A DISCUSSION OF THE FUNCTION

\[ w^2 = z^3 + z^2 + a \]

by

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A discussion of the function $W = Z^3 - 3tZ + a$. $W = u + vi$, $Z = x + yi$, $a$ is any constant and $t$ is a variable.

To determine the branch-points differentiate the equation and equate to zero the resulting derivative in $Z$.

$$2W = 3Z^2 - 6tZ = 0.$$  

$Z = 0$ or $2t$. are the critical points. Substituting these values for $Z$ in the original equation we find that when $Z = 0$, $W = \sqrt{a}$; and when $Z = 2t$, $W = \sqrt{a - 4t^3}$. The branch-points then are $W = \sqrt{a}$ and $W = \sqrt{a - 4t^3}$. Since $W$ becomes infinite when $Z$ is infinite there will be a third branch-point at infinity in the $W$-Plane. In the discussion that follows $a = 1$ and $t$ is a variable parameter. Thus we have $W = \sqrt{a}$ when $Z = 0$ and $W = \sqrt{1 - 4t^3}$ when $Z = 2t$ for the value of the branch-points. Also when $W = 1$, $Z = 0$ twice or $3t$; and when $W = \sqrt{1 - 4t^3}$, $Z = -t$ or $2t$ twice.

Substituting the values $u + vi$ for $W$ and $x + yi$ for $Z$ in the original equation we get:

$$(u + vi)^3 = (x + yi)^3 - 3t(x + yi)^3 + 1.$$  

Upon expanding this becomes:

$$u^3 + 2uvi - v^3 = x^3 + 3xyi - 3xy^2 - yi - 3tx^2 - 6txyi + 3ty^2 + 1.$$  

Equating real parts and imaginary parts we get the two equations:

1. $u - v = x - 3xy - 3tx^2 - 3ty^2 + 1$
2. $2uv = 3xy^2 - y - 6txy$. 
Let \( t \) take on real positive and negative values including zero and we notice from the above relations between \( W \) and \( Z \) that the branch-points in the \( W \)-Plane lie on either the \( u \)-axis or the \( v \)-axis according to the value of \( t \). To determine the relationship between the two planes we need only to set equation (2) to zero and we thus get the two axes in the \( W \)-plane and the corresponding curves in the \( Z \)-plane can be determined from the equation:

\[
3x^2y - y^3 - 6txy = 0.
\]

Thus when \( t = 0 \) we find the following relations:

- When \( Z = 0 \), \( W = \pm 1 \), are branch-points
- when \( Z = 2t = 0 \), \( W = \pm \sqrt{1 - 4t^2} = \pm 1 \),
- when \( W = \pm 1 \), \( Z = 0 \) twice or \( 3t = 0 \),
- when \( W = \pm \sqrt{1 - 4t^2} = \pm 1 \), \( Z = 0 \) three times,
- when \( W = 0 \), \( Z = -1 \).

The original equation becomes:

\[
W = Z^3 + 1.
\]

Setting equation (2) to zero we get in the \( W \)-plane the \( u \)-axis and the \( v \)-axis. In the \( Z \)-plane we get the three straight lines \( y = \pm \sqrt{3} x \) and \( y = 0 \). Since the original equation is of the third degree in \( Z \) we have a three sheeted Riemann surface in the \( W \)-plane and two sheets in the \( Z \)-plane. Also since \( Z = 0 \) three times for the value \( W = \pm 1 \) the three sheets in the \( W \)-plane hang together at the points \( W = \pm 1 \). We need to consider only the positive value because the negative is an exact duplicate of the positive. Similarly only the positive values on the axis of imaginaries need be considered.
Since there is a branch-point at $W = 1$ and another at $W = +\infty$ we can establish a branch-cut along the axis of reals from $W = 1$ to $W = +\infty$ and let, say, sheets II and III hang together along this line. Let us choose the other branch-cut between sheets I and II from the point $W = 1$ to $W = 0$ along the axis of reals and then along the axis of imaginaries to positive infinity. The Riemann surface is thus complete. To the point $W = 1$ there corresponds in the $Z$-plane the point $Z = 0$. By choosing a number of points on the lines $y = \pm \sqrt{3} x$ on the positive side of the $x$-axis such as $1 \pm \sqrt{3} i$ and substituting in the equation: $W = Z^2 + 1$, we find that the cross-cut between sheets I and II maps into the two lines $y = \pm \sqrt{3} x$ to the right of the axis of imaginaries. Similarly the line from $W = 1$ along the axis of reals to positive infinity maps into the lines $y = \pm \sqrt{3} x$ to the left of the axis of imaginaries.

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**Fig. 1.**

**Fig. 2.**
The results of the mapping from the \( W \)-plane upon the \( Z \)-plane are exhibited in Fig. 1 and 2, and the details of the mapping are exhibited in Fig. 3 and 4.

These Figures show the mapping of the \( W \)-plane upon the \( Z \)-plane when \( t = 0 \), and \( K \) varies positively and negatively. The blue curves represent the relation when \( K \) is positive, the green when \( K \) is negative and the red when \( K \) is zero. The whole of the \( W \)-plane maps into the whole of the \( Z \)-plane.
Let us think of the $W$-plane as consisting of three sheets. To the first sheet we associate the values of $Z$ in I, and to the second and third sheets we then associate the values of $Z$ in II and III respectively. As $W$ traverses a closed circuit about $W=1$ in the positive direction starting from a point in the first sheet, say in the first quadrant, $Z$ will pass from I to II as $W$ crosses the axis of imaginaries from sheet I to sheet II of the three-sheeted Riemann surface. And $Z$ will pass to III as $W$ continuing in its circuit, passes from sheet II to sheet III upon crossing the axis of reals, thus completing the circuit. Upon continuing a second time about this branch-point $W$ remains in sheet III until it reaches the axes of reals a second time and passes to sheet II upon crossing this axis. $Z$ correspondingly passes from III to II and back to I as $W$ continues its circuit in sheet II and passes back to sheet I upon crossing the axis of imaginaries a third time. It was impossible for $W$ to pass to sheet I while on its second circuit about the branch-point because sheet III is smooth along the axis of imaginaries. About the point $W=1$ the various sheets of the Riemann surface are connected as shown in Fig. 5.

\[\begin{array}{c}
I \\
II \\
III
\end{array}\]

Fig. 5.
The discussion of the Riemann surface which replaces the \( W \)-plane for the given function is now complete. Any continuous curve upon this surface maps into a continuous curve upon the \( Z \)-plane.

Let us next consider the relationship of the two planes for the value \( t = 1 \). It is seen that the branch-point \( W = 1 \) in the \( W \)-plane remains the same while the other branch-point \( W = \sqrt{1-4t^2} \) changes to \( W = \sqrt{3} \) i. Further:

When \( W = 1 \), \( Z = 3 \) or zero twice.
When \( W = \sqrt{3} \) i, \( Z = -1 \) or 2 twice.

Two sheets in the \( W \)-plane hang together at \( W = 1 \) and two at \( W = \sqrt{3} \) i. From equation (2) we have:

\[
2uv = 3x^3y - y^3 - 6xy.
\]

Equating these to zero, we get the \( u \)-axis and the \( v \)-axis in the \( W \)-plane on which the branch-points lie. In the \( Z \)-plane we get the corresponding curves, namely:

\[
3x^2y - y^3 - 6xy = 0.
\]

Or the straight line \( y = 0 \), and the hyperbola:

\[
\left(\frac{x-1}{1}\right)^2 - \frac{y^3}{3} = 1
\]

The vertices of this hyperbola are at \( x = 0 \) and \( x = 2 \), and its center at \( x = 1 \). By choosing different points such as \( 3 + 3i \) on the hyperbola and on the straight line \( y = 0 \) and substituting them in the equation:

\[
W = Z^3 - 3Z^2 + 1.
\]
we find that to the branch of the hyperbola with center at $Z = 2$ there corresponds in the $W$-plane the $v$-axis from $W = \sqrt{3}i$ in the positive direction to infinity. The branch of the hyperbola with center at $Z = 0$ maps into the $u$-axis from $W = 1$ to the right to infinity. The line from $Z = 1$ to $Z = 3$ maps into the line from $W = 0$ to $W = \sqrt{3}i$. And the line from $Z = 1$ to $Z = -1$ maps into the line from $W = 0$ to $W = 1$. The results of this mapping are exhibited in Fig. 6 and Fig. 7.

Fig. 6. Fig. 7.
We again have a three-sheeted Riemann surface in the \( W \)-plane. Let sheet I and sheet II hang together at the point \( W = \sqrt{3} \) i and sheet II and sheet III hang together at the point \( W = 1 \). Let us therefore establish a cross-cut between sheets I and II along the yellow line to infinity and a cross-cut between sheets II and III along the red line to infinity. Along the green line and the purple line the three sheets are smooth. To sheets I and II and III in the \( W \)-plane there corresponds I, II, and III in the \( Z \)-plane.

If \( W \) completes a closed circuit about the point \( W = \sqrt{3} \) i as a center and with a radius less than 3 beginning in sheet I it will pass to sheet II upon crossing the yellow cross-cut. Since the three sheets are smooth along the purple line \( W \) will remain in sheet II. Upon completing a second circuit about the same point \( W \) will again pass to sheet I. Sheet III is smooth along this cross-cut. \( Z \) will correspondingly pass from I to II and back to I again. A continuous curve about this point in the \( W \)-plane maps into a continuous curve in the \( Z \)-plane. About the point \( W = \sqrt{3} \) i the various sheets of the Riemann surface are connected as shown in Fig. 8.

\[
\begin{align*}
&\text{I} \\
&\text{II} \\
&\text{III}
\end{align*}
\]

\text{Fig. 8.}
Similarly if \( W \) completes a closed circuit in the positive direction about the point \( W = 1 \) beginning in sheet II it will pass to sheet III and back to sheet II after a second circuit. Sheet I in this case is smooth. \( Z \) will correspondingly pass from II to III and back to II. \( Z \) will not enter I. The three sheets of the Riemann surface in this case are connected as shown in Fig. 9.

![Fig. 9.](image)

If \( W \) completes a closed circuit in the positive direction inclosing both points \( W = \sqrt{3} \, i \) and \( W = 1 \) beginning in the first sheet it will pass to sheet II upon crossing the yellow line and to sheet III upon crossing the red line. In its second circuit beginning in sheet III the yellow line will not affect \( W \) as sheet III is smooth along this line. But it will pass to sheet II in its second circuit upon crossing the red line. Then in its third revolution it will come back to sheet one upon crossing the yellow line and then back to its starting place after completing the third closed circuit. \( Z \) likewise passes from I to II and from II to III as \( W \) completes one circuit. From III it will pass to II as \( W \) completes its
second circuit. While \( W \) makes its third revolution \( Z \) passes from II to I back to its starting point.

About the two points the various sheets of the Riemann surface are connected as shown in Fig. 10.

```
I
II
III
```

Fig. 10.

The discussion of the Riemann surface for the function when \( t = 1 \) is now complete. Any continuous curve upon this surface maps into a continuous curve upon the \( Z \)-plane.

The details of the mapping of the \( W \)-plane upon the \( Z \)-plane are exhibited in Fig. 11 and Fig. 12. By letting \( K \) vary in the equation:

\[
2uv = 3x^2y - y^3 - 6xy = K
\]

we get the corresponding curves in the two planes. If \( K = 0 \) we get the two axes in the \( W \)-plane and in the \( Z \)-plane we get the \( x \)-axis and the hyperbola with centers at \( x = 0 \) and \( x = 2 \), (Red curves). When \( K = 1, 2, 3, \ldots \) we get in the \( W \)-plane the blue hyperbolas in the first and third quadrants. For the corresponding values of \( K \) we get the blue curves in the \( Z \)-plane. Again if \( K \) varies negatively we get in the \( W \)-plane
the green hyperbolas in the second and fourth quadrants and in the Z-plane the corresponding green curves. The greater the value for K the farther out the corresponding curves will lie. It is thus evident that as K takes on all possible values the whole of the W-plane maps exactly once upon the whole of the Z-plane.

Let us next consider the given function as t varies negatively. First let \( t = -1 \). For this value the branch-point \( W = 1 \) remains stationary while the other lies on the axis of reals, \( W = \sqrt{5} \). The equation becomes:

\[
2uv = 3x^2 y - y^3 - 6xy.
\]

Equating these to zero we get the hyperbola:
\[
\frac{(x+1)^2}{4} - \frac{y^2}{3} = 1.
\]
in the \(Z\)-plane. When \(W = 1\), \(Z = -2\) twice or \(-3\) and when \(W = \sqrt{5}\), \(Z = -2\) twice or \(1\). These double values for \(Z\) are the centers of the hyperbola in the \(Z\)-plane. The axis of reals in the \(W\)-plane from \(W = \sqrt{5}\) to the right to infinity maps into the hyperbola with center at \(Z = -2\), and the \(x\)-axis from \(Z = 1\) to the right indefinitely. The positive end of the axis of imaginaries and that part of the axis of reals between \(W = 0\) and \(W = 1\) in the \(W\)-plane maps into the hyperbola with center at \(Z = 0\) and the \(x\)-axis from \(Z = -3\) to the left indefinitely in the \(Z\)-plane. Again the part of the axis of reals in the \(W\)-plane from \(W = 1\) to \(W = \sqrt{5}\) maps into the axis of reals between the points \(Z = -3\) and \(Z = 1\) in the \(Z\)-plane. If we choose the yellow line as a branch-cut between sheet I and sheet II of the three-sheeted Riemann surface in the \(W\)-plane and the red line as a branch-cut between sheets II and III and let the three sheets be smooth between \(W = 1\) to \(W = \sqrt{5}\) the Riemann surface is complete. Then to the three sheets in the \(W\)-plane there correspond sections I, II and III in the \(Z\)-plane. The results of this mapping are exhibited in Fig. 13 and Fig. 14.
As $W$ traverses a small closed circuit about the point $W = 1$ with radius less than $\sqrt{5}$ in the positive direction starting from a point in the first, sheet $Z$ will pass from I to II and consequently $W$ will pass from the first sheet to the second sheet. By going about $W = 1$ $Z$ never passes to III since only I and II come together at the corresponding point $Z = 0$. In a similar manner, it will be seen that as $W$ traverses once a small closed circuit about $W = \sqrt{5}$, beginning at a point in the second sheet, $Z$ passes from II into III and upon continuing
a second time about this branch-point Z returns to its original position in II. In a neighborhood of this branch-point it is impossible for W to pass from sheet two or three into one since the region I is not associated with II or III at the corresponding point \( Z = -2 \). It is evident that all three branches become identical at the branch-point \( W = \infty \). To the right of the branch-point \( W = 1 \) the various sheets of the Riemann surface are connected as shown in Fig. 15, and to left from this point to the origin and thence in the positive direction indefinitely along the axis of imaginaries the three sheets are connected as shown in Fig. 16.

![Diagram showing sheets of the Riemann surface](image)

**Fig. 15**

**Fig. 16.**

The discussion of the Riemann surface is now complete. Any continuous curve upon this surface maps into a continuous curve upon the \( Z \)-plane. For example, the closed curve upon the Riemann surface, of which the ellipse \( E_w \) about the points \( W = 1 \) and \( W = \sqrt{5} \) as foci are the traces, map into ellipses in the \( Z \)-plane. If the variable \( w \) describes the ellipse \( E_w \), com-
mencing with a point \( w_0^{(1)} \) in the first sheet, then \( z \) describes a corresponding path beginning at \( z_0 \) lying in \( \Pi^+ \). As \( w \) crosses the positive axis of reals to the left of \( W = 1 \) the point passes into the second sheet and \( z \) passes into \( \Pi^+ \). Upon crossing the same axis to the right of \( W = \sqrt{5} \) the point passes into the third sheet and \( z \) passes into \( \Pi^+ \). When \( w \) has completed one revolution it is in the third sheet and we denote its position by \( w_0^{(3)} \). By a second revolution of \( w \) about these points, \( w \) remains in the third sheet while crossing the yellow line and \( z \) passes from \( \Pi^+ \) to \( \Pi^- \). But upon crossing the red line during this revolution \( w \) passes from the third sheet to the second sheet and \( z \) correspondingly passes from \( \Pi^- \) to \( \Pi^- \). By a third revolution on this ellipse the point passes from the second sheet to the first sheet upon crossing the yellow line and \( z \) passes from \( \Pi^- \) into \( \Pi^- \). And as \( w \) continues in its path and crosses the red line back to its original position \( w_0^{(0)} \), \( z \) passes across the corresponding red line from \( \Pi^- \) to \( \Pi^+ \) to its original position \( z_0 \).

When \( t = -2 \) we have the equation:

\[ 2uv = 3x^3y - y^3 - 12xy \]

Upon equating this to zero we have in the \( Z \)-plane the hyperbola:

\[
\left(\frac{x - 2}{4}\right)^2 - \frac{y^3}{12} = 1.
\]
The centers of this hyperbola are at \( x = 0 \) and \( x = -4 \). The branch-points in the \( W \)-plane are at \( W = 1 \) and \( W = \sqrt{33} \). It is at once evident that as \( t \) increases negatively the variable branch-point moves along the axis of reals in the positive direction indefinitely, while the other branch-point remains fixed at \( W = 1 \).

Let us next investigate the branch-points when \( t \) takes on pure imaginary values. For the value \( t = i \), one branch-point remains at \( W = 1 \) while the other is at \( W = \sqrt{1 + 4i} \). It is at once evident that the variable branch-point no longer remains on either of the axes in the \( W \)-plane. It has wandered off into the plane and taken on a complex value. To determine its location let:

\[
W^2 = (1 + 4i).
\]

Or \((u + vi)^2 = 1 + 4i\).

Expanding and equating real parts and imaginary parts we have:

\[
u^2 - v^2 = 1,
\]

and \(uv = 2\).

Solving these two equations simultaneously will give the value of \( u \) and of \( v \), thus locating the point. Thus the branch-point is on the intersection of the two hyperbolas \( u^2 - v^2 = 1 \) and \( uv = 2 \). Hence we get:

\[
u = \sqrt{\frac{1 + 17}{2}},
\]

and \(v = \sqrt{\frac{7 + 17}{2}}\).
We see that when $W = 1$, $Z = 0$ twice or $3i$ and when $W = \sqrt{1 + 4i}$, $Z = 2i$ twice or $-i$. Therefore the critical points in the $Z$-plane are zero and $2i$. By setting the equation:

$$u^2 - v^2 = x^3 - 3xy^2 + 6xy + 1$$

equal to $K$ we find that when $K = 1$ we get an hyperbola in the $W$-plane which passes through the two branch-points. Corresponding to this curve in the $W$-plane we have in the $Z$-plane the hyperbolas:

$$\frac{(y - 1)^2}{1} - \frac{x^2}{3} = 1.$$
Furthermore, the centers of these hyperbolas are on the critical points $Z = 2i$ and $Z = 0$. We also find that the line from $Z = 0$ to $Z = 2i$ corresponds to that part of the hyperbola

$$u^2 - v^2 = 1$$

in the $W$-plane from $W = 1$, to $W = \sqrt{1 + 4i}$. The rest of this curve in the first quadrant of the $W$-plane corresponds to the hyperbola in the $Z$-plane whose center is at $Z = 2i$, and that part of the curve in the fourth quadrant of the $W$-plane, colored green, corresponds to the hyperbola whose center is at $Z = 0$. The results of this conformal mapping are shown in Fig. 17 and Fig. 18.

Let sheet I and sheet II hang together at the branch-point $W = \sqrt{1 + 4i}$ in the first quadrant and let sheet II and sheet III hang together at $W = 1$. The three sheets are smooth along the blue line joining these two points. Let us furthermore establish a branch-cut between sheets one and two from the point $W = \sqrt{1 + 4i}$ in the first quadrant along the red line out indefinitely and a branch-cut between sheets two and three from the point $W = 1$, along the green line out indefinitely in the fourth quadrant. Thus to the sheets one, two and three there correspond in the $Z$-plane I, II, and III. The Riemann surface which maps the three sheets of the $W$-plane upon the $Z$-plane is complete.

If $t = 2i$ the variable branch-point has changed to $W = \sqrt{1 + 32i}$. In solving for this point we find that it is on the intersection of the two hyperbolas $u^2 - v^2 = 1$ and
Again if $t = -i$ the variable branch-point changes to $W = \sqrt{1 - 4i}$. This point is on the intersection of $u^2 - v^2 = 1$ and $uv = -2$. The first of these is the same as for positive values of $t$ but the second hyperbola has changed to a negative, thus changing from quadrants one and three to quadrants two and four. Figures 19 and 20 show the results of mapping the $W$-plane upon the $Z$-plane.
The results of the Riemann surfaces obtained for the different values of the variable parameter \( t \) show that as \( t \) takes on all possible values, the branch-point \( W = 1 \) remains fixed. As \( t \) takes on real positive values the branch-point \( W = \sqrt{1 - 4t^3} \) varies along the line from \( W = 1 \) to \( W = 0 \) and from there to positive infinity along the axis of imaginaries. As \( t \) takes on all negative real values the variable branch-point \( W = \sqrt{1 - 4t^3} \) varies along the axis of reals from \( W = 1 \) indefinitely in the positive direction. Again, if \( t \) takes on pure imaginary positive values the branch-point \( W = \sqrt{1 - 4t^3} \) varies along that part of the hyperbola \( u^2 - v^2 = 1 \) which lies in the first quadrant. And if \( t \) takes on negative imaginary values the variable branch-point in the \( W \)-plane wanders along that part of the hyperbola \( u^2 - v^2 = 1 \) which lies in the fourth quadrant.

We can represent the relationship of the variable branch-point \( W = \sqrt{1 - 4t^3} \) and the different values of the variable parameter \( t \) by means of a Riemann surface. The relationship between \( W \) and \( t \) are expressed by equation

\[
W = \sqrt{1 - 4t^3}
\]

or

\[
W^2 = 1 - 4t^3
\]

To find the critical points differentiate the above equation and equate to zero the resulting derivative in \( t \).

\[
2W = -12t^2
\]
We see that \( t=0 \) twice is a critical point in the \( t \)-plane.
If \( t=0 \), \( W=\pm 1 \). Therefore \( W=\pm 1 \) are the branch points in
the \( W \)-plane. If \( W=\pm 1 \), \( t=0 \). Since we have an equation in
\( W^2 \) the branch-point \( W=\pm 1 \) will be an exact duplicate of \( W=\pm 1 \)
and therefore we need not consider it. Since we have a quadratic in \( W \) we will get a two-sheeted Riemann surface in the
\( W \)-plane. The two sheets hang together at \( W=1 \). Let the two
sheets be smooth along the \( u \)-axis and the \( v \)-axis, and let us
establish a branch-cut between them along the hyperbola
\( u^2 - v^2 = 1 \). Thus to sheet one in the \( W \)-plane there corresponds
I in the \( t \)-plane and to sheet two in the \( W \)-plane there corresponds II in the \( t \)-plane. The results of this discussion are
demonstrated in Fig. 21 and Fig. 22.

The discussion of the Riemann surface is now complete.
Any continuous curve upon this surface maps into a continuous
curve upon the \( t \)-plane. For example, the closed curve upon
the Riemann surface, of which the circle \( C_w \), with \( W=1 \) as cen-
ter are the traces, map into circles in the \( t \)-plane. If the
variable \( w \) describes the circle \( C_w \), commencing with a point \( w_0 \)
in the first sheet, then \( z \) describes a corresponding path
beginning at \( z_0 \) lying in \( I^+ \). As \( w \) rotates in the negative di-
rection and crosses that part of the hyperbola \( u^2 - v^2 = 1 \), col-
ored green, the point passes into the second sheet and \( z \)
passes into \( II^+ \). Upon crossing the axis of reals \( w \) remains in
the second sheet since the two sheets are smooth along this line and \( z \) passes into \( \Pi^- \). As \( w \) crosses the other part of the hyperbola \( u^2 - v^2 = 1 \) it passes from the second sheet back into the first sheet and \( z \) correspondingly passes from \( \Pi^- \) into \( \Pi^- \). As \( w \) completes its circuit upon crossing the blue line it remains in the first sheet and reaches its original position \( w_0 \) and \( z \) correspondingly crosses the axis of reals and reaches its original position \( z_0 \).

From this relationship the variable branch-point \( w = \sqrt{1 - 4t^3} \) may be located in general as \( t \) takes on all possible real, imaginary, and also complex values.

![Fig. 21](image1.png)  ![Fig. 22](image2.png)