The Irreducibility of Certain Sets of Assumptions

by

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Submitted to the Department of Mathematics and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Master of Arts.

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C O N T E N T S

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Introduction.

It is the purpose of this paper to discuss the irreducibility of two given sets of mathematical assumptions, one characterizing a finite projective geometry and the other characterizing a finite euclidian geometry.

The subject is of particular interest because of its bearing upon the relations between euclidian geometry and projective geometry.

The paper will be divided into two divisions. In order to get the proper perspective, there will be given a brief historical sketch. This sketch constitutes Part I.

In Part II, the irreducibility of each of the given sets of assumptions is established.

Part I.

HISTORICAL SKETCH.

A mathematical science, as defined by J. W. Young,[1] is "Any body of propositions so arranged that every proposition after a certain one is a logical consequence of some or all that precede it." These propositions must deal with some kind of concepts, called, for convenience, elements, and their relations.

Some of these elements are more simple than others. In order to have a clear definition, the more simple must be used in defining the others. A s a consequence, some elements must be left undefined, and certain propositions must be regarded as more simple and more fundamental from which the other propositions must be deduced. Therefore, in setting up any mathematical science, there must be a set of undefined terms and a set of unproved propositions. It has been customary to use the terms axioms, postulates, and definitions. In this paper, the term assumption will be used to cover both axiom and postulate.

In recent years, a considerable number of investigations have been made as to the number of assumptions required to establish euclidian geometry, and in particular, as to how far this number can be reduced.

The tendency to question the irreducibility of Euclid's assumptions seems to have begun with Euclid himself.[2] This is indicated by the fact that he avoided using his parallel axiom as long as possible. Accounts show that the ancient mathematicians did not accept Euclid's assumptions as final. Numerous attempts were made either to prove his parallel axiom or to show that it was not independent.

The first notable advance was made in 1733 by Saccheri, an Italian Jesuit priest. He constructed what he thought was a new proof for the parallel axiom by the method of reductio ad absurdum. He was the first to contemplate the possibility of an hypothesis contradicting Euclid's. Accordingly, it seems just to assign to him the honor of being the originator of non-euclidian geometry. His work, however, was forgotten until Beltrami called attention to it near the end of the nineteenth century.

1. Fundamental Concepts of Algebra and Geometry, page 2.

2. J. W. Young, Fundamental Concepts of Algebra and Geometry, page 26.

After Saccheri's time, some of the most brilliant minds of the day lent their time and energy to investigating the problem. C. F. Guass,[4] after careful thought and study, decided the parallel axiom could not be proved. He also discovered non-euclidian geometry but did not see fit to announce his discovery.

The greatest credit, however, as pioneers in this field seems to belong to John Bolyai, a Hungarian, and Nicholas Lobatchewsky, a Russian, since they not only did the most brilliant work along this line but also had the courage to give it to the public.[5]

Their work was done during the early part of the nineteenth century. They were the first to publish a new kind of geometry. While the work of these men was mainly in non-euclidian geometry, their efforts laid the foundation for showing the irreducibility of the Euclidian assumptions.

Little attention was paid to the work of Bolyai and Lobatchewsky by mathematicians until it was brought to their notice by Riemann in 1854 and R. Baltzer in 1866. There was an immediate revival of interest in the new possibilities.

It had been recognized that nine of Euclid's assumptions were arithmetical in character and so applicable to all mathematics.[7] But the researches of Beltrami in Italy and Helmnoltz and Riemann in Germany led to the conclusion that on surfaces of constant curvature there may be three geometries: Lobatchewskian on a surface of constant negative curvature, Riemannian on a surface of constant positive curvature, and Euclidian on a surface of constant zero curvature. These three geometries contradict each other logically but not practically, since, so far as yet known, all are applicable to our space.

These researches and investigations brought into prominence the underlying assumptions of mathematics. More recently, attempts have been made to formulate and discuss these assumptions by a number of leading mathematicians, notably Pasch, Hilbert, Veblen, Peano, Whitehead and Russell.

3. John Bolyai, The Science of Absolute Space, Translator's Introduction, George Bruce Halstead, XIII.

4. R. Bonola, Non-euclidian Geometry, Pages 64 - 75.

5. John Bolyai, The Science of Absolute Space, Translator's Introduction, George Bruce Halstead, XXIV-XXX.

The abstract formulation of mathematical science dates back to Moritz Pasch in 1882.[8] At least, he was the first to study in detail the point on a line and to establish the idea of betweenness. In 1893, Guiseppe Peano published a treatise called Formulario di Matematico in which the most important propositions were demonstrated in terms of pure logic. In 1899, David Hilbert[9] published his Foundations of Geometry in which he used the Euclidian assumptions to prove his propositions. In this work, he discussed the irreducibility of his set of assumptions for Euclidian geometry. This work is a valuable critical investigation of the foundations of Euclidian Geometry.

In 1904, Oswald Veblen[10] produced one of the simplest and most rigorous discussions of the subject that has been written. In a clear cut way, he bases Euclidian Geometry on two undefined terms and twelve assumptions and proves the irreducibility of his assumptions.

The most formidable attempt to examine and state the fundamental assumptions of mathematics is that of Whitehead and Russell in their four-volume work "Principia Mathematica", published by the Oxford University Press, volume one appearing in 1910.

A number of other mathematicians have devoted a great deal of time to these fundamental aspects of mathematical science, but these men seem to have made the greatest contributions in the general field and in the special field of geometry.

7. John Bolyai, The Science of Absolute Space, Translator's Introduction, George Bruce Halstead, XX-XXI.

8. Moritz Pasch, Vorlesungen über neuere Geometrie, Leipzig, 1882.

9. David Hilbert, Grundlagen der Geometrie, Göttingen, 1899. Translation by E. J. Townsend, Chicago, 1902.

10. Transactions American Mathematical Society, Vol. V, Pages 343-384.

Part II.

In order for any set of assumptions to be irreducible,[11] they must be consistent (not contradictory), independent and sufficient.

The question of consistency is very important. If they are not consistent, the structure breaks down. Our test for consistency is by means of concrete representation. As yet, no better test for consistency is known.[12]

Our assumptions are independent, if no one can be derived from any of the others by logical deductions. Our method of showing independence[13] of an assumption consists in setting up a concrete example for which the particular assumption in question is false while all of the other assumptions are true.

Assumptions are said to be sufficient if all that is desired as inferences from them can be established without the use of further assumptions. It will be assumed in this paper that the given sets of assumptions are sufficient.

The notions of class and belonging to a class are essential in any mathematical science.[14]

We assume a set S of elements E_1, E_2, ... E_n, finite in number, having certain sub-sets called m-classes and characterized in the first discussion by the set of assumptions A and in the second discussion by the set of assumptions B given below.

Assumptions A.

I. If E_1 and E_2 are distinct elements of S, there is at least one m-class containing both E_1 and E_2.

II. If E_1 and E_2 are distinct elements of S, there is not more than one m-class containing both E_1 and E_2.

III. Every two m-classes have at least one element of S in common.

11. Veblen and Young, Projective Geometry, Vol. I, Pages 2 - 6.

12. Ibid., Page 3, footnote, and references there cited.

13. J. W. Young, Fundamental Concepts of Algebra and Geometry, Page 43.

14. Ibid., Page 47.

IV. There exists at least one m-class.

V. The elements of S do not all belong to the same m-class.

VI. Every m-class contains at least P+1 elements of S, where P is an arbitrary prime number.

VII. No m-class contains more than P+1 elements of S.

Assumptions B.

I. If E_1 and E_2 are distinct elements of S, there is at least one m-class containing both E_1 and E_2.

II. If E_1 and E_2 are distinct elements of S, there is not more than one m-class containing both E_1 and E_2.

III. There exists at least one m-class.

IV. Every m-class contains at least P elements of S, where P is an arbitrary odd prime.

V. No m-class contains more than P elements of S.

VI. For every m-class, there exists at least one conjugate m-class.

Definition--Two m-classes with no element of S in common are called conjugates of each other.

We now proceed to the proof of the consistency and independence of each of these two sets of assumptions.

Assumptions A.

To illustrate clearly the method and for the sake of ease in following the argument in the general case, the proofs will be given in detail for the cases P=2, P=3, and P=5. These will be followed by the proof for an arbitrary prime P.

Case P=2.

We wish to show assumptions A are consistent and independent when P=2 and P+1=3.

Consistency. Take as the set S the numbers (1,2,3, 4,5,6,7) arranged in the 7 m-classes as follows:

1	1	2	2	3	3	1
2	4	4	5	4	5	6
3	5	6	7	7	6	7.

These 7 elements in the 7 m-classes indicated satisfy assumptions A, and the assumptions are therefore consistent when P=2.

Independence of I. Take as the set S the 5 elements $(E_1, E_2, E_3, E_4, E_5)$ arranged in the two m-classes $(E_1E_2E_3)$ and $(E_1E_4E_5)$. These five elements and two m-classes satisfy all of assumptions A except I and I is false.

Independence of II. Take as the set S the 4 elements (E_1, E_2, E_3, E_4) arranged in the 3 m-classes as follows: $(E_1E_2E_3)$, $(E_1E_2E_4)$ and $(E_2E_3E_4)$. These 4 elements and 3 m-classes satisfy all of assumptions A except II and II is false.

Independence of III. Take as the set S the 9 elements of the determinant,

$$\begin{matrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{matrix}$$

and the following 12 m-classes will be found to satisfy all of assumptions A except III and III will be false:

The three classes $(A_1A_2A_3)$, $(B_1B_2B_3)$, $(C_1C_2C_3)$ formed from columns;

The three classes $(A_1B_1C_1)$, $(A_2B_2C_2)$, $(A_3B_3C_3)$ formed from rows;

The six classes $(A_1B_2C_3)$, $(A_3B_2C_1)$, $(A_1B_3C_2)$, $(A_2B_1C_3)$, $(A_2B_3C_1)$ and $(A_3B_1C_2)$ formed from the determinant, each m-class containing one and only one element from each row and column.

Independence of IV. If we deny IV, IV is false but all of the others are satisfied vacuously.

Independence of V. Take as the set S the 3 elements (E_1, E_2, E_3) arranged in the single m-class $(E_1E_2E_3)$. These three elements and one m-class satisfy assumptions I, II, IV, VI and VII; III is satisfied vacuously and V is false.

Independence of VI. Take as the set S the 3 elements (E_1, E_2, E_3) arranged in the three m-classes (E_1E_2), (E_1E_3), (E_2E_3) and all assumptions of set A are satisfied except VI and VI is false.

Independence of VII. Take as the set S the numbers (1,2,3,4........,13) arranged in the 13 m-classes as follows:

1	1	2	2	2	3	3	3	4	4	4	1	1
2	5	5	6	7	5	6	7	5	6	7	8	9
3	6	8	10	12	10	12	8	12	8	10	10	11
4	7	9	11	13	13	9	11	11	13	9	12	13.

These 13 elements and the 13 m-classes indicated satisfy all of assumptions A except VII and VII is false.

Therefore assumptions A are independent when P=2. Since assumptions A are consistent and independent, they are irreducible when P=2.

Case P=3.

We wish to show assumptions A are consistent and independent when P=3 and P+1=4.

Consistency. Take as the set S the numbers (1,2,3, 4....,13) arranged in 13 m-classes as follows:

1	1	2	2	2	3	3	3	4	4	4	1	1
2	5	5	6	7	5	6	7	5	6	7	8	9
3	6	8	10	12	10	12	8	12	8	10	10	11
4	7	9	11	13	13	9	11	11	13	9	12	13.

These 13 elements and 13 m-classes indicated satisfy assumptions A, and the assumptions are therefore consistent when P=3.

Independence of I. Take as the set S the 7 elements $(E_1, E_2, E_3, E_4, E_5, E_6, E_7)$ arranged in the two m-classes $(E_1E_2E_3E_4)$ and $(E_1E_5E_6E_7)$. These 7 elements and 2 m-classes satisfy all of assumptions A except I and I is false.

Independence of II. Take as the set S the 6 elements $(E_1, E_2, E_3, E_4, E_5, E_6)$ arranged in the three m-classes $(E_1E_2E_3E_4)$, $(E_1E_2E_5E_6)$ and $(E_3E_4E_5E_6)$. These 6 elements and 3 m-classes satisfy all of assumptions A except II and II is false.

Independence of III. Take as the set S the 16 elements of the determinant:

$$\begin{matrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4, \end{matrix}$$

and the following 20 m-classes will be found to satisfy all of assumptions of set A except III and III will be false:

The four m-classes $(A_1B_1C_1D_1)$, $(A_2B_2C_2D_2)$, $(A_3B_3C_3D_3)$ and $(A_4B_4C_4D_4)$ formed from rows;

The four m-classes $(A_1A_2A_3A_4)$, $(B_1B_2B_3B_4)$, $(C_1C_2C_3C_4)$

and $(D_1D_2D_3D_4)$ formed from columns;

The twelve m-classes formed from the determinant, each m-class containing one and only one element from each row and column, each column an m-class:

$A_1A_1A_1$	$A_2A_2A_2$	$A_3A_3A_3$	$A_4A_4A_4$
$B_2B_3B_4$	$B_4B_3B_1$	$B_1B_2B_4$	$B_3B_2B_1$
$C_3C_4C_2$	$C_3C_1C_4$	$C_2C_4C_1$	$C_2C_1C_3$
$D_4D_2D_3$	$D_1D_4D_3$	$D_4D_1D_2$	$D_1D_3D_2$

Independence of IV. If we deny IV, IV is false but all of the others are satisfied vacuously.

Independence of V. Take as the set S the elements (E_1, E_2, E_3, E_4) arranged in the single m-class $(E_1E_2E_3E_4)$. The 4 elements and the m-class satisfy assumptions I, II, IV, VI and VII, III is satisfied vacuously and V is false.

Independence of VI. Take as the set S the numbers 1, 2,3,4,5,6,7 arranged in the 7-m-classes as follows:

1	1	2	2	3	3	1
2	4	4	5	4	5	6
3	5	6	7	7	6	7.

These 7 elements and the 7 m-classes indicated satisfy all assumptions of the set A except VI and VI is false.

Independence of VII. Take as the set S the numbers (1,2,3,4........,31) arranged in the 31 m-classes as follows:

1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
2	7	7	8	9	10	11	7	8	9	10	11	7	8	9	10	11
3	8	12	16	20	24	28	16	20	24	28	12	20	24	28	12	16
4	9	13	17	21	25	29	21	25	29	13	17	29	13	17	21	25
5	10	14	18	22	26	30	26	30	14	18	22	18	22	26	30	14
6	11	15	19	23	27	31	31	15	19	23	27	27	31	15	19	23

5	5	5	5	5	6	6	6	6	6	1	1	1	1
7	8	9	10	11	7	8	9	10	11	12	13	14	15
24	28	12	16	20	28	12	16	20	24	16	17	18	19
17	21	25	29	13	25	29	13	17	21	20	21	22	23
30	14	18	22	26	22	26	30	14	18	24	25	26	27
23	27	31	15	19	19	23	27	31	15	28	29	30	31.

These 31 elements and the 31 m-classes indicated satisfy all of the assumptions of the set A except VII and VII is false.

Therefore assumptions A are independent when P=3. Since they are consistent and independent, assumptions A are irreducible when P=3.

Case P=5.

We wish to show assumptions A are consistent and independent when P=5.

Consistency. Take as the set S the 31 numbers and the 31 m-classes used in proving independence of VII in case P=3.

These 31 elements and 31 m-classes indicated satisfy assumptions A, and assumptions A are therefore consistent when P=5.

Independence of I. Take as the set S the elements $(E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9, E_{10}, E_{11})$ arranged in the two m-classes $(E_1E_2E_3E_4E_5E_6)$ and $(E_1E_7E_8E_9E_{10}E_{11})$. These 11 elements and 2 m-classes satisfy all of assumptions A except I and I is false.

Independence of II. Take as the set S the elements $(E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9)$ arranged in the three m-classes $(E_1E_2E_3E_4E_5E_6)$, $(E_1E_2E_3E_7E_8E_9)$ and $(E_4E_5E_6E_7E_8E_9)$. These 9 elements and the 3 m-classes satisfy all of assumptions A except II and II is false.

Independence of III. Take as the set S the 36 elements of the determinant D_6 in which double subscripts are used:

$$D_6 = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} & A_{04} & A_{05} \\ A_{10} & A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{20} & A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{30} & A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{40} & A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{50} & A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{pmatrix}.$$

As a means of selecting the m-classes, we will make use of modular equations. In modular equations, the idea of congruence is used and the relation between the two members of the equation is expressed by the sign "$\equiv$" read "congruent to" instead of the sign "=". For example, all integers are congruent to 0 or 1, modulus 2, which means that all integers divided by 2 give a remainder of 0 or 1; and all integers are congruent to 0, 1 or 2, modulus 3, which means that all integers divided by 3 give a remainder of 0, 1 or 2. Thus we have $2\equiv 0$, $3\equiv 1$, $5\equiv 1$, modulus 2; $3 \equiv 0$, $5 \equiv 2$, modulus 3; $4 \equiv 0$, $7 \equiv 3$, modulus 4; $7 \equiv 1$, $13 \equiv 1$, $7 \equiv 13$, modulus 6, etc.

We select for the m-classes the terms in the expansion of the determinant D_6 such that the elements, A_{xy}, belonging to an m-class are all those which have subscripts satisfying a congruence equation of the form $X \equiv K$ or $Y \equiv MX+K$, modulus 6, where M and K are the numbers (0,1, 2,3,4,5); (i.e. residues modulus 6). There are exactly 42 of these equations. For assumptions VI and VII show that each m-class must have 6 and only 6 elements of S, and assumptions I and II show that there must be a distinct m-class for each pair of elements of D_6. Since the first element of the m-classes can be chosen in 36 different ways, and the second element in 35 different ways, and since there are 30 ways of choosing pairs of elements within an m-class, there must be $36 \times 35 \div 30$ or 42 distinct m-classes. This follows from permutations and combinations.

The following page gives a map of the concrete examples of m-classes so chosen that the modular equations $X \equiv K$ and $Y \equiv MX+K$, modulus 6, are satisfied for all the values of K and M where K and M are the numbers (0,1,2,3,4,5).

The pairs of numbers at the top of the page represent A_{xy} of the determinant D_6. The X equations on the left represent rows and the y equations represent columns in the determinant D_6. The crosses in the small squares represent incidence of rows and columns, while the rows of crosses represent the m-classes.

To show that the set S of 36 points satisfy all of the assumptions except III, it is only necessary to notice:

I and II are true since E_1 and E_2 are found once and only once in each m-class;

IV and V are obviously true;

VI and VII are true since each m-class contains exactly 6 elements.

Independence of IV. If we deny IV, IV is false and all of the others are satisfied vacuously.

Independence of V. Take as the set S the 6 elements $(E_1,E_2,E_3,E_4,E_5,E_6)$ arranged in the single m-class $(E_1E_2 E_3E_4E_5E_6)$. These 6 elements and the m-class satisfy assumptions I, II, IV, VI and VII, III is satisfied vacuous-

	0,0	0,1	0,2	0,3	0,4	0,5	1,0	1,1	1,2	1,3	1,4	1,5	2,0	2,1	2,2	2,3	2,4	2,5	3,0	3,1	3,2	3,3	3,4	3,5	4,0	4,1	4,2	4,3	4,4	4,5	5,0	5,1	5,2	5,3	5,4	5,5
X ≡ 1							x	x	x	x	x	x																								
X ≡ 2													x	x	x	x	x	x																		
X ≡ 3																			x	x	x	x	x	x												
X ≡ 4																									x	x	x	x	x	x						
X ≡ 5																															x	x	x	x	x	x
X ≡ 0	x	x	x	x	x	x																													
Y ≡ 1		x							x							x							x							x							x				
Y ≡ 2			x							x							x							x							x							x			
Y ≡ 3				x							x							x							x							x							x		
Y ≡ 4					x							x							x							x							x							x	
Y ≡ 5						x							x							x							x							x							x
Y ≡ 0	x						x						x						x						x						x					
Y ≡ X	x							x							x							x							x							x
Y ≡ X+1		x							x							x							x							x	x					
Y ≡ X+2			x							x							x							x	x							x				
Y ≡ X+3				x							x							x	x							x							x			
Y ≡ X+4					x							x	x							x							x							x		
Y ≡ X+5						x	x							x							x							x							x	
Y ≡ 2X	x								x							x		x								x									x	
Y ≡ 2X+1		x							x							x		x							x							x						x
Y ≡ 2X+2			x							x		x							x							x		x					
Y ≡ 2X+3				x							x		x							x							x		x				
Y ≡ 2X+4					x		x							x							x		x							x			
Y ≡ 2X+5						x		x							x							x		x							x		
Y ≡ 3X	x								x			x								x			x								x		
Y ≡ 3X+1		x							x			x								4			x								x	
Y ≡ 3X+2			x							x			x								x			x								x
Y ≡ 3X+3				x		x								x			x								x			x					
Y ≡ 3X+4					x		x								x			x								x			x				
Y ≡ 3X+5						x			x								x			x								x			x			
Y ≡ 4X	x									x					x			x									x					x			
Y ≡ 4X+1		x									x					x			x									x					x		
Y ≡ 4X+2			x			x									x					x			x									x	
Y ≡ 4X+3				x			x									x					x			x									x
Y ≡ 4X+4					x					x			x									4					x			x					
Y ≡ 4X+5						x					x			x									x					x			x				
Y ≡ 5X	x											x					x					x					x					x				
Y ≡ 5X+1		x					x										x					x					x					x			
Y ≡ 5X+2			x					x					x										x					x					x		
Y ≡ 5X+3				x					x					x					x										x					x	
Y ≡ 5X+4					x					x					x					x					x										x
Y ≡ 5X+5						x					x					x					x					x					x					

ly and V is false.

Independence of VI. Take as the set S the 57 numbers (1,2,3,4.........,57) arranged in the 57 m-classes as follows:

1	1	2	2	2	2	2	2	2	3	3	3	3	3	3	3
2	9	9	10	11	12	13	14	15	9	10	11	12	13	14	15
3	10	16	22	28	34	40	46	52	22	28	34	40	46	52	16
4	11	17	23	29	35	41	47	53	29	35	41	47	53	17	23
5	12	18	24	30	36	42	48	54	36	42	48	54	18	24	30
6	13	19	25	31	37	43	49	55	43	49	55	19	25	31	37
7	14	20	26	32	38	44	50	56	50	56	20	26	32	38	44
8	15	21	27	33	39	45	51	57	57	21	27	33	39	45	51

4	4	4	4	4	4	4	5	5	5	5	5	5	5
9	10	11	12	13	14	15	9	10	11	12	13	14	15
28	34	40	46	52	16	22	34	40	46	52	16	22	28
41	47	53	17	23	29	35	53	17	23	29	35	41	47
54	18	24	30	36	42	48	30	36	42	48	54	18	24
25	31	37	43	49	55	19	49	55	19	25	31	37	43
38	44	50	56	20	26	32	26	32	38	44	50	56	20
51	57	21	27	33	39	45	45	51	57	21	27	33	39

6	6	6	6	6	6	6	7	7	7	7	7	7	7
9	10	11	12	13	14	15	9	10	11	12	13	14	15
40	46	52	16	22	28	34	46	52	16	22	28	34	40
23	29	35	41	47	53	17	35	41	47	53	17	23	29
48	54	18	24	30	36	42	24	30	36	42	48	54	18
31	37	43	49	55	19	25	55	19	25	31	37	43	49
56	20	26	32	38	44	50	44	50	56	20	26	32	38
39	45	51	57	21	27	33	33	39	45	51	57	21	27

8	8	8	8	8	8	8	1	1	1	1	1	1
9	10	11	12	13	14	15	16	17	18	19	20	21
52	16	22	28	34	40	46	22	23	24	25	26	27
47	53	17	23	29	35	41	28	29	30	31	32	33
42	48	54	18	24	30	36	34	35	36	37	38	39
37	43	49	55	19	25	31	40	41	42	43	44	45
32	38	44	50	56	20	26	46	47	48	49	50	51
27	33	39	45	51	57	21	52	53	54	55	56	57.

These 57 elements and the 57 m-classes satisfy all of assumptions A except VII and VII is false.

Therefore assumptions A are independent when P=5. Since they are also consistent, assumptions A are irreducible when P=5.

Case P = An Arbitrary Prime.

We wish to show assumptions A are consistent and independent when P is an arbitrary prime.

Consistency. Take as the set S the P^2 elements A_{xy}($X=0,1,2.....,P-1$; $Y=0,1,2.....,P-1$) of D_{P-1} as given below and the elements $(A_{0P}, A_{1P}, A_{2P}.....A_{PP})$, and for the m-classes of the set S the P^2+P+1 classes as shown in a subsequent paragraph.

In arranging the general set S, each of whose m-classes contains P-1 elements, we will first expand the determinant D_{P-1} with double subscripts with only P elements to the row and column:

$$D_{P-1}=\begin{array}{cccccc} A_{00} & A_{01} & A_{02} & A_{03} & \cdots\cdots & A_{0(P-1)} \\ A_{10} & A_{11} & A_{12} & A_{13} & \cdots\cdots & A_{1(P-1)} \\ A_{20} & A_{21} & A_{22} & A_{23} & \cdots\cdots & A_{2(P-1)} \\ A_{30} & A_{31} & A_{32} & A_{33} & \cdots\cdots & A_{3(P-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ A_{(P-1)0} & A_{(P-1)1} & A_{(B-1)2} & A_{(P-1)3} & \cdots\cdots & A_{(P-1)(P-1)}. \end{array}$$

For the sake of convenience, we will let the set S' represent the elements of D_{P-1}.

We have a determinant of P^2 elements. From assumptions I and II, it follows there must be an m-class for each distinct pair of elements and, from assumptions VI and VII, each m-class must contain P and only P elements of D_{P-1}. Since the first element of each m-class of the set S' can be chosen in P^2 ways and the second element in P^2-1 ways, and since there are $(P-1)P$ ways of choosing pairs of elements within an m-class, there must be exactly $\frac{P^2(P^2-1)}{P(P-1)} = P(P+1)$ distinct m-classes in set S'.

To select the P^2+P m-classes of set S' from D_{P-1}, we make use of modular equations as was done in the independence proof of assumption III for the case $P=5$.

We take for our m-classes of the set S' the terms in the expansion of D_{P-1} such that the elements belonging to an m-class are those whose subscripts satisfy a congruence equation of the form $X\equiv K$ or $Y\equiv MX+K$, modulus P, where M and K represent the numbers $(0,1,2,3.....,P-1)$. For there are P^2+P congruence equations of the form $X\equiv K$ and $Y\equiv MX+K$; P^2 of the form $Y\equiv MX+K$ and P of form $X\equiv K$.

From the m-classes thus made up from the set S', we derive the general set of P^2+P+1 m-classes as follows: to each m-class determined by a congruence equation $X \equiv K$ modulus P, where K has the values $(0,1,2,3,\ldots\ldots,P-1)$, we add the element A_{0P}; to the m-classes determined by the congruence equations $Y \equiv MX+K$, modulus P, where M and K have the values $(0,1,2,3,\ldots\ldots,P-1)$, we add the elements $(A_{1P}, A_{2P}, A_{3P}, \ldots\ldots\ldots A_{PP})$ successively one to each. In other words, to the m-classes determined by the congruence equations $Y \equiv MX+K$, modulus P when M is 0 and K has the values $(0,1,2,3,\ldots\ldots,P-1)$, we add the element A_{1P}; to the m-classes determined by the congruence equations $Y \equiv MX+K$, modulus P, where M is 1 and K has the values $(0, 1,2,3,\ldots\ldots,P-1)$, we add the element A_{2P} and so on for the remaining congruence equations and the remaining elements, $(A_{3P}, A_{4P}, \ldots\ldots, A_{PP})$.

To show that these m-classes satisfy assumptions A, it is only necessary to notice:

Assumptions I and II are satisfied, since, in the m-classes determined by the congruence equations $X \equiv K$ and $Y \equiv MX+K$, modulus P, where M and K have the values $(0,1,2, 2,\ldots\ldots,P-1)$, each pair of elements is found once and only once in an m-class, and, after elements $(A_{0P}, A_1), A_2), \ldots\ldots \ldots\ldots, A_{PP})$ are added, each m-class will have exactly $P+1$ elements;

Assumption III is satisfied after the addition of the elements $(A_{0P}, A_{1P}, A_{2P} \ldots\ldots, A_{PP})$ since each pair of m-classes has one and only one element of S in common;

Assumptions VI and VII are satisfied, since every congruence equation of the form $X \equiv K$ or $Y \equiv MX+K$, modulus P, where M and K have the values $(0,1,2,3\ldots\ldots,P-1)$ are satisfied by exactly P pairs of values of (X,Y); and, when the extra element is added, each m-class will have exactly $P+1$ elements. Hence assumptions A are consistent for an arbitrary prime P.

Independence of I. Take as the set S the elements $E_1, E_2, E_3, \ldots\ldots, E_{P+1}, E_{P+2}, \ldots\ldots, E_{2P+1}$ and the two m-classes $(E_1 E_2 E_3 \ldots\ldots E_{P+1})$ and $(E_{P+1} E_{P+2} E_{P+3} \ldots\ldots E_{2P+1})$. These $2P+1$ elements and two m-classes satisfy all of assumptions A except I and I is false.

Independence of II. Take as the set S the elements $E_1, E_2, E_3, \ldots\ldots E_{P+1}, E_{P+2}, \ldots\ldots E_{2P+1}$ and the three m-classes $(E_1 E_2 E_3 E_4 \ldots E_{P+1})$, $(E_1 E_2 E_3 E_4 \ldots\ldots E_{2P+1})$ and $(\ldots\ldots E_{P+1} \ldots \ldots E_{2P+1})$ so arranged that each m-class has $P+1$ elements. These $2P+1$ elements and 3 m-classes satisfy all of assumptions A except II and II is false.

Independence of III. Take as the set S the $(P+1)^2$ elements of the determinant D_P,

$$D_P = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \cdots\cdots\cdots\cdots & A_{0P} \\ A_{10} & A_{11} & A_{12} & A_{13} \cdots\cdots\cdots\cdots & A_{1P} \\ A_{20} & A_{21} & A_{22} & A_{23} \cdots\cdots\cdots\cdots & A_{2P} \\ A_{30} & A_{31} & A_{32} & A_{33} \cdots\cdots\cdots\cdots & A_{3P} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{P0} & A_{P1} & A_{P2} & A_{P3} \cdots\cdots\cdots\cdots & A_{PP} \end{pmatrix},$$

and for the m-classes of the set S the $(P+1)(P+2)$ terms in the expansion of D_P.

From assumptions VI and VII, there must be exactly $P+1$ elements in every m-class of the set S; and from assumptions I and II, there must be an m-class for each distinct pair of elements A_{XY} of D_P. Since the first element can be selected in $(P+1)^2$ ways and the second in $(P+1)^2-1$ ways and since there are $P(P+1)$ ways of choosing pairs of elements within an m-class, there must be $\dfrac{((P+1)^2)\ ((P+1)^2-1)}{P(P+1)} = (P+1)(P+2)$ distinct m-classes.

We select for our m-classes the terms in the expansion of D_P such that the elements A_{XY} belonging to an m-class are all those whose subscripts satisfy a congruence equation of form $X \equiv K$ or $Y \equiv MX+K$, modulus $P+1$, where M and K are the numbers $(0,1,2,3,\ldots\ldots,P)$. For there are exactly $(P+1)(P+2)$ congruence equations of the form $X \equiv K$ or $Y \equiv MX+K$, modulus $P+1$; $P+1$ of the form $X \equiv K$ and $(P+1)^2$ of form $Y \equiv MX+K$ and each equation is satisfied by $P+1$ pairs of values $(X;Y)$.

To show that the set S of $(P+1)^2$ points (X,Y) and $(P+1)(P+2)$ lines as we have chosen them satisfy all of assumptions A except III, it is only necessary to notice:

Assumption I is satisfied since a congruence equation of the form $X \equiv K$ or $Y \equiv MX+K$ has the K, or the $M+K$, completely and uniquely determined by two pairs of subscripts (X,Y) which satisfy it;

(a) Assumption II is satisfied since, to the modulus $P+1$, no subscript (X,Y) which satisfy an equation of the form $X \equiv K_1$ can satisfy an equation $X \equiv K_2$ where $K_1 \not\equiv K_2$. (Similar argument applies to the equation $Y \equiv K$);

(b) No two distinct subscripts (X_1,Y_1) and (X_1,Y_2)

which satisfy an equation of the form $X \equiv X_1$ can both satisfy an equation of the form $Y \equiv MX+K$ since from $Y_1 \equiv MX_1+K$ and $Y_2 \equiv MX_1-K$ would follow $Y_1 \equiv Y_2$, contrary to the supposition that (X_1,Y_1) and (X_1,Y_2) are distinct.

(c) No two distinct subscripts (X_1,Y_1) and (X_2,Y_2) can both satisfy two distinct equations of the form $Y \equiv M_1X+K_1$, and $Y \equiv M_2X+K_2$ (M_1 and M_2 not both zero) for, if so, from $Y_1 \equiv M_1X_1+K_1$ and $Y_1 \equiv M_2X_1+K_2$ follows $X_1 \equiv \frac{K_2-K_1}{M_1-M_2}$, and from

$Y_2 \equiv M_1X_2+K_2$ and $Y \equiv M_2X_2+K_2$ similarly $X_2 \equiv \frac{K_2-K_1}{M_1-M_2}$, and hence

$X_1 \equiv X_2$ and from $Y_1 \equiv M_1X_1+K_1$ and $Y_2 \equiv M_1X_2+K_1$ would follow $Y_1 \equiv Y_2$ contrary to supposition. ($M_1=M_2$ gives immediately $K_1=K_2$ contrary to supposition);

Assumption III is false since the m-classes determined by $X \equiv K_1$ and $X \equiv K_2$ have no elements in common if $K_1 \neq K_2$, for example $X \equiv 1$ and $X \equiv 2$, modulus $P+1$;

Assumptions IV and V are obviously true;

Assumptions VI and VII are satisfied since every congruence equation of form $X \equiv K$ or $Y \equiv MX+K$, modulus $P+1$, is satisfied by exactly $P+1$ pairs of values (X,Y).

Independence of IV. If we deny IV, IV is false and all of the others are satisfied vacuously.

Independence of V. Take as the set S the elements $(E_1, E_2, E_3 \ldots\ldots E_{P+1})$ and the single m-class $(E_1E_2E_3\ldots\ldots E_{P+1})$. These $P+1$ elements and the one m-class satisfy assumptions I, II, IV, VI and VII, III is satisfied vacuously, and V is false.

Independence of VI. Take as the set S the 31 numbers $(1,2,3,4,\ldots\ldots,31)$ and the 31 m-classes as found on page 12, and assumption VI is false and all of the others are true.

Independence of VII. Since P is an arbitrary prime, we can choose P_1 a larger prime. Hence we can determine a set S'' of $P_1^2+P_1+1$ elements and $P_1^2+P_1+1$ m-classes that will satisfy all of assumptions A, except VII and VII will be false.

Therefore assumptions A are independent for an arbitrary prime P. Since they are also consistent, assumptions A are irreducible for an arbitrary prime P.

Assumptions B.

To illustrate clearly the method and for the sake of ease in following the argument in the general case, the proofs will be given in detail for the cases P=3 and P=5. These will be followed by the proof for an arbitrary prime.

Case P=3.

We wish to show that assumptions B are consistent and independent when P=3.

Consistency. Take as the set S the numbers (1,2,3, 4,5,6,7,8,9) arranged in the 12 m-classes as follows:

1	4	1	1	1	2	2	2	3	3	3	7
2	5	4	5	6	4	5	6	4	5	6	8
3	6	7	8	9	8	9	7	9	7	8	9.

These 9 elements and the 12 m-classes indicated satisfy assumptions B and the assumptions are therefore consistent when P=3.

Independence of I. Take as the set S the elements (1,2,3,4,5,6,7,8,9) and the m-classes as given in showing consistency above, omitting the last m-class. These 9 elements and 11 m-classes satisfy all of assumptions B except I and I is false.

Independence of II. Take as the set S the numbers (1,2,3, 4,5,6,7,8,9) and the m-classes as given in showing consistency above increased by the m-class (1,4,8). These 9 elements and 13 m-classes satisfy all of assumptions B except II and II is false.

Independence of III. If we deny III, III is false and all of the others are satisfied vacuously.

Independence of IV. Take as the set S the 4 elements (E_1,E_2,E_3,E_4) arranged in the 6 m-classes, (E_1E_2), (E_3E_4), (E_1E_3), (E_1E_4), (E_2E_3) and (E_2E_4). These 4 elements and 6 m-classes satisfy all of assumptions B except IV and IV is false.

Independence of V. Take as the set S the numbers (1,2,3,4,5........,25) arranged in the 30 m-classes as follows:

1	6	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3
2	7	6	7	8	9	10	6	7	8	9	10	6	7	8	9	10
3	8	11	14	17	20	23	14	17	20	23	11	17	20	23	11	14
4	9	12	15	18	21	24	18	21	24	12	15	24	12	15	18	21
5	10	13	16	19	22	25	22	25	13	16	19	16	19	22	25	13

4	4	4	4	4	5	5	5	5	5	11	12	13
6	7	8	9	10	6	7	8	9	10	14	15	16
20	23	11	14	17	23	11	14	17	20	17	18	19
15	18	21	24	12	21	24	12	15	18	20	21	22
25	13	16	19	22	19	22	25	13	16	23	24	25.

These 25 elements in the 30 m-classes satisfy all of the assumptions of set B except V and V is false.

Independence of VI. Take as the set S the 7 numbers (1,2,3,4,5,6,7) arranged in 7 m-classes as follows:

1	1	2	2	3	3	1
2	4	4	5	4	5	6
3	5	6	7	7	6	7.

These 7 elements and 7 m-classes satisfy all of assumptions B except VI and VI is false. Since they are also consistent, assumptions B are irreducible when P=3.

Case P = 5.

We wish to show in this case that assumptions B are consistent and independent.

Consistency. Take as the set S the numbers (1,2,3, 4,5,6,7,.........,25) and the 30 m-classes given in the independence proof of V, case P=3. These 25 elements and 30 m-classes satisfy assumptions B, and assumptions B are therefore consistent.

Independence of I. Take as the set S the numbers (1,2,3,4,5.....,25) arranged in the 30 m-classes as given in the independence proof of V, case P=3, omitting the the last m-class. These 25 elements and 29 m-classes satisfy all of assumptions B except I and I is false.

Independence of II. Take as the set S the numbers (1,2,3,4,5.....,25) and the m-classes given in showing consistency of P=5 increased by the m-class (1,6,16,17,24). These 25 elements and 31 m-classes satisfy all of assumptions B except II and II is false.

Independence of III. If we deny III, III is false and all of the others are satisfied vacuously.

Independence of IV. Take as the set S the numbers (1,2,3, 4,5,6,7,8,9) arranged in the m-classes as given showing consistency when P=3. These 9 elements and 12 m-classes satisfy all of assumptions B except IV and IV is false.

Independence of V. Take as the set S the numbers (1,2,3,4,5,6......,49) arranged in the following 56 m-classes:

1	8	1	1	1	1	1	1	1	2	2	2	2	2	2	2
2	9	8	9	10	11	12	13	14	8	9	10	11	12	13	14
3	10	15	20	25	30	35	40	45	20	25	30	35	40	45	15
4	11	16	21	26	31	36	41	46	26	31	36	41	46	16	21
5	12	17	22	27	32	37	42	47	32	37	42	47	17	22	27
6	13	18	23	28	33	38	43	48	38	43	48	18	23	28	33
7	14	19	24	29	34	39	44	49	44	49	19	24	29	34	39

3	3	3	3	3	3	3	4	4	4	4	4	4	4
8	9	10	11	12	13	14	8	9	10	11	12	13	14
25	30	35	40	45	15	20	30	35	40	45	15	20	25
36	41	46	16	21	26	31	46	16	21	26	31	36	41
47	17	22	27	32	37	42	27	32	37	42	47	17	22
23	28	33	38	43	48	18	43	48	18	23	28	33	38
34	39	44	49	19	24	29	24	29	34	39	44	49	19

5	5	5	5	5	5	5	6	6	6	6	6	6	6
8	9	10	11	12	13	14	8	9	10	11	12	13	14
35	40	45	15	20	25	30	40	45	15	20	25	30	35
21	26	31	36	41	46	16	31	36	41	46	16	21	26
42	47	17	22	27	32	37	22	27	32	37	42	47	17
28	33	38	43	48	18	23	48	18	23	28	33	38	43
49	19	24	29	34	39	44	39	44	49	19	24	29	34

7	7	7	7	7	7	7	15	16	17	18	19
8	9	10	11	12	13	14	20	21	22	23	24
45	15	20	25	30	35	40	25	26	27	28	29
41	46	16	21	26	31	36	30	31	32	33	34
37	42	47	17	22	27	32	35	36	37	38	39
33	38	43	48	18	23	28	40	41	42	43	44
29	34	39	44	49	19	24	45	46	47	48	49.

These 49 elements and 56 m-classes satisfy all of assumptions B except V and V is false.

Independence of VI. Take as the set S the numbers (1,2,3.....,21) arranged in the 21 m-classes as follows:

1	1	2	2	2	2	3	3	3	3	4	4	4	4
2	6	6	7	8	9	6	7	8	9	6	7	8	9
3	7	10	13	16	19	13	10	19	16	19	16	13	10

4	8	11	14	17	20	17	20	11	14	14	11	20	17
5	9	12	15	18	21	21	18	15	12	18	21	12	15

5	5	5	5	1	1	1
6	7	8	9	10	11	12
16	19	10	13	13	14	15
20	17	14	11	16	17	18
15	12	21	18	19	20	21

These 21 elements and 21 m-classes satisfy all of assumptions B except VI and VI is false. Therefore, assumptions B are independent when $P=5$. Since they are also consistent, assumptions B are irreducible when $P=5$.

General Case P = An Arbitrary Prime.

We are now prepared to show that assumptions B are consistent and independent when P is an arbitrary prime.

Consistency. Take as the set S the elements A_{xy}($x=0, 1, 2, \ldots, P-1$; $y=0, 1, 2, \ldots, P-1$) of D_{P-1}, and for m-classes of the set S the P^2+P terms in the expansion of D_{P-1} to be selected as explained later:

$$D_{P-1}: \left(\begin{array}{cccccc} A_{00} & A_{01} & A_{02} & A_{03} & \cdots\cdots\cdots & A_{0P-1} \\ A_{10} & A_{11} & A_{12} & A_{13} & \cdots\cdots\cdots & A_{1P-1} \\ A_{20} & A_{21} & A_{22} & A_{23} & \cdots\cdots\cdots & A_{2P-1} \\ A_{30} & A_{31} & A_{32} & A_{33} & \cdots\cdots\cdots & A_{3P-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ A_{P-10} & A_{P-11} & A_{P-12} & A_{P-13} & & A_{P-1P-1}. \end{array} \right)$$

From assumptions IV and V, every m-class must contain exactly P elements of the set S. From assumptions I and II, there must be an m-class for each distinct pair of elements A_{xy}. Since the first pair of elements can be chosen in P^2 ways and the second in P^2-1 ways and since there $P(P-1)$ ways of choosing pairs of elements within an m-class, there must be exactly $\frac{P^2(P^2-1)}{(P-1)P} = P^2+P$ distinct m-classes.

We select for our m-classes the terms in the expansion of D_{P-1} such that the elements A_{xy} belonging to an m-class are all those whose subscripts satisfy a congruence equation of form $X \equiv K$ or $Y \equiv MX+K$, modulus P, where M and K have the values $(0, 1, 2, 3, \ldots, P-1)$. For there are exactly P^2+P congruence equations of the form $X \equiv K$ or $Y \equiv MX+K$, modulus P; P of the form $X \equiv K$ and P^2 of the form $Y \equiv MX+K$ and each equation is satisfied by exactly P pairs of values (X,Y).

To show that the set S of P^2 elements (X,Y) and P^2+P m-classes as we have chosen them, satisfy all of assumptions B, it is only necessary to notice:

Assumptions I and II are satisfied, since, in the m-classes determined by the congruence equations $X\equiv K$ and $Y\equiv MX+K$, modulus P, where M and K have all of the possible values (0,1,2,3.....,P-1), every E_1 and E_2 can be found once and only once satisfying a congruence equation;

Assumptions III, IV, and V are obviously satisfied;

Assumption VI is satisfied since the different sets of m-classes $X\equiv K_1$ and $X\equiv K_2$ each have a conjugate and since the different sets of m-classes $Y\equiv M_1X+K_1$ and $Y\equiv M_2X+K_2$ each have a conjugate since $M_1\neq M_2$ and $K_1\neq K_2$.

Independence of I. Take as the set S the P pairs of elements A_{xy} and for m-classes those given in showing consistency above, omitting the class $X\equiv K_1$. Assumption I, then, is false and all of the others are true.

Independence of II. Take as the set S the P pairs of elements A_{xy} and the m-classes given in showing consistency except that we repeat an m-class. Assumption II is false and all of the others are true.

Independence of III. If we deny III, III is false and all of the others are satisfied vacuously.

Independence of IV. Take as the set S the numbers (0, 1,2,3,....,25) arranged in the 30 m-classes as given in showing the independence of V in case P=3. These 25 elements and 30 m-classes satisfy all of assumptions B except IV and IV is false.

Independence of V. Since P is an arbitrary prime, we can choose P_1 a larger prime. Hence we can determine a set S'' of $P_1{}^2$ elements and $P_1{}^2+P_1$ m-classes that will satisfy all of assumptions B except V and V is false.

Independence of VI. Take as the set S the P elements (E_1,E_2,E_3......,E_p) arranged in the one m-class ($E_1E_2E_3$... ...E_p). These P elements and the single m-class satisfy all of assumptions B except VI and VI is false.

Therefore assumptions B are independent when P is an arbitrary prime. Since they are also consistent, assumptions B are irreducible when P is an arbitrary prime.

Appendix

Generalization of the arrangement for the independence proof of assumption VII, Set A.

1	1	2	2	2	2	…………	2	2
2	P+2	P+2	P+3	P+4	P+5	…………	2P	2P+1
3	P+3	2P+2	3P+1	4P	5P-1	…………	P^2-P+4	P^2+3
4	P+4	2P+3	3P+2	4P+1	5P	…………	P^2-P+5	P^2+4
5	P+5	2P+4	3P+3	4P+2	5P+1	…………	P^2-P+6	P^2+5
⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮
						P^2-P	P^2-1	P^2+P-2
						P^2-P+1	P^2	P^2+P-1
						P^2-P+2	P^2+1	P^2+P
P-1	2P+1	3P	4P+1	5P-2	6P-3	P^2-P+3	P^2+2	P^2+P+1

3	3	3	3	…………	3	3
P+2	P+3	P+4	P+5	…………	2P	2P+1
3P+1	4P	5P-1	6P-2	…………	P^2+3	2P+2
4P+1	5P	6P-1	7P-2	…………	2P+3	3P+2
5P+1	6P	7P-1	8P-2	…………	3P+3	4P+2
⋮	⋮	⋮	⋮	…………	⋮	⋮
P^2+P+1	3P	4P-1	5P-2	…………	P^2-P+3	P^2+2

4	4	4	4	…………	4	4
P+2	P+3	P+4	P+5	…………	2P	2P+1
4P	5P-1	6P-2	7P-3	…………	2P+2	3P+1
6P-1	7P-2	8P-3	9P-4	…………	4P+1	5P
⋮	⋮	⋮	⋮		⋮	⋮
P^2+2	P^2+P+1	3P	4P-1		P^2-2P+4	P^2-P+3
⋮	⋮	⋮	⋮	⋮	⋮	⋮

•	•	•	•	•	•	•
•	•	•	•	•	•	•
P	P	P	P	•	P	P
$P+2$	$P+3$	$P+4$	$P+5$	•	$2P$	$2P+1$
P^2-P+4	P^2+3	$2P+2$	$3P+1$	•	$9P-5$	P^2-2P+5
$9P-4$	P^2-2P+6	P^2-P+5	•	•	$7P-2$	$8P-3$
$7P-1$	$8P-2$	•	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	$8P-7$
•	•	•	•	•	$5P-3$	$6P-4$
$5P-2$	$6P-3$	•	•	P^2-P+1	$3P$	$4P-1$
$P+1$	$P+1$	$P+1$	$P+1$	•	$P+1$	$P+1$
$P+2$	$P+3$	$P+4$	$P+5$	•	$2P$	$2P+1$
P^2+3	$2P+2$	$3P+1$	$4P$	•	P^2-2P+5	P^2-P+4
P^2-P+5	P^2+4	$2P+3$	$3P+2$	•	$8P-2$	$9P-3$
P^2-2P+7	P^2-P+6	P^2+5	•	•	•	•
$9P-2$	$P-2P+8$	•	•	•	•	•
$8P$	•	•	•	•	•	•
•	•	•	•	•	•	$6P-6$
•	•	•	•	•	$4P-3$	$5P-4$
•	•	•	•	P^2-P	$3P-1$	$4-2$
$4P-1$	$5P-2$	$6P-3$	•	P^2-2	P^2+P+1	$3P$