Projective Differential Geometry

of Space Curves

by

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Introduction.

In his study of the projective differential properties of space curves, Wilczynski[#] obtains certain invariants and then finds a geometrical interpretation for the invariants. In this paper we shall use the method, which is essentially that given by E.B. Stouffer,[*] of obtaining a unique or canonical expansion for the curve and then finding the invariants as they arise naturally in the study of the geometrical properties of the curve.

Most of the properties of space curves given in this paper are given by Wilczynski,[#] but the methods of study used in this paper are in many ways simpler than those used by him. Sannia[†] also has treated space curves by methods somewhat similar to those in this paper, using the Fubini method of approach.

[#] Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces".

[*] Stouffer, "Some Canonical Forms and Associated Canonical Expansions", in Bulletin of the American Mathematical Society for May-June, 1928, pages 290-302.

[†] Sannia, Annali di Matematica Pura ed Applicata: (4), vol. 1, pp. 1-18; (4), vol. 3, pp. 1-25.

A Canonical Expansion for the Space Curve.

Given a parametric representation for a space curve

$$y_i = f_i(x), \quad (i = 1,2,3,4),$$

there is a unique fourth order differential equation defining a class of projectively equivalent curves provided that

$$\begin{vmatrix} y_1 & y_1' & y_1'' & y_1''' \\ y_2 & y_2' & y_2'' & y_2''' \\ y_3 & y_3' & y_3'' & y_3''' \\ y_4 & y_4' & y_4'' & y_4''' \end{vmatrix} \neq 0.$$

If this determinant equals zero, the curve is a plane curve. The differential equation may be written in the form

$$(1) \qquad y^{IV} + 4p_1 y''' + 6p_2 y'' + 4p_3 y' + p_4 y = 0,$$

where p_i are functions of x.

To this equation we may apply two general transformations $y = \lambda(x)\, y$, and $\xi = \xi(x)$, which will not change the curve or the form of the equation. The transformations will, however, change the coefficients in (1) into some other functions of x. The functions of the coefficients unchanged by these transformations must then give projective differential properties of the curve. Such functions are called invariants.

If to (1) we apply the transformation $\xi = \xi(x)$, we obtain the equation

(2) $$\bar{y}^{IV} + 4\bar{p}_1\bar{y}''' + 6\bar{p}_2\bar{y}'' + 4\bar{p}_3\bar{y}' + \bar{p}_4\bar{y} = 0,$$

where $\bar{p}_1 = \frac{1}{\xi'}(p_1 + \frac{3}{2}\eta),$

(2') $$\bar{p}_2 = \frac{1}{(\xi')^2}(p_2 + 2p_1\eta + \frac{2}{3}\eta' + \frac{7}{6}\eta^2),$$

$$\bar{p}_3 = \frac{1}{(\xi')^3}\left[p_3 + \frac{3}{2}p_2\eta + (\eta' + \eta^2)p_1 + \frac{3}{4}\eta\eta' + \frac{1}{4}\eta'' + \frac{1}{4}\eta^3\right],$$

$$\bar{p}_4 = \frac{1}{(\xi')^4}p_4, \quad \text{where} \quad \eta = \frac{\xi''}{\xi'}$$

If, in turn, to (2) we apply the transformation $\bar{y} = \lambda\bar{\bar{y}}$, we obtain the equation

(3) $$\bar{\bar{y}}^{IV} + 4\bar{P}_1\bar{\bar{y}}''' + 6\bar{P}_2\bar{\bar{y}}'' + 4\bar{P}_3\bar{\bar{y}}' + \bar{P}_4\bar{\bar{y}} = 0,$$

where $\bar{P}_1 = \frac{1}{\lambda}(\lambda' + \lambda\bar{p}_1),$

(3') $$\bar{P}_2 = \frac{1}{\lambda}(\lambda'' + 2\bar{p}_1\lambda' + \bar{p}_2\lambda),$$

$$\bar{P}_3 = \frac{1}{\lambda}(\lambda''' + 3\bar{p}_1\lambda'' + 3\bar{p}_2\lambda' + \bar{p}_3\lambda),$$

$$\bar{P}_4 = \frac{1}{\lambda}(\lambda^{IV} + 4\bar{p}_1\lambda''' + 6\bar{p}_2\lambda'' + 4\bar{p}_3\lambda' + \bar{p}_4\lambda).$$

If in equation (3), we let $\frac{\lambda'}{\lambda} = -\bar{p}_1$, so that $\bar{P}_1 = 0$, we obtain the equation

(4) $$\bar{\bar{y}}^{IV} + 6\bar{P}_2\bar{\bar{y}}'' + 4\bar{P}_3\bar{\bar{y}}' + \bar{P}_4\bar{\bar{y}} = 0,$$

where $\bar{P}_2 = \bar{p}_2 - \bar{p}_1^2 - \bar{p}_1',$

(4') $$\bar{P}_3 = \bar{p}_3 - 3\bar{p}_1\bar{p}_2 - \bar{p}_1'' + 2\bar{p}_1^3,$$

$$\bar{P}_4 = \bar{p}_4 - 4\bar{p}_1\bar{p}_3 - 6\bar{p}_2\bar{p}_1' + 6\bar{p}_1^2\bar{p}_1' + 6\bar{p}_1^2\bar{p}_2 - \bar{p}_1''' + 3\bar{p}_1'^2 - 3\bar{p}_1^4.$$

These expressions for $\bar{P}_2$, $\bar{P}_3$, $\bar{P}_4$ are unchanged by the $y = \lambda\bar{y}$ transformation and are called seminvariants. By a substitution of (2') in (4'), we obtain the effect of the transformation of the independent variable upon the seminvariants. This effect is given by

$$\bar{P}_2 = \frac{1}{(\xi')^2}\left(P_2 - \frac{5}{6}\mu\right);$$

$$\bar{P}_3 = \frac{1}{(\xi')^3}\left(P_3 - 3\eta P_2 - \frac{5}{4}\mu' + \frac{5}{2}\eta\mu\right), \tag{4''}$$

$$\bar{P}_4 = \frac{1}{(\xi')^4}\left(P_4 - 6\eta P_3 - 9\eta' P_2 + \frac{27}{2}\eta^2 P_2 + \frac{15}{2}\eta\mu' - \frac{15}{2}\eta^2\mu - \frac{3}{2}\mu'' + \frac{21}{4}\mu^2\right),$$

$$\bar{P}_2' = \frac{1}{(\xi')^3}\left(P_2' - 2\eta P_2 - \frac{5}{6}\mu' + \frac{5}{3}\eta\mu\right),$$

$$\bar{P}_2'' = \frac{1}{(\xi')^4}\left(P_2'' - 5\eta P_2' - 2\mu P_2 + 5\eta^2 P_2 - \frac{5}{6}\mu'' + \frac{5}{3}\mu^2 - \frac{25}{6}\mu\eta^2 + \frac{25}{6}\mu'\eta\right),$$

$$\bar{P}_2''' = \frac{1}{(\xi')^5}\Big(P_2''' - 9\eta P_2'' - 5\eta' P_2' - 2\mu P_2' - 2\mu' P_2 + 25\eta^2 P_2' + 10\eta\eta' P_2 - \frac{5}{6}\mu'''$$
$$+ \frac{10}{3}\mu\mu' - \frac{125}{6}\mu'\eta^2 - \frac{25}{3}\mu\eta\eta' + \frac{25}{6}\mu'\eta' + \frac{15}{2}\mu''\eta + 8\eta\mu P_2 - 20\eta^3 P_2 - \frac{20}{3}\eta\mu^2 + \frac{50}{3}\eta^3\mu\Big),$$

$$\bar{P}_3' = \frac{1}{(\xi')^4}\left(P_3' - 3\eta P_2' - 3\eta P_3 - 3\mu P_2 + \frac{15}{2}\eta^2 P_2 - \frac{5}{4}\mu'' + \frac{25}{4}\eta\mu' - \frac{25}{4}\mu\eta^2 + \frac{5}{2}\mu^2\right),$$

$$\bar{P}_4' = \frac{1}{(\xi')^5}\Big(P_4' - 6\eta' P_3 - 6\eta P_3' - 9\eta'' P_2 - 9\eta' P_2' + \frac{27}{2}\eta^2 P_2' + 63\eta\eta' P_2 + \frac{27}{2}\eta\mu'' + \frac{15}{2}\eta'\mu'$$
$$- \frac{75}{2}\eta^2\mu' - 15\eta\eta'\mu - \frac{3}{2}\mu''' + \frac{21}{2}\mu\mu' - 4\eta P_4 + 24\eta^2 P_3 - 54\eta^3 P_2 + 30\eta^3\mu - 21\mu^2\eta\Big),$$

where $\mu = \eta' - \frac{1}{2}\eta^2$, and P_2, P_3, P_4 are the same functions of p_i as are $\bar{P}_2, \bar{P}_3, \bar{P}_4$ of $\bar{p}_i$.

When we chose $\frac{\lambda'}{\lambda} = -\bar{p}_1$, we made $\bar{P}_1 = 0$. An examination of (2') shows that we must make the $\xi = \xi(x)$ transformation first if $\bar{P}_1 = 0$ is to be maintained. We can maintain $\bar{P}_1 = 0$ under a further $y = \lambda\bar{y}$ transformation only if $\lambda' = 0$, that is, if λ equals a constant. The choice for the ξ for the independent variable transformation is not apparent, but we will assume that it has been determined so that we now have equation (4) with coefficients fixed except for a factor, which coefficients are unchanged by any transformation where λ equals a constant and $\eta = 0$. We will proceed, then, to find a canonical expansion for the given space curve. The expansion will be called canonical because it is unique, that is, it is unchanged by the permissible transformations.

By the fundamental theorem of differential equations, a solution of (4), at a point given by $x = 0$ where the coefficients are regular, will be of the form

$$(5) \qquad Y_{(0)} = \bar{\bar{y}}_{(0)} + \bar{\bar{y}}'_{(0)} x + \bar{\bar{y}}''_{(0)} \frac{x^2}{\underline{|2}} + \bar{\bar{y}}'''_{(0)} \frac{x^3}{\underline{|3}} + \cdots\cdots$$

From an examination of (4), we see that $\bar{\bar{y}}^{IV}$ and all higher derivatives of $\bar{\bar{y}}$ may be expressed in terms of $\bar{\bar{y}}$, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$ and $\bar{\bar{y}}'''$. Equation (5) will reduce to the form

$$(6) \qquad Y = x_1 \bar{\bar{y}} + x_2 \bar{\bar{y}}' + x_3 \bar{\bar{y}}'' + x_4 \bar{\bar{y}}''',$$

where

$$(6') \qquad \begin{aligned} x_1 = {} & 1 - \bar{P}_4 \frac{x^4}{\underline{|4}} - \bar{P}_4' \frac{x^5}{\underline{|5}} + (6\bar{P}_2 \bar{P}_4 - \bar{P}_4'') \frac{x^6}{\underline{|6}} \\ & + (6\bar{P}_2 \bar{P}_4' + 18\bar{P}_2' \bar{P}_4 + 4\bar{P}_3 \bar{P}_4 - \bar{P}_4''') \frac{x^7}{\underline{|7}} \\ & + (-36\bar{P}_2^2 \bar{P}_4 + 6\bar{P}_2 \bar{P}_4'' + 24\bar{P}_2' \bar{P}_4' + 36\bar{P}_2'' \bar{P}_4 + 4\bar{P}_3 \bar{P}_4' + 16\bar{P}_3' \bar{P}_4 \\ & + \bar{P}_4^2 - \bar{P}_4) \frac{x^8}{\underline{|8}} + \cdots\cdots, \end{aligned}$$

$$x_2 = x - 4\bar{P}_3 \frac{x^4}{4!} - (4\bar{P}_3' + \bar{P}_4)\frac{x^5}{5!} + (24\bar{P}_2\bar{P}_3 - 4\bar{P}_3'' - 2\bar{P}_4')\frac{x^6}{6!}$$

$$+ (24\bar{P}_2\bar{P}_3' + 72\bar{P}_2'\bar{P}_3 + 6\bar{P}_2\bar{P}_4 + 16\bar{P}_3^2 - 4\bar{P}_3''' - 3\bar{P}_4'')\frac{x^7}{7!}$$

$$+ (-144\bar{P}_2^2\bar{P}_3 + 24\bar{P}_2\bar{P}_3'' + 96\bar{P}_2'\bar{P}_3' + 144\bar{P}_2''\bar{P}_3 + 12\bar{P}_2\bar{P}_4'$$

$$+ 80\bar{P}_3\bar{P}_3' + 24\bar{P}_2'\bar{P}_4 + 8\bar{P}_3\bar{P}_4 - 4\bar{P}_3^{IV} - 4\bar{P}_4''')\frac{x^8}{8!} \cdots\cdots,$$

$$x_3 = \frac{x^2}{2} - 6\bar{P}_2\frac{x^4}{4!} - (6\bar{P}_2' + 4\bar{P}_3)\frac{x^5}{5!} + (36\bar{P}_2^2 - 6\bar{P}_2'' - 8\bar{P}_3' - \bar{P}_4)\frac{x^6}{6!}$$

$$+ (144\bar{P}_2\bar{P}_2' + 48\bar{P}_2\bar{P}_3 - 6\bar{P}_2''' - 12\bar{P}_3'' - 3\bar{P}_4')\frac{x^7}{7!}$$

$$+ (-216\bar{P}_2^3 + 252\bar{P}_2\bar{P}_2'' + 144\bar{P}_2'^2 + 144\bar{P}_2\bar{P}_3' + 120\bar{P}_2'\bar{P}_3$$

$$+ 12\bar{P}_2\bar{P}_4 - 6\bar{P}_2^{IV} + 16\bar{P}_3^2 - 16\bar{P}_3''' - 6\bar{P}_4'')\frac{x^8}{8!} + \cdots\cdots,$$

$$x_4 = \frac{x^3}{3!} - 6\bar{P}_2\frac{x^5}{5!} - (12\bar{P}_2' + 4\bar{P}_3)\frac{x^6}{6!} + (36\bar{P}_2^2 - 18\bar{P}_2'' - 12\bar{P}_3' - P_4)\frac{x^7}{7!}$$

$$+ (216\bar{P}_2\bar{P}_2' + 48\bar{P}_2\bar{P}_3 - 24\bar{P}_3'' - 24\bar{P}_2''' - 4\bar{P}_4')\frac{x^8}{8!} + \cdots\cdots.$$

If we let $\frac{x_2}{x_1} = \xi$, $\frac{x_3}{x_1} = \eta$, $\frac{x_4}{x_1} = \zeta$, we obtain

$$\xi = x - 4\bar{P}_3\frac{x^4}{4!} + (4\bar{P}_4 - 4\bar{P}_3')\frac{x^5}{5!} + (24\bar{P}_2\bar{P}_3 - 4\bar{P}_3'' + 4\bar{P}_4')\frac{x^6}{6!} + \cdots,$$

$$\eta = \frac{x^2}{2} - 6\bar{P}_2\frac{x^4}{4!} - (6\bar{P}_2' + 4\bar{P}_3)\frac{x^5}{5!} + (36\bar{P}_2^2 - 6\bar{P}_2'' - 8\bar{P}_3' + 14\bar{P}_4)\frac{x^6}{6!} \cdots,$$

$$\zeta = \frac{x^3}{6} - 6\bar{P}_2\frac{x^5}{5!} - (12\bar{P}_2' + 4\bar{P}_3)\frac{x^6}{6!} + \cdots.$$

By the elimination of x, we obtain

(7)
$$\eta = \frac{\xi^2}{2} - 6\bar{P}_2\frac{\xi^4}{4!} + (16\bar{P}_3 - 6\bar{P}_2')\frac{\xi^5}{5!} + (36\bar{P}_2^2 - 6\bar{P}_2'' + 16\bar{P}_3' - 10\bar{P}_4)\frac{\xi^6}{6!} + \cdots,$$

$$\zeta = \frac{\xi^3}{6} - 6\bar{P}_2\frac{\xi^5}{5!} + 4(14\bar{P}_3 - 3\bar{P}_2')\frac{\xi^6}{6!} + \cdots\cdots.$$

The two equations (7) will be the canonical expansion for the given space curve, when the independent variable transformation is determined so that the $\bar{P}$'s are fixed.

A Tetrahedron of Reference.

Connected with each point $\bar{\bar{y}}$ of the curve will be three other points, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$, $\bar{\bar{y}}'''$. The point $\bar{\bar{y}}'$ lies on the tangent to the given curve at the point $\bar{\bar{y}}$. We shall designate the given curve by $C_{\bar{y}}$. As $\bar{y}$ moves, $\bar{\bar{y}}'$ generates a curve $C_{\bar{y}'}$. The point $\bar{y}''$ lies on the tangent to this curve at $\bar{y}'$. Similarly the point $\bar{\bar{y}}'''$ lies on the tangent to $C_{\bar{y}''}$ at $\bar{\bar{y}}''$. We shall speak of $\bar{\bar{y}}'$, $\bar{\bar{y}}''$, $\bar{\bar{y}}'''$ as derivative points. Any other point Y may be represented by the expression

$$Y = x_1 \bar{\bar{y}} + x_2 \bar{\bar{y}}' + x_3 \bar{y}'' + x_4 \bar{\bar{y}}'''.$$

Therefore we may use x_1, x_2, x_3, x_4 as the coordinates of a point. The coordinates of $\bar{y}$, $\bar{\bar{y}}'$, $\bar{y}''$, $\bar{\bar{y}}'''$ will be $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ respectively.

The osculating plane of a curve at a point must contain the point and the first and second derivative points. The osculating plane of $C_{\bar{y}}$ at $\bar{y}$ is evidently $x_4 = 0$. Thus one face of the tetrahedron of reference is geometrically located.

Any cubic may be represented parametrically in the form

$$
\begin{aligned}
x_1 &= a_1 + b_1 t + c_1 t^2 + d_1 t^3, \\
x_2 &= a_2 + b_2 t + c_2 t^2 + d_2 t^3, \\
x_3 &= a_3 + b_3 t + c_3 t^2 + d_3 t^3, \\
x_4 &= a_4 + b_4 t + c_4 t^2 + d_4 t^3.
\end{aligned}
\tag{8}
$$

The parameter t is independent of $\bar{\bar{y}}$ and three different values of t may be assigned to three different points arbitrarily. We shall choose t so that $t = 0$ at $\bar{\bar{y}}$. If we wish to make the cubic be the osculating cubic at $\bar{\bar{y}}$, the cubic must have three point contact at $\bar{\bar{y}}$ with the osculating plane of the space curve. If t is to equal zero at $\bar{\bar{y}} = (1,0,0,0)$, we must have $a_2 = a_3 = a_4 = 0$. If we solve (8) with $x_4 = 0$, we see that we must have $b_4 = c_4 = 0$ for three point contact at $\bar{\bar{y}}$. Since we are dealing with homogeneous coordinates, we may divide thru by a_1 and may now write the cubic in the form

$$(9)\qquad \begin{aligned} x_1 &= 1 + b_1' t + c_1' t^2 + d_1' t^3, \\ x_2 &= b_2' t + c_2' t^2 + d_2' t^3, \\ x_3 &= b_3' t + c_3' t^2 + d_3' t^3, \\ x_4 &= d_4' t^3. \end{aligned}$$

Since the plane $x_3 = 0$ passes thru the line on $\bar{\bar{y}}$ and $\bar{\bar{y}}'$, it is tangent to the space curve and therefore tangent to the osculating cubic at $\bar{\bar{y}}$. If we solve $x_3 = 0$ with (9), we must have $b_3' = 0$ for two point contact at $\bar{\bar{y}}$. The osculating cubic has another intersection with $x_3 = 0$. When we solve $x_3 = 0$ with (9), we obtain $t^2(c_3' + d_3' t) = 0$. If we choose this second intersection to be the point for which t is infinite, we must have $d_3' = 0$.

We may now replace t by $\frac{T}{b_2'}$ and write the cubic as

$$(10)\qquad \begin{aligned} x_1 &= 1 + B_1 T + C_1 T^2 + D_1 T^3, \\ x_2 &= T + C_2 T^2 + D_2 T^3, \\ x_3 &= C_3 T^2, \\ x_4 &= D_4 T^3. \end{aligned}$$

From (10) we obtain

$$(11)\qquad \begin{aligned} \xi = \frac{x_2}{x_1} &= T + (C_2 - B_1)T^2 + (D_2 - C_1 - C_2 B_1 + B_1^2)T^3 \\ &\quad + (B_1^2 C_2 - B_1^3 + 2C_1 B_1 - B_1 D_2 - C_1 C_2 - D_1)T^4 + \cdots, \\ \eta = \frac{x_3}{x_1} &= C_3 T^2 - B_1 C_3 T^3 + (B_1^2 C_3 - C_1 C_3)T^4 + (2B_1 C_1 C_3 - B_1^3 C_3 - D_1 C_3)T^5 + \cdots, \\ \zeta = \frac{x_4}{x_1} &= D_4 T^3 - B_1 D_4 T^4 + (B_1^2 D_4 - C_1 D_4)T^5 + \cdots. \end{aligned}$$

From these three expressions we may eliminate T obtaining expressions for η and ζ in terms of ξ. Since this cubic is to be the osculating cubic of the space curve it must have six point contact with the space curve. This is possible only if the expansions for the cubic and the space curve are identical thru the first five powers of ξ. By the elimination of T from (11) we obtain

$$\eta = C_3 \xi^2 + (B_1 C_3 - 2C_2 C_3)\xi^3 + \cdots,$$

$$\zeta = D_4 \xi^3 + (2B_1 D_4 - 3C_2 D_4)\xi^4 + \cdots.$$

If we equate these coefficients to the corresponding coefficients in the expansion (7), we find that

$$C_3 = \frac{1}{2}, \quad D_4 = \frac{1}{6}, \quad B_1 = 0, \quad C_2 = 0.$$

By a substitution of these values in (10), we obtain the simpler form

$$\xi = \frac{x_2}{x_1} = T + (D_2 - C_1)T^3 - D_1 T^4 + (C_1^2 - C_1 D_2)T^5 + \cdots,$$

$$\eta = \frac{x_3}{x_1} = \frac{T^2}{2} - C_1\frac{T^4}{2} - D_1\frac{T^5}{2} + C_1^2\frac{T^6}{2} + \cdots,$$

$$\rho = \frac{x_4}{x_1} = \frac{T^3}{6} - C_1\frac{T^5}{6} - D_1\frac{T^6}{6} + \cdots.$$

If we continue with the elimination of T, we find that

$$(12) \quad \begin{aligned} \eta &= \frac{\xi^2}{2} + (\tfrac{1}{2}C_1 - D_2)\xi^4 + D_1\frac{\xi^5}{2} + (C_1^2 + \tfrac{7}{2}D_2^2 - 4C_1D_2)\xi^6 + \cdots, \\ \rho &= \frac{\xi^3}{6} + (\tfrac{1}{3}C_1 - \tfrac{1}{2}D_2)\xi^5 + D_1\frac{\xi^6}{3} + \cdots. \end{aligned}$$

If we equate the coefficients of (12) to the corresponding ones in the expansion (7), we find that

$$C_1 = \tfrac{9}{10}\bar{P}_2, \quad D_2 = \tfrac{7}{10}\bar{P}_2, \quad D_1 = \frac{(8\bar{P}_3 - 3\bar{P}_2')}{30}.$$

We may now substitute back in (10). If, after the substitution, we multiply by 6, the equation for the osculating cubic becomes

$$(13) \quad \begin{aligned} x_1 &= 6 + \tfrac{27}{5}\bar{P}_2 T^2 + (8\bar{P}_3 - 3\bar{P}_2')\frac{T^3}{5}, \\ x_2 &= 6T + \tfrac{21}{5}\bar{P}_2 T^3, \\ x_3 &= 3T^2, \\ x_4 &= T^3. \end{aligned}$$

If we substitute in equations (12), we obtain the expansion for the cubic as follows.

$$\eta = \frac{\xi^2}{2} - \frac{\bar{P}_2 \xi^4}{4} - (3\bar{P}_2' - 8\bar{P}_3)\frac{\xi^5}{60} + \frac{\bar{P}_2^2 \xi^6}{200} + \cdots\cdots,$$

$$\zeta = \frac{\xi^3}{6} - \frac{\bar{P}_2 \xi^5}{20} - (3\bar{P}_2' - 8\bar{P}_3)\frac{\xi^6}{90} + \cdots\cdots.$$

Halphen's theorem states that if two space curves have contact of order n, there exists a unique plane from any point of which the projections of the two curves have contact of order (n+1). This unique plane we shall speak of as the principal plane.

Will it be possible to have $x_3 = 0$, (non-homogeneously $\eta = 0$) as the principal plane for the given space curve and the osculating cubic?

The canonical expansion for the given curve may be thought of as the equations of two cylinders, the first with its elements parallel to the ζ axis, and the second with its elements parallel to the η axis. The curve itself is the intersection of the two cylinders. The first equation then may be considered as the projection of the curve from the infinite point on the ζ axis onto the $\zeta = 0$, plane. In a similar way, the first equation for the osculating cubic may be thought of as the projection of the cubic from the infinite point of the ζ axis onto the $\zeta = 0$ plane. If the ζ axis is to be in the principal plane, we must have sixth order contact between the projections of the space curve and the cubic, that is, we must have

$$\frac{18\bar{P}_2^2 - 3\bar{P}_2'' + 8\bar{P}_3' - 5\bar{P}_4}{360} = \frac{\bar{P}_2^2}{200}.$$

This reduces to the expression

$$81\bar{P}_2^1 - 15\bar{P}_2'' + 40\bar{P}_3' - 25\bar{P}_4 = 0,$$

which we shall designate as $\bar{M}$.

Is it possible to choose the ξ and λ transformations so as to make $\bar{M} = 0$? Since $\bar{P}_2$, $\bar{P}_3$, $\bar{P}_4$ are seminvariants, any transformation of the form $y = \lambda\bar{y}$ will not disturb $\bar{M}$. Under the ξ transformation, [substituting (4") in $\bar{M}$], however, we find that

(14) $\quad \bar{M} = M - 15\eta\,\theta_3$, where $\theta_3 = (2P_3 - 3P_2')$.

The expression $(2P_3 - 3P_2')$ is a relative invariant as may be verified by substitution from (4"). In obtaining the canonical expansion for the space curve, we assumed that ξ had been chosen so that the coefficients would be uniquely determined except for a factor and maintained by further ξ transformations for which $\eta = 0$. From (14), we see that we need to choose $\eta = \frac{M}{15\theta_3}$, where $\theta_3 \neq 0$, in order to have $\bar{M} = 0$. For this choice of ξ, the τ axis is in the principal plane. The principal plane always contains the tangent line $\bar{y}\bar{y}'$. Therefore, if we choose $\eta = \frac{M}{15\theta_3}$, where $\theta_3 \neq 0$, the principal plane is $\eta = 0$ $(x_3 = 0)$. The case $\theta_3 = 0$ will be discussed later. The only permissible ξ transformations after this are those for which $\eta = 0$, for a second transformation gives

$$\bar{\bar{M}} = \bar{M} - 15\eta\,\bar{\theta}_3,$$

from which we see that $\bar{\bar{M}}$ is zero only if $\eta = 0$. A second face of the tetrahedron of reference is now geometrically located as the principal plane.

If we consider the projections of the curve and the osculating cubic from the infinite point of the η axis onto the $\eta = 0$ plane, the two curves will have sixth order contact if

$$\frac{8\bar{P}_3 - 3\bar{P}_2'}{90} = \frac{14\bar{P}_3 \quad 3\bar{P}_2'}{180} .$$

This reduces to

$$2\bar{P}_3 - 3\bar{P}_2' = 0, \quad \text{or } \bar{\Theta}_3 = 0.$$

Since Θ_3 is a relative invariant, it is not affected by the ξ and λ transformations except for some multiplier. If $\bar{\Theta}_3 = 0$, the principal plane contains the η axis and the $\bar{y}\bar{y}'$ line, that is, it is the plane $x_4 = 0$. Then for $\bar{\Theta}_3 = 0$, the principal plane and the osculating plane of the space curve coincide.

If we solve equations (13) with $x_3 = 0$, we obtain $T = 0$, giving a double root $(1,0,0,0)$, thus showing that the osculating cubic is tangent to the plane $x_3 = 0$ at $\bar{\bar{y}}$. To find the third intersection we must examine the case when T is infinite. For this case, replace T in (13) by $\frac{1}{t}$, obtaining the equation

$$x_1 = 6t^3 + \frac{27}{5}\bar{P}_2\, t - \left(\frac{3\bar{P}_2' - 8\bar{P}_3}{5}\right),$$

$$
(15)\qquad
\begin{aligned}
x_2 &= 6t^2 + \tfrac{21}{5}\bar{P}_2,\\
x_3 &= 3t,\\
x_4 &= 1,
\end{aligned}
$$

for the osculating cubic. If we solve (15) with $x_3 = 0$, we obtain the point of intersection of the cubic and the principal plane. This point has the coordinates $\left[\frac{(8\bar{P}_3 - 3\bar{P}_2')}{5};\ \frac{21}{5}\bar{P}_2;\ 0;\ 1\right]$.

The osculating cubic has an osculating plane given by

$$
(16)\qquad
\begin{vmatrix}
x_1 & x_2 & x_3 & x_4\\
6t^3+\frac{27}{5}\bar{P}_2 t-\left(\frac{3\bar{P}_2'-8\bar{P}_3}{5}\right) & 6t^2+\frac{21}{5}\bar{P}_2 & 3t & 1\\
18t^2+\frac{27}{5}\bar{P}_2 & 12t & 3 & 0\\
36t & 12 & 0 & 0
\end{vmatrix} = 0.
$$

The determinant (16) may be solved with $x_4 = 0$ giving the intersection of the osculating plane of the cubic and the osculating plane of the space curve, which intersection is

$$(17)\qquad x_1 - 3tx_2 + \left(6t^2 - \tfrac{9}{5}\bar{P}_2\right)x_3 = 0.$$

Equation (17) together with $x_4 = 0$ is the equation of a variable line in the plane $x_4 = 0$. This line has an envelope which we find to be

$$(18)\qquad 8x_1x_3 - 3x_2^2 - \tfrac{72}{5}\bar{P}_2x_3^2 = 0.$$

Equation (18) is the equation of a conic which we shall designate as the osculating conic for this discussion.

By a substitution of the coordinates of the point $\bar{\bar{y}}$, we find that the osculating conic goes thru $\bar{\bar{y}}$. The polar of a conic with respect to a point is given by

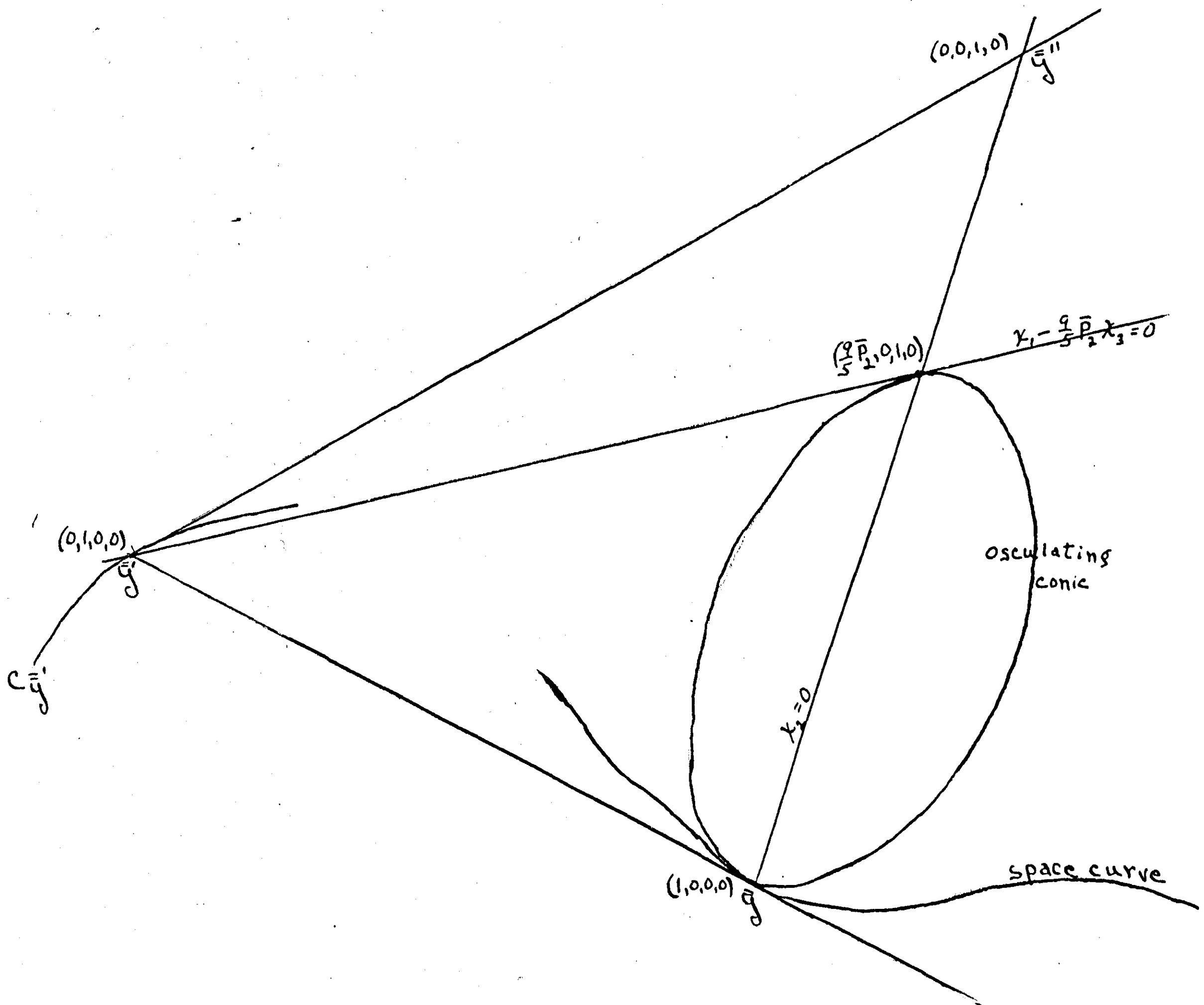
$$x_1' \frac{\partial f(x)}{\partial x_1} + x_2' \frac{\partial f(x)}{\partial x_2} + x_3' \frac{\partial f(x)}{\partial x_3} + x_4' \frac{\partial f(x)}{\partial x_4} = 0,$$

where x_i' $(i=1,2,3,4)$ are the running coordinates. The polar of the osculating conic with respect to $\bar{\bar{y}}' = (0,1,0,0)$ then becomes $6x_2 = 0$, $x_4 = 0$, which is the line on $\bar{\bar{y}}$ and $\bar{\bar{y}}''$.

When $t=0$ in (16), the osculating plane to the osculating cubic is a fixed plane and the variable line(17) becomes a fixed line

(19) $\qquad x_1 - \frac{1}{3}\bar{P}_2 x_3 = 0, \; x_4 = 0.$

This line obviously goes thru $\bar{\bar{y}}' = (0,1,0,0)$. Since $\bar{\bar{y}}'$ is on the tangent to $C_{\bar{\bar{y}}}$ at $\bar{\bar{y}}$, the point $\bar{\bar{y}}'$ is determined as the intersection of this tangent and the line given by equation (19). The $\bar{\bar{y}}\bar{\bar{y}}''$ line is determined as the polar of the osculating conic with respect to $\bar{\bar{y}}'$. The intersection of the tangent to the $C_{\bar{\bar{y}}'}$ curve at $\bar{\bar{y}}'$ and the $\bar{\bar{y}}\bar{\bar{y}}''$ line locates the $\bar{\bar{y}}''$ point. The $\bar{\bar{y}}'''$ point is located as the intersection of the tangent to $C_{\bar{\bar{y}}''}$ at $\bar{\bar{y}}''$ with the principal plane. Thus the local tetrahedron is completely determined geometrically.



$(0,0,1,0)$
$\bar{\bar{y}}''$
$(\frac{9}{5}\bar{p}_2,0,1,0)$
$x_1-\frac{9}{5}\bar{p}_2x_3=0$
$(0,1,0,0)$
$\bar{\bar{y}}'$
osculating
conic
$C_{\bar{\bar{y}}'}$
$x_2=0$
$(1,0,0,0)$
$\bar{y}$
space curve

By a substitution of $\eta = \frac{M}{15\theta_3}$, in (4"), we obtain expressions for the coefficients of (4) entirely in terms of the coefficients of (1). That these expressions for the coefficients of (4) are relative invariants may be shown by considering two equivalent equations

(A) $$y^{iv} + 4p_1 y''' + 6p_2 y'' + 4p_3 y' + p_4 y = 0,$$

(B) $$y^{iv} + 4q_1 y''' + 6q_2 y'' + 4q_3 y' + q_4 y = 0.$$

By two equivalent equations we mean two equations such that we may obtain one from the other by means of the transformations $y = \lambda \bar{y}$ and $\xi = \xi(x)$. If (A) and (B) are equivalent we must be able to express the q's of (B) in terms of the p's of (A) and functions of the transformations. Let us apply the particular transformations $y = \lambda \bar{y}$ and $\xi = \xi(x)$ to (A) and (B) which give the canonical forms

($\bar{A}$) $$\bar{y}^{iv} + 6\bar{P}_2 \bar{y}'' + 4\bar{P}_3 \bar{y}' + \bar{P}_4 \bar{y} = 0,$$

($\bar{B}$) $$\bar{\bar{y}}^{iv} + 6\bar{Q}_2 \bar{\bar{y}}'' + 4\bar{Q}_3 \bar{\bar{y}}' + \bar{Q}_4 \bar{\bar{y}} = 0.$$

As seen above we shall have expressions for the $\bar{P}$'s entirely in terms of the p's and for the $\bar{Q}$'s entirely in terms of q's, the expressions in the p's being the same as the expressions in the q's.

Is it possible to transform equation ($\bar{A}$) into equation ($\bar{B}$)? It is easily seen that by means of our two transformations we may go from equation ($\bar{A}$) to equation (A), from (A) to (B), and from (B) to ($\bar{B}$).

Therefore it must be possible to transform equation $(\overline{A})$ into equation $(\overline{B})$. We have seen however, that the only transformations which will maintain the form $(\overline{A})$ are those for which λ equals a constant and $\eta = 0$. These transformations will change the coefficients of $(\overline{A})$ by a multiplier only. Therefore the coefficients of $(\overline{A})$ can differ from those of $(\overline{B})$ by some multiplier only, and the transformations can change the expressions for the $\overline{P}$'s and $\overline{Q}$'s in terms of p's and q's by a multiplier only. Therefore the coefficients of $(\overline{A})$ and all their derivatives are relative invariants. Precisely the same argument shows that $\overline{\overline{y}}$ and its derivatives are relative covariants which may be expressed in terms of the y's and coefficients of (A).

A Second Tetrahedron of Reference.

In the previous work, the tetrahedron of reference has the points $\bar{\bar{y}}$, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$, $\bar{\bar{y}}'''$ as the vertices, which points were located geometrically. Another point, geometrically located as the intersection of the osculating cubic with the plane $x_3 = 0$, was found to have the coordinates $\left[\frac{(8\bar{P}_3 - 3\bar{P}_2')}{5}; \frac{21}{5}\bar{P}_2; 0; 1\right]$. In the plane $x_4 = 0$, we found an osculating conic with the equation (18). The polar of this conic with respect to $\bar{\bar{y}}'$ was found to be the line $x_2 = 0$. This line intersects the conic in the two points $(1,0,0,0)$ and $(\frac{9}{5}\bar{P}_2; 0; 1; 0)$. Thus besides the four vertices of the tetrahedron of reference, we have two other points geometrically located. Let us make a transformation which shall make $\left[\frac{(8\bar{P}_3 - 3\bar{P}_2')}{5}; \frac{21}{5}\bar{P}_2; 0; 1\right]$ be the vertex $(0,0,0,1)$ and $(\frac{9}{5}\bar{P}_2; 0; 1; 0)$ be the vertex $(0,0,1,0)$. We shall keep the other two vertices the same.

A general transformation in space is of the form

$$\begin{aligned}
\bar{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4, \\
\bar{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4, \\
\bar{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4, \\
\bar{x}_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4,
\end{aligned}$$

where x_i are the original coordinates and $\bar{x}_i$ are the transformed coordinates.

If we substitute the coordinates of the vertices in the general transformation above, we find that

$$\begin{array}{llll}
a_{11}=1, & a_{21}=0, & a_{31}=0, & a_{41}=0, \\
a_{12}=0, & a_{22}=1, & a_{32}=0, & a_{42}=0, \\
a_{13}=-\frac{9}{5}\bar{P}_2\,, & a_{23}=0, & a_{33}=1, & a_{43}=0, \\
a_{14}=\left(\frac{3\bar{P}_2'-8\bar{P}_3}{5}\right), & a_{24}=-\frac{21}{5}\bar{P}_2\,, & a_{34}=0, & a_{44}=1.
\end{array}$$

By a substitution of these values in the general transformation, we find that the necessary transformation is

$$(20)\qquad \begin{aligned}
\bar{x}_1 &= x_1-\tfrac{9}{5}\bar{P}_2\,x_3+\left(\frac{3\bar{P}_2'-8\bar{P}_3}{5}\right)x_4\,, \\
\bar{x}_2 &= x_2-\tfrac{21}{5}\bar{P}_2\,x_4\,, \\
\bar{x}_3 &= x_3\,, \qquad \bar{x}_4 = x_4.
\end{aligned}$$

Under this transformation, we have a new tetrahedron of reference with vertices $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, $\bar{y}_4$ where $\bar{y}_1$ is the $\bar{y}$ of the old tetrahedron, $\bar{y}_2$ is the $\bar{y}'$ of the old tetrahedron, $\bar{y}_3$ is the second intersection of the line $x_2=0$ with the osculating cubic, and $\bar{y}_4$ is the intersection of the osculating cubic with the plane $x_3=0$.

If we substitute equations (6') in equations (20), we obtain

$$\begin{aligned}
\bar{x}_1 = {} & 1-\tfrac{9}{10}\bar{P}_2\,x^2+(3\bar{P}_2'-8\bar{P}_3)\frac{x^3}{5\cdot 3!}+(54\bar{P}_2^2-5\bar{P}_4)\frac{x^4}{5\cdot 4!} \\
& +(-5\bar{P}_4'+36\bar{P}_2\,\bar{P}_2'+84\bar{P}_2\,\bar{P}_3)\frac{x^5}{5\cdot 5!} \\
& +(39\bar{P}_2\,\bar{P}_4-5\bar{P}_4''-324\bar{P}_2^3+54\bar{P}_2\,\bar{P}_2''+72\bar{P}_2\,\bar{P}_3'-36P_2'^2+84\bar{P}_2'\bar{P}_3+32\bar{P}_3^2)\frac{x^6}{5\cdot 6!} \\
& +(57\bar{P}_2\,\bar{P}_4'+87\bar{P}_2'\bar{P}_4+28\bar{P}_3\,\bar{P}_4-5\bar{P}_4'''-1188\bar{P}_2^2\,\bar{P}_2'-720\bar{P}_2^2\,\bar{P}_3+54\bar{P}_2'''\bar{P}_2 \\
& +108\bar{P}_2\,\bar{P}_3''-54\bar{P}_2'\bar{P}_2''-36\bar{P}_2'\bar{P}_3'+144\bar{P}_2''\bar{P}_3+96\bar{P}_3\,\bar{P}_3')\frac{x^7}{5\cdot 7!}+\cdots,
\end{aligned}$$

(21) $$\bar{x}_2 = x - \tfrac{7}{10}\bar{P}_2\, x^3 - \bar{P}_3 \frac{x^4}{6} + (126\bar{P}_2^2 - 20\bar{P}_3' - 5\bar{P}_4)\frac{x^5}{5\,|\underline{5}}$$

$$+ (204\bar{P}_2\bar{P}_3 - 20\bar{P}_3'' - 10\bar{P}_4' + 252\bar{P}_2\bar{P}_2')\frac{x^6}{5\,|\underline{6}}$$

$$+ (372\bar{P}_2\bar{P}_3' + 360\bar{P}_2'\bar{P}_3 + 51\bar{P}_2\bar{P}_4 + 80\bar{P}_3^2 - 20\bar{P}_3''' - 15\bar{P}_4''$$

$$- 756\bar{P}_2^3 + 378\bar{P}_2\bar{P}_2'')\frac{x^7}{5\,|\underline{7}} + \cdots,$$

$$\bar{x}_3 = x_3, \quad \bar{x}_4 = x_4.$$

If we let $\frac{\bar{x}_2}{\bar{x}_1} = \bar{\xi}$, $\frac{\bar{x}_3}{\bar{x}_1} = \bar{\eta}$, $\frac{\bar{x}_4}{\bar{x}_1} = \bar{\zeta}$, we obtain

$$\bar{\xi} = x + \bar{P}_2\frac{x^3}{5} + (\bar{P}_3 - \bar{P}_2')\frac{x^4}{10} + (20\bar{P}_4 - 20\bar{P}_3' - 36\bar{P}_2^2)\frac{x^5}{5\,|\underline{5}}$$

$$+ (216\bar{P}_2\bar{P}_3 - 360\bar{P}_2\bar{P}_2' - 20\bar{P}_3'' + 20\bar{P}_4')\frac{x^6}{5\,|\underline{6}}$$

$$+ (4440\bar{P}_2\bar{P}_3' - 5760\bar{P}_2'\bar{P}_3 + 3720\bar{P}_2\bar{P}_4 + 2640\bar{P}_3^2 - 100\bar{P}_3'''$$

(22) $$+ 100\bar{P}_4'' - 10584\bar{P}_2^3 + 2520\bar{P}_2'^2)\frac{x^7}{25\,|\underline{7}} + \cdots,$$

$$\bar{\eta} = \frac{x^2}{2} + \bar{P}_2\frac{x^4}{5} + (\bar{P}_3 - \bar{P}_2')\frac{x^5}{10} + (18\bar{P}_2^2 - 30\bar{P}_2'' - 40\bar{P}_3' + 70\bar{P}_4)\frac{x^6}{5\,|\underline{6}}$$

$$+ (-2808\bar{P}_2\bar{P}_2' + 2088\bar{P}_2\bar{P}_3 - 30\bar{P}_2''' - 60\bar{P}_3'' + 90\bar{P}_4')\frac{x^7}{5\,|\underline{7}} + \cdots,$$

$$\bar{\zeta} = \frac{x^3}{6} + \bar{P}_2\frac{x^5}{10} + (28\bar{P}_3 - 24\bar{P}_2')\frac{x^6}{|\underline{6}} + (558\bar{P}_2^2 - 90\bar{P}_2'' - 60\bar{P}_3' + 170\bar{P}_4)\frac{x^7}{5\,|\underline{7}} + \cdots$$

By eliminating x from the $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ expressions, we find that

$$\bar{\eta} = \frac{\bar{\xi}^2}{2} + (162\bar{P}_2^2 + 80\bar{P}_3' - 50\bar{P}_4 - 30\bar{P}_2'')\frac{\bar{\xi}^6}{5\,|\underline{6}}$$

(23) $$+ (216\bar{P}_2\bar{P}_2' + 72\bar{P}_2\bar{P}_3 - 30\bar{P}_2''' + 80\bar{P}_3'' - 50\bar{P}_4')\frac{\bar{\xi}^7}{5\,|\underline{7}} + \frac{C\bar{\xi}^8}{25\,|\underline{8}} + \cdots,$$

$$\bar{\zeta} = \frac{\bar{\xi}^3}{6} + 4(3\bar{P}_2' - 2\bar{P}_3)\frac{\bar{\xi}^6}{|\underline{6}} + (162\bar{P}_2^2 + 72\bar{P}_3' - 50\bar{P}_4 - 18\bar{P}_2'')\frac{\bar{\xi}^7}{|\underline{7}} + \frac{F\bar{\xi}^8}{5\,|\underline{8}} + \cdots,$$

where C and F stand for expressions in the $\bar{P}$'s.

The Osculating Quadric.

In the previous discussion, we found that the plane $\bar{x}_3 = 0$ was not defined for $\bar{\theta}_3 = 0$. We shall now show that when $\bar{\theta}_3 = 0$, the plane $\bar{x}_3 = 0$ may be defined as the tangent plane to the osculating quadric surface at $\bar{\bar{y}}$.

The general quadric surface equation will be of the form

$$a\bar{x}_1^2 + a'\bar{x}_1\bar{x}_2 + a_1\bar{x}_1\bar{x}_3 + a_2\bar{x}_1\bar{x}_4 + b_1\bar{x}_2^2 + b_2\bar{x}_2\bar{x}_3 + b_3\bar{x}_2\bar{x}_4$$

$$+ c_1\bar{x}_3^2 + c_2\bar{x}_3\bar{x}_4 + d_1\bar{x}_4^2 = 0.$$

By a consideration of expressions (21), we see that the $a\bar{x}_1^2$ term is the only term having x^0 power. Therefore we must have $a = 0$. The $a'\bar{x}_1\bar{x}_2$ term is the only one containing x, so we must have $a' = 0$. The general equation of the quadric then reduces to

$$(24) \qquad a_1\bar{x}_1\bar{x}_3 + a_2\bar{x}_1\bar{x}_4 + b_1\bar{x}_2^2 + b_2\bar{x}_2\bar{x}_3 + b_3\bar{x}_2\bar{x}_4 + c_1\bar{x}_3^2$$

$$+ c_2\bar{x}_3\bar{x}_4 + d_1\bar{x}_4^2 = 0,$$

which expressed non-homogeneously is

$$(25) \qquad a_1\bar{\eta} + a_2\bar{\zeta} + b_1\bar{\xi}^2 + b_2\bar{\xi}\bar{\eta} + b_3\bar{\xi}\bar{\zeta} + c_1\bar{\eta}^2 + c_2\bar{\eta}\bar{\zeta} + d_1\bar{\zeta}^2 = 0.$$

The osculating quadric surface will be determined by nine points. If we substitute the equations (22) in the equation (25) and set each of the coefficients of the first eight powers of xequal to zero, the coefficients of equation (25) may be determined. This is not an easy process and, since we are interested primarily in the condition that $\bar{x}_3 = 0$ be the tangent plane to the osculating quadric, we shall omit this calculation.

The tangent plane to a surface is given by

$$x_1' \frac{\partial f(x)}{\partial x_1} + x_2' \frac{\partial f(x)}{\partial x_2} + x_3' \frac{\partial f(x)}{\partial x_3} + x_4' \frac{\partial f(x)}{\partial x_4} = 0,$$

where x_i' are the running coordinates and $f(x)=0$ is the equation of the surface. From this we find that the equation of the tangent plane to the quadric at $\bar{y}$ is

$$a_2 \bar{x}_4 + a_1 \bar{x}_3 = 0.$$

If $\bar{x}_3 = 0$ is to be the tangent plane, we must have $a_2 = 0, a_1 \neq 0$.

To simplify the work we will write (23) in the form

$$\bar{\eta} = \frac{\bar{\xi}^2}{2} + A\bar{\xi}^6 + B\bar{\xi}^7 + C\bar{\xi}^8 + \cdots,$$

(26) $$\bar{\rho} = \frac{\bar{\xi}^3}{6} + D\bar{\xi}^6 + E\bar{\xi}^7 + F\bar{\xi}^8 + \cdots,$$

where A,B,C,D,E,F, represent the corresponding coefficients in (23). From the equations (26), we see that

$$\bar{\rho}^2 = \frac{\bar{\xi}^6}{36} + \cdots,$$

$$\bar{\eta} - \tfrac{1}{2}\bar{\xi}^2 = A\bar{\xi}^6 + B\bar{\xi}^7 + C\bar{\xi}^8 + \cdots,$$

$$3\bar{\rho} - \bar{\xi}\bar{\eta} = 3D\bar{\xi}^6 + (3E-A)\bar{\xi}^7 + (3F-B)\bar{\xi}^8 + \cdots,$$

$$4\bar{\eta}^2 - 6\bar{\xi}\bar{\rho} = -6D\bar{\xi}^7 + (4A-6E)\bar{\xi}^8 + \cdots.$$

We may then write the determinant condition for the elimination of $\bar{\xi}$, $\bar{\eta}$ and $\bar{\rho}$ from the expressions (27) as follows

$$\begin{vmatrix} \bar{\eta}-\frac{1}{2}\bar{\xi}^2 & A & B & C \\ 3\bar{\zeta}-\bar{\xi}\bar{\eta} & 3D & 3E-A & 3F-B \\ 4\bar{\eta}^2-6\bar{\xi}\bar{\zeta} & 0 & -6D & 4A-6E \\ \bar{\zeta}^2 & \frac{1}{36} & 0 & 0 \end{vmatrix} = 0.$$

The expansion of this determinant will be a quadratic expression and therefore represents a quadric surface. If we solve this with the equations of the curve by a substitution for $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ we obtain an expression starting with the ninth powers of x. This quadric must have nine point contact with the space curve at $\bar{\bar{y}}$ and is therefore the osculating quadric of the curve at the point $\bar{\bar{y}}$.# If we expand the determinant, we find that a_2, the $\bar{\zeta}$ coefficient is $-[B(4A-6E)+6DC]\times\frac{1}{12}$, and a_1, the $\bar{\eta}$ coefficient is $-[(3E-A)(4A-6E)+6D(3F-B)]\times\frac{1}{36}$. From equations (23) and (26) we find that D is $\bar{\theta}_3/_{180}$. If we let $\bar{\theta}_3=0$, a_2 becomes $-B(4A-6E)\times\frac{1}{12}$, and a_1 becomes $(A-3E)(4A-6E)\times\frac{1}{36}$.

The coefficient a_2 will be zero if $B=0$, or if $(4A-6E)=0$. Since we do not want a_1 to be zero, we will be interested only in $B=0$. From equations (23) and (26) we find that

$$B=(216\bar{P}_2\bar{P}_2'+72\bar{P}_2\bar{P}_3-30P_2'''+80P_3''-50\bar{P}_4')\times\frac{1}{517}.$$

Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces", page 278.

Since we have $\bar{\Theta}_3 = 3\bar{P}_2' - 2\bar{P}_3 = 0$, we find that

$$B = (216\bar{P}_2\bar{P}_3 + 90\bar{P}_2''' - 50\bar{P}_4')\times\frac{1}{517},$$

and that if B is to equal zero, we must have

$$\bar{P}_4' - \frac{9}{5}\bar{P}_2''' - \frac{108}{25}\bar{P}_2\bar{P}_3 = 0.$$

Wilczynski[#] has determined the relative invariant

$$\bar{\Theta}_4 = \bar{\underline{P}}_4 - 2\bar{P}_3' + \frac{6}{5}\bar{P}_2'' - \frac{6}{25}\bar{P}_2^2, \text{ where } \bar{\underline{P}}_4 = \bar{P}_4 - 3\bar{P}_2^2.$$

If we substitute the value for $\bar{\underline{P}}_4$ in $\bar{\Theta}_4$ and make use of $\bar{\Theta}_3 = 0$, we find that

$$\bar{\Theta}_4 = \bar{P}_4 - \frac{9}{5}\bar{P}_2'' - \frac{81}{25}\bar{P}_2^2,$$

$$\bar{\Theta}_4' = \bar{P}_4' - \frac{9}{5}\bar{P}_2''' - \frac{162}{125}\bar{P}_2\bar{P}_2', \text{ which is } \bar{\Theta}_4 = \bar{P}_4' - \frac{9}{5}\bar{P}_2''' - \frac{108}{25}\bar{P}_2\bar{P}_3 .$$

and that

$$B = -\frac{50}{517}\bar{\Theta}_4'$$

Can the ξ and λ transformations be chosen so as to make $B = 0$? Since $\bar{P}_2, \bar{P}_3, \bar{P}_4$ are seminvariants, the λ transformation has no effect on B. From a substitution of (4") in B, we find that

$$B = -\frac{50}{517}\Theta_4' + \frac{4\times 50}{517}\eta\,\Theta_4,$$

where Θ_4 and Θ_4' are in terms of the P's without the dashes. Therefore we can make $B = 0$, by choosing $\eta = \frac{\Theta_4'}{4\Theta_4}$, where $\Theta_4 \neq 0$.

We must now examine the coefficient a_1 and see if this transformation leaves $a_1 \neq 0$. Under the assumption

[#] Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces", page 240.

that $\bar{\Theta}_3 = 0$, a_1 becomes $(A-3E)(4A-6E)\cdot\frac{1}{36}$. From the equations (23 and (26) we find that

$$A-3E = \frac{400}{5\underline{|7}}\,\bar{\Theta}_4,$$

$$4A-6E = \frac{100}{5\underline{|7}}\,\bar{\Theta}_4.$$

Therefore a_1 is $\frac{400}{9\underline{|7}\cdot\underline{|7}}\,\bar{\Theta}_4^{\,2}$.

That the expression $\bar{\Theta}_4$ is a relative invariant may be verified by a substitution of (4") in $\bar{\Theta}_4$, which gives

$$\bar{\Theta}_4 = \frac{1}{(\xi')^4}\,\Theta_4.$$

Therefore a_1 will not be zero if $\Theta_4 \neq 0$. Thus we see that by choosing $\eta = \frac{\Theta_4'}{4\Theta_4}$, where $\Theta_4 \neq 0$, we may make $a_2 = 0$, $a_1 \neq 0$, and the tangent plane to the osculating quadric is the plane $x_3 = 0$.

Geometrical Interpretation of the Vanishing of Some of the Relative Invariants.

I. Interpretation of $\bar{\Theta}_3 = 0$.

A linear complex is in general determined by five lines. The osculating linear complex of the space curve will be determined by five consecutive tangents.

We may represent the point Y given by $x = 0$, by the expansion

$$Y = \bar{\bar{y}} + \bar{\bar{y}}'x + \bar{\bar{y}}''\frac{x^2}{2} + \bar{\bar{y}}'''\frac{x^3}{\lfloor 3} + (-6\bar{P}_2\bar{\bar{y}}'' - 4\bar{P}_3\bar{\bar{y}}' - \bar{P}_4\bar{\bar{y}})\frac{x^4}{\lfloor 4}$$
$$+ (-6\bar{P}_2\bar{\bar{y}}''' - 6\bar{P}_2'\bar{\bar{y}}'' - 4\bar{P}_3\bar{\bar{y}}'' - 4\bar{P}_3'\bar{\bar{y}}' - \bar{P}_4\bar{\bar{y}}' - \bar{P}_4'\bar{\bar{y}})\frac{x^5}{\lfloor 5} + \cdots\cdots$$

Any point Y' on the tangent to $C_{\bar{\bar{y}}}$ at Y is given by

$$Y' = \bar{\bar{y}}' + \bar{\bar{y}}''x + \bar{\bar{y}}'''\frac{x^2}{2} + (-6\bar{P}_2 y'' - 4\bar{P}_3\bar{\bar{y}}' - \bar{P}_4\bar{\bar{y}})\frac{x^3}{\lfloor 3}$$
$$+ (-6\bar{P}_2\bar{\bar{y}}''' - 6\bar{P}_2'\bar{\bar{y}}'' - 4\bar{P}_3\bar{\bar{y}}'' - 4\bar{P}_3'\bar{\bar{y}}' - \bar{P}_4\bar{\bar{y}}' - \bar{P}_4'\bar{\bar{y}})\frac{x^4}{\lfloor 4} + \cdots\cdots.$$

We may then write Y and Y' in the form

$$Y = x_1\bar{\bar{y}} + x_2\bar{\bar{y}}' + x_3\bar{\bar{y}}'' + x_4\bar{\bar{y}}''',$$
$$Y' = z_1\bar{\bar{y}} + z_2\bar{\bar{y}}' + z_3\bar{\bar{y}}'' + z_4\bar{\bar{y}}''',$$

where

$$x_1 = 1 - \bar{P}_4\frac{x^4}{\lfloor 4} - \bar{P}_4'\frac{x^5}{\lfloor 5} - \cdots\cdots,$$

$$x_2 = x - 4\bar{P}_3\frac{x^4}{\lfloor 4} + (-4\bar{P}_3' - \bar{P}_4)\frac{x^5}{\lfloor 5} + \cdots\cdots,$$

$$x_3 = \frac{x^2}{2} - 6\bar{P}_2\frac{x^4}{\lfloor 4} + (-6\bar{P}_2' - 4\bar{P}_3)\frac{x^5}{\lfloor 5} + \cdots\cdots,$$

$$x_4 = \frac{x^3}{\lfloor 3} - 6\bar{P}_2\frac{x^5}{\lfloor 5} + \cdots\cdots,$$

$$z_1 = -\bar{P}_4 \frac{x^3}{\lfloor 3} - \bar{P}_4' \frac{x^4}{\lfloor 4} + \cdots\cdots,$$

$$z_2 = 1 - 4\bar{P}_3 \frac{x^3}{\lfloor 3} + (-4\bar{P}_3' - \bar{P}_4)\frac{x^4}{\lfloor 4} + \cdots\cdots,$$

$$z_3 = x - 6\bar{P}_2 \frac{x^3}{\lfloor 3} + (-6\bar{P}_2' - 4\bar{P}_3)\frac{x^4}{\lfloor 4} + \cdots\cdots,$$

$$z_4 = \frac{x^2}{2} - 6\bar{P}_2 \frac{x^4}{\lfloor 4} + (-12\bar{P}_2' - 4\bar{P}_3)\frac{x^5}{\lfloor 5} + \cdots\cdots.$$

We shall follow the usual notation for the homogeneous line coordinates and let $w_{ik} = x_i z_k - x_k z_i$ $(i = 1,2,3,4;\ k = 1,2,3,4)$. If we carry these out to the first five powers of x, we obtain

$$w_{34} = \frac{x^4}{12} + \cdots\cdots,$$

$$w_{23} = \frac{x^2}{2} - 3\bar{P}_2 \frac{x^4}{4} + (\bar{P}_3 - 6\bar{P}_2')\frac{x^5}{30} + \cdots,$$

$$w_{14} = \frac{x^2}{2} - \bar{P}_2 \frac{x^4}{4} - (12\bar{P}_2' + 4\bar{P}_3)\frac{x^5}{\lfloor 5} + \cdots.$$

From these we may eliminate the first four powers of x, obtaining

$$w_{23} - w_{14} + 6\bar{P}_2 w_{34} = \frac{\bar{\theta}_3 x^5}{30} + \cdots\cdots.$$

Thus we see that

$$w_{23} - w_{14} + 6\bar{P}_2 w_{34} = 0,$$

is the equation of the osculating linear complex of the curve. If we have $\bar{\theta}_3 = 0$, the equation of the osculating linear complex contains no powers of x lower than the sixth and all the tangents to $C_{\bar{y}}$ belong to the osculating linear complex. This result shows that, <u>if, for a given curve $\bar{\theta}_3 = 0$, all the tangents to the curve belong to the osculating linear complex of the curve.</u>

II. Interpretation of $\bar{\theta}_3 = 0, \bar{\theta}_4 = 0$.

Wilczynski[#] has shown that equation (1) may be reduced to the form

(28) $\quad \bar{\bar{y}}^{IV} + 4\bar{P}_3\,\bar{\bar{y}}' + \bar{P}_4\,\bar{\bar{y}} = 0,$

by choosing $\mu = \frac{6}{5}\,p_2$. This form is called the Laguerre-Forsyth form and is characterized by $\bar{P}_1 = 0$, $\bar{P}_2 = 0$. Under these conditions, if $\bar{\theta}_3 = 0$, we see that $\bar{P}_3$ must be zero. If $\bar{\theta}_4 = 0$ also, we must have $\bar{P}_4 = 0$, and equation (28) becomes

(29) $\quad \bar{\bar{y}}^{IV} = 0.$

If we solve (29), we obtain the solution

$$\bar{\bar{y}} = ax^3 + bx^2 + cx + d,$$

where a, b, c, and d are arbitrary constants. The coordinates of any point $\bar{\bar{y}}$ of the curve are of the form

$$\bar{\bar{y}}_i = a_i x^3 + b_i x^2 + c_i x + d_i \quad (i = 1,2,3,4),$$

which is the parametric equation of the curve $C_{\bar{\bar{y}}}$. Any plane thru $\bar{\bar{y}}$ will have the equation

$$A\bar{\bar{y}}_1 + B\bar{\bar{y}}_2 + C\bar{\bar{y}}_3 + D\bar{\bar{y}}_4 = 0,$$

where A, B, C, and D are arbitrary constants. If we solve the equation of the plane with the parametric equation of the curve, we obtain a cubic expression in x, thus showing that the curve $C_{\bar{\bar{y}}}$ has three intersections with the plane. We may now state that, if, for a given space curve $\bar{\theta}_3 = 0, \bar{\theta}_4 = 0$, the curve is a space cubic. Since all the coefficients are zero, the space cubic has no projective differential properties.

[#] Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces", page 241.

III. Interpretation of the vanishing of the invariants $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$.

A. Under conditions $\bar{M} = 0, \bar{\Theta}_3 \neq 0$.

We have equation (1) in the form (4), that is,

$$\bar{y}^{IV} + 6\bar{P}_2\bar{y}'' + 4\bar{P}_3\bar{y}' + \bar{P}_4\bar{y} = 0.$$

We have also that

$$\bar{M} = 81\bar{P}_2^2 - 15\bar{P}_2'' + 40\bar{P}_3' - 25\bar{P}_4 = 0,$$

$$\bar{\Theta}_3 = 3\bar{P}_2' - 2\bar{P}_3 \neq 0.$$

1. If $\bar{P}_2 = 0$, $\bar{P}_3 \neq 0$, $P_4 \neq 0$, equation (4) becomes

$$\bar{y}^{IV} + 4\bar{P}_3\bar{y}' + \bar{P}_4\bar{y} = 0,$$

and we have $\bar{y}^{IV}$ as a linear combination of $\bar{y}$ and $\bar{y}'$, which shows that $\bar{y}^{IV}$ lies on the line $\bar{y}\bar{y}'$. Therefore $\bar{P}_2 = 0$ characterizes a set of curves such that the tangents to the $C_{\bar{y}'''}$ curve intersect the $\bar{y}\bar{y}'$ line, that is, the tangents lie in the plane $x_3 = 0$.

2. If $\bar{P}_3 = 0$, $P_2 \neq$ const., $P_4 \neq 0$, equation (4) becomes

$$\bar{y}^{IV} + 6\bar{P}_2\bar{y}'' + \bar{P}_4\bar{y} = 0,$$

and we see that $\bar{P}_3 = 0$ characterizes a set of curves such that the tangents to $C_{\bar{y}'''}$ intersect the line $\bar{y}\bar{y}''$, that is, the tangents lie in the plane $x_2 = 0$.

3. If $\bar{P}_4 = 0$, $\bar{P}_2 \neq 0$, $\bar{P}_3 \neq 0$, equation (4) becomes

$$\bar{y}^{IV} + 6\bar{P}_2\bar{y}'' + 4\bar{P}_3\bar{y}' = 0.$$

If we let $z = \bar{y}'$ and substitute in this equation, we obtain the new equation

$$z''' + 6\bar{P}_2 z' + 4P_3 z = 0.$$

Since this is a third order linear differential equation, it is the equation of a plane curve. Therefore $\bar{P}_4 = 0$ characterizes a set of curves such that the $C_{\bar{\bar{y}}'}$ curve is a plane curve. The tangents to the $C_{\bar{\bar{y}}'''}$ curve lie in the plane $x_1 = 0$.

4. If $\bar{P}_2 = 0$, $\bar{P}_4 = 0$, we must have $\bar{P}_3$ a constant different from zero. The equation (4) becomes

$$\bar{\bar{y}}^{iv} + 4\bar{P}_3\bar{\bar{y}}' = 0.$$

Therefore $\bar{P}_2 = 0$, $\bar{P}_4 = 0$ characterizes a set of curves such that $C_{\bar{\bar{y}}'}$ is a plane curve and the tangents to $C_{\bar{\bar{y}}''}$ go thru $\bar{\bar{y}}'$.

5. If $\bar{P}_3 = 0$, $\bar{P}_4 = 0$, $\bar{P}_2' \neq$ const., equation (4) becomes

$$\bar{\bar{y}}^{iv} + 6\bar{P}_2\bar{\bar{y}}'' = 0.$$

If we let $z = \bar{\bar{y}}''$ and substitute in this equation, we obtain the new equation

$$z'' + 6\bar{P}_2 z = 0.$$

Since this is a second order linear differential equation, it is the equation of a straight line. Therefore $\bar{P}_3 = 0$, $\bar{P}_4 = 0$ characterizes a set of curves such that the $C_{\bar{\bar{y}}''}$ curve is a straight line and the tangents to $C_{\bar{\bar{y}}''}$ go thru $\bar{\bar{y}}''$.

B. Under the conditions $\bar{\Theta}_4 \neq 0$, $\bar{\Theta}_3 = 0$.

We have

$$\bar{\Theta}_4 = \bar{P}_4 - \tfrac{9}{5}\bar{P}_2'' - \tfrac{81}{25}\bar{P}_2^2 \neq 0,$$

$$\bar{\Theta}_3 = 3\bar{P}_2' - 2\bar{P}_3 = 0.$$

By the same methods as used in (A), we find the following facts.

1. If $\bar{P}_2 = 0$, we must have $\bar{P}_3 = 0$, $\bar{P}_4 \neq 0$. Therefore

$\overline{P}_2 = 0$ is sufficient to characterize a set of curves such that the tangents to $C_{\bar{\bar{y}}}'''$ go thru $\bar{\bar{y}}$.

2. If $\overline{P}_3 = 0$, we must have $\overline{P}_2$ equal to zero or a constant different from zero. If $\overline{P}_3 = 0$, $\overline{P}_2 =$ const., $P_4 \neq 0$, we have a set of curves such that the tangents to the $C_{\bar{\bar{y}}}'''$ curve lie in the plane $x_2 = 0$.

3. If $\overline{P}_4 = 0$, we cannot have $\overline{P}_2 = 0$. We find that $\overline{P}_4 = 0$, $\overline{P}_2 =$ const. characterizes a set of curves such that $C_{\bar{\bar{y}}}'$ is a plane curve and the tangents to the $C_{\bar{\bar{y}}}'''$ curve lie in the plane $x_1 = 0$. We find also that $\overline{P}_4 = 0$, $\overline{P}_2 =$ const. characterizes a set of curves such that the $C_{\bar{\bar{y}}}'''$ curve is a straight line and the tangents to the $C_{\bar{\bar{y}}}'''$ curve go thru $\bar{\bar{y}}''$.

The Derivative Curves.

As $\bar{\bar{y}}$ moves along the curve, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$,... generate curves which we shall call derivative curves. The order of the derivative shall indicate the order of the derivative curve. Under certain conditions on the coefficients of equation (4)

$$\bar{\bar{y}}^{IV} + 6\bar{P}_2\bar{\bar{y}}'' + 4\bar{P}_3\bar{\bar{y}}' + \bar{P}_4\bar{\bar{y}} = 0,$$

some of the derivative curves may be plane curves or straight lines. We shall now find some of these conditions.

A. Under the conditions $\bar{M} = 0, \bar{\Theta}_3 \neq 0$.

1. In the previous section we found that $C_{\bar{\bar{y}}'}$, the first derivative curve would be a plane curve if $\bar{P}_4 = 0$. This curve cannot be a straight line if the given curve $C_{\bar{\bar{y}}}$ is a space curve.

2. If $\bar{P}_4 = 0$, $\bar{P}_3 = 0$, equation (4) reduces to

$$\bar{\bar{y}}^{IV} + 6\bar{P}_2\bar{\bar{y}}'' = 0.$$

From this we see that $C_{\bar{\bar{y}}''}$ is a straight line if $\bar{P}_4 = 0$, $\bar{P}_3 = 0$. If we differentiate (4) once, we obtain the equation

$$(30) \qquad \bar{\bar{y}}^{V} + 6\bar{P}_2\bar{\bar{y}}''' + (6\bar{P}_2' + 4\bar{P}_3)\bar{\bar{y}}'' + (4\bar{P}_3' + \bar{P}_4)\bar{\bar{y}}' + \bar{P}_4'\bar{\bar{y}} = 0.$$

From this equation we see that the $C_{\bar{\bar{y}}''}$ curve will be a plane curve if

$$\bar{P}_4' = 0, \quad 4\bar{P}_3' + P_4 = 0.$$

that is, if

$$\bar{P}_4 = \text{const.}, \quad \bar{P}_3 = mx + n, \text{ where } m = -\frac{c}{4}$$

3. From equation (30), we see that the $C_{\bar{\bar{y}}'''}$ curve will

be a straight line if

$$\bar{P}_4' = 0,\ 4\bar{P}_3' + \bar{P}_4 = 0,\ 3\bar{P}_2' + 2\bar{P}_3 = 0,$$

that is, if

$$\bar{P}_4 = \text{const.}, \bar{P}_3 = mx + n,\ \ P_2 = kx^2 + lx + p,$$

where $m = -\frac{c}{4}$, $k = \frac{c}{12}$, $l = -\frac{2}{3}\eta$. If we differentiate equation (30) we obtain the new equation

$$(31)\qquad \bar{\bar{y}}^{VI} + 6\bar{P}_2\bar{\bar{y}}^{IV} + (12\bar{P}_2' + 4\bar{P}_3)\bar{\bar{y}}''' + (6\bar{P}_2'' + 8\bar{P}_3' + \bar{P}_4)\bar{y}''$$

$$+ (4\bar{P}_3'' + 2\bar{P}_4')\bar{y}' + \bar{P}_4''\bar{y} = 0.$$

From this equation (31), we see that $C_{\bar{\bar{y}}}''$ will be a plane curve if

$$\bar{P}_4'' = 0,\ 4\bar{P}_3'' + 2\bar{P}_4' = 0,\ \ 6\bar{P}_2'' + 8\bar{P}_3' + \bar{P}_4 = 0,$$

that is, if

$$\bar{P}_4 = c_1x + c_2,\quad P_3 = a_1x^2 + a_2x + a_3,$$

$$P_2 = b_1x^3 + b_2x^2 + b_3x + b_4,$$

where $a_1 = -\frac{c_1}{4}$, $b_1 = \frac{c_1}{12}$, $b_2 = -\frac{(8a_2 + c_2)}{12}$.

B. Under the conditions $\bar{\theta}_3 = 0, \bar{\theta}_4 \neq 0$.

1. The curve $C_{\bar{\bar{y}}}'$ cannot be a straight line if $C_{\bar{\bar{y}}}$ is a space curve. The curve $C_{\bar{\bar{y}}}'$ will be a plane curve if $\bar{P}_4 = 0$.

2. From equation (4), we see that the $C_{\bar{\bar{y}}}''$ curve will be a straight line if $\bar{P}_4 = 0$, $\bar{P}_3 = 0$. Since $\bar{\theta}_3 = 0$, $\bar{P}_3 = 0$, implies that $\bar{P}_2' = 0$, that is that $\bar{P}_2 = \text{const.}$, or $\bar{P}_2 = 0$. If $\bar{P}_2 = 0$, we have $\bar{\bar{y}}^{IV} = 0$, which case we have discussed previously. If $\bar{P}_4 = 0$, $\bar{P}_3 = 0$, $\bar{P}_2 = \text{const.}$, the $C_{\bar{\bar{y}}}''$ curve will be a straight line.

We see from equation (30), that $C_{\bar{y}}''$ will be a plane curve if

$$\bar{P}_4' = 0, \quad 4\bar{P}_3' + \bar{P}_4 = 0,$$

that is, if

$$P_4 = c_1, \; P_3 = -\frac{c_1}{4}x + c_2, \quad P_2 = -\frac{c_1}{12}x^2 + \frac{2c_2}{3}x + c_3.$$

The conditions on $\bar{P}_3$ and $\bar{P}_4$ are the same as for the case $\bar{M} = 0, \bar{\Theta}_3 \neq 0$. In that case, however, there was no limitation on $\bar{P}_2$ but for the case $\bar{\Theta}_3 = 0, \bar{\Theta}_4 \neq 0$, the coefficient $\bar{P}_2$ must be a quadratic expression in x.

3. We see from the equation (30), that the curve $C_{\bar{y}}'''$ will be a straight line if

$$\bar{P}_4' = 0, \quad 4\bar{P}_3' + \bar{P}_4 = 0, \quad 3\bar{P}_2' + 2\bar{P}_3 = 0.$$

Since $\bar{\Theta}_3 = 0$, we have $3\bar{P}_2' - 2\bar{P}_3 = 0$, and accordingly we must have $\bar{P}_2' = \bar{P}_3 = 0$. Therefore we must have

$$\bar{P}_4 = 0, \; \bar{P}_3 = 0, \; \bar{P}_2 = \text{const.},$$

in order to have the $C_{\bar{y}}'''$ curve a straight line.

From equation (31), we see that the $C_{\bar{\bar{y}}}''$ curve will be a plane curve if

$$\bar{P}_4'' = 0, \; 2\bar{P}_3'' + \bar{P}_4' = 0, \quad 12\bar{P}_2'' + 8\bar{P}_3' + \bar{P}_4 = 0.$$

Under the condition that $\bar{\Theta}_3 = 0$, we find that this means

$$\bar{P}_4 = 0, \quad \bar{P}_3 = \text{const.}, \; \bar{P}_2 = \frac{2c}{3}x + c_2.$$

C. If we have $\bar{P}_4 = 0$, $\bar{P}_3 = 0$ under the conditions that $\bar{\Theta}_3 = 0, \bar{\Theta}_4 \neq 0$, we will have $\bar{P}_2' = 0$ and equation (4) becomes

(32) $\quad \bar{\bar{y}}^{IV} + 6\bar{P}_2\bar{\bar{y}}'' = 0.$

This equation (32) shows that the $C_{\bar{y}}''$ curve is a straight line under the above conditions. If we differentiate the equation (32), we obtain the equation

(33) $\quad \bar{\bar{y}}^{V} + 6\bar{P}_2\bar{\bar{y}}''' = 0,$

which shows that these conditions will make the $C_{\bar{\bar{y}}''}$ curve a straight line. If we continue the differentiation of equation (32) under these conditions,we will always obtain an equation of the form

$$\bar{\bar{y}}^{(k)} + 6\bar{P}_2\bar{\bar{y}}^{(k-2)} = 0,$$

where k stands for the order of the derivative.

Therefore, if all the tangents of a space curve belong to a linear complex, the necessary and sufficient conditions $\bar{P}_4 = 0$, $\bar{P}_3 = 0$, which make the second derivative ~~curve~~ a straight line,will make all the higher derivative curves be straight lines. The first derivative curve cannot be a straight line.

D. If we have $\bar{P}_4 = 0$, under the conditions that $\bar{\theta}_3 = 0, \bar{\theta}_4 \neq 0,$ equation (4) becomes

(34) $\quad \bar{\bar{y}}^{IV} + 6\bar{P}_2\bar{y}'' + 4\bar{P}_3\bar{y}' = 0.$

Therefore $\bar{P}_4 = 0$ is sufficient to make the $C_{\bar{\bar{y}}'}$ curve a plane curve. If we differentiate (34) once, we obtain the new equation

(35) $\quad \bar{y}^{V} + 6\bar{P}_2\bar{\bar{y}}''' + (6\bar{P}_2' + 4\bar{P}_3)\bar{y}'' + 4\bar{P}_3'\bar{y}' = 0.$

If in addition to $\bar{P}_4 = 0$, we have $\bar{P}_3 =$ const., we shall $\bar{P}_3' = 0$ and $\bar{P}_2'' = 0$, and the curve $C_{\bar{\bar{y}}''}$ will be a plane curve. If with $\bar{P}_4 = 0$, $\bar{P}_3 =$ const., we differentiate (34) twice, we obtain the equation

$$(36) \qquad \bar{\bar{y}}^{VI} + 6\bar{P}_2\,\bar{\bar{y}}^{IV} + (12\bar{P}_2' + 4\bar{P}_3\,)\bar{\bar{y}}''' = 0,$$

which shows that the $C_{\bar{\bar{y}}'''}$ curve is a plane curve. If we continue the differentiation of equation (34) under these conditions, we will always have an equation of the form

$$\bar{\bar{y}}^{(k)} + 6\bar{P}_2\,\bar{\bar{y}}^{(k-2)} + (\qquad)\bar{\bar{y}}^{(k-3)} = 0,$$

where the coefficient of $\bar{\bar{y}}^{(k-3)}$ is some function of $\bar{P}_2'$ and $\bar{P}_3$..

Therefore, if all the tangents of a space curve belong to a linear complex, the conditions $\bar{P}_4 = 0$, $\bar{P}_3 =$ const. which are sufficient but not necessary to make the first derivative curve a plane curve, will make all the higher derivative curves be plane curves. The conditions $\bar{P}_4 = 0$, $\bar{P}_3 =$ const., are necessary and sufficient conditions that the $C_{\bar{\bar{y}}'''}$ curve be a plane curve.

Summary of Interpretation of the Vanishing of Some of the Relative Invariants.

Table I. $\bar{\Theta}_3 = 2\bar{P}_3 - 3\bar{P}_2' \neq 0$, $\bar{M} = 81\bar{P}_2^2 - 15\bar{P}_2'' + 40\bar{P}_3' - 25\bar{P}_4 = 0$.

Conditions	Interpretation
$\bar{P}_2 = 0$, $\bar{P}_3 \neq 0$, $\bar{P}_4 \neq 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ lies in plane $x_3 = 0$.
$\bar{P}_3 = 0$, $(\therefore \bar{P}_2 \neq c)$, $\bar{P}_4 \neq 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ lies in plane $x_2 = 0$.
$\bar{P}_4 = 0$, $\bar{P}_2 \neq 0$, $\bar{P}_3 \neq 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ lies in plane $x_1 = 0$. $C_{\bar{\bar{y}}}'$ is a plane curve.
$\bar{P}_2 = 0$, $(\therefore \bar{P}_3 = c \neq 0)$, $\bar{P}_4 = 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ goes thru $\bar{\bar{y}}'$. $C_{\bar{\bar{y}}}'$ is a plane curve.
$\bar{P}_3 = 0$, $(\therefore P_2 \neq c)$, $\bar{P}_4 = 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ goes thru $\bar{\bar{y}}''$. $C_{\bar{\bar{y}}}''$ is a straight line.

Table II. $\bar{\Theta}_3 = 2\bar{P}_3 - 3\bar{P}_2' = 0$, $\bar{\Theta}_4 = P_4 - \frac{9}{5}\bar{P}_2'' - \frac{81}{25}\bar{P}_2^2 \neq 0$.

Conditions	Interpretation
$\bar{P}_2 = 0$, $(\therefore \bar{P}_3 = 0)$, $P_4 \neq 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ goes thru $\bar{\bar{y}}$.
$\bar{P}_3 = 0$, $(\therefore P_2 = c)$, $\bar{P}_4 \neq 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ lies in plane $x_2 = 0$.
$\bar{P}_4 = 0$, $(\therefore P_2 \neq 0)$, $\bar{P}_2 \neq c$, $P_3 \neq 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ lies in plane $x_1 = 0$. $C_{\bar{\bar{y}}}'$ is a plane curve.
$\bar{P}_4 = 0$, $\bar{P}_2 = c$, $\bar{P}_3 = 0$.	Line $\bar{\bar{y}}'''\bar{\bar{y}}^{IV}$ goes thru $\bar{\bar{y}}''$. $C_{\bar{\bar{y}}}''$ is a straight line.

Table III. $\bar{\Theta}_3 = 2\bar{P}_3 - 3\bar{P}_2' \neq 0$, $\bar{M} = 81\bar{P}_2^2 - 15\bar{P}_2'' + 40\bar{P}_3' - 25\bar{P}_4 = 0$.

Curve	Conditions for st. line	Conditions for plane curve
$C_{\bar{y}}'$		$\bar{P}_4 = 0$.
$C_{\bar{y}}''$	$\bar{P}_4 = 0$, $\bar{P}_3 = 0$.	$\bar{P}_4 = c_1$, $\bar{P}_3 = -\frac{c_1}{4}x + c_2$.
$C_{\bar{y}}'''$	$\bar{P}_4 = c_1$, $\bar{P}_3 = -\frac{c_1}{4}x + c_2$. $\bar{P}_2 = \frac{c_1}{12}x^2 - \frac{2}{3}c_2 x + c_3$.	$\bar{P}_4 = c_1 x + c_2$, $\bar{P}_3 = -\frac{c_1}{4}x^2 + c_3 x + c_4$. $\bar{P}_2 = \frac{c_1}{12}x^3 - \frac{(8c_3 + c_2)}{12}x^2 + c_5 x + c_6$

Table IV. $\bar{\Theta}_3 = 2\bar{P}_3 - 3\bar{P}_2' = 0$, $\bar{\Theta}_4 = \bar{P}_4 - \frac{9}{5}\bar{P}_2'' - \frac{81}{25}\bar{P}_2^2 \neq 0$.

Curve	Conditions for st. line	Conditions for plane curve
$C_{\bar{y}}'$		$\bar{P}_4 = 0$.
$C_{\bar{y}}''$	$\bar{P}_4 = 0$, $\bar{P}_3 = 0$, $(\therefore \bar{P}_2 = c_1)$.	$\bar{P}_4 = c_1$, $\bar{P}_3 = -\frac{c_1}{4}x + c_2$, $\bar{P}_2 = -\frac{c_1}{12}x^2 + \frac{2c_2}{3}x + c_3$.
$C_{\bar{y}}'''$	$\bar{P}_4 = 0$, $\bar{P}_3 = 0$, $\bar{P}_2' = 0$, $(\therefore P_2 = c_1)$.	$\bar{P}_4 = 0$, $\bar{P}_3 = c_1$, $\bar{P}_2 = \frac{2c_1}{3}x + c_2$.

Ruled Surfaces Connected with the Tetrahedron of Reference.

Each of the six edges of the tetrahedron of reference will define a ruled surface, which may or may not be a developable surface. However, before we can discuss these, it will be necessary to give briefly some of the facts concerning such surfaces.

A ruled surface is formed by the lines joining the corresponding points of two curves, which do not lie in the same plane. It is assumed that the curves are defined parametrically and the corresponding points are the points given by the same value of the parameter. A line joining two corresponding points is called a generator. A developable surface is a ruled surface formed by all the lines tangent to a given space curve, which curve is called the edge of regression. The curve completely determines the developable surface and accordingly, all the properties of the surface may be obtained from the equation of the curve.

Wilczynski[#] has shown that, if we consider two curves C_y and C_z generated by two points y and z, the differential equation for the non-developable ruled surface determined by the line joining y and z is of

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[#] Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces", chap. V, VI.

the form

$$y''+2p_{11}y'+2p_{12}z'+q_{11}y+q_{12}z=0,$$

(37) $$z''+2p_{21}y'+2p_{22}z'+q_{21}y+q_{22}z=0,$$

where the p's and q's are functions of x. If the tangents to the C_y curve intersect the tangents to the C_z curve, the ruled surface degenerates into a developable surface.

At every point of the surface there is one curve such that a tangent to the curve at the point intersects three consecutive generators. The tangent is called an asymptotic tangent. If all the tangents along a curve are asymptotic, the curve is called an asymptotic curve. If in the equation (37) we have $p_{12}=0$, the C_y curve is asymptotic. If $p_{21}=0$ the C_z curve is asymptotic.

In general there are two points, called flecnode points, on each generator at which an asymptotic tangent intersects four consecutive generators of the surface. There are four expressions in the p's and q's, given by

$$u_{ik}=p'_{ik}-q_{ik}+\Sigma p_{ij}p_{jk} \quad (k=1,2,\ i=1,2,\ j=1,2),$$

which we shall use. If $u_{12}=0$, the C_y curve is a flecnode curve, that is, it passes thru a flecnode point on each generator. If $u_{21}=0$, the C_z curve is a flecnode curve. If $u_{11}-u_{22}=0$, the two flecnode points on a generator are harmonic conjugates with respect to y and z.

In our study of the developable and ruled surfaces generated by the edges of the tetrahedron of reference obtained under the conditions $\overline{\Theta}_3\neq 0$, $\overline{M}=0$, we shall for convenience let

$$\bar{\bar{y}} = y,\ \bar{\bar{y}}' = r,\ \bar{\bar{y}}'' = z,\ \bar{\bar{y}}''' = s.$$

The line yr is tangent to the curve C_y and generates a developable with C_y as the edge of regression. Similarly the rz line generates a developable with C_r as the edge of regression, and the zs line generates a developable with C_z as the edge of regression.

If, in our study of the yz line, we make use of equation (4), we obtain by differentiation and the proper substitutions, the equations

$$(38)\qquad \begin{aligned} &y'' - z = 0, \\ &z'' + 4\bar{P}_3 y' + \bar{P}_4 y + 6\bar{P}_2 z = 0, \end{aligned}$$

which are of the form (37) and are the equations of a ruled surface, which we shall call S_1. Since $p_{12} = 0$, the curve C_y is an asymptotic curve on S_1. Since $p_{21} = 2\bar{P}_3$, we shall have C_z as an asymptotic curve on S when $\bar{P}_3 = 0$. From equations (38) we find that

$$u_{11} = 0,\ u_{22} = -6\bar{P}_2,\ u_{12} = -1,\ u_{21} = 2\bar{P}_3' - \bar{P}_4.$$

Therefore if $\bar{P}_2 = 0$, we have y as the harmonic conjugate of z with respect to the flecnode points on yz. We see also that, since u_{12} cannot be zero, the C_y curve cannot be a flecnode curve on S_1. The C_z curve will be a flecnode curve if $2\bar{P}_3' - \bar{P}_4 = 0$. It should be remembered that $\bar{P}_2, \bar{P}_3, \bar{P}_4$ are relative invariants.

By the use of equation (4), we find that if $\bar{P}_2 \neq 0$, the line on y and s generates a ruled surface S_2, given by the equations

$$
(39)\quad \begin{aligned}
&y'' + \frac{2\bar{P}_3}{3\bar{P}_2} y' + \frac{1}{6\bar{P}_2} s' + \frac{\bar{P}_4}{6\bar{P}_2} y = 0, \\
&s'' + \left(\bar{P}_4 + 4\bar{P}_3' - \frac{8\bar{P}_3^2}{3\bar{P}_2} - \frac{4\bar{P}_3\bar{P}_2'}{\bar{P}_2}\right) y' + \left(\frac{-2\bar{P}_3 - 3\bar{P}_2'}{3\bar{P}_2}\right) s' + \left(\bar{P}_4' - \frac{2\bar{P}_3\bar{P}_4}{3\bar{P}_2} - \frac{\bar{P}_2'\bar{P}_4}{\bar{P}_2}\right) y + 6\bar{P}_2 s = 0.
\end{aligned}
$$

The coefficients and the u expressions for this surface are too involved to yield any simple geometrical interpretation. If $\bar{P}_2 = 0$ we have the Laguerre-Forsyth equation

(40) $\bar{\bar{y}}^{iv} + 4\bar{P}_3 \bar{\bar{y}}' + \bar{P}_4 \bar{\bar{y}} = 0.$

Since $s = \bar{\bar{y}}''$ this equation may be written in the form

(41) $s' + 4\bar{P}_3 \bar{\bar{y}}' + \bar{P}_4 \bar{\bar{y}} = 0,$

which shows that s' lies on the $\bar{\bar{y}}\bar{\bar{y}}'$ line, that is, that the tangents to C_y intersect the tangents to C_s. Therefore the ys line generates a developable surface. There is some point T on the ys line at which the ys line touches the edge of regression of the developable. We may represent the point T by the expression

$$T = ay + bs.$$

The point T' must be on the ys line, and therefore must be a linear combination of y and s. If we differentiate $T = ay + bs$, we find that

$$T' = a'y + ay' + bs' + b's,$$

which after a substitution for s' of the value obtained from equation (41) becomes

(42) $T' = a'y + ay' + b's - 4\bar{P}_3 by' - \bar{P}_4 by.$

This equation (42) shows that we must have $a = 4\bar{P}_3 b$, if T' is to be on the ys line. Therefore

$$T = 4\bar{P}_3 by + bs = b(4\bar{P}_3 y + s).$$

Therefore, if $\overline{P}_2 = 0$, the ys line generates a developable surface having C_T as the edge of regression, where

$$T = 4P_3\,y + s.$$

When we study the rs line, we obtain the equations

$$(43)\qquad \begin{aligned} &r'' - s = 0,\\ &s'' + \Big(6\overline{P}_2' + 4\overline{P}_3 - \frac{6\overline{P}_2\overline{P}_4'}{\overline{P}_4}\Big)r' - \frac{\overline{P}_4'}{\overline{P}_4}s' + \Big(4\overline{P}_3' + \overline{P}_4 - \frac{4\overline{P}_3\overline{P}_4'}{\overline{P}_4}\Big)r + 6\overline{P}_2 s = 0, \end{aligned}$$

which shows that the rs line generates a ruled surface S_3 when $\overline{P}_4 \neq 0$, and that the C_r curve is an asymptotic curve on the surface S_3.

If $\overline{P}_4 = 0$, we have the equation (4) in the form

$$(44)\qquad \bar{\bar{y}}^{iv} + 6\overline{P}_2\,\bar{y}'' + 4\overline{P}_3\,\bar{y}' = 0.$$

If we differentiate equation (44) once and make use of the relations $r = \bar{y}'$, $s = \bar{y}'''$, we obtain the equations

$$(45)\qquad \begin{aligned} &r'' - s = 0,\\ &s'' + 6\overline{P}_2 s + (6\overline{P}_2' + 4\overline{P}_3)r' + 4\overline{P}_3' r = 0, \end{aligned}$$

which show that the rs line generates a ruled surface when $\overline{P}_4 = 0$. In equation (45) $p_{12} = 0$, and $p_{21} = 3\overline{P}_2' + 2\overline{P}_3$. Therefore C_r is an asymptotic curve on the ruled surface. We shall have C_s as an asymptotic curve if $2\overline{P}_3 + 3\overline{P}_2' = 0$. We find from equation (45) that $u_{12} = 1$, and $u_{21} = \overline{\Theta}_3'$. Therefore C_r cannot be a flecnode curve, but C_s will be a flecnode curve if $\overline{\Theta}_3 = \text{const}$.

For the tetrahedron of reference determined under the conditions $\bar{\theta}_3 = 0, \bar{\theta}_4 \neq 0$, we have

$$
(46) \qquad
\begin{aligned}
\bar{y}_1 &= \bar{\bar{y}}, \\
\bar{y}_2 &= \bar{\bar{y}}', \\
\bar{y}_3 &= \tfrac{9}{5}\bar{P}_2\bar{\bar{y}} + \bar{\bar{y}}'', \\
\bar{y}_4 &= \left(\frac{8\bar{P}_3 - 3\bar{P}_2'}{5}\right)\bar{\bar{y}} + \tfrac{21}{5}\bar{P}_2\bar{\bar{y}}' + \bar{\bar{y}}'''.
\end{aligned}
$$

From these relations (46) we see that the $\bar{y}_1\bar{y}_2$ line generates a developable surface having $C_{\bar{y}_1}$ as the edge of regression. We also see that $\bar{y}_3$ is on the $\bar{\bar{y}}\bar{\bar{y}}''$ line of the tetrahedron, so that the $\bar{y}_1\bar{y}_3$ line generates the same ruled surface as the $\bar{\bar{y}}\bar{\bar{y}}''$ line. The equation of this ruled surface is

$$
(47) \qquad
\begin{aligned}
&\bar{y}_1'' + \tfrac{9}{5}\bar{P}_2\bar{y}_1 - \bar{y}_3 = 0, \\
&\bar{y}_3'' + \tfrac{12}{5}\bar{P}_2'\bar{y}_1' + (\bar{P}_4 - \tfrac{9}{5}\bar{P}_2'' - \tfrac{189}{25}\bar{P}_2^2)\bar{y}_1 + \tfrac{21}{5}\bar{P}_2\bar{y}_3 = 0.
\end{aligned}
$$

From equation (47) we see that $p_{12} = 0$, $p_{21} = \frac{6}{5}\bar{P}_2'$, $u_{11} - u_{22} = \frac{12}{5}\bar{P}_2$. Therefore $C_{\bar{y}_1}$ is an asymptotic curve on the surface. If $\bar{P}_2 = 0$, $\bar{y}_1$ will be the harmonic conjugate of $\bar{y}_3$ with respect to the flecnode points on the $\bar{y}_1\bar{y}_3$ line, and $C_{\bar{y}_3}$ will be an asymptotic curve on the surface. The $C_{\bar{y}_3}$ curve will be asymptotic if $\bar{P}_2$ equals a constant different from zero.

When we consider the $\bar{y}_2\bar{y}_3$ line, we obtain the equations

$$
(48) \qquad
\begin{aligned}
&\bar{y}_2'' - \frac{\bar{P}_2'}{\bar{P}_2}\bar{y}_2' - \bar{y}_3' + \tfrac{9}{5}\bar{P}_2\bar{y}_2 + \frac{\bar{P}_2'}{\bar{P}_2}\bar{y}_3 = 0, \\
&\bar{y}_3'' + \left(\tfrac{21}{5}\bar{P}_2 + \frac{\bar{P}_2''}{\bar{P}_2} - \frac{5\bar{P}_4}{9\bar{P}_2}\right)\bar{y}_2' + \tfrac{12}{5}\bar{P}_2'\bar{y}_2 + \left(\frac{5\bar{P}_4 - 9\bar{P}_2''}{9\bar{P}_2}\right)\bar{y}_3 = 0.
\end{aligned}
$$

Therefore if $\bar{P}_2 \neq 0$, the $\bar{y}_2\bar{y}_3$ line generates a ruled surface.

If $\bar{P}_2 = 0$, we see from relations (46) that $\bar{y}_3$ coincides with $\bar{\bar{y}}''$ and the $\bar{y}_2\bar{y}_3$ line is the $\bar{\bar{y}}'\bar{\bar{y}}''$ line and generates a developable surface which has $C_{\bar{y}_2}$ as the edge of regression.

The $\bar{y}_1\bar{y}_4$ line generates a ruled surface if $\bar{P}_2 \neq 0$. The equations for this surface do not yield any simple geometrical facts depending on the vanishing of the invariants. If $\bar{P}_2 = 0$, we must have $\bar{P}_3 = 0$, and equation (4) becomes $\bar{\bar{y}}^{iv} + \bar{P}_4\bar{\bar{y}} = 0$. From relations (46) we see that $\bar{y}_4$ coincides with the derivative point $\bar{\bar{y}}'''$ and therefore the tangent to $C_{\bar{y}_4}$ goes thru $\bar{y}_1$. Therefore if $\bar{P}_2 = 0$, the $\bar{y}_1\bar{y}_4$ line generates a developable surface of which $C_{\bar{y}_4}$ is the edge of regression.

The $\bar{y}_3\bar{y}_4$ line will in general generate a ruled surface, but if $\bar{P}_2 = 0$, we have $\bar{y}_3$ and $\bar{y}_4$ coinciding with the derivative points $\bar{\bar{y}}''$ and $\bar{\bar{y}}'''$, so that we then have a developable surface of which $C_{\bar{y}_3}$ is the edge of regression.

The $\bar{y}_2\bar{y}_4$ line will generate a ruled surface if $5\bar{P}_4 - 9\bar{P}_2'' \neq 0$. We find also that

$$\bar{y}_4' = \left(\frac{9\bar{P}_2'' - 5\bar{P}_4}{5}\right)\bar{\bar{y}} - \frac{9}{5}\bar{P}_2\bar{y}_2'.$$

Therefore if $9\bar{P}_2'' - 5\bar{P}_4 = 0$, we see that the $\bar{y}_2\bar{y}_4$ line generates a developable surface.

From relations (46), we see that $\bar{P}_2 = 0$ will make the vertices of this tetrahedron of reference coincide with the derivative points, that is $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, $\bar{y}_4$ coincide with $\bar{\bar{y}}$, $\bar{\bar{y}}'$, $\bar{\bar{y}}''$, and $\bar{\bar{y}}'''$, respectively.