HYPERBOLIC FUNCTIONS

by

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Table of Contents

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Introduction

The writer's purpose in this thesis is to collect, digest, and arrange the most interesting, fundamental, and useful facts relating to the history, theory, and applications of hyperbolic functions into concise form, a sort of miniature hyperbolic trigonometry, prefaced by a brief history and bibliography. No origionality or research is to be attempted except in the manner of presentation.

A list of books and of articles bearing on various phases of the subject is appended, all of which have been consulted in some degree. The writer is, however, principally indebted to the two works on Hyperbolic Functions, one a compendium by Professor James McMahan, of Cornell University and appearing as one of the numbers in the series of mathematical monographs edited by Merriman and Woodward, the other a volume prepared by Becker and Van Orstand of the United States Geological Survey and published by The Smithsonian Institution.

The latter volume is in the main a working manual for use in both theoretical and practical physics.

It contains a five place table of logarithms of the hyperbolic sine,cosine,tangent,and cotangent of the argument u expressed in radians,a table of the natural values of the same functions of u,a table of natural and logarithmic circular functions of u,a table of the ascending and descending exponentials e^{u} and e^{-u} and of $\log_{10}e^{u}$ for use in equations representing natural phenomena where hyperbolic functions play a part.This last named table appears for the reason,as we shall see later,that the hyperbolic and exponential functions are mutually definable in terms of each other. The volume contains also a table of natural logarithms of numbers,a table giving the values of gudermanian u the latter being a function connecting the hyperbolic and circular functions,a table which is a reversion of the latter,giving the anti-gudermanian in terms of the gudermanian and furnishing the foundation of navigation with Mercator Charts,and,finally,a table for the conversion of radians into circular measure and the converse.These tables are prefaced by the fundamental definitions and formulae of hyperbolic trigonometry,some geometric illustrations,and some historical notes.

Though Osgood makes the statement that hyperbolic functions are coming into general use,there seems to be,so far as the writer can learn,only one complete treatment in English.This is the monograph first referred to above by McMahan and consists of 70 pages with an appendix.It is what the author claims for it, a compendium of hyperbolic trigonometry and was first published as a chapter in Merriman and Woodward's Higher Mathematics. The hyperbolic relations are simply and comprehensively presented in the first half of the work,after which the more general trigonometry of the complex plane is introduced,showing the circular and hyperbolic functions merging into a single class of transcendents,the singly periodic functions having either a real or a pure imaginary period. In the latter part of the brochure an opportunity is presented to view the subject in some of its practical relations in connection with physics and engineering. While most writers define the hyperbolic functions in relations to a sector of the rectangular hyperbola whose initial radius is coincident with the principal radius of the curve,Professor McMahan discards these restrictions "in the interest of analogy and generality with a gain in symmetry and simplicity", defining the functions "as certain characteristic ratios belonging to any sector of any hyperbola".

By the aid of the notion of the correspondence of points on conics he obtains from these definitions simple and general proofs of the addition theorems, the conversion formulae, the derivatives, the Maclaurin expansions and the exponential expressions. He arranges his proofs so they will apply also to the circular functions when regarded as the characteristic ratios belonging to any elliptic sector. In this method of approach he differs also from the more usual one of starting with the exponential expressions as the definitions of the hyperbolic functions.

History and Literature

The actual inventor of hyperbolic trigonometry was Vincenzo Riccati (Opuscula ad res Phys. et Math. pertinens, Bononiae, 1757). He adopted the notation Sh.ø and Ch.ø for the hyperbolic sine and cosine and Sc.ø and Co.ø for the respective circular functions. He proved the addition theorem of hyperbolic trigonometry geometrically and derived a construction for the solution of a cubic equation. It should be remarked that there had been at least two suggestions antedating Riccati, one calling attention to the analogy between the sector of a circle and that of the hyperbola, on the geometric side, due to Sir Isaac Newton, and one on the analytic side as to a certain use of the operator $\sqrt{-1}$ by means of which the area of the prolate sphere can be turned into an expression for that of the oblate sphere(Roger Cotes'Harmonica Mensurarum,1722).

Following Riccati's publication by a few years came the first and most important application of the hyperbolic functions,viz.-that of Gerhard Mercator in his map on Mercator's Projection,the exact date of which is uncertain. "To this day",say Becker and Van Orstand,"substantially all the deep sea navigation of the world is carried on by the help of this projection,which has been modified only to the extent of correcting the meridional parts for the ellipticity of the meridian." Mercator's problem was to find a projection on which the loxodrome would be a straight line,a problem which Mercator must have solved,but he published his map without explanation and the formal solution was left to Edward Wright.The loxodrome is a curve starting from the equator in a given direction and cutting all the meridians at the same angle.

A little later Daviet de Foncenex showed how to interchange circular and hyperbolic functions by the use of $\sqrt{-1}$ and produced the analogue of De Moivre's theorem. Johann Lambert systematized hyperbolic trigonometry,developed the functions into the form of series and derived exponential expressions for them.

He adopted the notation sinh u,etc.,and introduced the transcendent angle,now called the gudermannian, using it in computation and in the construction of tables. In 1830 C.Gudermann published an important memoir on Potential or Cyclic-Hyperbolic Functions, followed by extended tables and it was in recognition of this contribution that Caley in 1862 proposed the name "gudermannian" for the angle which Lambert called transcendent. In 1881 appeared Siegmund Günther's Lehre von den Hyperbelfunctionen and in 1906 Professor James McMahan's Hyperbolic Functions,Fourth Edition.

Probably the first large table of hyperbolic functions is Legendre's,about 1816,followed by those of Gudermann,in 1831,of Gronau,1862,of Newman and Glaisher in 1883,of Ligowski in 1890,of Forti in 1892 and finally those of Becker and Van Orstand,first referred to above,in 1908.

Epitome of Theory

Generation and Measurement of Circular Angles:

The magnitude of a circular angle may be defined in either of two ways: (1) By the ratio of the circular arc length generated by any point P on the generating line OP to the length r of the radius vector OP. (2) By the area of the circular sector swept out by the radius vector OP during the motion.

Generation and Measurement of Hyperbolic Angles:

In the case of the hyperbola if OA be the semi-axis and P is any point on the hyperbola the line OP generates both a circular and an hyperbolic angle. The magnitude of the hyperbolic angle may be defined in either of two ways: (1) By the ratio of the hyperbolic arc distance described to the length r of the radius vector. (2) By the area of the hyperbolic sector OAP swept over.

Analytic Definition of Any Angle, Circular or Hyperbolic: In either case, $ds=\sqrt{(dx)^2+(dy)^2}$ and $r=\sqrt{(x^2+y^2)}$. Whence, $d\phi = \frac{ds}{r}$, expressed in circular or hyperbolic radiand as the case may be. If the motion continues we have in either case: $\phi = \int\frac{ds}{r}$, expressed in the appropriate radian, circular or hyperbolic. Since in the circle r is constant and assumed equal to unity we have: $\phi_c = s$ radians. In the hyperbola however the radius vector varies, whence: $\phi_h = \int\frac{ds}{r} = \frac{s}{r'}$ hyperbolic radians and where r' is the integrated mean value of r during the motion. For a proof of this fact the reader is referred to Professor A.E.Kennelly's work on "Applications of Hyperbolic Functions to Electrical Engineering".

Geometric Definitions of the Hyperbolic Functions:

We make use of the second geometric approach to the development of the theory of hyperbolic trigonometry: Let the right triangle OAB have for its legs the semi-axes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ and let the two triangles OAP and OBP, whose areas vary with the point P, form, with the constant triangle OAB, ratios defined as follows:

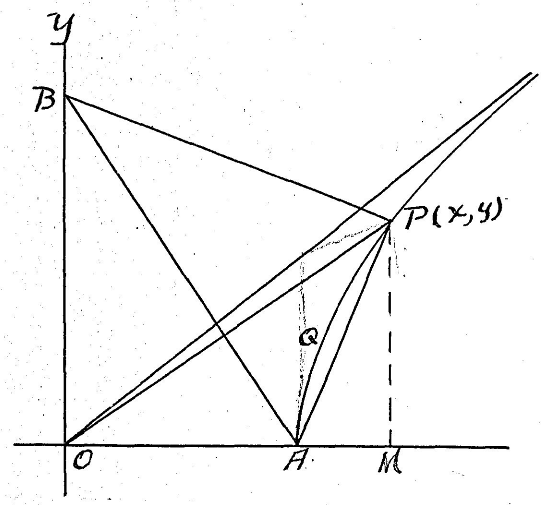


Fig.1

Sinh u = triangle OAP : triangle OAB

Cosh u = triangle OBP : triangle OAB

where u is a function defined thus:

$$u = \text{sector OAP} : \text{triangle OAB} = \frac{\text{triangle OMP} - \text{AMPQ}}{\text{triangle OAB}}$$

$$\text{or } u = \frac{1/2\, xy - b/a \int_a^x \sqrt{x^2-a^2}\, dx}{1/2\, ab} = \log(x/a + y/b)$$

Thus the sector OAP = $ab/2 \log(x/a + y/b)$.

Furthermore $\sinh u = 1/2ay / 1/2\, ab = y/b$

$$\cosh u = 1/2\, bx \;/\; 1/2\, ab = x/a$$

where sinh and cosh designate the hyperbolic sine and hyperbolic cosine respectively.

It is of interest to note the analogy between these functions and the sine and cosine functions of circular trigonometry which are capable of similar definitions. Thus, if $a = b = r$ and if P be any point on the circle whose center is O and radius is r, we have $\sin \phi = y/r = y/b = 1/2\, by / 1/2\, b^2$ = the ratio of the area of the triangle OAP to the area of the triangle OAB. In a similar way $\cos \phi$ can be defined as the precise analogue of cosh u and will be seen to be equal to the ratio of the triangle OBP to the triangle OAB. Pressing this development a little further it is easily seen that analogous functions can be defined for the ellipse and a corresponding trigonometry erected thereon. Returning to the hyperbola it is to be noted that in further analogy with circular trigonometry, the hyperbolic tangent, h-cotangent, h-secant and h-cosecant are defined as follows:

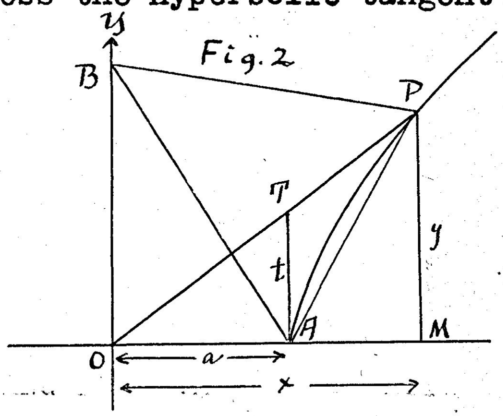
$\tanh u = \sinh u / \cosh u$, $\coth u = \cosh u / \sinh u$, $\operatorname{sech} u = 1/\cosh u$, and $\operatorname{csch} u = 1/\sinh u$.

Relations between the Functions

In addition to the four defining relations given in the last paragraph a fifth one is seen to follow immediately in view of the equation of the hyperbola and the fundamental definitions of sinh u and cosh u, viz: $\cosh^2 u - \sinh^2 u = 1$ an equation which bears a striking analogy to the trigonometric equation of the circle. It will be observed that, as in circular trigonometry the six hyperbolic functions are not independent of each other, so that, a numerical value of one of them being given those of the other five can be found. It is also clear that by manipulating these relations various other less fundamental ones can be obtained thus enabling one to express each function in terms of any other. Among these should be noted: $1-\tanh^2 u=\operatorname{sech}^2 u$ and $\coth^2 u-1=\operatorname{csch}^2 u$, from which follow immediately $\operatorname{sech} u=\sqrt{(1-\tanh^2 u)}$, etc, etc.

It is desirable to express the hyperbolic tangent as the ratio of two lines.

Fig. 2



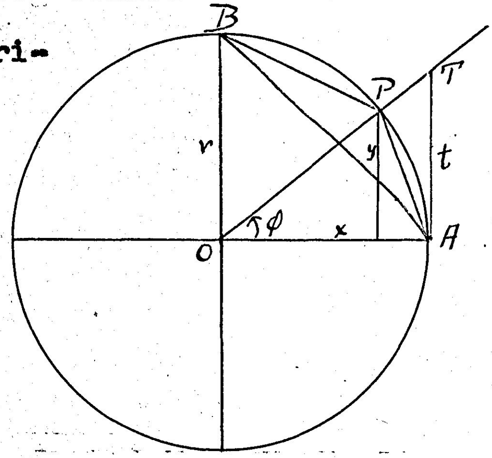
By definition,

$$\tanh u = \sinh u/\cosh u$$
$$= y/b/x/a = ay/bx$$
$$=(a/b)(y/x)$$
$$=(a/b)(t/a)=t/b$$

= triangle OAT/triangle OAB, a result entirely in harmony with the geometric definition of circular trigonometry.

We note that the sector OAP and the triangles OAP, POB, OAT are proportional to the functions u, sinh u, cosh u, and tanh u. By an inspection of the figure it thus appears that sinh $u > u >$ tanh u, a relation just the reverse of that prevailing in circular trigonometry as can be easily seen by a comparison of the corresponding sector and triangles of the circle. We have already seen that sin ϕ and cos ϕ are capable of geometric definitions entirely consonant with those taken for the hyperbolic sine and h-cosine.

We now observe that tan ϕ can be defined as equal to the ratio of the triangle OAT to the triangle OAB, which, in turn, is equal to the ratio of 1/2 tr to 1/2 r^2=t/r. It can be easily seen that the corresponding function in elliptical trigonometry would give t/b as in the case of the hyperbola. Similarly, the elliptic sine would be the ratio of triangle OAP to triangle OAB, which reduces to y/b while the elliptic cosine becomes x/a thus completing the analogy. If in the ellipse a = b these functions become identical with the corresponding circular ones.



Variations of the Hyperbolic Functions

By reference to Fig.1 and to the analytic definition of the u function as log(x/a + y/b) one can easily see that, as the point P traces the curve, the sectorial measure u assumes all values from $-\infty$ through 0 to $+\infty$. Since a and b are constant, by use of the definitions sinh u = y/b, cosh u = x/a, and tanh u = t/b (Fig. 2), we see that sinh 0 = 0, cosh 0 = 1, and tanh 0 = 0, since sector OAP = 0 if u = 0 and thus the point P must be at A, making x = a, y = 0, and t = 0. When u is negative, since triangle OAB is positive we must have the sector OAP negative, which by the usual convention, requires the point P to be below the axis of X . Thus y is negative ranging from 0 to $-\infty$.

If we confine ourselves to the right hand branch of the curve as evidently yielding all possible values of the functions we see that cosh u is positive since x is positive and ranges from 1 to $+\infty$, while tanh u is negative with u negative and decreases to -1 when the radius vector OP reaches the limiting asymptotic position.

As u increases positively we infer from considerations similar to those indicated above that the point P traces out the right hand upper part of the curve, making x, y, and t positive and of increasing value, whence we see that the h-cosine, h-sine, and h-tangent functions increase from 1, 0, and 0 to $+\infty$, $+\infty$, and +1 respectively.

Intermediate values can be had by consulting a table of natural hyperbolic functions, or, they can be computed, as will be seen later, to any desired degree of accuracy by use of the serial or exponential expressions for these functions.

Functions of -u in Terms of Functions of u

As seen in the last paragraph u is positive when the sector AOP is on the same side of the X axis as the triangle of reference. Thus when u is positive x, y, and t are positive, and when u is negative y and t are negative. Thus it appears clear that:

$\sinh(-u) = -\sinh u$ and $\operatorname{csch}(-u) = -\operatorname{csch} u$

$\tanh(-u) = -\tanh u$ and $\coth(-u) = -\coth u$

Since a is negative when x is negative we see that cosh u is always positive and thus we conclude that

$\cosh(-u) = \cosh u$ and $\operatorname{sech}(-u) = \operatorname{sech} u$.

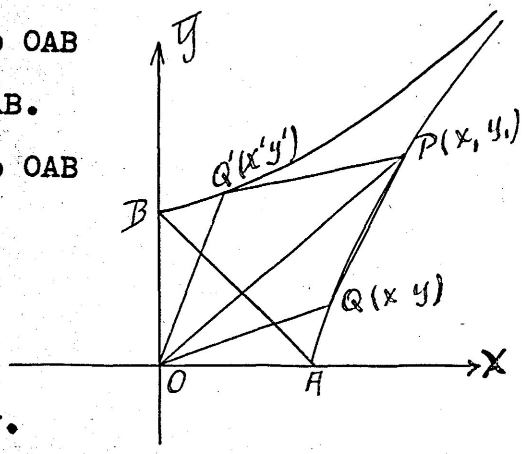
Inverse Hyperbolic Functions

The relations $\sinh u = y/b$, $\cosh u = x/a$, and $\tanh u = t/b$, etc., are also expressed by the notations $u = \sinh^{-1} y/b$, $u = \cosh^{-1} x/a$, $u = \tanh^{-1} t/b$, etc., which are read: "u is the sectorial measure whose h-sine is the ratio of y to b", or, "u is the anti-h-sine of y/b", etc.

As seen in the next to the last paragraph, u has two values of opposite signs that correspond to a given value of cosh u or sech u. By convention, therefore, the symbols $\cosh^{-1} q$ and $\operatorname{sech}^{-1} q$ denote the positive values of u that satisfy the equations $\cosh u = q$ and $\operatorname{sech} u = q$. The signs of the other anti-h-functions are the same as the signs of q. Thus, by virtue of the above convention all the inverse hyperbolic functions of real numbers are single valued.

Functions of Sums and Differences

Let u =sector OAP/triangle OAB and v = sector OAQ/triangle OAB. Then u-v = sector OQP/triangle OAB and $\sinh(u-v)=\Delta OQP/\Delta OAB$ $=1/2(xy_1-yx_1)/(1/2\ ab)$ $=(y_1/b)(x/a)-(x_1/a)(y/b)$ $=\sinh u \cosh v - \cosh u \sinh v$.



Also $\cosh(u-v)=\Delta OQ'P/\Delta OAB=1/2(x_1 y' - y_1 x')/(1/2\ ab)$ $=x_1/a(y'/b)-(y_1/b)(x'/a)$. Since Q' is the correspondent of Q on the conjugate hyperbola when a=b and since the conjugate semi-diameters are equally inclined to the axes, we have $y'/b=x/a$ and $x'/a=y/b$ and we conclude therefore, that

$$\cosh(u-v)=x_1/a(x/a)-y_1/b(y/b)=\cosh u \cosh v-\sinh u \sinh v$$

It may be noted at this point that it is possible to so define the h-functions as to make them independent of the position of the sectorial areas as does Professor McMahan. Furthermore the use of his definitions would enable one to use any hyperbola in making the foregoing proof or indeed any ellipse while we have confined ourselves to the equilateral hyperbola. It can be easily seen, however, that for the special case of the circle our argument will readily apply, giving, for the sine function, sin u cos v−cos u sin v. In the case of the cosine function for the ellipse or circle however the point Q' is in the second quadrant, the slopes of the conjugate diameters being opposite in sign. Since for the circle a=b the lines OQ and OQ' are perpendicular and we have $x'/a=-y/b$. Therefore after substitution we have:

$$\cos(u-v)=\cos u \cos v+ \sin u \sin v .$$

Since the above formulae are identities we may put $v=-v'$ thus deriving the sum formulae of hyperbolic trigonometry, viz:

$$\sinh(u+v')=\sinh u \cosh v' + \cosh u \sinh v'$$

$$\cosh(u+v')=\cosh u \cosh v' + \sinh u \sinh v',$$ in obtaining which we observe that $\sinh(-v')=-\sinh v'$ and $\cosh(-v')=\cosh v'$. Upon dropping primes we obtain the identities in the usual form.

To develope $\tanh(u\pm v)$ in terms of functions of u and v we may apply analytic methods as in circular trigonometry, making use of the above identities obtaining readily that:

$$\tanh(u\pm v)=(\tanh u \pm \tanh v)/(1 \pm \tanh u \tanh v)$$

If v=u we obtain at once $\sinh 2u=2 \sinh u \cosh u$

$$\cosh 2u=\cosh^2 u+\sinh^2 u=1+2 \sinh^2 u=2 \cosh^2 u-1 .$$

$\tanh 2u = 2\tanh u/(1+\tanh^2 u)$ and other multiple formulae. If in these formulae we put $u=1/2\,v$, we get readily: $\sinh v/2 = (\cosh v-1)^{1/2}/\sqrt{2}$

$$\cosh v/2 = (1+\cosh v)^{1/2}/\sqrt{2}$$

$$\tanh v/2 = (\cosh v-1)^{1/2}/(\cosh v+1)^{1/2}, \text{etc.}$$

Conversion Formulae

By manipulating the addition formulae we obtain:

$$\sinh(u+v)+\sinh(u-v)=2\sinh u\cosh v$$

$$\sinh(u+v)-\sinh(u-v)=2\cosh u\sinh v$$

$$\cosh(u+v)+\cosh(u-v)=2\cosh u\cosh v$$

$$\cosh(u+v)-\cosh(u-v)=2\sinh u\sinh v$$

If we make the transformations $s=u+v$ and $t=u-v$, or $u=(s+t)/2$ and $v=(s-t)/2$ the above identities become:

$$\sinh s + \sinh t = 2\sinh(s+t)/2\,\cosh(s-t)/2$$

$$\sinh s - \sinh t = 2\cosh(s+t)/2\,\sinh(s-t)/2$$

$$\cosh s + \cosh t = 2\cosh(s+t)/2\,\cosh(s-t)/2$$

$$\cosh s - \cosh t = 2\sinh(s+t)/2\,\sinh(s-t)/2$$

Limiting Ratios

We have seen that sinh 0 and tanh 0 are each 0. What then are the limiting values of $\sinh u/u$ and of $\tanh u/u$ as u approaches zero?

In discussing the relations between the functions we noted that $\sinh u > u > \tanh u$. Dividing by sinh u and taking reciprocals we obtain:

$1<\sinh u/u<\cosh u$. Again, dividing the first relation by tanh u and taking reciprocals: $\operatorname{sech} u<\tanh u/u<1$, Now when $u \doteq 0$ we have $\cosh u \doteq 1$ and $\operatorname{sech} u \doteq 1$ and hence $\lim_{u \doteq 0} \sinh u/u = 1$ and $\lim_{u \doteq 0} \tanh u/u = 1$.

Derivatives of the Functions

By making use of the limits just found Professor McMahon determined $d/du(\sinh u)=\cosh u$ and

$$d/du(\tanh u)=\operatorname{sech}^2 u$$

We proceed in a similar way to obtain the derivatives of the h-cosine and the cotangent: Let $y=\cosh u$, then, $\Delta y=\cosh(u+\Delta u)-\cosh u=2\sinh 1/2(2u+\Delta u)\sinh 1/2\Delta u$, the last step being obtained by the aid of the conversion formulae. Dividing by u and reducing right side,- $\Delta y/\Delta u = (\sinh(u+1/2\,\Delta u)\sinh 1/2\,\Delta u)/(1/2\,\Delta u)$. Taking the limit of both sides as $u \doteq 0$ and noting that:

$$\lim_{\Delta u \doteq 0} \sinh(1/2\,\Delta u)/1/2\,\Delta u = 1, \text{ we obtain:}$$

$$dy/du = \lim_{\Delta u \doteq 0} \Delta y/\Delta u = d/du(\cosh u) = \sinh u \ .$$

Again, putting $y=\coth u=\cosh u/\sinh u$, we get:

$$dy/du = d/du(\cosh u/\sinh u)=\sinh^2 u-\cosh^2 u)/\sinh^2 u=-\operatorname{csch}^2 u.$$

By taking the derivatives of the reciprocals of cosh u and sinh u we find the derivatives of sech u and csch u to be $-\operatorname{sech} u \tanh u$ and $-\operatorname{csch} u \coth u$ respectively.

It is worth noting that these are the formulae of circular trigonometry except for the signs of the cosine and secant derivatives.This discrepancy we trace to the conversion formula used which involves the h-cosine of the difference of two sectorial measures, which in turn goes back as before remarked to the property that in the hyperbola conjugate diameters lie in the same quadrant while in the ellipse their slopes are opposite in sign. Another comparison between these derivatives and those of the circular functions shows that,while the h-sine and h-cosine reproduce themselves in two differentiations,the circular sine and cosine produce their opposites in two differentiations. It has been suggested that these facts account for the frequent appearance of these functions in the solution of physical problems. To quote McMahon,-"They answer the question: what function has its second derivative equal to a positive(or negative)constant multiple of the function itself?" We proceed to attack this suggestion from a different direction: Let y be a function of x and put $d^2y/dx^2=mx$ according to the first condition. Integrating,we get:

$$y=c'e^{mx}+c''e^{-mx}=A \cosh mx + B \sinh mx$$

Again,putting $d^2y/dx^2= -m^2x$ and integrating we get:

$$y=c'e^{mix}+c''e^{-mix}=A \cos mx + B \sin mx$$,both

of which results harmonize with the facts about the derivatives.We have here anticipated a relation between the h-functions and the exponential functions to be developed later on.

Derivatives of the Inverse Functions

Professor McMahon finds that $d/dx(\sinh^{-1}x)=1/\sqrt{(x^2+1)}$ $d/dx(\tanh^{-1}x)=1/(1-x^2)$ where $|x|\leq 1$ and $d/dx(\text{sech}^{-1}x)=-1/x\sqrt{(1-x^2)}$ .We proceed to find the rest.

Put $u = \cosh^{-1}x$, then $x = \cosh u$ and $dx = \sinh u\ du = \sqrt{(\cosh^2 u-1)}du = \sqrt{(x^2-1)}du$.
Whence $du = dx/\sqrt{(x^2-1)}$ and $d/dx(\cosh^{-1}x)=1/\sqrt{(x^2-1)}$.

In a similar manner we have found:
$d/dx(\coth^{-1}x)=1/(1-x^2)$ where $|x|\geq 1$ since $\tanh u=1/x\leq 1$, and $d/dx(\text{csch}^{-1}x)=1/x\sqrt{(x^2+1)}$.

If $x = y/a$ the above formulae would become:
$d/dy(\sinh^{-1}y/a)=1/\sqrt{(y^2+a^2)}$, $d/dy(\cosh^{-1}y/a)=1/\sqrt{(y^2-a^2)}$,etc.

Since $d/dx(\sinh^{-1}x)=1/\sqrt{(1+x^2)}=(1+x^2)^{-1/2}$,which,by expansion,becomes $1-(1/2)x^2+(1/2)(3/4)x^4-(1/2)(3/4)(5/6)x^6+..$ we may integrate,securing $\sinh^{-1}x=x-(1/2)(x^3/3)+ \ldots-\ldots$ where the constant of integration is 0 since both sides vanish with x. The series is convergent for $x<1$ but it is possible to write another convergent for $x>1$.

In a similar manner the other inverse functions may be expressed in serial form.

Expansion of Hyperbolic Functions

As in the case of the circular functions, sinh u and cosh u can be profitably expressed in the form of series by the aid of the derivatives and Maclaurin's theorem. Thus $\sinh u = u+u^3/3!+u^5/5!+ \ldots\ldots$ Cosh u can be found in similar fashion or more simply by differentiation giving $1+u^2/2!+u^4/4!+\ldots\ldots$ in both of which $|r|<1$ for any u real or complex provided n be taken great enough. Hence these series may be taken as definitions of the h-sine and h-cosine respectively. They can evidently be adapted to corresponding circular uses merely by changing alternate signs. Series can be found for the other functions by division but the results give no evident law among the coefficients and their values are therefore to be computed from those of the h-sine and h-cosine.

Exponential Definitions of Hyperbolic Functions

By adding and subtracting the above series we get:

$$\sinh u + \cosh u = 1+u+u^2/2!+u^3/3!+\ldots\ldots= e^u$$

$$\sinh u - \cosh u =-1+u-u^2/2!+u^3/3!-\ldots\ldots=-e^{-u}$$

Adding and dividing by 2 we have $\sinh u=(e^u-e^{-u})/2$

Subtracting and dividing by 2, $\cosh u=(e^u+e^{-u})/2$

from which we get $\tanh u=(e^u-e^{-u})/(e^u+e^{-u})$, etc.

Starting with the above relations as definitions one should be able to derive the several formulae of hyperbolic trigonometry. For examples:

(a) $\sinh(-u)=(e^{-u}-e^{u})/2=-\sinh u$, etc.

(b) $\cosh^2u-\sinh^2u=(e^{2u}+2+e^{-2u})/4-(e^{2u}-2+e^{-2u})/4=1$, etc.

(c) $\sinh(u+v)=(e^{u+v}-e^{-u-v})/2 =(2e^{u+v}-2e^{-u-v})/4$

$=(e^{u+v}+e^{u-v}-e^{v-u}-e^{-u-v})/4 +(e^{u+v}-e^{u-v}+e^{v-u}-e^{-v-u})/4$

$=(e^{u}-e^{-u})/2(e^{v}+e^{-v})/2 + (e^{u}+e^{-u})/2(e^{v}-e^{-v})/2$

$= \sinh u \cosh v + \cosh u \sinh v$,etc.

(d) $d^2/du^2(\sinh mu)=d^2/du^2(e^{mu}-e^{-mu})/2$

$=m/2.d/du(e^{mu}+e^{-mu})=m^2/2(e^{mu}-e^{-mu})=m^2\sinh mu$.

(e) $(\cosh u + \sinh u)^n=(e^{u}+e^{-u}+e^{u}-e^{-u})^n/2^n = e^{nu}$

(f) $e^u=(e^{u}+e^{-u}+e^{u}-e^{-u})/2 = \cosh u + \sinh u$

The last formula gives $u = \log_e(\cosh u + \sinh u)$ or expressed in terms of Figure 1, $u=\log_e(x/a + y/b)$, which becomes, in the case of the equilateral hyperbola with $a=b=1, u=\log_e(x+\sqrt{x^2-1})$.

We have already pointed out that u becomes infinite as the point P moves to infinity along the hyperbola. We here return by way of exponentials to the definition of u assumed at the beginning of the discussion of the theory of hyperbolic trigonometry. We may now readily infer that, given a table of h-sines and h-cosines a table of exponential functions might be constructed and conversely.

Logarithmic Expressions for Inverse Functions

Let $x=\sinh u$, then $\cosh u=\sqrt{x^2+1}$ and by addition,

$x+\sqrt{x^2+1}=\sinh u + \cosh u = e^u$. Comparing results:

$\sinh^{-1}x = u = \log_e(x + \sqrt{x^2+1})$. Similarly,-

$\cosh^{-1}x = \log_e(x + \sqrt{x^2-1})$, etc.

The Gudermannian Function

If on two central conics of different kinds, as for examples, the circle and the equilateral hyperbola two points P' and P" are so related that $x'/a'=a''/x''$, the ordinants y' and y'' having the same sign, then there exists a fixed functional relation between their sectorial measures, v and u. The elliptic sectorial measure being called the gudermanian of the corresponding hyperbolic sectorial measure, the latter is called the anti-gudermanian of the former, viz,- $v=\text{gd } u$ and $u=\text{gd}^{-1}v$. In the circle $v=\phi$.

Circular Functions of Gudermanian

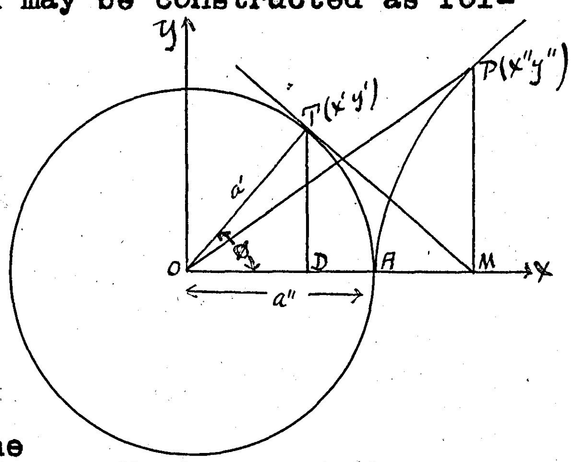
Since $x'/a'=\cos v$ and $x''/a''=\cosh u$ we have, in view of the foregoing definitions: $\cosh u = \sec v$, $\sinh u = \sqrt{\sec^2 v-1} = \tan v$, $\tanh u=\tan v/\sec v=\sin v$, $\coth u=\csc v$, $\text{sech } u=\cos v$, and $\text{csch } u = \cot v$.

It can easily be seen that the gudermanian is useful in computation. If a certain hyperbolic function be known v can be found by the aid of a table of circular functions, whereupon the other circular functions of v will give the remaining hyperbolic functions of u.

It may be seen by a comparison of the foregoing relations that $v = \text{gd } u = \sec^{-1}(\cosh u)$, also,- $v = \tan^{-1}\sinh u = \cos^{-1}\text{sech } u = \sin^{-1}\tanh u$. Again,- $u = \text{gd}^{-1}v = \cosh^{-1}\sec v = \sinh^{-1}\tan v = \text{sech}^{-1}\cos v$, etc. Finally,- gd 0=0, since when u=0, v=0; $\text{gd}(\infty) = \pi/2$ because when $u = \infty$ $\sinh u = \infty$ and hence $\tan^{-1}\infty = \pi/2$; $\text{gd}(-\infty) = -\pi/2$, and $\text{gd}(-u) = -\text{gd}u$. Conversely: $\text{gd}^{-1}0 = 0$, $\text{gd}^{-1}\pi/2 = \infty$, and $\text{gd}^{-1}(-\pi/2) = -\infty$.

The Gudermanian Angle

We have remarked that if the ellipse be a circle gd u = v = ϕ radians. Now the angle ϕ is called the gudermanian angle and may be constructed as follows: Since $\cosh u = \sec\phi$ because of $x''/a'' = a'/x'$, as we saw at the beginning of the definition, we must have for the circle $a'' = r = a'$ and hence if OT be the terminal line



of ϕ where OA is the initial line of ϕ and of the sector whose measure is u, and if OP be the terminal line of the sector and OM the abscissa of P, then OTM must be a right triangle because of the above proportion, since if a perpendicular be dropped from T upon OA, the proportion would be satisfied. As M moves from A where it parts company with P and T the angle ϕ, which is gudermanian u evidently varies from 0 to $\pi/2$.

Derivatives of Gudermanian and Inverse

These may be obtained as follows, the argument being essentially that of Professor McMahon: since $v = gd\ u$ and $U = gd^{-1}v$, we have seen that $\sec v = \cosh u$, whence $\sec v \tan v\ dv = \sinh u\ du$. This, in view of $\tan v = \sinh u$, gives $\sec v\ dv = du$, or, $d(gd^{-1}v) = \sec v\ dv = du$. Again, in view of the last equation, we have $dv = \cos v\ du = \text{sech}\ u\ du$, or $d(gd\ u) = \text{sech}\ u\ du$.

It is convenient to note at this point an interesting relation between the elliptic and hyperbolic functions, though we do not imply that the hyperbolic functions received their name because of any analogy with the elliptic functions, although they do touch at the point we now mention, viz.— for the special case where the modulus $k = 1$ in the elliptic integral of the "first kind", we have:

$$u = \int_0^\phi d\phi/\sqrt{(1-k^2\sin^2\phi)} = \int_0^\phi \sec\phi\ d\phi = \int_0^\phi d(gd^{-1}\phi) = gd^{-1}\phi.$$

i.e.— $am\ u = \phi = gd\ u$, $\sin am\ u = \sin\phi = \tanh u$, etc, etc.

Serial and logarithmic expressions for the gudermanian function and its inverse can be obtained, the former by use of the series for sech u and sec v in the formula $d(gd\ u)=sech\ u\ du$ and $d(gd^{-1}v)=sec\ v\ dv$, and integrating; the latter by noting that sec v + tan v is the equivalent of cosh u + sinh u, which is in turn equal to e^u. Hence, $e^u=(1+\sin v)/\cos v=(1-\cos(\pi/2+v))/\sin(\pi/2+v)$. Since the latter is equal to $\tan(\pi/4+v/2)$, we have that: $u = gd^{-1}v = \log_e \tan(\pi/2+v/2)$.

Elementary Integrals

$\int \sinh u\ du=\cosh u$, $\int \cosh u\ du = \sinh u$ and $\int sech\ u\ du$ clearly follow as the inverses of the corresponding derivatives.

$\int \tanh u\ du=\int \sinh u\ du/\cosh u=\int d(\cosh u)\ du/\cosh u$, which becomes log cosh u. In a similar way it can be shown that $\int \coth u\ du \equiv \log \sinh u$, etc.

$\int dx/\sqrt{(x^2+a^2)}=\sinh^{-1}x/a$ and $\int dx/\sqrt{(x^2-a^2)}=\cosh^{-1}x/a$, etc., are the immediate consequences of integrating both sides of the several anti-h-functions. They are seen to be similar to the corresponding integrals in circular trigonometry.

Functions of Complex Numbers

Obviously the geometric definitions of sinh u and cosh u would not be applicable to such functional symbols as sinh(x+iy) and cosh(x+iy) where $i=\sqrt{-1}$. Since vector quantities are of importance in mathematical physics it is necessary to assign to these functions suitable analytical meanings which shall be consistent with known algebraic values sinh x and cosh x when y = 0 and also to allow the addition theorems to be made general.

Since the series for sinh u and cosh u are convergent for all values of u real or complex the following definitions are assumed:

$$\sinh(x+iy)=(x+iy)+1/3!(x+iy)^3+1/5!(x+iy)^5+\ldots\ldots$$

$$\cosh(x+iy)=1+1/2!(x+iy)^2+1/4!(x+iy)+\ldots\ldots$$

When x and y are given these results can be carried out to any desired degree of accuracy in the form a+bi. The other functions are defined as before, viz.-

$$\tanh(x+iy) = \sinh(x+iy)/\cosh(x+iy), \text{ etc.}$$

Checking these definitions up somewhat we may note as examples: sinh(-u)=-sinh u, cosh(-u)=cosh u, d/du(sinh u)=cosh u, and d/du(cosh u)=sinh u, the formulae holding for u either real or complex.

Addition Theorems for Complex Numbers

We have seen, that with u and v real,-

sinh(u+v) = sinh u cosh v + cosh u sinh v , Hence,-

$$(u+v)+1/3!(u+v)^3+\ldots=(u+1/3!u^3+\ldots)(1+v^2/2!+\ldots)$$
$$+(1+u^2/2!+\ldots)(v+v^3/3!+\ldots)$$

This formula is identically true for all real values of u and v and we may therefore equate any function on the left of the form $1/n!(u+v)^n$ with the terms on the right which, when collected form an nth. degree function which is numerically equal to the former for more than n values of v when u is constant and for more than n values of u when v is constant. Hence these terms form an algebraic identity and therefore the whole equation holds for all values of u real or complex. Hence the theorem. The other sum and difference formulae can be found after cosh(u+v) has been developed in a similar manner. We therefore conclude that:

$$\sinh(x \pm iy) = \sinh x \cosh iy \pm \cosh x \sinh iy$$

$$\cosh(x \pm iy) = \cosh x \cosh iy \pm \sinh x \sinh iy$$

It is furthermore easily shown by a substitution of the series for the functions that for any u:

$\cosh^2 u - \sinh^2 u = 1$, $\cosh u \pm \sinh u = \exp(\pm u)$ and hence by an easy manipulation of the latter, we get

$\cosh u = (\exp u + \exp -u)/2$ and $\sinh u = (\exp u - \exp -u)/2$

Functions of Pure Imaginaries

If in the series we put for u the pure imaginary iy we have: $\sinh iy = iy + (iy)^3/3! + (iy)^5/5! + \ldots = i\sin y$

$$\cosh iy = 1 - y^2/2! + y^4/4! + \ldots = \cos y$$

whence, $\tanh iy = \sinh iy/\cosh iy = i\sin y/\cos y = i\tan y$.

These formulae evidently serve as a medium of exchange between the hyperbolic and circular functions. In a similar way if we put x+iy for u in the series we readily obtain: $\sinh(x\pm iy)=\pm i\ \sin(y\mp ix)$

$\cosh(x\pm iy)= \cos(y\mp ix)$

Functions of x+iy in the Form U+Vi

By the addition formulae:

$$\sinh(x+iy)=\sinh x \cosh iy + \cosh x \sinh iy$$

$$\cosh(x+iy)=\cosh x \cosh iy + \sinh x \sinh iy$$

Since $\sinh iy = i \sin y$, and $\cosh iy = \cos y$, these become: $\sinh(x+iy)=\sinh x \cos y + i \cosh x \sin y$

$$\cosh(x+iy)=\cosh x \cos y + i \sinh x \sin y$$

If $\sinh(x+iy)=U+iV$ and $\cosh(x+iy)=U'+V'$, we shall have by equating coefficients of reals and imaginaries: $U=\sinh x \cos y$, $V=\cosh x \sin y$

$U'=\cosh x \cos y$, $V'=\sinh x \sin y$

Putting $W=U+iV=\sinh z=\sinh(x+iy)$ we proceed to trace on the w plane the movements corresponding to certain movements of the point z:

First let the point z move along a line parallel to the axis of y, tracing the line $x=k$. We shall then have $U=\sinh k \cos y$ and $V=\cosh k \sin y$, whence,– $\cos y = U/\sinh k$, and $\sin y = V/\cosh k$, from which eliminating y by squaring and adding we obtain:

$$1 = U^2/(\sinh k)^2 + V^2/(\cosh k)^2$$

which evidently represents a family of ellipses whose parameter is k and with major semi-axis cosh k, since $\cosh k > \sinh k$. Again, since the abscissae of the foci are given by $a^2e^2 = a^2-b^2$, we shall have: $\pm ae = \sqrt{(\cosh^2 k-\sinh^2 k)} = \pm 1$, whatever real value k may have. Therefore the ellipses are confocal.

As k approaches zero from either side the ellipse degenerates into the line joining the foci in view of $U = \sinh 0 \cos y = 0$, while as $k \doteq \pm\infty$, the curve becomes a circle since $\sinh^2 k = \cosh^2 k = \infty$.

Next let z move along a line parallel to the axis of x, tracing out a line y=k. We then have $U=\sinh x \cos k$ and $V=\cosh x \sin k$, whence $\sinh x=U/\cos k$, $\cosh x =V/\sin k$. Eliminating x by squaring and subtracting we have:

$1 = V^2/\sin^2 k - U^2/\cos^2 k$, which represents a family of hyperbolas whose parameter is k and semi-transverse axis is sin k. Since the abscissae of the foci of an hyperbola are given by $a^2e^2=a^2+b^2$, we have: $ae=\pm\sqrt{(\cos^2 k+\sin^2 k)}=\pm 1$, whatever the real value of k. Since the coordinates of the foci of each curve are $(0,\pm 1)$, each is orthogonal to the other for a given k, each system being confocal with itself and with the other system.

In the case of the hyperbola, as $k \doteq 0$ from either above or below the curve degenerates into the u axis in view of $V=\cosh x \sin 0=0$, the asymptotes closing scissor like upon the u axis. When $|k|$ increases we have a periodicity set up in the hyperbolas due to the periodicity of the sine and cosine functions which form their semi-axes. In view of $U=\sinh x \cos k$ we see that $U=0$ for $k=(2n+1)\pi/2$ (n=any integer), indicating that all such lines on the z plane map into the axis of imaginaries on the w plane. Again, in view of $V=\cosh x \sin k$, we see that $V=0$ for $k=n\pi$ (n an integer), indicating that all such lines map into the axis of reals. Thus the period of sinh z seems to be 2π, a conclusion which is confirmed by recalling that $\sinh z = (e^z - e^{-z})/2$ and that $e^z = e^x(\cos y + i \sin y)$ an expression which is evidently the same as: $e^x(\cos(y+2\pi)+i \sin(y+2\pi))=e^{x+i(y+2\pi)}= e^{z+2\pi}$. Since e^z has the period 2π and since sinh z is expressible in terms of z, our conclusion seems clear that a strip 2π wide parallel to the axis of x is the correspondent of the entire w plane. Thus for a given $z = a + bi$ the function W is single valued as we shall see below. But a given $W = c + di$ would not map into an unique z. On the other hand it can be seen that there would be an infinity of points z corresponding to a given W.

To find the correspondent of a particular point a+bi on the w plane we observe first that there are just four possible points of intersection of the ellipse $1=U^2/\sinh^2 a+V^2/\cosh^2 a$ and the hyperbola

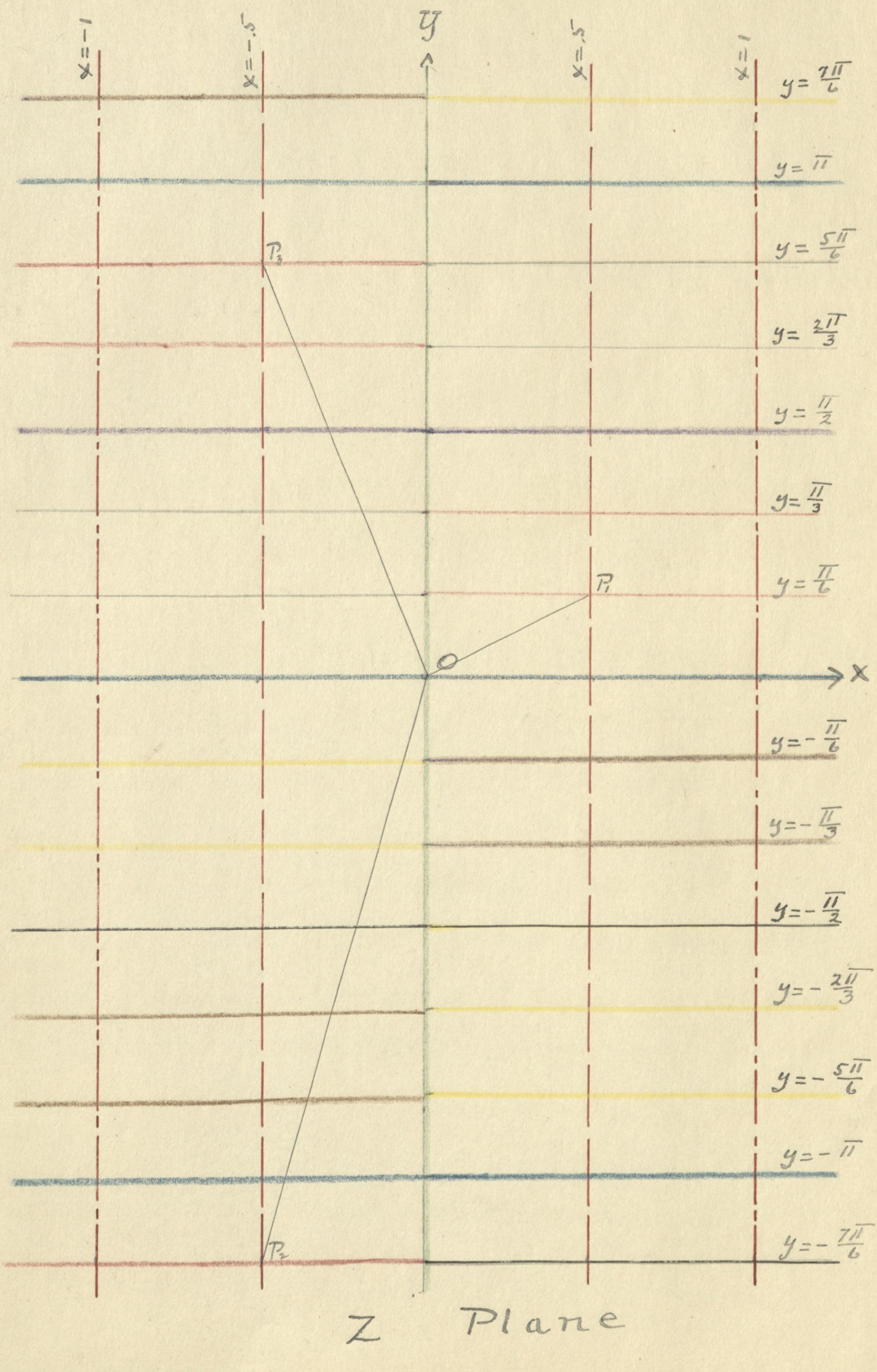
$1=V^2/\sin^2 b-U^2/\cos^2 b$. Again there are of course four quadrants in which the point a+bi may lie corresponding to the four combinations which may be effected in view of the consideration that either or both a and b may be either positive or negative. We therefore direct attention to the equations: U=sinh a cos b and V=cosh a sin b from which it appears clear that:

(1) If a is positive and b lies between zero and $\pi/2$ U and V are both positive and W is in the first quanrant; if however, with a positive b lies between $\pi/2$ and π, we have U negative with V positive. Thus with a constant and b varying from zero to π we find the point W tracing out the upper half of the ellipse from the point(sinh a,0) to the point(-sinh a,0).

Again, with a positive and b lying between π and $3\pi/2$ we have both U and V negative, but if b assumes values between $3\pi/2$ and 2π with a remaining positive we have the point W in the fourth quadrant. Thus if a has remained constant the entire ellipse has been traced in counter-clock-wise fashion. If b continues to increase the ellipse is retraced. If b assumes negative values the curve is traced in reverse order again and again. By allowing a to vary from zero to positive infinity we can get all ellipses. It thus appears that any strip 2π wide and extending to the right from the y axis maps into the entire w plane.

(2) If a is negative and b varies from zero to 2π in the positive direction we have in view of the same equations used in (1) that the point W would move in a clock-wise direction, with a constant, along the ellipse from the point(-sinh a,o) and returning thereto. We thus secure no new points and it is evident that a strip 2π wide and extending to the left from the axis of y would also map into the entire w plane.

(3) Summing up: Any point of the z plane in the region to the right of the y axis and bounded by the lines y=o and $y=\pi/2$ maps into the point of intersection of the corresponding curves in the first quadrant of the w plane; any point similarly located as to the y axis but lying between the lines $y=\pi/2$ and $y=\pi$ will map into the point of intersection of the curves in the second quadrant; the next strip above gives the third, and the one following, the fourth quadrant of the w plane. To repeat the w plane we have but to take more strips. Thus the first quadrant alone gives the w plane infinitely many times. The same remark might be made of the other quadrants but the order in which the various areas on the w plane would be secured would vary as we have seen if we repeat this process in the second quadrant on the z plane.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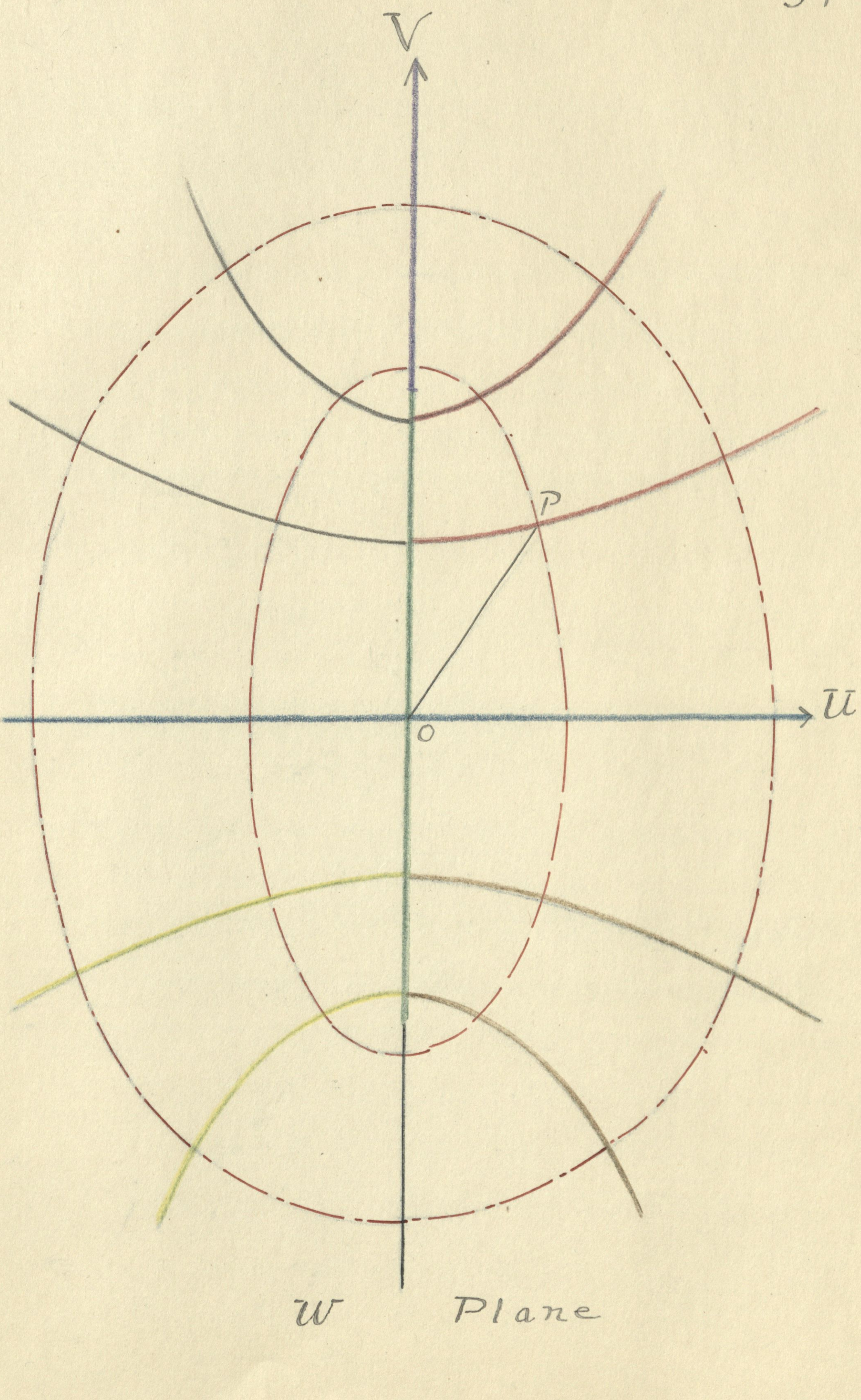
P₁
P₂
P₃
O

Z Plane



W Plane

Bibliography

"Applications of Hyperbolic Functions to Electrical Engineering Problems" by A.E.Kennelly

"Differential and Integral Calculus" by W.F.Osgood

"Hyperbolic Functions" by James McMahon

"Hyperbolic Functions" - in Smithsonian Mathematical Tables — by G.F.Becker and C.E.Van Orstrand

"Jahr Buch"- Berlin-Notes on and estimates of various articles not available in the origional.

"Les fonctions hyperboliques dans l'enseignement moyen" par P.Barbarin,in Ens.Math.2-443-447

"Tables of Complex and Circular Functions"- A.E.Kennelly

"Theorie des fonctions hyperboliques" par M.G.Fontene in Nouv.Annales de Mathematiques, 10-481-501