples of Banach lattices. If X_+ is the positive cone of a Banach lattice X, then we will define the lattice interior of X_+ as

(2.6) $$X_+^o = \{u \in X_+ | \text{ for all } v \in X_+ \text{ if } u \wedge v = 0 \text{ then } v = 0\}.$$

Note that $X_+^o \cup \{0\}$ is a cone and if L_p is a Banach function space, then $f \in L_{p+}^o$ if and only if $f(x) > 0$ a.e. A Banach lattice X will be called non-trivial if there exists $u \in X_+^o$ such that $||u|| > 0$.

If X is a Banach lattice, then the dual space X^* is also a Banach lattice with

(2.7) $$u^* \leq v^* \text{ if and only if } u^*(w) \leq v^*(w) \text{ for all } w \in X_+,$$

since we can easily show that if $u^* \leq v^*$ then $||u^*|| \leq ||v^*||$ for $u^*, v^* \in X_+^* = \{w^* | w^* \geq 0\}$ and $(u^* \vee v)(w) = \sup\{u^*(w_1) + v^*(w_2) | w_1, w_2 \in X_+$ and $w = w_1 + w_2\}$ for all $w \in X_+$. See Kelley and Namioka [13 p. 232].

Let X and Y be ordered Banach spaces. Then a linear partial ordering $\leq$ in $X \times Y$ is defined by the positive cone.

(2.8) $$(X \times Y)_+ = \{(u,v) \in X \times Y | u \in X_+ \text{ and } v \in Y_+\}.$$

A norm for $X \times Y$ can be chosen in many ways. For example,

(2.9) $$||(u,v)|| = \max\{||u||, ||v||\}$$

or for $1 \leq p < \infty$

$$(2.10) \qquad ||(u,v)|| = (||u||^p + ||v||^p)^{1/p} .$$

With either of these norms $X \times Y$ is an ordered Banach space. Moreover, if X and Y are Banach lattices, then $X \times Y$ is a Banach lattice.

We will now introduce some terminology concerning mappings (in general non-linear) between ordered Banach spaces.

Let X and Y be ordered Banach spaces and let Φ be a function defined on $D(\Phi) \subset X$ into Y. Then

(i) Φ is positive if and only if $\Phi(X_+ \cap D(\Phi)) \subset Y_+$

(ii) Φ is non-decreasing if and only if $0 \leq u \leq v$ implies $\Phi(u) \leq \Phi(v)$

(iii) Φ is stable if and only if $X_+ \subset D(\Phi)$, Φ is positive, non-decreasing, continuous and

$$(2.11) \qquad ||u||_X \leq 1 \text{ implies } ||\Phi(u)||_Y \leq 1.$$

If X and Y are Banach lattices we have

(iv) Φ is strictly positive if and only if $\Phi(X_+^\circ \cap D(\Phi)) \subset Y_+^\circ$

(v) Φ is strictly stable if and only if $X_+^\circ \subset D(\Phi)$, Φ is strictly positive, non-decreasing, continuous and satisfies (2.11).

Finally,

(vi) Φ is a power function if and only if Φ is positive, continuous and satisfies (2.11) with $\Phi(0) = 0$, and in addition, (2.12) $\Phi(\alpha u) = \hat{\Phi}(\alpha)\Phi(u)$ for all $\alpha \in R^1_+$ and $u \in D(\Phi)$

defines a function $\hat{\Phi}: R^1_+ \to R^1_+$ independent of u, such that $\hat{\Phi}$ is non-decreasing and not constant.

We notice if Φ is a power function, then $\hat{\Phi}$ is both stable and a power function. Moreover, $\hat{\Phi}$ is a continuous, multiplicative homomorphism of R^1_+ into itself with $\hat{\Phi}(1) = 1$. Hence, there exists a real number $p > 0$ such that $\hat{\Phi}(\alpha) = \alpha^p$ for all $\alpha \in [0,\infty)$ and we will define $Ex(\Phi) = p$. Finally, if Φ is a 1-1 power function onto Y_+, then Φ^{-1} is defined and $\hat{\Phi}^{-1} = 1/\hat{\Phi}$ satisfies condition (2.12) although Φ^{-1} may not be a power function.

We will now consider two examples.

Example 1: Let X, Y be ordered Banach spaces and $K: X \to Y$ a bounded positive linear transformation with $||K|| \leq \alpha$. Then $(1/\alpha)K$ is stable and a power function with $\widehat{\frac{1}{\alpha}K}(\beta) = \beta$ for all $\beta \in [0,\infty)$ so $Ex(\frac{1}{\alpha}K) = 1$.

Example 2: Let $(A, \mathcal{A}, \mu)$ be a σ-finite measure space, $p,q \in [1,+\infty)$. For $f \in L^p_+$ define $\Phi(f(x)) = f(x)^{p/q}$ for all $x \in A$. It is easily verified that Φ is a stable function and in fact a power function with $Ex(\Phi) = p/q$. Thus, we see that stable and power functions do not have to be linear and $Ex(\Phi)$ may assume any positive value.

An analogue of the function Φ in Example 2 for L^P spaces where P is a multi-index will be important in the study of integral transformations between these spaces as carried out in the sequel.

Note that if α is a real number, $0 < \alpha \leq 1$, then $\alpha\Phi$ is stable whenever Φ is stable and $\alpha\Phi$ is a power function whenever Φ is a power function. Moreover, if $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$ and Φ and Ψ

are stable functions with the same domains and into a common range, then $\alpha\Phi + \beta\Psi$ is a stable function.

One can verify immediately the following.

Proposition 1: Let X, Y and Z be ordered Banach spaces, $\Phi : X_+ \to Y$ and $\Psi : Y_+ \to Z$

(i) if Φ and Ψ are (strictly) stable functions, then $\Psi\circ\Phi : X_+ \to Z$ is a (strictly) stable function,

(ii) if Φ and Ψ are power functions, then $\Psi\circ\Phi$ is a power function, $\widehat{\Psi\circ\Phi} = \hat{\Psi}\circ\hat{\Phi}$ and $Ex(\Psi\circ\Phi) = Ex(\Psi)\cdot Ex(\Phi)$.

(iii) if $\Phi : X_+ \times Y_+ \to Z$ is a stable (power) function and $I : X_+ \to X_+ \times Y_+$ is the natural injection, then $\Phi\circ I : X_+ \to Z$ is a stable function (power function with $Ex(\Phi\circ I) = Ex(\Phi)$).

Thus, we see that compositions and restrictions of stable and power functions give functions of the same type. The norm on $X \times Y$ can be calculated by either equation (2.9) or (2.10) in the above proposition. This norm can also be used in the following proposition, which gives some examples of how to construct stable and power functions on product spaces from such functions on the original spaces.

Proposition 2: Let X_1, X_2, Y_1 and Y_2 be ordered Banach spaces, $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, $\Phi_1 : X_{1+} \to Y_1$ and $\Phi_2 : X_{2+} \to Y_2$.

(i) if Φ_1 and Φ_2 are (strictly) stable functions, then

$$\Psi_1(u_1, u_2) = \frac{1}{2}(\Phi_1(u_1),\ \Phi_2(u_2)) \tag{2.13}$$

defines a (strictly) stable function $\Psi_1 : X_+ \to Y$.

Moreover, if $Y_1 = Y_2$, then

(2.14) $$\Psi_2(u_1,u_2) = \tfrac{1}{2}[\Phi_1(u_1) + \Phi_2(u_2)]$$

defines a (strictly) stable function $\Psi_2 : X_+ \to Y_1$

(ii) If Φ_1 and Φ_2 are power functions, then for $u = (u_1,u_2) \in X_+$ we define

(2.15) $$\Psi_1(u) = \tfrac{1}{2}\hat{\Phi}_1 \circ \hat{\Phi}_2(\|u\|)(\Phi_1(u_1/\|u\|),\ \Phi_2(u_2/\|u\|))$$

if $\|u\| \neq 0$ and $\Psi_1(u) = 0$ if $\|u\| = 0$. If $Ex(\Phi_1) \geq 1$ and $Ex(\Phi_2) \geq 1$, then Ψ_1 is a power function with a range in Y. Moreover, if $Y_1 = Y_2$,

(2.16) $$\Psi_2(u) = \tfrac{1}{2}\hat{\Phi}_1 \circ \hat{\Phi}_2(\|u\|)(\Phi_1(u_1/\|u\|) + \Phi_2(u_2/\|u\|))$$

with $\Psi_2(u) = 0$ if $\|u\| = 0$ is a power function with range in Y_1.

Proof: (i) If Φ_1 and Φ_2 are (strictly) stable functions, then Ψ_1 and Ψ_2 are clearly continuous, (strictly) positive and non-decreasing. $\|\Psi_1(u_1,u_2)\| \leq \tfrac{1}{2}(\|\Phi_1(u_1)\| + \|\Phi_2(u_2)\|)$ so if $\|(u_1,u_2)\| \leq 1$, then $\|u_1\| \leq 1$ and $\|u_2\| \leq 1$ and hence $\|\Phi_1(u_1)\| \leq 1$ and $\|\Phi_2(u_2)\| \leq 1$ and thus $\|\Psi_1(u_1,u_2)\| \leq 1$. Therefore, Ψ_1 is a (strictly) stable function and so is Ψ_2 by a similar argument.

(ii) Let Φ_1 and Φ_2 be power functions. Then clearly Ψ_1 and Ψ_2 defined by (2.15) and (2.16) are positive or strictly positive and continuous except possibly at 0. Moreover, $\Psi_1(0) = 0$, $\Psi_2(0) = 0$, and $\hat{\Psi}_1$ and $\hat{\Psi}_2$ are well defined, non-constant and non-decreasing since $\hat{\Phi}_1$ and $\hat{\Phi}_2$ satisfy these conditions. To verify condition (2.11) for Ψ_1 consider $u = (u_1,u_2) \in X_+$ with $\|u\| \leq 1$.

Then, $0 \leq \hat{\Phi}_1 \circ \hat{\Phi}_2(||u||) \leq 1$ since $\hat{\Phi}_1$ and $\hat{\Phi}_2$ are stable functions and $||(\Phi_1(u_1/||u||), \Phi_2(u_2/||u_2||))|| \leq ||\Phi_1(u_1/||u||)|| + ||\Phi_2(u_2/||u||)|| \leq 1 + 1 = 2$ since $||(u_1/||u||)|| \leq 1$ and $||(u_2/||u||)|| \leq 1$. Therefore, $||\Psi_1(u)|| \leq 1$. By a similar argument Ψ_2 also satisfies condition (2.11). Thus, both Ψ_1 and Ψ_2 are p functions if they are continuous at 0. But this follows from the assumption that $Ex(\Phi_1) = p > 1$ and $Ex(\Phi_2) = 2 > 1$ since

$$\Psi_1 = \tfrac{1}{2}(||u||^{p(q-1)}\Phi_1(u_1), \ ||u||^{q(p-1)}\Phi_2(u_2))$$

and

$$\Psi_2 = \tfrac{1}{2}(||u||^{p(q-1)}\Phi_1(u_1) + ||u||^{q(p-1)}\Phi_2(u_2)) \blacksquare$$

In a paper by Gagliardo [9 p. 431] we find the following:

Lemma 1: Let X be an Mc ordered Banach space and $\Phi : X_+ \to X$ be a stable function. Then for every $\delta > 0$ there exists $u \in X_+$ such that

$$||u|| \leq 1, \ u \geq 0, \ u \neq 0$$

and

$$(1 + \delta)u \geq \Phi(u).$$

Proof: Choose $u_1 \in X_+$ such that $u_1 \neq 0$ and $||u_1|| \leq (\delta/1+\delta))$. Define $u_n \in X_+$ for $n = 2,3,\ldots,$ by

$$u_n = u_1 + (1/(1+\delta))\Phi(u_{n-1}).$$

By induction $||u_n|| \leq 1$ and $u_{n+1} \geq u_n$ for $n = 1,2,3,\ldots$. Thus, there exists $u \in X_+$ such that $u_n \to u$ and $||u|| \leq 1$ by the monotone convergence (Mc) property of X. By continuity $u = u_1 + (1/(1+\delta))\Phi(u)$. Hence $u \neq 0$ and $(1+\delta)u - \Phi(u) = (1+\delta)u_1 \geq 0 \blacksquare$

The following theorem is related to a result of Gagliardo [9 p. 430] but has a more general setting.

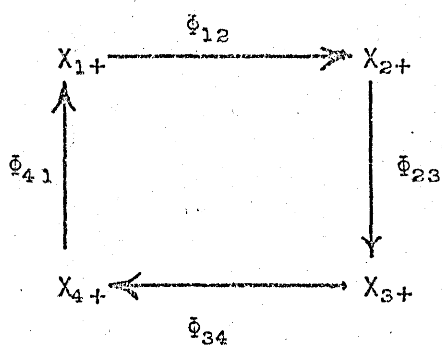
<u>Theorem 1</u>: Let X_1, X_2, X_3 and X_4 be ordered Banach spaces and suppose X_4 is an Mc ordered Banach space. Let $\Phi_{12}:X_{1+} \to X_2$, $\Phi_{23}:X_{2+} \to X_3$, $\Phi_{34}:X_{3+} \to X_4$ and $\Phi_{41}:X_{4+} \to X_1$ and suppose Φ_{41} and Φ_{23} are 1-1 stable functions and for some numbers $\alpha > 0$ and $\beta > 0$ we have $(1/\alpha)\Phi_{12}$ and $(1/\beta)\Phi_{34}$ are stable functions. Then for each number $\epsilon > 0$ there exists $\varphi_1 \in X_{1+}$ and $\varphi_3 \in X_{3+}$ such that

(i) $\Phi_{12}(\varphi_1) \leq (\alpha + \epsilon)\Phi_{23}^{-1}(\varphi_3)$

(ii) $\Phi_{34}(\varphi_3) \leq (\beta + \epsilon)\Phi_{41}^{-1}(\varphi_1)$

(iii) $||\varphi_1||_{X_1} \leq 1$, $||\varphi_3||_{X_3} \leq 1$,

$\varphi_1 \neq 0$.



<u>Proof</u>: Let $\delta > 0$ be such that $\beta(1+\delta)^2 \leq \beta + \epsilon$. For $u \in X_{4+}$ define $\Phi(u) = (1/\beta)\Phi_{34}(\Phi_{23}((1/(\alpha+\epsilon))\Phi_{12}[\Phi_{41}((1/(1+\epsilon))u)]))$.

Note that $\Phi:X_{4+} \to X_4$ is a stable function so by Lemma 1, there exists $u \in X_{4+}$ such that $||u||_{X_4} \leq 1$, $u \neq 0$ and

$$(1/\beta)\Phi_{34}(\Phi_{23}((1/(\alpha+\epsilon))\Phi_{12}[\Phi_{41}((1/(1+\delta))u)])) \leq (1+\delta)u.$$

Let $\varphi_1 = \Phi_{41}((1/(1+\delta))u)$ and $\varphi_3 = \Phi_{23}((1/(\alpha+\epsilon))\Phi_{12}\varphi_1)$. Note that $||\varphi_1||_{X_1} \leq 1$ since $||(1/(1+\delta))u||_{X_4} \leq 1$

and

$$||\varphi_3||_{X_3} \leq 1 \text{ since } ||(1/(\alpha+\epsilon))\Phi_{12}\varphi_1||_{X_2} \leq 1.$$

Now

$$\Phi_{12}(\varphi_1) = (\alpha + \epsilon)\Phi_{23}^{-1}(\varphi_3)$$

and

$$\begin{aligned}\Phi_{34}(\varphi_3) &\leq \beta(1 + \delta)u \\ &\leq \beta(1 + \delta)^2\Phi_{41}^{-1}(\varphi_1) \\ &\leq (\beta + \epsilon)\Phi_{41}^{-1}(\varphi_1)\end{aligned}$$

and finally

$\varphi_1 \neq 0$ since Φ_{41} is 1-1 and $u \neq 0$ ∎

Of more special interest in the sequel is the following immediate

Corollary: Let X and Y be Banach lattices and X^* (which together with Y^* are hence Banach lattices) satisfy the monotone convergence principle. Let $\Phi:X_+^* \to X$ and $\Psi:Y_+ \to Y^*$ be 1-1 stable functions and $T:X \to Y$ be a bounded, positive linear transformation such that $||T|| \leq \alpha$. Then for every real $\epsilon > 0$ there exists $\varphi \in X_+$ and $\psi \in Y_+^*$ such that

(i) $T(\varphi) \leq (\alpha+\epsilon)\Psi^{-1}(\psi)$

(ii) $T^*(\psi) \leq (\alpha+\epsilon)\Phi^{-1}(\varphi)$

(iii) $||\varphi||_x \le 1$, $||\psi||_y \le 1$ (iv) $\varphi \ne 0$.

In a manner different than that of Aronszajn and Szeptycki [2 p. 143], we will now generalize a theorem of S. Banach [3 p.87] (concerning the continuity of integral transformations) to positive transformations between Banach lattices.

Theorem 2: Let X be a Banach lattice such that for any sequence $u_n \in X_+$ such that $0 \le u_n \le u_{n+1}$ and $||u_n|| \le \alpha$ for all $n = 1,2,3,\ldots$, and some $\alpha > 0$ there exists $u \in X_+$ such that $u_n \le u$ for all $n = 1,2,3,\ldots$ and $||u||$ finite. Let Y be an ordered Banach space. Then a positive linear transformation $T:X \to Y$ (defined for all $u \in X$) is necessarily bounded.

Proof: Suppose T is unbounded. Then there will exist $u_n \in X$ such that

$$||Tu_n|| \ge 2^{2n} ||u_n|| \text{ for } n = 1,2,\ldots \; .$$

Let

$$v_n = (2^{-n}/||u_n||)\, |u_n|.$$

Note

$$v_n \in X_+ \text{ and } ||v_n|| = 2^{-n}.$$

Clearly

$$T(|u_n|) \ge T(u_n) \text{ since } |u_n| = u^+ + u^- \ge u^+ - u^- = u.$$

Hence,

$$\begin{aligned} ||Tv_n|| &= (2^{-n}/||u_n||)\, ||T(|u_n|)|| \\ &\ge (2^{-n}/||u_n||)\, ||T(u_n)|| \\ &\ge (2^{-n}/||u_n||) 2^{2n} ||u_n|| \\ &= 2^n = 2^{2n} ||v_n||. \end{aligned}$$

Let

$$w_1 = v_1 \text{ and } w_n = v_n \vee w_{n-1} \text{ for } n = 2,3,\ldots .$$

Note

$$||w_1|| = ||v_1|| = 1/2$$

and by induction

$$||w_{n+1}|| \leq ||w_n|| + ||v_{n+1}|| \leq ((2^n - 1)/2^n) + (1/2^{n+1})$$

$$= ((2^{n+1} - 1)/2^{n+1}) < 1.$$

Now

$$0 \leq w_n \leq w_{n+1} \text{ for all } n = 1,2,\ldots$$

so there exists $w \in X_+$ such that $w_n \leq w$ and $||w|| < +\infty$. Thus, $Tw \in Y$ and $||Tw||$ is finite. However, $0 \leq v_n \leq w_n \leq w$ so $0 \leq Tv_n \leq Tw$. Hence, $||Tw|| \geq ||Tv_n|| \geq 2^n$ for all $n = 1,2,\ldots$ which is a contradiction. Therefore, T is bounded∎

Remark 1: If X is an Mc Banach lattice or a Banach function space satisfying the Fatou property, then X satisfies the conditions of Theorem 2. In particular, a positive linear transformation $T:L^p \to L^q$ (defined for all $f \in L^p$) is bounded for any $p \geq 1$ and $q \geq 1$.

Remark 2: If $T = T^+ - T^-$ where T^+, T^- are positive linear transformations $T^+:X \to Y$, $T^-:X \to Y$ (both defined on all of X), then T is bounded. In particular, this remark applies to integral transformations.

CHAPTER THREE

THE SPACES L^P WITH MIXED NORMS

In the next few sections we will be studying functions of several variables. Let (A_i, Λ_i, μ_i) for $i = 1,2,\ldots,n$ and (B_j, Ω_j, ν_j) for $j = 1,2,\ldots,m$ be σ-finite measure spaces and

$$A = \prod_{i=1}^{n} A_i \qquad \text{and} \quad B = \prod_{i=1}^{m} B_i$$

be their product measure spaces. The corresponding spaces of measurable functions will be M_i, N_i, M and N. Corresponding to the vector variables $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_m)$ we will use the multi-indices $P = (p_1,\ldots,p_n)$ and $Q = (q_1,\ldots,q_m)$ where $1 \le p_i, q_j \le \infty$. For $f \in M$ we define $||f||_{\binom{P}{\mu}} =$

$$||f||_{\binom{p_1,\ldots,p_n}{\mu_1,\ldots,\mu_n}} = ||f||_{p_1,\ldots,p_n} = ||f||_P$$

to be the value obtained by successively taking the p_1 norm in x_1, the p_2 norm in $x_2,\ldots,$ the p_n norm in x_n in that order. Thus, if $p_i < \infty$ for all $i = 1,\ldots,n$ then

$$(3.1) \qquad ||f||_P = \left(\int\cdots\left(\int\left(\int |f(x_1,\ldots,x_n)|^{p_1} d\mu_1\right)^{p_2/p_1} d\mu_2\right)^{p_3/p_2}\cdots d\mu_n\right)^{1/p_n}$$

Following Benedek and Panzone [4 p. 301], we define

$$(3.2) \qquad L^P(A) = L^P = \{f \in M \mid ||f||_P < \infty\}.$$

These are Banach function spaces with norms satisfying the Fatou property [4 p. 302]. If $1 \le p_i < \infty$ for $i = 1,\ldots,n$ then the function norm $||\ ||_P$ is smooth [4 p.312]. We define $||g||_Q$ for

$g \in N$ and $L^Q = L^Q(B)$ in the same manner.

If $\chi(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is an algebraic relation among the real numbers $\alpha_1, \ldots, \alpha_k$ and $P^{(1)}, P^{(2)}, \ldots, P^{(k)}$ are multi-indices with the same number of components, then $\chi(P^{(1)}, P^{(2)}, \ldots, P^{(k)})$ means $\chi(p_i^{(1)}, \ldots, p_i^{(k)})$ for each i. In particular $P' = P(P-1)^{-1}$ means $p_i' = p_i/(p_i - 1)$. The "Generalized Holder Inequality" can now be stated as follows:

$$(3.3) \qquad \left|\int \cdots \int f(x_1, \ldots, x_n) g(x_1, \ldots, x_n) \, d\mu_1(x_1) \cdots d\mu_n(x_n)\right| \le \|f\|_P \|g\|_{P'}$$

for any $f, g \in M$ and $1 \le P \le \infty$. Moreover, Benedek and Panzone [4 p. 304] have proved that if $1 \le P < \infty$, then the dual space of L^P is $L^{P'}$. Finally, if P and Q are multi-indices of length n and m respectively and $r \in R^1$, then $Q < r < P$ will be taken to mean $q_j < r < p_i$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. By $L^{P,Q}$ or $L^{(P,Q)}$ we will mean the space of functions of $m + n$ variables with mixed norm corresponding to the multi-index $(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m)$.

Since we will not always use the integrals and variables in the same order, at times it will be necessary to replace the usual, simpler expression $\|f\|_P$ by $\|f\|_{\binom{P}{\mu}}$ or $\|f\|_{\binom{p_1, p_2, \ldots, p_n}{\mu_1, \mu_2, \ldots, \mu_n}}$ to avoid confusion or ambiquity. If $r \in [1, +\infty)$, then the norm for $L^r(A)$ in the product space is

$$(3.4) \qquad \|f\|_{\binom{r}{\mu}} = \|f\|_{\binom{r, r, \ldots, r}{\mu_1, \mu_2, \ldots, \mu_n}}$$

It can also be easily shown that

$$(3.5)\qquad ||f^r||_P = ||f||^r_{\begin{pmatrix}p_1 r, p_2 r, \ldots, p_n r \\ \mu_1, \mu_2, \ldots, \mu_n\end{pmatrix}} .$$

Finally, by Tonelli's theorem, for $r \in [1,\infty)$ and f measurable on $A \times B$,

$$(3.6)\qquad ||f(x,y)||_{\begin{pmatrix}r & , r \\ \mu(x) & , \nu(y)\end{pmatrix}} = ||f(x,y)||_{\begin{pmatrix}r & , r \\ \nu(y) & , \mu(x)\end{pmatrix}} .$$

<u>Definition</u>: If $\alpha, \beta \in R^e_+ = [0, +\infty]$, then

$$(3.7)\qquad a(\alpha,\beta) = \frac{\alpha\beta}{\alpha-\beta} = \frac{1}{(1/\beta)-(1/\alpha)}$$

with the convention

$$(3.8)\qquad a(\alpha,\alpha) = +\infty.$$

We note

$$(3.9)\qquad a(\alpha,\beta) = -a(\beta,\alpha) = a(\beta',\alpha')$$

where $(1/\alpha) + (1/\alpha') = (1/\beta) + (1/\beta') = 1$ and $a(\alpha,+\infty) = -\alpha$ while $a(+\infty,\alpha) = \alpha$.

An obvious but useful extension of a well known result is the following:

<u>Proposition 3</u>: Let P and Q be multi-indices with $1 \leq P$ and

$1 \leq Q < +\infty$, $\alpha \in (0,\infty)$ and $K(x,y) \geq 0$ a function, then $K: L^P \to L^Q$ with $||K|| \leq \alpha$ if and only if

$$\iint K(x,y) f(x) g(y)\, d\mu dv \leq \alpha ||f||_P ||g||_{Q'}$$

for all $f \in L_+^P$ and $g \in L_+^{Q'}$.

See [9 p. 429].

Lemma: If R_1, R_2, G and F are measurable functions of $m + n$ variable on $A \times B$ and $1 < Q \leq r \leq P < \infty$ where $r \in (1,\infty)$ then

(3.10)
$$J \equiv \left|\iint G F R_1 R_2 \, d\mu dv\right|$$

$$\leq ||R_1||_{\binom{a(q_1',r'),\, a(q_1',r'),\, \ldots,\, a(q_m',r')}{\mu \quad ,\quad \nu_1 \quad ,\ldots,\quad \nu_m}}$$

$$\cdot\ ||R_2||_{\binom{a(p_1,r),\, a(p_1,r),\, \ldots,\, a(p_n,r)}{\nu \quad ,\quad \mu_1 \quad ,\ldots,\quad \mu_n}}$$

$$\cdot\ ||G||_{\binom{q_1',\, q_1',\, \ldots,\, q_m'}{\mu,\, \nu_1,\, \ldots,\, \nu_m}}$$

$$\cdot\ ||F||_{\binom{p_1,\, p_1,\, \ldots,\, p_n}{\nu_1,\, \mu_1,\, \ldots,\, \mu_n}}$$

Proof: By the generalized Holder inequality

$$J \leq ||R_1||_{\binom{a(q_1',r'),\, a(q_1',r'),\, \ldots,\, a(q_m',r')}{\mu \quad ,\quad \nu_1 \quad ,\ldots,\quad \nu_m}}$$

$$\cdot\ ||G||_{\binom{q_1',\, q_1',\, \ldots,\, q_m'}{\mu,\, \nu_1,\, \ldots,\, \nu_m}}$$

$$\cdot\ ||R_2 F||_{\binom{r,\, r}{\mu,\, \nu}}$$

since

$$[(1/r') - (1/q_j')] + (1/q_j') + (1/r) = 1.$$

By applying Tonelli's theorem followed by the generalized Holder inequality we get,

$$||R_2F||_{\binom{r,r}{\mu,\nu}} = ||R_2F||_{\binom{r,r}{\nu,\mu}}$$

$$\leq ||R_2||_{\binom{a(p_1,r),a(p_1,r),\ldots,a(p_n,r)}{\nu,\mu_1,\ldots,\mu_n}}$$

$$\cdot ||F||_{\binom{p_1,p_1,\ldots,p_n}{\nu,\mu_1,\ldots,\mu_n}}$$

since

$$[(1/r) - (1/p_1)] + (1/p_1) = (1/r).$$

The Lemma follows immediately.

The following theorem is a generalization of a theorem due to Aronszajn. See Gagliardo [9 p.429].

<u>Theorem 3</u>: Suppose $K(x,y) \geq 0$, $r \in R^1$, $1 < Q \leq r \leq P < \infty$, P and Q are multi-indices and suppose there exists functions $\varphi(x)$, $\varphi_1(x)$, $\psi(y)$, $\psi_1(y)$ all greater than 0 a.e. and a real number $c > 0$ such that

(i) $[K\varphi](y) \leq c\, \psi_1(y)$

(ii) $[K^*\psi](x) \leq c\, \varphi_1(x)$

(iii) $$\|(\psi_1 \psi^{1-r'})^{(1/r')}\|_{\binom{a(q_1{}', r'), \ldots, a(q_m{}', r')}{\nu_1(y_1), \ldots, \nu_m(y_m)}}$$

$$\leq 1$$

(iv) $$\|(\varphi_1 \varphi^{1-r})^{(1/r)}\|_{\binom{a(p_1, r), \ldots, a(p_n, r)}{\mu_1(x_1), \ldots, \mu_n(x_n)}}$$

$$\leq 1.$$

Then

$K: L^P \to L^Q$

is a well defined, bounded integral transformation and

$\|K\| \leq c.$

<u>Proof:</u> First consider the case

$q_1 < r < p_1.$

Let

$$R_1 = (K\varphi)^{\frac{1}{q_1} - \frac{1}{r}} \psi_1^{\frac{1}{q_1{}'}} \psi^{-\frac{1}{r'}}$$

$$R_2 = (K\psi)^{\frac{1}{r} - \frac{1}{p_1}} \varphi_1^{\frac{1}{p_1}} \varphi^{-\frac{1}{r}}$$

and for

$f \in L^{P}$ and $g \in L^{Q'}$,

$$F = K^{\frac{1}{p_1}} \psi^{\frac{1}{p_1}} \varphi_1^{-\frac{1}{p_1}} f$$

$$G = K^{\frac{1}{q_1'}} \varphi^{\frac{1}{q_1'}} \psi_1^{-\frac{1}{q_1'}} g.$$

Since, adding the powers of K, φ and ψ we get

$$\frac{1}{q_1} - \frac{1}{r} + \frac{1}{r} - \frac{1}{p_1} + \frac{1}{p_1} + \frac{1}{q_1'} = 1, \quad \frac{1}{q_1} - \frac{1}{r} - \frac{1}{r'} + \frac{1}{q_1'}$$

$$= 1 - 1 = 0,$$

and

$$-\frac{1}{r} + \frac{1}{r} - \frac{1}{p_1} + \frac{1}{p_1} = 0,$$

we have

$$J \equiv \left|\iint K f g\, dx\, dy\right| = \left|\iint G F R_1 R_2\, dx\, dy\right|$$

$$\leq \|R_1\|_{\binom{a(q_1',r'),\ a(q_1',r'),\ \ldots,\ a(q_m',r')}{\mu(x),\ \nu_1(y_1),\ \ldots,\ \nu_m(y_m)}}$$

$$\cdot \|R_2\|_{\binom{a(p_1,r),\ a(p_1,r),\ \ldots,\ a(p_n,r)}{\nu(y),\ \mu_1(x_1),\ \ldots,\ \mu_n(x_n)}}$$

$$\cdot \|G\|_{\binom{q_1',\ q_1',\ \ldots,\ q_m'}{\mu_1,\ \nu_1,\ \ldots,\ \nu_m}}$$

$$\cdot \|F\|_{\binom{p_1,\ p_1,\ \ldots,\ p_n}{\nu_1,\ \mu_1,\ \ldots,\ \mu_n}}$$

$$\equiv J_1 \cdot J_2 \cdot J_3 \cdot J_4$$

by the above Lemma.

Now

$$J_1 = ||(K\varphi)^{\frac{1}{a(q_1',r')}}[\psi_1^{\frac{1}{q_1'}}\psi^{-\frac{1}{r}}]||_{(a(q_1',r'),a(q_1',r'),\ldots,a(q_n',r'))_{\mu,\nu_1,\ldots,\nu_n}}$$

$$= ||(\int K\varphi d\mu)^{\frac{1}{a(q_1',r')}}[\psi_1^{\frac{1}{q_1'}}\psi^{-\frac{1}{r}}]||_{(a(q_1',r'),\ldots,a(q_n',r'))_{\nu_1,\ldots,\nu_n}}$$

$$\leq c^{\frac{1}{a(q_1',r')}}||\psi_1^{\frac{1}{r'}}\psi^{-\frac{1}{r}}||_{(a(q_1',r'),\ldots,a(q_n',r'))_{\nu_1,\ldots,\nu_n}}$$

(by condition i of the theorem and for the power of ψ_1 we have

$$\frac{1}{q_1'} + \frac{q_1'-r'}{q_1'r'} = \frac{r'+q_1'-r'}{q_1'r'} = \frac{1}{r'} \quad)$$

so $J_1 \leq c^{\frac{1}{r'}-\frac{1}{q_1'}}$ by condition iii of the theorem.

Similary, we can show that

$$J_2 = ||(K\psi)^{\frac{1}{a(p_1,r)}}[\varphi_1^{\frac{1}{p_1}}\varphi^{-\frac{1}{r'}}]||_{(a(p_1,r),a(p_1,r),\ldots,a(p_n,r))_{\nu,\mu_1,\ldots,\mu_n}}$$

$$= ||(\int K\psi dy)^{\frac{1}{a(p_1,r)}}[\varphi_1^{\frac{1}{p_1}}\varphi^{-\frac{1}{r'}}]||_{(a(p_1,r),a(p_2,r),\ldots,a(p_n,r))_{\mu_1,\mu_2,\ldots,\mu_n}}$$

$$\leq c^{\frac{1}{a(p_1,r)}}||\varphi_1^{\frac{1}{r}}\varphi^{-\frac{1}{r'}}||_{(a(p_1,r),\ldots,a(p_n,r))_{\mu_1,\ldots,\mu_n}}$$

(by condition ii of the theorem and since for the powers of φ_1 we have

$$\frac{1}{r} - \frac{1}{p_1} + \frac{1}{p_1} = \frac{1}{r}).$$

So $J_2 \leq c^{\frac{1}{r} - \frac{1}{p_1}}$ by condition iii of the theorem.

But

$$J_3 = ||[(K\varphi)/\psi_1]^{\frac{1}{q_1}'} g||_{\binom{q_1', q_1', \ldots, q_m'}{\mu, \nu_1, \ldots, \nu_m}}$$

$$= ||[(\int K\varphi d\mu)/\psi_1]^{\frac{1}{q_1}'} g||_{\binom{q_1', \ldots, q_m'}{\nu_1, \ldots, \nu_m}}$$

$$\leq c^{\frac{1}{q_1}'} ||g||_{\binom{q_1', \ldots, q_m'}{\nu_1, \ldots, \nu_m}}$$

by condition i of the theorem.

And

$$J_4 \leq c^{\frac{1}{p_1}} ||f||_{\binom{p_1, \ldots, p_n}{\mu_1, \ldots, \mu_n}}.$$

Therefore

$$J \leq c\, ||f||_P ||g||_{Q'}$$

since

$$\frac{1}{r}' - \frac{1}{q_1}' + \frac{1}{r} - \frac{1}{p_1} + \frac{1}{p_1} + \frac{1}{q_1}' = 1$$

and the conclusion follows from proposition 3. The proof of the case $p_1 = r = q_1$ is similar and may be outlined briefly as follows. Using the same F, G, R_1 and R_2,

$$J \leq ||FGR_1R_2||_{(\mu,\nu)}^{(1,1)}$$

$$\leq ||GR_1||_{(\mu\ ,\nu\ )}^{(r',r')} \cdot ||FR_2||_{(\mu,\nu)}^{(r,r)} \text{ by Holder's inequality}$$

$$\leq ||\ |G|^{r'} R_1^{\ r'}\ ||_{(\mu,\nu)}^{(1,1)\,(1/r')} \cdot ||\ |F|^{r} R_2^{\ r}||_{(\nu,\mu)}^{(1,1)\,(1/r)}$$

by Tonelli's theorem

$$\leq ||[K\varphi]\psi_1^{-1}|g|^{r'} R_1^{\ r'}||_{(\nu)}^{(1)\,(1/r')} \cdot$$

$$||[K^*\psi]\varphi_1^{-1}|f|^{r}R_2^{\ r}||_{(\mu)}^{(1)} \quad \text{since } p_1 = r = q_1$$

$$\leq c^{\frac{1}{r'}} \cdot ||\ |g|^{r'} R_1^{\ r'}||_{(\nu)}^{(1)\,(1/r')} \cdot$$

$$c^{\frac{1}{r}}||\ |f|^{r}R_2^{\ r}||_{(\mu)}^{(1)\,(1/r)} \quad \text{by i and ii}$$

$$\leq c^{\frac{1}{r}+\frac{1}{r'}} \cdot ||\ |g|^{r'}||_{(\nu)}^{(Q/r')\,(1/r')} \cdot ||R_1^{\ r'}||_{(\nu)}^{(Q'/(Q'-r'))\,(1/r')}$$

$$\cdot\ ||\ |f|^{r}||_{(\mu)}^{(P/r)\,(1/r)} \cdot ||R_2^{\ r}||_{(\mu)}^{(P/(P-r))\,(1/r)}$$

$$\leq c||g||_{(\nu)}^{(Q')} \cdot ||f||_{(\mu)}^{(P)} \cdot ||R_2||_{(\mu)}^{(a(P,r))} \cdot$$

$$||R_1||_{(\nu)}^{(a(Q',r'))}$$

$$\leq c||g||_Q \cdot ||f||_P \text{ by conditions iii and iv.}$$

Thus, the theorem follows from the preceding proposition. Note that the first iterated norm on R_1 and R_2 and hence for conditions iii and iv is the ∞-norm on x_1 and y_1 respectively∎

CHAPTER FOUR

ORDERING WITHIN MULTI-INDICES

One of the important facts about L^P spaces is that their norms are dependent on the ordering of P. In particular

$$||f||_{\binom{p_1,p_2}{\mu_1,\mu_2}} \quad ,$$

$$||f||_{\binom{p_2,p_1}{\mu_2,\mu_1}} \quad ,$$

and

$$||f||_{\binom{p_2,p_1}{\mu_1,\mu_2}}$$

may all be different. It will be necessary for our further work to develop some notations and rules regarding the permutation of multi-indices and integrals. For σ a permutation of $\{1,\ldots,n\}$, we define

$$(4.1) \qquad \sigma(P) = \sigma(p_1,\ldots,p_n) = (p_{\sigma(1)},\ldots,p_{\sigma(n)})$$

and

$$(4.2) \qquad \sigma\binom{P}{\mu} = \sigma\binom{p_1,\ldots,p_n}{\mu_1,\ldots,\mu_n} = \binom{p_{\sigma(1)},\ldots,p_{\sigma(n)}}{\mu_{\sigma(1)},\ldots,\mu_{\sigma(n)}} \ .$$

Note that $[\sigma(P)]_i = p_{\sigma(i)}$. We will need the following special permutations.

<u>Defintion</u>: Let P be a multi-index. Then

(1) π_P is a permutation on $\{1,\ldots,n\}$ such that

$$[\pi_p(P)]_i \leq [\pi_p(P)]_j$$

whenever $i \leq j$ and if $p_i = p_j$ and $i \leq j$, then

$$\pi_p^{-1}(i) \leq \pi_p^{-1}(j).$$

(2) η_p is the permutation on $\{1,\ldots,n\}$ such that

$$[\eta_p(P)]_i \geq [\eta_p(P)]_j$$

whenever $i \leq j$ and if $p_i = p_j$ and $i \leq j$, then

$$\eta_p^{-1}(i) \leq \eta_p^{-1}(j).$$

Thus, π_p is a permutation that will reorder P in increasing order and η_p reorders P in decreasing order with equal elements left in the same order.

<u>Proposition 4</u>: If P is a multi-index, then

(i) $[\pi_p(P)]' = \eta_{p'}(P') = \pi_p(P')$

and

(ii) $[\eta_p(P)]' = \pi_{p'}(P') = \eta_p(P')$.

Moreover, if $r \in (1,\infty)$ and $r < P$ then

(iii) $\pi_p = \eta_{a(p,r)}$

and

(iv) $\eta_p = \pi_{a(p,r)}$.

<u>Proof</u>: Note if

$$p_i \leq p_j$$

then

$$p_i{}' \geq p_j{}' .$$

Thus, parts (i) and (ii) follow immediately. Suppose $1 < r < p_i < p_j$. Then

$$0 < (1/r) - (1/p_i) < (1/r) - (1/p_j)$$

$$0 < (1/a(p_i,r)) < (1/a(p_j,r))$$

and hence

$$a(p_j,r) < a(p_i,r).$$

Thus, (iii) and (iv) follow immediately∎

We can state a further relationship between the permutation π_P and the multi-index P by using the following.

<u>Definition</u>: Let σ be a permutation on $\{1,\ldots,n\}$ and $P = (p_1,\ldots,p_n)$. Then

(1) $\sigma \downarrow P$ if and only if for all i and $j = 1,\ldots,n$, $i \leq j$ and $p_i \leq p_j$ implies $\sigma^{-1}(i) \leq \sigma^{-1}(j)$.

(2) $\sigma \uparrow P$ if and only if for all i and $j = 1,\ldots,n$, $i \leq j$ and $p_i \geq p_j$ implies $\sigma^{-1}(i) \leq \sigma^{-1}(j)$.

Thus, we notice that $\pi_P \downarrow P$ while $\Pi_P \uparrow P$. Moreover, $\sigma \downarrow P$ if and only if $\sigma \uparrow P'$. Later in Proposition 6 we will see that if $\sigma \downarrow P$ then $L^{\binom{P}{\mu}}$ continuously contains $L^{\sigma\binom{P}{\mu}}$.

Minkowski's inequality can be stated as $||\Sigma f_n||_p \leq \Sigma||f_n||_p$. In the case when p is a single number, we have the Generalized Minkowski's inequality, (4.3) $$||\int f(x,y)\,d\mu(x)||_{\binom{p}{\nu(y)}} \leq \int ||f(x,y)||_{\binom{p}{\nu(y)}}\,d\mu(x)$$

See [4 p.592]. We may restate this as:

<u>Proposition 5</u>: Let K be a measurable function on A x B and $p\in[1,\infty]$, then

(4.4) $$||K||_{\binom{1,p}{\mu,\nu}} \leq ||K||_{\binom{p,1}{\nu,\mu}} \quad .$$

<u>Proof</u>: For p = 1 this is Tonelli's theorem since

$$||K||_{\binom{1,1}{\mu,\nu}} = ||K||_{\binom{1,1}{\nu,\mu}} \quad .$$

For $p = \infty$ we have

$$|K(x,y)| \leq ||K(x,\xi)||_{\binom{\infty}{\nu(\xi)}} \text{ a.e. x and y}$$

so

$$\int_X |K(x,y)|\,d\mu$$

$$\leq \int_X ||K(x,y)||_{\binom{\infty}{\nu(y)}}\,d\mu \quad \text{a.e. y}$$

and hence

$$\|K(x,y)\|_{(\mu,\nu)}^{1,\infty} = \left\|\int_X |K(x,y)|\,d\mu\right\|_{(\nu)}^{\infty}$$

$$\leq \int_X \|K(x,y)\|_{(\nu)}^{\infty}\,d\mu$$

$$= \|K\|_{(\nu(y);\mu(x))}^{\infty,\ 1}.$$

For $1 < P < \infty$ let $g \in L^{P'}$ and $\|g\|_{P'} \leq 1$. Then

$$\int g(y)\int |K(x,y)|d\mu(x)\,d\nu(y) = \int\left(\int g(y)\,|K(x,y)|\,d\nu d\mu\right.$$

$$\leq \int \|K(x,y)\|_{(\nu)}^{(P)}\,d\mu$$

$$= \|K\|_{(\nu,\mu)}^{(P,1)} \quad .$$

Hence

$$\|K\|_{(\mu,\nu)}^{(1,P)} = \sup_{\|g\|_{P'} \leq 1} \int g(y)\left(\int |K(x,y)|\,d\mu\right)d\nu$$

$$\leq \|K\|_{(\nu,\mu)}^{(P,1)} \quad \blacksquare$$

In connection with this proposition, it should be noted that if $P > 1$ and $K(x,y)$ cannot be expressed as $K(x,y) = \varphi(x)\psi(y)$, then the inequality (4.4) is a strict one [10, p. 148].

We can easily derive the following:

Lemma: If $K(x,y)$ is a measurable function on $A \times B$ and $r, q \in [1, +\infty)$ with $1 \leq r \leq q$ then, (4.5) $$\|K\|_{\binom{r,q}{\mu,\nu}} \le \|K\|_{\binom{q,r}{\nu,\mu}} .$$

<u>Proof</u>: Suppose $1 \le r < \infty$. Applying Proposition 5 to $|K|^r$ with $p = q/r$,

$$\|\,|K|^r\|_{\binom{1,\frac{q}{r}}{\mu,\nu}} \le \|\,|K|^r\|_{\binom{\frac{q}{r},1}{\nu,\mu}}$$

and hence

$$\|K\|^r_{\binom{r,q}{\mu,\nu}} \le \|K\|^r_{\binom{q,r}{\nu,\mu}} .$$

If $r = +\infty$, then $q = +\infty$ and hence

$$\|K\|_{\binom{\infty,\infty}{\mu,\nu}} = \|K\|_{\binom{\infty}{\mu\times\nu}} = \|K\|_{\binom{\infty,\infty}{\nu,\mu}}$$ ∎

Now using this lemma and mathematical induction we can extend the theorem to cover the case of P a multi-index and σ a permutation.

<u>Proposition 6</u>: If P is a multi-index, $f \in L^P$ and σ is a permutation such that $\sigma \downarrow P$, then

(4.6) $$\|f\|_{\sigma\binom{P}{\mu}} \le \|f\|_{\binom{P}{\mu}} \text{ and } L^{\binom{P}{\mu}} \subset L^{\sigma\binom{P}{\mu}}$$

and if $\sigma \uparrow P$, then

(4.7) $$\|f\|_{\binom{P}{\mu}} \le \|f\|_{\sigma\binom{P}{\mu}} \text{ and } L^{\sigma\binom{P}{\mu}} \subset L^{\binom{P}{\mu}} .$$

Remark 1: Theorems of this type are generally attributed to Jessen [8 p.530], [10 p. 150,169].

Remark 2: Let $P = (p_1,\ldots,p_n)$, $Q = (q_1,\ldots,q_m)$, σ and $\tilde{\tilde{\sigma}}$ be permutations of $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$ respectively, and T be a bounded linear transformation;

$$T:L^P \to L^Q,\ \tilde{\tilde{\sigma}}\downarrow Q \text{ and } \sigma\uparrow P.$$

Then

$$T:L^{\sigma(P)} \to L^{\tilde{\tilde{\sigma}}(Q)}.$$

Moreover,

$$||T|| \leq \alpha$$

as an operator from

$$L^{\sigma(P)} \to L^{\tilde{\tilde{\sigma}}(Q)}$$

whenever the same is true as an operator from

$$L^P \to L^Q .$$

Proposition 7: Let P and Q be multi-indices, and $K(x,y)\in L^{P',Q}$. Then for the integral transformation K we have $L^P \subset D(K)$ and $KL^P \subset L^Q$. Moreover $K:L^P \to L^Q$ is bounded with norm $||K|| \leq ||K(x,y)||_{P',Q}$.

Proof: Let $f\in L^P$ and $g\in L^{Q'}$. Then

$$|(Tf,g)| = \left|\int K(x,y)\,f(x)\,g(y)\,d\mu(x)\,d\nu(y)\right|$$

$$\leq \ ||K||_{\binom{P',Q}{\mu,\nu}} \ ||f(x)g(y)||_{\binom{P,Q'}{\mu,\nu}}$$

$$\leq \ ||K||_{\binom{P',Q}{\mu,\nu}} \ ||f||_{\binom{P}{\mu}} \ ||g||_{\binom{Q'}{\nu}} \ \blacksquare$$

CHAPTER FIVE

AN EXISTENCE THEOREM

In applying Theorems 1 and 3 to the special case of L^P spaces, we will use some non-linear transformations between L_+^P and $L_+^{P'}$ given by the

<u>Definition</u>: Let $P = (p_1, \ldots, p_n)$ and $1 \leq P < \infty$. For any function $f \in L_+^P$ we define the function $\Gamma_P(f)$ by

$$(5.1) \qquad \Gamma_P(f) = f^{p_1 - 1} \prod_{i=1}^{n-1} ||f||^{p_{i+1} - p_i}_{\binom{p_1, \ldots\ldots, p_i}{\mu_1(x_1), \ldots, \mu_i(x_i)}}$$

with the agreement that $\Gamma_P(f) = 0$ whenever f or any of the above partial norms are equal to zero. We define Γ_Q similarly.

<u>Proposition 8</u>: Let $f \in L_+^P$ with $1 < P < +\infty$ and $\alpha \in [0, +\infty)$. Then

(i) $||\Gamma_P(f)||_{P'} = ||f||_P^{p_n - 1}$

(ii) $\Gamma_P : L_+^P \to L^{P'}$

(iii) $||\Gamma_P(f)||_{P'} \leq 1$ if any only if $||f||_P \leq 1$

(iv) $\Gamma_P(0) = 0$

(v) $\Gamma_P(\alpha f) = \alpha^{p_n - 1} \Gamma_P(f)$

(vi) If $g = \Gamma_P(f)$ then $f = \Gamma_{P'}(g)$

(vii) $\Gamma_P^{-1} = \Gamma_{P'}$

(viii) $\Gamma_P : \overset{\circ}{L}{}_+^P \to \overset{\circ}{L}{}_+^{P'}$ is 1-1, onto, continuous and open

(ix) Γ_P is a power function.

<u>Proof</u>:

(i) We proceed by induction. For n = 1

$$\|\Gamma_p(f)\|_{p'} = \|f^{p_1-1}\|_{p_1'}$$

$$= \|f^{p_1/p_1'}\|_{p_1'}$$

$$= \|f\|_{p_1}^{p_1/p_1'}$$

$$= \|f\|_{p_1}^{p_1-1}$$

Assume the theorem is true for $n = k$ and let f be a function of $k + 1$ variables. Then applying our induction hypothesis to $\|f\|_{p_1}$, we have

$$\Big\| \, \|f\|_{p_1}^{p_2-1} \prod_{i=2}^{k} \Big\| \, \|f\|_{p_1} \Big\|_{(p_2,\ldots,p_i)}^{p_{i+1}-p_i} \Big\|_{(p_2',\ldots,p_{k+1}')}$$

$$= \Big\| \, \|f\|_{p_1} \Big\|_{(p_2,\ldots,p_{k+1})}^{p_{k+1}-1}$$

or

$$\Big\| \, \|f\|_{p_1}^{p_2-1} \prod_{i=2}^{k} \|f\|_{(p_1,\ldots,p_i)}^{p_{i+1}-p_i} \Big\|_{(p_2',\ldots,p_{k+1}')}$$

$$= \|f\|_{(p_1,\ldots,p_{k+1})}^{p_{k+1}-1} .$$

Thus,

$$\|\Gamma_p(f)\|_{p'} =$$

$$\Big\| \, \|f^{p_1-1}\|_{p_1'} \prod_{i=1}^{k} \|f\|_{(p_1,\ldots,p_i)}^{p_{i+1}-p_i} \Big\|_{(p_2',\ldots,p_{k+1}')}$$

$$= \Big\| \, \|f\|_{p_1}^{p_1-1} \|f\|_{p_1}^{p_2-p_1} \prod_{i=2}^{k} \|f\|_{(p_1,\ldots,p_i)}^{p_{i+1}-p_i} \Big\|_{(p_2',\ldots,p_{k+1}')}$$

$$= \Big\| \, \|f\|_{p_1}^{p_2-1} \prod_{i=2}^{k} \|f\|_{(p_1,\ldots,p_i)}^{p_{i+1}-p_i} \Big\|_{(p_2',\ldots,p_{k+1}')}$$

$$= \|f\|_{(p_1,\ldots,p_{k+1})}^{p_{k+1}-1}$$

Now parts (ii), (iii) and (iv) follow immediately from part (i). Since

$$\Gamma_p(\alpha f) = \Gamma_p(f) \cdot \alpha^{p_n - 1 + \sum_{i=1}^{n-1}(p_{i+1} - p_i)}$$

$$= \Gamma_p(f) \cdot \alpha^{p_n - 1}$$

we have part (v). Now for part (vi) we note $g \in L_+^{p'}$ by part (ii) above. We proceed by induction. For n = 1,

$$g = f^{p_1 - 1} = f^{p_1/p_1'}$$

so
$$f = g^{p_1'/p_1} = g^{p_1' - 1}$$

$$= \Gamma_{p'}(g).$$

For n= 2, we have

$$g = f^{p_1 - 1}||f||_{p_1}^{p_2 - p_1} = f^{p_1/p_1'}||f||_{p_1}^{(p_2/p_2') - (p_1/p_1')}$$

so
$$||g||_{p_1'} = ||f^{p_1/p_1'}||_{p_1'}\,||f||_{p_1}^{(p_2/p_2') - (p_1/p_1')}$$

$$= ||f||_{p_1}^{p_1/p_1'}\,||f||_{p_1}^{(p_2/p_2') - (p_1/p_1')}$$

$$= ||f||_{p_1}^{p_2/p_2'}$$

and

$$f^{p_1/p_1'} = g||f||_{p_1}^{(p_1/p_1') - (p_2/p_2')}$$

$$= g||g||_{p_1'}^{(p_2'/p_2)((p_1/p_1') - (p_2/p_2'))}$$

Hence,

$$f = g^{p_1'/p_1} ||g||_{p_1'}^{(p_1'/p_1)(p_2'/p_2)((p_1/p_1')-(p_2/p_2'))}$$

$$= g^{p_1'/p_1} ||g||_{p_1'}^{(p_2'/p_2)-(p_1'/p_1)}$$

$$= g^{p_1'-1} ||g||_{p_1'}^{p_2'-p_1'}$$

$$= \Gamma_P(g).$$

Now assume part (vi) is true for some $n=k \geq 2$. Let

$$P = (p_1, \ldots, p_{k+1})$$

and $f \in L^P$.

Then

$$||g||_{\binom{p_1'}{\mu_1}} = ||f^{p_1/p_1'}||_{p_1'} \prod_{i=1}^{k} ||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i}$$

$$= ||f||_{p_1}^{p_1/p_1'} ||f||_{p_1}^{(p_2/p_2')-(p_1/p_1')} \prod_{i=2}^{k} ||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i}$$

$$= ||f||_{p_1}^{p_2/p_2'} \prod_{i=2}^{k} ||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i}$$

$$= \Gamma_{p_2,p_3,\ldots,p_{k+1}}(||f||_{\binom{p_1}{\mu_1}})$$

Hence,

$$g = f^{p_1/p_1'} ||f||_{p_1}^{p_2-p_1} \prod_{i=2}^{k} ||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i}$$

$$= f^{p_1/p_1'} ||f||_{p_1}^{(p_2/p_2')-(p_1/p_1')} ||f||_{p_1}^{-(p_2/p_2')} ||g||_{p_1'}$$

$$= f^{p_1/p_1'} ||f||_{p_1}^{-(p_1/p_1')} ||g||_{p_1'}$$

$$= f^{p_1/p_1'} [\, ||g||_{p_1}^{p_2'/p_2} \prod_{i=2}^{k} ||g||_{p_1',\ldots,p_i'}^{p_{i+1}'-p_i'}]^{-(p_1/p_1')} ||g||_{p_1'}$$

by the induction hypothesis applied to $||f||_{p_1}$.

Finally, solving for f, we get

$$f = g^{p_1'/p_1} [\, ||g||_{p_1'}^{p_2'/p_2} \prod_{i=2}^{k} ||g||_{p_1,\ldots,p_i'}^{p_{i+1}'-p_i'}] ||g||_{p_1'}^{-(p_1'/p_1)}$$

$$= g^{p_1'/p_1} ||g||_{p_1'}^{(p_2'/p_2)-(p_1'/p_1)} \prod_{i=2}^{k} ||g||_{p_1',\ldots,p_i'}^{p_{i+1}-p_i}$$

$$= g^{p_1'-1} \prod_{i=1}^{k} ||g||_{p_1',\ldots,p_i'}^{p_{i+1}'-p_i'}$$

$$= \Gamma_{p'}(g).$$

Now part (vii) simply restates (vi). The continuity of Γ_p for positive function follows from part (i) since Γ_p preserves convergence almost everywhere and the function norms are smooth on L^P. Thus, (viii) and (ix) follow immediately ∎

Remark 1: We will prove later for $2 < P < \infty$ that Γ_p is Lipschitzian on bounded subsets of $L_+^{\circ P}$.

Remark 2: Part (iii) is the same as (2.11) in the definition of stable functions, etc.

Remark 3: The function Γ_p is not in general monotone. For example, consider Lebsegue measure on $A_1 = (0,2)$ and $A_2 = (0,1)$ with $n = 2$, $p_1 = 2$, $p_2 = 1$,

$$f_1(x_1,x_2) = 1$$

for $x_1 \in A_1$ and $x_2 \in A_2$ and

$$f_2(x_1,x_2) = \begin{cases} 1 \text{ if } x_1 \in (0,1) \\ 2 \text{ if } x_1 \in (1,2) \end{cases} \text{ for } x_2 \in A_2.$$

Then $f_1 \leq f_2$ but

$$\Gamma_{p_1,p_2}(f_1) = (1/\sqrt{2})$$

for all $x_1 \in A_1$ and $x_2 \in A_2$ while

$$\Gamma_{p_1,p_2}(f_2)(x_1,x_2) = \begin{cases} (1/\sqrt{5}) \text{ if } x_1 \in (0,1] \\ (2/\sqrt{5}) \text{ if } x_1 \in (1,2) \end{cases}$$

Thus for any $x_2 \in A_2$,

$$\Gamma_{p_1,p_2}(f_2) \leq \Gamma_{p_1,p_2}(f_1) \text{ on } (0,1]$$

but

$$\Gamma_{p_1,p_2}(f_1) \leq \Gamma_{p_1,p_2}(f_2) \text{ on } (1,2) .$$

However, it should be noted that if $p_i \leq p_{i+1}$ for all $i = 1,2,\ldots,$ $n-1$ then Γ_p is non-decreasing. This is easily seen from the definition of Γ_p.

Proposition 9: Let $1 < P < +\infty$. Then $\Gamma_{\pi_p(P)}: \mathring{L}_+^{\pi_p(P)} \to \mathring{L}_+^{\pi_p(P)'}$ is

(i) non-decreasing and continuous,

(ii) stable and strictly stable and

(iii) a power function.

<u>Proof</u>: Note

$$L_+^{P} \subset L_+^{\pi_p(P)}$$

and

$$L_+^{\eta_p'(P)} \subset L_+^{P'}$$

and the imbeddings

$$\tilde{I}_1 : L_+^{P} \to L_+^{\pi_p(P)}$$

and

$$\tilde{I}_2 : L_+^{\eta_p'(P')} \to L_+^{P'}$$

are continuous, non-decreasing and 1-1. Then

$$\Phi \equiv \tilde{I}_2 \circ \Gamma_{\pi_p P} \circ \tilde{I}_1 = \Gamma_{\pi_p P} : L_+^{P} \to L_+^{P'}$$

is continuous and non-decreasing.

Let

$$\beta = -1 + \text{Max}\,\{p_i \,|\, i=1,\ldots,n\}.$$

Then,

$$\|\Phi(f)\|_{P'} \leq \|\Phi(f)\|_{\eta_p'(P')} = \|f\|^{\beta}_{\pi_p(P)} \leq \|f\|^{\beta}_{P}\,.$$

Thus if

$$\|f\|_P \leq 1$$

then,

$$||\Phi(f)||_{p'} \leq 1.$$

Hence Φ is stable and strictly stable, since

$$[\pi_p(P)]_{i+1} - [\pi_p(P)]_i \geq 0$$

for $i = 1,2,\ldots,n-1$.

Also

$$\Phi(\alpha f) = \alpha^{\beta}\Phi(f) \text{ and } \Phi(0) = 0$$

so Φ is a power function.∎

Since $\Gamma_{\pi_p(P)}$ is strictly stable, we can prove a variation of Theorem 1 for L^P spaces which generalizes a theorem due to Gagliardo [9:430]. The following diagram may facilitate the reading of the proof. Note

$$\pi_Q(Q') = \eta_{Q'}(Q')$$

and

$$\eta_P(P') = \pi_{p'}(P').$$

Let

$$\Psi = \Gamma_{\pi_Q(Q)}$$

$$\Phi = \Gamma_{\eta_P(P')}$$

$$
\begin{array}{ccc}
L_+^{P} & \xrightarrow{\quad T \quad} & L_+^{Q} \\
\tilde{I}\uparrow & & \downarrow\tilde{I} \\
L_+^{\eta_P(P)} & & L_+^{\pi_Q(Q)} \\
\Phi\uparrow & & \downarrow\Psi \\
L_+^{\eta_P(P')} & & L_+^{\pi_Q(Q')} \\
\tilde{I}\uparrow & & \downarrow\tilde{I} \\
L_+^{P'} & \xleftarrow{\quad T^* \quad} & L_+^{Q'}
\end{array}
$$

<u>Theorem 4</u>: Let $1 < P < \infty$ and $1 < Q < \infty$,

$$\Phi = \Gamma_{\eta_P(P')} \quad \text{and} \quad \Psi = \Gamma_{\pi_Q(Q)} \quad .$$

If $T: L^P \to L^Q$ is a positive, bounded linear transformation with $||T|| \le c$, then for every $\epsilon > 0$ there exist functions $\varphi \in L_+^{\circ P}$ and $\psi \in L_+^{\circ Q'}$ such that

(i) $T\varphi(y) \le (c + \epsilon)\Psi^{-1}(\psi)(y)$

(ii) $T^*\psi(x) \le (c + \epsilon)\Phi^{-1}(\varphi)(x)$

(iii) $||\varphi||_{\eta_P(P)} \le 1$

and

$$||\psi||_{\pi_Q(Q')} \le 1 \quad .$$

Proof: Let $\delta > 0$ be such that

$$c(1+\delta)^2 \leq c+\epsilon$$

and choose $V_o \in L^Q$ and $U_o \in L^{P'}$ such that

$$V_o(y) > 0 \text{ a.e.}$$

but

$$||V_o||_Q \leq \frac{\epsilon}{c+\epsilon}$$

and

$$U_o(x) > 0 \text{ a.e.}$$

but

$$||U_o||_{p'} \leq \frac{\delta}{1+\delta}$$

For $U \in L_+^{P'}$ we define

$$S(U) = \left(\frac{1}{c}\right) T^* \Psi \left(V_o + \left(\frac{1}{c+\epsilon}\right) T \Phi \left(U_o + \left(\frac{1}{1+\delta}\right) U\right)\right)$$

Note $S: L_+^{P'} \to L_+^{P'}$ is a stable function since $(1/c)T^*$, Ψ, $(1/c)T$ and Φ are stable and $||S(U)||_{p'} \leq 1$ whenever $||U||_{p'} \leq 1$ can be checked by a simple computation.

By the lemma to theorem 1

there exists $U \in L_+^{p'}$ such that $U \neq 0$, $||U||_{p'} \leq 1$

and

$$S(U) = \frac{1}{c} T^* \Psi (V_o + \frac{1}{c+\varepsilon} T\Phi(U_o + \frac{1}{1+\delta} U)) \leq (1+\delta)U.$$

Let

$$\varphi = \Phi(U_o + \frac{1}{1+\delta} U)$$

and

$$\psi = \Psi(V_o + \frac{1}{c+\varepsilon} T\varphi).$$

Then $\varphi \geq 0$ a.e. and $\psi > 0$ a.e. since Φ and Ψ are strictly positive.

$$T\varphi = (c+\varepsilon)[\Psi^{-1}(\psi) - V_o] \leq (c+\varepsilon)\Psi^{-1}(\psi),$$

$$T^*\psi \leq c(1+\delta)U$$

$$\leq c(1+\delta)^2[\Phi^{-1}(\varphi) - U_o]$$

$$\leq (c+\varepsilon)\Phi^{-1}(\varphi),$$

$||\varphi||_{\eta_p(P)} \leq 1$ since Φ is a stable function and

$$||U_o + \frac{1}{1+\delta} U||_{\eta_p(P')} \leq ||U_o + \frac{1}{1+\delta} U||_{p'} \leq \frac{\delta}{1+\delta} + \frac{1}{1+\delta} = 1$$

and $||\psi||_{\pi_Q(Q)} \leq 1$ since Ψ is a stable function and

$$||V_o + \frac{1}{c+\varepsilon} T\varphi||_{\pi_Q(Q)} \leq ||V_o + \frac{1}{c+\varepsilon} T\varphi||_Q \leq \frac{\varepsilon}{c+\varepsilon} + \frac{c}{c+\varepsilon} = 1 \blacksquare$$

Remark 4: Condition (iii) of Theorem 4 implies

$$||\varphi||_p \leq 1$$

and

$$||\psi||_{Q'} \leq 1$$

by Proposition 6.

CHAPTER SIX

THE BOUNDEDNESS OF INTEGRAL TRANSFORMATIONS

We will now use the transformations Γ_P to reduce the conditions (i) through (iv) of Theorem 3 to a form similar to that of conditions (i) through (iii) of theorem 4. The resulting theorem is a generalization of a theorem due to N. Aronszajn. See Gagliardo [9 p. 429]. We will need several lemmas related to the simplification of conditions (iii) and (iv) of Theorem 3.

Lemma 1: If

$f(x) > 0$ a.e., $r \in R^1$ and $1 < r \leq P < \infty$

then

$$||(\Gamma_P(f)f^{1-r})^{\frac{1}{r}}||_{(a(p_1,r),\ldots,a(p_n,r))} = ||f||_P^{\frac{p_n-r}{r}} . \tag{6.1}$$

Proof: We will induct on n where $P = (p_1,\ldots,p_n)$. For $n = 1$,

$$||(f^{p_1-1}f^{1-r})^{\frac{1}{r}}||_{\frac{p_1 r}{p_1-r}} = ||f^{(p_1-r)/r}||_{\frac{p_1 r}{p_1-r}}$$

$$= ||f||_{p_1}^{(p_1-r)/r} .$$

Again for $n = 2$, we have

$$||(f^{p_1-1}||f||_{p_1}^{p_2-p_1}f^{1-r})^{\frac{1}{r}}||_{(\frac{p_1 r}{p_1-r},\frac{p_2 r}{p_2-r})}$$

$$= \| \, \|f^{(p_1-r)/r}\|_{\frac{p_1 r}{p_1-r}} \; \|f\|_{p_1}^{(p_2-p_1)/r} \|_{\frac{p_2 r}{p_2-r}}$$

$$= \| \, \|f\|_{p_1}^{(p_1-r)/r} \|f\|_{p_1}^{(p_2-p_1)/r} \|_{\frac{p_2 r}{p_2-r}}$$

$$= \| \, \|f\|_{p_1}^{(p_2-r)/r} \|_{\frac{p_2 r}{p_2-r}}$$

$$= \|f\|_{p_1,p_2}^{(p_2-r)/r} \, .$$

Now assume the theorem is true for $n = k$. Then

$$\|(f^{p_1-1}[\prod_{i=1}^{k} \|f\|_{p_1,\ldots,p_i}^{p_{i+1}-p_i}]f^{1-r})^{\frac{1}{r}}\|_{(a(p_1,r),\ldots,a(p_{k+1},r);\ \mu_1,\ldots,\mu_{k+1})}$$

$$= \| \, \|f^{(p_1-r)/r}\|_{\frac{p_1 r}{p_1-r}} \prod_{i=1}^{k} (\|f\|_{p_1,\ldots,p_i}^{p_{i+1}-p_i})^{\frac{1}{r}}$$

$$\|_{(a(p_2,r),\ldots,a(p_{k+1},r);\ \mu_2,\ldots,\mu_{k+1})}$$

$$= \| \, \|f\|_{p_1}^{(p_1-r)/r} \|f\|_{p_1}^{(p_2-p_1)/r} \cdot$$

$$\prod_{i=2}^{k} (\|f\|_{p_1,\ldots,p_i}^{p_{i+1}-p_i})^{\frac{1}{r}} \|_{(a(p_2,r),\ldots,a(p_{k+1},r);\ \mu_2,\ldots,\mu_{k+1})}$$

$$= \| ([\|f\|_{p_1}^{p_2-1} \prod_{i=2}^{k} \|f\|_{p_1,\ldots,p_1}^{p_{i+1}-p_i}]$$

$$\|f\|_{p_1}^{1-r})^{\frac{1}{r}} \|_{\left(\substack{a(p_2,r),\ldots,a(p_{k+1},r)\\ \mu_2,\ldots,\mu_{k+1}}\right)}$$

$$= \| \; \|f\|_{p_1} \|_{p_2,\ldots,p_{k+1}}^{(p_{k+1}-r)/r}$$

by the induction hypothesis is applied to $\|f\|_{\binom{p_1}{\mu_1}}$,

$$= \|f\|_{p}^{(p_{k+1}-r)/r} .$$

We note that the above statements hold immediately for $p_i > r$ and each step will also hold if some $p_i = r$ since we take $f^\circ = 0$ when $f = 0$ and $f^\circ = 1$ otherwise and $a(r,r) = +\infty$ to give the sup norm∎

Corollary: If $f(x) > 0$, $r \in R^1$ and $1 < Q \leq r < \infty$, then

$$(6.2) \qquad \|(\Gamma_Q{}'(f) f^{1-r'})^{\frac{1}{r'}}\|_{\left(\substack{a(q_1',r'),\ldots,a(q_m',r')\\ \nu_1,\ldots,\nu_m}\right)}$$

$$= \|f\|_{\binom{Q'}{\nu}}^{(q_m'-r')/r'}$$

Theorem 5: Suppose $K(x,y) \geq 0$ a.e., P and Q are multi-indices,

$1 < Q \leq P < \infty$, $c > 0$ is a real number, $\Phi = \Gamma_{P'}$ and $\Psi = \Gamma_Q$. If there exist functions $\varphi \in L_+^P$ and $\psi \in L_+^Q$ such that $\varphi(x) > 0$ and $\psi(y) > 0$ a.e. and for the integral transformation with kernel K,

(i) $[K\varphi](y) \leq c\Psi^{-1}(\psi)(y)$

(ii) $[K^*\psi](x) \leq c\Phi^{-1}(\varphi)(x)$

(iii) $||\varphi||_P \leq 1$ and $||\psi||_{Q'} \leq 1$

then the kernel K defines a bounded linear transformation $K: L^P \to L^Q$ with $||K|| \leq c$.

Proof: Since $Q \leq P$, we can choose $r \in R^1$ such that $1 < Q \leq r \leq P < \infty$ and apply Theorem 3 with $\psi_1 = \Psi^{-1}(\psi)$ and $\varphi_1 = \Phi^{-1}(\varphi)$. The preceding lemmas show that conditions (iii) above imply conditions (iii) and (iv) of Theorem 3 so $||K|| \leq c$ ∎

Remark 1: For $P < Q$, Theorem 5 can be false as indicated in Remark 1.IV of Gagliardo [9 p. 430]. It should be remembered in connection with Theorem 5, that $\Phi^{-1} = \Gamma_P$ and $\Psi^{-1} = \Gamma_{Q'}$.

If we consider Theorem 5 and note that the theorem remains true if we replace c by $c + \epsilon$ for any $\epsilon > 0$, then under the additional assumptions that $\pi_P(P) = P$ and $\pi_Q(Q) = Q$, we see that Theorem 4 can be used to prove a converse to Theorem 5. Thus, we have immediately, Proposition 10: Suppose $K(x,y) \geq 0$, P and Q are multi-indices $1 < Q \leq P < \infty$, the p_i's are decreasing and the q_j's are increasing, $c > 0$ is a real number, $\Phi = \Gamma_{P'}$ and $\Psi = \Gamma_Q$. Then the integral transformation with kernel K is defined on L^P and $K: L^P \to L^Q$ is bounded with $||K|| \leq c$ if and only if the conditions (i) through (iii) of Theorem 4 are satisfied for all $\epsilon > 0$.

The relationship of Theorems 4 and 5 may be expressed as follows. If the positive integral transformation K is such that $K: L^P \to L^Q$ with $||K|| \leq c$, then Theorem 4 asserts the existence of the functions φ and ψ needed to apply Theorem 5 to prove the weaker statement that $K: L^{\eta_P(P)} \to L^{\pi_Q(Q)}$ with $||K|| \leq c$. From the diagram immediately preceding Theorem 4 we see that

$$L^{\eta_P(P)} \xrightarrow{\tilde{I}} L^P \xrightarrow{K} L^Q \xrightarrow{\tilde{I}} L^{\pi_Q(Q)} .$$

In the next section we will return to the question of finding a converse and more general theorems furnishing partial converses for Theorem 5.

To conclude this section, we will give some remarks and a theorem that may be useful in applying Theorem 5 to prove that a given integral transformation with positive kernel is bounded with bound not exceeding c.

Remark 2: In Theorem 5 if Q = P = r, then conditions iii) can be deleted and the theorem will still be true. This can be seen by considering the proof of Theorem 3 and the fact that

$$(f^{(r/r')} f^{1-r})^{\frac{1}{r}} = f^{\frac{1}{r'}} f^{\frac{1}{r}-1} = f^{\frac{1}{r'}+\frac{1}{r}-1}$$

$$= f^{\circ} \leq 1.$$

The hypothesis of Theorem 5 may be difficult to check when a large number of variables are involved. If n = 1 or m = 1, then Theorem 5 is easier and Theorem 3 may be simplified sufficiently to become useful by setting $p_1 = r$ or $q_1 = r$ respectively. We will now establish some sufficient conditions for boundedness of an integral transformation that may be much easier to use, when applicable. By induction we can easily prove the

Lemma 2: If $\varphi_i \in L_+^{\circ p_i}$ for i = 1,...,n and

$$\varphi(x) = \prod_{i=1}^{n} \varphi_i(x_i) ,$$

then

$$\Gamma_P(\varphi)(x) = C_1 \prod_{i=1}^{n} \varphi_i^{(p_i/p_i')}(x_i) \tag{6.3}$$

where

$$C_1 = \prod_{i=1}^{n-1} ||\varphi_i||_{p_i}^{p_n - p_i} .$$

Proof: We will induct on n where $P = (p_1,\ldots,p_n)$.

For $n = 1$ we have

$$\Gamma_P(\varphi) = 1 \cdot \varphi_1^{p_1/p_1'}$$

so $C_1 = 1$.

For $n = 2$ we have

$$\Gamma_P(\varphi) = \varphi^{p_1/p_1'} ||\varphi||_{p_1}^{(p_2/p_2') - (p_1/p_1')}$$

$$= \varphi_1^{p_1/p_1'} \varphi_2^{p_1/p_1'} ||\varphi_1||_{p_1}^{(p_2/p_2') - (p_1/p_1')} \varphi_2^{(p_2/p_2') - (p_1/p_1')}$$

$$= ||\varphi_1||_{p_1}^{p_2 - p_1} \varphi_1^{p_1/p_1'} \varphi_2^{p_2/p_2'}$$

so $C_1 = ||\varphi_1||_{p_1}^{p_2 - p_1}$.

Assuming the theorem for $n = k$, let $P = (p_1,\ldots,p_{k+1})$.

$$\Gamma_P(\varphi) = \Gamma_P\left(\varphi_{k+1} \prod_{i=1}^{k} \varphi_i\right)$$

$$= \Gamma_P\left(\prod_{i=1}^{k} \varphi_i\right) \cdot \varphi_{k+1}^{p_k - 1} ||\varphi||_{(p_1,\ldots,p_k)}^{p_{k+1} - p_k}$$

$$= \left[\prod_{i=1}^{k-1} ||\varphi_i||_{p_i}^{p_k - p_i}\right]\left[\prod_{i=1}^{k} \varphi_i^{p_i/p_i'}\right] \varphi_{k+1}^{p_k/p_k'} .$$

$$\cdot \left[\prod_{i=1}^{k} ||\varphi_i||_{p_i} \right]^{p_{k+1}-p_k} \cdot \varphi_{k+1}^{p_{k+1}-p_k}$$

$$= \left[\prod_{i=1}^{k} ||\varphi_i||_{p_i}^{p_{k+1}-p_i} \right] \prod_{i=1}^{k+1} \varphi_i^{p_i/p_i'}$$

using the definition of Γ_p, the induction hypothesis and the fact that

$$p_{k+1}-p_k = (p_{k+1}/p_{k+1}') - (p_k/p_k').$$

Hence

$$C_1 = \prod_{i=1}^{k} ||\varphi_i||_{p_i}^{p_{k+1}-p_i}$$

∎

<u>Theorem 6</u>: Suppose $K(x,y) \geq 0$, P and Q are multi-indices, $1 < Q \leq P < \infty$ and $c > 0$ is a real number. Let σ and $\tilde{\sigma}$ be any permutations on $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$. If for some $c > 0$, there exists functions $\varphi_i > 0$ a.e. for $i=1,\ldots,n$ and $\psi_j > 0$ a.e. for $j=1,\ldots,m$ such that the integral transform K with kernal $K(x,y)$ and the functions

$$\varphi(x) = \prod_{i=1}^{n} \varphi_i(x_i), \text{ and } \quad \psi(y) = \prod_{j=1}^{m} \psi_j(y_j)$$

satisfy

(i) $K\varphi(y) \leq c \prod_{j=1}^{m} \psi_j(y_j)^{q_j'/q_j}$

(ii) $K^*\psi(x) \leq c \prod_{i=1}^{n} \varphi_i(x_i)^{p_i/p_i'}$

(iii) $||\varphi_i||_{p_i} \leq 1$ and $||\psi_j||_{q_j} \leq 1$ $\}$ $\begin{matrix} i=1,\ldots,n \\ j=1,\ldots,m \end{matrix}$

Then $K: L^{\sigma}(P) \to L^{\tilde{\sigma}}(Q)$ is a bounded integral transformation with $||K|| \leq c$.

Proof: By Jessen's theorem, Proposition 6 above, if this theorem is true for $\sigma = \eta_p$ and $\tilde{\sigma} = \pi_Q$, then it is true for all σ and $\tilde{\sigma}$. Thus, without loss of generality, we assume $\sigma = \eta_p$ and $\tilde{\sigma} = \pi_Q$. Note that conditions iii) and iv) above immediately imply the corresponding conditions of Theorem 5. Let

$$\beta = \operatorname{Min}\{p_i \,|\, i=1,\ldots,n\} = [\eta_p(P)]_n .$$

Then by lemma 3, condition ii) above implies condition ii) of Theorem 5 since the constant

$$C_1 = \prod_{i=1}^{n-1} ||\varphi_i||_{p_i}^{\beta - p_i} \geq 1$$

because $\beta \leq p_i$ for $i=1,\ldots,n-1$. We can prove that condition i) above implies condition i) of Theorem 5 in a similar manner. Thus, this theorem follows immediately from Theorem 5 ∎

CHAPTER SEVEN

FURTHER EXISTENCE THEOREMS

We will now present some results that are similar in form in Theorem 1.

<u>Theorem 7</u>: Let X and Y be nontrivial Banach lattices and

$$\Phi: X_+^{\circ *} \to X_+^{\circ} \quad \text{and} \quad \Psi: Y_+^{\circ} \to Y_+^{\circ *}$$

be continuous, 1 to 1, positive or strictly positive functions such that

$$||u||_{X^*} \leq 1$$

implies

$$||\Phi(u)||_X \leq 1$$

and

$$||v||_Y \leq 1$$

implies

$$||\Psi(v)||_{Y^*} \leq 1.$$

Let $T: X \to Y$ be a compact, positive linear transformation such that $||T|| \leq c$. Then for all real $\epsilon > 0$ there exist $\varphi \in X_+^{\circ}$ and $\psi \in Y_+^{\circ *}$ such that

(i) $T\varphi \leq (c+\epsilon)\Psi^{-1}(\psi)$

(ii) $T^*\psi \leq (c+\epsilon)\Phi^{-1}(\varphi)$

(iii) $||\varphi||_X \leq 1$ and $||\psi||_{Y^*} \leq 1$

(iv) $\varphi \neq 0$ and $\psi \neq 0$.

Proof: Let $\delta > 0$ be a real number such that

$$c(1+\delta)^2 \leq c+\epsilon$$

and choose

$$v_o \in \overset{\circ}{Y}_+$$

and $$u_o \in \overset{\circ}{X}{}^*_+$$

such that

$$||v_o||_Y \leq \frac{\epsilon}{c+\epsilon}$$

and

$$||u_o||_{X^*} \leq \frac{\delta}{1+\delta} \quad .$$

For $u \in \overset{\circ}{X}{}^*_+$ define

$$S(u) = \frac{1}{c} T^*\Psi(v_o + \frac{1}{c+\epsilon} T\Phi(u_o + \frac{1}{1+\delta} u))$$

Now S is continuous, positive and if $||u||_{X^*} \leq 1$ then $||Su||_{X^*} \leq 1$. Let $w \in \overset{\circ}{X}{}^*_+$ be such that $||w||_{X^*} \leq \frac{\delta}{1+\delta}$ and define u_n and v_n as follows for $n = 1,2,\ldots,$:

$$u_1 = w,\ v_1 = S(w),\ u_n = w + \frac{1}{1+\delta} v_{n-1},\ \text{and}\ v_n = S(u_n).$$

Note for $n = 1,2,\ldots$ we have $||u_n|| \leq 1$, $||v_n|| \leq 1$ and $u_n \geq w$.

Thus there exists $v \in X_+^*$ such that some subsequence $v_{n_k} \to v$ since this sequence is the image of a bounded sequence under the compact operator $(1/c)T^*$. Note $||v||_{x^*} \leq 1$. Define

$$u = w + \frac{1}{1+\delta}v. \quad \text{Then } u_{n_k} \to u \text{ and by continuity}$$

of S,

$$u = w + \frac{1}{1+\delta} S(u)$$

or

$$(1+\delta)u - S(u) = (1+\delta)w \geq 0$$

where

$$||u||_{x^*} \leq 1 \text{ and } u \in \overset{\circ}{X}{}_+^*.$$

Using

$$(1+\delta)u \geq S(u)$$

with

$$\varphi = \Phi(u_o + \frac{1}{1+\delta} u)$$

and

$$\psi = \Psi(v + \frac{1}{c+\epsilon} T\varphi)$$

we obtain the conclusion of this theorem exactly as in Theorem 4 ∎

Thus, if the positive integral transformation $K:L^P \to L^Q$ with $||K|| \leq c$ is compact then theorem 7 asserts the existence of the functions φ and ψ needed to apply theorem 5 to prove that $K:L^P \to L^Q$ is bounded with $||K|| \leq c$.

Another method of trying to prove theorems similar to theorem 1 might be to use a Lipschitz condition. Rather than using Lemma 1 before theorem 1, we will consider the following:

Lemma 1: Let X be a non-trivial ordered Banach lattice and $\Phi: X_+ \to X$ be a power function (Φ not necessarily non-decreasing) with $Ex(\Phi) = q$ which is Lipschitzian on any set of the form

(7.1) $$A_w = \{u \in \overset{\circ}{X}_+ \mid u \geq w \text{ and } ||u|| \leq 1\}$$

where

$$w \in \overset{\circ}{X}_+ \text{ and } ||w|| \geq \frac{1}{2} \quad .$$

Thus, there exists α_w such that for any $u, v \in A_w$,

(7.2.) $$||\Phi u - \Phi v|| \leq \alpha_w ||u-v||.$$

If any one of the following conditions holds,

(i) $q > 1$

(ii) $\alpha_w \leq \alpha$ for all $w \in \overset{\circ}{X}_+$, $||w|| \geq \frac{1}{2}$

(iii) for each $r \in (0,1)$ there exists $w \in \overset{\circ}{X}_+$ such that

$$||w|| \leq r\alpha_w^{(1/(q-1))}$$

then for every $\delta > 0$ there exists $u \in \overset{\circ}{X}_+$ such that

$$||u|| \leq 1, \quad u \geq 0, \quad u \neq 0 \text{ and} .$$

(7.3) $(1+\delta)u \geq \Phi(u)$.

<u>Proof:</u> Choose $u_1 \in X_+^\circ$ such that

$$||u_1|| \leq \frac{\delta}{1+\delta} \quad \text{and } u_1 \neq 0.$$

Let $A = A_{u_1}$ and $\alpha = \alpha_{u_1}$ as above so $||\Phi(u)-\Phi(v)|| \leq \alpha||u-v||$ for all $u, v \in A$. Note that we can assume $\delta \in (0,1)$. Let

$$\beta = \text{Min}\{\frac{1}{\alpha}, 1\}.$$

Define the sequence $\{u_n\}$ by,

$$u_n = u_1 + \frac{\beta}{1+\delta}\Phi(u_{n-1}) \quad \text{for } n = 2,3,\ldots .$$

Note

$$||u_1|| \leq \frac{\delta}{1+\delta} \leq 1$$

and if

$$||u_{n-1}|| \leq 1$$

then

$$||u_n|| \leq ||u_1|| + \frac{\beta}{1+\delta}||\Phi(u_{n-1})|| \leq \frac{\delta}{1+\delta} + \frac{1}{1+\delta} = 1.$$

Also

$$u_n \geq u_1.$$

Thus

$$u_n \in A \quad \text{for } n = 1,2,\ldots .$$

Now

$$||u_{n+1}-u_n|| = \frac{\beta}{1+\delta}\,||\Phi(u_n)-\Phi(u_{n-1})|| \leq \frac{\frac{1}{\alpha}}{1+\delta}\,\alpha||u_n-u_{n-1}||$$

$$\leq \frac{1}{1+\delta}\,||u_n-u_{n-1}|| \text{ for } n > 2.$$

Since

$$||u_2-u_1|| \leq \frac{\beta}{1+\delta}\,||\Phi u_1|| \leq \frac{1}{1+\delta}$$

we have

$$||u_{n+1}-u_n|| \leq \frac{1}{(1+\delta)^n}$$

Thus clearly $\{u_n\}$ is convergent so there exists $u \in A$ such that $u_n \to u$. Since Φ is continuous,

$$u = u_1+\frac{\beta}{1+\delta}\,\Phi(u) \text{ or } \frac{1}{\beta}(1+\delta)u-\Phi(u)= \frac{1}{\beta}(1+\delta)u_1 \geq 0.$$

If $\beta = 1$ (ie $\alpha \leq 1$) we are through. Thus, assume $\alpha > 1$. Let $Ex(\Phi) = q$ so that for any real number $C > 0$ we have $\Phi(Cv) = C^q\Phi(v)$. Hence if $v = \alpha^{\frac{1}{1-q}}u$

then

$$(1+\delta)v \geq \Phi(v)$$

since

$$\alpha\,\alpha^{\frac{-1}{1-q}}(1+\delta)v - \alpha^{\frac{-q}{1-q}}\,\Phi(v) = \alpha(1-\delta)u_1 \geq 0.$$

Note $v \geq 0$ and $v \neq 0$ and in particular $v \in \mathring{X}_+$. Thus all we need

to prove is that $||v|| \leq 1$. But $\alpha > 1$ so if $q > 1$, then $\alpha^{\frac{1}{1-q}} < 1$.
Hence

$$||v|| \leq \alpha^{\frac{1}{1-q}} \; ||u|| \leq 1.$$

for $q > 1$.

On the other hand, if $\alpha_w \leq \alpha$ for all $w \in \overset{\circ}{X}_+$ with $||w|| \leq \frac{1}{2}$,

then we can choose $\alpha_w = \alpha$ and u_1 such that it also satisfies the additional conditions

$$||u_1|| \leq \alpha^{\frac{1}{q-1}} \frac{\delta}{1+\delta}$$

and

$$||\Phi u_1|| \leq \alpha^{\frac{q}{q-1}} \frac{\delta}{1+\delta}$$

since

$$||\Phi(w)|| = ||w||^q$$

for all $w \in \overset{\circ}{X}_+$. Thus,

$$||u_2 - u_1|| = \frac{\beta}{1+\delta} \; ||\Phi(u_1)|| \leq \frac{1}{1+\delta} \alpha^{-1} \alpha^{\frac{q}{q-1}} \frac{\delta}{1+\delta} = \alpha^{\frac{1}{q-1}} \frac{\delta}{(1+\delta)^2}$$

and

$$||u_{n+1} - u_n|| \leq \frac{\beta}{1+\delta} \; ||\Phi(u_n) - \Phi(u_{n-1})|| \leq \frac{1}{1+\delta} ||u_n - u_{n-1}|| \;..$$

$$\leq \alpha^{\frac{1}{q-1}} \frac{\delta}{(1+\delta)^{n+1}}$$

while

$$||u_1|| \leq \alpha^{\frac{1}{q-1}} \frac{\delta}{1+\delta}$$

Hence

$$||u|| \leq ||u_1|| + \sum_{n=1}^{\infty} ||u_{n+1}-u_n||$$

$$\leq \alpha^{\frac{1}{q-1}} \left(\frac{\delta}{1+\delta} + \sum_{n=1}^{\infty} \frac{\delta}{(1+\delta)^{n+1}}\right)$$

$$\leq \alpha^{\frac{1}{q-1}}$$

and therefore

$$||v|| \leq \alpha^{\frac{1}{1-q}} ||u|| \leq 1.$$

Finally, if condition iii) holds then the last computations are again valid∎

Thus, using essentially the same proof as that of Theorem 1 we can prove:

<u>Proposition 11</u>: Let X and Y be nontrivial Banach lattices and

$\Phi : X_+^* \to X_+$ and $\Psi : Y_+ \to Y_+^*$

be power functions, with $Ex(\Psi) = q$ and $Ex(\Phi) = p'$, which are Lipschitzian on any set

$A_w = \{u \in X_+^\circ \mid u \geq w \text{ and } ||u|| \leq 1\}$

where $w \in X_+^\circ$ and $||w|| \leq \frac{1}{2}$. Assume they are Lipschitzian on all of X_+^* and Y_+ or that $p'q > 1$. Let $T: X \to Y$ be a bounded, positive linear transformation with $||T|| \leq c$. Then for every $\varepsilon > 0$ there exist $\varphi \in X_+$ and $\psi \in Y_+^*$ such that

(i) $T\varphi \leq (c+\varepsilon)\Psi^{-1}(\psi)$

(ii) $T^*\psi \leq (c+\varepsilon)\Phi^{-1}(\varphi)$

(iii) $||\varphi||_X \leq 1$ and $||\psi||_{Y^*} \leq 1$

(iv) $\varphi \neq 0$.

Moreover, if Φ, Ψ and T are strictly positive, then $\varphi \in X_+^\circ$ and $\psi \in Y_+^{\circ *}$.

We will now attempt to find some sufficient conditions for Γ_P to be Lipschitzian. These will be obtained by computing the Gateaux differential of Γ_P. We will need the following:

Lemma 2: If $p_1 > 2$ then

$$||f^{p_1-2}g||_{p_1'} \leq ||f||_{p_1}^{p_1-2}\,||g||_{p_1}.$$

Proof:

$$|| \, |f|^{p_1-2} g||_{p_1'} = \{\int |f|^{(p_1-2)p_1'} |g|^{p_1'} dx\}^{(1/p_1')}$$

$$\leq \{||f^{p_1(p_1-2)/(p_1-1)}||_{\frac{p_1-1}{p_1-2}} ||g^{p_1'}||_{p_1-1}\}^{(1/p_1')}$$

$$= \{||f||_{p_1}^{p_1(p_1-2)/(p_1-1)} ||g||_{p_1}^{p_1'}\}^{(1/p_1')}$$

$$= ||f||_{p_1}^{p_1-2} \, ||g||_{p_1}$$

We shall now compute the first Gateaux variation of the transformations Γ_P and then show that this is in fact a Gateaux differential which is uniformly bounded on appropriate sets. For the remainder of this section, let $P = (p_1, \ldots, p_n)$ and for $f \in L^P$ let $\Phi(f) = \mathrm{sgn}(f)\Gamma_P(|f|)$. For $P > 1$ we use elementary calculus to compute

(7.4)
$$\partial\Phi(f,h) = \mathrm{sign}(f)|f|^{p_1-2} \prod_{i=1}^{n-1} ||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i-1} \cdot$$
$$\left[(p_1-1)\left(\prod_{i=1}^{n-1} ||f||_{p_1,\ldots,p_i}\right)h + |f| \sum_{i=1}^{n-1} (p_{i+1}-p_i)\left(\prod_{j=1}^{n-1}{}' ||f||_{p_1,\ldots,p_j}\right) df_i\right]$$

where the prime on the product sign indicates that i-th factor has been deleted and

(7.5)
$$df_i = ||f||_{p_1,\ldots,p_i}^{1-p_i} \iint\cdots\int (\Gamma_{p_1,\ldots,p_i})(|f|)h\,d\mu_1,\ldots,d\mu_i$$

is the Gateaux differential of the partial norm. All we need to know about df_i is that it is linear in h and that $|df_i| \leq ||h||_{p_1,\ldots,p_i}$. However, these facts follow immediately from the formula (7.5).

<u>Lemma 3</u>: If $2 < p_i < \infty$ for all $i=1,2,\ldots,n$, then

(7.7)
$$||\partial\Phi(f,h)||_P \leq ||f||_P^{p_n-2}\left(p_1-1+\sum_{i=1}^{n-1}|p_{i+1}-p_i|\right)||h||_P \; .$$

Proof:

$$||\partial\Phi(f,h)||_{p'}$$

$$\leq ||\prod_{i=1}^{n-1}||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i-1}[(p_1-1)(\prod_{i=1}^{n-1}||f||_{p_1,\ldots,p_i})||f^{p_1-2}h||_{p_1'} +$$

$$||f^{p_1-1}||_{p_1'}\sum_{i=1}^{k}|p_{i+1}-p_i|(\prod_{j=1}^{n-1}{}'||f||_{p_1,\ldots,p_j})]|df_i|||_{p_2',\ldots,p_n'}$$

$$\leq ||\prod_{i=1}^{n-1}||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i-1}[(p_1-1)(\prod_{i=1}^{n-1}{}'||f||_{p_1,\ldots,p_i})||f||_{p_1}^{p_1-2}||h||_{p_1} +$$

$$||f||_{p_1}^{p_1-1}\sum_{i=1}^{n-1}|p_{i+1}-p_i|(\prod_{i=1}^{n-1}{}'||f||_{p_1,\ldots,p_i})]|df_i|||_{p_2',\ldots,p_n'}$$

$$\leq ||\prod_{i=2}^{n-1}||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i-1}[||f||_{p_1}^{p_2-2}((p_1-1)+|p_2-p_1|)(\prod_{i=2}^{n-1}||f||_{p_1,\ldots,p_i})||h||_{p}$$

$$+ ||f||_{p_1}\sum_{i=2}^{n-1}|p_{i+1}-p_i|(\prod_{j=2}^{n-1}{}'||f||_{p_1,\ldots,p_j})]|df_i|\ ||_{p_2',\ldots,p_n'}$$

now continuing by induction, suppose for some k that $1 < k < n$

and $||\partial\Phi(f,h)||_{p'}$

$$\leq ||\prod_{i=k}^{n-1}||f||_{p_1,\ldots,p_i}^{p_{i+1}-p_i-1}[||f||_{p_1,\ldots,p_{k-1}}^{p_k-2}((p_1-1)+\sum_{i=1}^{k-1}|p_{i+1}-p_i|)(\prod_{i=k}^{n-1}||f||_{p_1,\ldots p_i})$$

$$||h||_{p_1,\dots,p_{k-1}} + ||f||^{p_k-1}_{p_1,\dots,p_{k-1}} \sum_{i=k}^{n-1} |p_{i+1}-p_i|\,(\prod_{j=k}^{n-1}{}' ||f||_{p_1,\dots,p_j})\,|df_i|]||_{p_k',\dots,p_n'} .$$

Then

$$||\partial\Phi(f,h)||_{p'}$$

$$\leq ||\prod_{i=k}^{n-1} ||f||^{p_{i+1}-p_i-1}_{p_1,\dots,p_i} [((p_1-1)+\sum_{i=1}^{k-1} |p_{i+1}-p_i|)(\prod_{i=k}^{n-1} ||f||_{p_1,\dots,p_i}) \cdot$$

$$||\ ||f||^{p_k-2}_{p_1,\dots,p_{k-1}} ||h||_{p_1,\dots,p_{k-1}} ||_{p_k'} + ||\ ||f||^{p_k-1}_{p_1,\dots,p_{k-1}} ||_{p_k'}$$

$$\sum_{i=k}^{n-1} |p_{i+1}-p_i|\,(\prod_{i=k}^{n-1}{}' ||f||_{p_1,\dots,p_j})\,|df_i|]||_{p_{k+1}',\dots,p_n'}$$

$$\leq ||\prod_{i=k}^{n-1} ||f||^{p_{i+1}-p_i-1}_{p_1,\dots,p_i} [((p_1-1)+\sum_{i=1}^{k-1} |p_{i+1}-p_i|)(\prod_{i=k}^{n-1} ||f||_{p_1,\dots,p_i}) \cdot$$

$$||f||^{p_k-2}_{p_1,\dots,p_k} ||f||_{p_1,\dots,p_k} + ||f||^{p_k-1}_{p_1,\dots,p_{k-1}} \cdot$$

$$\sum_{i=k}^{n-1} |p_{i+1}-p_i|\,(\prod_{j=k}^{n-1}{}' ||f||_{p_1,\dots,p_j})\,|df_i|]||_{p_{k+1}',\dots,p_n'}$$

$$\leq ||\prod_{i=k+1}^{n-1} ||f||^{p_{i+1}-p_i-1}_{p_1,\dots,p_i} [((p_1-1)+\sum_{i=1}^{k} |p_{i+1}-p_i|)\prod_{i=k+1}^{n-1} ||f||_{p_1,\dots,p_i}$$

$$||f||^{p_k-1}_{p_1,\ldots,p_k}||h||_{p_1,\ldots,p_k} + ||f||^{p_k}_{p_1,\ldots,p_{k-1}} \quad .$$

$$\sum_{i=k+1}^{n-1} |p_{i+1}-p_i|\Big(\prod_{j=k}^{n-1}||f||_{p_1,\ldots,p_j}\Big)|df_i|\Big]\Big|\Big|_{p'_{k+1},\ldots,p'_n}$$

$$\leq \Big|\Big|\prod_{i=k+1}^{n-1}||f||^{p_{i+1}-p_i-1}_{p_1,\ldots,p_i}\Big[\Big((p_1-1)+\sum_{i=1}^{k}|p_{i+1}-p_i|\Big)\prod_{i=k+1}^{n-1}||f||_{p_1,\ldots,p_i}$$

$$||f||^{p_{k+1}-2}_{p_1,\ldots,p_k}||h||_{p_1,\ldots,p_k} + ||f||^{p_k}_{p_1,\ldots,p_{k-1}} \quad .$$

$$\sum_{i=k+1}^{n-1} |p_{i+1}-p_i|\Big(\prod_{j=k+1}^{n-1}||f||_{p_1,\ldots,p_j}\Big)|df_i|\Big]\Big|\Big|_{p'_{k+1},\ldots,p'_n} \quad .$$

Thus inequality (7.7) follows by induction ending with $k+1 = n$ ∎

Now clearly $\partial\Phi(f,h)$ is well defined, linear and continuous in h. Moreover, we can show:

Proposition 12: If $2 < (p_1,\ldots,p_n) < \infty$, then $\partial\Phi(f,h)$ is the Gateaux differential of Φ.

Proof: For some fixed $f,h \in L^P$ consider

$$\Delta(\alpha) \equiv \frac{1}{\alpha}[\Phi(f+\alpha h) - \Phi(f) - \partial\Phi(f,\alpha h)]$$

$$= \frac{1}{\alpha}[\Phi(f+\alpha h) - \Phi(f)] - \partial\Phi(f,h)$$

for α real. By the definition of $\partial\Phi(f,h)$ (see equation 7.5) we see that

$$\lim_{\alpha\to 0} \frac{1}{\alpha}[\Phi(f+\alpha h)(x) - \Phi(f)(x)] = \partial\Phi(f,h)(x) \text{ a.e.}$$

so $\Delta(\alpha) \to 0$ a.e. If we restrict $|\alpha| \leq 1$ then by the mean value theorem,

$$\begin{aligned}\frac{1}{\alpha}[\Phi(f+\alpha h)(x) - \Phi(f)(x)] &\leq \partial\Phi(f,\alpha^* h)\\ &\leq \alpha^*\partial\Phi(f,h)\\ &\leq \partial\Phi(f,h)\end{aligned}$$

where $|\alpha^*| \leq |\alpha| \leq 1$ for almost every x. Hence by the Lebesgue Dominated Convergence theorem [4,p.302] we have $||\Delta(\alpha)||_{P'} \to 0$ proving that $\partial\Phi(f,h)$ is the Gateaux differential of Φ∎

__Proposition 13__: Let $X = L^P$ where $P = (p_1,\ldots,p_n) > 2$, and $w \in X_+^\circ$ such that $||w||_P \leq \frac{1}{2}$. Then the function

$$\Phi(f) = \Gamma_P(f)$$

is defined and Lipschitzian on

$$A_w = \{u \in X_+^\circ \mid u \geq w \text{ and } ||u||_P \leq 1\}$$

<u>Proof</u>: This follows immediately since Φ has a uniformly bounded Gateaux differntial. See Kantorovich and Akilov [12,p.660]. ∎

Thus, we see that Proposition 11 can be applied to prove a theorem similar to Theorem 4 if $p > 2$ and $Q' > 2$ regardless of the ordering within P and Q.

CHAPTER EIGHT

REDUCING THE NUMBER OF VARIABLES

Applying the methods used in theorems 2 and 5 and using the lemmas preceding those theorems, we can prove the following theorem about reducing the number of variables.

<u>Theorem 8</u>: Let $P^{(1)}, P^{(2)}, Q^{(1)}$ and $Q^{(2)}$ be multi-indices for the variables $x^{(1)}, x^{(2)}, y^{(1)}$ and $y^{(2)}$. Suppose $K(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)}) \geq 0$ a.e., $r \in R^1_+$, $1 < P^{(2)}, Q^{(2)} < \infty$, $1 < Q^{(1)} \leq r \leq P^{(1)} < \infty$ and that $K_2(x^{(2)}, y^{(2)}) \geq 0$ a.e. defines a bounded integral transformation $K_2: L^{P^{(2)}} \to L^{Q^{(2)}}$ with $c = ||K_2||$. If there exist functions $\varphi(x^{(1)}, x^{(2)}, y^{(2)})$, $\varphi_1(x^{(1)}, x^{(2)}, y^{(2)})$, $\psi(y^{(1)}, x^{(2)}, y^{(2)})$ and $\psi_1(y^{(1)}, x^{(2)}, y^{(2)})$ all positive almost everywhere, satisfying

(i) $\int K\varphi \, dx^{(1)} \leq K_2\psi_1$

(ii) $\int K\psi \, dy^{(1)} \leq K_2\varphi_1$

(iii) $$||\psi_1^{\frac{1}{r'}} \psi^{-\frac{1}{r}}||_{(a(Q^{(1)\prime}, r')_{y^{(1)}})} \leq 1$$

(iv) $$||\varphi_1^{\frac{1}{r}} \varphi^{-\frac{1}{r'}}||_{(a(P^{(1)}, r)_{x^{(1)}})} \leq 1$$

then $K: L^{P^{(1)}, P^{(2)}} \to L^{Q^{(1)}, Q^{(2)}}$ is bounded with

$||K|| \leq c$. Moreover if $\psi_1 = \Gamma_{Q^{(1)}}{}'(\psi)$ and

$\varphi_1 = \Gamma_{P^{(1)}}(\varphi)$; then we can replace conditions (iii)

and (iv) by

(v) $P^{(1)} = Q^{(1)}$ or both $||\varphi||_{P^{(1)}} \leq 1$

and $||\psi||_{Q^{(1)'}} \leq 1$.

<u>Proof</u>: Let $f \in L_+^{P^{(1)},P^{(2)}}$ and $g \in L_+^{Q^{(1)},Q^{(2)}}$. Define

$$F = K^{1/p_1^{(1)}} \psi^{1/p_1^{(1)}} \varphi_1^{-1/p_1^{(1)}} f$$

$$G = K^{1/q_1^{(1)'}} \varphi^{1/q_1^{(1)'}} \psi_1^{-1/q_1^{(1)'}} g$$

$$R_1 = (K\varphi)^{(1/q_1^{(1)}) - \frac{1}{r}} \psi_1^{1/q_1^{(1)'}} \psi^{-\frac{1}{r}}$$

$$R_2 = (K\psi)^{\frac{1}{r} - (1/p_1^{(1)})} \varphi_1^{1/p_1^{(1)}} \varphi^{-\frac{1}{r}},$$

Then as in theorem 2 we have

$$J = \iiiint K f(x^{(1)}, x^{(2)}) g(y^{(1)}, y^{(2)}) dx^{(1)} dx^{(2)} dy^{(1)} dy^{(2)}$$

$$= \iiiint R_1 R_2 F G dx^{(1)} dy^{(1)} dx^{(2)} dy^{(2)}$$

$$\leq \iint \|R_1\|_{\left(a(q_1{}^{(1)\prime}_{x^{(1)}},\, r'),\ a(Q^{(1)\prime}_{y^{(1)}},\, r')\right)}$$

$$\cdot\ \|R_2\|_{\left(a(p_1{}^{(1)}_{y^{(1)}},\, r),\ a(P^{(1)}_{x^{(1)}},\, r)\right)}$$

$$\cdot\ \|G\|_{\left(q_1{}^{(1)\prime}_{x},\ Q^{(1)\prime}_{y}\right)}$$

$$\cdot\ \|F\|_{\left(p_1{}^{(1)}_{y},\ P^{(1)}_{x}\right)}\ dx^{(2)}\,dy^{(2)}$$

$$\equiv \iint J_1 \cdot J_2 \cdot J_3 \cdot J_4\, dx^{(2)}\,dy^{(2)}$$

where

$$J_1 = \left\| \left(\int K\varphi\, dx^{(1)}\right)^{1/(a(q_1{}^{(1)\prime},\, r'))} [\psi_1{}^{1/q_1{}^{(1)\prime}}\, \psi^{-\frac{1}{r}}] \right\|_{\left(a(Q^{(1)\prime}_{y^{(1)}},\, r')\right)}$$

$$\leq K_2{}^{1/(a(q_1{}^{(1)\prime},\, r'))} \left\|\psi_1{}^{\frac{1}{r'}}\ \psi^{\frac{1}{r}}\right\|_{\left(a(Q^{(1)\prime}_{y^{(1)}},\, r')\right)}$$

and similarly

$$J_2 \leq K_2{}^{1/(a(p_1{}^{(1)},\, r)}\, \left\|\varphi_1{}^{\frac{1}{r}}\ \varphi^{-\frac{1}{r'}}\right\|_{\left(a(P^{(1)}_{x^{(1)}},\, r)\right)}$$

while

$$J_3 = ||[(\int K\varphi dx^{(1)})/\psi_1]^{1/q_1^{(1)\prime}} g||_{(Q^{(1)\prime}_{y^{(1)}})}$$

$$\leq K_2^{1/q_1^{(1)\prime}} ||g||_{(Q^{(1)\prime}_{y^{(1)}})}$$

and similarly

$$J_4 \leq K_2^{1/p_1^{(1)}} ||f||_{(P^{(1)}_{x^{(1)}})} .$$

Hence

$$J \leq \iint K_2 ||\psi_1^{\frac{1}{r'}} \psi^{-\frac{1}{r}}||_{a(Q_1^{(1)\prime}, r')} .$$

$$||\varphi_1^{\frac{1}{r}} \varphi^{-\frac{1}{r'}}||_{a(P^{(1)}, r)} \cdot ||g||_{Q^{(1)\prime}} .$$

$$||f||_{P^{(1)}} dx^{(2)} dy^{(2)} .$$

Thus, if conditions (iii) and (iv) hold, then

$$J \leq \iint K_2 ||g||_{Q^{(1)\prime}} ||f||_{P^{(1)}} dx^{(2)} dy^{(2)}$$

$$\leq c ||f||_{P^{(1)}, P^{(2)}} ||g||_{Q^{(1)\prime}, Q^{(2)\prime}}$$

so $K: L^{P^{(1)},P^{(2)}} \to L^{Q^{(1)},Q^{(2)}}$ is bounded with $||K|| \leq c$.

Let $\psi_1 = \Gamma_{Q^{(1)'}}(\psi)$ and $\varphi_1 = \Gamma_{P^{(1)}}(\varphi)$. Then by lemma 1 to theorem 5 we have

$$J \leq \iint K_2 ||\psi||_{Q^{(1)'}}^{(q^{(1)'}_{m(1)} - r')/r'} ||\varphi||_{P^{(1)}}^{(p^{(1)}_{n(1)} - r)/r} \cdot$$

$$||f||_{P(1)} ||g||_{Q(1)} dx^{(2)} dy^{(2)}$$

$$\leq c||f||_{P(1),P(2)} ||g||_{Q(1)',Q(2)'}$$

if $q^{(1)}_{m(1)} = r = p^{(1)}_{n(1)}$ or both $||\varphi||_{P(1)} \leq 1$ and $||\psi||_{Q(1)} \leq 1$.

Thus, if condition (v) holds then our conclusion follows without checking (iii) and (iv) ∎

<u>Remark 1</u>: In theorem 8 if $P^{(1)} = (\alpha, \alpha, \ldots, \alpha)$ then setting $r = \alpha$ we can replace condition (v) by only $||\psi||_{Q(1)'} \leq 1$. Similarly if $Q^{(1)} = (\alpha, \alpha, \ldots, \alpha)$ then we only need $||\varphi||_{P(1)} \leq 1$.

<u>Remark 2</u>: Theorem 5 is essentially a corollary of theorem 8 if we think of a constant c as an integral transformation from R^1 into R^1 where $c(\alpha) = c \cdot \alpha$ for all $\alpha \in R^1$.

Theorem 8 allows us to reduce the problem of whether $K: L^P \to L^Q$ is bounded to a similar problem with less variables. As examples, we have the following:

<u>Corollary 1</u>: Let $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ be multi-indices of the same length. Suppose $1 < q_i \leq p_i < \infty$ for $i = 1, \ldots, n$ and $K(x,y) \geq 0$ a.e. is a measurable function. Let $K_{n+1} \in (0,\infty)$ be a real constant. If there exist functions

$K_i(x_i, \ldots, x_n,\ y_i, \ldots, y_n) \geq 0$ a.e.,

$\varphi_i(x_i, \ldots, x_n,\ y_{i+1}, \ldots, y_n) > 0$ a.e., and

$\psi_i(x_{i+1}, \ldots, x_n,\ y_i, \ldots, y_n) > 0$ a.e. for $i=1, \ldots, n$

with $K_1 = K$ and satisfying

$$\text{(i)} \quad \int K_i \varphi_i \, dx_i \leq K_{i+1} \psi_i^{\,q_i'-1}$$

$$\text{(ii)} \quad \int K_i \psi_i \, dy_i \leq K_{i+1} \varphi_i^{\,p_i-1}$$

$$\text{(iii)} \quad q_i = p_i \text{ or } ||\varphi_i||_{p_i} \leq 1 \text{ or } ||\psi_i||_{q_i'} \leq 1$$

for all $i=1, \ldots, n$. Then $K_1 : L^{p_1, \ldots, p_n} \to L^{q_1, \ldots, q_n}$ is bounded with $||K_1|| \leq K_{n+1}$. In particular $K: L^P \to L^Q$ is bounded with $||K|| \leq K_{n+1}$.

<u>Remark 3</u>: Note if $P = Q$ then condition (iii) is automatically satisfied. If $i=n$ then $\varphi_i(x_i, \ldots, x_n, y_{i+1}, \ldots, y_n) = \varphi(x_n)$ and similarly for ψ_i.

Corollary 2: Suppose $1 < q \leq p_1 < \infty$ and $p_2 \in (1,\infty)$, and $K(x_1,x_2,y) \geq 0$ a.e. If there exist functions $K_2(x_2) \in L_+^{p_2'}$, $\varphi(x_1,x_2) > 0$ a.e. and $\psi(x_2,y) > 0$ a.e. satisfying

(i) $\int K\varphi dx_1 \leq K_2(x_2)\psi^{q'-1}$

(ii) $\int K\psi dy \leq K_2(x_2)\varphi^{p_1-1}$

(iii) $p_1 = q$ or $||\varphi||_{\binom{p_1}{x_1}} \leq 1$ or $||\psi||_{\binom{q'}{y}} \leq 1$

Then $K: L^{p_1,p_2} \to L^q$ is bounded with $||K|| \leq ||K_2||_{p_2'}$.

Corollary 3: Let P and Q be multi-indices. Assume $1<P,Q<\infty$, $q_1 \leq p_1$, $K(x,y) \geq 0$ a.e., and $K_2(x_2,x_3,\ldots,x_n,y_2,y_3,\ldots,y_m) \geq 0$ a.e. defines a bounded integral transformation $K_2: L^{p_2,\ldots,p_n} \to L^{q_2,\ldots,q_m}$ with $||K_2|| = c$. If for every $K_\varepsilon \in L_+^{p_2,\ldots,p_n,q_2',\ldots,q_m'}$ such that $K_\varepsilon > 0$ a.e. there exist functions $\varphi(x,y_2,\ldots,y_m) > 0$ a.e., and $\psi(x_2,\ldots,x_n,y) > 0$ a.e. satisfying

(i) $\int K\varphi dx_1 \leq (K_2 + K_\varepsilon)\psi^{q_1'-1}$

(ii) $\int K\psi dy_1 \leq (K_2 + K_\varepsilon)\varphi^{p_1-1}$

(iii) $p_1 = q_1$ or $||\varphi||_{p_1} \leq 1$ or $||\psi||_{q_1'} \leq 1$

then $K: L^P \to L^Q$ is bounded with $||K|| \leq c$.