SIMPLE AND COMPLETE K-POINTS IN CONTINUOUS AND IN MODULAR PROJECTIVE SPACES

by

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INTRODUCTION.

It is the purpose of this paper,

(1) To trace the development of the concepts of the simple *k*-point and the complete *k*-point as they appear in mathematical literature and to formulate their definitions in a modular or an ordinary projective *n*-space (Sections 1 and 2); and

(2) To determine the number of complete *k*-points in a modular projective plane (Section 3).

Throughout this paper both the common terms and S_0, S_1, S_2, S_3, and S_r are used to mean a point, line, plane, 3-space and *r*-space respectively.

Section 1. The Simple K-Point.

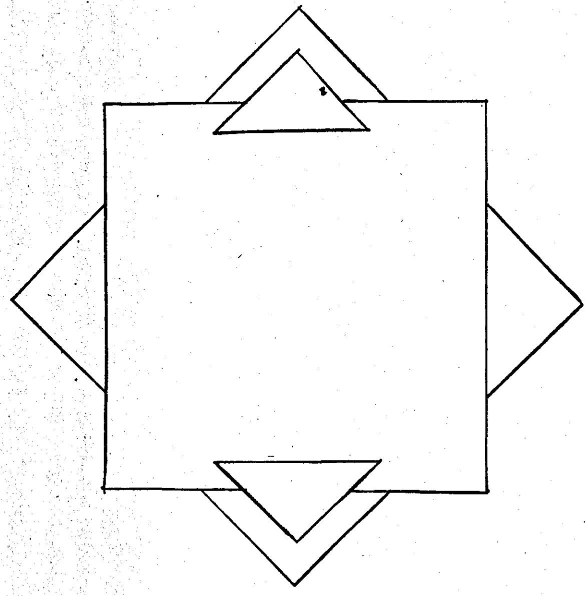
History of the simple k-point. The concepts of the simple 3-point and the simple 4-point date back at least to the Rhind papyrus presumably written by the Egyptian priest Ahmes as early as 1700 B.C. There the square, the oblong, the isosceles triangle and the isosceles trapezium appear.[1] Very early, too, the harpadonaptae or "rope-stretchers" were familiar with the right triangle whose sides are 3, 4 and 5.[2] Such figures as the square, the triangle, (probably the right triangle), the irregular quadrilateral, the regular hexagon and combinations of these were employed by the Assyrians and Babylonians in their mural decorations at a very early date.[3] Also on the walls of the earliest Egyptian temples may be found figures similar to these:[4]

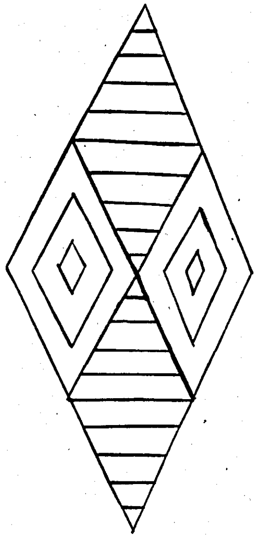
[1] Eisenlohr, p. II and pp. 126-9.

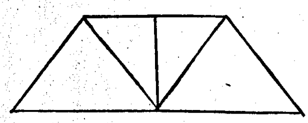
[2] Gow, pp. 129-30.

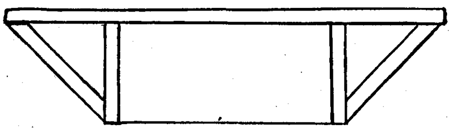
[3] Cantor, Vol. I, p. 98.

[4] Cantor, p. 66.









When Thales carried geometry to the Greek Colonies about 600 B.C., he mentioned several properties of triangles of various kinds; and Pythagoras added others to these besides the famous Pythagorean theorem. He also considered squares, quadrilaterals, regular pentagons and regular hexagons. He knew that the plane about a point can be completely filled by triangles, squares or regular hexagons, and knew the five regular solids inscribable in a sphere.[1] Moreover, it was from the tendency of the Pythagoreans to connect the ideas of number and magnitude that figurate numbers developed,[2] for they knew at least the triangular and square numbers.[3] Antiphon and other early Greek geometers who attempted the quadrature of the circle knew how to construct regular polygons where the number of sides is $2^n \cdot 3$ or $2^n \cdot 4$.[4] Euclid classified the 3-points and 4-points and assembled the properties of the simple figures due to earlier geometers. Proclus says Euclid first used the word parallelogram to include the square, rectangle, rhombus and rhomboid.[5]

1 Gow, p. 154.

2 Davies' Dictionary.

3 Gow, p. 70

4 Allman, pp 65-6.

5 Tropfke, Vol. II. pp 45 et seq.

He also presented the regular pentagon, which may have been known to the Pythagoreans, and the regular quindecagon, so that by his time any regular polygon whose number of sides is 2^n, $3\cdot 2^n$, $5\cdot 2^n$ or $15\cdot 2^n$ was understood.[1] Concave quadrilaterals were recognized by Proclus of Zenodorus about 180 B.C.[2]

From this it appears that only the 3-point and 4-point and the regular k-points for $k>4$ were considered; and no considerable further progress was made with regular polygons until the time of Gauss 2000 years later. Thru his analytic methods, the possibility of the construction of the inscribed seventeen-sided polygon was discovered, and later constructions have been given by Poncelet, Steiner, Von Staudt and others.[3] Gauss also generalized and proved that "a regular polygon of p sides where p is a prime of the form 2^h+1 is geometrically inscriptible".[4] The efforts of Kronecker, Eisenstein and Arndt finally led to the general theorem that "a regular polygon of n sides can be inscribed by ruler and compasses if and

1 Heath, Vol. II, pp. 97 and 110.

2 Tropfke, Vol. II, p. 52.

3 Klein, pp. 33-4

4 Young's Monographs - p. 386.

only if $n = 2^{\ell} \cdot p_1 \cdot p_2 \ldots$ where $p_1, p_2 \ldots$ are distinct primes of the form $2^{2^t} + 1$". Richelot worked out the construction for the polygon where one of the primes is 257.[1]

<u>Definition of a simple k- point</u>. The simple _k_-point has been defined by Veblen and Young for the plane and the 3-space;[2] but this definition permits two _r_-spaces to be identical and this seems undesirable. A simple _k_-point in an _n_-space may be defined as _k_ points, $P_1, P_2 \ldots P_k$, in the _n_-space, taken in a certain cyclic order,[3] together with the n-1 sets of _r_-spaces (r = 1,2 3 ...n-1), each set consisting of _k_ _r_-spaces, subject to the following conditions:

1. The _k_ points are so chosen that no r+1 consecutive points lie on the same (r-1)-space, and no two sets of _r_ + 1 points determine the same _r_-space.
2. Each _r_-space is determined by two consecutive (r-1)-spaces.

1 Tropfke, Vol. II, p. 107.

2 Veblen & Young, Vol. II, p. 107.

3 By cyclic order is meant that the first point (or first _r_-space) is considered as the successor of the k th point (or k th _r_-space).

Section 2. The Complete K-point.

History of the complete k-point. The history of the complete k-point in ordinary geometry is more difficult to trace, but we know that the Pythagorean brotherhood had as a secret sign the pentagram, or what we call the complete 5-line.[1] Menelaus used the complete 4-line in the first century A.D.; and Girard in examining the triangle and the diagonals of the simple quadrilateral, pentagon and hexagon was led to the consideration of the complete k-point in the plane for $k<7$, but Carnot (1753-1823) was the one who first used the expression "Complete four-side".[2] Lachlan in 1893 used the terms polystigm and polygram to denote the complete k-line in a plane.[3] In 1910, Veblen and Young published the definition of the complete k-point in a 3-space.

Definition of independent points. K points are said to be independent in an n-space (S_n) if there is no S_{n-1} which contains $n+1$ of them.

[1] Heath, Vol. II. p 99.

[2] Tropfke, Vol. II, pp. 52-3.

[3] Lachlan, p. 83.

Definition of a complete k-point in an S_n.

Obviously, a complete *k*-point in an S_n is the figure formed by *k* independent points, together with the $\binom{k}{2}$[1] S_1s joining every pair of the points, the $\binom{k}{3}$ S_2s joining every set of three of the points, the $\binom{k}{4}$ S_3s joining every set of four of the points, ... and the $\binom{k}{n}$ S_{n-1}s joining every set of *n* points.

1 The binomial coefficient $\binom{k}{n}$ is here used, of course, to mean $\frac{k(k-1)(k-2)\ldots\ldots(k-n+1)}{n!}$

Section 3. Number of Complete K-points In A Modular Projective Plane.

Projective and Modular spaces. The assumptions which characterize a projective *n*-space were given by Veblen & Young in "The American Journal of Mathematics", Vol. 30, pp. 348-52. The definition and general properties of finite projective spaces were given by Veblen and Bussey in "Transactions of the American Mathematical Society", Vol. 7, pp. 241-259. There a modular *n*-space is considered as containing $s^n + s^{n-1}$ ..... $s + 1$ points and, in particular, a line contains $s + 1$ points.

Number of complete *k*-points in a plane.

Let us first consider the number of points excluded from further choice as points of a *k*-point when an *r*-point has been constructed. It follows immediately from the definition that a 1-point excludes 1-point and a 2-point excludes $s + 1$ points, or s more points than were excluded by a 1-point. The number of additional points excluded by constructing a 3-point is clearly $2(s-1) + 1$, for the third point is a new point and from this point to each of the original two points there is a line which contains, besides the third point and a point of the 2-point, S-1 points. Now take a fourth point which has not been counted. Besides this point and a vertex of the original 3-point, each of the three new lines contains S-1 points, but since each line meets only two of the lines of the 3-point on a vertex, it will meet the third one in another point, and this point has already been counted. Hence the number of additional points used is $3(S-2) + 1$.

Likewise, the number of additional points excluded by an _r_-point after an (r-1)-point has been constructed is $(r-1)\left[S - \frac{(r-1)(r-2)}{2} - (r-2) - 1\right] + 1$, for in an (r-1)-point there are, by definition, r-2 lines on each vertex and a total of $\frac{(r-1)(r-2)}{2}$ lines. Therefore, each of the r-1 lines on the r^{th} point will contain, besides the r^{th} point, $S - \left\{\left[\frac{(r-1)(r-2)}{2} - (r-2)\right]\right\} - 1 = S - \frac{r^2 - 5r + 8}{2}$ points.

$\therefore$ the total number of points excluded by an _r_-point from further choice is:

$$N_r = \left[0(s-0) + 1\right] + \left[1(s-1) + 1\right] + \left[2(s-1) + 1\right] + \left[3(s-2) + 1\right]$$
$$+ \left[(r-1)(S - \frac{r^2 - 5r + 8}{2}) + 1\right] =$$

$$\frac{r(r-1)s}{2} + r - \left[0 + 1.1 + 2.1 + 3.2 + 4.4 \ldots\ldots (r-1)(\frac{r^2-5r+8}{2})\right.$$

$$= \frac{r(r-1)}{2} s + r - \frac{r(r-1(2r-1)}{6} - \left[0 + 0 - 2 - 3 - 0 + 5.2 \ldots\right.$$
$$\left.\frac{(r-1)\ (r-2)\ (r-5)}{2}\right] \qquad [1]$$

It is interesting to note that N_r in the form $N_r =$
$(1) + (s) + (2s-1) + (3s-5) + (4s-15) + (5s-34) \ldots$

$$(r-1)\ s - (\frac{r-1}{2})\ (r^2 - 4r + 5)$$

is a recurring series of the fourth order, where every five consecutive terms are connected by the following identity:

$$a_n - 4a_{n-1} + 6a_{n-2} - 4a_{n-3} + a_{n-4} = 0$$

Again, let us find the total number of points used in constructing an r-point by finding the number of points on the $\frac{r\ (r-1)}{2}$ lines. There would be $\frac{r(r-1)}{2}(s + 1)$ points if the intersections of the lines were counted on only one line. Each vertex is counted on r-1 lines or r-2 times too often and each diagonal point, the intersection of two sides not on a common vertex, is counted twice. The line on any two points is met in points other than the vertices by all of the lines on all of the other points or by $\frac{(r-2)(r-3)}{2}$ lines. This line may be determined in

$\frac{r(r-1)}{2}$ ways. Therefore, the total number of diagonal points is $1/2 \cdot \frac{r(r-1)}{2} \cdot \frac{(r-2)(r-3)}{2} = \frac{r(r-1)(r-2)(r-3)}{8}$,

and the total number of points used in constructing an r-point is

$$\frac{r(r-1)}{2}(s+1) - \left[r(r-2) + \frac{r(r-1)(r-2)(r-3)}{8}\right]$$

$$= \frac{r}{8}\left[4(r-1)(s+1) - (r-2)(r^2-4r+11)\right] \qquad [2]$$

Equating [1] and [2], we find they are equal, providing

$$\frac{-3r^4 + 26r^3 - 57r^2 + 34r}{24} = -\left[-2-3+0+5+2 \ldots \frac{(r-1)(r-2)(r-5)}{2}\right]$$

This is easily proved by mathematical induction, for, adding $\frac{r(r-1)(r-4)}{2}$ to both sides, we have

$$\frac{-3r^4 + 14r^3 + 3r^2 - 14r}{24} = -\left[-2-3+0+5+2 \ldots \frac{(r-1)(r-2)(r-5)}{2} + \frac{r(r-1)(r-4)}{2}\right]$$

which is the same result as that obtained by substituting $r + 1$ for r in the equation.

The number of complete k-points in a plane is, therefore,

$$\frac{(s^2+s+1)(s^2+s)(s^2)(s^2-2s+1)\ldots\ldots(s^2+s+1-N_{k-1})}{\lfloor k}.$$

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