CERTAIN PLANE VECTORIAL LOCI.

by

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INTRODUCTION.

Since the time of the ancients,problems,the nature of which was to find the locus of a point moving under given conditions,have been investigated by mathematicians.

It is the purpose of this paper to investigate systematically,the loci arising from constant sums,differences,quotients and products of distances from two fixed points,two lines,point and line,point and circle,line and circle and two circles.The first five may be considered as special cases of the last,where one or both circles have become circles of radii zero or infinity.Arising from this investigation are some well known loci that have been obtained in various other ways before, and some which are new so far as the author has been able to determine.

Before the time of Des Cartes,the so called conic sections were defined as sections of a cone,from which their name was derived.But, on the invention of Analytical Geometry it was found that the sections were readily expressed by equations which were the locus of a moving point, moving so that the ratio of its distances from a fixed line and a fixed point is a constant.By varying the constant the different conic sections were obtained.

It is a well known fact that the ellipse and hyperbola are also obtainable as the locus of a point which moves so that the sum and difference respectively, of its distances from two fixed points is a constant.Likewise the Ovals of Cassini are defined as the locus of a point which moves so that the product of its distances from two fixed points is a constant.

The Conchoid of Nicomedes,invented by Nicomedes about 100 A.D.,is defined as the locus formed by measuring on aline which revolves about a fixed point,without a fixed line,a constant lenghh in either direction from the point where it intersects the given fixed line.

The Limacon,invented by Blaise Pascal 1643, is defined as the locus of a point on a variable secant of a circle through a fixed point where the length of the secant determined by the circle is measured on the secant in either direction from the circle.

These curves,and others were obtained in an entirely different manner,in this paper.

While the proofs of theorems I, and II are well known they are included here for the sake of completeness in the discussion.

<u>THEOREM I</u> : The locus of a point which moves so that the sum of its distances from two fixed points is a constant, is an ellipse.

PROOF : Choose the y-axis perpendicular to and bisecting the line joining the two points. Let P be the moving point whose locus is to be found, and let the constant be 2a.

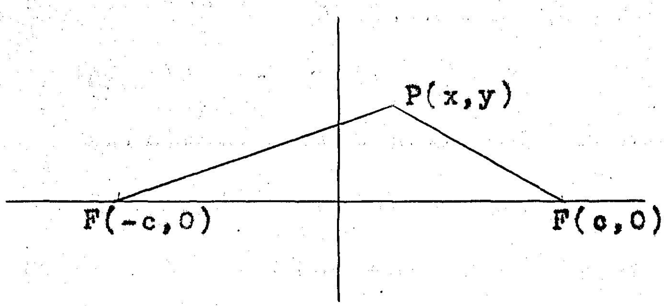


Fig. I

The equation of the locus then is

$$\sqrt{y^2+(c+x)^2} + \sqrt{y^2+(c-x)^2} = 2a$$

or

$$\sqrt{y^2+(c+x)^2} = -\sqrt{y^2+(c-x)^2} + 2a$$

squaring and reducing this becomes

$$x^2(a^2-c^2) + a^2y^2 = a^2(a^2-c^2)$$

and if we let $a^2-c^2=b^2$, the equation becomes *

$$b^2x^2 + a^2y^2 = a^2b^2$$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is the equation of the ellipse.

<u>THEOREM II</u> : The locus of a point which moves so that the difference of its distances from two fixed points is a constant, is an hyperbola.

* this is permissible for $2a > 2c$, or $a > c$, therefore $a^2-c^2 > 0$.

PROOF : Use Fig.I. The equation is

$$\sqrt{y^2+(c+x)^2} = 2a + \sqrt{y^2+(c-x)^2} \text{ .}$$

Squaring and reducing, this becomes

$$x^2(a^2-c^2) - a^2y^2 = a^2(a^2-c^2) \text{ ;}$$

and if we let a-c b, the equation becomes *

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

<u>THEOREM III</u> : The locus of a point which moves so that the quotient of its distances from two fixed points is a constant, is a circle.

PROOF : Using Fig.I.

$$\frac{\sqrt{y^2+(c+x)^2}}{\sqrt{y^2+(c-x)^2}} = k \text{ .}$$

Squaring and reducing, the equation becomes

$$(y^2+c^2+x^2)(1-k^2) - 2cx(1+k^2) = 0 \text{ .}$$

To remove the x-term, substitute $x = x' - \frac{c(1+k^2)}{1-k^2}$, and the locus becomes

$$x'^2 + y^2 = \frac{4c^2k^2}{(1-k^2)^2}$$

which is readily seen to be a circle of radius $\frac{2ck}{1-k^2}$, and center at the new origin.

<u>THEOREM IV</u> : The locus of a point which moves so that the product of its distances from two fixed points is a constant, is a Cassian Oval. If $c^4 = k^2$, it is a lemniscate; if $c^4 < k^2$, the locus consists of one oval, and if $c^4 > k^2$, of two ovals.

PROOF : <u>Use Fig.I.</u>

$$\sqrt{y^2+(c+x)^2}\,\sqrt{y^2+(c-x)^2} = k$$

Simplifying, the locus becomes

$$y^4+x^4+c^4+2(c^2y^2+x^2y^2-c^2x^2) = k^2$$

* this is permissible, for FP-F'P' < FF' or 2a < 2c, therefore $a^2-c^2<0$.

Changing to polar coordinates $x=\rho\cos\theta$
$y=\rho\sin\theta$, the locus becomes

$$\rho^4(\sin^2\theta+\cos^2\theta)+2\rho^2c^2(\sin^2\theta-\cos^2\theta)+c^4-k^2=0$$

or $\rho^4-2\rho^2c^2(1-2\cos^2\theta)+c^4-k^2=0$

from which
$$\rho=\pm\sqrt{\frac{-2c^2(1-2\cos^2\theta)\pm\sqrt{4c^4(1-2\cos^2\theta)-4(c^4-k^2)}}{2}}$$

From this it may be sen at once, that given a certain value of c, the nature of the locus depends on the value of ~~the~~ k These will be discussed under three heads(a),(b),(c), which will include all cases.

(a) when $c^4=k^2$ $\rho=0$; $\rho=c\sqrt{2(2\cos^2\theta-1)}$. Fig.II.

(b) when $c^4<k^2$ $\rho=\pm\sqrt{2}\sqrt{-c^2(1-2\cos^2\theta)\pm\sqrt{c^4(1-2\cos^2\theta)^2-(c^4-k^2)}}$

in orderto plat let $c^2=2$, $k=3$. Fig.III.

(c) when $c^4>k^2$ $\rho=\pm\sqrt{2}\sqrt{-c^2(1-2\cos^2\theta)\pm\sqrt{c^4(1-2\cos^2\theta)^2-(c^4-k^2)}}$

Refer to values of ρ (in first column) under Fig.III.(1) for $\theta=0°$, ρ is always real.(2) for $\theta=30°$, ρ will be real if $k^2\geqq 3/4c^4$, and if $c^2\geqq\sqrt{4k^2-3c^4}$, which on squaring reduces to $k^2\leqq c^4$(assumption started with).(3) for $\theta=45°$, ρ can never be real for $k^2-c^4<0$ always.(4) for $\theta=60°$, ρ is real if $k^2>3/4c^4$ and if $c^2<\sqrt{4k^2-3c^4}$ which is impossible.(5) for $\theta=90°$, ρ cannot possibly be real. CONCLUSION: Real values of ρ cannot occur, in the first quadrant, between 45° and 90°, and if it does not follow that $k^2\geqq 3/4c^4$, real values will not occur between 30° and 45°, so that in this case the locus is between 0° and 30°, but if $k^2\geqq 3/4c^4$, the locus may lie between 0° and 45°. The symmetry of the locus, as seen by Fig, IV, makes it unnecessary to discuss the other three quadrants.

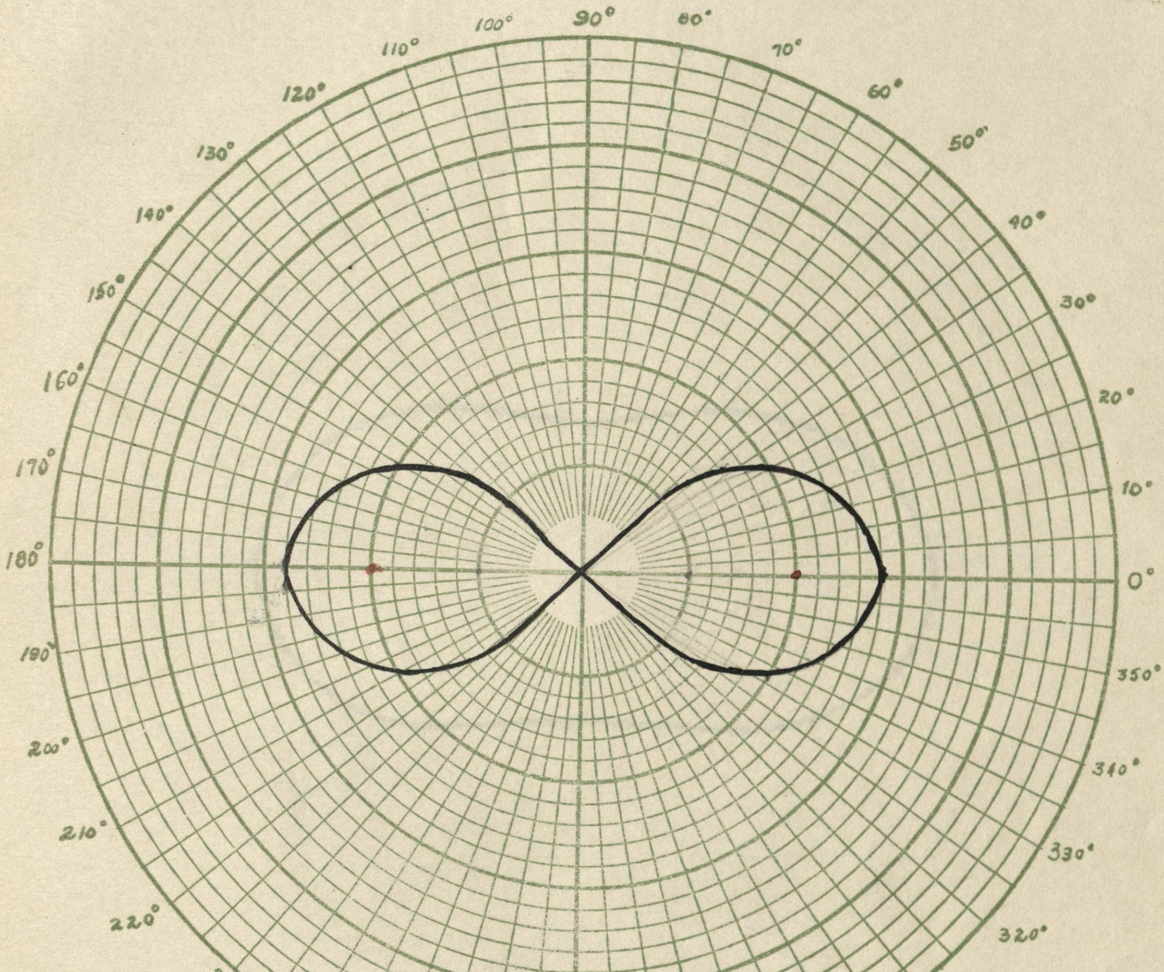


Fig. II.

θ	ρ
0	$\pm\sqrt{2}\,c$
30	$\pm c$
45	0
60	$f(i)$
90	"
120	"
135	0
150	$\pm c$
180	$\pm\sqrt{2}\,c$

$$\rho = c\sqrt{2(2\cos^2\theta - 1)}$$

Locus —

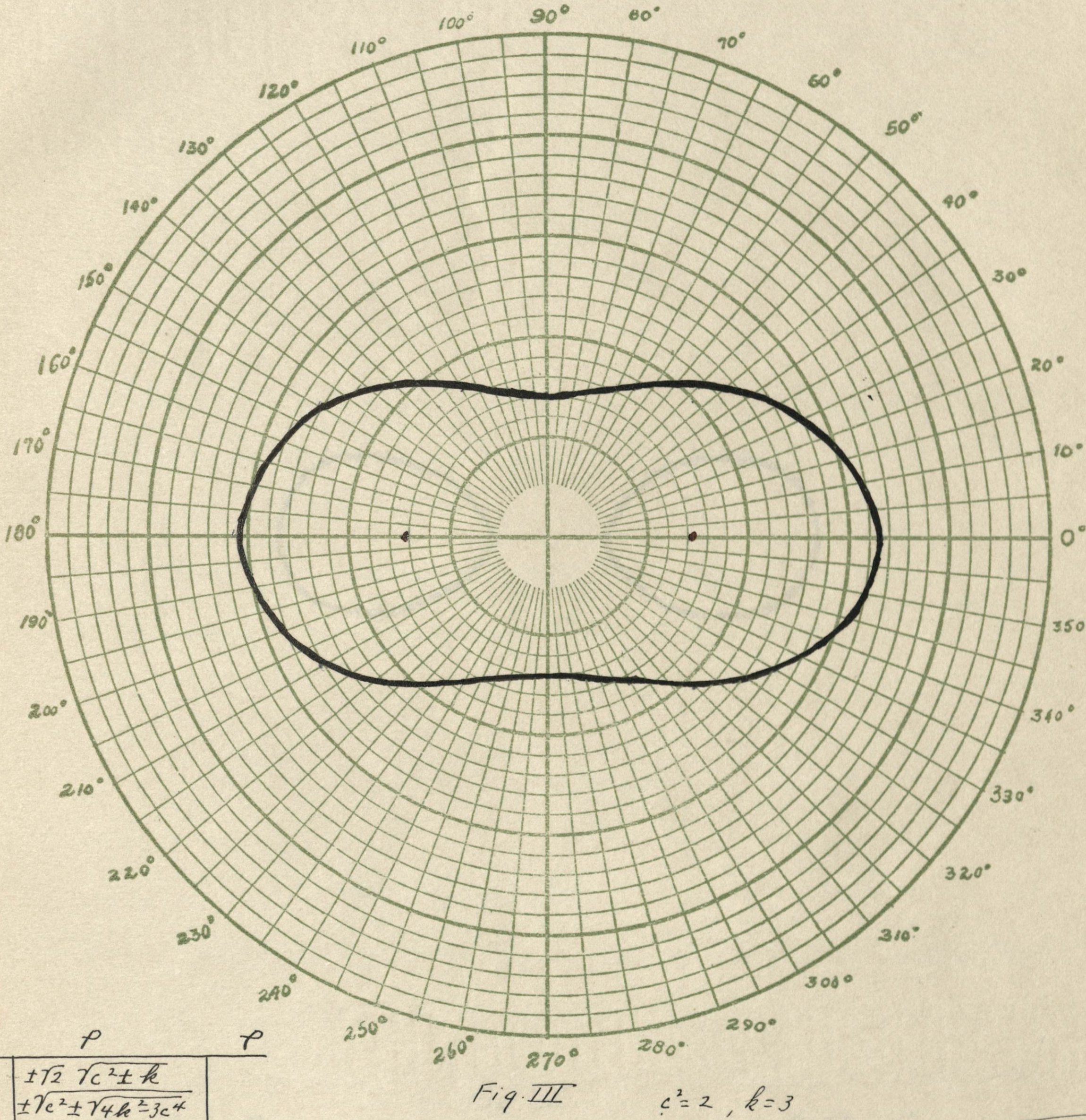


Fig. III

θ	ρ	ρ
0	$\pm\sqrt{2}\sqrt{c^2 \pm k}$	
30	$\pm\sqrt{c^2 \pm \sqrt{4k^2 - 3c^4}}$	
45	$\pm\sqrt{2}\sqrt[4]{k^2 - c^4}$	
60	$\pm\sqrt{-c^2 \pm \sqrt{4k^2 - 3c^4}}$	
90	$\pm\sqrt{2}\sqrt{-c^2 \pm k}$	
120	$\pm\sqrt{-c^2 \pm \sqrt{4k^2 - 3c^4}}$	
135	$\pm\sqrt{2}\sqrt[4]{k^2 - c^4}$	
150	$\pm\sqrt{c^2 \pm \sqrt{4K^2 - 3c^4}}$	
180	$\pm\sqrt{2}\sqrt{c^2 \pm k}$	

$c^2 = 2$, $k = 3$

$$\rho = \pm\sqrt{2}\sqrt{-2(1-2\cos^2\theta) \pm \sqrt{4(1-2\cos^2\theta)^2 + 5}}$$

Locus —

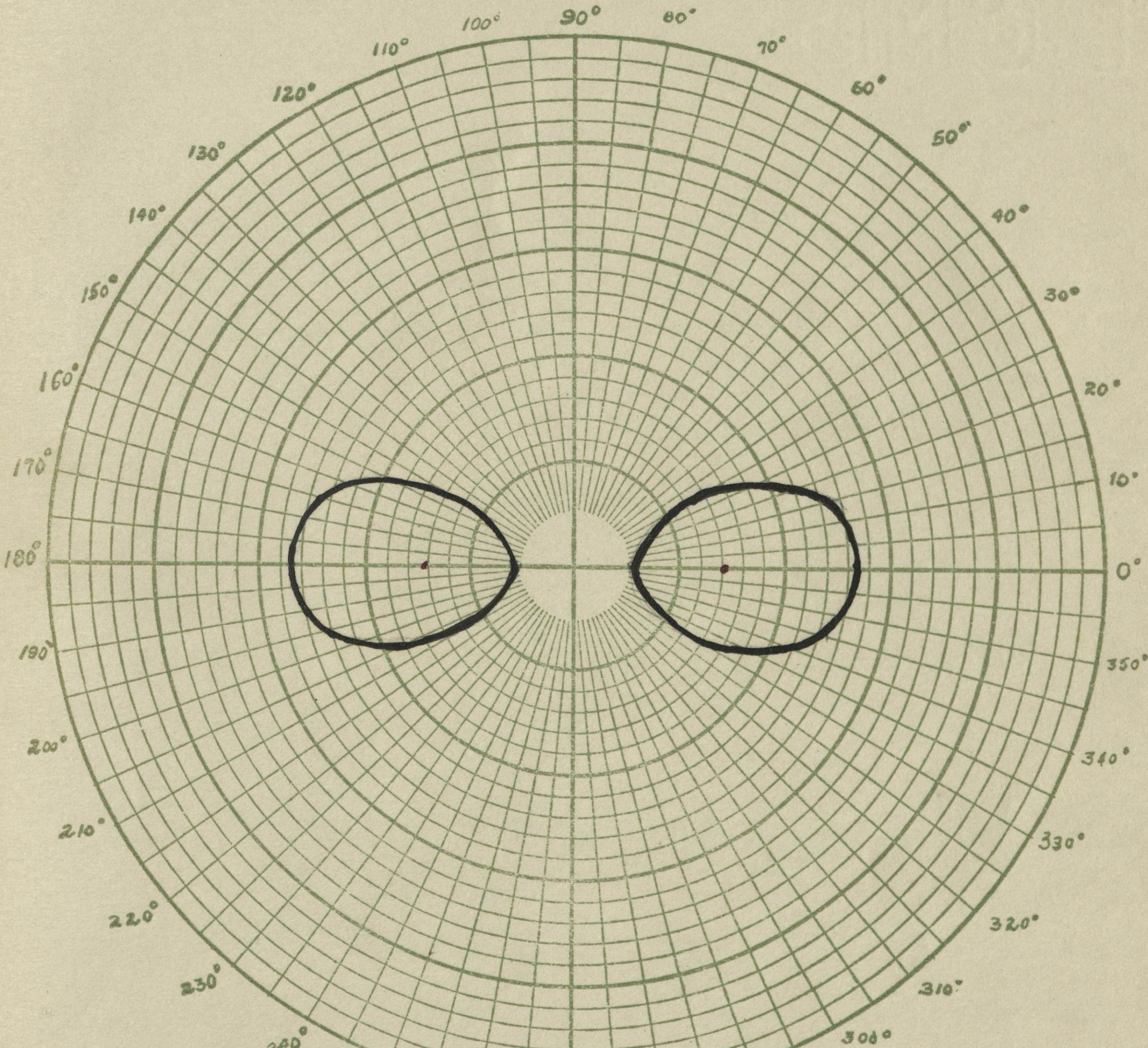


Fig IV

θ	ρ	ρ
0	±2.7	±.5
30	±1.4	
:	f(i)	
'	"	
	"	
150	±1.4	
180	±2.7	±.5

$k=\sqrt{3}\ ,\ c^2=2$

$\rho = \pm\sqrt{2}\sqrt{-2(1-2\cos^2\theta) \pm \sqrt{4(1-2\cos^2\theta)^2-1}}$

Locus —

THEOREM V: The locus of a point, which moves so that the sum of its distances from two fixed straight lines is a constant is, (a) if the lines intersect, a straight line, (b) if the lines are parallel, all points between the two lines when the constant is equal to the distance between the two lines, and a parallel straight line when the constant is greater that the distance between the two lines.

PROOF :

(a)

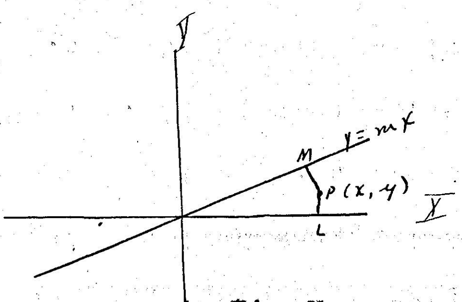


Fig.V.

(b)

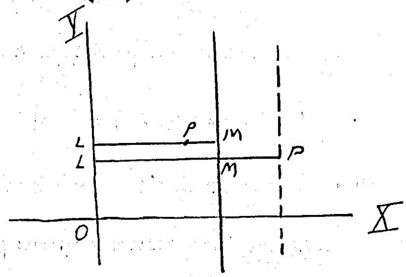


Fig.VI.

(a) Choose the x-axis as one line, and let the equation of the other line be y=mx. The locus is obtained then by letting

$$LP + PM = k$$

or

$$\frac{mx - y}{\sqrt{m^2+1}} + y = k$$

which reduces to

$$mx + y(\sqrt{m^2+1} - 1) = k\sqrt{m^2+1}$$

which is seen to be a straight line.

(b) Use Fig.VI. If P be any point between the lines, the sum of its distances from the two lines is the constant LM. If the point is outside, if it is not a parallel straight line as LP increases PM must decrease. This is impossible so that the locus must be a parallel straight line.

THEOREM VI: The locus of a point which moves so that the differences of its distances from two fixed lines is a constant, is (a) a line not through the intersection of the two lines, (b) a line parallel to the two lines.

PROOF : Use Fig.V. (a)

$$-y+mx\pm y\sqrt{m^2+1} = k\sqrt{m^2+1}$$

which is a straight line not through the origin.

(b) $LP-PM=k$

if LP decreases MP increases, and therefore LP and PM must both be constants. Thus the locus is a line parallel to the two given lines, its location depending on the value of k.

THEOREM VII: The locus of a point which moves so that the quotient of its distances from two fixed lines is a constant, is (a) if the lines intersect, a line through their intersection, (b) if the lines are parallel, a line parallel to the given lines.

PROOF: (a) $$\frac{mx-y}{y\sqrt{m^2+1}} = k$$ Using Fig.V.

which reduces to, $mx-y(1+k\sqrt{m^2+1})=0$.

This is a line thru the intersection of the given lines.

(b) $\frac{x}{c-x}=k$ or $x=\frac{ck}{1+k}$

This is a line parallel to the two given lines.

THEOREM VIII : The locus of a point which moves so that the product of its distances from two fixed lines is a constant, is (a) if the lines intersect, any hyperbola, (b) if the lines are parallel, two lines parallel to the two given lines if $c^2-4k>0$, two imaginary lines if $c^2<4k$, and two coincident lines if $c^2=4k$.

PROOF: (a) $(mx-y)y = k\sqrt{m^2+1}$ Using Fig.V.

or $y^2 = mxy\ k\sqrt{m^2+1}$

which is an hyperbola.

(b) $x(c-x) = k$

which is two parallel lines if $c^2 > 4k$, two imaginary lines if $c^2 < 4k$ and two coincident lines if $c^2 = 4k$.

THEOREM IX : The locus of a point which moves so that the sum of its distances from a fixed line and point is a constant, is (a) if $k = c$, the x-axis twice; if (b) $k > c$, a parabola.

PROOF:

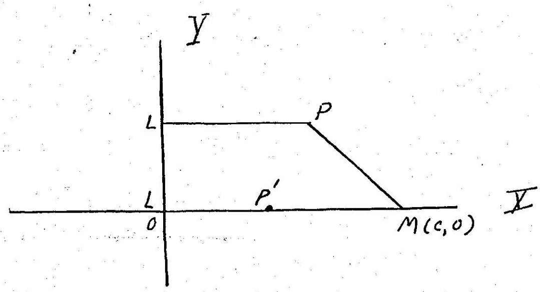


Fig.VII.

$$LP + PM = k$$

$$x + \sqrt{y^2 + (c-x)^2} = k$$

or $$y^2 = (k-c)(k+c-2x)$$

(a) if $k = c$, the locus becomes $y^2 = 0$, which is the x-axis twice.*

(b) if $k > c$, $k - c > 0$, and the locus is seen to be a parabola open to the left and vertex at, $\left(\frac{k+c}{2}, 0\right)$. #

*For P', distance is LP'+P'M=k (this is where point, whose locus is to be found, is between given point and line). If P' is to the right of M, P'M must be considered as < 0; if P' is to the left of y-axis, LP' is considered as < 0.

To consider entire parabola as locus, for points on the parabola to the left of the y-axis LP' must be taken as negative, otherwise, sum of the distances could not k.

(c) $k \nless c$, for the perpendicular is the shortest distance from a point to a line.

THEOREM X: The locus of a point which moves so that the difference of its distances from a fixed line and a fixed point is a constant, is a parabola, or x-axis if $k<0$.

PROOF: $PL - PM = k$ Using Fig.VII.

or $\sqrt{y^2+(c-k)^2} - x = k$, which on squaring becomes,

$y^2 = (k+c)(k-c+2x)$.

(a) If $k=c$, the constant term drops out and the locus is,

$y^2 = 4cx$.

This is a parabola with vertex at the orågin and open to the right.

(b) If $k>c$, locus is a parabola open to the right and vertex at $\left(\frac{c-k}{2}, 0\right)$.

(c) If $k<c$, and $|k|<|c|$, the locus is a parabola open to the right and vertex at $\left(\frac{c-k}{2}, 0\right)$. If $|k|>|c|$, the locus is a parabola open to the left and vertex at $\left(\frac{c-k}{2}, 0\right)$.

THEOREM XI: The locus of a point which moves so that the quotient of its distance from a fixed line and a fixed point is a constant, is a conic section.

PROOF:

$$\sqrt{\frac{y^2+(c-x)^2}{x}} = k \qquad \text{Using Fig.VII.}$$

or $y^2+x^2(1-k^2)-2cx+c^2=0$.

(a) If $1-k^2=0$, i.e. if $k^2=1$, or $k=\pm 1$

the locus becomes,

$$y^2 = 2cx - c^2.$$

This is the equation of a parabola with vertex at $(c/2, 0)$ and open to the right.

(b) If $1-k^2>0$, i.e. $k<\pm1$, the equation becomes, if $1-k^2=m^2$,

$$y^2+m^2x^2-2cx+c^2=0.$$

Substituting $x = x'+\frac{c}{m^2}$, the locus becomes,

$$y^2+m^2x^2-\frac{c^2(m^2-1)}{m^2}=0,$$

which is the equation of an ellipse.

(c) If $1-k^2<0$, i.e. $k>\pm1$, the equation becomes, if $1-k^2=m^2$,

$$y^2-m^2x^2-2cx+c^2=0.$$

Substituting $x=x'-\frac{c}{m^2}$, the locus becomes,

$$y^2-m^2x^2+\frac{c^2(1-m^2)}{m^2}=0,$$

which is the equation of an hyperbola.

<u>THEOREM XII</u>: The locus of a point which moves so that the product of its distances from a fixed point and a fixed line is a constant, is, $y^2=\frac{k^2}{x^2}-c^2-x^2+2cx$. *

Proof: $x\sqrt{y^2+(c-x)^2}=k$, Using Fig. VII.

or $y^2=\frac{k^2}{x^2}-c^2-x^2+2cx$.

(a) If $k^2=1$, the locus is, if $c=1$,

$$y^2=\frac{1}{x^2}-1-x^2+2x.$$ See Fig. VIII.

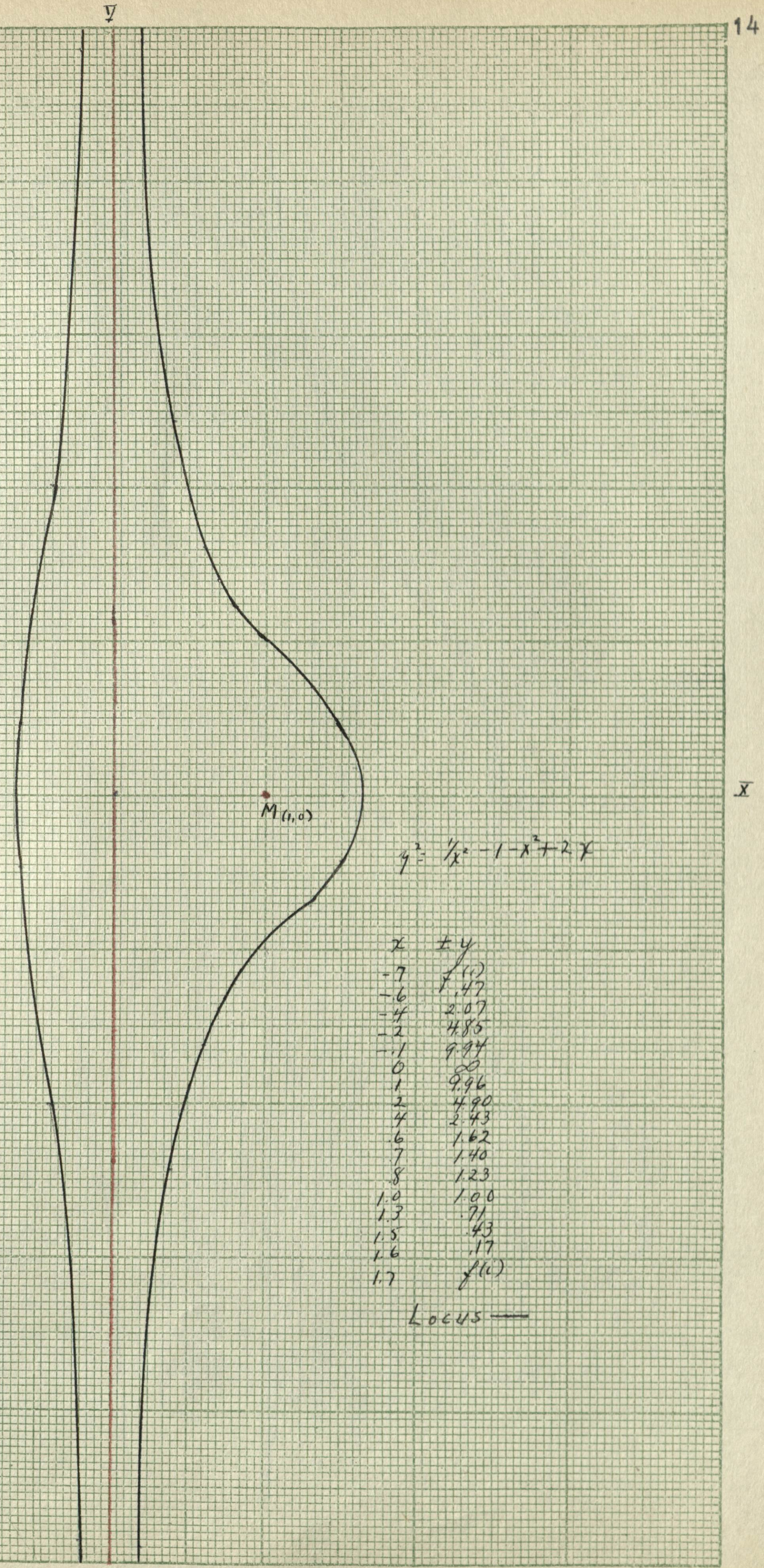
(b) If $k^2>1$. To plot let $k=2$ and $c=1$. The locus becomes,

$$y^2=\frac{4}{x^2}-1-x^2+2x.$$ See Fig. IX.

(c) If $k^2<1$. Let $k=1/2$, $c=1$, and the locus becomes,

$$y^2=\frac{.25}{x^2}-1-x^2+x.$$ See Fig. X.

* This is similar to the general equation of the Concoid of Nicomedes ($x^2y^2=-x^4-2ax^3+(b^2-a^2)x^2+2ab^2x+b^2a^2$), where the x-term is missing.



$y^2 = 1/x^2 - 1 - x^2 + 2x$

x	$\pm y$
-7	f(i)
-.6	1.47
-4	2.07
-2	4.85
-.1	9.94
0	∞
1	9.96
2	4.90
4	2.43
.6	1.62
.7	1.40
.8	1.23
1.0	1.00
1.3	.71
1.5	.43
1.6	.17
1.7	f(i)

Locus —

Fig. VIII.

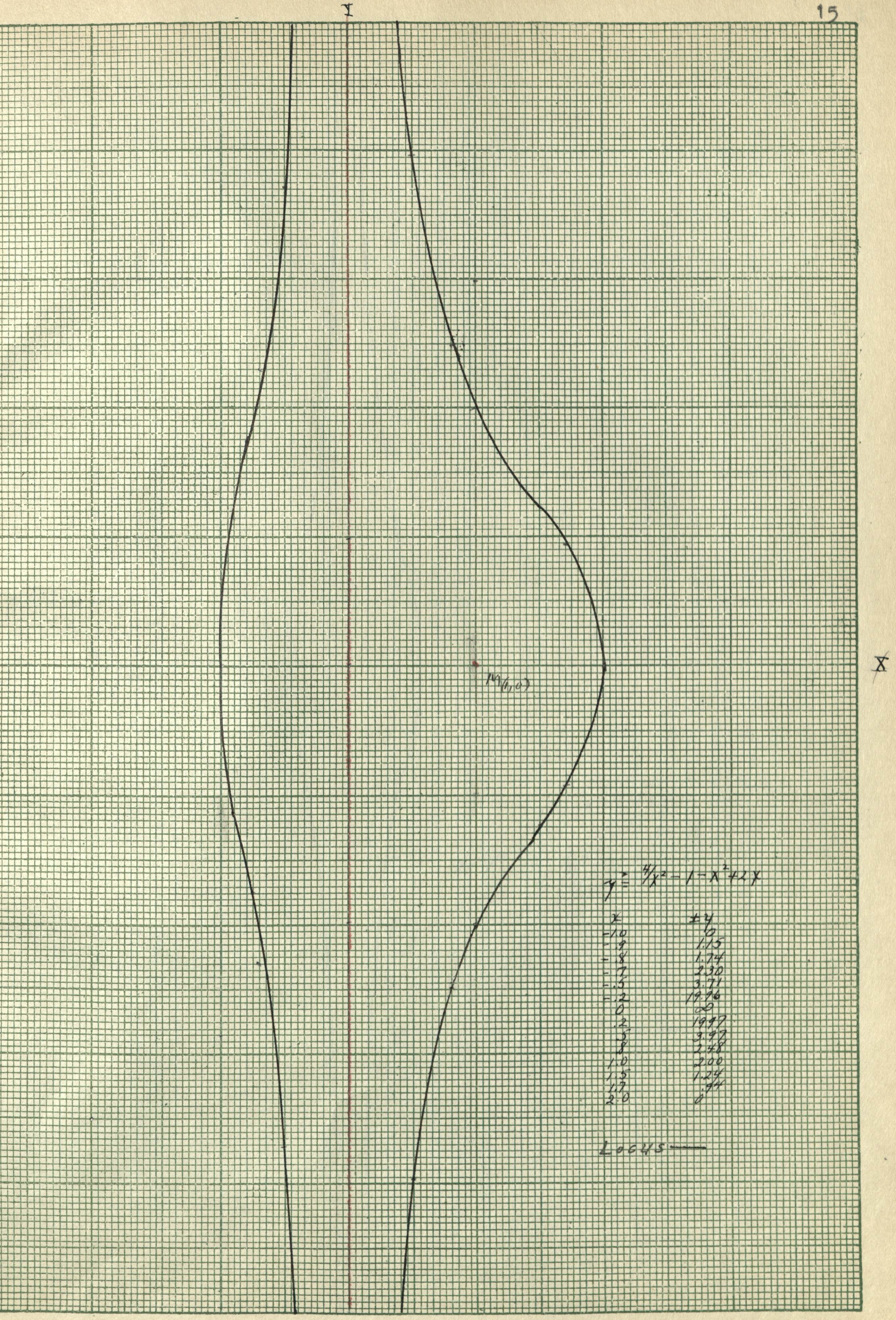

Y
X
M(1,0)
$y^2 = 4/x^2 - 1 - x^2 + 2x$
x ±y
-1.0 0
-.9 1.15
-.8 1.74
-.7 2.30
-.5 3.71
-.2 19.96
0 ∞
.2 19.97
.5 3.97
.8 2.48
1.0 2.00
1.5 1.24
1.7 .94
2.0 0
Locus ——

Fig IX.

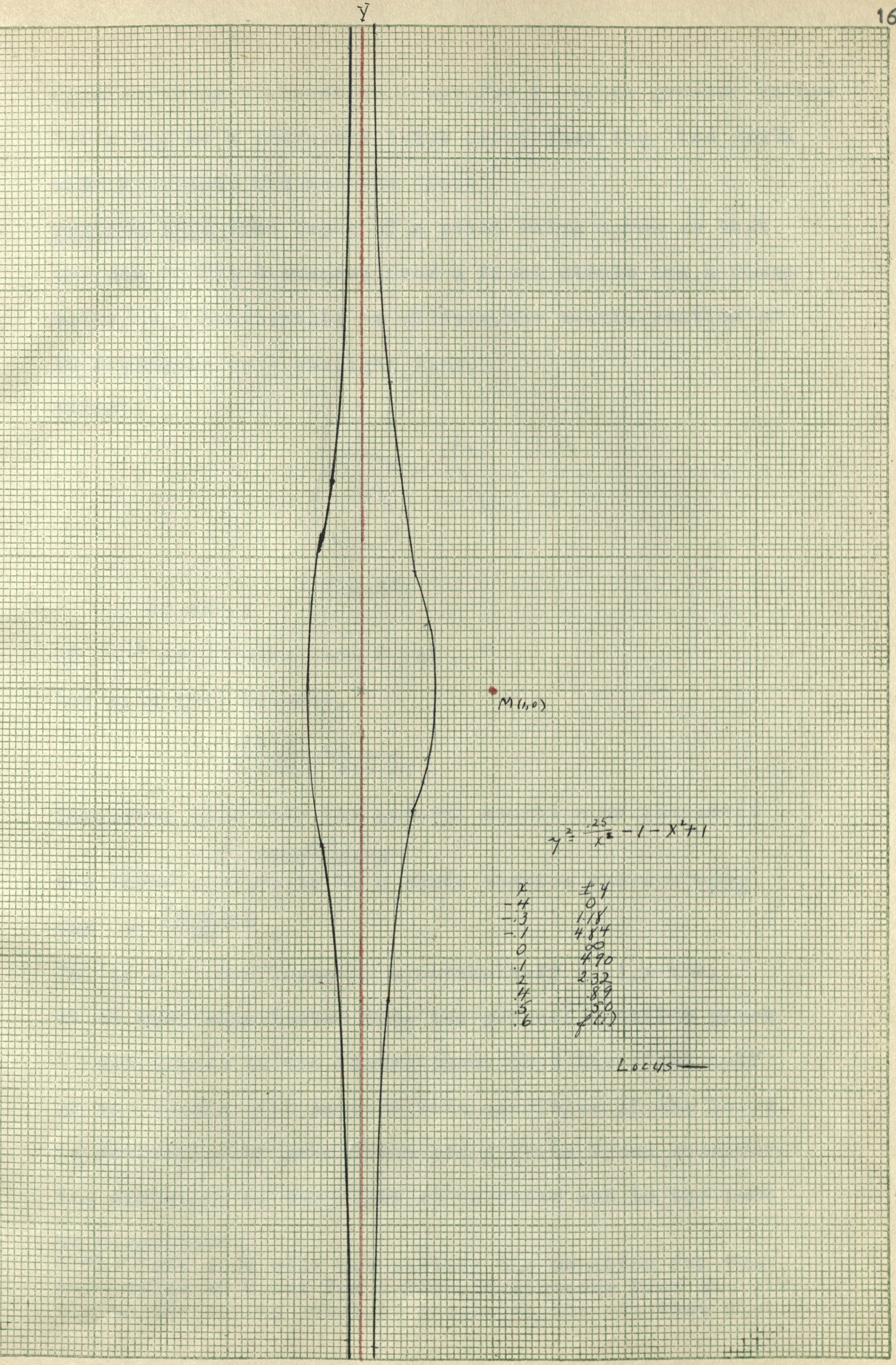

Y
X
M(1,0)
y² = .25/x² − 1 − x² + 1
x ±y
−.4 0
−.3 1.18
−.1 4.84
0 ∞
.1 4.90
.2 2.32
.4 .89
.5 .50
.6 f(i)
Locus ——

Fig. X.

Conclusion: As k increases, locus, for $y = 0$, has greater values for x; and as k decreases, locus approaches the line untill when $k=0$, locus becomes the line.

THEOREM XIII : The locus of a point which moves so that the sum of its distances from a fixed circle and a fixed point is a constant, is an ellipse, the x-axis, section of an hyperbola or a straight line.

PROOF :

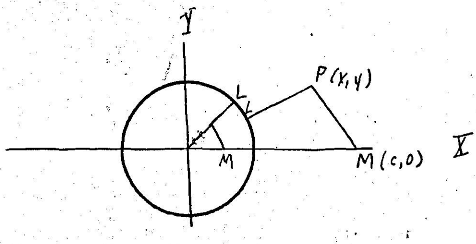


Fig. XI.

$$LP + PM = k$$

or $$\rho - r + \sqrt{\rho^2 + c^2 - 2c\rho\cos\theta} = k.$$

Reducing this becomes,

$$\rho = \frac{(r+k)^2 - c^2}{2(r+k-c\cos\theta)}.$$

Comparing this with the general equation of the conic

$$\rho = \frac{em}{1-e\cos\theta}$$

the locus appears to be a conic section where $e = \frac{c}{r+k}$,

and $m = \frac{(r+k)^2 - c^2}{2c}$.

(a) if $e = 1$, the locus is a parabola. If $e = 1$, $r+k = c$, which when substituted into the general equation gives $\rho = 0$. A glance at the figure shows that the right half of the x-axis will satisfy this condition if the locus is determined by going from L to P + P to M, and considering direction to the right as positive and to the left as negative.*

* The left side will not satisfy the equation for the same value of k, but for $k = k+2r$, if distance from the circle to the moving point is chosen as the shortest dist.

(b) If $e > 1$, namely $c > r+k$. This is impossible, for one side of a triangle may not be greater than the sum of the other two sides.

(c) If $e < 1$, namely $c < r+k$. This is possible, therefore the locus is an ellipse.

The above discussion holds for P exterior to the circle. If, then, P is inside LP becomes $(r-\rho)$, and locus is

$$\rho = \frac{(r-k)^2-c^2}{2(r-k-c\cos\theta)}.$$

This may be compared to the general equation of the conic, as was done before.

(a) If $e=1$. $c=r-k$, and locus becomes $\rho=0$.

(b) If $e>1$. $c>r-k$, and locus becomes an hyperbola, and since P was chosen inside of the given circle, those values inside of the circle only, should be used. Fig. X. *

(c) If $e<1$. $c<r-k$. This is impossible if c is outside of the circle, for $c>r$. For c inside of the circle, it is also impossible, for $c+k \not< r$. #

<u>THEOREM XIV</u>: The locus of a point which moves so that the difference of its distances from a fixed point and fixed circle is a constant, is an ellipse, the x-axis, a section of an hyperbola or a straight line. Using Fig. XI.

PROOF: If the locus is obtained by taking $LP-PM=k$, the

* As a special case under (b); if $r=k$, locus becomes a straight line.

\# Discussion of all other cases in this theorem, where c is inside of the circle, is not given, for the results are the same as if c were outside of the circle.

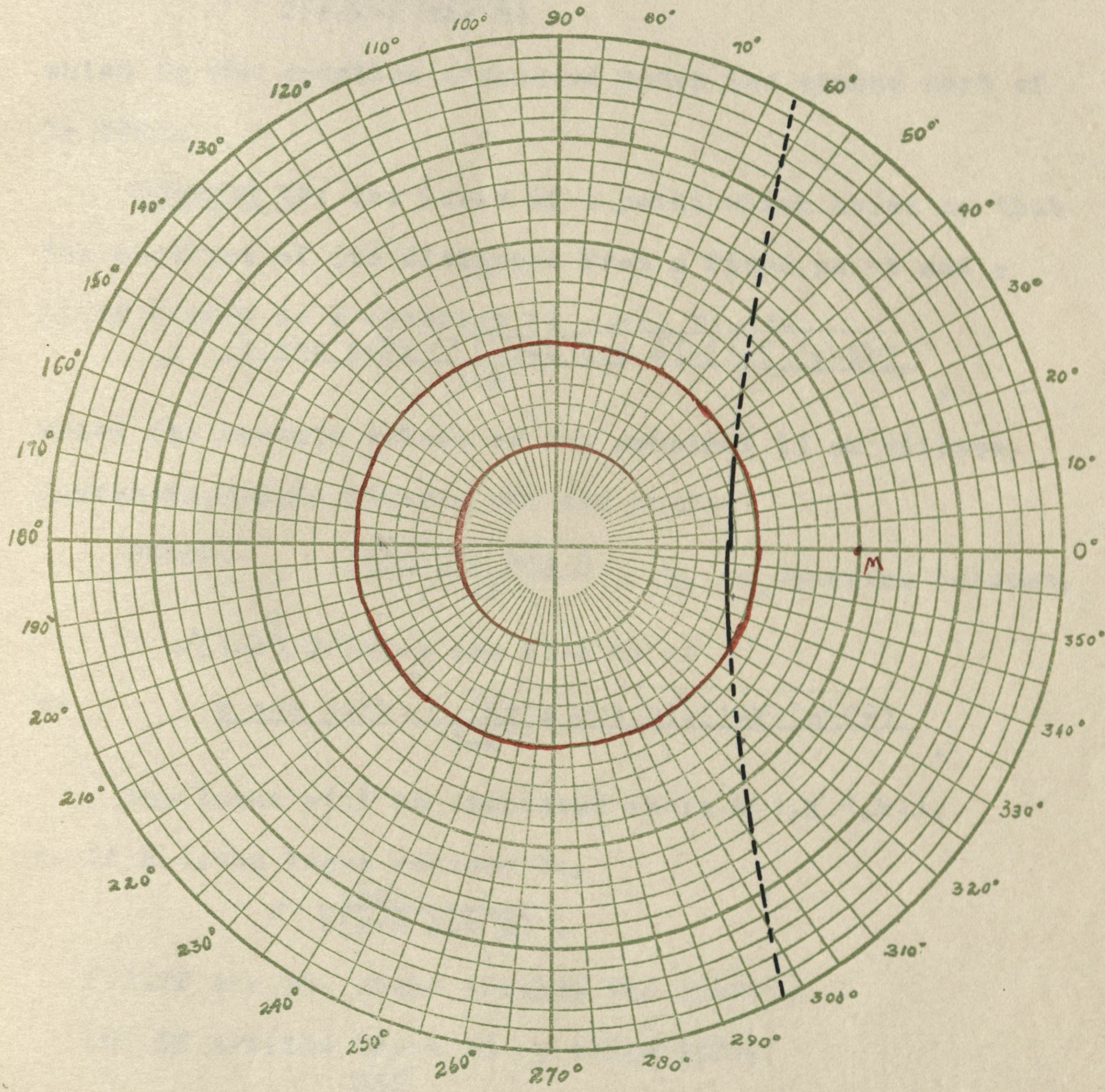


Fig. XII

θ	ρ
0°	1.75
15	1.82
30	2.07
45	2.68
60	4.38
70	8.41
80	218.75
81	-145.83
90	-8.75

$c = 3$
$k = 1.5$
$r = 2$

$$\rho = \frac{-8.75}{1 - 6\cos\theta}$$

Locus——

equation is the same as that for Th.XIII, for the only difference is that the negative sign of the radical is used and this disappears on squaring. If the locus is obtained as PM-LP = k, the equation becomes,

$$\rho = \frac{(r-k)^2 - c^2}{2(r-k-c\cos\theta)}$$

which is the equation discussed under the second part of Th.XIII.

THEOREM XV: The locus of a point which moves so that the quotient of its distances from a fixed point and a fixed circle is a constant is using Fig.XI,

$$\rho = \frac{c\cos\theta - k^2r \pm\sqrt{(k^2r - c\cos\theta)^2 - (1-k^2)(c^2-k^2r^2)}}{1-k^2},$$

which for certain values of the constant is an ellipse, hyperbola, x-axis, limacon or two ovals.

PROOF: $\dfrac{\sqrt{\rho^2+c^2-2\rho c\cos\theta}}{\rho - r} = k$, or, squaring becomes,

$$\rho^2(1-k^2)+2\rho(k^2r - c\cos\theta)+c^2-k^2r^2=0$$

or

$$\rho = \frac{c\cos\theta - k^2r \pm\sqrt{(k^2r - c\cos\theta)^2 - (1-k^2)(c^2-k^2r^2)}}{1-k^2}.$$

The locus will be discussed where $k^2=1, k^2>1, k^2<1$.

A. If $k^2=1$, the locus reduces to,

$$\rho = \frac{r^2-c^2}{2(r-c\cos\theta)}.$$

(1). If $c=r$, the locus becomes the x-axis.

(2) If $c<r$, the locus is an ellipse, for

$$\rho = \frac{\frac{r^2-c^2}{2r}}{1-(c/r)\cos\theta},$$ is the locus, and $c/r<1$,

so that the equation is an ellipse.

(3). If $c>r$, the locus is an hyperbola, for $c/r>1$.

B. If $k^2\neq 1$ and $c^2=k^2r^2$, or $kr=\pm c$, the locus becomes,

$$\rho = 0;\ \text{or}\ \rho = \frac{2}{1-k^2}(c\cos\theta - k^2r)$$

B.

(1). If $|k|>1$, the locus is a limacon without the loop, the equation being, if $k>1$, $\rho=\frac{2c}{k-1}(k-\cos\theta)$; and if $k>-1$, $\frac{-2c}{k-1}(k+\cos\theta)$. For $|k|>1$, $c>r$.

(2). If $|k|<1$, the locus is a limaconwith the loop, the equation being when $0<k<1$, $\rho=\frac{2c}{k^2-1}(k-\cos\theta)$; and if $0>k>-1$, $\rho=\frac{-2c}{k^2-1}(k+\cos\theta)$. For $|k|<1$, $c<r$.

C. If $k^2>1$

(1). If $r=c$, it follows that $k^2r>c$. The locus becomes

$$\rho=c\,\frac{\cos\theta-k\pm\sqrt{(k^2-\cos\theta)^2-(1-k^2)^2}}{1-k^2}$$

. See Fig. XIII.

(2). If $c>r$, it follows that $k^2c>r$, or $k^2>r/c$; but it cannot be concluded as to whether the ratio $c/r \gtreqless k^2$, so each will be discussed separately.

(a). If $k^2r=c$, the locus becomes,

$$\rho=\frac{c(\cos\theta-1)\pm\sqrt{c(1-\cos\theta)^2-(1-k^2)(c-r)c}}{1-k^2}$$

$-c(1-k^2)(c-r)$ is always positive, so that the radical is always real and $>c(\cos\theta-1)$, which shows that as θ varies from $0°$ to $180°$ that one-half of one oval is above and one-half of the other oval is below the x-axis. See Fig. XIV.

(b). If $rk^2>c$, The radical is always real so that the locus goes around the origin.* See Figs. XV and XV'.

(c). If $k^2r<c$, the radical is always real. See Fig. XVI.

(3). If $c<r$, it follows that $k^2r>c$. The radical is always real.** See Fig. XVII.

D. If $k^2<1$

(1). If $r=c$, $k^2r<c$, and the locus becomes,

$$\rho=c\,\frac{\cos\theta-k^2\pm\sqrt{(k^2-\cos\theta)^2-(1-k^2)^2}}{1-k^2}$$

* See Note I.

**See Note II.

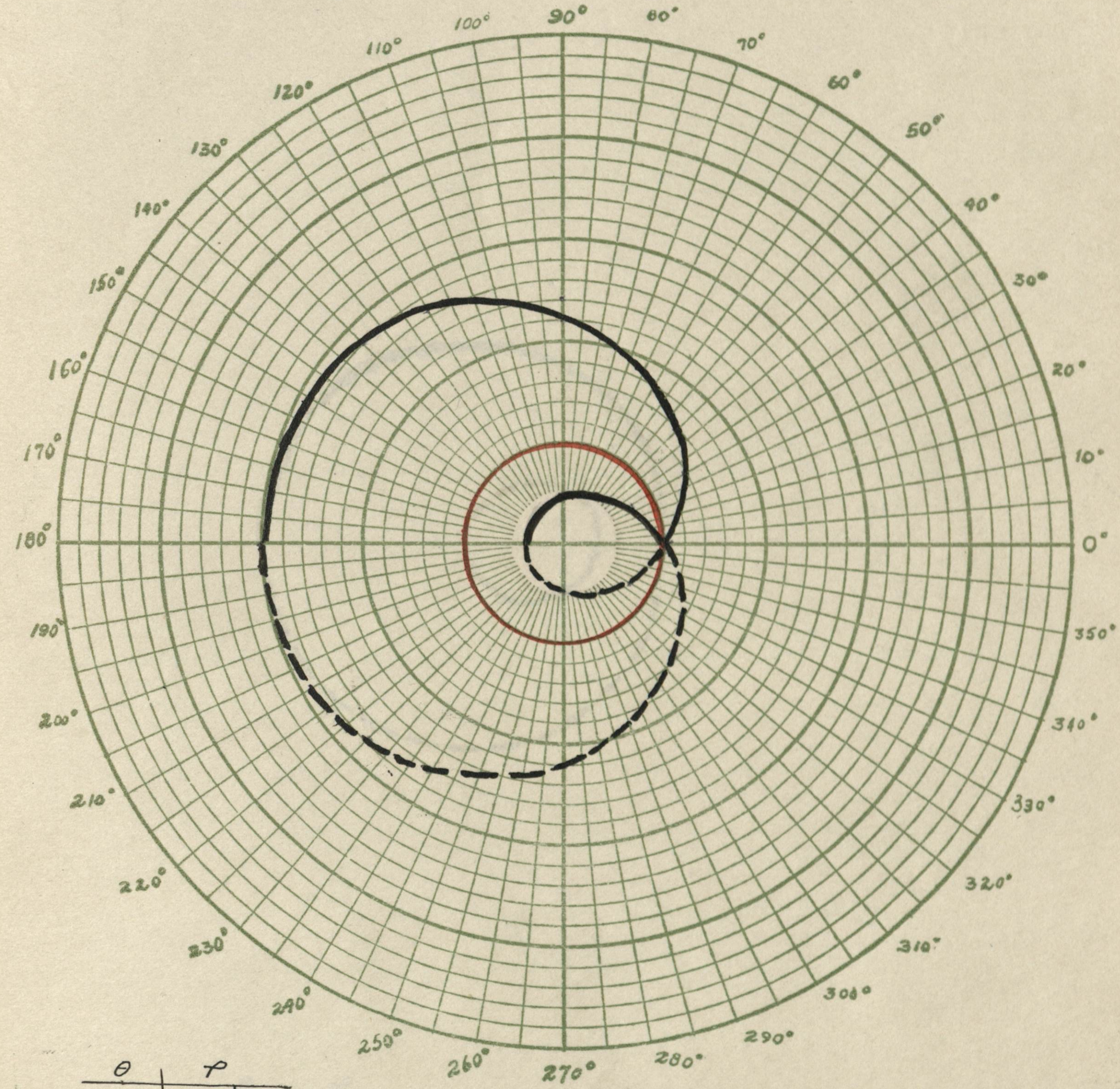


Fig. XIII

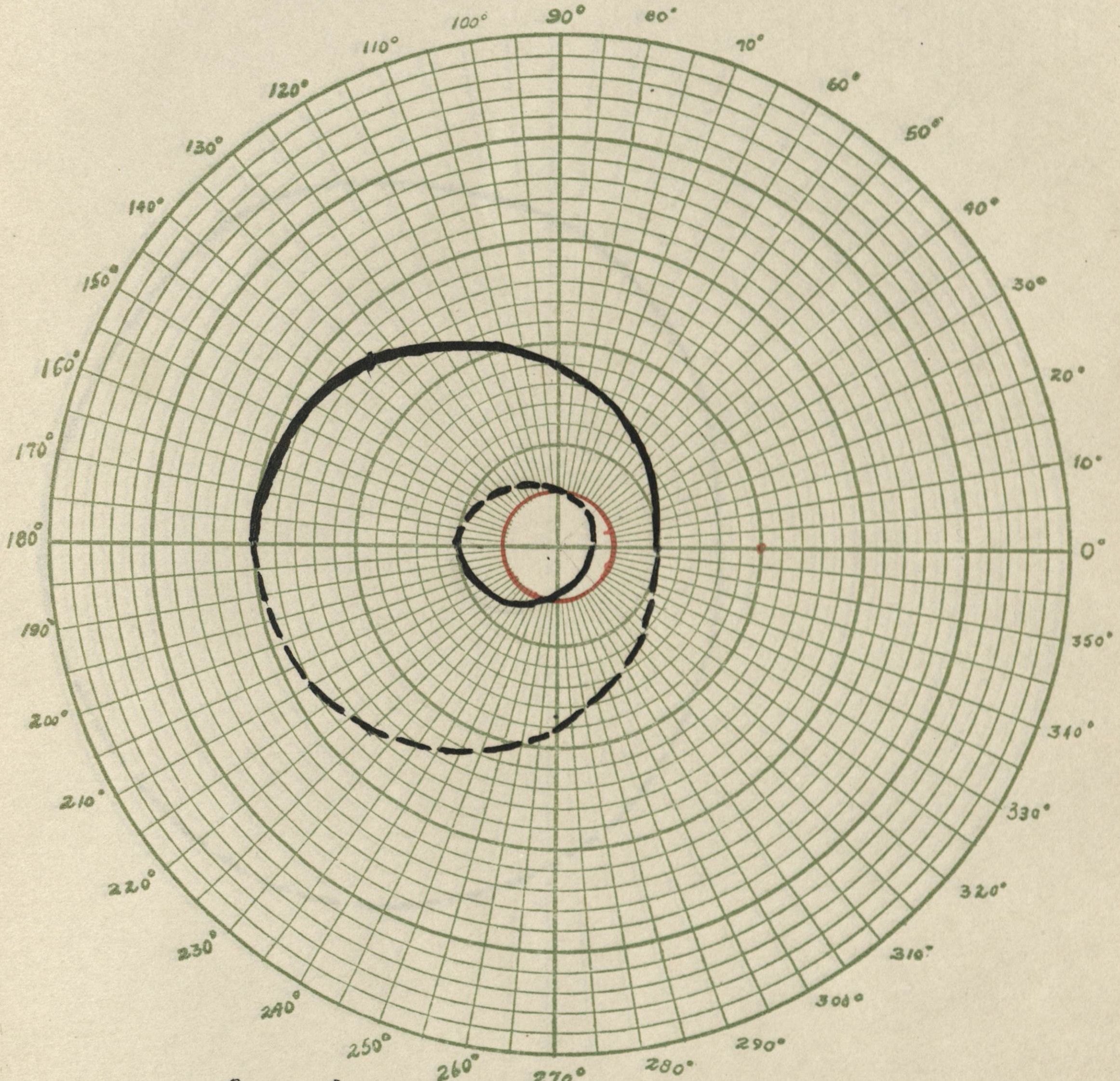
θ	ρ	
0°	1.00	1.00
30	.74	1.35
45	.65	1.56
60	.56	1.77
90	.45	2.21
120	.38	2.62
135	.36	2.78
150	.35	2.89
180	.33	3.00

$k=2$
$r=1$
$c=1$

$$\rho = \frac{4 - \cos\theta \mp \sqrt{(4-\cos\theta)^2 - 9}}{3}$$

Locus —— $0° < \theta < 180°$

– – – $180° > \theta < 360°$



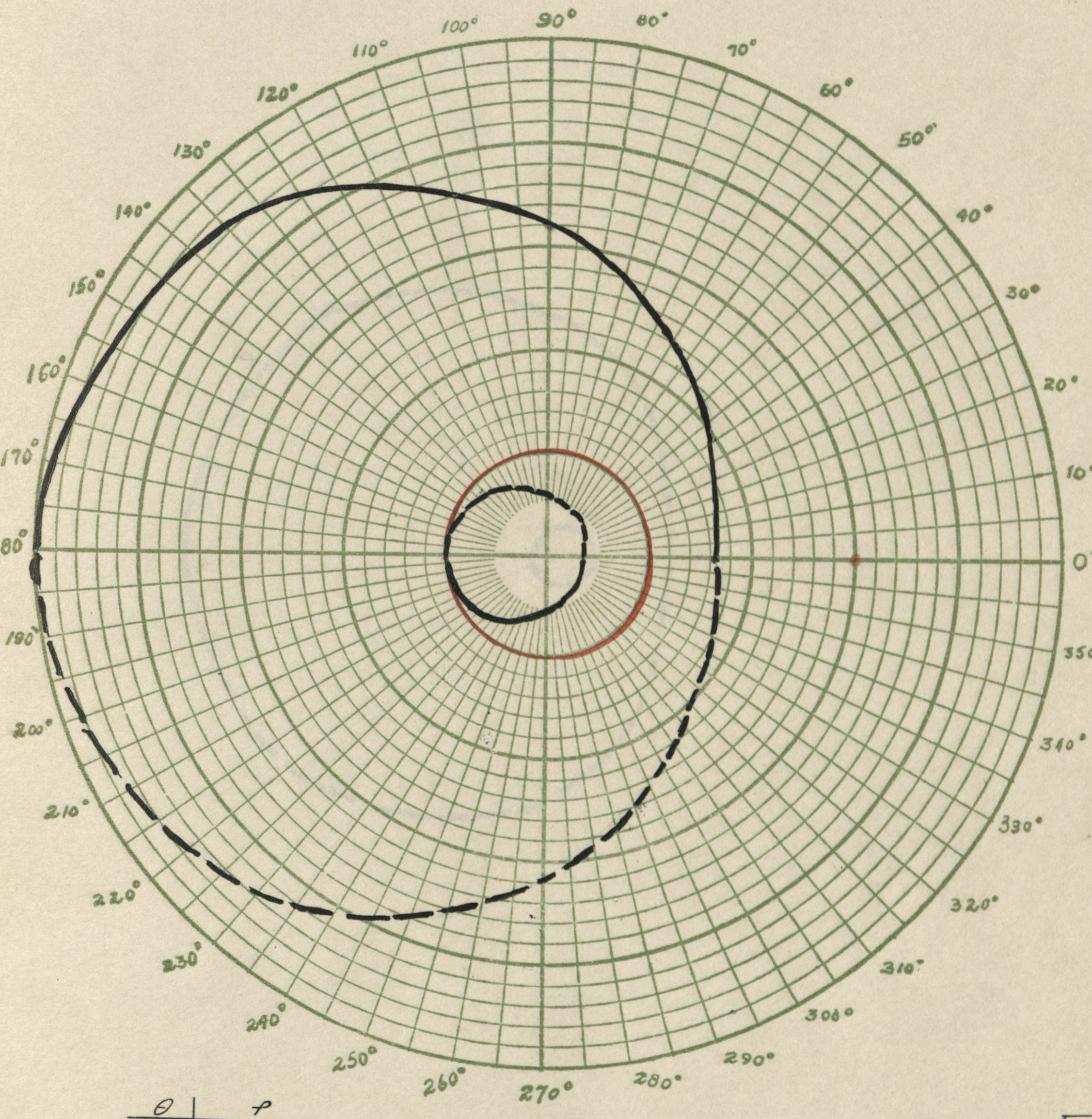
θ	ρ	
0°	2.00	-2.00
45	2.43	- 1.65
90	3.73	- 1.07
135	5.31	- .75
180	6.00	- .67

Fig. XIV

$k = 2$
$n = 1$
$c = 4$

$$\rho = \tfrac{4}{3}\left[1 - \cos\theta \mp \sqrt{(1-\cos\theta)^2 + \tfrac{9}{4}}\right]$$

Locus ——



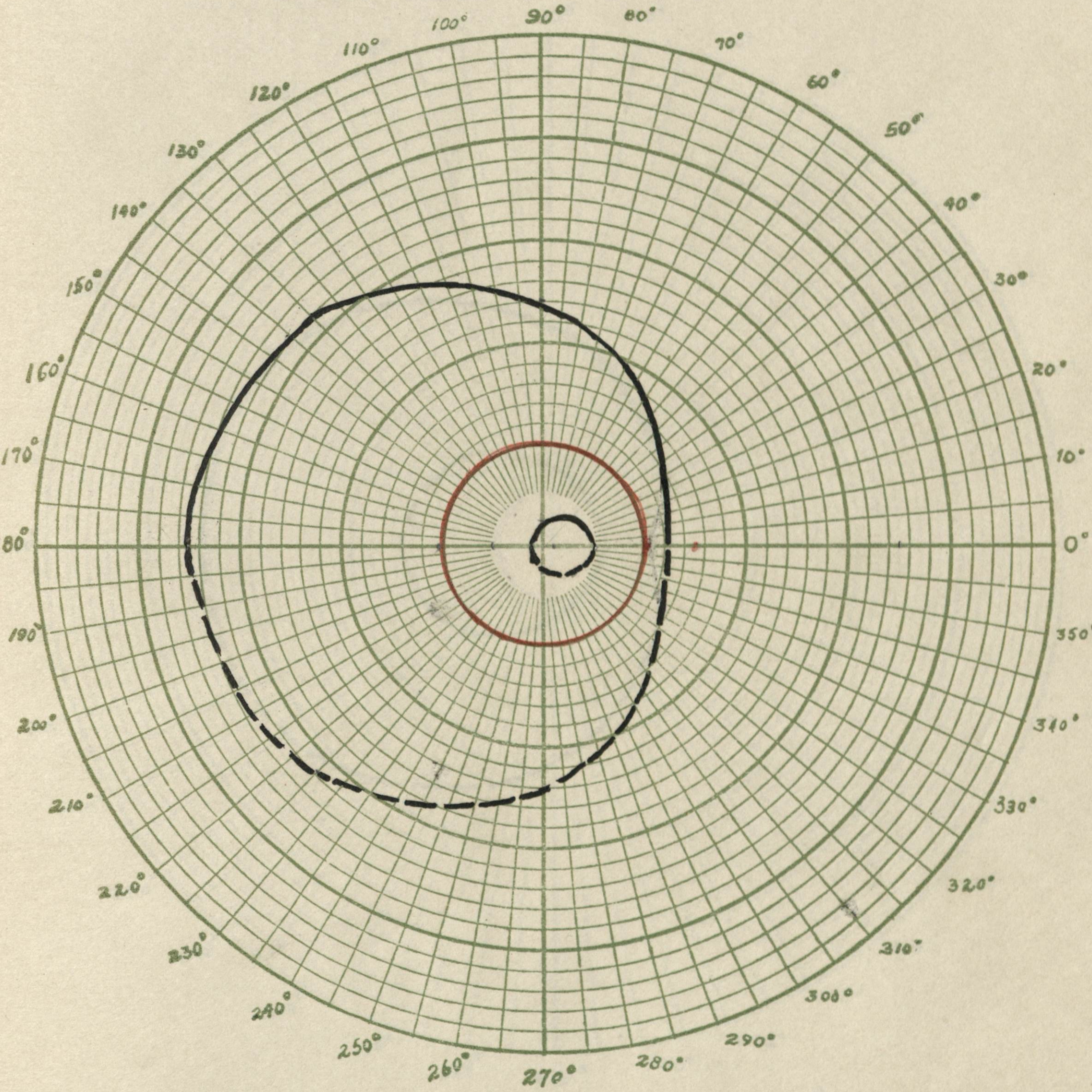
θ	ρ	
0°	-1.00	1.66
45	-.81	2.06
90	-.52	3.19
135	-.38	4.46
180	-.33	5.00

Fig XV

$k = 2$
$n = 1$
$c = 3$

$$\rho = \frac{4 - 3\cos\theta \mp \sqrt{(4 - 3\cos\theta)^2 + 15}}{3}$$

Locus ——



θ	ρ	
0°	.50	1.20
45	.37	1.59
90	.24	2.43
135	.18	3.19
180	.17	3.50

Fig. XV'

$k = 2$
$r = 1$
$c = 1.5$

$$\rho = \frac{4 - 1.5\cos\theta \mp \sqrt{(4 - 1.5\cos\theta)}}{3}$$

Locus —

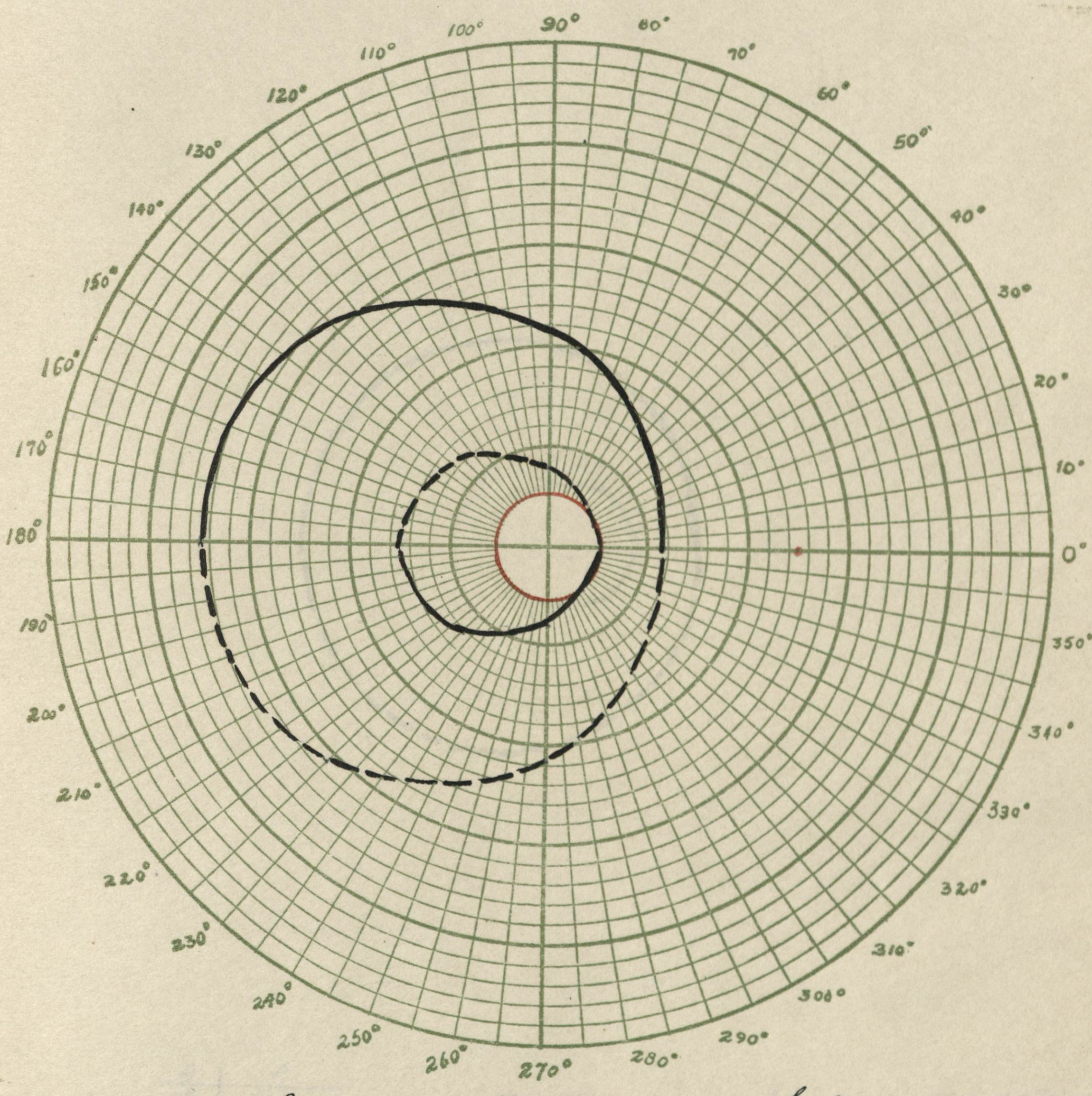


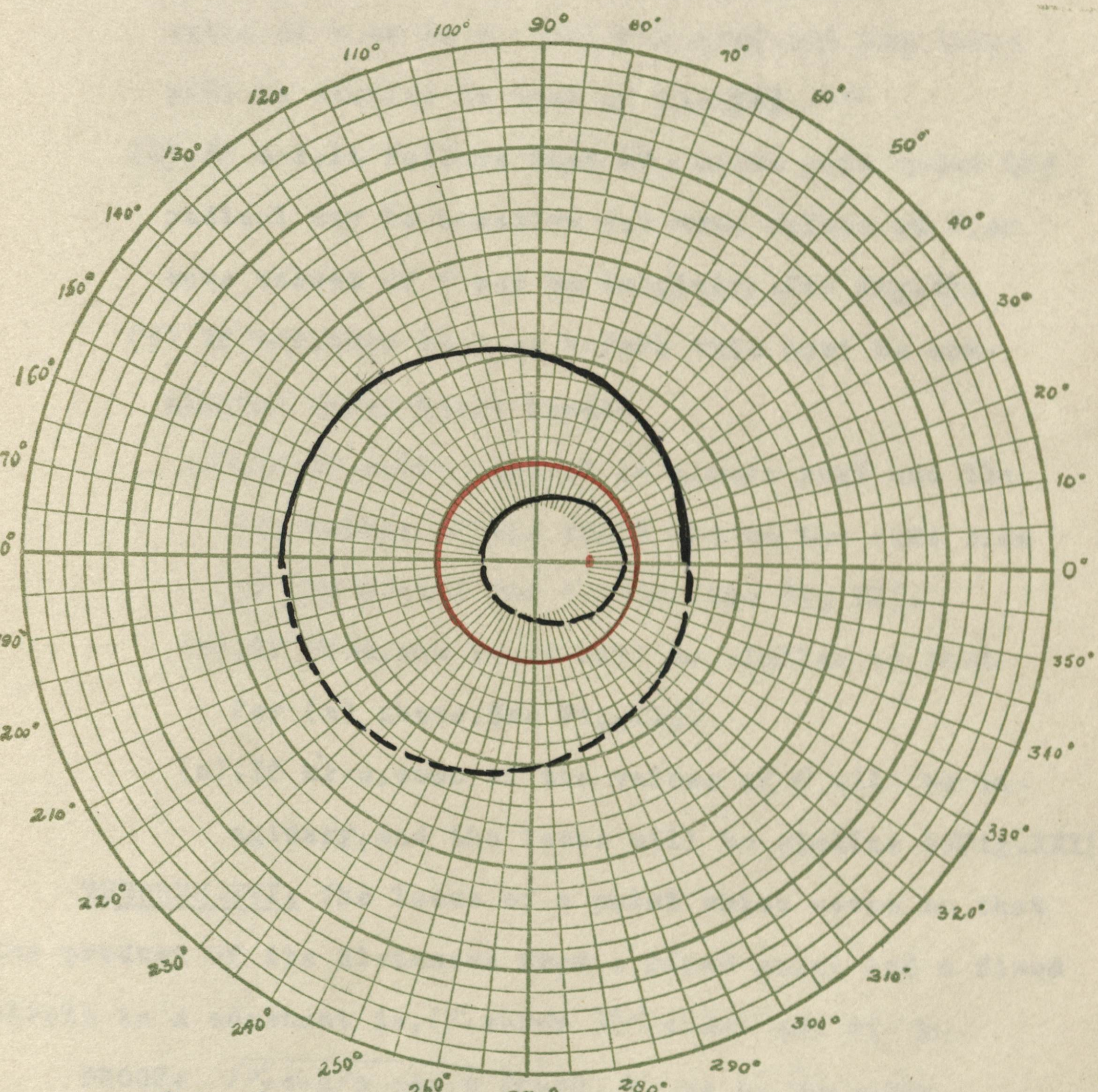
Fig. XVI

θ	ρ	
0°	-3.00	2.33
45	-2.50	2.80
90	-1.63	4.30
135	-1.13	6.17
180	-1.00	7.00

$k = 2$
$n = 1$
$c = 5$

$$\rho = \frac{4 - .5\cos\theta \mp \sqrt{(4 - .5\cos\theta)^2 + 63}}{3}$$

Locus —



θ	ρ	
$0°$	.83	1.50
30	.78	1.60
45	.74	1.70
60	.70	1.80
90	.61	2.06
120	.55	2.29
135	.53	2.37
150	.51	2.44
180	.50	2.50

Fig XVII

$k = 2$
$n = 1$
$c = \frac{1}{2}$

$$\rho = \frac{8 - \cos\theta \mp \sqrt{(8-\cos\theta)^2 - 45}}{6}$$

Locus —

For some values of θ between 0° and 180°, $(k^2-\cos\theta)^2 < (1-k^2)^2$. When $\theta = 0$, ρ is real, but for values of $\theta = 0$ up to a certain angle (depending on the values of the constants chosen), ρ is imaginary; from that value of θ on to $\theta = 180^\circ$, ρ is real, and the locus will be similar to that of Fig. XVIII.

(2). If $c>r$, it follows that $k^2r < c$. The part under the radical may be negative for some values of θ, so some values of ρ may be imaginary. See Fig. XIX.

(3) If $c<r$, then $k^2c<r$. As before this must be considered under three heads.

(a). If $k^2r=c$, the radical is always real and the two halves of the ovals are on the same side of the x-axis for $0^\circ<\theta<180^\circ$. See Fig XXI.

(b). If $k^2r>c$, the locus will be similar to that for (a) above. See Fig. XXII.

(c). If $k^2r<c$, some of the values of ρ will be imaginary, and the locus will be similar to Fig. XXIII.

<u>THEOREM XVI:</u> The locus of a point which moves so that the product of its distances from a fixed point and a fixed circle is a constant is, $(\rho^2+r^2-2\rho r)(\rho^2+c^2-2c\rho\cos\theta) = k$.

PROOF: $\sqrt{\rho^2+c^2-2\rho c\cos\theta}\,(\rho-r) = k$, or on squaring

$$(\rho^2+r^2-2\rho r)(\rho^2+c^2-2c\rho\cos\theta) = k.$$

In order to determine the nature of the locus, values will be assigned to the various constants, and the locus plotted. Points on the locus will be approximated in the following manner, for a given value of the variable θ, roots

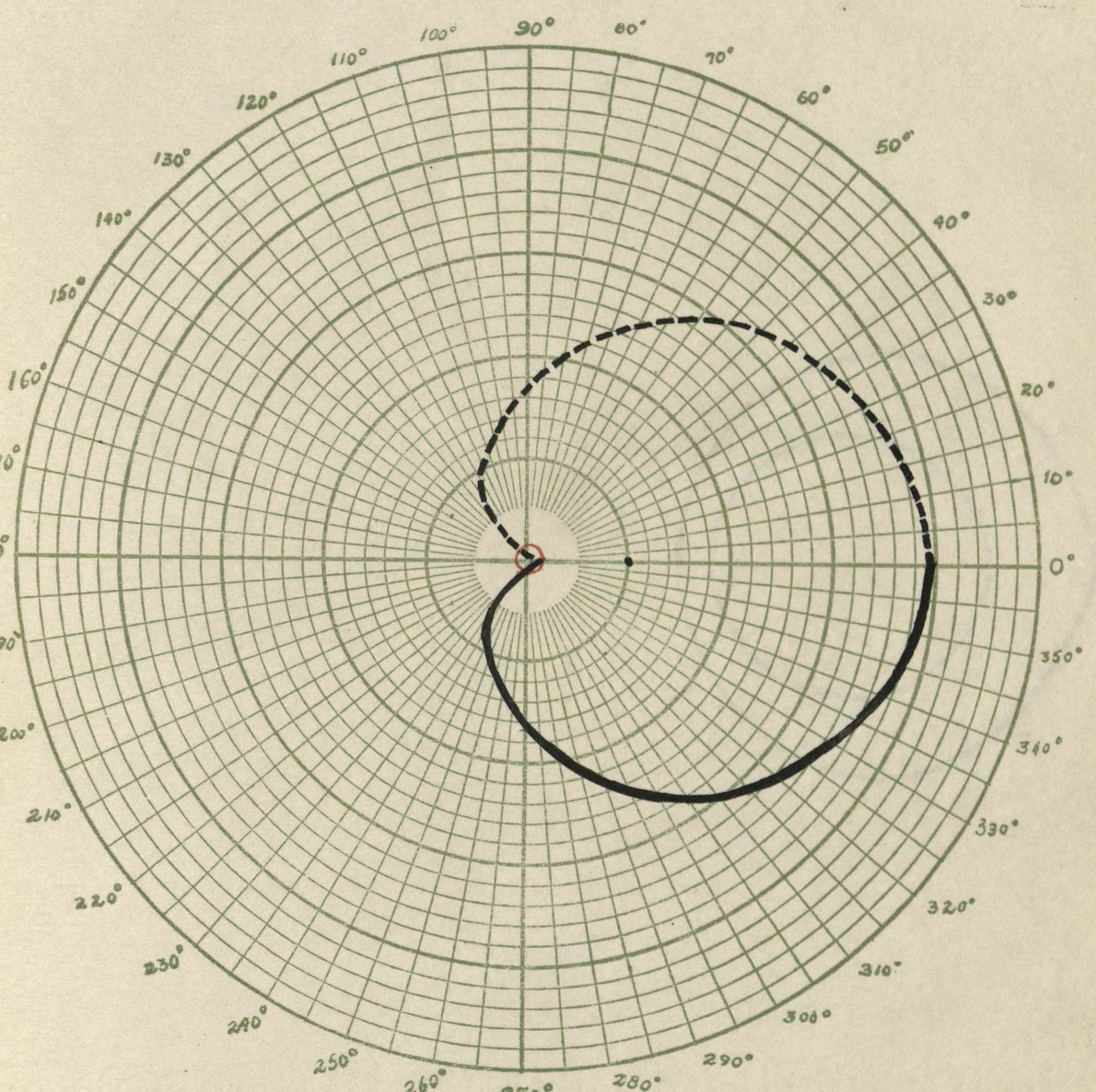


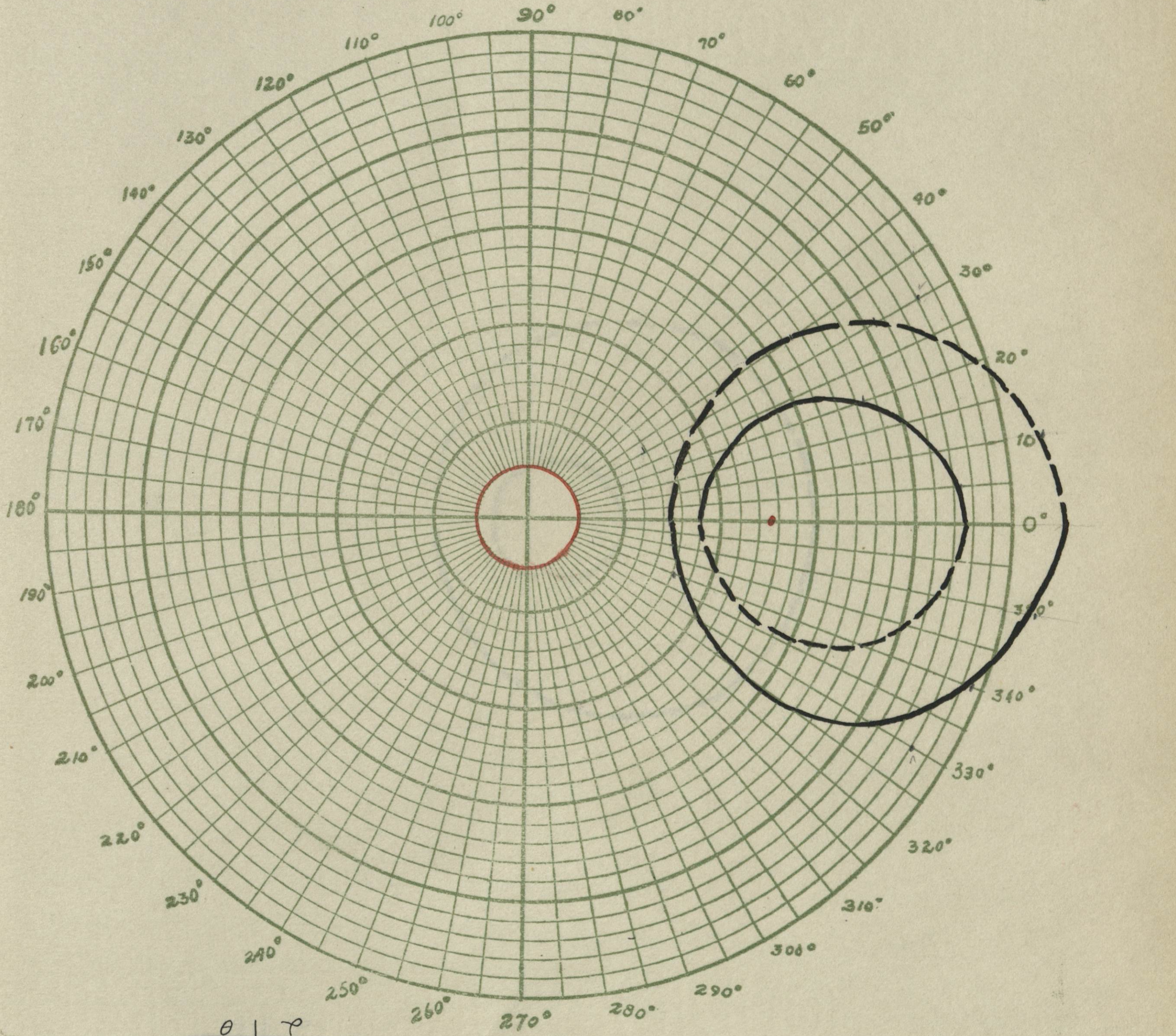
Fig. XVIII

θ	ρ	
0	1.00	
⋮	$f(i)$	
⋮	⋮	
51°41'	−1	
60	−4	−.21
90	−8.42	−.[illegible]
120	−13.68	−.05
135	−15.95	−.05
150	−17.63	−.05
180	−20.05	−.05

$k = .9$
$n = 1$
$c = 1$

$$\rho = \frac{-.81 + \cos\theta \pm \sqrt{(.81 - \cos\theta)^2 - (.19)^2}}{.19}$$

Locus—



θ	ρ	
0°	9.00	3.67
5	8.92	3.69
10	8.65	3.81
20	7.40	4.47
23	6.60	5.00
24	f(i)	
:	:	
,	,	
144	:	
145	-6.60	-5.00
150	-8.27	-4.00
160	-9.87	-3.33
170	-10.75	-3.08
180	-11.00	-3.00

Fig $\overline{\underline{XIX}}$

$k = .5$
$c = 5$
$r = 1$

$$\rho = \frac{\cos\theta - .05 \pm \sqrt{(.05 - \cos\theta)^2 - .7425}}{.15}$$

Locus ——

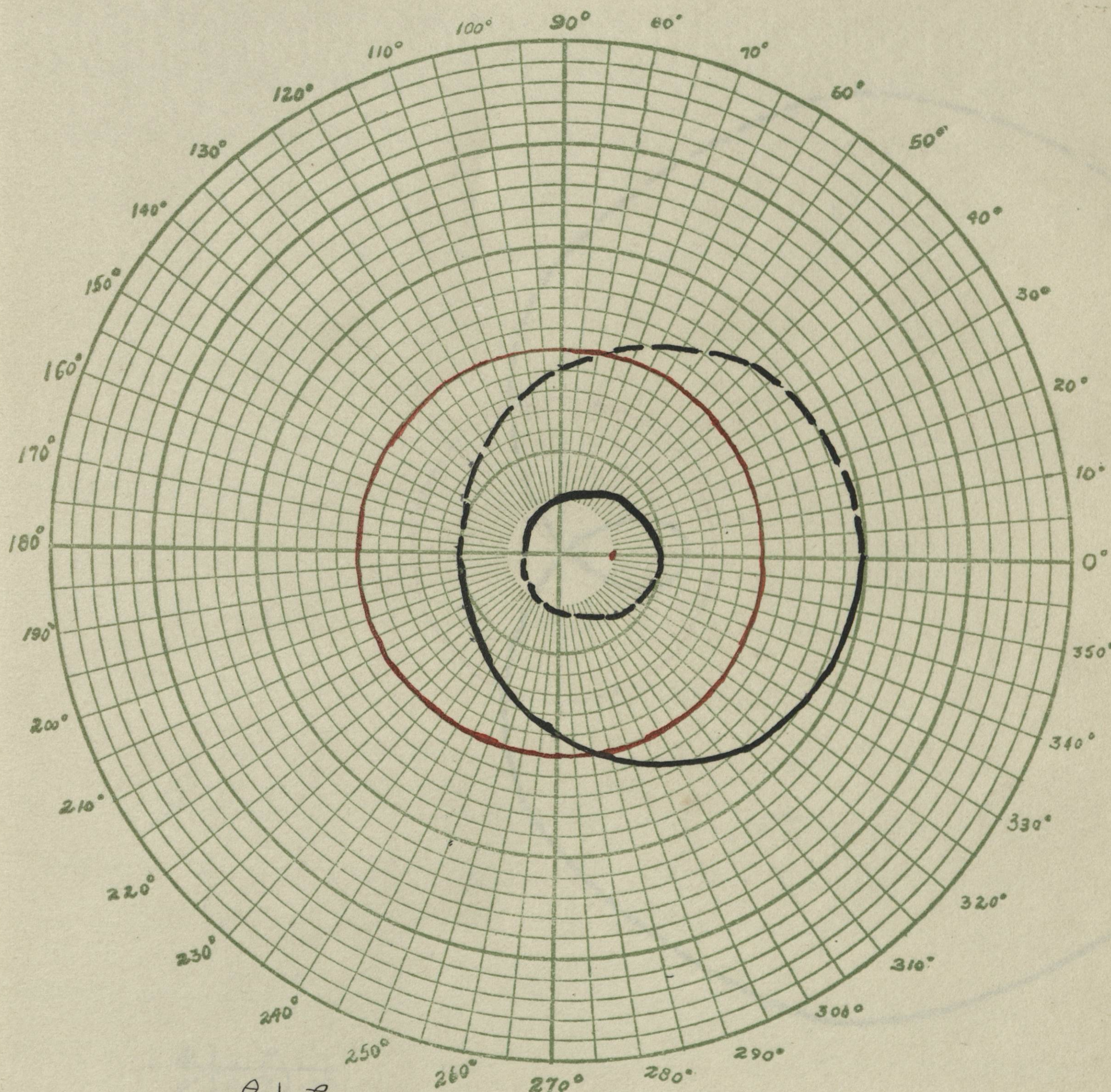


Fig XX

θ	ρ	
0	2.00	-2.00
15	1.96	-2.64
30	1.88	-2.23
45	1.65	-2.43
90	1.07	-3.73
135	.75	-5.31
150	.69	-5.68
180	.67	-6.00

$k = .5$
$n = 4$
$c = 1$

$$\rho = \frac{\cos\theta - 1 \pm \sqrt{(1-\cos\theta)^2 + 2.25}}{.75}$$

Locus —

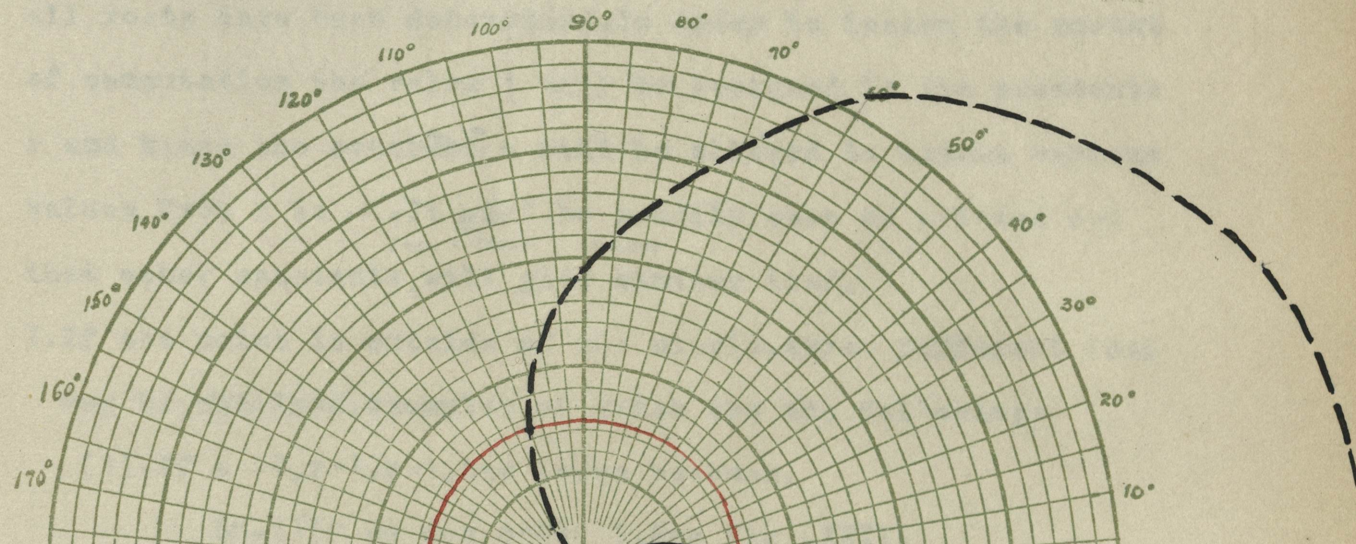


Fig. XXI

θ	ρ	
0°	2.53	.27
10	2.33	.29
20	1.93	.35
28	1.15	.59
29	1.06	.64
30	f(i)	
⋮	⋮	
60		
61	-.59	-1.15
62	-.40	-1.68
65	-.35	-1.97
90	-.18	-5.89
120	-.07	-10.33
135	-.05	-12.19
155	-.05	-13.95
180	-.05	-14.75

$k^2 = 5$
$n = 3$
$c = 2.2$

$$\rho = 4.4\cos\theta - 3 \pm \sqrt{(3 - 4.4\cos\theta)^2 - 68}$$

Locus ——

of the resulting fourth degree equation will be approximated according to the following theorem, "If f(a) and f(b) have contrary signs, a root of $f(\rho)=0$ lies between a and b".* Sturm's theorem will be employeed throughout to show that all roots have been determined. In order to lessen the amount of computation the value 1 will be assigned to the constants r and k; and the constant c will be allowed to assume various values from 0 to ∞. It will be readily seen or pointed out that other constants than those chosen will give similar loci.

I. If the point is outside of the circle, three different loci may be obtained, examples of which are the following:

(1). If $c=2, r=1, k=1$, the locus becomes,

$$(\rho-1)^2(\rho^2-4\cos\theta+4)=1. \text{ See Fig. XXII.}$$

(2). If $c=3, r=1$ and $k=1$, the locus becomes,

$$(\rho-1)^2(\rho^2-6\cos\theta+9)=1. \text{ See Fig. XXIII.}$$

(3). If $c=4, r=1$ and $k=1$, the locus becomes,

$$(\rho-1)^2(\rho^2-6\cos\theta+16)=1. \text{ See Fig. XXIV.}$$

From the figure it may be seen at once that if the constant, k, is greater than half of the distance between the point and the circle; i.e. if $\frac{c-r}{2}<k$, for $\theta=0$ there exists only two real roots of the equation and the locus must assume a shape similar to that of Fig. XXIII. Fig. XXIII will occur whenever the following relationship exists between the constants, $\frac{c-r}{2}=k$. If $\frac{c-r}{2}>k$, the locus will be similar to that of Fig. XXIV.

II. If the point is inside of the circle, i.e. if $c<r$, the

* Fine's, "College Algebra", pp. 450.

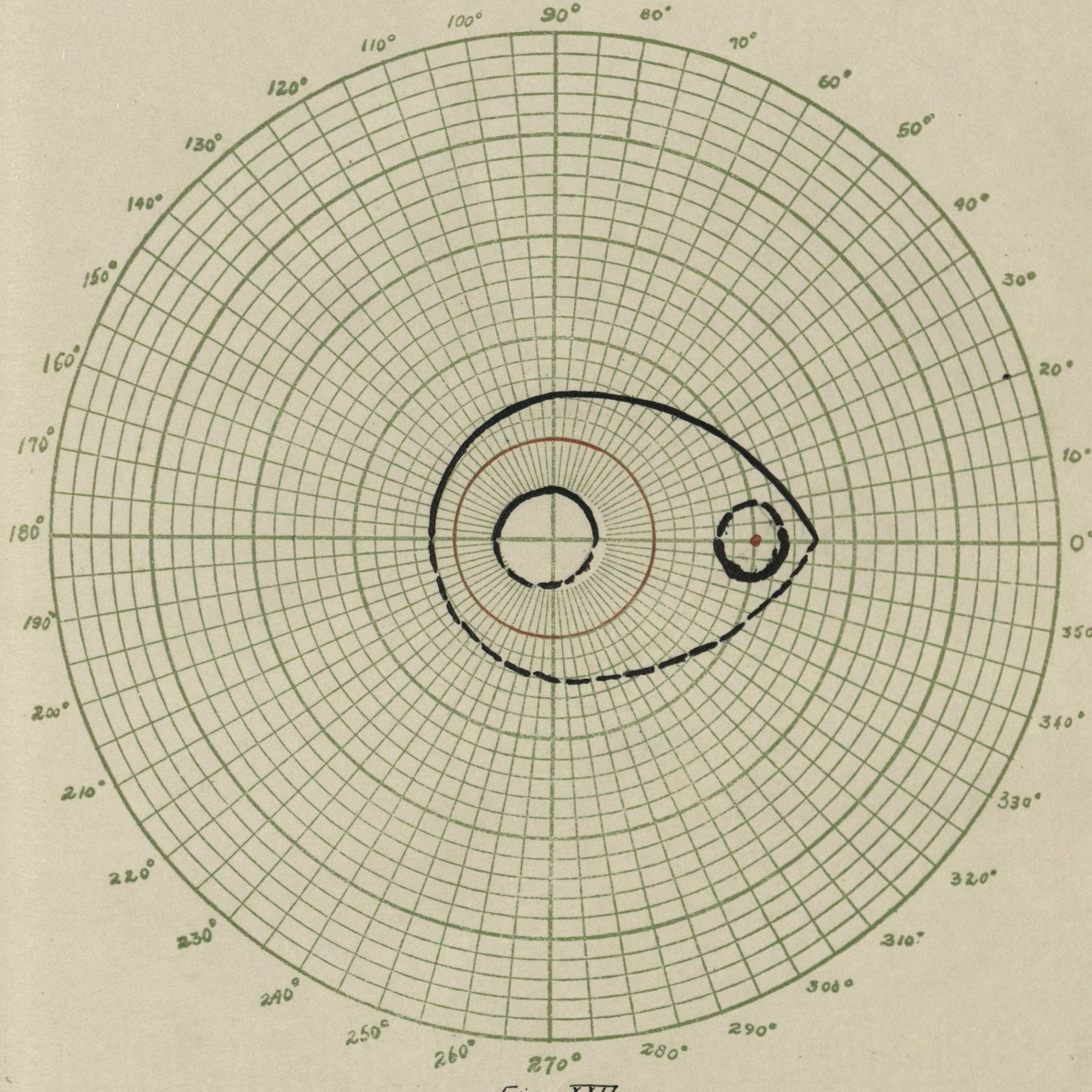


Fig. XXII

θ	ρ			
0	2.62	.38		
30	2.0	.4+		
45	1.7 −	.4+		
60	1.6 −	.4+		
90	1.4+	.5+		
120	1.3+	.6−		
135	1.2+	.6−		
150	1.2+	.6+		
170	1.2+	.6+	−2.1+	−2.2+
175	—	—	−1.7+	−2.2+
180	1.2+	.62	−1.62	−2.30

$k=1$
$c=2$
$n=1$

$$(\rho-1)^2(\rho^2-4\rho\cos\theta+4)=1$$

Locus ——

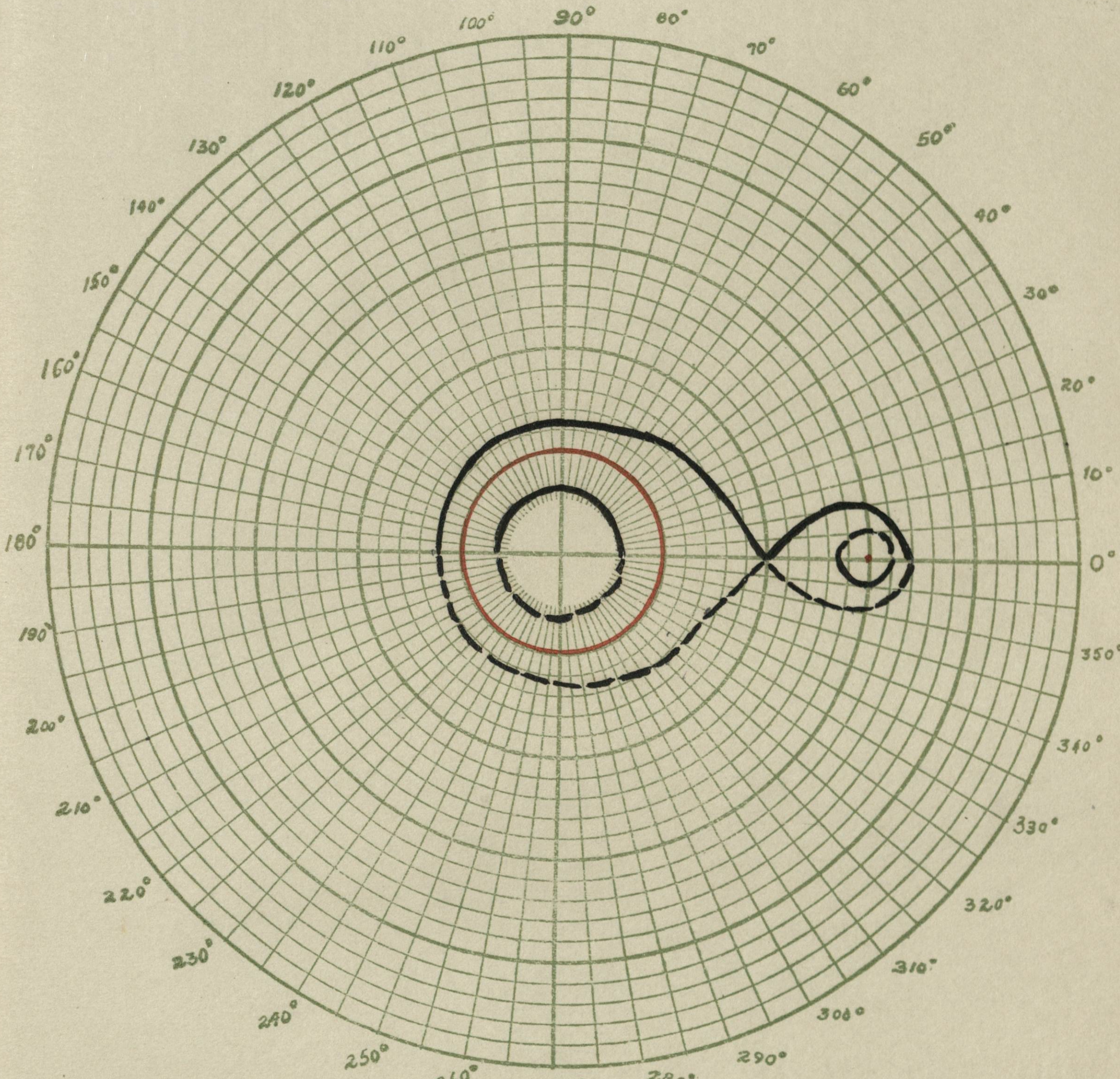


Fig. $\boxed{XXIII}$

θ	ρ					
0	2	—	3.4	.6		
7	—	2.3−	3.2+	—		
10	—	2.6+	2.9+	=		
20	1.7			.6+		
45	1.5+					
90	1.3+			.7−		
135	1.3+			.7−		
175	—			—	−3.0+	−3.1+
180	1.24			.73	−2.73	−3.24

$k=1$
$c=3$
$r=1$

$$(\rho-1)^2(\rho^2+9-6\rho\cos\theta)=1$$

Locus ——

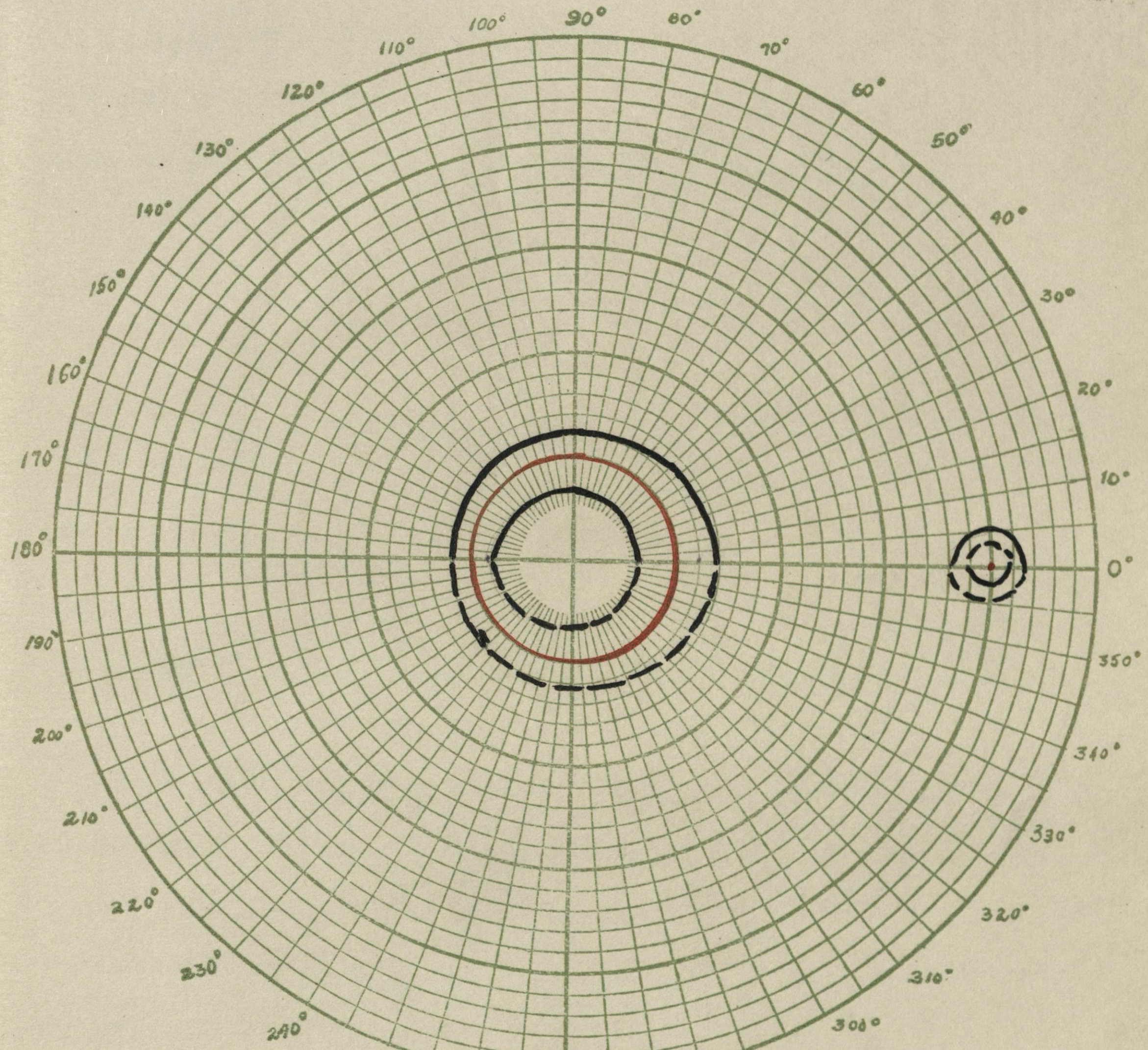


Fig. XXIV

θ	ρ					
0	4.30+	.70−	3.62	1.38		
45		.7−		1.3+		
90		.7+		1.2+		
135		.7+		1.2+		
180		.79		1.19	−4.19	−3.79

$k=1$
$c=4$
$n=1$

$$(\rho-1)^2(\rho^2+16-8\rho\cos\theta)=1$$

Locus —

locus will be similar to that of Fig.XXV, except when the point is at the center of the circle, when the locus is a circle.

III. If the point is on the circle, i.e. $c=r$, the two ovals within the larger contour reduce to points. See Fig.XXVI.

<u>THEOREM XVII</u>: The locus of a point which moves so that the sum of its distances from a fixed circle and a fixed line is a constant is a parabola, or the x-axis.

PROOF :

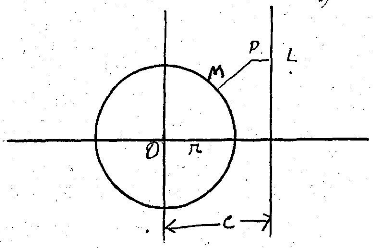


Fig.XXVII.

$MP+LP=k$

or $\rho-r+c-\rho\cos\theta=k$, which gives,

$$\rho=\frac{k+r-c}{1-\cos\theta}.$$

Comparing this with the general equation of the conic,

$$\rho=\frac{em}{1-e\cos\theta},$$

it is seen at once that the only possible case of the conic is the parabola, since e is 1. Then m is $k+r-c$. *
If $r+k=c$, the locus becomes the x-axis between the point and the circle, and if direction is taken into consideration, it may be the entire x-axis.

<u>THEOREM XVIII</u>: The locus of a point which moves so that the difference of its distances from a fixed line and a fixed circle is a constant is a parabola. Using Fig.XXVII.

PROOF: $-\rho+r+c-\rho\cos\theta=k$, from which

* If P is inside of the circle, the equation is obtained by replacing $r-\rho$ for $\rho-r$. Result is similar.

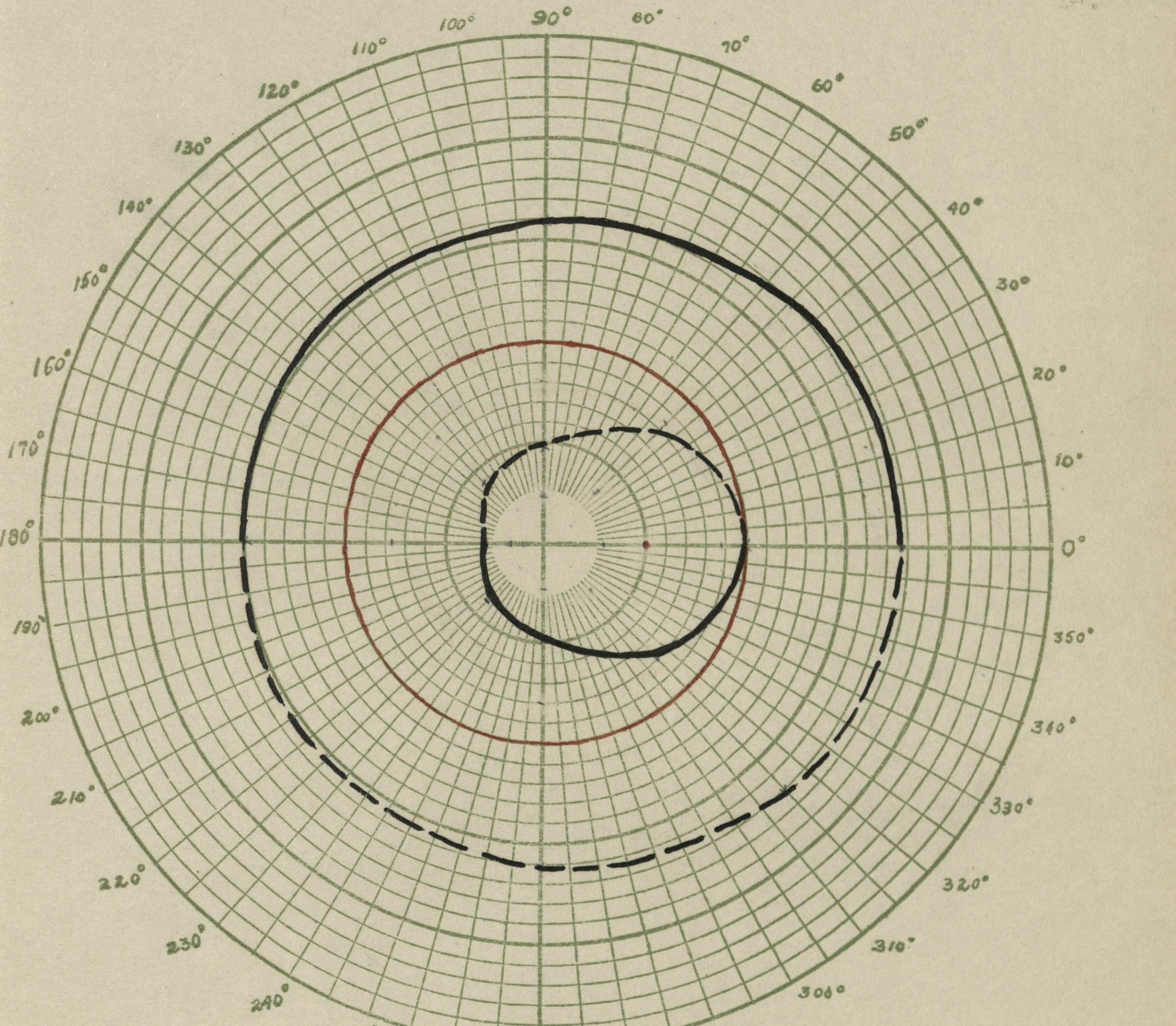


Fig. XXV

$k = 1$
$r = 1$
$c = .5$

$$(\rho - 1)^2(\rho^2 - \rho\cos\theta + .25) = 1$$

Locus —

θ	ρ	
0	1.78	−.28
45	1.7+	−.4+
90	1.6	−.5+
135	1.5+	−.8+
180	1.5	−1.0

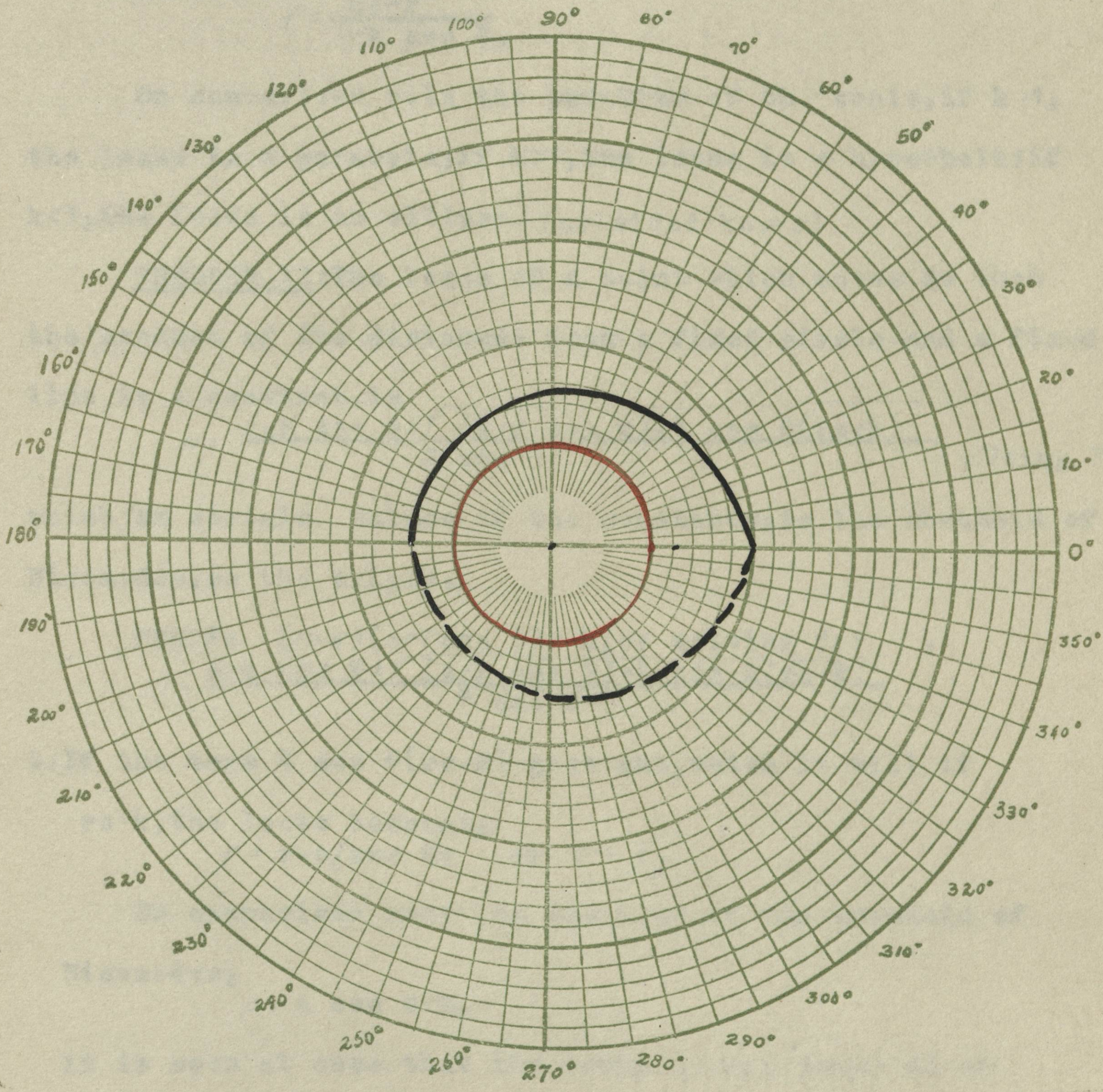


Fig. XXVI

θ		ρ
0	0	2.0
45	0	1.7+
90	0	1.5+
180	0	$\pm\sqrt{2}$

$k=1$
$n=1$
$c=.1$

$$(\rho-1)^2(\rho^2-2\rho\cos\theta+.01)=1$$

Locus ——

or $\rho = \dfrac{c+r-k}{1+\cos\theta}$, which is a parabola.

THEOREM XIX: The locus of a point which moves so that the quotient of its distances from a fixed circle and fixed line is a constant, is a conic section. Using Fig.XXVII.

PROOF: $\dfrac{\rho - r}{c-\rho\cos\theta} = k$, from which

$$\rho = \frac{kc-r}{1+k\cos\theta}.$$

On comparison with the equation of the conic, if $k=1$, the locus is a parabola; if $k>1$, the locus is a hyperbola; if $k<1$, the locus is an ellipse. (provided $kc \neq r$)

THEOREM XX: The locus of a point which moves do that the product of its distances from a fixed circle and a fixed line is a constant, is

$$\rho = \frac{c+r\cos\theta \pm\sqrt{(c+r\cos\theta)^2-4\cos\theta(rc+k)}}{2\cos\theta}$$, Using Fig.XXVII.

which for certain values of the constants, is the conchoid of Nicomedes, or the origin.

PROOF: $(\rho-r)(c-\rho\cos\theta) = k$, or solving for ,

$$\rho = \frac{c+r\cos\theta \pm\sqrt{(c+r\cos\theta)^2-4\cos\theta(rc+k)}}{2\cos\theta}$$

A. If the term $4\cos\theta(rc+k)$ goes out, which it will if $rc=k$, the locus becomes,

$$\rho = r-c/\cos\theta; \quad \text{or } \rho = 0.$$

On comparison with the equation of the conchoid of Nicomedes,

$$\rho = a\csc\theta + b,$$

it is seen at once that the locus is the conchoid of Nicomedes. If $r=c$, $r>c$, or $r<c$ the three different types of conchoids is obtained. The first has the greater cusp touching the origin, the second has the loop, and the last has the least prominent cusp.*

* Refer to "Plane and Solid Analytic Geometry", Osgood and Graustein, pp.215.

B. If the term 4 cos $\theta(rc+k)$ does not drop out.

I. If $r=c$, locus is similar to Fig. XXVIII.

II. If $r<c$, locus is similar to Figs. XXIX and XXX.

III. If $r>c$, locus is similar to Fig. XXXI.

<u>THEOREM XXI</u>: The locus of a point which moves so that the sum of its distances from two fixed circles is a constant is,

$$4x^2[(c+d)^2-(k+a+b)^2]-4(k+a+b)^2y^2+x[4d^3+4d^2c-4d(k+a+b)^2+4(k+a+b)^2c-4c^2d-4c^3]+[c^2-(k+a+b)^2]^2+d^2[d^2-2c^2-2(k+a+b)^2]=0.$$ (From Fig. XXXII).

This is an ~~general~~ equation of the second degree and therefore is a conic section, the nature of the conic depending on the relationship between the constants. It will be shown that Theorems I, V, IX, XIII, and XVII, are special cases of this theorem where the circles have become point circles or straight lines, which means that the radius has become zero or infinity.

A. Th. I, where both circles have shrunken to points.

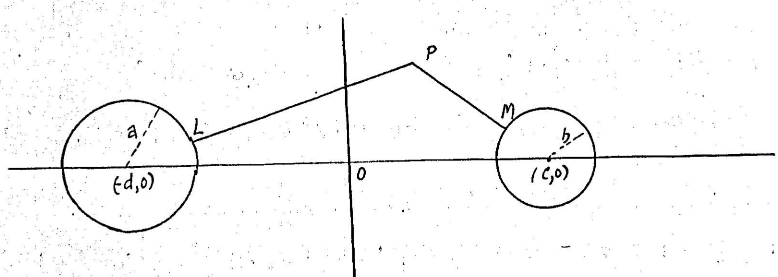


Fig. XXXII

$PL=\sqrt{(d+x)^2+y^2}-a$

$OM=\sqrt{(c-x)^2+y^2}-b$

$PL+PM=k$, then becomes

$\sqrt{(d+x)^2+y^2}=k+a+b-\sqrt{(c-x)^2+y^2}$, which on squaring to remove radicals becomes:

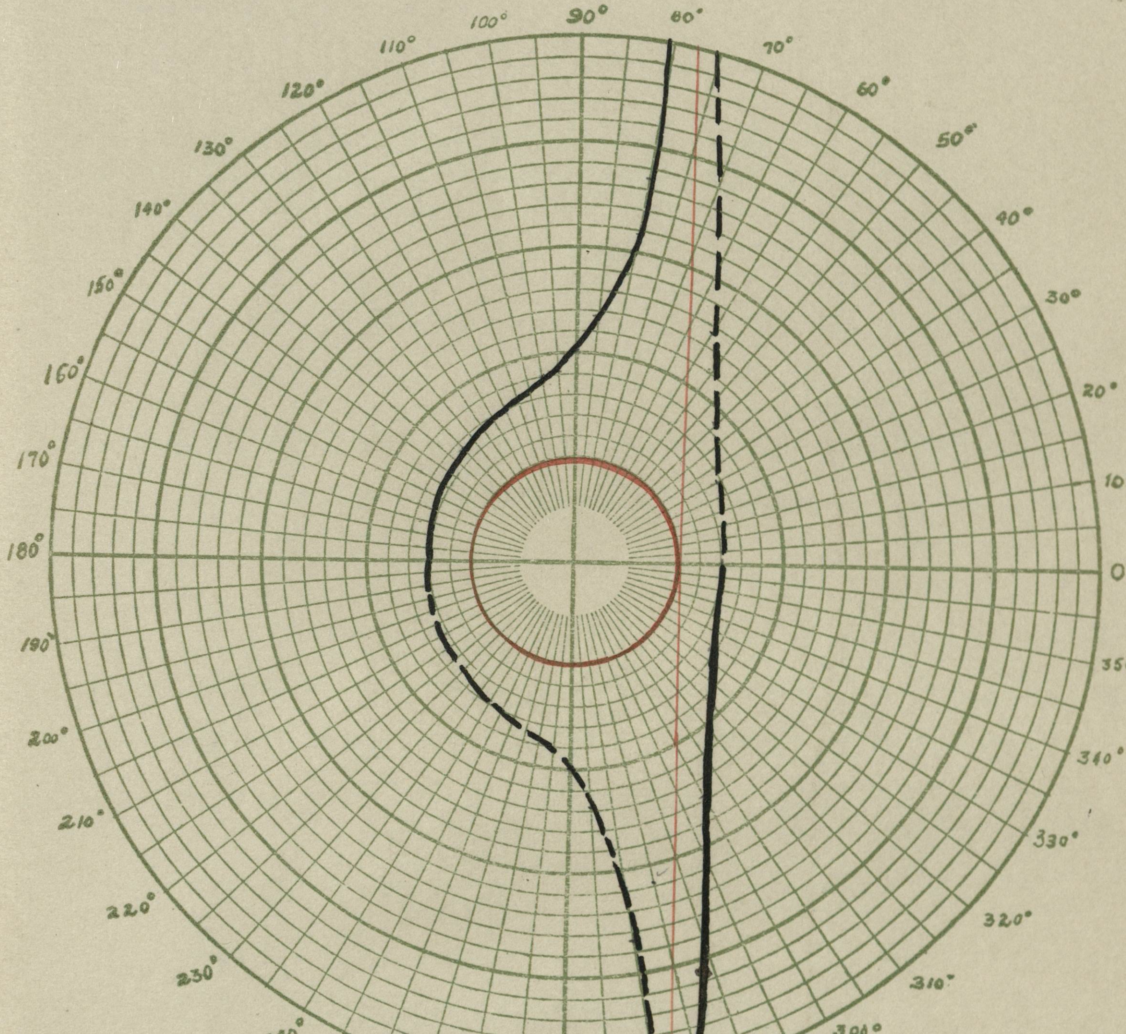


Fig. XXVIII

θ	ρ	
0	$\rho(i)$	
⋮	⋮	
79	$\rho(i)$	
80	3.71	3.18
85	9.83	2.38
88	27.50	2.07
90	∞	2.0
95	−11.94	1.83
120	−2.56	1.56
135	−1.88	1.49
150	−1.56	1.44
180	−1.41	1.41

$k = 1;$
$c = 1;$ $\rho = \dfrac{1+\cos\theta \pm \sqrt{(1+\cos\theta)^2 - 8\cos\theta}}{2\cos\theta}$
$n = 1$

Locus ——

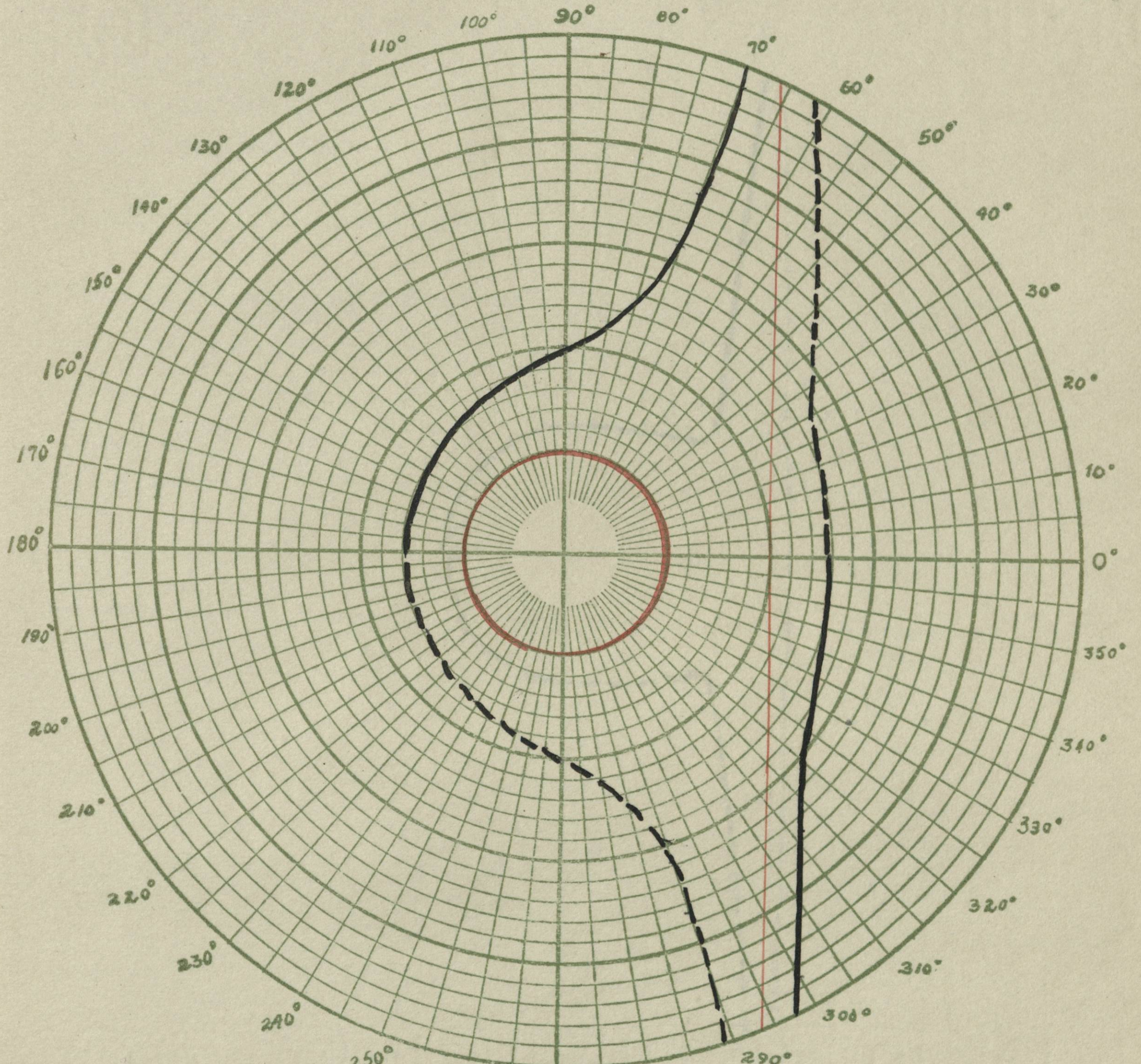


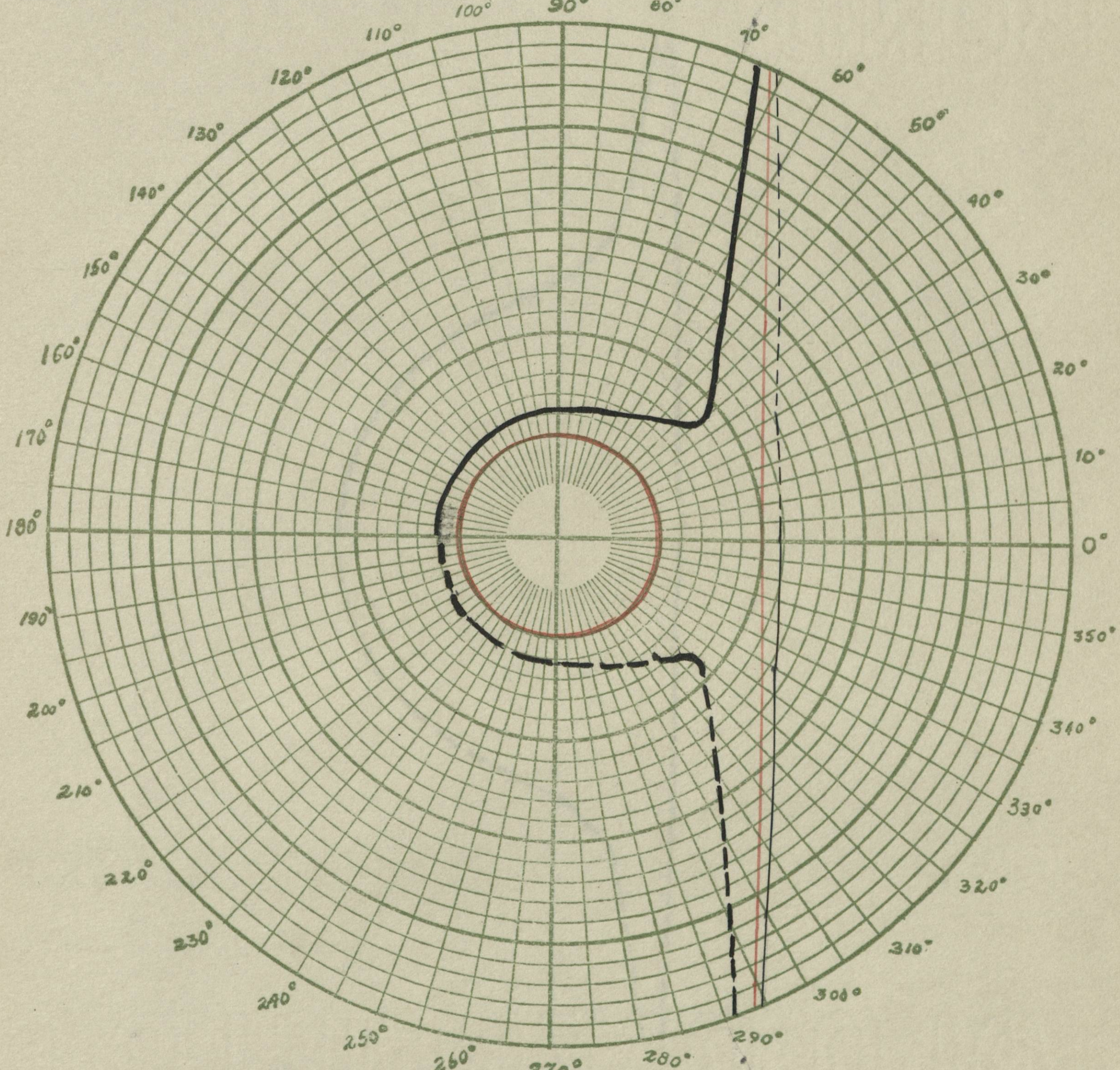
Fig. XXIX

$k = 2$
$r = 1$
$c = 2$

$$\rho = \frac{2+\cos\theta \pm \sqrt{(2+\cos\theta)^2 - 16\cos\theta}}{2\cos\theta}$$

Locus —

θ	ρ	
0	$f(i)$	
⋮	⋮	
69		
70	3.72	3.17
80	10.53	2.24
90	∞	2.0
95	-23.16	1.94
120	-4.70	1.70
150	-2.88	1.58
180	-2.56	1.56



θ	ρ	
0	$f(i)$	
⋮	⋮	
39	⋮	
40	1.88	1.74
70	5.55	1.32
80	11.14	1.26
90	∞	1.25
120	-4.20	1.20
150	-2.47	1.17
180	-2.16	1.16

Fig. XXX

$k = .5$
$r = 1$
$c = 2$

$$\rho = \frac{2 + \cos\theta \pm \sqrt{(2+\cos\theta)^2 - 10\cos\theta}}{2\cos\theta}$$

Locus ——

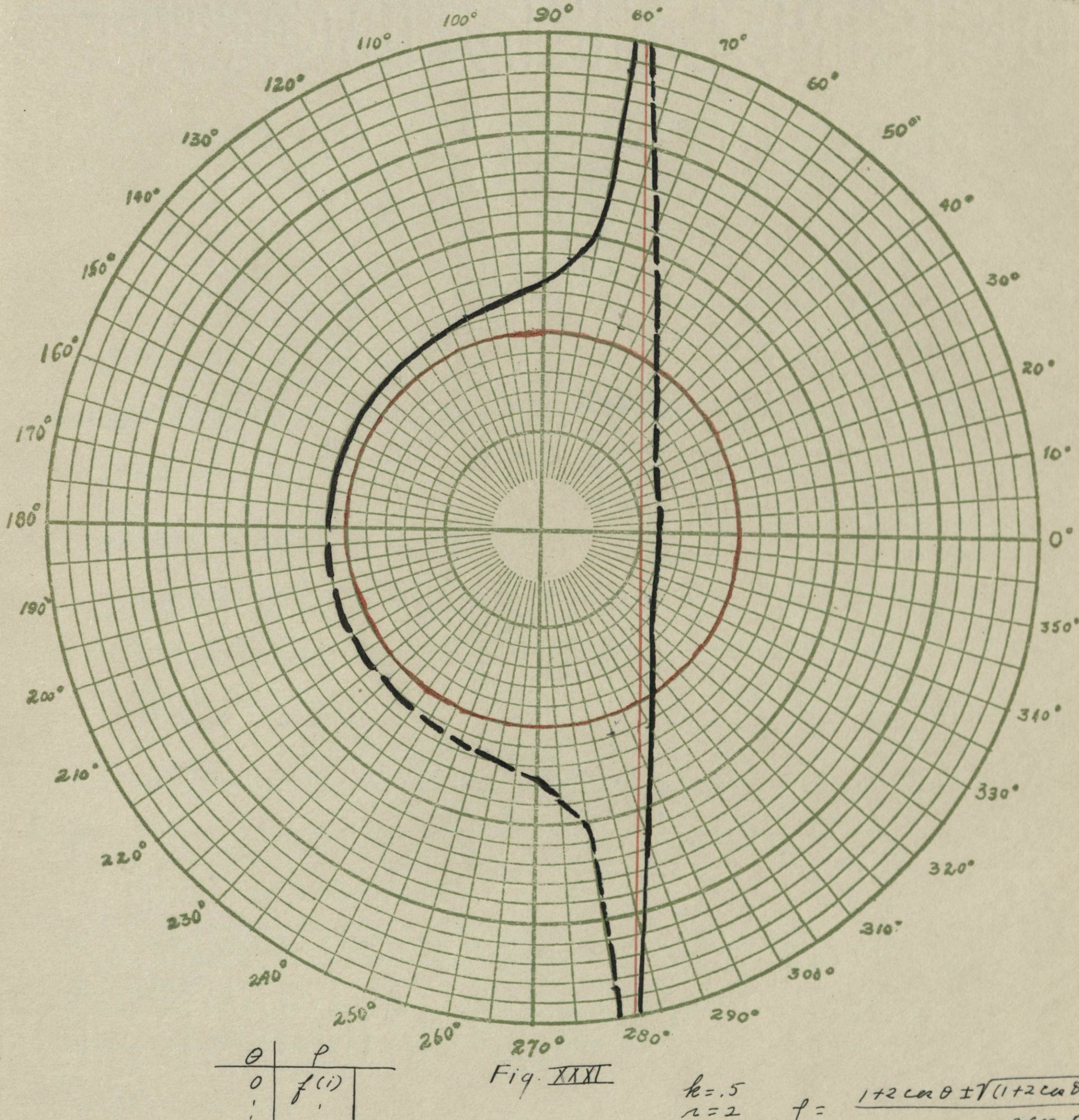


Fig. XXXI

θ	ρ	
0	$f^{(i)}$	
⋮	⋮	
79	⋮	
80	4.69	3.03
85	11.00	2.77
90	∞	2.5
95	-12.24	2.47
105	-4.15	2.31
110	-3.24	2.29
120	-2.24	2.24
135	-1.61	2.18
150	-1.33	2.17
180	-1.16	2.16

$k = .5$
$r = 2$
$c = 1$

$$\rho = \frac{1 + 2\cos\theta \pm \sqrt{(1+2\cos\theta)^2 - 10\cos\theta}}{2\cos\theta}$$

Locus ——

$$(a)\ 4x^2\left[(c+d)^2-(k+a+b)^2\right]-4y^2(k+a+b)^2+4x\left[d^3+d^2c-d(k+a+b)^2+c(k+a+b)^2-c^2d-c^3\right]$$
$$+\left[c-(k+a+b)^2\right]^2+d^2\left[d^2-2c^2-2(k+a+b)^2\right]=0$$

If $d=c$, $a=0$ and $b=0$, the circles become points, and locus is,

$x^2(k^2-c^2)+ky^2=k^2(k^2-c^2)$, which is the equation obtained befoee.*

B.Th.V, where both circles have become straight lines.

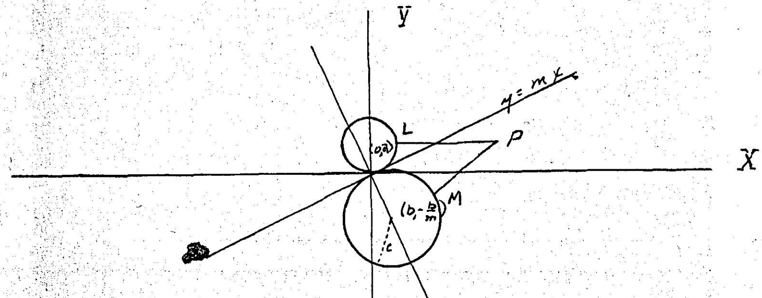


Fig.XXXIII.

$\sqrt{x^2+(a-y)^2}-a+\sqrt{(x-b)^2+(y-b/m)^2}-c=k$, is the general equation obtained for two circles from Fig.XXXIII.

If we let $c=a$, then $a^2=b^2+b^2/m^2$. Then the equation becomes, on squaring twice to remove radicals,

$$(k+2a)^2(x^2+y^2-bx+by/m-ay-k^2/4-ak)=\left[bx-(b/m+a)y\right]^2.$$

Dividing both sides through by a^3, the equation becomes,

$$(k/a+2)^2(x^2/a+y^2/a-bx/a+by/am-yk^2/4a-k)=1/a\left[bx/a-(by/am+1)y\right]^2$$

Letting $a\to\infty$, the equation reduces to,

$mx+y(\sqrt{m^2+1}-1)=-k\sqrt{m^2+1}$, which is the equation gotten before where here k is the negative of k in Th.V.

C.Th.IX, where the left-hand circle has become a line and the other circle becomes a point. In equation(a) let $d=a$, then let $d\to\infty$, $b\to 0$, and c c. The equation then becomes,

$$(b).\ 4x^2\left[(c+d)^2-(k+d)^2\right]-4y^2(k+d)^2+4x\left[d^3+d^2c-d(d+k)^2+c(d+k)^2-c^2d-c^3\right]$$
$$+\left[c^2-(d+k)^2\right]^2+d^2\left[d^2-2c^2-2(d+k)^2\right]=0$$

If $k=c$, (b) becomes,

$-4(c+d)y^2=0$, which is the equation obtained before.**

* See Pg.3
** See Pg.11

If $k \neq c$, (b) becomes,

$$4x^2\left[c^2-k^2+2d(c-k)\right] -4(k^2+d^2-2kd)y^2+4x(-2d^2k-dk^2+2cd^2-ck^2+2cdk-c^2d-c^3) + c^4+k^4+4d^2k^2-4c^2d^2-2c^2k^2-4c^2dk+4dk^3=0.$$

Dividing through by d, and since $d \to \infty$ the equation becomes $y^2=(k^2-c^2)-2(k-c)x$, which is the equation obtained before.*

D. Th. XIII, where the right-hand circle has become a point.

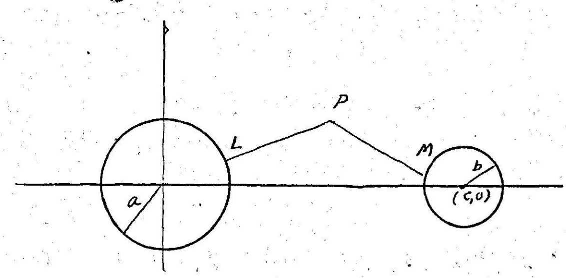


Fig. XXXIV.

$PL = \rho - r.$

$PM = \sqrt{(c-\rho\cos\theta)^2+\rho^2\sin^2\theta} - b$, or $\sqrt{\rho^2+c^2-2\rho c\cos\theta} - b$.

The sum becomes $\sqrt{\rho^2+c^2-2c\cos\theta} = k+b-\rho+r$, which on squaring gives,

$$\rho = \frac{k^2+b^2+r^2-c^2+2r(k+b)+2kb}{2(k+b+r-c\cos\theta)}$$

Let $b \to 0$, and the equation becomes that gotten before,

$$\rho = \frac{(k+r)^2-c^2}{2(k+r-c\cos\theta)} \quad ^{**}.$$

E. Th. XVII, where the right-hand circle has become a straight line. The line should not go through the circle, therefore the equation of the right-hand circle must be changed slightly to insure this.

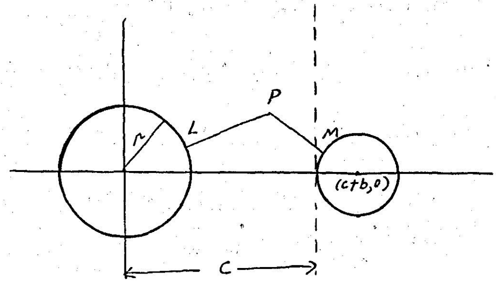


Fig. XXXV.

* See Pg. 11

** See Pg. 18

$PL = \rho - r$

$PM = \sqrt{(c+b)^2 - 2(c+b)\rho\cos\theta + \rho^2} - b.$

The sum then becomes,

$$\sqrt{(c+b)^2 + \rho^2 - 2\rho\cos\theta(c+b)} = r - \rho + b + k.$$

Reducing this becomes,

$$\rho = \frac{c^2 + 2bc - r^2 - 2r(b+k) - k^2 - 2bk}{2\left[(c+b)\cos\theta - r - b - k\right]}$$

and if $b \to \infty$, the circle becomes a line through $(c,0)$ parallel to the y-axis, and the equation becomes that gotten before,

$$\rho = \frac{r+k-c}{1-\cos\theta} \quad . \quad *$$

THEOREM XXII: The locus of a point which moves so that the difference of its distances from two fixed circles is a constant is,

$$4x^2\left[(d+c)^2 - (k+a-b)^2\right] - 4y^2(k+a-b)^2 + 4x\left[d^2(d+c) - c^2(d+c) + (d-c)(k+a\ b)^2\right]$$
$$+\left[d^2 - (k+a-b)^2\right]^2 + d^2\left[d^2 - 2c^2 - 2(k+a-b)^2\right] = 0. \text{ (From Fig.XXXII).}$$

This likewise is ~~the~~ an ~~general~~ equation of the second degree and therefore is a conic section, the nature of the conic depending on the relationship between the constants. It will be shown that Theorems II, VI, XIV and XVIII are ~~limiting~~ cases of this theorem, where the circles have become points or straight lines.

A. Th. II, where both circles have shrunken to points. See Fig. XXXII

$\sqrt{(d+x)^2 + y^2} = k + a - b + \sqrt{(c-x)^2 + y^2}$ is the locus, and on simplifying becomes,

$$(c).\ 4x^2\left[(d+c)^2 - (k+a-b)^2\right] - 4(k+a-b)^2y^2 + 4x\left[d^2(d+c) - c^2(d+c) + (d-c)(k+a-b)^2\right]$$
$$+\left[c^2 - (k+a-b)^2\right]^2 + d^2\left[d^2 - 2c^2 - 2(k+a-b)^2\right] = 0.$$

* See Pg. 37

and if $a \to 0$, $b \to 0$, and d becomes c, the locus is that obtained as before *

$$x^2(c^2-a^2) - a^2y^2 = a^2(c^2-a^2).$$

B. Th. VI, where both circles have become straight lines. Refer to Fig. XXXIII. The locus for two circles is,

$$\sqrt{x^2+(a-y)^2} - a - \sqrt{(x-b)^2+(y+b/m)^2} + c = k$$

Reducing, this becomes, if we let $c=a$,

$$[bx-y(a+b/m)]^2 = k^2(x^2+y^2+a^2-bx-k^2/2-ay+by/m).$$

Substituting the value for $b = \frac{am}{\sqrt{m^2+1}}$, and dividing by a^2,

$$\left[\frac{mx}{\sqrt{m^2+1}} - y\left(1+\frac{1}{\sqrt{m^2+1}}\right)\right]^2 = k^2\left(\frac{x^2}{a}+\frac{y^2}{a}+1-\frac{amx}{\sqrt{m^2+1}}-\frac{k^2}{2}-ay+\frac{ay}{\sqrt{m^2+1}}\right).$$

Letting $a \to \infty$, the equation becomes,

$mx-y(\sqrt{m^2+1}+1) = \pm k\sqrt{m^2+1}$, which is the equation gotten before if the positive value of k is used.**

C. Th. X, where left-hand circle has become a line and the other circle becomes a point. Equation (c) becomes if we let $b \to 0$ and $a=d$,

$$4x^2(c^2+2cd-k^2-2kd)-4y^2(k+d)^2+4x(2d^2c-c^2d-c^3-2d^2k+k^2c+2kdc-dk^2)+[c-(k+d)^2]^2$$
$$+d^2[d^2-2c^2-2(k+d)^2]=0.$$

Dividing by d^2 and letting $d \to \infty$, the equation becomes,

$y^2=(k^2-c^2)-2x(k-c)$, or $y^2(k-c)[(k+c)-2x]$, which is the equation of the parabola obtained before. If $k=c$, the locus becomes the y-axis.

D. Th. XIV, where the right-hand circle has become a point. From Fig. XXXIV shows that the locus is,

$$\sqrt{c-2\rho\cos\theta+\rho^2} = k+\rho-r+b.$$

Squaring this becomes,

$$\rho = \frac{k^2+r^2-c^2+2(kb-kr-br)}{2(r-k-b)-2\cos\theta}$$

If we let $b=0$, the circle becomes a point, The equation becomes,

$(r-k)-c$

* Pg. 4

10

$$\rho = \frac{(r-k)^2 - c^2}{2(r-k-\cos\theta)}$$

E.Th.XVIII, where the right-hand circle has become a straight line. Refer to Fig.XXXV. The equation for the two circles is,

$$\sqrt{(c+b)^2+\rho^2-2\rho\cos\theta(c+b)} = \rho+k-r+b.$$

Squaring and reducing this becomes,

$$\rho = \frac{k^2+r^2+2kb-2kr-2rb-c^2-2cb}{2(r-k-b)-2(c+b)\cos\theta}.$$

Dividing numerator and denomenator by b, and since $b \to \infty$, the locus becomes that obtained before,

$$\rho = \frac{c+r-k}{1 \quad \cos\theta}. \;*$$

<u>THEOREM XXIII</u> :The locus of a point which moves so that the quotient of its distances from two fixed circles is a constant is,

$$\rho = \frac{-(bk-k^2a+c\cos\theta) \pm \sqrt{(bk-k^2a+c\cos\theta)-(k^2-1)\left[(b-ka)^2-c^2\right]}}{k^2-1}.$$

A.Th.III, where both circles have become points. The equation for the two circles from Fig.XXXII is,

$$\sqrt{(d+x)^2+y^2} = a-kb+\sqrt{k(c-x)^2+y}.$$

Squaring and reducing this becomes,

$$\text{(d).}\quad (x^2+y^2)^2(1-k^2)^2+(a-kb)^4+(d^2-k^2c^2)^2+4x^2(d+k^2c)^2+2(x^2+y^2)(1-k^2)\left[2x(d+k^2c)\right.$$
$$\left.-(a-kb)^2+(d^2-k^2c^2)\right] -2(a-kb)^2\left[(d^2-k^2c^2) \quad 2x(d+k^2c)\right]+4x(d+k^2c)(d^2-k^2c^2)$$
$$-4k^2(a-kb)^2(c^2+x^2-2cx+y^2) = 0.$$

If we let a=0, b=0, and d=c, the circles become points and the locus becomes,

$$(x^2+y^2)^2(1-k^2)^2+c^4(1-k^2)^2+4xc(1+k^2)^2+2(x^2+y^2)(1-k^2)\left[2xc(1+k^2)+c^2(1-k^2)\right]+$$
$$4xc^3(1+k^2)(1-k^2) = 0, \quad \text{or}$$

$\left[(1-k^2)(x^2+y^2+c^2)+2c(1+k^2)x\right]^2 = 0$, which is the equation obtained before.**

* See Pg. 38
** See Pg. 4

B. Theorem VII, where both circles have become straight lines. Refer to Fig. XXXIII. The locus for two circles is,

$$\sqrt{x^2+(a-y)^2} = a-kc+k\sqrt{(x-b)^2+(y\ b/m)^2}.$$

Squaring and reducing this becomes, letting $a=c$, and expressing b in terms of c,

$$(x^2+y^2)^2(1-k^2)^2\quad 4y^2c^2\left(1+\frac{k^2}{\sqrt{m^2+1}}\right)^2+\frac{4c^2m^2k^4x^2}{\sqrt{m^2+1}}+2(x^2+y^2)(1-k^2)\left[-2yc\left(1+\frac{k^2}{\sqrt{m^2+1}}\right)\right.$$
$$\left.+\frac{2cmk^2x}{\sqrt{m^2+1}}+2kc^2-2k^2c^2\right]-4ky\left(1+\frac{k^2}{\sqrt{m^2+1}}\right)\left[\frac{2cmk^2x}{\sqrt{m^2+1}}+2c^2k-2k^2c^2\right]+$$
$$\frac{4mck^2x}{\sqrt{m^2+1}}(2c^2k-2k^2c^2)-4k^2c^2(1-k)^2\left(x^2-\frac{2cmx}{\sqrt{m^2+1}}+y^2+\frac{2cy}{\sqrt{m^2+1}}\right)+4k^2c^4+$$
$$4k^4c^4-8ak^3c^4-4k^2c^6(1-2k+k^2)=0.$$

Dividing the equation by c^3, and letting $c\to\infty$, it becomes

$mkx-y(k+\sqrt{m^2+1})=0$, and dividing by k,

$mx-y\left(1+\dfrac{\sqrt{m^2+1}}{k}\right)=0$, which is the equation gotten before with the exception that the constant is different. (reciprocal)

C. Theorem XI, where left-hand circle becomes a line and other circle becomes a point. In equation (d), if $b=0$, $a=d$, and $d\to\infty$, the locus becomes,

$$(x^2+y^2)^2(1-k^2)^2+d^4+(d^2-k^2c^2)^2+4x^2(d+k^2c^2)^2+2(x^2+y^2)(1-k^2)\left[2x(d+k^2c)-d^2+\right.$$
$$\left.(d^2-k^2c^2)\right]-2d^2\left[(d^2-k^2c^2)\ 2x(d+k^2c)\right]+4x(d+k^2c)(d^2-k^2c^2)-4k^2d^2(c^2+x^2-2cx+y^2)=0$$

Dividing by d^2, and let $d\to\infty$, the locus becomes, after dividing by k^2,

$x^2(1-1/k^2)+y^2-2cx+c^2=0$,

which is the equation gotten before except that this constant is the reciprocal of that k used before.

D. Theorem XV, where the right-hand circle has become a point. The locus for two circles is, using Fig. XXXIV,

$$\frac{\sqrt{(c-\rho\cos\theta)^2+\rho^2\sin^2\theta}-b}{\rho-a}=k\text{, which becomes,}$$

$\rho^2(k^2-1)+2\rho(kb-k^2a+c\cos\theta)+(b-ka)^2-c^2=0$, whence

$$\rho=\frac{-(kb-k^2a\ c\cos\theta)\pm\sqrt{k(bk-k^2a+c\cos\theta)^2-(k^2-1)\left[(b-ka)^2-c^2\right]}}{k^2-1}$$

and if $b=0$, one circle becomes a point and locus becomes,

$$\rho = \frac{c\cos\theta - ka \mp \sqrt{(c\cos\theta - ak)^2 + (k^2a^2 - c^2)(1-k^2)}}{-k^2+1},$$

the equation gotten before.*

E. Theorem XIX, where the right-hand circle has become a straight line. The locus for two circles is, using Fig. XXXV,

$$\frac{\rho - r}{\sqrt{(c+b)^2 - 2(c+b)\cos\theta + \rho^2} - b} = k.$$

Reducing this becomes,

$$\rho^2(1-1/k^2) - 2\rho\left[(c+b)\cos\theta + b/k - r/k^2\right] + c^2 + 2cb - r^2/k^2 + 2br/k = 0.$$

Dividing by b, and letting $b \to \infty$, the equation is that gotten before,

$$\rho = \frac{kc + r}{1 + k\cos\theta}.$$

DISCUSSION OF THE LOCUS, WHEN THE TWO CIRCLES REMAIN AS CIRCLES.

Several different forms of the locus have been obtained, so that the most simple equation will be chosen for discussion, namely, using Fig. XXXIV.

$$\rho = \frac{-(bk - k^2a + c\cos\theta) \mp \sqrt{(bk - k^2a + c\cos\theta)^2 - (k^2-1)\left[(b-ka)^2 - c^2\right]}}{k^2 - 1}.$$

A. If $k=1$, the locus becomes, as obtained from Fig. XXXII.

$$\sqrt{(d+x)^2 + y^2} = a - b + \sqrt{(c-x)^2 + y^2}.$$

Reducing this, becomes,

$$4x^2\left[(d+c)^2 - (a-b)^2\right] - 4(a-b)^2y^2 + 4x\left[(d+c)(d^2-c^2) - (d+c)(a-b)^2 + 2c(a-b)^2\right] + \left[(d^2-c^2) - (a-b)^2\right]^2 - 4(a-b)^2c^2 = 0.$$

This is a general equation of the second degree where, if $B^2 - 4AC = 0$, the locus is a parabola, if $B^2 - 4AC < 0$, the locus is an ellipse, and if $B^2 - 4AC > 0$, the locus is an hyperbola. Here $B = 0$, $A = 4\left[(d+c)^2 - (a-b)^2\right]$, and $C = -4(a-b)^2$. The relationship between the constants, therefore to get the different

* If $a=b$, the locus is the mid-perpendicular to the line joining the centers.

conic sections is, for a parabola, $(a-b)^2=(d+c)^2$; for the ellipse $(a-b)^2 > (d+c)^2$; for the hyperbola, $(a-b)^2 < (d+c)^2$.

B. If $k^2>1$. (Using the equation from Fig. XXIV)

I. If $b=a$. To simplify platting let a b 1. (other const. similar

(1). If $c=1$, locus will be similar to Figs. XXXVI, XXXVI'.

(2). If $c>1$, locus will be similar to Figs. XXXVII, ~~XXXVII~~!

(3). If $c<1$, locus will be similar to Figs. XXXVIII, XXXVIII'.

II. If $a \neq b$, locus will be similar to those above or an example worked out is that for Fig. XXXIX, XXXIX '.

C. If $k^2<1$.

I. If $b=a$. Here likewise choose a b 1.

(1). If $c=1$, locus will be similar to Fig. XXXX.

(2). If $c>1$, locus will be similar to Fig. XXXXI.

(3). If $c<1$, locus will be si,ilar to Fig. XXXXII.

II. If $a \neq b$, the locus will be similar to those under C.I.

<u>CONCLUSION</u>: The locus, obtained for the quotient of a moving point's distance as a constant, appears to be two ovals, one within the other, when two distinct circles are used. Sometimes (with certain constants) the ovals are touching, as the lomacon, so that the author is led to believe that these are general curves under which the limacon is a special curve. It should be noticed that the limacon and conic sections including the straight line were obtained when the radii of both or one circle has become zero or infinity.

<u>THEOREM XXIV</u>: The locus of a point which moves so that the product of its distances from two fixed circles is a constant is,

$$\rho^4-2\rho^3(c\cos\theta+r)+\rho^2(c^2+r^2+4rc\cos\theta-b^2)-2\rho(cr^2\cos\theta+rc^2+bk-b^2r)+$$
$$r^2c^2-k^2-b^2r^2+2bkr=0.$$

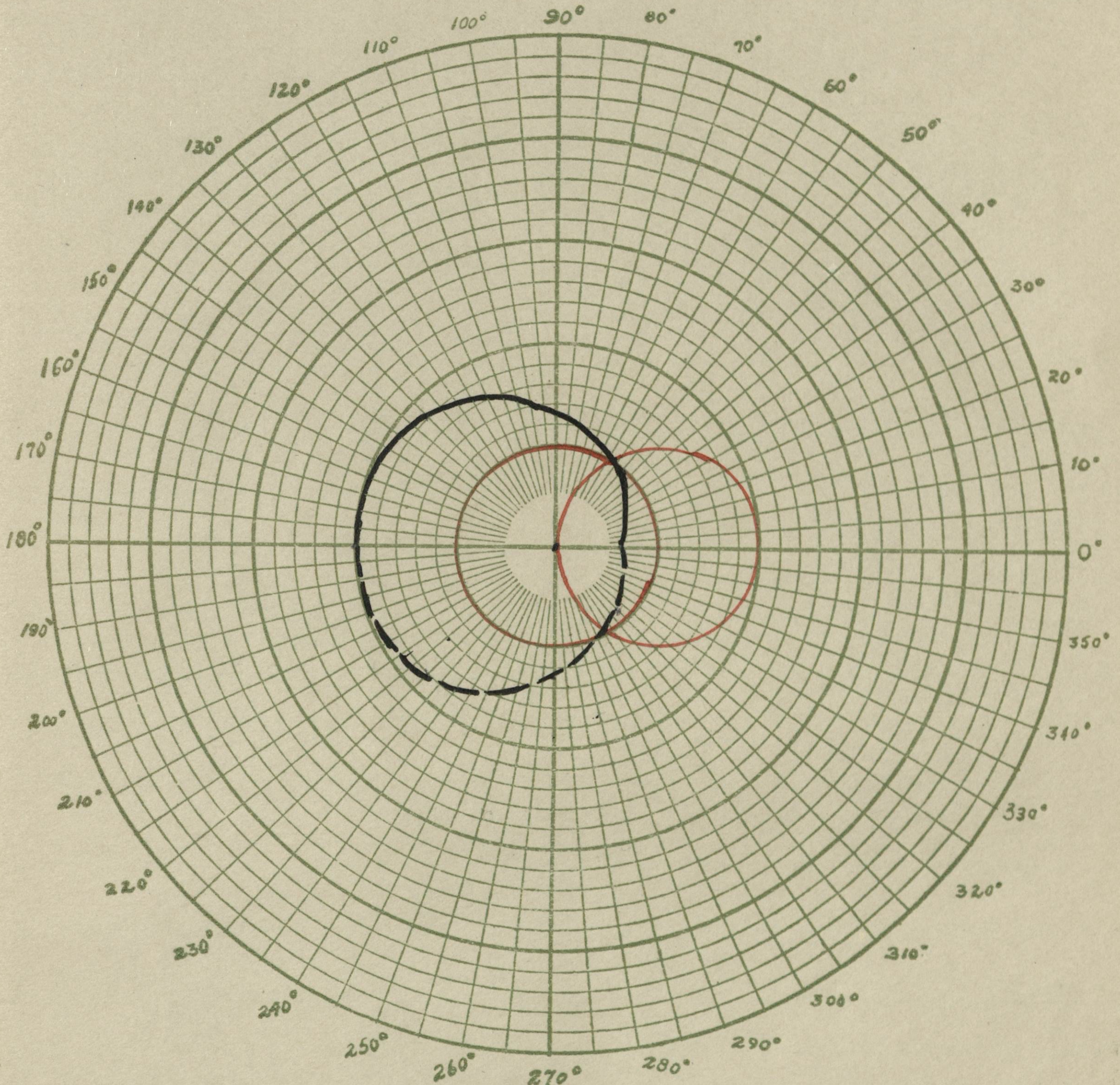


Fig. XXXVI

θ	ρ	
0	.66	0
45	.86	
90	1.33	
135	1.80	
180	2.00	

$k = 2$
$a = 1$
$b = 1$
$c = 1$

$$\rho = \frac{2 - \cos\theta \mp (\cos\theta - 2)}{3} = 0,\ \tfrac{2}{3}\left[2 - \cos\theta\right]$$

Locus ——

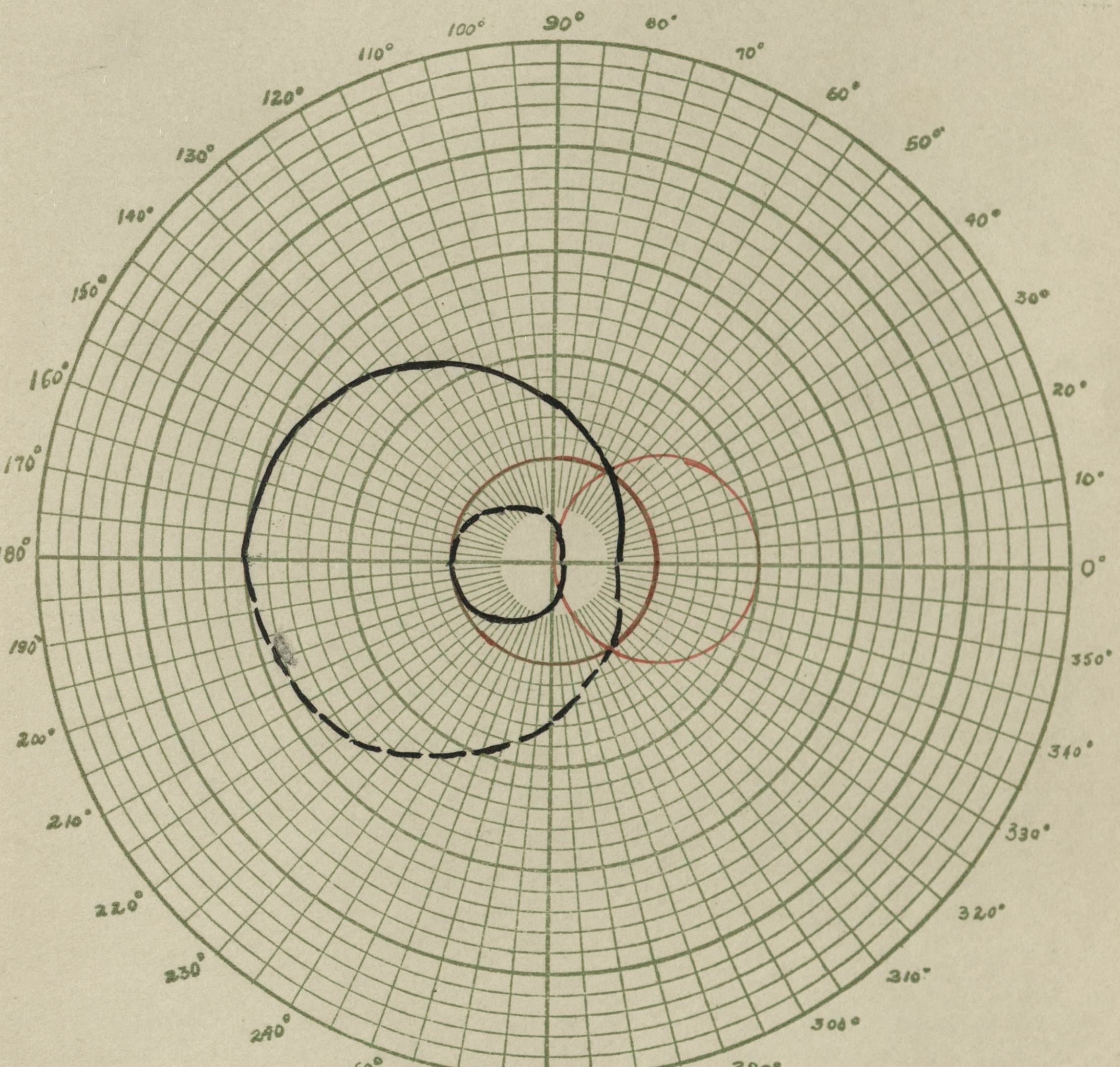


Fig. XXXVI'

θ	ρ	
0	-1.6	.61
45	-.74	.81
90	-.38	1.58
135	-.23	2.57
180	-.20	3.00

$k = 1.5$
$c = 1$
$a = 1$
$b = 1$

$$\rho = \frac{.75 - \cos\theta \mp \sqrt{(\cos\theta - .75)^2 + .94}}{1.25}$$

Locus ——

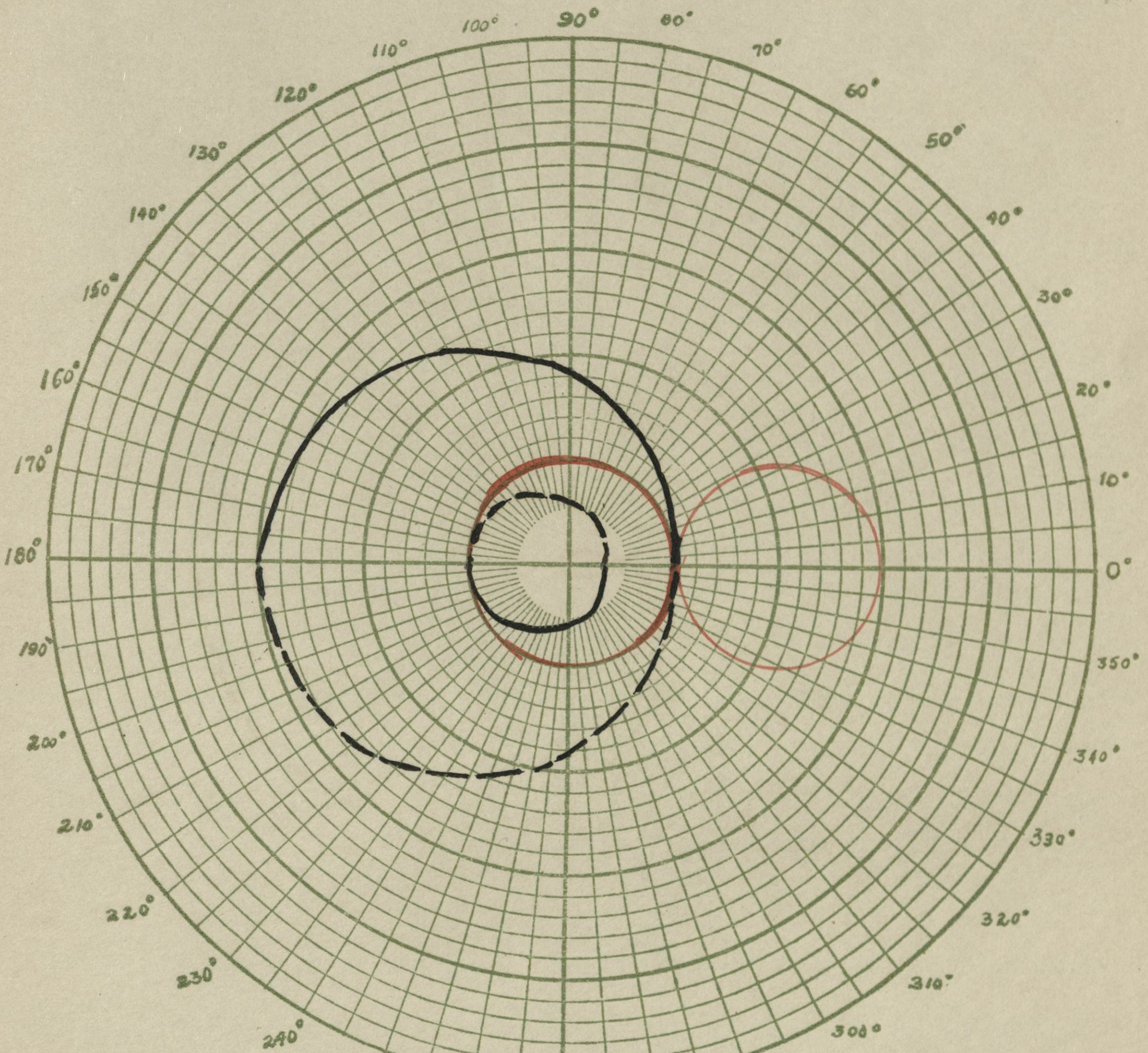


Fig XXXVII

θ	ρ	
0	1.0	-1.0
45	1.21	-.83
90	1.87	-.53
135	2.65	-.37
180	3.0	-.33

$k = 2$
$c = 2$
$a = 1$
$b = 1$

$$\rho = \tfrac{2}{3}\left[(1-\cos\theta) \pm \sqrt{(\cos\theta-1)^2+2.25}\right]$$

Locus —

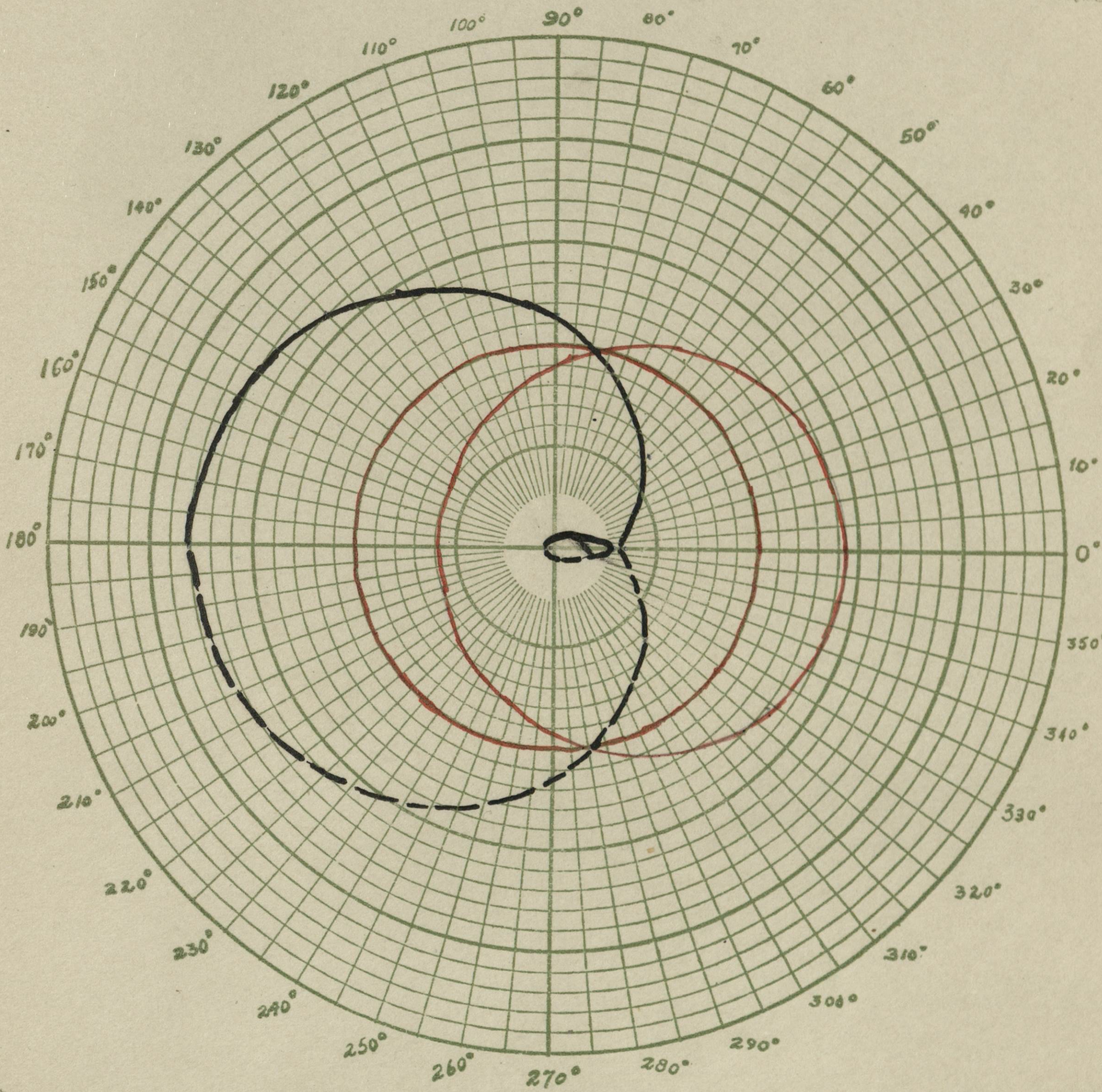


Fig. XXXVIII

θ	ρ	
0	.27	.29
45	.11	.64
90	.06	1.14
135	.05	1.60
180	.04	1.80

$k = 1.5$
$c = .4$
$a = 1$
$t = 1$

$$\rho = \frac{.75 - .4\cos\theta \mp \sqrt{(.4\cos\theta - .75)^2 - .11}}{1.25}$$

Locus ——

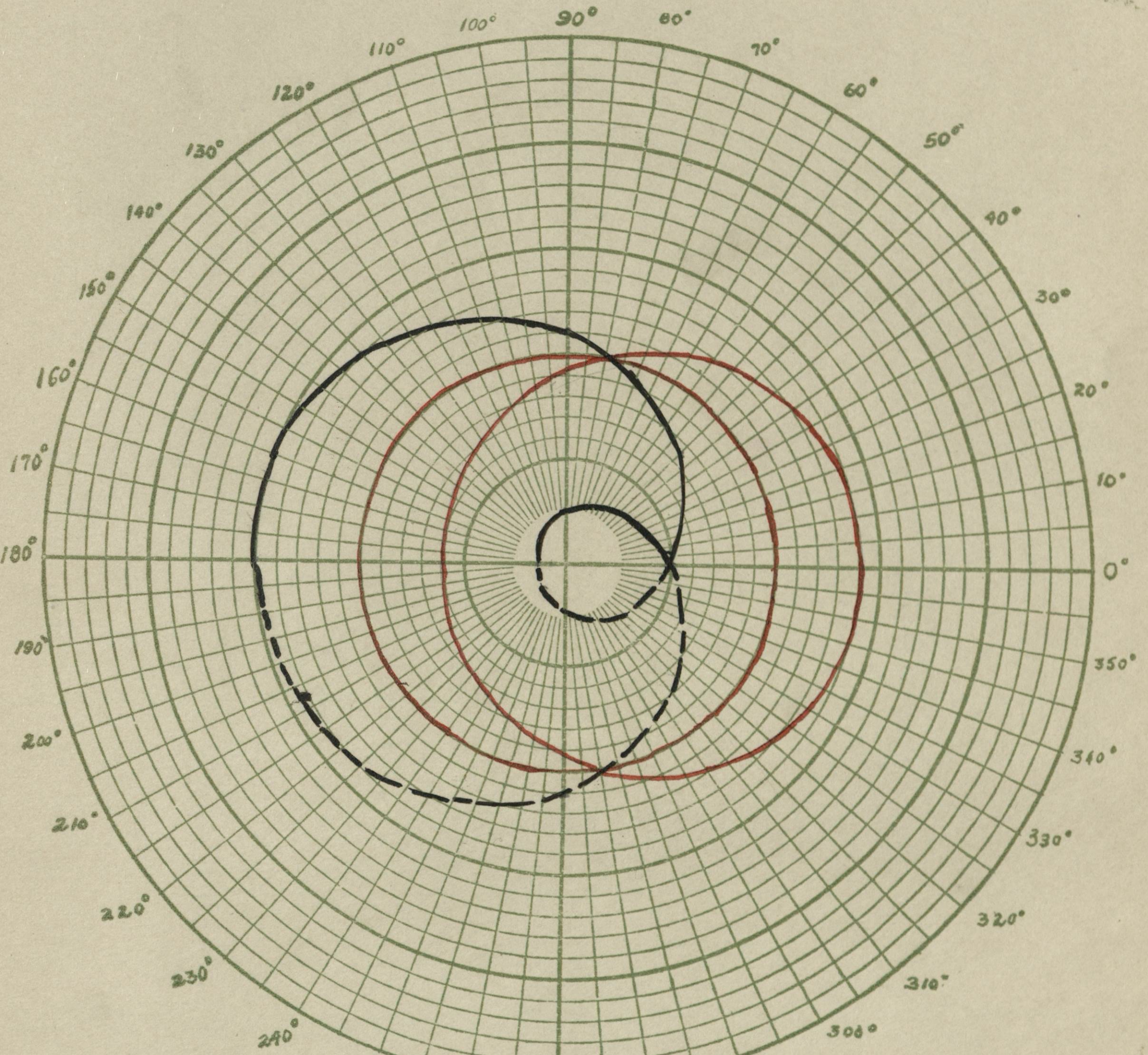


Fig. XXXVIII'

θ	ρ	
0	.50	.
45	.33	.77
90	.29	1.11
135	.17	1.40
180	.17	1.50

$k = 2$
$c = .5$
$a = 1$
$b = 1$

$$\rho = \frac{2 - .5\cos\theta \mp \sqrt{(.5\cos\theta - 2)^2 - 2.25}}{3}$$

Locus ——

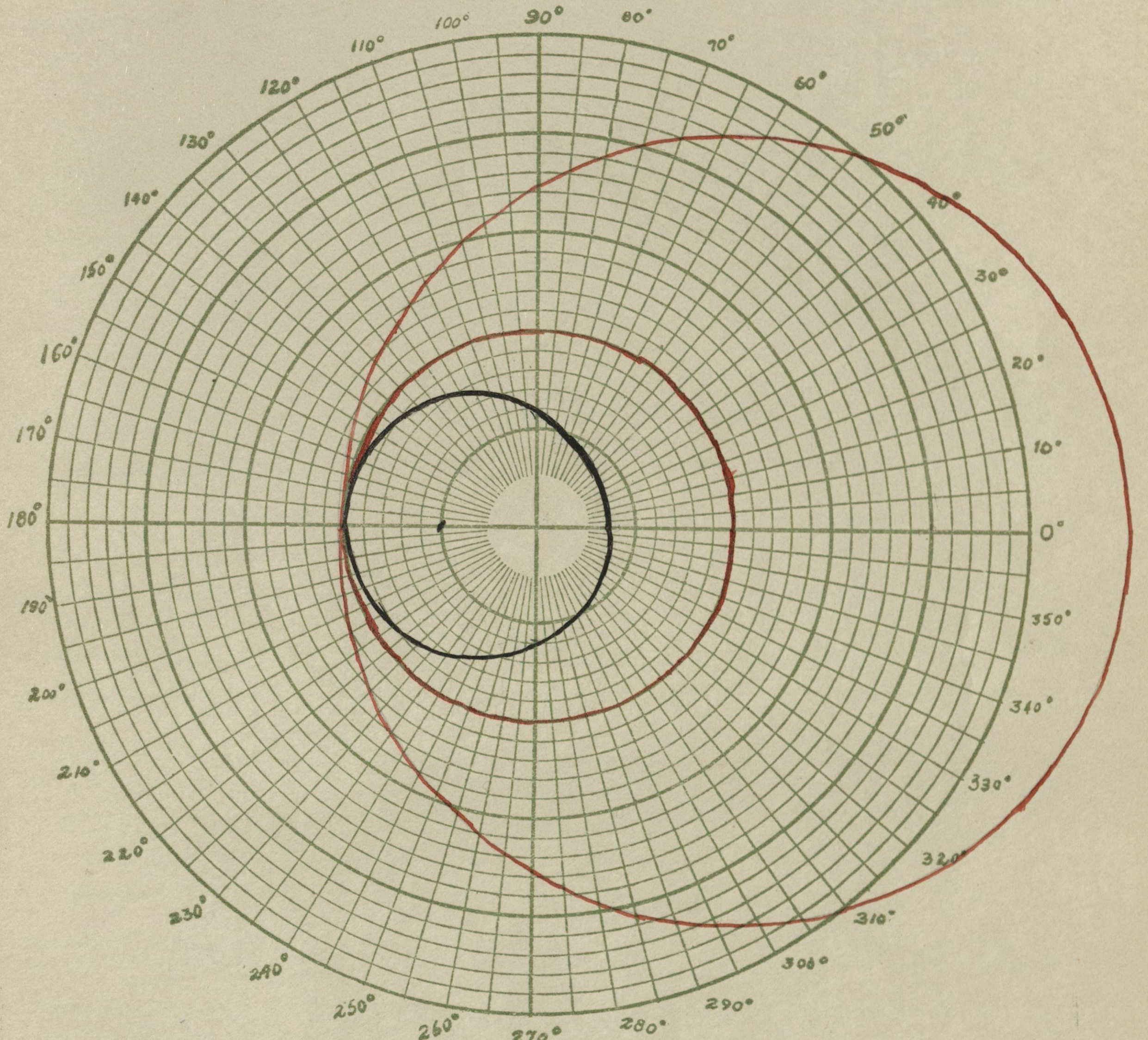


Fig. XXXIX

θ	ρ	
0	-1.0	.33
45	-.86	.39
90	-.58	.58
135	-.39	.86
180	-.33	1.00

$k=2$
$c=1$
$a=1$
$b=2$

$$\rho = \frac{-\cos\theta \mp \sqrt{3+\cos^2\theta}}{3}$$

Complete locus is obtained as θ varies from 0° to 180°.

Locus ——

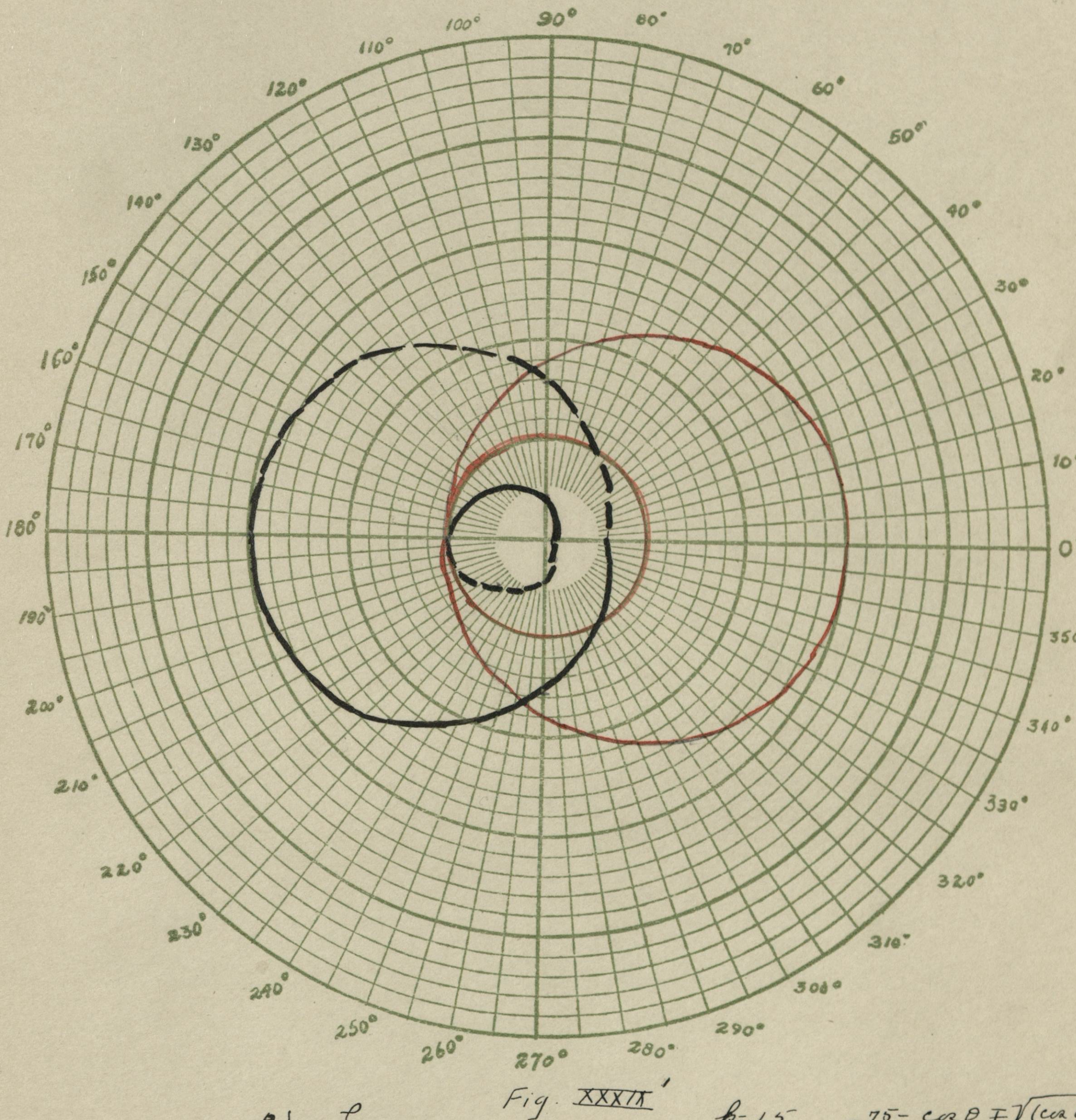


Fig. XXXIX'

θ	ρ	
0	-3.00	.20
45	-2.57	.23
90	-1.58	.38
135	-.81	.74
180	-.60	1.00

$k = 1.5$
$c = 1$
$a = 1$
$b = 2$

$$\rho = \frac{-.75 - \cos\theta \mp \sqrt{(\cos\theta + .75)^2 + .94}}{1.25}$$

Locus ——

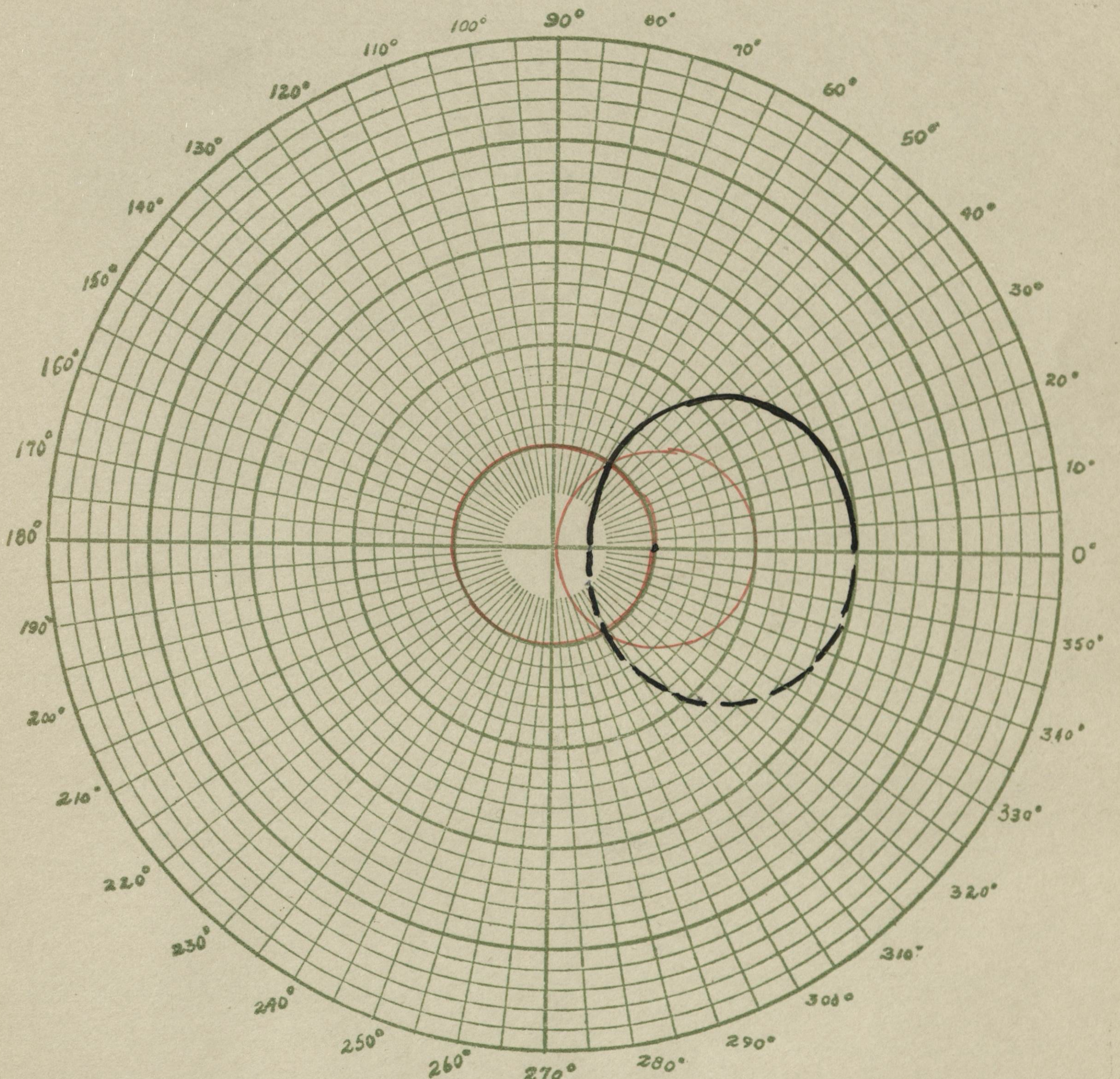


Fig. XXXV

θ	ρ	
0	3.0	.33
45	2.08	.48
60	1.0	
61	f(i)	
:	:	
180	-1	

$k = .5$
$c = 1$
$a = 1$
$b = 1$

$$\rho = \frac{\cos\theta + .25 \pm \sqrt{(\cos\theta + .25)^2 - (.75)^2}}{.75}$$

Locus ——

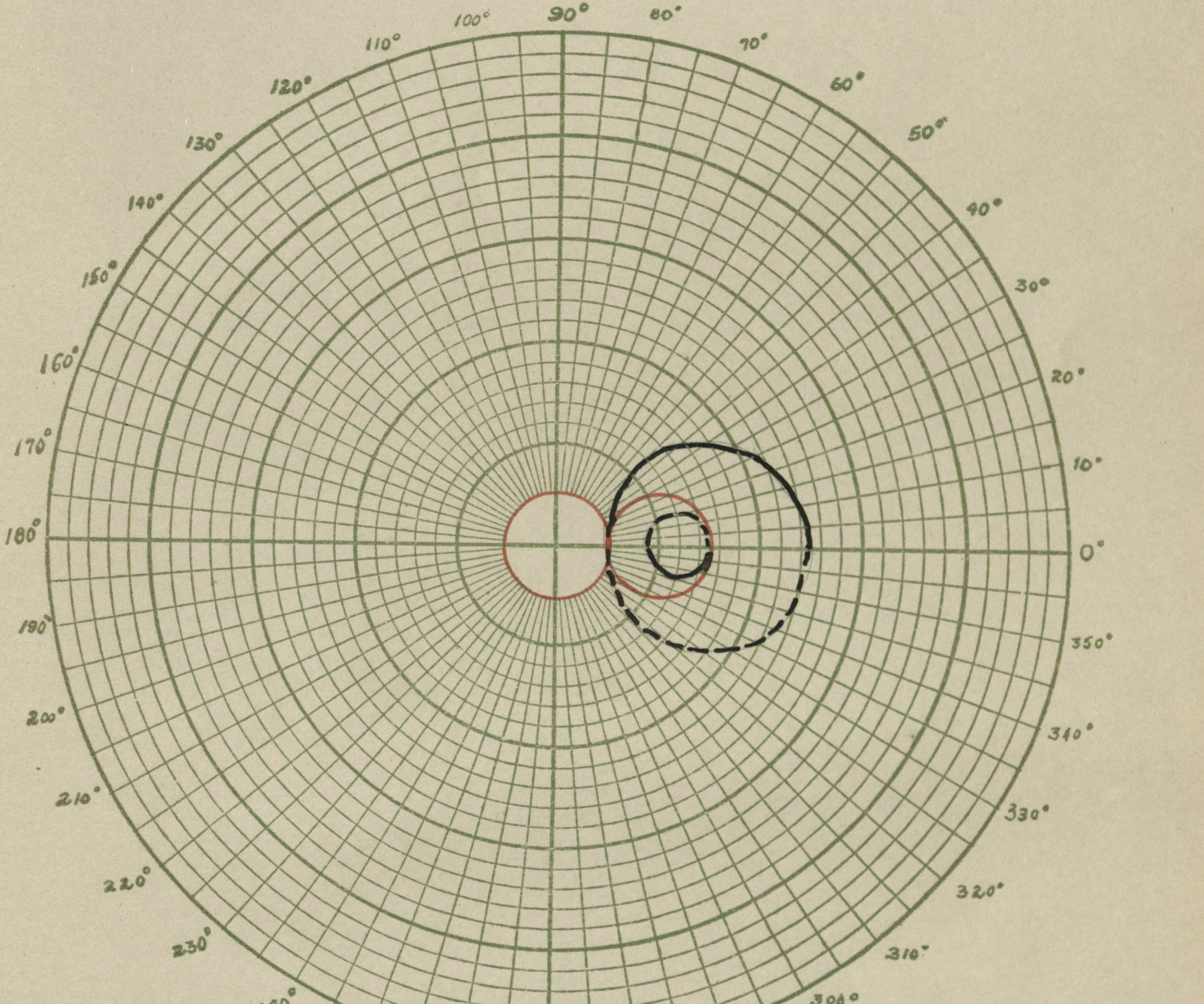


Fig. XXXXI

θ	ρ	
0	5.00	1.00
15	4.73	1.05
44	2.53	1.96
45	$f^{(i)}$	
⋮	⋮	
164		
165	-2.43	-2.72
180	-1.67	-3.00

$k = .5$
$c = 2$
$a = 1$
$b = 1$

$$\rho = \frac{2\cos\theta + .25 \pm \sqrt{(2\cos\theta + .25)^2 - 2.81}}{.75}$$

Locus ——

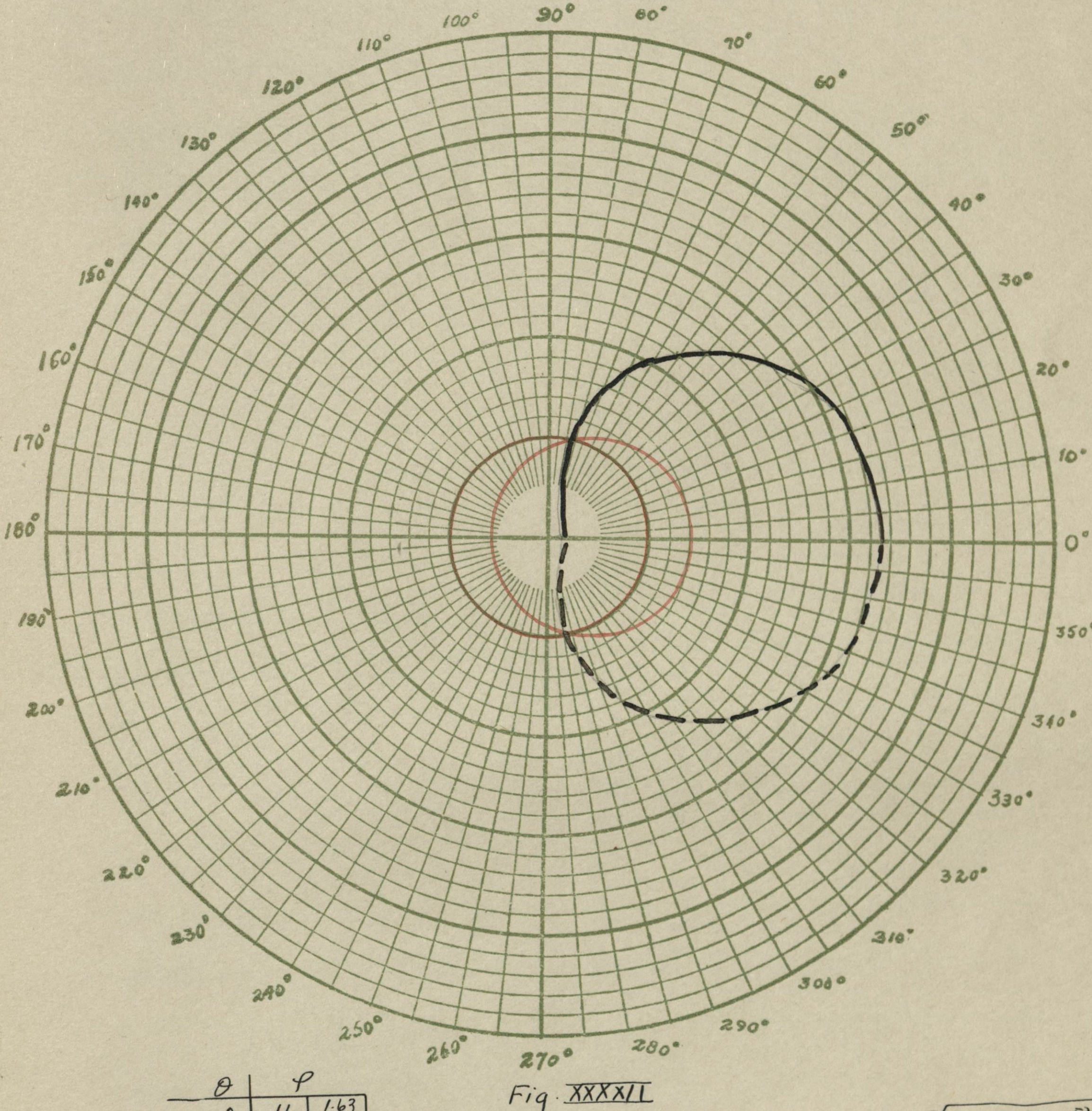


Fig. XXXXII

θ	ρ	
0	.11	1.63
30	.12	1.48
45	.16	1.25
60	.19	1.01
75	.27	.77
80	.43	.43
81	$f(i)$	
⋮	⋮	
180	⋮	

$k = .5$
$c = .4$
$a = 1$
$f = 1$

$$\rho = \frac{.4\cos\theta + .25 \pm \sqrt{(.4\cos\theta + .25)^2 - .10}}{.75}$$

Locus ——

A. Theorem IV, where both circles have shrunken to points.

From Fig. XXXII the locus for two circles is,

$$\left(\sqrt{(d+r)^2+y^2}-a\right)\left(\sqrt{(c-x)^2+y^2}-b\right)=k.$$

Letting r_1 represent $\sqrt{(d+x)^2+y^2}$, and $r_2=\sqrt{(c-x)^2+y^2}$, the locus becomes on squaring and reducing,

$$(r_1^4+a^4-2r_1^2a^2)(r_2^8-4r_2^6b^2+6r_2^4b^4-4r_2^2b^6+b^8)+k^4(r_2^4+b^4+2r_2^2b^2)+4a^2k^2(r_2^6+b^2r_2^4+b^6+2br_2^5-2r_2^3b^3-2b^4r_2^2)$$
$$-2(r_1^2-a^2)(r_2^4-2r_2^2b^2+b^4)\left[k^2(r_2^2+b^2)+2ak(r_2^3+br_2^2-b^3)\right]+4ak^3(r_2^2+b^2)(r_2^3+br_2^2-b^3)$$
$$-4k^2b^2r_2^2(k^2-2abk+a^2b^2)=0.$$

For the circles to become points, $a=0$, $b=0$, and locus is, $r_2^4(r_1^2r_2^2-k^2)^2=0$, or substituting in the values for r_1 and r_2,

$$(d^2+x^2+y^2+2dx)(c^2+x^2+y^2-2cx)-k^2=0.$$

Letting $c=d$,

$$c^4+x^4+y^4-2c^2x^2+2c^2y^2+2x^2y^2=k,$$ which is the equation gotten before. When $r_2=0$, the locus is imaginary.

B. Theorem VIII, where both circles have become straight lines.

The locus of the point from two circles from Fig. XXXIII is,

$$\left(\sqrt{x^2+(a-y)^2}-a\right)\left(\sqrt{(x-b)^2+(y+b/m)^2}-c\right)=k.$$

If $r_1=\sqrt{x^2+(a-y)^2}$ and $r_2=\sqrt{(x-b)^2+(y+b/m)^2}$, the locus becomes,

$$(r_1^2-a^2)^2(r_2^2-c^2)^4+k^4(r_2^2+c^2)^2+4a^2k^2c^2(r_2^2-c^2)^2-2(r_1^2-a^2)(r_2^2-c^2)\left[k^2(r_2^2+c^2)+2ack(r_2^2-c^2)\right]+$$
$$4ak^3c(r_2^4-c^4)=4k^4cr_2^2+4a^2k^2r_2^2(r_2^2-c^2)^2+8ack^3r_2^2(r_2^2-c^2).$$

Substituting in the values for the r's, and letting $a=c$ the locus becomes, expressing b in terms of c,

$$(x^4+y^4+4c^2y^2+2x^2y^2-4x^2cy-4cy^3)\left[(x^2+y^2)^4+4(x^2+y^2)^3\frac{8c}{\sqrt{m^2+1}}(y-x)+6(x^2+y^2)\frac{4c^2}{m^2+1}(y-mx)^2+\right.$$
$$4(x^2+y^2)\frac{8c^3}{(m^2+1)^{3/2}}(y-mx)^3+\frac{16c^4}{(m^2+1)^2}(y-mx)^4+k^4(x^4+y^4+4c^4+\frac{4mc^2x^2}{m^2+1}+\frac{4c^2y^2}{m^2+1}+2x^2y^2+4x^2c^2-$$
$$\left.\frac{4mcx^3}{\sqrt{m^2+1}}+\frac{4cyx^2}{\sqrt{m^2+1}}+4y^2c^2-\frac{4mcxy^2}{\sqrt{m^2+1}}+\frac{4cy^3}{\sqrt{m^2+1}}-\frac{8mc^3x}{\sqrt{m^2+1}}+\frac{8c^3y}{\sqrt{m^2+1}}-\frac{8mc^2yx}{m^2+1}\right]+4c^4k^2(x^2+y^2-\frac{2mcx}{\sqrt{m^2+1}}$$

$$+\frac{2cy}{\sqrt{m^2+1}}-2(x^2+y^2-2cy)\Big(x^4+y^4+\frac{4m^2c^2x^2}{m^2+1}+\frac{4c^2y^2}{m^2+1}+2x^2y^2-\frac{4mcx^3}{\sqrt{m^2+1}}+\frac{4cx^2y}{\sqrt{m^2+1}}-\frac{4mcxy^2}{\sqrt{m^2+1}}+$$

$$\frac{4cy^3}{\sqrt{m^2+1}}-\frac{8mc^2xy}{m^2+1}\Big]\Big[k^2\Big(x^2+y^2+2c^2-\frac{2mcx}{\sqrt{m^2+1}}+\frac{2cy}{\sqrt{m^2+1}}\Big)+2ck^2\Big(x^2+y^2-\frac{2mcx}{\sqrt{m^2+1}}+\frac{2cy}{\sqrt{m^2+1}}\Big)\Big]+$$

$$4c^2k^3\Big\{x^4+y^4-\frac{4mx^3}{\sqrt{m^2+1}}+x^2\Big[\frac{4m^2c^2}{m^2+1}+2y^2+2c^2+\frac{4cy}{\sqrt{m^2+1}}\Big]-4x\Big[\frac{mcy^2}{\sqrt{m^2+1}}+\frac{c^3m}{\sqrt{m^2+1}}+\frac{2mc^2y}{m^2+1}\Big]+\frac{4c^2y^2}{m^2+1}+$$

$$2y^2c^2+\frac{4cy^3}{\sqrt{m^2+1}}+\frac{4c^3y}{\sqrt{m^2+1}}\Big\}-4k^4c^2\Big(x^2+y^2+c^2-\frac{8mcx}{\sqrt{m^2+1}}+\frac{2cy}{\sqrt{m^2+1}}\Big)-4ck^2\Big(x^2+y^2+c^2-\frac{2mcx}{\sqrt{m^2+1}}+\frac{2cy}{\sqrt{m^2+1}}\Big)$$

$$\Big(x^4+y^4+\frac{4m^2c^2x^2}{m^2+1}+\frac{4c^2y^2}{m^2+1}+2x^2y^2-\frac{4mcx^3}{\sqrt{m^2+1}}+\frac{4cyx^2}{\sqrt{m^2+1}}-\frac{4mcxy^2}{\sqrt{m^2+1}}+\frac{4cy^3}{\sqrt{m^2+1}}-\frac{8mc^2xy}{m^2+1}\Big)-$$

$$8c^2k^3\Big(x^2+y^2+c^2-\frac{2mcx}{\sqrt{m^2+1}}+\frac{2cy}{\sqrt{m^2+1}}\Big)\Big(x^2+y^2-\frac{2mxc}{\sqrt{m^2+1}}+\frac{2cy}{\sqrt{m^2+1}}\Big)=0.$$

The highest terms occuring in powers of c are c . Then collecting these terms and merely indicating the others the locus becomes,

$$c^6\Big[4y^2\frac{16}{(m^2+1)^2}(y-mx)^4-4k^2\Big(\frac{4m^2x^2}{m^2+1}+\frac{4y^2}{m^2+1}-\frac{8mxy}{m^2+1}\Big)\Big]+Ac^5+Bc^4+Dc^3+Ec^2+Fc+G=0$$

Dividing by c^6 and letting $c\to\infty$, the locus becomes,

$$\frac{4y^2}{(m^2+1)^2}(y-mx)^4=\Big(\frac{kmx}{\sqrt{m^2+1}}-\frac{ky}{\sqrt{m^2+1}}\Big)^2.$$

Taking the square root the equation becomes,

$y^2=k/2\sqrt{m^2+1}+mxy$, or if $K=k/2$, the locus is that obtained before,

$$y^2=K\sqrt{m^2+1}+mxy.$$

C. Theorem XII, where left-hand circle has become a line and the other circle has become a point. From Fig. XXXII the locus for two circles is,

$$(r_1-a)(r_2-b)=k,\quad\text{where } r_1=\sqrt{(d+x)^2+y^2},\ r_2=\sqrt{(c-x)^2+y^2}.$$

On squaring this becomes,

$$(r_1^2-a^2)^2(r_2^2-b^2)^2+k^4(r_2^2+b^2)^2+4a^2k^2b^2(r_2^2-b^2)^2-2(r_1^2-a^2)(r_2^2-b^2)^2\Big[k^2(r_2^2+b^2)+2abk(r_2^2-b^2)\Big]+$$

$$4ak^3b(r_2^4-b^4)=4k^4b^2r_2^2+4a^2k^2r_2^2(r_2^2-b^2)^2+8abk^3r_2^2(r_2^2-b^2).$$

To let one circle become a point let b=0, and letting a=d, substituting in the value for r_1, the locus becomes,

$$r_2^8\left[6d^2x^2+4dx^3+x^4+y^4+2y^2d^2+4y^2dx+2y^2x^2\right]-2r_2^8d^2(x^2+y^2)+k^4r_2^4-2r_2^6k^2(d^2+2dx+x^2+y^2)$$
$$-2d^2k^2r_2^6=0.$$

Dividing by d^2 and letting $d\to\infty$, the other circle becomes a line and the locus becomes,

$$4r_2^6\left[r_2^2x^2-k^2\right]=0.$$

$r_2=0$, gives an imaginary locus, but a real locus is obtained when the other factor is put equal to 0. Substituting for r_2, the locus becomes,

$$(c-x)^2+y^2=\frac{k^2}{x^2},$$ which is the locus gotten before.

D. Theorem XVI, where the right-hand circle has become a point. The locus when both are circles is, from Fig. XXXIV

$$(\rho-r)\left(\sqrt{c^2-2\rho c\cos\theta+\rho^2}-b\right)=k.$$

Squaring this becomes,

$$\rho^4-2\rho^3(c\cos\theta+r)+\rho^2(c^2+r^2+4rc\cos\theta-b^2)-2\rho(cr^2\cos\theta+rc^2+bk-b^2r)$$
$$+r^2c^2-k^2-b^2r^2+2bkr=0.$$

If b 0, one of the circles becomes a point and the locus becomes,

$$\rho^4+\rho^3(4rc\cos\theta+r^2+c^2)-2\rho^3(c\cos\theta+r)-2\rho(rc^2+r^2c\cos\theta)+c^2r^2-k^2=0.$$

This is the equation gotten before.

E. Theorem XX, where right-hand circle has become a straight line, from Fig. XXXV, the locus for two circles is,

$$(\rho-r)\left(\sqrt{(c+b)^2-2(c+b)\rho\cos\theta+\rho^2}-b\right)=k.$$

Simplifying this becomes,

$$\rho^4-2\rho^3\left[(c+b)\cos\theta+r\right]+\rho^2\left[c^2+2bc-r^2+4(c+b)r\cos\theta\right]-2\rho\left[r(c+b)^2\right.$$
$$\left.+r^2(c+b)\cos\theta+kb-b^2r\right]+r^2(c+b)^2-k^2+2kbr-b^2r^2=0.$$

Dividing by b and letting $b \to \infty$, so one circle may become a line, the locus becomes,

$\rho^3 \cos\theta - \rho^2(c+2r\cos\theta) + \rho(2cr+r^2\cos\theta+k) - r^2c - kr = 0$, or

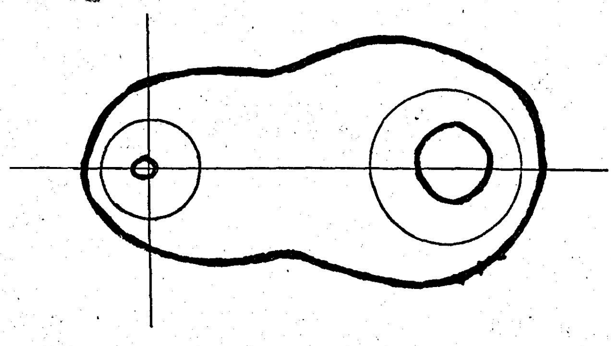
$(\rho-r)\left[\rho^2\cos\theta - \rho(c+r\cos\theta) + rc + k\right] = 0$.

The second factor is the locus gotten before. The first factor is the original circle, which is seen at once to satisfy the given conditions, for if the moving point is on the circle, the product of its distance from the circle and the line is always zero.

DISCUSSION OF THE LOCUS WHEN THE TWO CIRCLES REMAIN AS CIRCLES.

The following discussion is carried on in a general manner. The loci shown are not numerically exact, but will give an idea of the nature of the locus.

(1) When the constant k is such that no value of the locus may lie on the x-axis between the two circles, the locus will appear similar to Fig. XXXXIII.



Since a' is farther from circle I than any other point of the locus it is readily seen that it must be nearer circle II than any other part of the locus for the product of the two distances to remain constant. Thus, following the arrow to (1), since the locus approaches nearer and nearer the circle I, it must therefore be farther from circle II. A similar discussion holds for the locus from (1) to a.The locus below is similar to that above the x-axis.Ovals may appear within the circles as in th figure.It may be seen at once that the distance ab bc,ab de and bc de,for d being nearer circle II than a orc, it follows that it must be farther from circle I,than a or c.

(2) When the constant k is such that one value of the locus may lie on the x-axis between the two circles,the locus will be similar to Fig.XXXXIV.The discussion for the inner ovals is similar to that given in (1) above.

(3)

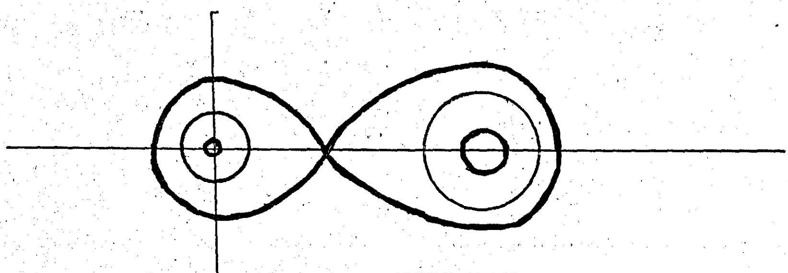


Fig. XXXXIV

(3) when the constant k is small enough in comparison to the size and distance apart of the two circles there may be two values of the locus on the x-axis betweenthe two circles.The locus then will appear similar to Fig.XXXXV.

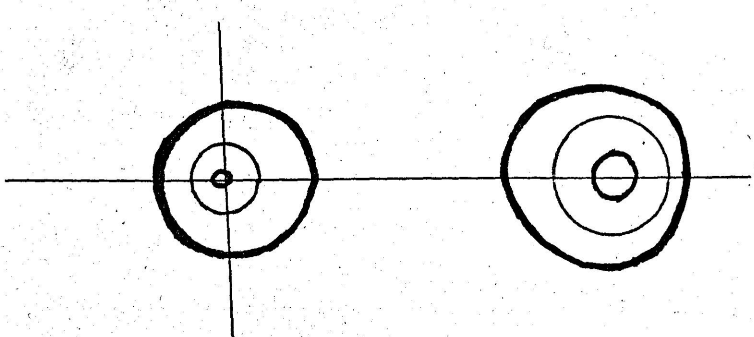


Fig. XXXXV

(4) If the circles are tangent externally,the locus may appear similar to Fig.XXXXVI.However,if the constant is too great one or both of the inner ovals may not appear.

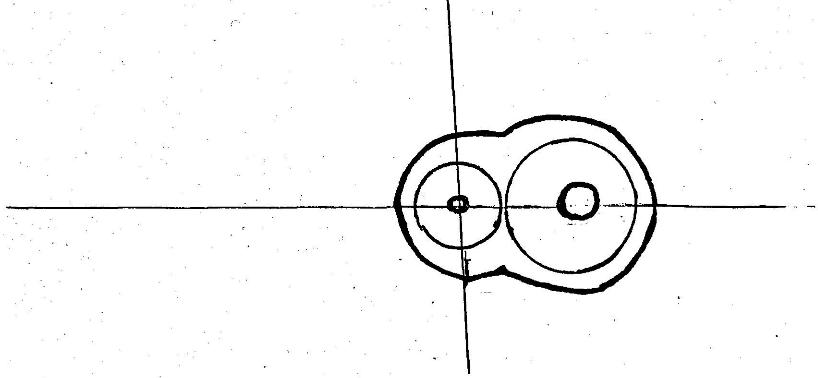


Fig. XXXXVI

(5) If the circles are tangent internally, the locus may appear similar to Fig. XXXXVII. As in (4) it may be that the constant k is so great that the inner ovals may not appear.

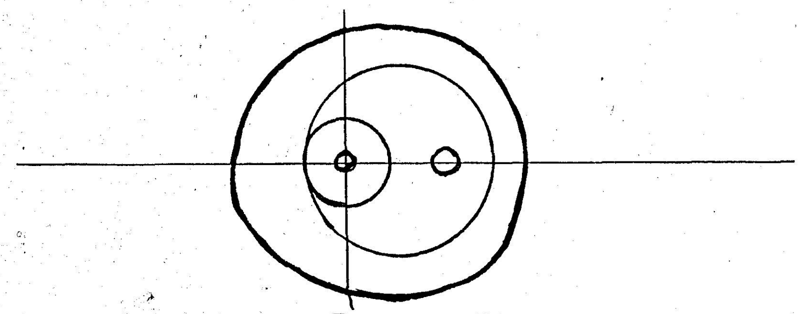


Fig. XXXXVII

(6) If the circles intersect only slightly the locus may appear as Fig. XXXXVII, one or both of the inner ovals not appearing. If the intersection is greater the locus may appear as Fig. XXXXVIII, the inner oval may or may not appear.

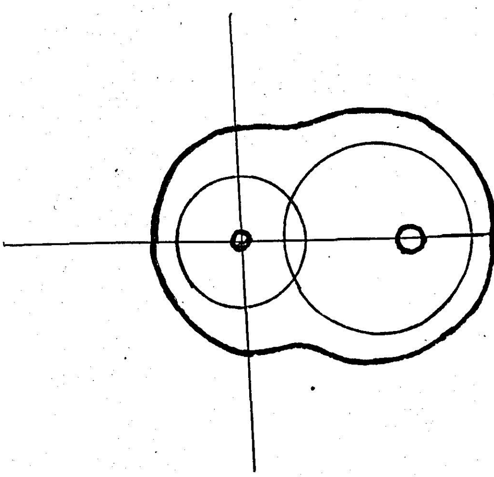


Fig. XXXXVIII

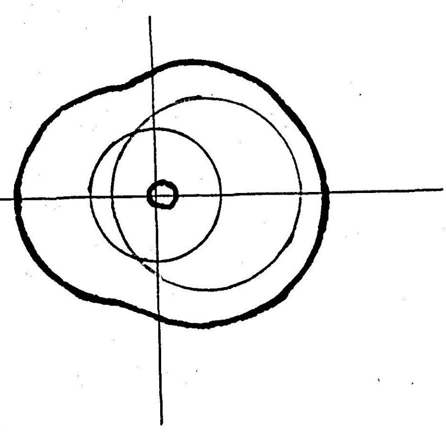


Fig. XXXXVIII′

SUMMARY.

It has been pointed out that the pointe and line are limiting cases of the circle,where the radius has become zero or infinity.Thus the loci obtained for constant sums,differences quotients and products of two points,two lines,a line and point, a point and circle,and a line and a circle,may be thought of as limiting loci obtained from two circles.Thus there is obtained a relation between different curves that were thought to have no relationship what so ever before.By varying the constants in different ways,the conic sections and other well known curves as well as some other curves have been obtained in a similar manner.

Considering constant sums,the general case of two circles gave the conic sections;two points--an ellipse;two lines a--straight line ;a point and line--the x-axis or a parabola; a pointand circle--an ellipse,x-axis,section of an hyperbola or a straight line; a line and circle--a parabola or the x-axis.

For constant differences,the general case of the two circles gave the conic sections;two points--an hyperbola;two lines--a straight line;a point and line--a parabola or the x-axis; a point and circle--an ellipse,x-axis,section of hyperbola or a straightline; a line and circle--a parabola.

For constant quotients,the general equation gives by varying the constants-- the conic sections,a limacon and a figure composed of two contours,often appearing as two ovals while it has been obtained to approach the limacon(Fig.XXXVIII)M two points--a circle;two lines--a straight line thru their intersection;a line and point--the conic sections;a circle and point--loci similar to those obtained for the general case of two circles;a circle and line--the conic sections.

Considering constant Products,the general equation gives loci simialr to Figs.XXXXIII to XXXXVIII;two points give the Cassinian ovals;two lines the hyperbola;a point and line--loci similar to the Conchoid of Nicomedes;a circle and point--loci similar to Figs.XXII,XXIII and XXXIV;a line and circle--the conchoid of Nicomedes or figures similar to the conchoid.

Note I.

(1) To show $(k^2r-c\cos\theta)^2<(k^2-1)(k^2r^2-c^2)$ is false.To do this asdume (1) is true and prove it false.
The following are the assumptions made;$k^2>1$,$r>c$. And it is known that $r^2>2rc-c^2$.
Squaring and reducing(1) becomes,using this fact,

(2) $k^2(2rc-c^2+c^2-2rc\cos\theta)<c^2(1-\cos^2\theta)$.
Collecting terms(2) becomes,

(3) $2k^2rc\ (1-\cos\theta)<c^2(1-\cos^2\theta)$.
Dividing by $c(1-\cos\theta)$ *

(4) $2rk^2<c(1+\cos\theta)$.
The greatest value that the right-hand side may have is when $\cos\theta$ is 1,therefore (4) becomes,

(5) $2rk^2<2c$ or

(6) $rk^2<c$,which is false.Therefore by increasing the right hand side,and decreasing the left-hand side the equation was prowen false,so that it follows at once that (1) is false.

Note II.

(1) To show $(k^2r-c\cos\theta)^2<(k^2-1)(k^2r^2-c^2)$;is false.The same method will be employed as for note I.
The following asumptions were made; $k^2>1$, $c>r$.And it is known that $c^2>2rc-r^2$.
Squaring ahd reducing (1) becomes,

(2) $k^2(r^2+c^2-2rc\cos\theta)<c^2(1-\cos^2\theta)$,or

(3) $2rck^2\ (1-\cos\theta)<c^2(1-\cos^2\theta)$.
Dividing out $(1-\cos\theta)$** the equatin becomes,

(4) $2rk^2<c(1+\cos\theta)$.
The greatest value right-hand side may assume is $2c$, therefore,

(5) $2rk^2<2c$,or

(6) $rk^2<c$, which is false under the assumptions made.It has been shown that by increasing the right-hand side of(1) and decreasing the left hand-side that the inequality is false;therefore without the increaseor decrease it follows that (1) is false.

* If $1-\cos\theta$ 0,it could not have been divided out.Therefore, it must be shown that it cannot equal zerc.Substitute $\cos\theta=1$ in the original equation,it becomes,

$k^4r^2+c^2-2k^2rc<k^4r^2-k^2(c^2+r^2)+c^2$,or

$c^2+r^2<2rc$,which is false.

**