ON CURVES DETERMINED BY
INTRINSIC EQUATIONS.

By

Raymond Hamilton Carpenter.

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Approved by: E. B. Stouffer
Instructor in Charge.

C. H. Ashton
Head of Department.

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In the subsequent discussion we shall make use of the direction-cosines of three mutually perpendicular lines uniquely connected with every point of a space-curve. The first is the **tangent**, the limiting position of a secant as the points of intersection approach coincidence. The direction-cosines of the tangent we denote by $\alpha, \beta, \gamma$. In order to define the other two lines, we need the definition of the osculating plane.

Any three points of the curve determine a plane. If the three points approach coincidence, a plane is determined, uniquely associated with the point of coincidence. It is called the **osculating plane**.

The second of the lines, mentioned above, is that normal to the curve, at the point of contact of the tangent, which lies in the osculating plane. We call it the **principal normal** and indicate its direction-cosines by $l, m, n$. The third line is that normal to the curve, at the point of contact of the tangent, which is at the same time normal to the osculating plane. We call it the **binormal** and indicate its direction-cosines by $\lambda, \mu, \nu$.

We shall now define the terms **first curvature**, **radius of curvature**, **second curvature** or **torsion**, and **radius of torsion**, which are used throughout the discussion.

Let $P$ and $P'$ be two points of a space-curve, $\Delta s$ the length of arc between these points, and $\Delta t$ the angle be-
tween the tangents at these two points. The limiting
value of $\frac{\Delta t}{\Delta s}$ as $P'$ approaches $P$, namely $\frac{dt}{ds}$, we define as
the first curvature. We denote it by $\frac{1}{r}$. Its reciprocal
$r$ is called the radius of curvature and is the radius of
the circle determined by three points of the curve as the
points approach coincidence. This is the osculating cir-
cle.

If we call the angle between the osculating planes
at two neighboring points of the curve $\Delta \gamma$, and the arc $\Delta s$,
the limit of the ratio $\frac{\Delta \gamma}{\Delta s}$, namely $\frac{d\gamma}{ds}$ gives the rate of
change of direction of the osculating plane as the point
moves along the curve. We define this as the second cur-
vature or torsion and indicate it by $\frac{1}{p}$. The reciprocal
we call the radius of torsion. It has no geometrical mean-
ing analogous to that of $r$. The analytical definitions\(^1\)
of the two curvatures are:

$$\frac{1}{r^2} = \frac{d\alpha}{ds} + \frac{d\beta}{ds} + \frac{d\gamma}{ds}$$

where $\alpha, \beta, \gamma$, are defined as above. If the parameter is
the length of arc measured from a point of the curve, we
may write

$$\frac{1}{r^2} = (X''')^2 + (Y''')^2 + (Z''')^2$$

For the second curvature, we have

$$\frac{1}{p^2} = \frac{d\lambda}{ds} + \frac{d\mu}{ds} + \frac{d\nu}{ds}$$

\(^1\)Eisenhart, "Differential Geometry", p. 16, 17.
or if the parameter is the length of arc measured from a point of the curve, we may write

\[
\frac{1}{p} = -r^2 \begin{vmatrix} \dot{x}' & \dot{y}' & \dot{z}' \\ \ddot{x}' & \ddot{y}' & \ddot{z}' \\ \dddot{x}' & \dddot{y}' & \dddot{z}' \end{vmatrix}
\]

We are now ready to state the theorem which forms the basis for this discussion.

**Theorem 1.** A space-curve is uniquely determined except for its position in space, whenever its radius or curvature and its radius of torsion are given as functions of the arc. The two equations, \( r = f(s) \), \( \varphi = \varphi(s) \), are called the **intrinsic equations** of the space-curve.

In determining the equation in Cartesian Coordinates, of a curve defined by means of its intrinsic equations, we shall make use of the derivatives of the direction-cosines of the tangent, normal and binormal, expressed in terms of the other cosines and the two radii of curvature. They are called collectively the **Frenet-Serret formulas**.

\[
\begin{align*}
\frac{d\alpha}{ds} &= \frac{1}{r}, & \frac{d\beta}{ds} &= \frac{m}{r}, & \frac{d\gamma}{ds} &= \frac{n}{r} \\
\frac{d\lambda}{ds} &= \frac{1}{p}, & \frac{d\mu}{ds} &= \frac{m}{p}, & \frac{d\nu}{ds} &= \frac{n}{p} \\
\frac{d\xi}{ds} &= -\left(\frac{\alpha}{r} + \frac{\lambda}{p}\right), & \frac{d\eta}{ds} &= -\left(\frac{\beta}{r} + \frac{\mu}{p}\right), & \frac{dn}{ds} &= \frac{\gamma}{r} + \frac{\nu}{p}
\end{align*}
\]


2Eisenhart, D. R., p. 17.
The equations
\[
\begin{align*}
\frac{du}{ds} &= \frac{v}{r}, \\
\frac{dv}{ds} &= -\left(\frac{u}{r} + \frac{w}{p}\right), \\
\frac{dw}{ds} &= \frac{v}{p}
\end{align*}
\]
may have three sets of solutions, namely, \(u = \alpha, v = l, w = \lambda; u = \beta, v = m, w = \mu; u = \gamma, v = n, w = \nu;\) which are such that for each value of \(s, \alpha, \beta, \gamma; l, m, n; \lambda, \mu, \nu,\) are direction-cosines as defined above. The quantities are therefore subject to the condition
\[
\begin{align*}
u^2 + v^2 + w^2 &= 1.
\end{align*}
\]

In order to integrate the equations (2), we introduce two functions \(\sigma\) and \(\omega\) which satisfy (3) and are defined by
\[
\begin{align*}
\sigma &= \frac{u + iv}{1 - w} \text{ or } \frac{1 + w}{u - iv}, \\
\omega &= \frac{u + iv}{1 - w} \text{ or } \frac{1 + w}{u + iv}.
\end{align*}
\]

Since the functions \(\sigma\) and \(-\frac{1}{\omega}\) differ only in the sign of the imaginary parts, they are conjugate imaginaries. By solving simultaneously equations (3) and (4) we obtain
\[
\begin{align*}
\sigma &= \frac{1 - \sigma \omega}{\sigma - \omega}, \\
v &= \frac{1 + \sigma \omega}{\sigma - \omega}, \\
w &= \frac{\sigma + \omega}{\sigma - \omega}
\end{align*}
\]
We shall now show that if \(\sigma\) and \(\omega\) are subject to the Frenet-Serret Formulae, they are solutions of the equation,
\[
\begin{align*}
\frac{d\sigma}{ds} &= \frac{i}{2p} = \frac{1}{r} \theta - \frac{i}{2p} \omega^2
\end{align*}
\]
and conversely that to obtain either \(\sigma\) or \(\omega\) we must solve a differential equation of this form. The method will be indicated for \(\sigma\) alone since for \(\omega\) it would be essentially the same.
Differentiating (4), we have

$$\frac{d\sigma}{ds} = \left[ (1 - w) \left( \frac{du}{ds} + iv \frac{dv}{ds} \right) + (u + iv) \frac{dw}{ds} \right] + (1 - w)^2$$

After substituting the values of \( \sigma \) and \( \frac{d\sigma}{ds} \) for \( \theta \) in (6) we obtain

$$\frac{(1 - w)\left( \frac{du}{ds} + iv \frac{dv}{ds} \right) + (u + iv) \frac{dw}{ds}}{(1 - w)^2} = \frac{1}{2p} - \frac{1}{r(1 - w)} - \frac{1}{2p} \frac{(u + iv)^2}{1 - w}$$

When we equate the real parts of this equation, we have

$$\frac{(1 - w)\frac{du}{ds} + u \frac{dw}{ds}}{(1 - w)^2} = \frac{v}{r(1 - w)} + \frac{2uv}{2p(1 - w)^2}$$

or

$$\frac{(1 - w)\frac{du}{ds} + u \frac{dw}{ds}}{r} = \frac{v}{r(1 - w)} + \frac{u}{p}$$

If we substitute for \( \frac{du}{ds} \) and \( \frac{dw}{ds} \) from (2), we have the identity

$$\frac{(1 - w)\frac{v}{r} + u \frac{v}{p}}{r} = \frac{(1 - w)\frac{v}{r} + u \frac{v}{p}}{r}$$

In similar manner, we equate the imaginary parts and obtain

$$\frac{dv}{ds} \frac{(1 - w)}{r} = \frac{w(w - 1)}{p} - \frac{u(1 - w)}{r} + \frac{v^2}{p}$$

By substituting for \( \frac{dv}{ds} \) and \( \frac{dw}{ds} \) from (2) we have the identity

$$\frac{w(1 - w)}{p} - \frac{u(1 - w)}{r} + \frac{v^2}{p} = \frac{w(1 - w)}{p} - \frac{u(1 - w)}{r} + \frac{v^2}{p}$$

We shall show that in order to obtain \( \sigma \) we must solve an equation of the type (6). First, we substitute in (7) the values of \( \frac{du}{ds}, \frac{dv}{ds}, \) and \( \frac{dw}{ds} \) from (2) and have

$$\frac{d\sigma}{ds} = \frac{v}{r} - \frac{1}{r} \left( \frac{u}{r} + \frac{w}{p} \right) + \frac{u + iv}{(1 - w)2p}$$

After substituting the values of \( u, v, \) and \( w \) from (5) we get
which simplifies to
\[
\frac{d\sigma}{ds} = \frac{1}{2p} - \frac{1}{r} \sigma - \frac{1}{2p} \sigma^2
\]
an equation exactly of the form (6). It is seen now that our problem is one of finding solutions of (6) from which we can work back to (1) and thence to the values of \(X, Y, Z\) of any point of the curve.

Equation (6) is a special form of the general Riccati Equation\(^1\)

\[
(8) \quad \frac{d\theta}{ds} = L + 2M\theta + N\theta^2
\]

Where \(L, M,\) and \(N\) are functions of \(S\). Three general theorems on the properties of the Riccati equation are of particular interest and will be given here.

**Theorem.** When a particular integral of a Riccati equation is known, the general integral can be obtained by two quadratures.

Let \(\theta\) be the known integral of (8). If we place \(\theta = 0_1 + \frac{1}{\varphi}\) we determine \(\varphi\) as follows:

Differentiate \(\theta\) with respect to \(S\) we obtain
\[
\frac{d\theta}{ds} = \frac{d\theta_1}{ds} - \frac{1}{\varphi^2} \frac{d\varphi}{ds}
\]

Substituting for \(\theta\) and \(\frac{d\theta}{ds}\) in (8) we have
\[
\frac{d\varphi}{ds} - \frac{1}{\varphi^2} \frac{d\varphi}{ds} = L + 2M(\theta_1 + \frac{1}{\varphi}) + N\left(\frac{1}{\varphi^2} + 2\theta_1 + \theta_1^2\right)
\]

\(^1\)Eisenhart, D. G., p. 25.
which reduces to

\[ \frac{d\varphi}{ds} + 2(M + N\Theta_1)\varphi + N = 0 \]  

Let \( \varphi = \varphi_0 V \). The value of \( V \) we shall determine later.

Then

\[ \frac{d\varphi}{ds} = \varphi \frac{dv}{ds} + \varphi \frac{d\varphi}{ds} \]

After substituting the values of \( \varphi \) and \( \frac{d\varphi}{ds} \) in (9), we have

\[ \varphi \frac{dv}{ds} + \left[ \frac{d\varphi}{ds} + 2(M + N\Theta_1)\varphi \right] v + N = 0 \]

We now determine the value of \( \varphi \), which will make the coefficient of \( V \) in (10) vanish, by setting

\[ \frac{d\varphi}{ds} + 2(M + N\Theta)\varphi = 0 \]

which gives

\[ \varphi = e^{-2\int[M + N\Theta]ds} \]

substituting this value of \( \varphi \) in (10) we have

\[ \frac{dv}{ds} e^{-2\int[M + N\Theta]ds} + N = 0 \]

Integration gives

\[ V = -\int N e^{2\int[M + N\Theta]ds} + a \]

Where \( a \) is the constant of integration.

Since \( \varphi = \varphi_0 V \), we have (11)

\[ \varphi = a e^{-2\int[M + N\Theta]ds} - \left( \int N e^{2\int[M + N\Theta]ds} \right) \left( e^{-2\int[M + N\Theta]ds} \right) \]

which can be abbreviated to the form

\[ \varphi = a f_0(S) + f_a(S) \]

where \( a \) is a constant of integration.

Since \( \theta = \Theta_0 + \frac{1}{\varphi} \) we can write the general integral in the
form

$$\theta = \frac{\alpha \phi, f_1(S) + \phi, f_2(S) + 1}{a f_1(S) + f_2(S)}$$

or more briefly

$$\theta = \frac{a \phi + Q}{a R + S}$$

where $P$, $Q$, $R$ and $S$ are functions of $s$.

Equation (11) shows that the general solution $\theta$ can be obtained by only two integrations.

**Theorem II**. When two particular integrals of a Riccati equation are known, the general integral can be found by one quadrature.

Let $\theta_1$ and $\theta_2$ be known solutions of equation (8). If we make the substitutions $\theta = \frac{1}{\varphi} + \theta_1$ and $\theta = \frac{1}{\psi} + \theta_2$, we have by comparison with (9)

$$\frac{d\varphi}{ds} + 2(M + N\theta_1)\varphi + N = 0$$

and

$$\frac{d\psi}{ds} + 2(M + N\theta_2)\psi + N = 0$$

If we multiply these equations by $\frac{1}{\varphi}$ and $\frac{1}{\psi}$ respectively and subtract, we have

$$\frac{d\psi}{ds} \frac{1}{\psi} - \frac{d\varphi}{ds} \frac{1}{\varphi} + 2N(\theta_1 - \theta_2) + N\left(\frac{1}{\varphi} - \frac{1}{\psi}\right) = 0$$

But

$$\frac{1}{\psi} - \frac{1}{\varphi} = \theta_1 - \theta_2$$

Hence we have, after multiplication by $\frac{\psi}{\varphi}$

$$\frac{1}{\varphi} \frac{d\psi}{ds} - \frac{\psi}{\varphi^2} \frac{d\varphi}{ds} = N(\theta_1 - \theta_2) \frac{\psi}{\varphi}$$

or

$$\frac{d}{ds} \left(\frac{\psi}{\varphi}\right) = N(\theta_1 - \theta_2) \frac{\psi}{\varphi}$$

which integrates into

$^{1}$Eisenhart, D. G. p. 20.
\[ \frac{\psi}{\varphi} = a e^{N(\theta - \theta_0)} ds \]

where \( a \) is the constant of integration.

Since \( 1 = \theta - \theta_0 \) and \( 1 = \theta - \theta_1 \), we can write

\[ (13) \quad \frac{\psi}{\varphi} = \frac{\theta - \theta_1}{\theta - \theta_0} = a e^{N(\theta - \theta_0)} ds \]

Thus it appears from (13) that \( \theta \) can be found by a single quadrature. This theorem we used later in finding the equation of the cylindrical helix.

**THEOREM III.** THE CROSS-RATIO OF ANY FOUR PARTICULAR INTEGRALS OF A RICCATI EQUATION IS A CONSTANT.

We use the general solution (12) and assign values to the particular integrals, \( \theta_1, \theta_2, \theta_3, \theta_4 \) indicated

\[ \theta_j = \frac{a_jP + Q}{a_jR + S} \quad (j = 1, 2, 3, 4) \]

We wish to prove \( \frac{\theta_4 - \theta_1}{\theta_3 - \theta_1} = \frac{\theta_4 - \theta_2}{\theta_3 - \theta_2} \) constant. We begin by finding the value of \( \theta_4 - \theta_1 \). When the above values are substituted for \( \theta_4 \) and \( \theta_1 \)

\[ \theta_4 - \theta_1 = \frac{a_4P + Q}{a_4R + S} - \frac{a_1P + Q}{a_1R + S} = \]

\[ \frac{a_1a_4PR + a_1QR + a_4PS + QS - (a_1a_4PR + a_4QR + QS + a_4PS)}{(a_4R + S)(a_1R + S)} \]

which reduces to

\[ \theta_4 - \theta_1 = \frac{(PS - QR)(a_4 - a_1)}{(a_4R + S)(a_1R + S)} \]

By symmetry

\[ \theta_4 - \theta_2 = \frac{(PS - QR)(a_4 - a_2)}{(a_4R + S)(a_2R + S)} \]

Whence we have for the quotient

\[ \#Eisenhart, D. G., p. 26 \]
\[
\frac{\theta_3 - \theta_1}{\theta_3 - \theta_2} = \frac{(a_3 - a_1)(a_2 R + S)}{(a_3 - a_2)(a_1 R + S)}
\]

Hence \( \frac{\theta_4 - \theta_1}{\theta_4 - \theta_2} = \frac{a_4 - a_1}{a_4 - a_2} = \text{constant.} \)

From this theorem it follows that if three particular integrals are known, the general integral can be obtained without quadrature.

We shall now determine the coordinates of a curve defined by its intrinsic equations. We have shown that \( \sigma \) and \( \omega \) are solutions of the differential equation (6) and, by reference to equation (12), we can write

\[
(14) \quad \sigma_j = \frac{a_j P + Q}{a_j R + S}, \quad \omega_j = \frac{b_j P + Q}{b_j R + S} \quad (j = 1, 2, 3)
\]

which are six particular integrals of (6). From them, we obtain three sets of solutions of the Frenet-Serret formulæ, which are

\[
(15) \quad \alpha = \frac{1 - \sigma_1 \omega_1}{1 - \omega_1}, \quad 1 = \frac{1 + \sigma_1 \omega_1}{1 - \omega_1}, \quad \lambda = \frac{\sigma_1 + \omega_1}{\omega_1 - \omega_3}
\]

and similar expressions in \( \sigma_2 \) and \( \omega_2 \) for \( \beta, m, \mu \), and in \( \sigma_3 \) and \( \omega_3 \) for \( \gamma, n, \nu \).

These values satisfy the condition

\[
\alpha^2 + 1^2 + \lambda^2 = 1
\]

\[
\beta^2 + m^2 + \mu^2 = 1
\]

\[
\gamma^2 + n^2 + \nu^2 = 1
\]

and in order that they may be the direction-cosines of three mutually perpendicular lines, they must satisfy in addition the conditions
\[
\alpha (\beta + lm + \lambda \mu) = 0 \\
(\beta \gamma + mn + \mu \gamma) = 0 \\
\alpha \gamma + nl + \lambda \gamma = 0
\]

By substituting the values of \(\alpha, \beta, l, m, \lambda, \) and \(\mu\) in the first of these we have

\[
\frac{(1-\sigma_1 \psi_1)(1-\sigma_2 \psi_2) - (1+\sigma_1 \psi_1)(1+\sigma_2 \psi_2) + (\sigma_1 + \psi_1)(\sigma_2 + \psi_2)}{(\sigma_1 - \psi_1)(\sigma_2 - \psi_2)} = 0
\]

which reduces to

\[
2\sigma_1 \psi_1 + 2\sigma_2 \psi_2 - \sigma_1 (\sigma_2 - \sigma_2) - \sigma_1 (\psi_1 + \psi_1) = 0
\]
or expressed as a ratio, to

\[
\frac{\sigma_1 - \sigma_2}{\sigma_1 - \psi_2} : \frac{\psi_1 - \psi_2}{\psi_1 - \psi_2} = -1
\]

We write by symmetry

\[
\frac{\sigma_j - \sigma_k}{\sigma_j - \psi_k} : \frac{\psi_j - \psi_k}{\psi_j - \psi_k} = -1 \quad (j = 1, 2, 3 \quad j \neq k)
\]

When the values of \(\sigma_j\) and \(\psi_j\) from (14) are substituted we have

\[
\frac{ajP + Q}{ajR + S} = \frac{akP + Q}{akR + S} \quad \frac{bjP + Q}{bjR + S} = -1
\]

which reduces to

\[
\frac{a_j - a_k}{a_j R + S} \quad \frac{a_k - b_j}{a_k R + S} = -1
\]
or

\[
\frac{a_j - a_k}{a_j R + S} : \frac{b_j - a_k}{b_j R + S} = -1 \quad (k = 1, 2, 3 \quad j \neq k)
\]

Hence, in order that \(\alpha, \beta, \gamma\) etc. may satisfy the condition for perpendicular lines, we have that each two, of
the three pairs of constants \( a, b; a_2, b_2; a_3, b_3 \); form a harmonic range.

When we substitute in (15) the values of \( \sigma_j \) and \( \nu_j \) from (14) we find that

\[
\alpha = \frac{1}{a_i} \left[ \frac{a_i P + Q}{a_i R + S} \cdot \frac{b_i P + Q}{b_i R + S} \right]
\]

which simplifies to

\[
\alpha = \frac{1 - a_i b_i}{a_i - b_i} \cdot \frac{(P^2 - R^2) - (Q^2 - S^2)}{2(PS - QR)} + \frac{1 + a_i b_i}{a_i - b_i} \cdot \frac{(P^2 - R^2) + (Q^2 - S^2) + \frac{a_i + b_i}{a_i + b_i} \cdot \frac{RS - PQ}{PS - QR}}{2(PS - QR)}
\]

or when \( U, V, \) and \( W \) are introduced as defined below we have

\[
\alpha = \frac{U}{a_i - b_i} + \frac{V}{a_i - b_i} + \frac{W}{a_i - b_i}
\]

and by symmetry

\[
\beta = \frac{1 - a_2 b_2}{a_2 - b_2} \cdot \frac{(P^2 - R^2) - (Q^2 - S^2)}{2(PS - QR)} + \frac{1 + a_2 b_2}{a_2 - b_2} \cdot \frac{(P^2 - R^2) + (Q^2 - S^2) + \frac{a_2 + b_2}{a_2 + b_2} \cdot \frac{RS - PQ}{PS - QR}}{2(PS - QR)}
\]

\[
\gamma = \frac{1 - a_3 b_3}{a_3 - b_3} \cdot \frac{(P^2 - R^2) - (Q^2 - S^2)}{2(PS - QR)} + \frac{1 + a_3 b_3}{a_3 - b_3} \cdot \frac{(P^2 - R^2) + (Q^2 - S^2) + \frac{a_3 + b_3}{a_3 + b_3} \cdot \frac{RS - PQ}{PS - QR}}{2(PS - QR)}
\]

where for brevity we have made the following substitution.

\[(17a)\]

\[
U = \frac{(P^2 - R^2) - (Q^2 - S^2)}{2(PS - QR)}
\]

\[
V = \frac{(P^2 - R^2) + (Q^2 - S^2)}{2(PS - QR)}
\]

\[
W = \frac{RS - PQ}{PS - QR}
\]

Remembering that \( \alpha = \frac{dx}{ds}, \beta = \frac{dy}{ds}, \) and \( \gamma = \frac{dz}{ds} \) we obtain
the coordinate $x$, by integrating the equations (17) and likewise $y$ and $z$ from $G$ and $Y$. It should be noticed, however, that the above solution depends upon our knowledge of solutions of the differential equation (6). If no solution is obtainable, the problem cannot be solved by this means.

We shall now investigate some of the known solutions in order to see how the radius of curvature and the radius of torsion determine the curve.

Suppose the torsion is zero or in other words that $\mathcal{P}$ is infinite. Equation (6) immediately reduces to
\[
\frac{d\theta}{ds} = \frac{19}{r}
\]
or
\[
\log \Theta = i \int \frac{ds}{r} + \log a
\]
where $\log a$ is the constant of integration. If we set
\[
\int \frac{ds}{r} = \eta
\]
we have
\[
\theta = ae^{-i\eta}
\]
where $\eta$ is the measure of the arc of the spherical indicatrix.

But this value of $\theta$ is of the form (12), $\Theta = \frac{\alpha r + q}{\alpha r + s}$ in which $\mathcal{P} = e^{-i\eta}$, $Q = R = 0$, $S = 1$. If we substitute these in (17a), we have
\[
U = \frac{1 + e^{2i\eta}}{2e^{-i\eta}}, \ V = \frac{1 - e^{2i\eta}}{2e^{i\eta}}, \ W = 0
\]
If we choose the values of $a$ and $b$ so that

\#Eisenhart, D. G., p. 9
a_1 = 1 \quad b_1 = -1
\quad a_2 = 1 \quad b_2 = -i
\quad a_3 = \cos \theta \quad b_3 = 0

we find that condition (16) is satisfied and that the coefficients of U, V, and W in (17) are such that
\alpha = U, \beta = V, \gamma = W.

Since \frac{dx}{ds} = U, \frac{dy}{ds} = V, \frac{dz}{ds} = W

we have for \(X, Y, Z\)
\[X = \int \frac{(1 + e^{2i\eta})}{2e^{i\eta}} ds = \int \frac{e^{i\eta} - e^{-i\eta}}{2i} ds = \int \cos \eta ds\]
\[Y = \int \frac{(1 - e^{2i\eta})}{2e^{i\eta}} ds = \int \frac{e^{i\eta} - e^{-i\eta}}{2i} ds = \int \sin \eta ds\]
\[Z = 0 ds = 0.\]

It is evident from the value of \(Z\) that the curves, for which the torsion is zero, are plane curves. Since \(r\) remains an arbitrary function of \(s\), the only limitation placed on the curves is that they lie in the plane \(Z = C\).

We shall now show that the curves, for which the first curvature is zero or for which the radius \(r\) is infinite, are straight lines. Since
\[X'' = \frac{d^2 \alpha}{ds^2} = \frac{1}{r}, \quad Y'' = \frac{d^2 \beta}{ds^2} = \frac{m}{r}, \quad \text{and} \quad Z'' = \frac{d^2 \gamma}{ds^2} = \frac{n}{r},\]
we have when \(r\) is infinite
\[X'' = Y'' = Z'' = 0.\]

When these values are substituted in the equation for the torsion
it assumes the indeterminate form

\[ \frac{1}{p} = \infty \cdot 0 \]

We therefore find it convenient to disregard equation (6) and integrate directly from the Frenet-Serret formula. The derivatives of \( \alpha, \beta, \) and \( \gamma, \) as we have seen are all zero and we have

\[ \alpha = C, \quad \beta = C_a, \quad \gamma = C_s \]

where \( C, C_a, C_s \) are constants of integration. These are equivalent to the equations

\[ \frac{dx}{ds} = C, \quad \frac{dy}{ds} = C_a, \quad \frac{dz}{ds} = C_s \]

which became after integration

\[ x = C_s s + C_s', \quad y = C_a s + C_a', \quad z = C_s s + C_s' \]

These are the parametric equations of the straight line.

A **cylindrical helix** is a curve which lies upon a cylinder and cuts the elements of the cylinder under constant angle.

**THEOREM IV**\(^{\#}\). **EVERY CURVE WHOSE RADII OF FIRST AND SECOND CURVATURE ARE IN A CONSTANT RATIO IS A CYLINDRICAL HELIX.**

We shall give the proof of this theorem after we have found the equations of the curve defined by the intrinsic equations,

\[ \rho = C \cdot f(s), \quad r = f(s) \]

from which it is evident that \( \frac{\rho}{r} = C. \)

\(^{\#}\)Eisenhart, D. G., p. 20
If we set \( \gamma = re \) equation (6) reduces to the form
\[
\frac{d\theta}{ds} = \frac{1}{2f} \left( 1 - 2c\theta - \theta^2 \right)
\]
Evidently two particular integrals of this equation are the roots of the quadratic
\[
\theta^2 + 2c\theta - 1 = 0
\]
They are \( \theta_1 = -c - \sqrt{c^2 + 1}, \theta_2 = -c + \sqrt{c^2 + 1} \) and are real and unequal if \( c \) is real.

By Theorem II, the general solution may be obtained by one quadrature and is of the form
\[
(13) \quad \frac{\theta - \theta_1}{\theta - \theta_2} = a \int N(\theta, - \theta_2) \, ds
\]
where
\[
N = \frac{1}{2f}
\]
when (13) is solved for \( \theta \) we have
\[
\theta = \frac{\theta_2 a e^{\int \frac{1}{2f}(\theta, - \theta_2) \, ds}}{a e^{\int \frac{1}{2f}(\theta, - \theta_2) \, ds} - 1} - \theta_1
\]
In order to simplify the value of \( \theta \) we set
\[
(18) \quad t = \frac{\theta_2}{2} \frac{ds}{f} = \frac{\sqrt{c^2 + 1}}{c} \int \frac{ds}{A}
\]
Then we have
\[
(19) \quad \theta = \frac{a e^{\int_{\theta_1}^{\theta_2} ds}}{\int_{\theta_1}^{\theta_2} ds} - 1
\]
Our general solutions of (6) have the condition placed on them that \( \sigma \) and \( -\frac{1}{q} \) are conjugate imaginaries. We take
\[
\sigma = \frac{a e^{\int_{\theta_1}^{\theta_2} ds}}{\int_{\theta_1}^{\theta_2} ds} - 1, \quad q = \frac{b e^{\int_{\theta_1}^{\theta_2} ds}}{\int_{\theta_1}^{\theta_2} ds} - 1
\]
Then \( a_1 \) and \( b_1 \) must be so chosen that
are conjugate imaginaries. We shall let \( b_j = u + vi \) and set its conjugate \( u - vi = b_j^* \). By applying Euler's formula to the second of these quantities, changing the sign of the imaginary parts, we have the equation:
\[
\frac{a_j \theta_a e^{it} - \theta}{a_j e^{it} - 1} = \frac{1 - (u - iv)(\cos t - i \sin t)}{(u - iv)\theta_a (\cos t - i \sin t) - \theta},
\]
or
\[
\frac{a_j \theta_a e^{it} - \theta}{a_j e^{it} - 1} = \frac{1 - b_j^* \theta}{b_j^* \theta_a e^{it} - \theta},
\]
When cleared of fractions, the equation is
\[
a_j b_j^* \theta_a - a_j \theta_a e^{it} - \theta - b_j^* \theta, \theta_a e^{it} = 1 - b_j^* \theta, \theta_a e^{it} = a_j e^{it} - a_j b_j^* - 1 + b_j^* \theta
\]
Since \( \theta, \theta_a = -1 \), this reduces to
\[
a_j b_j^* = -\frac{1 + \theta^2}{1 + \theta^2}
\]
If the values of \( \theta, \) and \( \theta_a \) are substituted in this equation we have
\[
a_j b_j^* = -\frac{1 + 2c^2 + 2c \sqrt{c^2 + 1} + c^2 + 1}{1 + c^2 - 2c \sqrt{c^2 + 1} + c^2 + 1}
\]
Multiply both terms of the right member by \( 1 + c^2 + c \sqrt{c^2 + 1} \). We then have
\[
a_j b_j^* = -\frac{1 + 3c^2 + 2c^2 + 2c(1 + c^2) \sqrt{c^2 + 1}}{1 + c^2}
\]
which in terms of \( \theta \) becomes
If we set \(a_3 = \infty\) and \(b_3 = 0\), this condition is satisfied and when these values are substituted in (16) we find

\[a_j + b_j = 0 \quad \text{where } j = 1, 2.\]

When we substitute \(-b_j\) for \(a_j\) in (20), we have

\[b_j b_{j'} = \theta^2, \quad (j = 1, 2)\]

The solutions of this equation are

\[b_1 = \theta, \quad b_2 = -i\theta,\]

whence

\[a_1 = -\theta, \quad a_2 = i\theta.\]

Having determined the values of the \(a's\) and \(b's\), it remains only to find the values of \(P, Q, R\) and \(S\) in order to have the general solution.

When we compare equations (12) and (19) we see at once

\[P = e^{it\theta_2}, \quad Q = -\theta, \quad R = e^{it}, \quad S = 1\]

so that from (17a) we have

\[U = \frac{e^{2it\theta_2} - e^{2it\theta_1} + 1}{2(-e^{it\theta_2} + e^{it\theta_1})} \]

\[= \frac{e^{2it}}{2\theta} \frac{(\theta^2 - 1) - (\theta_1^2 - 1)}{(\theta - \theta_1)(\theta - \theta_2)}\]

since \(\theta, \theta_2 = -1\) we can write

\[U = \frac{e^{2it}}{2\theta} \frac{(\theta^2 + \theta_2\theta_1) - (\theta_1^2 + \theta_1\theta_2)}{(\theta - \theta_1)(\theta - \theta_2)}\]

which by factoring becomes

\[U = \frac{(\theta_2 + \theta_1)(e^{2it} \theta_2 - \theta_1)}{2\theta (\theta - \theta_2)}\]
After substituting for \( \theta_2 + \theta_1 \) and \( \theta_2 - \theta_1 \) in terms of \( \alpha \), we have

\[
U = \frac{c}{\sqrt{\alpha^2 + 1}} \left( \alpha - \theta_2 \right)
\]

In the same way, we obtain values for \( V \) and \( W \) which are

\[
V = \frac{i c}{\sqrt{\alpha^2 + 1}} \left( \alpha' \right)
\]

\[
W = \frac{1}{\sqrt{\alpha^2 + 1}}
\]

We now set the values of \( U \), \( V \), \( W \) and of the \( a' \)'s and \( b' \)'s already determined into equation (17) and obtain in succession \( \alpha \), \( \beta \), and \( \gamma \).

\[
\alpha = \frac{1 + \theta^2}{-2\theta} U + \frac{1 - \theta^2}{-2\theta} V + 0
\]

Introducing the values of \( U \) and \( V \), we have

\[
\alpha = \frac{c}{\sqrt{\alpha^2 + 1}} \left[ 1 + \theta_2 \frac{\theta e^{it} - \theta e^{-it}}{2} + \frac{\theta^2 (1 - \theta^2) (\theta e^{it} + \theta e^{-it})}{4\theta} \right]
\]

which simplifies to

\[
\alpha = \frac{c}{\sqrt{\alpha^2 + 1}} \left[ \frac{e^{it} + e^{-it}}{2} \right]
\]

or

\[
\alpha = \frac{c}{\sqrt{\alpha^2 + 1}} \cdot \text{cost.}
\]

Whence

\[
(21) \quad x = \frac{c}{\sqrt{\alpha^2 + 1}} \int \text{cost} \, ds
\]

In like manner, we have for \( \beta \).

\[
\beta = \frac{1 + \theta^2}{2\theta} U + \frac{1 - \theta^2}{2\theta} V + 0
\]

\[
\beta = \frac{c}{\sqrt{\alpha^2 + 1}} \left[ \frac{1 - \theta^2}{2\theta} \frac{\theta e^{it} - \theta e^{-it}}{2} + \frac{1 + \theta^2}{2\theta} \frac{(\theta e^{it} + \theta e^{-it})}{2} \right]
\]
which simplifies to
\[ \beta = \frac{c}{\sqrt{c^2 + 1}} \left[ \frac{e^{it} - e^{-it}}{2i} \right] \]
or
\[ \beta = \frac{c}{\sqrt{c^2 + 1}} \sin t \]
whence (22)
\[ \gamma = \frac{c}{\sqrt{c^2 + 1}} \int \sin t \, ds. \]

From equation (17), we have for \( \gamma \)
\[ \gamma = c + c + \theta \]
or
\[ \gamma = \frac{1}{\sqrt{c^2 + 1}} \]
whence (23)
\[ z = \frac{3}{\sqrt{c^2 + 1}}. \]

Since in determining \( X, Y, Z \) of the point \( P \) of the curve we find \( \gamma \) is a constant, it follows that the angle made by the tangent, and hence by the curve, with the \( Z \)-axis is a constant. The curve then evidently lies on a cylinder whose elements are parallel to the \( Z \)-axis and are cut by the curve at a constant angle. Therefore, we can say that if the ratio of the two curvatures is a constant the curve is a cylindrical helix.

We shall turn our attention now to the discussion of four particular cases of cylindrical helices defined by their intrinsic equations. In obtaining the coordinates of a point \( P \) of the curves we shall find it only necessary to integrate equations (18), (21) and (22) and to use (23).
We shall use them in the order named without making specific references each time.

The Curve Whose Equations are \( r = \rho = \sqrt{2} \left( \frac{s^2 + a^2}{a} \right) \)

Consider the curve represented by the intrinsic equations \( r = \rho = \sqrt{2} \left( \frac{s^2 + a^2}{a} \right) \) and for which \( c = 1 \). Then

\[
t = \int \frac{ads}{s^2 + a^2} = \arctan \frac{s}{a}
\]

whence

\[
X = \frac{a}{\sqrt{2}} \int \frac{ds}{\sqrt{s^2 + a^2}} = \frac{a}{\sqrt{2}} \log \left( s + \sqrt{s^2 + a^2} \right),
\]

\[
Y = \frac{1}{\sqrt{2}} \int \frac{sds}{\sqrt{s^2 + a^2}} = \frac{s^2 + a^2}{2},
\]

and

\[
Z = \frac{s}{\sqrt{2}}.
\]

We shall show that the projection of the curve upon the XY-plane is a catenary. From the parametric equation for \( X \) we have

\[
\frac{\sqrt{2}X}{a} = s + \sqrt{s^2 + a^2}
\]

which solved for \( s \) gives

\[
s = \frac{\sqrt{2}X}{a} - a^2 \frac{e^a}{2} - \frac{\sqrt{2}X}{a}
\]

and

\[
\sqrt{s^2 + a^2} = \frac{\sqrt{2}X}{a} + a^2 \frac{e^a}{2}.
\]

The latter when substituted in the equation for \( Y \), gives
In order to change this equation to the standard form of the catenary, we replace $X$ by $X_1 + k$ for which the value of $k$ will be determined later. We then have

$$Y = \frac{1}{2\sqrt{2}} \left( e^{\frac{\sqrt{2}X}{a}} + a^2 e^{-\frac{\sqrt{2}X}{a}} \right).$$

We now place the coefficient of $e^{-\frac{\sqrt{2}X}{a}}$ equal to zero and determine the value of $k$,

or

$$a^2 e^{\frac{-\sqrt{2}k}{a}} = 1,$$

$$e^{\frac{2\sqrt{2}k}{a}} = a^2,$$

and

$$e^{\frac{-\sqrt{2}k}{a}} = a.$$

We use only the positive value of $a$ so that $k$ may be real and have

$$k = \frac{a}{\sqrt{2}} \log a.$$

When this value of $k$ is substituted in our equation for $Y$ we have

$$Y = \frac{a}{2\sqrt{2}} \left( e^{\frac{\sqrt{2}X}{a}} + e^{-\frac{\sqrt{2}X}{a}} \right).$$

This evidently is of the standard form

$$Y = \frac{a}{2} \left( e^X + e^{-X/a} \right).$$

The projection on the $YZ$-plane is an hyperbola and is given by the equation

$$Y = \sqrt{2a^2 + a^2}.$$

However, $Y$ is restricted to the positive value of the radi-
al and we have only that branch of the hyperbola lying on the right of the Z-axis.

The projection on the X Z-plane is given by the equation (3)

\[ X = \frac{a}{\sqrt{2}} \log(l2z + \sqrt{2z^2 + a^2}) \]

or

\[ X = \frac{a}{\sqrt{2}} \left[ \log \left( \frac{l2z + \sqrt{2z^2 + a^2}}{a} \right) + \log a \right] \]

or

\[ X = \frac{a}{\sqrt{2}} \left[ \sin^{-1} \frac{l2z}{a} + \log a \right]. \]

The accompanying drawings of the projections are for the curve for which \( a = 1 \). See figs. 1, 2, 3 at end.

**The Curve Whose Equations Are**

\[ r = \rho = \sqrt{2(a^2 - s^2)} \]

Consider the curve represented by the equations

\[ r = \rho = \sqrt{2(a^2 - s^2)} \]

for which \( c = 1 \).

Then

\[ t = \frac{1}{2} \sqrt{\frac{ds}{2(s^2 - a^2)}} = \text{arc sin} \frac{s}{a} \]

from which

\[ s = a \sin t \]

and

\[ ds = a \cos t \, dt, \]

whence

\[ X = \frac{a}{\sqrt{2}} \int \cos^2 t \, dt = \frac{a\sqrt{2}}{8} (2t + \sin 2t), \]

\[ Y = \frac{a}{\sqrt{2}} \int \sin t \cos t \, dt = \frac{a\sqrt{2}}{4} \sin^2 t = \frac{a\sqrt{2}}{8} (1 - \cos 2t), \]

and

\[ Z = \frac{a \sin t}{\sqrt{2}}. \]

The equations of the projections on the coordinate planes are

\[ \begin{cases} 
X = \frac{a\sqrt{2}}{8} (2t + \sin 2t) \\
Y = \frac{a\sqrt{2}}{8} (1 - \cos 2t) 
\end{cases} \]

These are the parametric equations of the inverted cycloid

\[ # \text{Granville, Differential and Integral Calculus, p. 272.} \]
Evidently, \( y \) and \( a \) have the same sign and the curve will lie above or below the \( x \)-axis according as \( a \) is chosen positive or negative.

\[
Y = \frac{Z^2}{\sqrt{2a}}
\]

It is worthy of note that since \( y \) cannot be greater than \( \frac{\sqrt{2}}{4} \), the projection is only a part of the parabola.

\[
\begin{align*}
X &= \frac{a\sqrt{2}}{6} (2t + \sin 2t) \\
Z &= \frac{a \sin t}{\sqrt{2}}
\end{align*}
\]

In this case \( z \) must lie in the interval \( \frac{a}{\sqrt{2}} \geq z \geq -\frac{a}{\sqrt{2}} \).

The drawings show the projections of the curve for which \( a = 1 \). See figures 4, 5, 6 at end.

**The Curve Whose Equations Are**

\[
\begin{align*}
X &= \frac{a\sqrt{2}}{6} (2t + \sin 2t) \\
Y &= \frac{1}{\sqrt{2}} \int s \sqrt{s^2 - a^2} ds = \frac{a}{\sqrt{2}} \log s, \\
Y &= \frac{1}{\sqrt{2}} \int \left( s^2 - a^2 \right) ds = \frac{1}{\sqrt{2}} \left[ s^2 - a^2 - a \arccos \frac{a}{s} \right], \\
Z &= \frac{s}{\sqrt{2}}
\end{align*}
\]

The equations of the projections on the coordinate planes are

\[
\begin{align*}
X &= \frac{a}{\sqrt{2}} \log s \\
Y &= \frac{1}{\sqrt{2}} \left( s^2 - a^2 - a \arccos \frac{a}{s} \right)
\end{align*}
\]
It is apparent from these equations that for real values of X and Y we must have \( S \geq a \) and \( S > 0 \).

\[
\begin{align*}
Y &= \frac{1}{\sqrt{2}} \left[ \sqrt{S^2 - a^2} - a \arccos \frac{a}{S} \right] \\
Z &= \frac{S}{\sqrt{2}}
\end{align*}
\]

(2)

Because of the limitations on \( S \) mentioned above, we find that \( Z \geq \frac{a}{\sqrt{2}} \). The slope of the tangent to the curve, \( \frac{dy}{dz} \), is \( \sqrt{1 - \frac{a^2}{2S^2}} \). If \( Z = \frac{a}{\sqrt{2}} \) the slope is zero but for values of \( Z \) very large in comparison with \( a \), the slope approaches unity and the curve approaches a straight line.

\[
\frac{\sqrt{2X}}{a} = \sqrt{2}Z.
\]

(3)

This is the exponential curve but the limitations on \( S \) eliminate that part of the curve lying on the left of the Z-axis.

The drawings show the projections when \( a = 1 \). Fig. 7, 8, 9.

The Curve Whose Equations are \( r = as \), \( \rho = bs \).

Consider the curve represented by the intrinsic equations \( r = as \), and \( \rho = bs \), for which

\[
\rho = c = b \frac{a}{a}
\]

Then

\[
t = \sqrt{\frac{a^2 + b^2}{b}} \int \frac{ds}{as} = \frac{\sqrt{a^2 + b^2}}{ab} \log s
\]

whence

\[
s = e^{ht}
\]

where \( h = \sqrt{\frac{a}{a^2 + b^2}} \) and \( ds = he^{ht} dt \).

It follows that

\[
X = \frac{h}{a^2} \int e^{ht} \cos t \, dt,
\]

\[
= \frac{h^2}{a(h^2 + 1)} \left[ e^{ht} (\sin t + h \cos t) \right].
\]
\[ Y = \frac{h^2}{a} \int e^{ht} (\sin t) \, dt \]
\[ = \frac{h^2}{a(h^2 + 1)} e^{ht} (\sin t - h \cos t), \]

and \( Z = \frac{h}{b} S = \frac{h}{b} e^{ht}. \)

In order to simplify these values of \( X, Y, \) and \( Z \) we set
\[ A_0 = \frac{h^2}{a(h^2 + 1)}, \quad B = \frac{h}{b} \]
and have for \( X, Y, Z, \)
\[ X = A_0 e^{ht} (\sin t + h \cos t), \]
\[ Y = A_0 e^{ht} (\sin t - h \cos t), \]
\[ Z = B e^{ht}. \]

The expressions for \( X \) and \( Y \) admit further simplification by rotating the axes in the \( XY \)-plane. In order to determine the angle \( \varphi \), through which we rotate the axes, we proceed as follows: multiply both \( X \) and \( Y \) by \( h \) and form the sum and difference:
\[ X + hY = A_0 e^{ht} (h^2 + 1) \sin t \]
and
\[ hX - Y = A_0 e^{ht} (h^2 + 1) \cos t. \]

After multiplying both equations by \( k \) where \( k = \frac{1}{\sqrt{h^2 + 1}} \)
we have
\[ kX + hky = A_0 e^{ht} / \sqrt{h^2 + 1} \sin t \]
\[ hkX - ky = A_0 e^{ht} / \sqrt{h^2 + 1} \cos t. \]

and by setting \( A = A_0 \sqrt{h^2 + 1} \) we have
\[ kX + hky = Ae^{ht} \sin t \]
\[ hkX - ky = Ae^{ht} \cos t. \]

We now define \( \varphi \) as arc \( \sin k = \) arc \( \cos \) \( hk. \) Indicating the new coordinates by \( X', Y', \) and \( Z' \) we have after rotation
\[ X' = Ae^{ht}\cos t \]
\[ Y' = Ae^{ht}\sin t \]
\[ Z' = Z = Be^{ht} \]

We shall show that the curve lies upon a circular cone whose axis coincides with the Z-axis and cuts the elements of the cone under constant angle.

The equation of the circular cone formed by rotating the line \( Z = LX \) about the Z-axis is
\[ X^2 + Y^2 = L^2Z^2 \]

If \( L = \frac{A}{B} \), the above values of \( X' \), \( Y' \) and \( Z' \) satisfy the equation and hence the curve lies on the cone.

The equation of the element of the cone through the point \( (X', Y', Z') \) of the curve is
\[ \frac{X'}{X'} = \frac{Y'}{Y'} = \frac{Z'}{Z'} \]

Its direction-cosines are
\[ \frac{X'}{\sqrt{X'^2 + Y'^2 + Z'^2}} \quad \frac{Y'}{\sqrt{X'^2 + Y'^2 + Z'^2}} \quad \text{and} \quad \frac{Z'}{\sqrt{X'^2 + Y'^2 + Z'^2}} \]

The direction-cosines of the tangent to the curve are
\[ a = \frac{\frac{dX'}{dt}}{\sqrt{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2}} \]
\[ \beta = \frac{\frac{dY'}{dt}}{\sqrt{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2}} \]
\[ \gamma = \frac{\frac{dz'}{dt}}{\sqrt{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2}} \]

If we indicate by \( w \) the angle between the element of the cone and the tangent, or between the element and the curve, we have

\[ \cos w = \frac{x'\frac{dx'}{dt} + y'\frac{dy'}{dt} + z'\frac{dz'}{dt}}{\sqrt{x'^2 + y'^2 + z'^2}} \frac{\sqrt{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2}}{\sqrt{A^2 + B^2}} \]

which reduces to

\[ \cos w = \frac{h(A^2 + B^2)}{\sqrt{A^2 + B^2}} \frac{\sqrt{A^2 + B^2}}{\sqrt{A^2(1 + h^2) + B^2h^2}} \]

Since \( A, B, \) and \( h \) are functions of \( a \) and \( b \), the angle \( w \) at which the curve cuts any element of the cone is constant.

We show the projection on the coordinate planes for the case where \( a = b = \sqrt{2} \). The equations of the projections are

\[ \begin{align*}
(1) & \quad \begin{cases}
X = e^t\sin t + \cos t \\
Y = e^t\sin t - \cos t
\end{cases} \\
(2) & \quad \begin{cases}
Y = e^t\sin t - \cos t \\
Z = \frac{e^t}{\sqrt{2}}
\end{cases} \\
(3) & \quad \begin{cases}
X = e^t\sin t + \cos t \\
Z = \frac{e^t}{\sqrt{2}}
\end{cases}
\end{align*} \]

See figures 10, 11 and 12 at the end.
Fig. 1.

\[ y = \frac{1}{2\sqrt{2}} \left( e^{\sqrt{2}x} + e^{-\sqrt{2}x} \right) \]

1 inch = \( 2\sqrt{2}x \)

1 inch = \( 2\sqrt{2}x \)
Fig. 2.

\[ y = \sqrt{z^2 + \frac{1}{2}} \]

1 inch = \(2 \sqrt{y}\)

1 inch = \(2 \sqrt{z}\)
Fig. 3.

\[ x = \frac{1}{\sqrt{2}} (\sinh \sqrt{2} z) \]

1 inch = \( 2\sqrt{2} \) x

1 inch = \( 2\sqrt{2} \) z.
Fig. 4.

\[ x = \frac{1}{4\sqrt{2}} (2t + \sin 2t) \]
\[ y = \frac{1}{4\sqrt{2}} (1 - \cos 2t) \]

1 inch = 3\sqrt{2}x
1 inch = 40\sqrt{2}y
Fig. 5.

\[ y = \frac{x^2}{\sqrt{2}} \]

1 inch = 40\sqrt{2}y

1 inch = 8\sqrt{2}z.
Fig. 6.

\[ x = \frac{1}{4\sqrt{2}} (2t + \sin 2t) \]
\[ z = \frac{1}{\sqrt{2}} \sin t. \]

1 inch = 8\sqrt{2}x
1 inch = 8\sqrt{2}z.
Fig. 7.

\[ x = \frac{1}{\sqrt{2}} \log s \]

\[ y = \frac{1}{\sqrt{2}} \left[ \frac{u^2 + 1}{u} = \arccos \frac{1}{u} \right] \]

1 inch = 20\sqrt{2}x

1 inch = 20\sqrt{2}y.
Fig. 8.

\[ y = \frac{1}{\sqrt{2}} \left( s^2 + 1 + \arccos \frac{1}{s} \right) \]

\[ z = \frac{s}{\sqrt{2}} \]

1 inch = \(20\sqrt{2}y\)

1 inch = \(10\sqrt{2}z\)
Fig. 9.

\[ e^{\sqrt{2x}} = \sqrt{z} \]

1 inch = 20\sqrt{2x}

1 inch = 10\sqrt{z}

\[ z = e^x \]
Fig. 10.

\[ x = e^t(\sin t + \cos t) \]
\[ y = e^t(\sin t - \cos t) \]

1 inch = 2x
1 inch = 2y.
Fig. 11.

\[ y = e^t (\sin t - \cos t) \]

\[ z = \frac{e^t}{\sqrt{2}} \]

1 inch = 2y
1 inch = 2z.
Fig. 12.

\[ x = e^t (\sin t + \cos t) \]
\[ z = \frac{e^t}{\sqrt{2}}. \]

1 inch = 2x
1 inch = 2z.
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