PARTIALLY REGULAR POLYGONS INSCRIPTIBLE IN A CIRCLE.

by

Edith Gladys Jones

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Approved By:

W. G. Mitchell

Department of Mathematics.

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PARTIALLY REGULAR POLYGONS INSCRIPTIBLE IN A CIRCLE.

Introduction.

The subject of simple, convex[1] polygons which are inscriptible in a circle, and which are regular, that is, whose sides are equal and whose angles are equal, has been studied[2] very thoroughly for many centuries, but the subject of simple, convex polygons which are inscriptible in a circle and which are only partially regular, that is, which have only certain elements equal, has received very little attention. It is the purpose of this paper to investigate and determine properties of some of these latter polygons.

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[1]"A convex polygon is one no side of which when produced can enter within the space enclosed by the perimeter." Chauvenet, Treatise on Elementary Geometry, Philadelphia, 1875, page 35.
[2]See article by Leonard Eugene Dickson on the subject of regular polygons in Mathematical Monographs, New York, 1911, by J. W. A. Young, page 379, and also references to other writings on the same subject given at the end of that article.

They will be discussed in the following order:

I. Polygons inscriptible in a circle and having their opposite sides equal. These we shall call contra-regular polygons.

II. Polygons inscriptible in a circle and having their alternate sides equal. These we shall call altra-regular polygons.

III. Polygons inscriptible in a circle and having pairs of adjacent sides equal. These we shall call juxta-regular polygons.

All constructions are assumed made in an Euclidean plane and the theorems of elementary Euclidean geometry are assumed.

I. CONTRA-REGULAR POLYGONS.

DEFINITION. Opposite vertices of an inscribed polygon are vertices between which are the same number of other vertices in either direction.

DEFINITION. Opposite sides of an inscribed polygon are those sides between which are the same number of other sides in either direction.

DEFINITION. Opposite angles of a polygon are the angles at opposite vertices of the polygon.

DEFINITION. A diagonal of a polygon is a line joining two non-consecutive vertices.

THEOREM 1. <u>A diagonal joining opposite vertices of a contra-regular polygon is a diameter.</u>

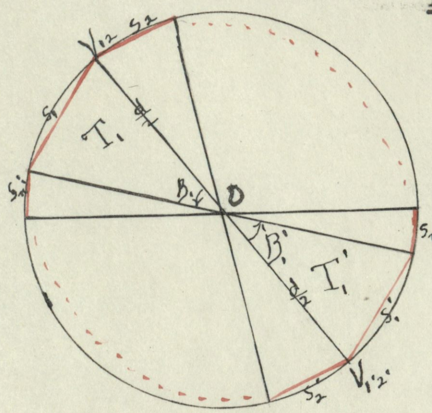

Fig. 1.

<u>Proof:</u> $s_1 = s_1'$ by hypothesis.

$\frac{d}{2} = \frac{d}{2}$.

$\triangle T_1 \cong \triangle T_1'$ (Triangles having three sides of one respectively equal to three sides of the other are congruent.)

$\angle B_1 = \angle B_1'$ (Corresponding parts of congruent triangles are equal.)

Similarly $\angle B_2 = \angle B_2'$ and so on to $\angle B_n = \angle B_n'$. But

$\angle B_1 + \angle B_2 + \cdots\cdots + \angle B_n + \angle B_1' + \angle B_2' + \cdots\cdots + \angle B_n' = 360°$ and

$2(\angle B_1 + \angle B_2 + \cdots + \angle B_n) = 360°$. Therefore, $\angle B_1 + \angle B_2 + \cdots + \angle B_n = \frac{1}{2}(360°) = 180° = \angle V_{12}\ OV_{1'2'}$. Thus $V_{12}\ OV_{1'2'}$ is a straight line, and a diameter.

CONSTRUCTION. <u>The above theorem gives us a method for constructing a contra-regular polygon.</u>

In a semi-circle inscribe a semi-polygon of n sides. Draw diameters from each vertex of this semi-polygon. The vertices of the other half of a contra-regular polygon of 2n sides are determined where the diameters intersect the remaining semi-circle.

THEOREM 2. <u>The opposite angles of a contra-regular polygon are equal.</u>

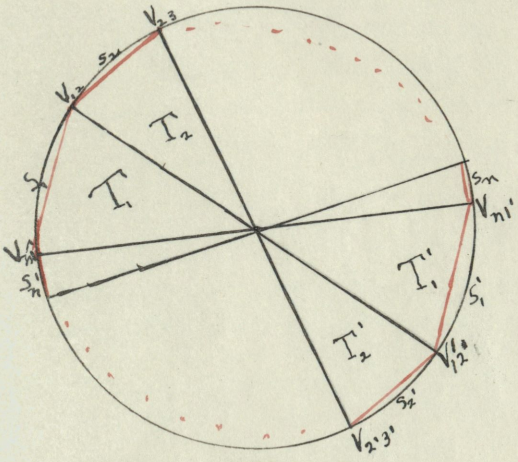

Fig. 2.

Proof: $\Delta T_1 \cong \Delta T_1'$ and $\Delta T_2 \cong \Delta T_2'$ (See Theorem 1.)

Hence quadrilateral $\frac{d}{2} s_1 s_2 \frac{d}{2} \cong \frac{d}{2} s_1' s_2' \frac{d}{2}$. Therefore, angle $(\alpha_1 + \alpha_2)$ equals angle $(\alpha_1' + \alpha_2')$, i.e. $\angle V_{12} = \angle V_{1'2'}$. Similarly the remaining opposite angles, V_{23} and $V_{2'3'}$, V_{34} and $V_{3'4'}$, and so on to $V_{n'1}$ and $V_{n1'}$, may be shown to be equal.

THEOREM 3. <u>The opposite sides of a contra-regular polygon are parallel.</u>

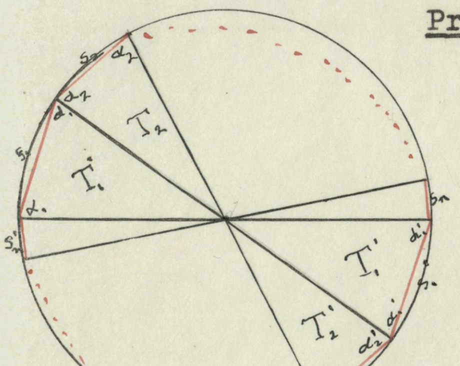

Fig. 3.

Proof: Triangles T_1, T_2, ... T_n, T_1', T_2', ... T_n' are isosceles. $T_1 \cong T_1'$. (See Theorem 1.) $\angle \alpha_1 = \angle \alpha_1'$ Thus $s_1 \parallel s_1'$. (If, when two lines are crossed by a third, the alternate interior angles are equal, the lines are parallel.) Similarly $s_2 \parallel s_2'$, - - - and $s_n \parallel s_n'$.

THEOREM 4. <u>The area of a contra-regular polygon of 2n sides is equal to</u> $\frac{1}{2}(s_1\sqrt{d^2 - s_1^2} + s_2\sqrt{d^2 - s_2^2} + \dots + s_n\sqrt{d^2 - s_n^2})$.

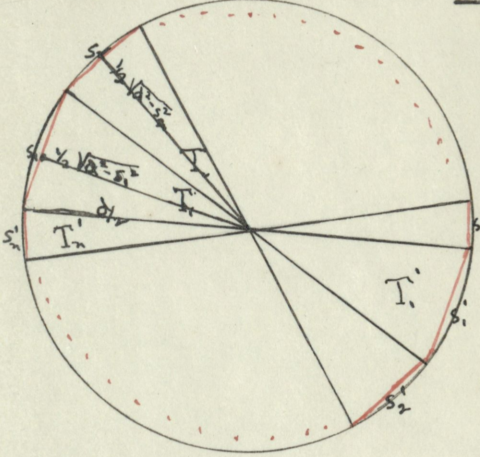

Fig. 4.

<u>Proof:</u> The altitude of T_1, an isosceles triangle of sides $\frac{d}{2}$, $\frac{d}{2}$, s_1 is $\frac{1}{2}\sqrt{d^2 - s_1^2}$ and its area $\frac{1}{4}s_1\sqrt{d^2 - s_1^2}$. The area of $T_1 + T_1' = \frac{1}{2}s_1\sqrt{d^2 - s_1^2}$. Similarly the area of $T_2 + T_2' = \frac{1}{2}s_2\sqrt{d^2 - s_2^2}$, and the area of $T_n + T_n' = \frac{1}{2}s_n\sqrt{d^2 - s_n^2}$. Hence the area of the polygon is

$$\frac{1}{2}(s_1\sqrt{d^2 - s_1^2} + s_2\sqrt{d^2 - s_2^2} + \cdots + s_n\sqrt{d^2 - s_n^2}).$$

THEOREM 5. <u>A contra-regular polygon of four sides is a rectangle whose area is $s_1 s_2$.</u>

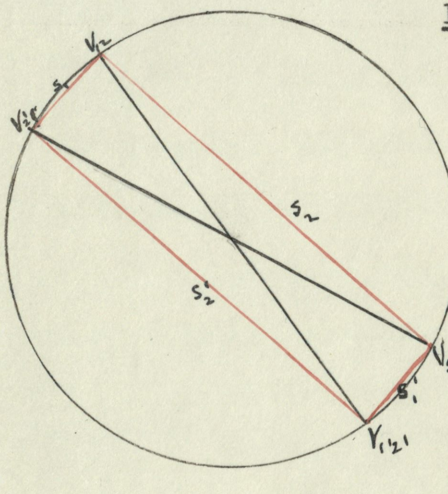

Fig. 5.

<u>Proof:</u> $\triangle V_{2'1} V_{12} V_{21'}$ has two vertices on a diameter. Hence, $\angle V_{12}$ is a right angle, and since the opposite $\angle V_{1'2'} = \angle V_{12}$, $\angle V_{1'2'}$ is a right angle. Similarly, $\angle V_{21'}$ and $\angle V_{2'1}$ are right angles. $s_1 \parallel s_1'$, and $s_2 \parallel s_2'$. Therefore, $V_{12} V_{21'} V_{1'2'} V_{2'1}$ is a rectangle and consequently the area of the quadrilateral is $s_1 s_2$.

THEOREM 6. <u>The area of a contra-regular polygon of six sides may be expressed as</u> $A = s_1\sqrt{d^2 - s_1^2} + 2\sqrt{s(s - s_2)(s - s_3)(s - \sqrt{d^2 - s_1^2})}$

<u>where</u> $s = \frac{s_2+s_3+\sqrt{d^2-s_1^2}}{2}$.

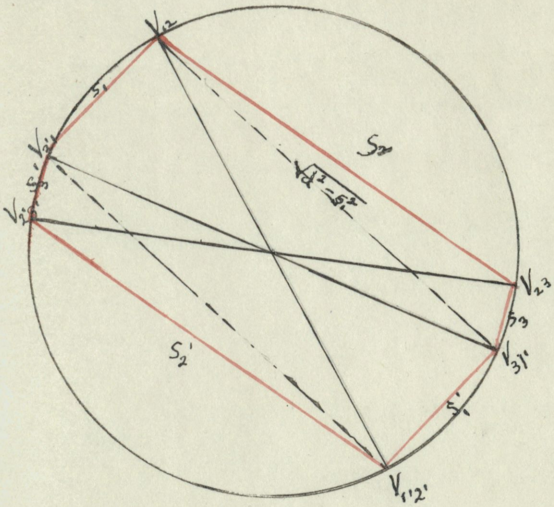

Fig. 6.

<u>Proof:</u> $V_{12}V_{31'}V_{1'2'}V_{3'1}$ is a rectangle of area equal to $s_1\sqrt{d^2-s_1^2}$. Removing $V_{12}V_{31'}V_{1'2'}V_{3'1}$ from the polygon leaves two congruent triangles $V_{12}V_{23}V_{31'}$ and $V_{1'2'}V_{2'3'}V_{3'1}$, each having an area equal to $\sqrt{s(s-s_2)(s-s_3)(s-\sqrt{d^2-s_1^2})}$, s being equal to one-half the sum of the three sides. (Formula from Trigonometry.) Hence $A = s_1\sqrt{d^2-s_1^2} + 2\sqrt{s(s-s_2)(s-s_3)(s-\sqrt{d^2-s_1^2})}$.

THEOREM 7. <u>The relation between the sides of a contra-regular hexagon and the diameter of the circumscribed circle is expressed by the equations</u> $d^3 - d(s_1^2+s_2^2+s_3^2) - 2s_1s_2s_3 = 0$, and $d^3 - d(s_1^2+s_2^2+s_3^2) + 2s_1s_2s_3 = 0$.

<u>Proof:</u> In any triangle of sides a, b, c inscribed in a circle of diameter d, $d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}$ where $s = \frac{a+b+c}{2}$. In Fig. 6

$$d = \frac{s_2s_3\sqrt{d^2-s_1^2}}{\frac{2}{4}\sqrt{(s_2+s_3+\sqrt{d^2-s_1^2})(-s_2+s_3+\sqrt{d^2-s_1^2})(s_2-s_3+\sqrt{d^2-s_1^2})(s_2+s_3-\sqrt{d^2-s_1^2})}} =$$

$$\frac{2s_2s_3\sqrt{d^2-s_1^2}}{\sqrt{[d^2-s_1^2-(s_2-s_3)^2]\,[(s_2+s_3)^2-d^2+s_1^2]}} =$$

$$\frac{2s_2 s_3\sqrt{d^2-s_1^2}}{\sqrt{(d^2-s_1^2-s_2^2+2s_2s_3-s_3^2)(s_2^2+s_3^2+2s_2s_3-d^2+s_1^2)}}$$

$$d^2=\frac{4s_2^2s_3^2d^2-4s_1^2s_2^2s_3^2}{4s_2^2s_3^2-(s_1^2+s_2^2+s_3^2-d^2)^2}=$$

$$\frac{-4s_2^2s_3^2d^2+4s_1^2s_2^2s_3^2}{-4s_2^2s_3^2+s_1^4+s_2^4+s_3^4+d^4+2s_1^2s_2^2+2s_1^2s_3^2-2s_1^2d^2+2s_2^2s_3^2-2s_2^2d^2-2s_3^2d^2}$$

$$d^6+d^4(-2s_1^2-2s_2^2-2s_3^2)+d^2(-4s_2^2s_3^2+s_1^4+s_2^4+s_3^4+2s_1^2s_2^2$$

$$+2s_1^2s_3^2+2s_2^2s_3^2+4s_2^2s_3^2)-4s_1^2s_2^2s_3^2=0.$$

$$d^6-2d^4(s_1^2+s_2^2+s_3^2)+d^2(s_1^2+s_2^2+s_3^2)^2-4s_1^2s_2^2s_3^2=0.$$

$$[d^3-d(s_1^2+s_2^2+s_3^2)-2s_1s_2s_3][d^3-d(s_1^2+s_2^2+s_3^2)+2s_1s_2s_3]=0.$$

This relation is also shown by evaluating $\sin^2 V_{23}$ and $\cos^2 V_{23}$, and simplifying the equation $\sin^2 V_{23}+\cos^2 V_{23}=1$. A third method of securing this result is as follows: $\cos V_{23}=\frac{s_2^2+s_3^2-d^2+s_1^2}{2s_2s_3}=$ $\cos(\alpha+90°)=-\sin\alpha=-\frac{s_1}{d}$. Simplifying, the equation reduces to $d^3-d(s_1^2+s_2^2+s_3^2)-2s_1s_2s_3=0$.

THEOREM 8. <u>In a contra-regular polygon the values of the angles at the vertices are</u>

$$\angle V_{12}=\text{arc sin }\frac{s_1\sqrt{d^2-s_2^2}+s_2\sqrt{d^2-s_1^2}}{d^2}$$

$$\angle V_{23}=\text{arc sin }\frac{s_2\sqrt{d^2-s_3^2}+s_3\sqrt{d^2-s_2^2}}{d^2}$$

$$\vdots$$

$$\angle V_{n-1,n}=\text{arc sin }\frac{s_{n-1}\sqrt{d^2-s_n^2}+s_n\sqrt{d^2-s_{n-1}^2}}{d^2}$$

$$\angle V_{n,1}=\text{arc sin }\frac{s_n\sqrt{d^2-s_1^2}+s_1\sqrt{d^2-s_n^2}}{d^2}$$

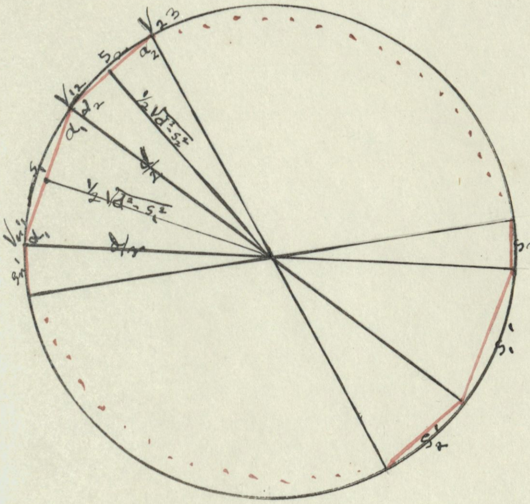

Fig. 7.

Proof: $\sin\alpha_1 = \frac{\sqrt{d^2 - s_1^2}}{d}$; $\cos\alpha_1 = \frac{s_1}{d}$

$\sin\alpha_2 = \frac{\sqrt{d^2 - s_2^2}}{d}$; $\cos\alpha_2 = \frac{s_2}{d}$

$\sin(\alpha_1 + \alpha_2) = \sin\alpha_1\cos\alpha_2 + \cos\alpha_1\sin\alpha_2 =$

$\frac{\sqrt{d^2 - s_1^2}}{d}\frac{s_2}{d} + \frac{s_1}{d}\frac{\sqrt{d^2 - s_2^2}}{d} =$

$\frac{s_2\sqrt{d^2 - s_1^2} + s_1\sqrt{d^2 - s_2^2}}{d^2}$. There-

fore, $V_{12} = (\alpha_1 + \alpha_2) = \text{arc sin}$

$\frac{s_1\sqrt{d^2 - s_2^2} + s_2\sqrt{d^2 - s_1^2}}{d^2}$.

In like manner,

$$V_{23} = \text{arc sin } \frac{s_2\sqrt{d^2 - s_3^2} + s_3\sqrt{d^2 - s_2^2}}{d^2}$$

$$\vdots$$

$$V_{n-1,n} = \text{arc sin } \frac{s_{n-1}\sqrt{d^2 - s_n^2} + s_n\sqrt{d^2 - s_{n-1}^2}}{d^2}$$

$$V_{n,1} = \text{arc sin } \frac{s_n\sqrt{d^2 - s_1^2} + s_1\sqrt{d^2 - s_n^2}}{d^2}$$

THEOREM 9. In a contra-regular polygon the values of the angles between the radii of the circumscribed circle which join the center of the circle and the vertices of the polygon are:

$$\angle B_1 = \text{arc sin } \frac{2\, s_1\sqrt{d^2 - s_1^2}}{d^2}$$

$$\angle B_2 = \text{arc sin } \frac{2\, s_2\sqrt{d^2 - s_2^2}}{d^2}$$

$$\vdots$$

$$\angle B_n = \text{arc sin } \frac{2\, s_n\sqrt{d^2 - s_n^2}}{d^2}$$

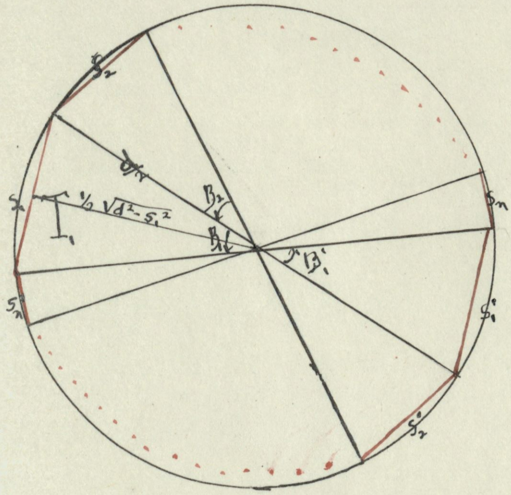

Fig. 8.

Proof: The altitude of the isosceles triangle T_1 bisects the vertex angle B_1. $\sin \frac{B_1}{2} = \frac{s_1}{d}$; $\cos \frac{B_1}{2} = \frac{\sqrt{d^2 - s_1^2}}{d}$

$$\sin B_1 = 2 \sin \frac{B_1}{2} \cos \frac{B_1}{2} = \frac{2s_1\sqrt{d^2 - s_1^2}}{d^2}.$$

$$B = \text{arc} \sin \frac{2s_1\sqrt{d^2 - s_1^2}}{d^2}.$$

In like manner

$$B_2 = \text{arc} \sin \frac{2s_2\sqrt{d^2 - s_2^2}}{d^2}$$

$$\vdots$$

$$B_n = \text{arc} \sin \frac{2s_n\sqrt{d^2 - s_n^2}}{d^2}.$$

II. ALTRA-REGULAR POLYGONS.

THEOREM 1. An altra-regular polygon of 2n sides is, if $n=2k$, a contra-regular polygon, and hence has all of the properties of a contra-regular polygon.

Proof: Denoting the sides of an altra-regular polygon as s_1, s_2, $\cdots\cdots s_n$, we have (1). $s_1 = s_3 = s_5 = \cdots = s_{2n-1}$ and (2) $s_2 = s_4 = s_6 = \cdots = s_{2n}$. The side opposite a given side s_1 is s_{n+1}. If n is odd, s_{n+1} belongs to (2) and hence does not equal s_1, but if n is even, i.e. if $n = 2k$, s_{n+1} belongs to (1) and hence equals s_1, and the polygon is a contra-regular polygon.

THEOREM 2. <u>The angles of an altra-regular polygon are equal.</u>

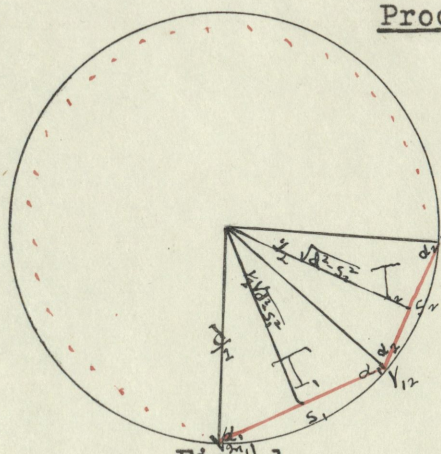

Fig. 1.

Proof: $\Delta T_1 \cong \Delta T_3 \cong \Delta T_5 \cong \dots \cong \Delta T_{2n-1}$ and $\Delta T_2 \cong \Delta T_4 \cong \Delta T_6 \cong \dots \cong \Delta T_{2n}$. (Triangles having three sides of one respectively equal to three sides of the other are congruent.)

$\angle\alpha_1 = \angle\alpha_3 = \angle\alpha_5 = \dots = \angle\alpha_{2n-1}$ and $\angle\alpha_2 = \angle\alpha_4 = \dots = \angle\alpha_{2n}$

Thus, $\angle(\alpha_1+\alpha_2) = V_{12} = V_{23} = V_{34} = V_{45} = \dots = V_{2n,1}$.

THEOREM 3. <u>The value of each angle at the circumference of an altra-regular polygon is</u> $\sin^{-1}\dfrac{(s_1\sqrt{d^2-s_2^2}+s_2\sqrt{d^2-s_1^2})}{d^2}$.

Proof: (See Fig. 1.) $\sin\alpha_1 = \dfrac{\sqrt{d^2-s_1^2}}{d}$; $\sin\alpha_2 = \dfrac{\sqrt{d^2-s_2^2}}{d}$

$\cos\alpha_1 = \dfrac{s_1}{d}$. $\cos\alpha_2 = \dfrac{s_2}{d}$. $\sin(\alpha_1+\alpha_2) = \dfrac{s_1\sqrt{d^2-s_2^2}+s_2\sqrt{d^2-s_1^2}}{d^2}$

But $\angle(\alpha_1+\alpha_2) = V_{12} = V_{23} = V_{34} = V_{45} = \dots = V_{2n,1}$. Hence,

$$V_{i,i+1} = \sin^{-1}\frac{s_1\sqrt{d^2-s_2^2}+s_2\sqrt{d^2-s_1^2}}{d^2}$$

($i = 1, 2, \dots\dots\dots\dots 2n$, and $2n+1 = 1$).

THEOREM 4. <u>Opposite sides of an altra-regular polygon are parallel.</u>

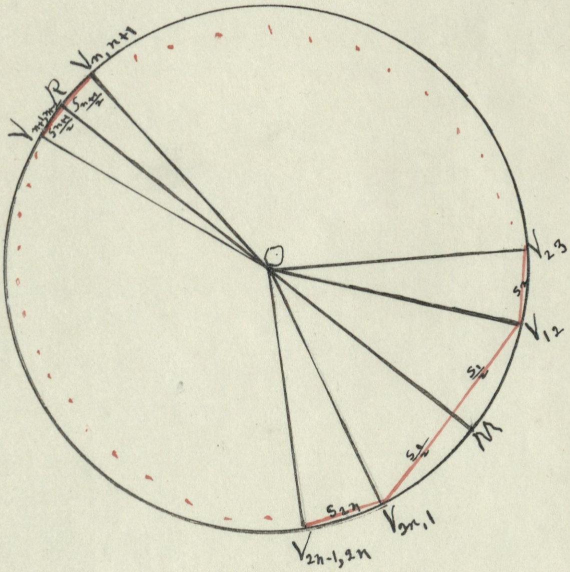

Fig. 2.

Proof: In an altra-regular polygon of 2n sides erect the mid-perpendicular to a given side s_1 and the mid-perpendicular to s_{n+1}, the side opposite s_1. Join their point of intersection, which is the center of the circle, with the vertices of the polygon. arc $MV_{2n,1}$ = arc MV_{12} (The bisector of an angle bisects the arc subtended by it.)

arc $V_{2n-1,2n}V_{2n,1}$ = arc V_{12} V_{23} = ... =

arc $V_{n-1,n}V_{n,n+1}$ = arc $V_{n+1,n+2}$ $V_{n+2,n+3}$.

(Equal chords subtend equal arcs.)

arc $RV_{n+1,n+2}$ = arc $RV_{n,n+1}$. Hence,

arc RM = arc MR. But, arc RMR subtends a 360° angle. Therefore, arc RM and arc MR each subtend a 180° angle, and ROM is a straight line. Hence s_1 is parallel to s_{n+1}.

THEOREM 5. <u>Diagonals between opposite vertices of an altra-regular polygon are parallel to the sides opposite.</u>

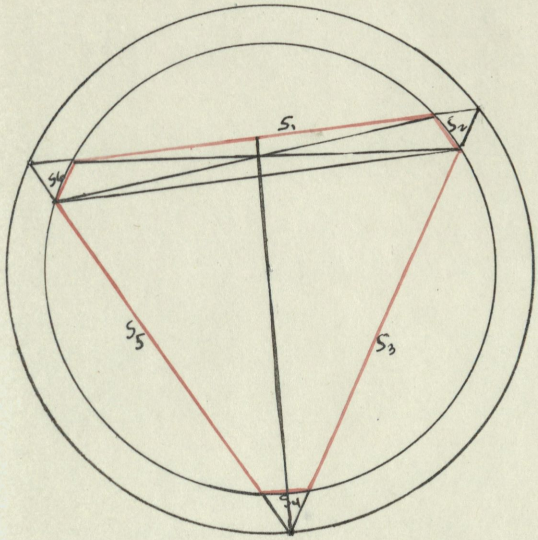

Fig. 3.

<u>Proof:</u> Erect the mid-perpendicular to the diagonal and the mid-perpendicular to one of the sides opposite it, say s_1. Join their intersection with each vertex of the part of the polygon included within the diagonal and s_1. The two triangles whose bases are $\frac{s_1}{2}$ are congruent. Those whose bases are s_2 and s_{2n}, and so on to those whose bases are one-half the diagonal are also congruent since each pair has two sides and the included angle equal. The sum of the angles about the intersection of the mid-perpendiculars is $360°$. The angle opposite $\frac{s_1}{2}$ equals the angle opposite $\frac{s_1}{2}$, that opposite s_2 equals that opposite s_{2n} and so on. Therefore, the sum of the central angles at O on the right of the mid-perpendiculars equals the sum of those on the left and hence is equal to $180°$. Thus, the mid-perpendicular to s_1 is the mid-perpendicular to the diagonal, and the diagonal is parallel to s_1. Since s_1 is parallel

to s_{n+1}, the diagonal is also parallel to s_{n+1}.[1]

THEOREM 6. *In an altra-regular polygon the value of each angle at the center opposite a side equal to s_1 is $\sin^{-1}\dfrac{2s_1\sqrt{d^2-s_1^2}}{d^2}$ and of each angle opposite a side equal to s_2 is $\sin^{-1}\dfrac{2s_2\sqrt{d^2-s_2^2}}{d^2}$.*

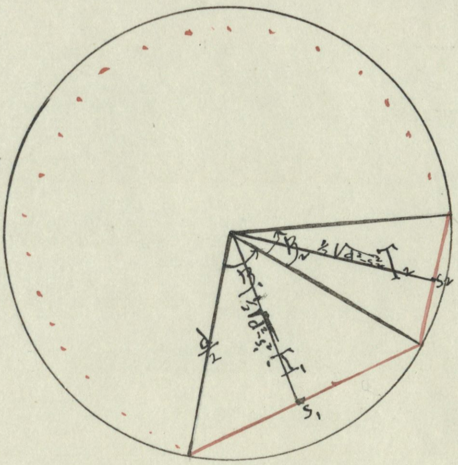

Fig. 4.

Proof: The altitude of isosceles triangle T_1 bisects the vertex angle B_1. $\sin\dfrac{B_1}{2}=\dfrac{s_1}{d}$; $\sin\dfrac{B_2}{2}=\dfrac{s_2}{d}$.

$\cos\dfrac{B_1}{2}=\dfrac{\sqrt{d^2-s_1^2}}{d}$; $\cos\dfrac{B_2}{2}=\dfrac{\sqrt{d^2-s_2^2}}{d}$

$\sin B_1=\dfrac{2s_1\sqrt{d^2-s_1^2}}{d^2}$; $\sin B_2=\dfrac{2s_2\sqrt{d^2-s_2^2}}{d^2}$.

Therefore, since $T_1\cong T_3\cong\cdots\cong T_{2n-1}$ and $T_2\cong T_4\cong\cdots\cong T_{2n}$, $B_1=\sin^{-1}\dfrac{2s_1\sqrt{d^2-s_1^2}}{d^2}=B_3=B_5=\cdots=B_{2n-1}$ and

$$B_2=\sin^{-1}\frac{2s_2\sqrt{d^2-s_2^2}}{d^2}=B_4=B_6=B_8=\cdots=B_{2n}.$$

THEOREM 7. *The area of an altra-regular polygon of 2n sides is $\dfrac{n}{4}\left(s_1\sqrt{d^2-s_1^2}+s_2\sqrt{d^2-s_2^2}\right)$.*

[1]Polygons in which $n=4k$ are not included in this discussion, since a vertex, and not a side, is opposite the diagonal.

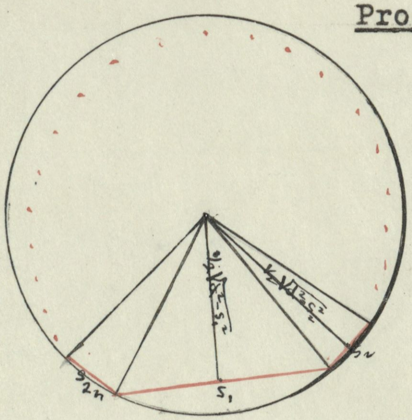

Fig. 5.

Proof: The polygon is composed of n triangles of area $\frac{s_1}{4}\sqrt{d^2 - s_1^2}$ and of n triangles of area $\frac{s_2}{4}\sqrt{d^2 - s_2^2}$. Hence the total area of the polygon is $\frac{n}{4}(s_1\sqrt{d^2 - s_1^2} + s_2\sqrt{d^2 - s_2^2})$.

CONSTRUCTION 1. To construct any altra-regular polygon which is inscriptible in a circle by means of ruler and compass.

Method 1.

Construction: Inscribe in a circle a regular polygon[1] of n sides. Moving in a counter-clockwise direction at each vertex of the regular polygon as a center, and with a radius less than the side of the regular polygon, describe an arc cutting the circle. With the vertices of the regular polygon and the points where the arcs cut the circle as

[1] "A regular polygon of n sides can be inscribed by ruler and compasses if, and only if, $n = 2^{\ell} p_1 p_2 \ldots.$, where $p_1 p_2 \ldots$ are distinct primes of the form $2^{2^t} + 1$." J. W. A. Young, Mathematical Monographs, New York, 1911, page 379.

vertices construct a polygon. This is an altra-regular polygon of $n = 2^{\ell+1} p_1 p_2 \ldots\ldots$ sides where $p_1 p_2 \ldots\ldots$ are distinct primes of the form $2^{2^t}+1$.[1] This construction leads us to

THEOREM 8. <u>Only altra-regular polygons of $2^{\ell+1} p_1 p_2 \ldots$ sides where $p_1 p_2 \ldots\ldots$ are distinct primes of the form $2^{2^t}+1$ can be inscribed in a circle by means of a ruler and compass.</u>

<u>Proof:</u> Suppose it is possible to inscribe in a circle by means of ruler and compass an altra-regular polygon of 2k sides where 2k is not equal to $2^{\ell+1} p_1 p_2 \ldots\ldots$, $p_1 p_2 \ldots$ being distinct primes of the form $2^{2^t}+1$. Draw $V_{2n,1}V_{23}$, $V_{23}V_{45}$, ... to $V_{2n-2,2n-1}V_{2n,1}$. Since $s_1 = s_3 = s_5 = \ldots\ldots = s_{2n-1}$; $s_2 = s_4 = \ldots = s_{2n}$, and $\angle V_{12} = \angle V_{34} = \ldots = V_{2n-1,2n}$, $\Delta V_{2n,1} V_{12} V_{23} \cong \Delta V_{23} V_{34} V_{45} \cong \ldots \cong \Delta V_{2n-2,2n-1} V_{2n-1,2n} V_{2n,1}$. Hence, $V_{2n,1}V_{23} = V_{23}V_{45} = \ldots = V_{2n-2,2n-1}V_{2n,1}$, and the polygon formed by them is a regular polygon of k sides inscribed in a circle. But it is possible to inscribe in a circle by means of ruler and compass regular polygons of only $2^{\ell} p_1 p_2 \ldots$ sides, $p_1 p_2 \ldots\ldots$ being distinct primes of the form

[1] See Plates I. and II. for illustrations of this method of construction of altra-regular polygons.

$2^{2^r}+1$. Hence, it is possible to inscribe in a circle altra-regular polygons of only $2^{l+1}p_1 p_2 \ldots\ldots$ sides, $p_1 p_2 \ldots\ldots$ being distinct primes of the form $2^{2^t}+1$.

CONSTRUCTION 1. <u>To construct any altra-regular polygon which is inscriptible in a circle by means of ruler and compass.</u>

<u>Method 2.</u>

<u>Construction:</u> As in Method 1. inscribe in a circle a regular polygon of n sides. With the center of the circle, O, as a center, and a radius greater than the radius of a circle inscribed in the polygon and less than the radius of the circumscribed circle, describe a circle. The points at which the circle cuts the regular polygon are vertices of an altra-regular polygon of 2n sides.

<u>Proof:</u> Drop the perpendiculars p_1, $p_2 \ldots\ldots p_n$ respectively to the sides S_1, S_2, ... S_n of the regular polygon. $p_1 = p_2 = \ldots = p_n$. (In the same circle, equal chords are equally distant from the center.) $\frac{d}{2} = \frac{d}{2}$. Therefore triangles of bases $\frac{s_1}{2}$, $\frac{s_3}{2}$, ... $\frac{s_{2n-1}}{2}$ are congruent. (Right triangles having

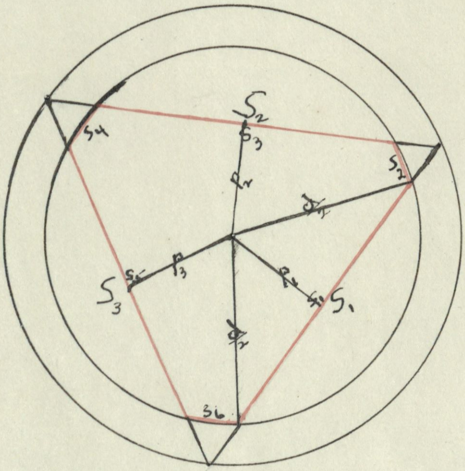

Fig. 6.

the hypotenuse and one side of one respectively equal to the hypotenuse and one side of the other, are congruent.) Hence, $\frac{s_1}{2} = \frac{s_3}{2} = \frac{s_5}{2} = \cdots = \frac{s_{2n-1}}{2}$, and $s_1 = s_3 = \cdots\cdots = s_{2n-1}$. The small triangles at the vertices of the regular polygon have two sides equal to $\frac{S_1}{2} - \frac{s_1}{2}$, and hence equal to each other, and their included angles, being angles of the regular polygon are equal. Hence the triangles are congruent. (Triangles having two sides and the included angle of one respectively equal to two sides and the included angle of the other, are congruent.) Hence, $s_2 = s_4 = s_6 = s_8 = \cdots\cdots = s_{2n}$. Thus the polygon of sides s_1, s_2, ... s_{2n} is an altra-regular polygon.

LEMMA 1. <u>The corners cut off from an equilateral triangle in the construction of an altra-regular hexagon (Method 2, Construction 1) are themselves equilateral triangles of sides equal to s_2.</u>

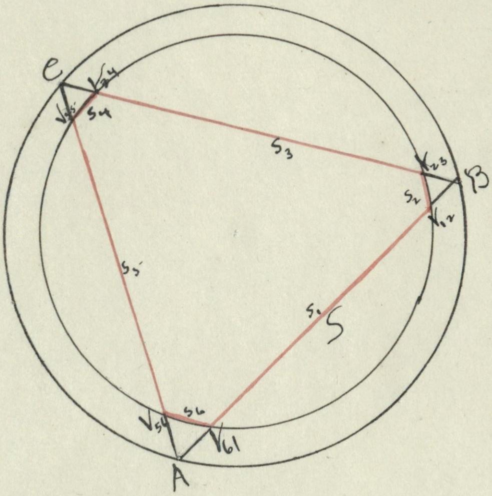

Fig. 7.

Proof: In the Proof, Construction 1, Method 2, it was shown that the corners cut off from a regular polygon in the construction of an altra-regular polygon are congruent triangles having two sides equal to $\frac{S}{2} - \frac{s_1}{2}$. Hence, in Fig. 7, triangles $AV_{61}V_{56}$, $BV_{12}V_{23}$, and $CV_{34}V_{45}$ are congruent isosceles triangles. But $\angle A = \angle B = \angle C = 60°$. Therefore, $\angle AV_{61}V_{56} = \angle AV_{56}V_{61} = \angle BV_{12}V_{23} = \angle BV_{23}V_{12} = \angle CV_{34}V_{45} = \angle CV_{45}V_{34} = 60°$. Thus triangles $AV_{61}V_{56}$; $BV_{12}V_{23}$; and $CV_{34}V_{45}$ are equiangular and hence equilateral. But one side of $\Delta BV_{12}V_{23}$ is s_2. Therefore each side of each of the three triangles equals s_2.

COROLLARY. The sides of the equilateral triangle used as a base of an altra-regular hexagon in Method 2, Construction 1 are equal to $s_1 + 2s_2$.

LEMMA 2. The diameter D of the circle circumscribed about an equilateral triangle S S S equals $\frac{2}{\sqrt{3}}$ S.

Proof: $\frac{2}{3} A = \frac{D}{2}$ where A is the altitude of the triangle. (The medians, which are also altitudes of an isosceles triangle, meet in a point, which is two-thirds of the distance from a vertex to the mid-point of the opposite side.)

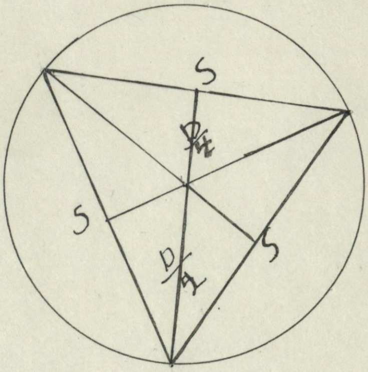

Fig. 8.

The area of the triangle is $\frac{3}{8}$ DS.

$D = \dfrac{S^3}{2\,\frac{3}{8}DS} = \dfrac{S^2}{\frac{3}{4}D}$. $D^2 = \frac{4}{3}S^2$.

$D = \dfrac{2}{\sqrt{3}}S$.

THEOREM 9. <u>In an altra-regular hexagon the diameter of the circumscribing circle is</u> $d = \frac{2\sqrt{3}}{3}\sqrt{s_1^2 + s_1 s_2 + s_2^2}$.

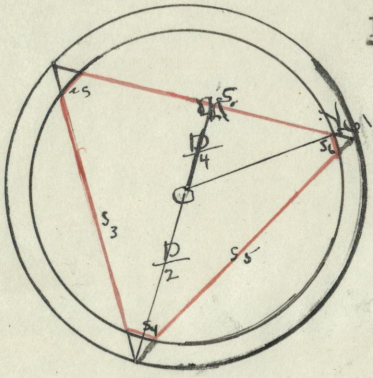

Fig. 9.

<u>Proof:</u> $ON = \frac{D}{4}$ (D equals the diameter of the circumscribing circle of triangle ABC.) $\overline{OV}_{61}^2 = \overline{ON}^2 + \overline{NV}_{61}^2$.

$\frac{d^2}{4} = \frac{D^2}{16} + \frac{s_1^2}{4}$. But $D = \frac{2S}{\sqrt{3}} = \frac{2}{\sqrt{3}}(s_1 + 2s_2)$.

Therefore, $\frac{d^2}{4} = \frac{4}{3}\,\frac{(s_1^2 + 4s_1 s_2 + 4s_2^2)}{16} + \frac{s_1^2}{4}$.

$d^2 = \dfrac{s_1^2 + 4s_1 s_2 + 4s_2^2 + 3s_1^2}{3} = \dfrac{4(s_1^2 + s_1 s_2 + s_2^2)}{3}$.

Therefore, $d = \frac{2\sqrt{3}}{3}\sqrt{s_1^2 + s_1 s_2 + s_2^2}$.

COROLLARY. <u>The area of an altra-regular hexagon in terms of the sides is:</u> $\frac{\sqrt{3}}{4}\left(s_1\sqrt{s_1^2 + 4s_1 s_2 + 4s_2^2} + s_2\sqrt{4s_1^2 + 4s_1 s_2 + s_2^2}\right)$.

CONSTRUCTION 2. <u>To construct an altra-regular hexagon when the sides s_1 and s_2 are given.</u>

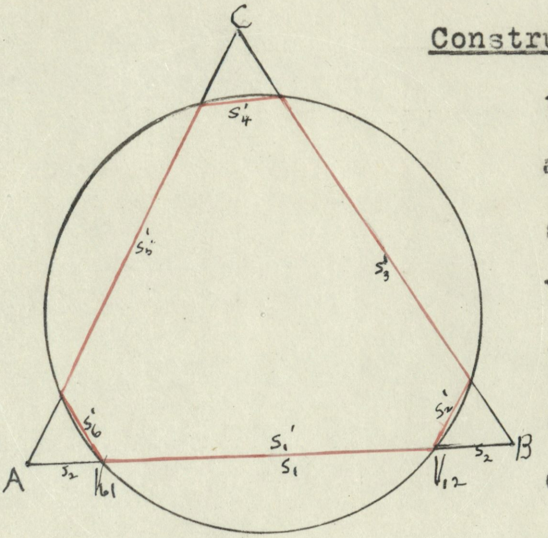

Fig. 10.

Construction: Construct an equilateral triangle ABC of base s_1+2s_2. With a center at A and an arc equal to s_2 describe an arc cutting AB at V_{61}. Determine O, the center of the circle which can be circumscribed about triangle ABC. With O as a center and OV_{61} as a radius describe a circle. The intersections of the circle with ΔABC are the vertices of an altra-regular hexagon of sides s_1 and s_2.

Proof: $AV_{61} = s_2' = s_4' = s_6' = s_2$. (Lemma 1). $s_1 = AB - (AV_{61} + BV_{12}) = s_1' = s_3' = s_5'$. Therefore, $V_{61}\ V_{12}\ V_{23}\ V_{34}\ V_{45} V_{56}$ is the required altra-regular hexagon.

LEMMA. The corners cut off from a square by an altra-regular octagon (Construction 1, Method 2) are congruent isosceles right triangles of sides equal to $\frac{\sqrt{2}}{2}s_2$ and hypotenuse s_2.

Proof: In the Proof, Construction 1, Method 2, it was shown that the corners cut off from a regular polygon in the construction of an altra-regular polygon are congruent triangles having two sides equal to $\frac{S}{2} - \frac{s_1}{2}$. Hence in Fig. 11 triangles $AV_{81}\ V_{78}$;

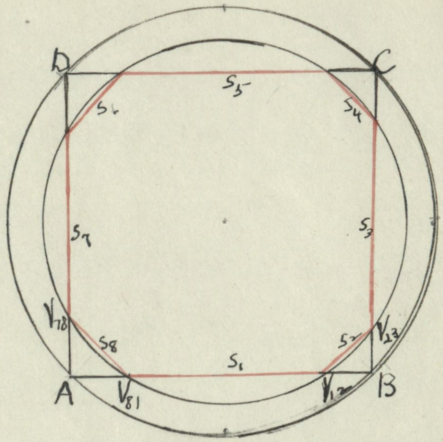

Fig. 11.

$BV_{12}V_{23}$ etc. are congruent isosceles triangles, and since $\angle A = \angle B = \angle C = \angle D = 90^\circ$, they are right triangles each of base equal to s_2. $2\,\overline{AV_{81}}^2 = s_2^2$. Therefore, $\frac{\sqrt{2}}{2}\, s_2 = AV_{81} = BV_{12} = BV_{23}$ etc.

THEOREM 10. <u>In an altra-regular octagon the diameter of the circumscribing circle is</u> $d = \sqrt{2}\,\sqrt{s_1^2 + \sqrt{2}\, s_1 s_2 + s_2^2}$.

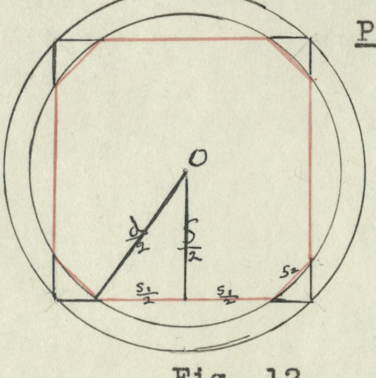

Fig. 12.

<u>Proof:</u> $\frac{d^2}{4} = \frac{S^2}{4} = \frac{s_1^2}{4}$ (S is a side of the square). But $S = s_1 + 2\frac{\sqrt{2}}{2}\, s_2$ (Lemma). $d^2 = (s_1 + \sqrt{2}\, s_2)^2 + s_1^2 = s_1^2 + 2\sqrt{2}\, s_1 s_2 + 2\, s_2^2 + s_1^2$. $d = \sqrt{2}\,\sqrt{s_1^2 + \sqrt{2}\, s_1 s_2 + s_2^2}$.

COROLLARY. <u>The area of an altra-regular octagon in terms of its sides is:</u> $s_1\sqrt{s_1^2 + 2\sqrt{2}\, s_1 s_2 + 2s_2^2} + s_2\sqrt{2\, s_1^2 + 2\sqrt{2}\, s_1 s_2 + s_2^2}$.

DEFINITION. A <u>regular star polygon</u> or <u>cross polygon</u> is a polygon produced by the lines joining alternate vertices of a regular polygon.

Thus only regular star polygons[1] of $2^{\ell} p_1 p_2 \ldots$ points where $p_1 p_2 \ldots$ are distinct primes of the form $2^{2^t}+1$, can be constructed by means of ruler and compass.

CONSTRUCTION 3. <u>To construct any altra-regular polygon having as a basis a regular star-polygon of $2^{\ell}5$; $2^{\ell}6$; $2^{\ell}8$ sides.</u>[2]

<u>Construction:</u> Inscribe in a circle a regular star polygon of $2^{\ell}5$; $2^{\ell}6$; or $2^{\ell}8$ points. Describe a second circle concentric with the first, but smaller, cutting the points of the star polygon. With the intersections of this circle and the star polygon as vertices construct a polygon.

<u>Proof:</u> Draw OP_6, OP_1 etc. and $OV_{12,1}$, $OV_{1,2}$, etc. (Fig. 13) $ON_1 = ON_3 = ON_5$ etc. $ON_9 \perp P_6 P_7$; $ON_1 \perp P_6 P_2$ etc. $OV_{10,11} = OV_{11,12} = OV_{12,1}$ etc. Therefore, $\Delta ON_9 V_{10,11} \cong \Delta ON_1 V_{11,12} \cong \Delta ON_3 V_{1,2}$ etc. Therefore, $N_9 V_{10,11} = N_1 V_{11,12} = N_1 V_{2,3}$ etc. $N_9 P_6 = N_{11} P_5 = N_{11} P_1 = N_1 P_2$ etc. Therefore, $P_6 V_{10,11} = P_6 V_{11,12} = P_1 V_{12,1} = P_1 V_{1,2}$ etc. $\angle P_6 = \angle P_1 = \angle P_2$ etc. Therefore, $\Delta V_{10,11} P_6 V_{11,12} \cong \Delta V_{12,1} P_1 V_{1,2}$ etc. Therefore,

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[1] See <u>History of Mathematics</u>, Macmillan, N.Y., 1894, by Cajori, pages 22, 135, 156 for references to work on star polygons by the Pythagoreans, Thomas Bradwardine, and Kepler.

[2] This method of construction produces figures which are of interest chiefly in "Design".

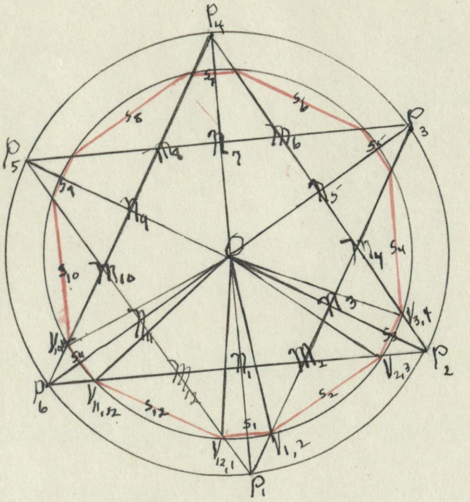

Fig. 13.

$s_{11} = s_1 = s_3$ etc. $P_6M_{12} = P_1M_{12} = P_1M_2$ etc. Thus, $V_{11,12}M_{12} = V_{12,1}M_{12} = V_{1,2}M_2$ etc. $\angle P_6M_{12}P_1 = \angle P_1M_2P_2$ etc. Therefore, $\Delta V_{11,12}M_{12}V_{12,1} \cong \Delta V_{1,2}M_2V_{2,3}$ etc. Therefore, $s_{12} = s_2 = s_4$ etc. and the polygon of sides s_1, s_2, s_3, s_4 etc. is an altra-regular polygon.[1]

Altra-Regular Polygons as Found in Mineralogy.

There are found in the isometric system of mineralogy a number of substances which tend to crystallize in forms which are combinations of positive and negative tetrahedrons; of the tetrahedron and dodecahedron; of the cube and octahedron; of the cube, octahedron, and dodecahedron; and of the cube and trapezohedron. In the first two cases it very frequently happens that there are produced solids whose faces are altra-regular hexagons, and in the last three cases, solids whose faces are altra-regular octagons. On

[1] Although the figure used here is the dodecagon, the proof itself is general and will apply to all cases in which the construction is possible.

crystals of galena, fluorite, boracite, sphalerite, tetrahedrite, and pyrite are found faces in the shape of altra-regular hexagons; and on crystals of gold, halite, sylvite, analcite, galena, fluorite, cuprite, and magnetite are found faces which are altra-regular octagons. It is interesting to note that some crystals of galena and fluorite, for example, show altra-regular hexagons, and that others show altra-regular octagons. Altra-regular polygons seem to be limited to the crystals of the isometric system.

Altra-Regular Polygons as Found in Design.

In the work in Design produced during a long period of time, and in many countries, we find a wide range of examples of designs having as their bases altra-regular polygons, especially the altra-regular octagon. There are few historical examples of the altra-regular hexagon although it makes a basis for attractive designs. The polygons of a greater number of sides than eight do not lend themselves very readily to effective use in Design, and hence are seldom found there. The designs based on the altra-regular polygons range from the purely geometric, thru those based on plant forms, to the purely imaginative.

In architecture altra-regular polygons, and especially those produced by Construction 3, are found as outlines of rose windows and as medallions in stone and wood carvings.

In designs used in the interior of buildings, especially in homes, altra-regular polygons are found on wall and fire-place tiles, in mosaic flooring, on the single teapot tiles, on the tops of decorative boxes of porcelain or wood in color and in carving, on floor coverings such as rugs and linoleums, and on textile fabrics both in the cases where the designs are reproduced by machinery and where they are worked by hand.

Of special interest are certain old Coptic and Byzantine designs, those woven in Chinese brocades, those of the Renaissance period, and Russian embroidery designs, all of which use the altra-regular octagon as a basis but not as a dominant feature. These figures are sometimes used in a large pattern, sometimes as small figures they are repeated on a border, and sometimes they are repeated in such a way as to cover the fabric. In one particular Chinese-Japanese design the altra-regular octagon itself was an outstanding feature. A Japanese pattern combined in an interesting manner the altra-regular octagon with other geometric designs and a flower design. Another example of the altra-regular octagon was as a part of a design used on a woollen carriage cushion from Skåne, Sweden. In this the octagon stood out prominently. In another case, the octagon was used as an outline of a flower pattern in an old Persian prayer carpet.

A very unusual design which was of primitive origin, was one of sixteen sides based on a regular eight point star polygon. This showed the radii of both circles which are used in the construction of the polygon by Construction 3.

Several examples of the modern usage of altra-regular polygons in Design are given in Plates I, II, and III.

ALTRA-REGULAR HEXAGONS WITH DESIGNS BASED THEREON.

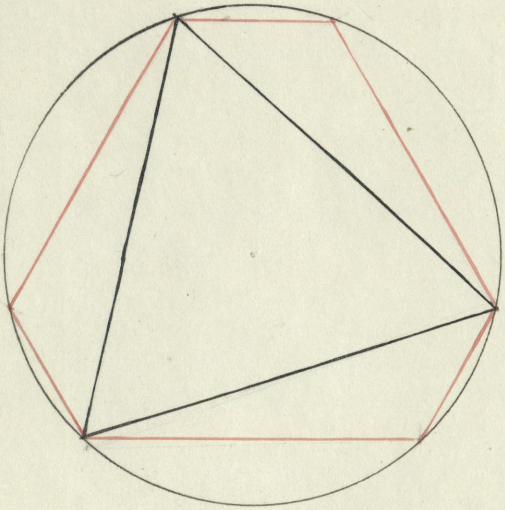

Fig. 14a.

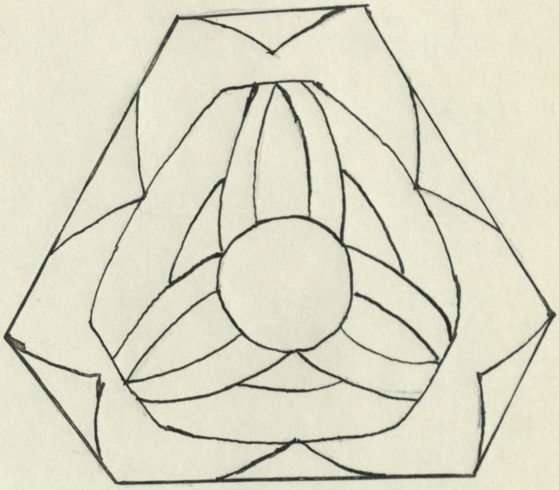

Fig. 14b.

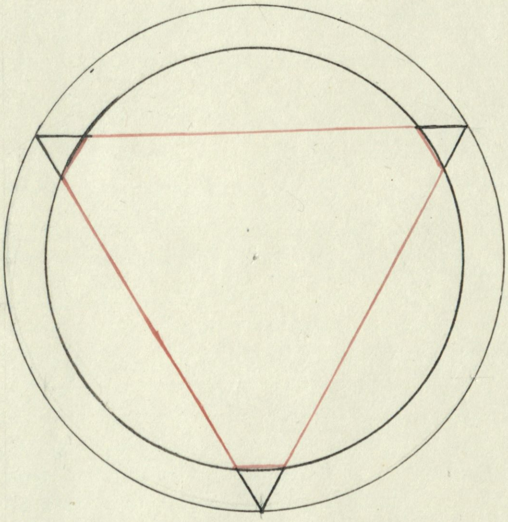

Fig. 15a.

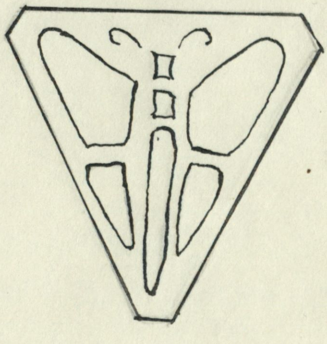

Fig. 15b.

PLATE I.

ALTRA-REGULAR OCTAGONS WITH DESIGNS BASED THEREON.

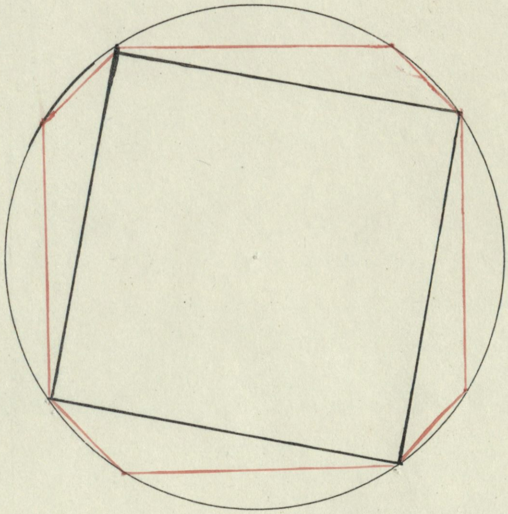

Fig. 16a.

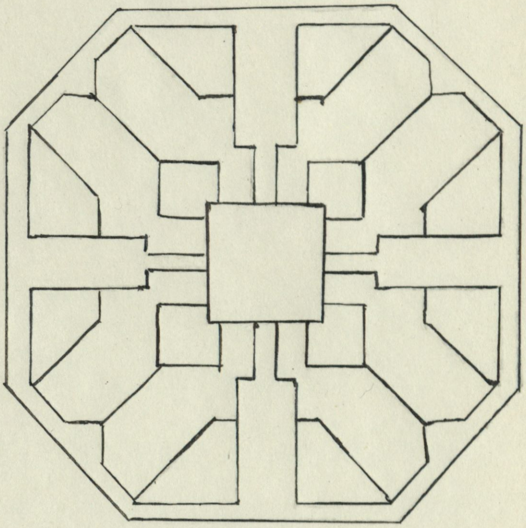

Fig. 16b.

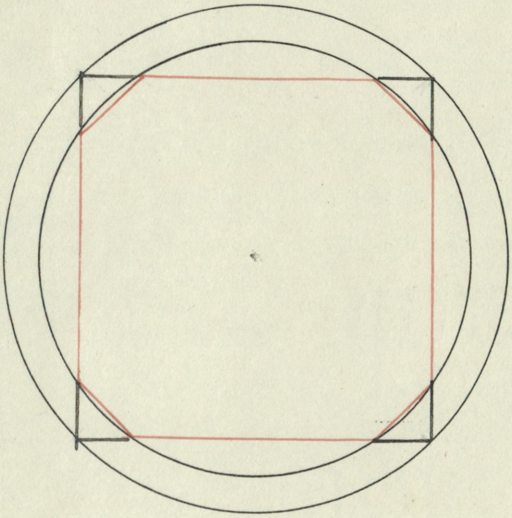

Fig. 17a.

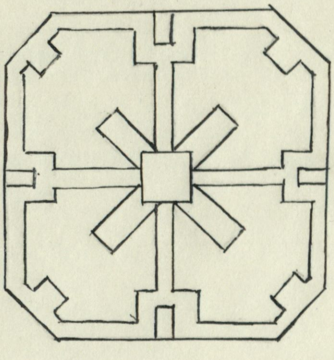

Fig. 17b.

PLATE II.

ALTRA-REGULAR POLYGONS WITH DESIGNS BASED THEREON.

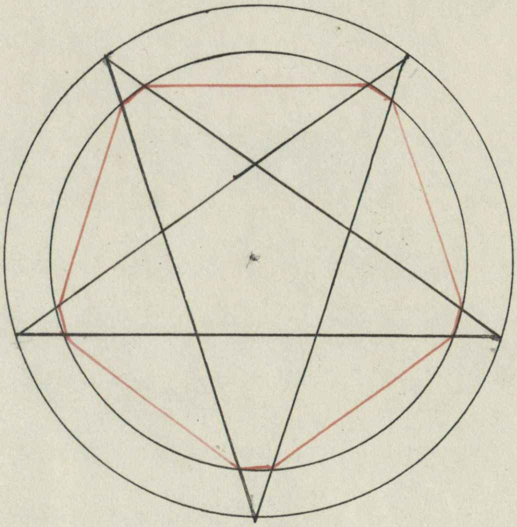

Fig. 18a.

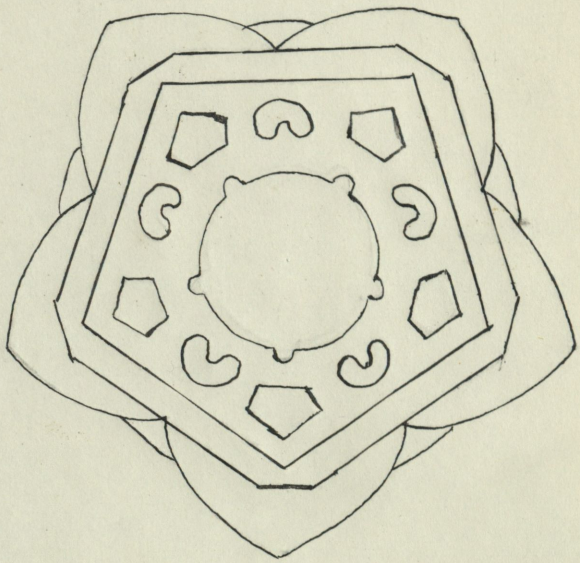

Fig. 18b.

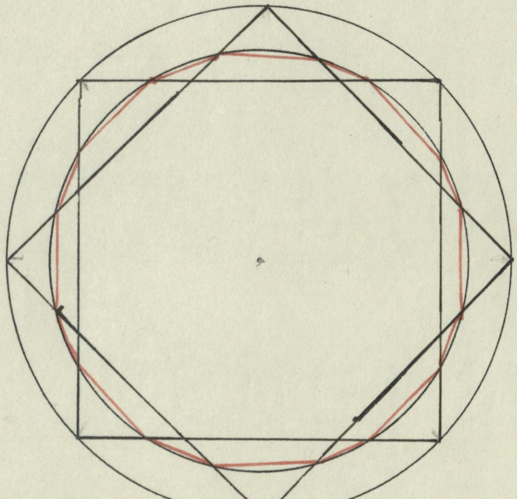

Fig. 19a.

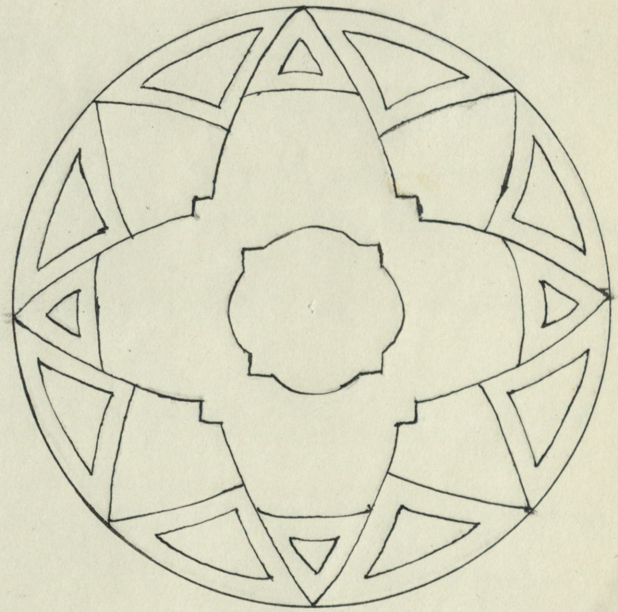

Fig. 19b.

PLATE III.

III. JUXTA-REGULAR POLYGONS.

A juxta-regular polygon may be constructed by the bisection of the arcs subtended by the sides of any convex polygon inscribed in a circle, and then using the vertices of the original polygon and the points where the arcs were bisected as vertices for the new polygon, which is thus juxta-regular, since equal arcs are subtended by equal chords. Since the original polygon which is the basis for the juxta-regular polygon has no limitations placed upon it except that it be convex and cyclic, the properties of the juxta-regular polygon can be determined only when the character of its basis is known. Thus, no generalizations can be made on the subject of juxta-regular polygons as a class.

If the basis of a juxta-regular polygon is a regular polygon, the juxta-regular polygon is also a regular polygon, and hence further discussion is out of the field of this paper.

If the juxta-regular polygon is a quadrilateral, it is known as a cyclic kite[1], a subject which is given con-

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[1]See "Kite" page 49, Second-Year Mathematics, Chicago, 1910, by Breslich.

siderable attention, especially in the form of problems, in our text-books of Elementary Euclidean Geometry. Thus, it need not be discussed in this paper.

THEOREM 1. <u>If the basis of a juxta-regular polygon is a contra-regular polygon, the juxta-regular polygon is also contra-regular, and consequently has all of the properties of a contra-regular polygon.</u>

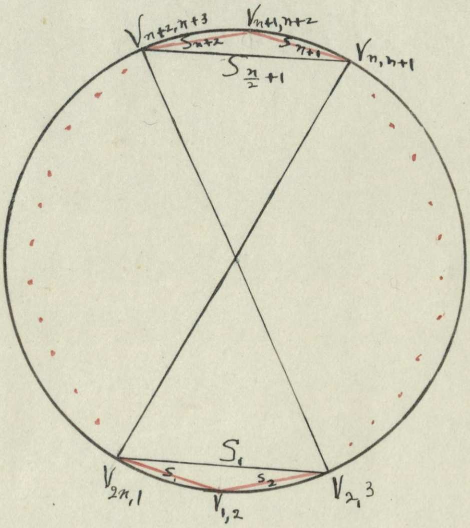

Fig. 1.

<u>Proof:</u> $S_1 = S_{\frac{n}{2}+1}$ and arc $V_{2n,1}V_{2,3} =$ arc $V_{n,n+1}V_{n+2,n+3}$. But $\frac{\text{arc } V_{2n,1}V_{2,3}}{2} =$ arc $V_{2n,1}V_{1,2} =$ arc $V_{1,2}V_{2,3} =$ arc $V_{n,n+1}V_{n+1,n+2} =$ arc $V_{n+1,n+2}V_{n+2,n+3}$. Hence, $s_1 = s_2 = s_{n+1} = s_{n+2}$. (Equal arcs are subtended by equal chords.) Similarly, all other opposite sides of the polygon may be shown to be equal.

THEOREM 2. <u>The area of a juxta-regular polygon which is based on a contra-regular polygon is</u>

$$\underline{\left(s_1\sqrt{d^2 - s_1^2} + s_3\sqrt{d^2 - s_3^2} + s_5\sqrt{d^2 - s_5^2} + \ldots + s_{n-1}\sqrt{d^2 - s_{n-1}^2}\right).}$$

<u>Proof:</u> A juxta-regular polygon based on a contra-

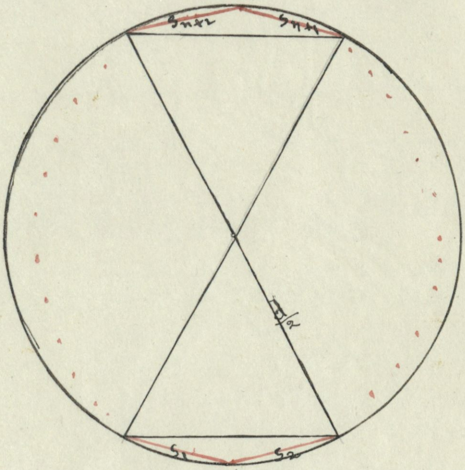

Fig. 2.

regular polygon is a contra-regular polygon. Hence, its area is

$$\tfrac{1}{2}(s_1\sqrt{d^2-s_1^2}+s_2\sqrt{d^2-s_2^2}+\dots+s_n\sqrt{d^2-s_n^2}).$$

But $s_1=s_2$, $s_3=s_4$, etc. Hence, the area of the polygon is

$$(s_1\sqrt{d^2-s_1^2}+s_3\sqrt{d^2-s_3^2}+s_5\sqrt{d^2-s_5^2}+\dots+\dots\dots+s_{n-1}\sqrt{d^2-s_{n-1}^2}).$$

THEOREM 3. <u>The number of sides possessed by a juxta-regular polygon based on either a contra-regular polygon or an altra-regular polygon must be divisible by four, and the least number of sides possible is eight.</u>

<u>Proof:</u> It is possible for a contra-regular, altra-regular, or juxta-regular polygon to exist only when the polygon has 2n sides. If the contra-regular or altra-regular polygon used as a basis for the juxta-regular polygon has n sides where $n=2k$, then the juxta-regular polygon has 4k sides. Since the basic polygon must have at least four sides in order to exist, the smallest number of sides which it is possible for the

juxta-regular polygon to have is eight.

THEOREM 4. <u>The area of a juxta-regular polygon of 2n sides based on an altra-regular polygon is</u> $\frac{n}{4}\left(s_1\sqrt{d^2 - s_1^2} + s_2\sqrt{d^2 - s_2^2}\right)$.

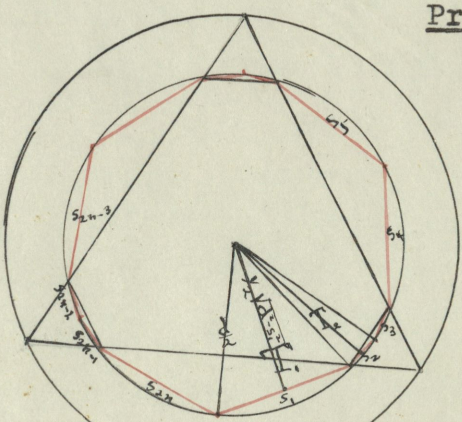

Fig. 3.

<u>Proof:</u> $s_{2n} = s_1 = s_4 = s_5 = s_8 = s_9 = \cdots = s_{2n-4} = s_{2n-3}$, and $s_2 = s_3 = s_6 = s_7 = \cdots = s_{2n-2} = s_{2n-1}$. (Equal arcs are subtended by equal chords.) Therefore, the polygon is composed of n isosceles triangles of bases equal to s_1 and n triangles of bases equal to s_2. The area of $T_1 = \frac{1}{4}s_1\sqrt{d^2 - s_1^2}$ and of $T_2 = \frac{1}{4}s_2\sqrt{d^2 - s_2^2}$. Hence, the total area of the polygon is $\frac{n}{4}(s_1\sqrt{d^2 - s_1^2} + s_2\sqrt{d^2 - s_2^2})$.

COROLLARY. <u>The area of a juxta-regular polygon based on an altra-regular polygon equals the area of an altra-regular polygon of the same number of sides and having its sides equal to those of the juxta-regular polygon.</u>

THEOREM 5. <u>In a juxta-regular polygon based on an altra-regular polygon, those vertices which are not also vertices of the basic polygon are the vertices of a regular polygon.</u>

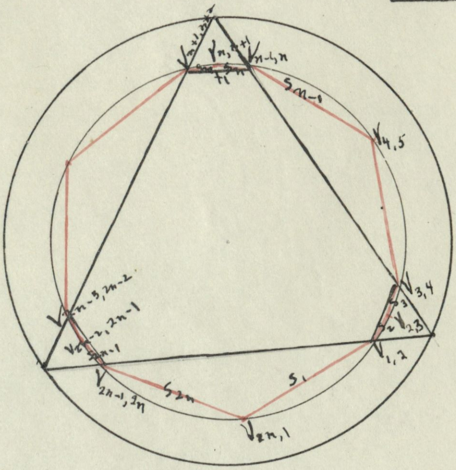

Fig. 4.

<u>Proof:</u> arc $V_{2n-1,2n}V_{2n,1}$ = arc $V_{2n,1}V_{1,2}$ = arc $V_{34}V_{45}$ = arc $V_{45}V_{56}$ etc. and arc $V_{12}V_{23}$ = arc $V_{23}V_{34}$ = arc $V_{56}V_{67}$ = ...= arc $V_{2n-3,2n-2}V_{2n-2,2n-1}$. Hence, arc $(V_{2n-2,2n-1}V_{2n-1,2n} + V_{2n-1,2n}V_{2n,1})$ = arc $(V_{2n,1}V_{12} + V_{12}V_{23})$ etc. Hence, chords $V_{2n-2,2n-1}V_{2n,1} = V_{2n,1}V_{23} = V_{2,3}V_{4,5}$ etc.

THEOREM 6. <u>In a juxta-regular polygon based on an altra-regular polygon, the lines joining opposite vertices which are not also vertices of the basic polygon, are diameters.</u>

<u>Proof:</u> The juxta-regular polygon has 4k sides. Hence, the regular polygon (Theorem 5) has 2k sides, and opposite sides are equal, making it contra-regular. Therefore, the joins of the vertices of this regular polygon are diameters.

I N D E X .