# Orbital Stability of Ground State Solutions to the Nonlinear Schrödinger Equation

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# Abstract

In this paper we study ground state solutions to the focusing, nonlinear Schrödinger equation

$$iu_t = -\Delta u - |u|^{2\sigma}u, \qquad u \in C\big(\mathbb{R};\ H^1(\mathbb{R})\big),\ \sigma \geq 0$$

and determine the values of $\sigma$ for which such solutions exist, and are stable.

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# Chapter 1

# Introduction

Even an abbreviated perusal through various scientific journals reveals that the nonlinear Schrödinger equation (NLS)

$$iu_t = -\Delta u - \gamma |u|^{2\sigma} u, \qquad \gamma = \pm 1,\ \sigma \geq 0,$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$, arises in a large number of scientific investigations, including

- propagation of light in nonlinear optical fiber,
- small amplitude gravity waves on surfaces of zero viscosity water,
- propagation of Davydov's alpha-helix solutions,
- oscillations in hot plasmas,

and numerous others—see Malomed (2005) and Balakrishnan (1985) for more examples. In this paper, we examine the stability of the ground state solutions of the NLS. Strictly speaking, a ground state solutions $\phi \in H^1(\mathbb{R}^n)$ is a time independent function which is not actually a solution to the NLS. However, the time evolution $e^{i\lambda^2 t}\phi(x)$ of $\phi$ is a solution to the NLS. In general, any $H^1(\mathbb{R}^n)$ function $\psi$ whose time evolution $e^{i\lambda^2 t}\psi(x)$ solves an NLS equation is called a bound state solution. Ground state solutions form a special class of bound state solutions and which are both positive and radially symmetric, and they are of particular interest, as they represent the standing wave (bound state) solutions with lowest possible “energy.”

Nonlinear Schrödinger equations are often classified by their value for $\gamma$. If $\gamma = -1$, the resulting partial differential equation (PDE) is called a “defocusing” NLS, and if $\gamma = 1$, the resulting PDE is referred to as a “focusing.” While outside the scope of this paper, it can be shown that ground state solutions fail to exist for a defocusing ($\gamma = -1$) NLS equation. Fortunately, as we show in Chapter 3, ground state solutions do exist for a focusing ($\gamma = 1$) NLS equation—provided that we place the restriction on the nonlinear term that $\sigma < 2/n$. As such, through out this paper, we restrict our attention to focusing NLS equations.

In Chapter 2, we establish that NLS initial value problems (IVP) are locally well-posed over $H^1(\mathbb{R}^1)$. That is, for each in initial condition, we can find a solution to the NLS satisfying said initial condition which exists at least for some finite time. Moreover, given two solutions to an NLS IVP whose initial data are close to each other, then the two solutions will stay close for at least a small window of time. This later fact is particularly important, as it provides us with a starting point for our stability analysis of ground state solutions. In particular, for some initial data $u_0 \in H^1(\mathbb{R}^n)$ which is sufficiently close to a ground state solution $\phi \in H^1(\mathbb{R}^n)$ then the fact that NLS IVP's are locally well-posed means that we know the solution $u$ corresponding to $u_0$ will stay close to $\phi$ for at least some interval of time. Ultimately, for stability, we will want to show that $u$ stays close to the time evolution of $\phi$ for the entire time domain on which $u$ is defined.

Currently, the two major tools for establishing the stability of a nonlinear PDE are spectral analysis of the linearizion of the PDE and Lyapunov stability theory. Unfortunately, as we see in Chapter 4, the first approach fails to yield any rigorous results. However, it does provide us with some crucial insights into the stability of ground state solutions which we exploit in Chapter 6, where we apply a modified version of the Lyapunov functional method involving Vakhitov-Kolokolov projection to determine the conditions under which ground state solutions are stable.

One thing which is important to note is that most of the arguments presented in this paper are not dependent $n$, the number of spatial dimensions. As such, throughout this paper, we tend to use notation which is consistent with working in an arbitrary number of spatial dimensions (*i.e.*, using $\nabla u$ to denote the first spatial derivative of $u$ as opposed to $\partial_x u$). However, there are a few places where, in order to simplify our argument, we restrict our attention to the case where $n = 1$ in order to use tools which are only applicable in one spatial dimension. Nonetheless, in each one of these cases where we restrict our attention to a single spatial dimension, the result we present actually does hold in higher dimensions—only the proof for higher dimensions is well beyond the scope of this paper.

# Chapter 2

# Local Well-Posedness

A natural starting point for our stability analysis of the NLS is to determine the conditions for which the general nonlinear Schrödinger (NLS) initial value problem (IVP)

$$\begin{cases} iu_t = -\Delta u - \gamma|u|^{2\sigma}u \\ u(0) = u_0 \in H^1(\mathbb{R}^n) \end{cases}, \qquad \gamma = \pm 1,\ \sigma > 0 \tag{2.0.1}$$

is locally well-posed. That is, the NLS IVP satisfies the conditions given in the following definition:

**Definition 2.0.1** (Local Well-Posedness)**.** An initial value problem

$$\begin{cases} u' = f(t,u) \\ u(0) = u_0 \end{cases} \tag{2.0.2}$$

on a Banach space $\big(X, \|\cdot\|_X\big)$ is said to be locally well-posed if it satisfies the following conditions:

1) The IVP (2.0.2) has a unique, local solution for each initial data $u_0 \in X$. That is, (2.0.2) admits a unique solution $u(x,t)$ for $t$ in the interval $[-T,T]$, for some $T = T(u_0) > 0$.

2) The map

$$u_0 \in X \mapsto u \in C\big([-T,T];X\big)$$

which takes initial data $u_0$ to the corresponding unique solution of (2.0.2) is continuous in the sense that for each given $\varepsilon > 0$ and initial data $u_0 \in X$, we can find a $\delta = \delta\left(\varepsilon, u_0\right) > 0$ so that if $v_0 \in X$ satisfies

$$\|u_0 - v_0\|_X < \delta,$$

then, for some $\hat{T} = \hat{T}(\varepsilon, u_0, v_0)$ with $0 < \hat{T} \leq T$,

$$\sup_{t \in [-\hat{T}, \hat{T}]} \|u(t) - v(t)\|_X < \varepsilon,$$

where $u$ and $v$ denote the unique solutions of (2.0.2) corresponding to the initial data $u_0$ and $v_0$, respectively.

As we mention in the introduction, the second condition of local well-posedness is really a sort of "mini" stability condition. For stability, we want $u$ and $v$ to stay close together over their entire time domains—provided that $u$ and $v$ start sufficiently close. On the other hand, condition 2 merely requires that $u$ and $v$ stay close together for *some* time interval—no matter how short that time interval might be. As such, we actually use the fact that the NLS is locally well-posed to eventually analyze the stability of ground state solutions.

In this chapter, we present an argument showing that the NLS IVP is locally well-posed over one spatial dimension—that is, for an NLS IVP taken over $H^1(\mathbb{R}^1)$. However, it can also be shown that an NLS IVP over $H^1(\mathbb{R}^2)$ is locally well-posed for all values of $\sigma \geq 0$, and for an NLS IVP taken over $H^1(\mathbb{R}^n)$, Sulem & Sulem (1999) show that it is only locally well-posed for $0 \leq \sigma < \frac{2}{n-2}$. While the proof of the local well-posedness for NLS IVP of spatial dimension $n > 1$ is outside the scope of this paper, the interested reader can find such a proof in Chapter 3.2 of Sulem & Sulem (1999).

## 2.1 Preliminary Results and Definitions

Before we verify that the NLS IVP is locally well-posed, we need a few definitions and results from general ordinary differential equations (ODE) and partial differential equations (PDE) theory. Throughout the remainder of this section, we use the notation $D(A)$ to denote the domain of a densely defined linear operator $A : D(A) \subseteq X \to X$, where $X$ is some arbitrary Banach space. Recall, that by definition, $D(A)$ is dense in $X$. Further, for a Banach space $X$, we will also use the notation $\mathscr{L}(X)$ to represent the Banach space of all bounded linear operators acting on $X$, and let $\|\cdot\|_{\mathscr{L}(X)}$ denote the corresponding induced operator norm on $\mathscr{L}(X)$. Recall that $\|\cdot\|_{\mathscr{L}(X)}$ is defined by

$$\|A\|_{\mathscr{L}(X)} := \sup_{u \in X\setminus\{0\}} \frac{\|Au\|_X}{\|u\|_X} = \sup_{\|u\|_X=1} \|Au\|_X$$

for every $A \in \mathscr{L}(X)$.

**Definition 2.1.1** ($C_0$-continuous group)**.** Let $X$ be a Banach space and $A$ be a linear operator acting on $X$. Then if the family of operators $\{T(t)\}_{t\in\mathbb{R}}$ satisfies

1) $T(0) = \mathbb{1}_X$

2) $T(s)T(t) = T(s+t)$ for every $t, s \in \mathbb{R}$

3) $\|T(h)f - f\|_X \to 0$ as $h \to 0^+$ for every $f \in X$.

then we say the family $\{T(t)\}_{t\in\mathbb{R}}$ is a $C_0$ *group of operators*.

If, in addition, $T$ also satisfies the condition that $T(h) \to \mathbb{1}_X$ uniformly in $\mathscr{L}(X)$ as $h \to 0^+$, then we say that the family $\{T(t)\}_{t\in\mathbb{R}}$ is a *uniformly continuous group of operators*.

**Definition 2.1.2** ($C_0$−semigroup)**.** If we relax the requirement that $t \in \mathbb{R}$ to $t \geq 0$ in Definition 2.1.1, then we say $\{T(t)\}_{t\geq 0}$ is a $C_0$ *semigroup*, if $\{T(t)\}_{t\geq 0}$ satisfies the relaxed conditions

1) $T(0) = \mathbb{1}_X$

2) $T(s)T(t) = T(s+t)$ for every $t, s \geq 0$

3) $\|T(h)f - f\|_X \to 0$ as $h \to 0^+$ for every $f \in X$.

Similarly, if the semigroup $\{T(t)\}_{t\geq 0}$ satisfies the condition that $T(h) \to \mathbb{1}_X$ uniformly in $\mathscr{L}(X)$ as $h \to 0^+$, then $\{T(t)\}_{t\in\mathbb{R}}$ is said to be a *uniformely continuous semigroup of operators.*

*Remark.* Note that under the relaxed conditions of Definition 2.1.2, $\{T(t)\}_{t\geq 0}$ is an algebraic semigroup in the sense that no element of $\{T(t)\}_{t\geq 0}$ has an additive inverse in $\{T(t)\}_{t\geq 0}$.

If $A \in \mathscr{L}(X)$ for some Banach space $X$, then we can define the operator exponential of $tA$ as

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

In fact, there is a result from semigroup theory which states that a one parameter family of operators $\{T(t)\}_{t\in\mathbb{R}}$ is a uniformly continuous group of operators if and only if there exists some $A \in \mathscr{L}(X)$ so that

$$T(t) = e^{tA}$$

for each $t \in \mathbb{R}$. In such a case, we say that $A$ is the *generator* for the uniformly continuous group of operators $\{T(t)\}_{t\in\mathbb{R}}$. Of course, the analogous result (and definition) also holds for uniformly continuous semigroups.

However, if $A$ is an unbounded linear operator on the Banach space $X$, while we still have a concept what it means for $A$ to generate either a $C_0-$semigroup or $C_0-$group, it's definition requires a little more motivation. From ODE theory, we know that for $A \in \mathscr{L}(X)$, the ODE

$$\begin{cases} u_t = Au, \quad u \in C^\infty(\mathbb{R};\ X) \\ u(0) = u_0, \quad u_0 \in X \end{cases} \tag{2.1.1}$$

has a unique solution for each $u_0$. As such, we can define a map $\Phi$

$$\Phi : X \to C^1(\mathbb{R};\ X)$$
$$u_0 \mapsto u(t),$$

which sends each initial data $u_0$ to its corresponding unique solution $u$. In fact, it can be shown that $u(t) = \Phi(u_0) = e^{tA}u_0$.

On the other hand, if $A$ is an unbounded linear operator, then, under the right circumstances, we can still find a one parameter $C_0$-semigroup $\{T(t)\}_{t\geq 0}$ satisfying

$$u(t) = \Phi(u_0) = T(t)u_0.$$

In such a case, $A$ is called the generator of the $C_0-$semigroup $\{T(t)\}_{t\geq 0}$, and, in an mildly egregious abuse of notation, it is common practice to write $e^{tA}$ in place of $T(t)$.

More formally:

**Definition 2.1.3** (Generator of a $C_0-$semigroup)**.** Let $X$ be a Banach space and $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ be a $C_0-$semigroup acting on $X$. Define the operator $A$ acting on $X$ by

$$Au := \lim_{h\to 0^+} \frac{1}{h}\big(T(h)u - u\big) = \frac{d}{dt}\Big|_{t=0} T(t)u,$$

where $u \in X$. The domain $D(A)$ of $A$ is defined to be

$$D(A) := \left\{ u \in X \;\middle|\; \text{the limit } \lim_{h\to 0^+} \frac{1}{h}\big(T(h)u - u\big) \text{ exists} \right\}.$$

Provided that $D(A)$ is dense in $X$, we say that $A$ is the *generator* of $\{T(t)\}_{t\geq 0}$ (*i.e.* $A$ generates a $C_0-$semigroup).

Definition 2.1.3 is sometimes referred to as the "infinitesimal generator" definition. While Definition 2.1.3 seems to have little relation to our preceding discussion, an important result from semigroup theory implies that if $A$ is the generator for a $C_0-$semigroup $\{T(t)\}_{t\geq 0}$, then for all $u_0 \in D(A)$, the function

$$u(t) = T(t)u_0$$

is the classical solution (see Definition 2.1.4) of the IVP (2.1.1) corresponding to the initial data $u_0$.

**Definition 2.1.4** (Classical Solution)**.** Let $X$ be a Banach space and $A : D(A) \subseteq X \to X$ be a closed, densely defined linear operator. Consider the IVP

$$\begin{cases} u_t = Au + f(t), \quad t > 0 \\ u(0) = u_0 \in X \end{cases}, \tag{2.1.2}$$

where $f$ is a continuous map from $\mathbb{R}^{>0}$ to $X$ $\big(i.e.,\ f \in C^1(\mathbb{R}^{>0}; X)\big)$. We say that a solution $u$ to the IVP (2.1.2) is a *classical solution* if

$$u \in C^1\left(\mathbb{R}^{>0};\ X\right) \cap C\left(\mathbb{R}^{>0} \cup \{0\};\ D(A)\right)$$

and $u$ satisfies (2.1.2) pointwise for $t \geq 0$.

While classical solutions are the holy grail of PDE theory, we are not always given that a particular PDE might admit strictly classical solutions. However, if we relax our expectations slightly, then we can often hope for something almost as good, a so-called "mild solution." The following proposition provides us some insight into how we might define a mild solution.

**Proposition 2.1.5.** *If $A$ is the generator of a $C_0-$semigroup $\{T(t)\}_{t\geq 0}$, then any classical solution $u$ of* (2.1.2) *can be written as*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\ ds, \tag{2.1.3}$$

*where Equation* (2.1.3) *is commonly referred to as* Duhamel's Formula.

*Proof.* If we rewrite (2.1.2) as $u_s = Au + f(s)$, then

$$u_s - Au = f(s). \tag{2.1.4}$$

Since the operators $A$ and $T(t-s) = e^{(t-s)A}$ commute, by multiplying both sides of (2.1.4) by $T(t-s)$, we obtain

$$T(t-s)u_s - AT(t-s)u = T(t-s)f(s). \tag{2.1.5}$$

Now, since

$$\frac{d}{ds}T(t-s)u = T(t-s)u_s + \left(\frac{d}{ds}T(t-s)\right)u(s) = T(t-s)u_s + AT(t-s)u, \tag{2.1.6}$$

the Fundamental Theorem of Calculus in conjunction with (2.1.5) implies

$$\int_0^t T(t-s)f(s) = T(t-s)u(s)\Big|_0^t = u(t) - T(t)u_0. \tag{2.1.7}$$

The desired result is an immediate consequence of (2.1.7). ❏

In fact, if the operator $A$ in Proposition 2.1.5 generates a uniformly continuous semigroup, then any $u \in L^2\big(R^{>0}; X\big)$ which satisfies DuHamel's formula is a classical solution to the IVP (2.1.2)—thus motivating the definition of a "mild solution":

**Definition 2.1.6** (Mild Solution)**.** A function $u$ which satisfies (2.1.3) ($i.e$ Duhamel's Formula) is called a mild solution of the IVP (2.1.2).

**Theorem 2.1.7.** *Let $X$ be a Banach space and let $f : X \to X$ be locally Lipschitz. If $A$ is the generator of a $C_0$-semigroup on $X$, then for every $u_0$ we can find a $t_{max} \in (0, \infty]$ so that* *the initial value problem*

$$\begin{cases} u_t = Au + f(u), \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{2.1.8}$$

*has a unique mild solution* $u \in C\big([0, t_{max});\ X\big)$*.*

*Moreover, if* $t_{max} < \infty$*, then* $\|u(t)\|_X \to \infty$ *as* $t$ *converges to* $t_{max}$ *from below.*

The proof we present for Theorem 2.1.7 is adapted from a lecture by Johnson (2014).

*Proof.* Recall that a function $f : X \to X$ is said to be locally Lipschitz if, for every $R > 0$, we can find a constant $M = M(R) > 0$, called the "Lipschitz constant" of $f$, so that

$$\|f(u) - f(v)\|_X \leq M\|u - v\|_X$$

for every $u,\ v \in X$ with $\|u\|_X,\ \|v\|_X \leq R$.

Now suppose $A$ generates the $C_0-$semigroup $\{T(t)\}_{t\geq 0}$. Then, without loss of generality, we may assume $\|T(t)\|_{\mathscr{L}(X)} \leq 1$ for each $t \geq 0$. For some fixed $\tau > 0$, set

$$Y_\tau := Y := C\big([0,\tau]; X\big) = \big\{u : [0,\tau] \to X \ \big|\ u \text{ is continuous}\big\}$$

with norm $\|u\|_Y := \sup_{t\in[0,\tau]} \|u\|_X$. Note that $\big(Y, \|\cdot\|_Y\big)$ is a Banach space. Let $u_0 \in X$ be fixed, and consider the map

$$\begin{aligned} F\ &:\ Y \to Y \\ u(t) &\mapsto T(t)u_0 + \int_0^t T(t-s) f\big(u(s)\big)\ ds. \end{aligned}$$

It is easy to see that mild solutions of (2.1.8) are fixed points of $F$. Define

$$R := 2\|u_0\|_X \qquad \text{and} \qquad \mathcal{W}_R := \big\{u \in Y \ \big|\ \|u\|_Y \leq R\big\},$$

and let $M = M(R) > 0$ be the Lipschitz constant of $f$ corresponding to $R$. Given $u \in \mathcal{W}_R$, for all $t \in [0, \tau]$ we have

$$\begin{aligned}
\|F(u)(t)\|_X &\leq \|u_0\|_X + \int_0^\tau \left\|f\big(u(s)\big)\right\|_X \, dx \\
&\leq \|u_0\|_X + \int_0^\tau \left(\left\|f\big(u(s)\big) - f(0)\right\|_X + \|f(0)\|_X\right) \, ds \\
&\leq \|u_0\|_X + \int_0^\tau \left(M \|u(s)\|_X + \|f(0)\|_X\right) \, ds \\
&\leq \|u_0\|_X + \tau \left(MR + \|f(0)\|_X\right) \\
&< 2 \|u_0\|_X \, . \qquad (2.1.9)
\end{aligned}$$

To make the final inequality in (2.1.9), we can either take $\tau$ to be sufficiently small, or take $R$ to be sufficiently small. We assume $\tau$ is small. Since $f$ is locally Lipschitz, for all $u, v \in \mathcal{W}_R$ and for every $t \in [0, \tau)$ we have

$$\begin{aligned}
\|F(u)(t) - F(v)(t)\|_X &\leq \int_0^t \left\|T(t-s)\Big(f\big(u(s)\big) - f\big(v(s)\big)\Big)\right\| \, ds \\
&\leq \int_0^t \left\|f\big(u(s)\big) - f\big(v(s)\big)\right\|_X \, ds \\
&\leq \int_0^t M\|u(s) - v(s)\|_X \, ds \\
&\leq \tau M \|u - v\|_Y \\
&< \|u - v\|_Y \qquad (2.1.10)
\end{aligned}$$

Observe that the right-hand sides of both (2.1.9) and (2.1.10) and independent of $t$. Hence, we can take the supremum in $t$ of both sides of (2.1.9) and (2.1.10) to obtain

$$\|F(u)\|_Y = \sup_{t \in [0,\tau]} \|F(u)(t)\| < 2\|u_0\|_X = 2 \|u_0\|_Y$$

and

$$\|F(u) - F(v)\|_Y = \sup_{t \in [0,\tau]} \|F(u)(t) - F(v)(t)\|_X < \|u - v\|_Y$$

Set

$$\tau_1 := \min\left\{\frac{R - \|u_0\|_X}{MR + \|f(0)\|_X}, \frac{1}{2M}\right\}.$$

By taking $\tau = \tau_1$ as above, we obtain

i) $F \ : \ \mathcal{W}_R \to \mathcal{W}_R$, and

ii) $F$ is a strict contraction on $\mathcal{W}_R$.

Thus, by the Banach fixed point theorem, there exists a unique $u \in \mathcal{W}_R = \mathcal{W}_R(\tau_1)$ such that $F(u) = u$. That is, $u$ is a unique mild solution of (2.1.8) on $[0, \tau_1]$. By repeating the above argument, we can extend $u$ to a solution on $[0, \tau_1 + \tau_2]$, with $\tau_2 > 0$, by deforming $u \in [\tau_1, \tau_1 + \tau_2]$ as $u(t) = w(t)$, where $w(t)$ solves

$$w(t) = T(t - \tau_1)u(\tau_1) + \int_{\tau_1}^{t} T(t-s)f\big(w(s)\big)\ ds,$$

for $\tau_1 \leq t \leq \tau_1 + \tau_2$. Hence, we can continue this process to extend $u$ to a solution on the maximal time interval $[0, t_{max}]$ for the solution.

Lastly, to see that

$$\lim_{t \to t_{max}} \|u(t)\|_X = \infty,$$

whenever $t_{max} < \infty$, suppose, by way of contradiction, that there exists a strictly increasing sequence $\{t_j\}_{j=1}^{n}$ converging to $t_{max}$ for which there exists some $C > 0$ so that $\|u(t_j)\|_X \leq C$ for every $j \in \mathbb{N}$. From our previous argument, it follows that for $j \gg 1$, if $u$ is defined on $[0, t_j]$, then it can be extended to a solution on $[0, t_j + \delta)$, where $\delta > 0$ is independent of $j$, as the sequence $\{u(t_j)\}_{j=1}^{\infty}$ is uniformly bounded. However, this fact implies that we could extend the temporal domain on which the solution $u$ is valid beyond $[0, t_{max})$, which contradicts the maximality of $t_{max}$. ❏

*Remark.* If the operator $A$ from Theorem 2.1.7 actually generates a $C_0-$group—and not just a $C_0-$semigroup—then by applying Theorem 2.1.7 to the operator $-A$, we can find a $t_{min} < 0$ so that (2.1.8) has a unique solution which is valid on $[t_{min}, t_{max}]$.

## 2.2 Local Well-Posedness of the NLS IVP

The first step in showing an NLS IVP is locally well-posed is to verify that it satisfies the first condition of Definition 2.0.1. To do so, we make use of the remark following the proof of Theorem 2.1.7. Note that (2.0.1) can be rewritten as

$$\begin{cases} u_t = i\Delta\phi + i\gamma|u|^{2\sigma}u \\ u(0) = u_0 \end{cases},$$

and take $A := i\Delta$ and $f : u \mapsto i\gamma|u|^{2\sigma}u$. In which case, if we assume $A$ generates a $C_0$-group, then the aforementioned remark implies that for each $u_0 \in H^1(\mathbb{R}^1)$, there exists $t_{min} < 0$ and $t_{max} > 0$ for which the corresponding unique solution $u(t)$ is defined on $[t_{min}, t_{max}]$. If we take $T := \min\{|t_{min}|,\ t_{max}\}$, then $u$ is clearly defined on $[-T, T]$, as required by the first local well-posedness condition. Consequently, to show that the NLS IVP is locally well posed, we first show that it satisfies the hypotheses of Theorem 2.1.7.

We begin by showing that $f$ is locally Lipschitz. For simplicity, we consider the case where $\sigma = 1$. The general case for $\sigma > 0$ is similar. Recall that the Sobolev embedding theorem implies $H^k(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ if $k > \frac{n}{2}$. Thus, for $k = n = 1$, we can find a constant $C > 0$ so that[1]

$$\|u\|_{L^\infty(\mathbb{R}^1)} \leq C\|u\|_{H^1(\mathbb{R}^1)}. \tag{2.2.1}$$

Using (2.2.1) allows us to prove the following claim which will aid in verifying that $f$ is locally Lipschitz when $n = 1$ and $\sigma = 1$.

[1]Unfortunately, since $k \not> \frac{n}{2}$ for $k = 1$ and $n \geq 2$, the Sobolev embedding theorem does not imply an analogous result to (2.2.1) over $H^1(\mathbb{R}^n)$, $n \geq 2$. Thus, to apply the Sobolev embedding theorem, we are forced to fix $n = 1$. As we remark at the end of this Section, this is the only place in our discussion on the well posedness of the NLS IVP where we cannot immediately generalize our argument to $H^1(\mathbb{R}^n)$ for $n \geq 2$.

**Claim 2.2.1.** *For $n = \sigma = 1$ and $u, v \in H^1(\mathbb{R}^1)$ the following two inequalities hold*

$$\|f(u)\|_{H^1(\mathbb{R})} \leq C \|u\|_{L^\infty(\mathbb{R})}^2 \|u\|_{H^1(\mathbb{R})} \tag{2.2.2}$$

*and*

$$\|f(u) - f(v)\|_{H^1(\mathbb{R})} \leq C \left(\|u\|_{H^1(\mathbb{R})} + \|v\|_{H^1(\mathbb{R})}\right)^2 \|u - v\|_{H^1(\mathbb{R})}, \tag{2.2.3}$$

*where $C$ is some constant—though, not necessarily the same constant in both* (2.2.2) *and* (2.2.3).

*Proof.* Verifying inequality (2.2.2) is simply a matter of direct computation:

$$\begin{aligned}
\|f(u)\|_{H^1(\mathbb{R})}^2 &= |\gamma|^2 \left(\left\|Du^2\overline{u}\right\|_{L^2(\mathbb{R})}^2 + \left\|u^3\right\|_{L^2(\mathbb{R})}^2\right) \\
&\leq \left(\left(\left\|\overline{u}Du^2\right\|_{L^2(\mathbb{R})} + \left\|u^2 D\overline{u}\right\|_{L^2(\mathbb{R})}\right)^2 + \|u\|_{L^\infty(\mathbb{R})}^4 \|u\|_{L^2(\mathbb{R})}^2\right) \\
&\leq \left(\|u\|_{L^\infty(\mathbb{R})}^4 \left(2\|Du\|_{L^2(\mathbb{R})} + \|D\overline{u}\|_{L^2(\mathbb{R})}\right)^2 + \|u\|_{L^\infty(\mathbb{R})}^4 \|u\|_{L^2(\mathbb{R})}^2\right) \\
&\leq C \|u\|_{L^\infty(\mathbb{R})}^4 \left(\|Du\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2\right) \\
&= C \|u\|_{L^\infty(\mathbb{R})}^4 \|u\|_{H^1(\mathbb{R})}^2
\end{aligned}$$

Hence, $\|f(u)\|_{H^1(\mathbb{R})} \leq C \|u\|_{L^\infty(\mathbb{R})}^2 \|u\|_{H^1(\mathbb{R})}$, as claimed.

To verify the second inequality, observe that

$$\begin{aligned}
f(u) - f(v) &= \gamma\Big(|u|^2 u - |v|^2 v \pm |u|^2 v\Big) \\
&= \gamma\Big(|u|^2 (u - v) + \left(|u|^2 - |v|^2\right) v\Big).
\end{aligned} \tag{2.2.4}$$

Since

$$\begin{aligned}
\left\| |u|^2 (u-v) \right\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |u|^4 (u-v)^2 \ dx \\
&\leq \|u\|_{L^\infty(\mathbb{R})}^4 \|u-v\|_{L^2(\mathbb{R})}^2 \\
&\leq C \|u\|_{H^1(\mathbb{R})}^4 \|u-v\|_{L^2(\mathbb{R})}^2 \\
&\leq C \left( \|u\|_{H^1(\mathbb{R})}^4 + \|v\|_{H^1(\mathbb{R})}^4 \right) \left( \|u-v\|_{L^2(\mathbb{R})}^2 \right), \qquad (2.2.5)
\end{aligned}$$

by Sobolev embedding, where $C > 0$ is some constant, and

$$\begin{aligned}
\left\| \left( |u|^2 - |v|^2 \right) v \right\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} v^2 \left( |u|^2 - |v|^2 \right)^2 \ dx \\
&= \int_{\mathbb{R}} v^2 \big( (|u| + |v|)(|u| - |v|) \big)^2 \ dx \\
&\leq \int_{\mathbb{R}} v^2 (|u| + |v|)^2 |u-v|^2 \ dx \\
&\leq \int_{\mathbb{R}} v^2 C \left( |u|^2 + |v|^2 \right) |u-v|^2 \ dx \\
&\leq C \|v\|_{H^1(\mathbb{R})}^2 \left( \|u\|_{H^1(\mathbb{R})}^2 + \|v\|_{H^1(\mathbb{R})}^2 \right) \left( \|u-v\|_{L^2(\mathbb{R})}^2 \right) \\
&\leq C \left( \|u\|_{H^1(\mathbb{R})}^4 + \|v\|_{H^1(\mathbb{R})}^4 \right) \left( \|u-v\|_{L^2(\mathbb{R})}^2 \right). \qquad (2.2.6)
\end{aligned}$$

it follows that

$$\|f(u) - f(v)\|_{L^2(\mathbb{R}^n)} \leq C \left( \|u\|_{H^1(\mathbb{R})}^4 + \|v\|_{H^1(\mathbb{R})}^4 \right) \left( \|u-v\|_{L^2(\mathbb{R})}^2 \right). \qquad (2.2.7)$$

Thus, by applying chain rule to $f(u) - f(v)$ and repeating the above argument, one can ultimately show that

$$\|f(u) - f(v)\|_{H^1(\mathbb{R})} \leq C \left( \|u\|_{H^1(\mathbb{R})} + \|v\|_{H^1(\mathbb{R})} \right)^2 \|u-v\|_{H^1(\mathbb{R})},$$

as claimed. ❏

At this point, showing that $f$ is locally Lipschitz when $n = \sigma = 1$, is simply a matter of applying inequality (2.2.3). Without loss of generality, we may consider any neighborhood $B(0, R) \subset H^1(\mathbb{R}^1)$ of the zero function with radius $R > 0$ and take $u, v \in B(0, R)$. In which case, (2.2.3) implies

$$\|f(u) - f(v)\|_{H^1(\mathbb{R})} \leq C \left( \|u\|_{H^1(\mathbb{R})} + \|v\|_{H^1(\mathbb{R})} \right)^2 \|u - v\|_{H^1(\mathbb{R})} \leq 4CR^2 \|u - v\|_{H^1(\mathbb{R})}, \quad (2.2.8)$$

verifying that $f$ is locally Lipschitz in this case. As mentioned above, $f$ can still be shown to be locally Lipschitz for general $\sigma > 0$.

We now turn our attention to showing $A$ generates a $C_0$ unitary group of operators. Recall the following important result from functional analysis:

**Theorem 2.2.2** (Stone's Theorem). *Let $H$ be a Hilbert space and $A$ be an operator on $H$. Then there exists a one parameter, uniformly continuous group of unitary operators $\{T(t)\}_{t \in \mathbb{R}}$ generated by $A$ if and only if $A$ is skew adjoint.*

While the proof of Stone's Theorem is outside the scope of this project, it can be found in Conway (1997).

Using $A := i\Delta$ as above, since, for every $u, v \in H^1(\mathbb{R}^1)$,

$$\langle Au, v \rangle_{L^2(\mathbb{R}^1)} = \int_{R^1} -i\Delta \overline{u} v \, dx = \int_{R^1} i\nabla \overline{u} \cdot \nabla v \, dx = - \int_{R^1} \overline{u} \left( i\Delta u \right) \, dx = - \langle u, Av \rangle_{L^2(\mathbb{R}^1)},$$

we see that

$$\begin{aligned} \langle Au, v \rangle_{H^1(\mathbb{R}^1)} &= \langle Au, v \rangle_{L^2(\mathbb{R}^1)} + \langle ADu, Dv \rangle_{L^2(\mathbb{R}^1)} \\ &= - \langle u, Av \rangle_{L^2(\mathbb{R}^1)} - \langle Du, ADv \rangle_{L^2(\mathbb{R}^1)} \\ &= - \langle u, Av \rangle_{H^1(\mathbb{R}^1)} . \end{aligned}$$

Thus, $A$ is skew symmetric on $H^1(\mathbb{R}^1)$ and therefore, by Stone's Theorem, generates a uniformly continuous group of unitary operators, which is an even stronger result than is necessary to apply Theorem 2.1.7. Moreover, since $A$ generates a uniformly continous unitary group, not only can we conclude the existence and uniqueness of local mild solutions for each initial data $u_0$ (as granted by Theorem 2.1.7), but we can also conclude that these unique local, mild solutions are actually *classical* solutions. As such, to show that the NLS IVP is locally well posed, it only remains to show that the NLS IVP is continuous with respect to initial data. The following lemma is a significant step in that direction:

**Lemma 2.2.3.** *For initial data $u_0,\ v_0 \in H^1(\mathbb{R}^1)$, there exist constants $T > 0$ and $M > 0$ so that for $t \in [-T, T]$*

$$\|u(t) - v(t)\|_{H^1(\mathbb{R}^1)} \leq e^{Mt} \|u_0 - v_0\|_{H^1(\mathbb{R}^1)}, \tag{2.2.9}$$

*where $u$, and $v$ are the respective solutions to the NLS IVP (in one spatial dimension) corresponding to $u_0$ and $v_0$.*

*Proof.* Let $u,\ v$ be solutions to the NLS IVP corresponding to the respective initial data $u_0,\ v_0 \in H^1(\mathbb{R}^n)$. Then, from our previous work in showing that the NLS IVP satisfies the first condition of local well posedness, there exists $T > 0$ so that $u,\ v \in \big(Y, \|\cdot\|_Y\big)$ and $\|u\|_Y,\ \|v\|_Y < \infty$, where

$$Y := C\Big([-T,T];\ H^1\left(\mathbb{R}^1\right)\Big), \qquad \text{and} \qquad \|w\|_Y := \sup_{t\in[-T,T]} \|w(t)\|_{H^1(\mathbb{R})},$$

for $w \in Y$. We may set $R = 2\|u - v\|_Y$, as $\|u - v\|_Y \leq \|u\|_Y + \|v\|_Y < \infty$, and take $M$ to be the local Lipschitz constant of $f$ with respect to $R$. For $t \in [-T, T]$, define

$$W(t) = \|u(t) - v(t)\|_{H^1(\mathbb{R}^1)}.$$

Using Duhamel's formula (Proposition 2.1.5) and the fact that the operator group generated by $A$ is unitary in $H^1(\mathbb{R}^1)$, we see that

$$\begin{aligned}
W(t) &= \left\| u_0 - v_0 + \int_0^t e^{tA} f\big(u(s)\big)\ ds - \int_0^t e^{tA} f\big(v(s)\big)\ ds \right\|_{H^1(\mathbb{R}^1)} \\
&\leq \|u_0 - v_0\|_{H^1(\mathbb{R}^1)} + \int_0^t \left\|e^{tA}\right\|_{H^1(\mathbb{R}^1)} \left\|f\big(u(s)\big) - f\big(v(s)\big)\right\|_{H^1(\mathbb{R}^1)}\ ds \\
&\leq \|u_0 - v_0\|_{H^1(\mathbb{R}^1)} + M \int_0^t \|u(s) - v(s)\|_{H^1(\mathbb{R}^1)}\ ds \\
&= \|u_0 - v_0\|_{H^1(\mathbb{R}^1)} + M \int_0^t W(t)\ ds
\end{aligned}$$

Thus, by Grönwall's inequality (See Corollary 2.2.5),

$$\|u(t) - v(t)\|_{H^1(\mathbb{R}^1)} \leq e^{Mt} \|u_0 - v_0\|_{H^1(\mathbb{R}^1)},$$

for every $t \in [-T, T]$ ❏

In the proof of Lemma 2.2.3, we use a corollary to an important theorem from ODE theory called Grönwall's inequality. Since Grönwall's inequality is nearly ubiquitous in ODE and PDE research, we state without proof Grönwall's inequality and its corollary, as presented in Gerald Teschl's textbook *Ordinary Differential Equations and Dynamical Systems*.[2]

**Theorem 2.2.4** (Gronwall)**.** *Suppose $\psi(t)$ satisfies*

$$\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)\ ds, \qquad t \in [0, T]$$

*with $\alpha(t) \in \mathbb{R}$ and $\beta(t) \geq 0$. Then*

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)\ dr\right)\ ds, \qquad t \in [0, T].$$

[2]See Teschl (2012).

**Corollary 2.2.5.** *If*

$$\psi(t) \leq \alpha + \int_{-t}^{t} \left(\beta\psi(s) + \gamma\right) \, ds, \qquad t \in [-T, T],$$

*then*

$$\psi(t) \leq \alpha \exp(\beta t) + \frac{\gamma}{\beta}(\exp(\beta(t) - 1)), \qquad t \in [-T, T].$$

Finally, to finish of the argument that the NLS IVP is locally well posed, let $\varepsilon > 0$ be arbitrary, and choose $T > 0$ and $M > 0$ as in the proof of Lemma 2.2.3. Set $\delta := \varepsilon / e^{MT}$, and assume $\|u_0 - v_0\|_{H^1(\mathbb{R}^1)} < \delta$. Then Lemma 2.2.3 implies that

$$\sup_{t \in [-T,T]} \|u(t) - v(t)\|_{H^1(\mathbb{R}^1)} \leq e^{MT} \|u_0 - v_0\|_{H^1(\mathbb{R}^1)} < \varepsilon,$$

for respective solutions $u,\ v$ to the NLS IVP corresponding to $u_0,\ v_0 \in H^1(\mathbb{R}^1)$. We therefore conclude that the NLS IVP in one spatial dimension is locally well posed.

*Remark.* The only place where the argument presented in this section cannot be immediately generalized to the NLS IVP in $n$ spatial dimensions, is the argument where we show that the function $f$ is locally Lipschitz. Specifically, over $H^1(\mathbb{R}^n)$, where $n \geq 2$, we do not have a convenient Sobolev embedding theorem which allows us to conclude

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C \|u\|_{H^1(\mathbb{R}^n)},$$

for some constant $C > 0$, as we do in the $n = 1$ case. However, while it can still be shown that the NLS IVP is still well posed over $H^1(\mathbb{R}^n)$—provided $0 < \sigma < \frac{2}{n-2}$—it is significantly more difficult to do without the use of the aforementioned embedding, and is beyond the scope of this project. Again, for more information about how to show that the NLS is locally well posed over $H^1(\mathbb{R}^n)$ for $n > 1$, see Sulem & Sulem (1999).

# Chapter 3

# Existence of Ground State Solutions

Bound state solutions are time independent functions $\phi \in H^1(\mathbb{R}^n)$ for which time dependent functions $u(x,t) \in C\big(\mathbb{R};\ H^1(\mathbb{R}^n)\big)$ of the form

$$u(x,t) = e^{i\lambda^2 t}\phi(x), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{3.0.1}$$

where $\lambda \in \mathbb{R}$ is some spectral parameter, are solutions to the NLS. Assuming, for the sake of argument, that bound state solutions to the NLS exist, then, by substituting (3.0.1) into the NLS we obtain,

$$-\lambda^2 e^{-i\lambda^2 t}\phi = -e^{-i\lambda^2 t}\Delta\phi - e^{-i\lambda^2 t}\gamma|\phi|^{2\sigma}\phi. \tag{3.0.2}$$

Canceling the $e^{i\lambda^2 t}$ term on both sides of the above equation and rearranging terms gives us the nonlinear, elliptic equation

$$\Delta\phi = \lambda^2\phi - \gamma|\phi|^{2\sigma}\phi, \tag{3.0.3}$$

called the *profile equation*. Hence, bound state solutions of the NLS exist, if and only if solutions to the PDE (3.0.3) exist.

While the general existence theory for bound state solutions lies outside the purview of this paper, in this chapter, we do examine the existence theory of a special class of bound state solutions called "ground state solutions." In essence, ground state solutions are bound state solutions of minimum possible energy—that is, a ground state solution is a function in $H^1(\mathbb{R}^n)$ which minimizes the Hamiltonian operator corresponding to the NLS under the constraint discussed in Section 3.3.

## 3.1 Special Case

We begin our discussion on the existence of solutions to (3.0.3) by first considering the special case where we assume $\phi \in H^1(\mathbb{R}^1)$ and $\sigma = 1$. Later, in Section 3.3, we consider the general case.

In one dimension, (3.0.3) can be written as

$$\phi_{xx} = \lambda^2\phi - \gamma|\phi|^2\phi. \tag{3.1.1}$$

Multiplying both sides of (3.1.1) by $\phi_x$ gives us

$$\phi_x\phi_{xx} = \lambda^2\phi\phi_x - \gamma|\phi|^2\phi\phi_x. \tag{3.1.2}$$

Since $\phi_x\phi_{xx} = \left(\frac{1}{2}(\phi_x)^2\right)'$ and $\lambda^2\phi\phi_x = \left(\frac{\lambda^2}{2}\phi^2\right)'$, we see that

$$\left(\frac{1}{2}(\phi_x)^2\right)' = \left(\frac{\lambda^2}{2}\phi^2\right)' - F(\phi), \tag{3.1.3}$$

where $F(\phi)$ denotes some function of $\phi$. Hence, by integrating both sides of (3.1.3), the equation becomes

$$\frac{1}{2}(\phi_x)^2 = \frac{\lambda^2}{2}\phi^2 - F(\phi) + E, \tag{3.1.4}$$

where $F(\phi)$ again denotes some function of $\phi$ and $E$ is a constant of integration. Observe that since $\phi, \phi_x \in L^2(\mathbb{R})$, we know that $\phi, \phi_x \to 0$ as $x \to \pm\infty$. This fact forces the constant of integration $E$ to be zero. Thus, if $u(x,t)$ is a bound state solution of the NLS, then $\phi$ must satisfy the O.D.E

$$\phi_x^2 = G(\phi) := \lambda^2\phi^2 - 2F(\phi). \tag{3.1.5}$$

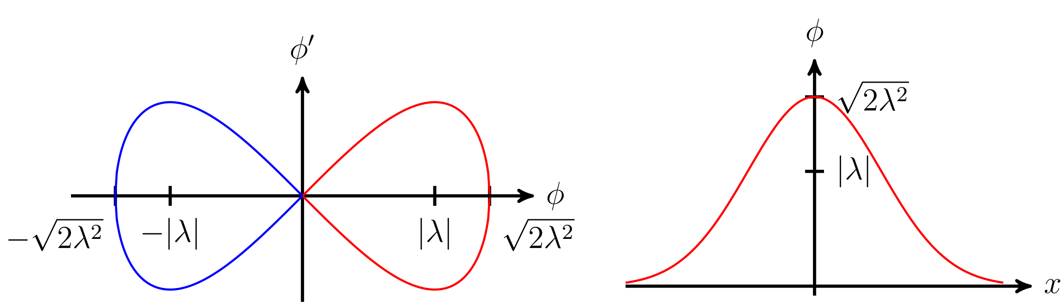
Now, if we make the additional assumption that $\phi$ is real valued, then we can rewrite equation (3.1.3) as

$$\left(\frac{1}{2}(\phi_x)^2\right)' = \left(\frac{\lambda^2}{2}\phi^2\right)' - \left(\frac{\gamma}{4}\phi^4\right)', \tag{3.1.6}$$

which, after integration, becomes

$$(\phi_x)^2 = \lambda^2\phi^2 - \frac{\gamma}{2}\phi^4, \tag{3.1.7}$$

as the constant of integration is zero (as discussed above). It easy to see from (3.1.7) that for the case where $\phi$ is assumed to be real, $\phi_x = 0$ when $\phi = 0$ or $\phi = \pm\sqrt{2\lambda^2}$. Moreover, for a focusing NLS (*i.e.* $\gamma = 1$), looking at the phase plane for (3.1.7), as shown below in Figure 3.1a, we see that (3.1.7) has a solution which is monomodal, strictly positive, and unique up to translations.




(a) Focusing NLS Phase Portrait (b) Real Bound State Solution

Figure 3.1: Phase portrait (a) and solution (b) for (3.1.7) when $\gamma = 1$

Alternatively, for a defocusing ($\gamma = -1$) NLS, the phase portrait for (3.1.7) blows-up as $\phi \to \pm\infty$, as shown in Figure 3.2. In such a case, (3.1.7) cannot possibly admit a solution in $L^2(\mathbb{R}^1)$.

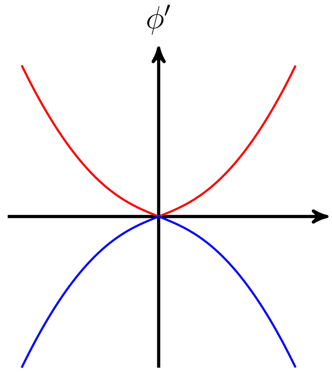



Figure 3.2: Phase portrait for (3.1.7) when $\gamma = -1$

The general proof that $L^2$-integrable bound state solutions do not exist for a defocusing NLS equation exceeds the scope of this paper. Nonetheless, throughout the remainder of this paper, we focus exclusively on focusing NLS equations.

## 3.2 Preliminary Results Needed to Prove the Existence of Ground State Solutions

Determining the conditions on $\sigma > 0$ for which the PDE (3.0.3) has a solution involves the use of concentration compactness. In the absence of compactness or some sort of uniform bound or control on a sequence, concentration compactness provides us with an alternative to the Bolzano-Weierstrass theorem:

**Theorem 3.2.1** (Concentration Compactness)**.** *Let* $\{u_k\}_{k=1}^{\infty}$ *be a bounded sequence in* $H^1(\mathbb{R}^n)$ *with* $\|u_k\|_{L^2(\mathbb{R}^n)} = \lambda > 0$ *for every* $k > 0$*. Then, there exists subsequence* $\left\{u_{k_j}\right\}_{j=1}^{\infty}$ *satisfying exactly one of the following:*

*i)* **Convergence of Translates:** *There exists a sequence* $\{y_j\}_{j=1}^{\infty}$ *in* $\mathbb{R}^n$ *so that* $\left\{u_{k_j}(\cdot - y_j)\right\}_{j=1}^{\infty}$

*is convergent in* $L^p(\mathbb{R}^n)$ *for all* $p$ *satisfying* $2 \leq p < \frac{2n}{n-2}$ *if* $n \geq 2$ *and* $2 \leq p \leq \infty$ *if* $n = 1$.

*ii)* **Vanishing:** *The subsequence* $\left\{u_{k_j}\right\}_{j=1}^{\infty}$ *converges to* $0$ *as* $k_j \to \infty$ *in* $L^p(\mathbb{R}^n)$ *for all* $p$ *satisfying* $2 < p \leq \infty$ *if* $n = 1$ *and* $2 < p < \frac{2n}{n-2}$ *if* $n \geq 2$.

*iii)* **Splitting:** *There exists* $\nu \in (0, \lambda)$ *and bounded sequences* $\{v_j\}_{j=1}^{\infty}$ *and* $\{w_j\}_{j=1}^{\infty}$ *in* $H^1(\mathbb{R}^n)$ *with* $\|v_j\|_L^2 (\mathbb{R}^n)^2 \to \nu$, $\|w_j\|_L^2 (\mathbb{R}^n)^2 \to \lambda - \nu$, *and dist* $\big(\, spt(v_j), spt(w_j)\big) \to \infty$ *so that* $\left\|u_{k_j} - (v_j + w_j)\right\|_{L^p(\mathbb{R}^n)} \to 0$ *for every* $p$ $F$ *satisfying* $2 \leq p < \frac{2n}{n-2}$ *if* $n \geq 2$ *and* $2 < p \leq \infty$ *if* $n = 1$. *Moreover,*

$$\liminf_{j\to\infty} \int_{\mathbb{R}^n} \left(\left|Du_{k_j}\right|^2 - |Dv_j|^2 - |Dw_j|^2\right)\, dx \geq 0.$$

*Remark.* The concentration compactness theorem means that provided we can rule out vanishing and splitting, then, for a sequence satisfying the hypotheses of Theorem 3.2.1, we still find a subsequence which can be made to converge through the use of a sequence of translates. As such, one common approach to applying concentration compactness involves first ruling out both vanishing and splitting—just as we do later in this section.

Another important theorem from functional analysis which we use in our discussion on the existence of ground state solutions is the Banach-Alaoğlu theorem, which depends on the concept of weak convergence:

**Definition 3.2.2** (Weak Convergence)**.** Let $X$ be a Banach space with associated field of scalars $\mathbb{F}$. We say that a sequence $\{x_k\}_{k=1}^{\infty} \subset X$ converges weakly to $x \in X$ if $T(u_k) \to T(u)$ in $\mathbb{F}$ for every functional $T$ in the dual space $X^* := \mathcal{L}(X,\ \mathbb{F})$ of $X$.

**Theorem 3.2.3** (Banach-Alaoğlu)**.** *Let* $X$ *be a reflexive Banach space—i.e.* $X$ *is isomorphic to the dual* $X^{**}$ *of its dual space* $X^*$*—and suppose that the sequence* $\{x_k\}_{k=1}^{\infty} \subset X$ *is bounded. Then there exists a subsequence* $\left\{x_{k_j}\right\}_{j=1}^{\infty}$ *of* $\{x_k\}_{k=1}^{\infty}$ *which converges weakly to some* $x \in X$.

For a proof of the Banach-Alaoğlu, see Conway (1997).

## 3.3 General Proof of the Existence of Ground State Solutions

Since (3.0.3) is a nonlinear, elliptic equation, using variational methods to determining the existence to (3.0.3) is a very natural approach. To this end, define functionals $\mathcal{H}$, $\mathcal{N}$ : $H^1(\mathbb{R}^n) \to \mathbb{R}$ by

$$\mathcal{H}(u) := \frac{1}{2}\int_{\mathbb{R}^n} |Du|^2 \, dx - \frac{1}{2(\sigma+1)}\int_{\mathbb{R}^n} |u|^{2(\sigma+1)} \, dx,$$

and

$$\mathcal{N}(u) := \frac{1}{2}\int_{\mathbb{R}^n} |u|^2 \, dx = \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^n)}.$$

Note that $\mathcal{H}$ is the Hamiltonian operator associated with the NLS and is well defined on $H^1(\mathbb{R}^n)$ if $H^1(\mathbb{R}^n) \subset L^{2(\sigma+1)}(\mathbb{R}^n)$. As we show later, since $H^1(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $p$ satisfying $2 \le p \le \frac{2n}{n-p}$ and $n \ge 2$, it follows that we need $2(\sigma+1) \le \frac{2n}{n-2}$. That is, $\mathcal{H}$ is well defined for $\sigma \le \frac{2}{n-2}$.

**Theorem 3.3.1.** *If the constrained variational problem*

$$I_\mu := \inf_{\substack{u \in H^1(\mathbb{R}^n) \\ \mathcal{N}(u)=\mu}} \mathcal{H}(u), \qquad \mu > 0$$

*has a minimizer $\phi \in H^1(\mathbb{R}^n)$, then $\phi$ is a weak solution of* (3.0.3) *for some $\lambda^2 \in \mathbb{R}$.*

Essentially, Theorem 3.3.1 is a Lagrange multiplier type statement in that if $\phi$ satisfies $I_\mu$, then it minimizes the operator $\mathcal{H}$ subject to the constraint $\mathcal{N}(\phi) = \mu$. The Lagrangian corresponding the the variational problem given in Theorem 3.3.1 is $\mathcal{E}(u) := \mathcal{H}(u) + \mu\mathcal{N}(u)$. In Section 6.1 we show that $u$ is a bound state solution if and only if it is a critical point of $\mathcal{E}$—thus establishing the validity of Theorem 3.3.1.

The next theorem provides us with some conditions on $\sigma$ for which it may be possible for the variational problem given in Theorem 3.3.1 to have a solution:

**Theorem 3.3.2.** *For $\sigma$ satisfying $0 < \sigma \leq \frac{2}{n-2}$ and $I_\mu$ is as defined in Theorem 3.3.1, then*

*i)* **Sub-critical case:** *If $0 < \sigma < \frac{2}{n}$, then $-\infty < I_\mu < 0$.*

*ii)* **Critical case:** *If $\sigma = \frac{2}{n}$, then there exists $\mu_0$ so that*

$$I_\mu = \begin{cases} 0, & \text{if } 0 < \mu \leq \mu_0 \\ -\infty, & \text{if } \mu > \mu_0 \end{cases}.$$

*iii)* **Super-critical case:** *If $\sigma > \frac{2}{n}$, then $I_\mu = -\infty$ for every $\mu > 0$.*

*Proof.* Fix $\mu > 0$ and take

$$u \in C_c^\infty(\mathbb{R}^n) := \left\{ v \in C^\infty(\mathbb{R}^n) \ : \ v \text{ has compact support} \right\}$$

with $\|u\|_{L^2(\mathbb{R}^n)} = \mu$. In order to prove both the sub-critical and super-critical cases, consider the dilations

$$u_L(x) := L^{n/2} u(Lx), \quad L > 0.$$

Note that for all $L > 0$,

$$\|u_L\|_{L^2(\mathbb{R}^n)} = \|u_L\|_{L^2(\mathbb{R}^n)},$$

and

$$\mathcal{H}(u_L) = \frac{L^2}{2} \int_{\mathbb{R}^n} |Du|^2 \, dx - \frac{L^{n\sigma}}{2(\sigma+1)} \int_{\mathbb{R}^n} |u|^{2(\sigma+1)} \, dx$$

Since both

$$\int_{\mathbb{R}^n} |Du|^2 \, dx, \qquad \text{and} \qquad \int_{\mathbb{R}^n} |u|^{2(\sigma+1)} \, dx$$

are positive and fixed, if $n\sigma > 2$ $\left(i.e.\ \sigma > \frac{2}{n}\right)$, then $\mathcal{H}(u_L) \to -\infty$ as $L \to \infty$—thus verifying (iii).

Similarly, if $n\sigma < 2$ $\left(i.e.\ \sigma < \frac{2}{n}\right)$, then $\mathcal{H}(u_L) < 0$ for $L > 0$ sufficiently small. Thus $I_\mu < 0$. Now to show that $I_\mu > -\infty$ when $\sigma < \frac{2}{n}$, we need the Gagliardo-Nirenberg-Sobolev

(GNS) inequality, which states that there exists a constant $C_{n,p} > 0$

$$\|v\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} \|Dv\|_{L^p(\mathbb{R}^n)}^{n\left(\frac{1}{2}-\frac{1}{p}\right)} \|v\|_{L^p(\mathbb{R}^n)}^{1-n\left(\frac{1}{2}-\frac{1}{p}\right)} \tag{3.3.1}$$

for every $v \in L^p(\mathbb{R}^n)$, where $p$ satisfies

$$\begin{cases} 2 \leq p < \infty, & \text{for } n = 1,\ 2 \\ 2 \leq p \leq \frac{2n}{n-2}, & \text{for } n > 2 \end{cases}$$

Since $2\sigma < \frac{4}{n-2}$ implies $2(\sigma+1) < \frac{2n}{n-2}$, by taking $p = 2(\sigma+1)$ in the GNS inequality yeilds

$$\begin{aligned} \mathcal{H}(u) &= \frac{1}{2}\int_{\mathbb{R}^n} |Du|^2 \ dx - \frac{1}{2(\sigma+1)}\int_{\mathbb{R}^n} |u|^{2(\sigma+1)} \ dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^n} |Du|^2 \ dx - \frac{C_{n,p}^{2(\sigma+1)}}{2(\sigma+1)} \|Du\|_{L^2(\mathbb{R}^n)}^{n\sigma} \mu^b, \end{aligned}$$

for $b = \frac{1}{2}\left(1 - n\left(\frac{1}{2} - \frac{1}{2(\sigma+1)}\right)\right)$. Hence, $\mathcal{H}(u) \geq g\left(\|Du\|_{L^2(\mathbb{R}^n)}\right)$, where

$$g(R) = \frac{1}{2}R^2 - \frac{C_{n,p}^{2(\sigma+1)}}{2(\sigma+1)}\mu^b R^{n\sigma},$$

for $\sigma < \frac{2}{n}$. Now, since $n\sigma < 2$, the function $g$ is “quadratic-like” for $R > 0$. In particular, $g$ has a minimum $g_{min}$. Thus,

$$\mathcal{H}(u) \geq g_m in > \infty,$$

which proves the sub-critical case.

Finally, to verify the critical case, define

$$\mathcal{A}_\mu := \left\{u \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} u^2 dx = \mu\right\}$$

and note that $I_\mu = \inf_{u \in \mathcal{A}_\mu} \mathcal{H}(u)$. From our previous work with the GNS inequality, recall that if $p = 2(\sigma+1)$, then there exists some constant $C_n > 0$ so that

$$\|u\|_{L^{2(\sigma+1)}(\mathbb{R}^n)}^{2(\sigma+1)} \leq C_n^{2(\sigma+1)} \|Du\|_{L^2(\mathbb{R}^n)}^{n\sigma} \mu^d, \tag{3.3.2}$$

where $d = 2(\sigma+1) - n\sigma$, and $u \in \mathcal{A}_\mu$. If we set $\sigma = \frac{2}{n}$, then (3.3.2) becomes

$$\|u\|_{L^{4/n+2}(\mathbb{R}^n)}^{4/n+2} \leq C_n^{4/n+2} \|Du\|_{L^2(\mathbb{R}^n)}^2 \mu^{4/n}. \tag{3.3.3}$$

From (3.3.3), we therefore see that

$$\begin{aligned} \mathcal{H}(u) &\geq \frac{1}{2}\|Du\|_{L^2(\mathbb{R}^n)}^2 - \frac{C_n^{4/n+2}}{\frac{4}{n}+2}\|Du\|_{L^2(\mathbb{R}^n)}^2 \mu^{4/n} \\ &= \left(\frac{1}{2} - \left(\frac{C_n^{4/n+2}}{\frac{4}{n}+2}\right)\mu^{4/n}\right)\|Du\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \tag{3.3.4}$$

As such, provided that

$$\frac{1}{2} - \left(\frac{C_n^{4/n+2}}{\frac{4}{n}+2}\right)\mu^{4/n} \geq 0, \tag{3.3.5}$$

then $\mathcal{H}(u) > -\infty$ for every $u \in \mathcal{A}_\mu$. Using this result as a guide, set

$$\mu_0 := \left(\frac{K_n^{4/n+2}}{2/n+1}\right)^{\frac{n}{4}}, \tag{3.3.6}$$

where

$$K_n^{-1} := \inf\left\{\frac{\|Du\|_{L^2(\mathbb{R}^n)}^{n/(n+2)}\|u\|_{L^2(\mathbb{R}^n)}^{2/(n+2)}}{\|u\|_{L^{4/n+2}(\mathbb{R}^n)}} \;:\; u \in H^1(\mathbb{R}^n)\backslash\{0\}\right\} > 0.$$

With a little algebra, it is easy to see that every $\mu \in (0, \mu_0]$ satisfies the inequality (3.3.5), which means that $I_\mu > -\infty$ for every such $\mu$.

Now, to see that $I_\mu \geq 0$ for every $\mu \in (0, \mu_0]$, we take any $u \in \mathcal{A}_\mu$ and consider the function $u_\ell(x) := \ell^{n/2}u(\ell u)$. Since

$$\begin{aligned}\mathcal{H}(u_\ell) &= \frac{\ell}{2}\|Du\|^2_{L^2(\mathbb{R}^n)} - \frac{\ell^2}{4/n+2}\|u\|^{4/n+2}_{L^{4/n+2}(\mathbb{R}^n)} \\ &= \ell^2\left(\frac{1}{2}\|Du\|^2_{L^2(\mathbb{R}^n)} - \frac{1}{4/n+2}\|u\|^{4/n+2}_{L^{4/n+2}(\mathbb{R}^n)}\right) \\ &= \ell^2\mathcal{H}(u),\end{aligned}$$

if we take $\ell \to \infty$, then $\mathcal{H}(u_\ell)$ converges to either zero or $\pm\infty$. However, given that $u_\ell \in \mathcal{A}_\mu$, we know that $\lim_{\ell\to\infty}\mathcal{H}(u_\ell)$ can only be 0 or $\infty$. This implies that $\mathcal{H}(u) \geq 0$. As such, since $u$ was chosen arbitrarily, we need only find a specific $u \in \mathcal{A}_\mu$ for which $\mathcal{H}(u) = 0$ to verify the desired result. ❏

Theorem 3.3.2 demonstrates that while it is impossible for $I_\mu$ to have a minimizer if $\sigma > \frac{2}{n}$, it may be possible for $I_\mu$ to have minimizer if $\sigma < \frac{2}{n}$. Lemma 3.3.3 and Theorem 3.3.4 tells us that in fact, if $\sigma < \frac{2}{n}$, then $I_\mu$ does have a minimizer.

**Lemma 3.3.3.** *The map $\mu \mapsto I_\mu$ is strictly subadditive. In particular,*

$$I_\mu < I_\nu + I_{\mu-\nu}$$

*for every $\nu \in (0, \mu)$.*

*Proof.* We begin by observing that for every $\theta > 1$ and $u \in H^1(\mathbb{R}^n)$ with $\|u\|_{L^2(\mathbb{R}^n)} = \mu$, we have

$$\mathcal{H}(\theta u) = \frac{\theta^2}{2}\int_{\mathbb{R}^n}|Du|^2\,dx - \frac{\theta^{2(\sigma+1)}}{2(\sigma+1)}\int_{\mathbb{R}^n}|u|^{2(\sigma+1)}\,dx < \theta^2\mathcal{H}(u),$$

which implies that

$$I_{\theta^2\mu} < \theta^2 I_\mu,$$

for ever $\theta > 1$. Thus for all $\nu \in \left(0, \frac{\mu}{2}\right]$, using symmetry we find that

$$\begin{aligned} I_\mu &= I_{\left(\frac{\mu}{\nu}\nu\right)} \\ &< \frac{\mu}{\nu} I_\nu \\ &= I_\nu + \frac{\mu-\nu}{\nu} I_\nu \\ &\leq I_\nu + I_{\left(\frac{\mu-\nu}{\nu}\nu\right)} \\ &= I_\nu + I_{\mu-\nu}. \end{aligned}$$

Similarly, if $\nu \in \left(\frac{\mu}{2}, \mu\right)$, we can repeat the above process using the map $\nu \mapsto \mu - \nu$ to obtain the desired result. ❏

**Theorem 3.3.4.** *If $0 < \sigma < \frac{2}{n}$ and $\mu > 0$, then the problem*

$$I_\mu = \inf_{\substack{u \in H^1(\mathbb{R}^n) \\ \mathcal{N}(u)=\mu}} \mathcal{H}(u), \qquad \mu > 0 \tag{3.3.7}$$

*has a minimizer.*

*Proof.* Since $-\infty < I_\mu$, we can use the infimum approximation property to find a minimizing sequence $\{u_k\}_{k=1}^{\infty}$ in $H^1(\mathbb{R}^n)$ so that $\mathcal{N}(u_k) = \mu$ for each $k \in \mathbb{N}$, and

$$\mathcal{H}(u_k) \to I_\mu$$

as $k \to \infty$. Let the function $g$ be defined as in the previous proof. That is,

$$g(R) := \frac{1}{2} R^2 - \frac{C_{n,p}^{2(\sigma+1)}}{2(\sigma+1)} \mu^b R^{n\sigma}.$$

Then, for all sufficiently large $k$, the GNS inequality[1] implies that

$$g\left(\|Du\|_{L^2(\mathbb{R}^n)}\right) \leq \mathcal{H}(u_k) < 0,$$

as $I_\mu < 0$, by Theorem 3.3.2. Without loss of generality, we may assume that for each $k \in \mathbb{N}$,

$$\|Du\|_{L^2(\mathbb{R}^n)} \leq R_0,$$

where $R_0$ is the unique strictly positive root of $g$. Thus, $\{u_k\}_{k=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^n)$. As such, our objective is now to apply concentration compactness to show the existence of a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ which satisfies convergence of translates. To do so, we need to first rule out the possibility of $\{u_{k_j}\}_{j=1}^{\infty}$ either vanishing or splitting. We consider the former possibility first.

By definition,

$$\mathcal{H}(u_k) \leq \frac{I_\mu}{2} < 0,$$

for $k$ sufficiently large. Hence,

$$\frac{1}{2}\|Du_k\|_{L^2(\mathbb{R}^n)}^2 - \frac{1}{2(\sigma+1)}\|u_k\|_{L^{2(\sigma+1)}(\mathbb{R}^n)}^{2(\sigma+1)} \leq \frac{I_\mu}{2} < 0,$$

which implies that

$$\frac{1}{2(\sigma+1)}\|u_k\|_{L^{2(\sigma+1)}(\mathbb{R}^n)}^{2(\sigma+1)} \geq -\frac{I_\mu}{2} > 0 \tag{3.3.8}$$

for every $k \gg 1$. Now, suppose by way of contradiction that there exists some subsequence

[1]Equation (3.3.1)

$\{u_{k_j}\}_{j=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ which vanishes. That is, $u_{k_j} \to 0$ in $L^p(\mathbb{R}^n)$ for all $p$ satisfying

$$\begin{cases} 2 < p \leq \infty, & \text{for } n = 1 \\ 2 < p < \frac{2n}{n-2}, & \text{for } n \geq 2 \end{cases}.$$

Since

$$0 < \sigma < \frac{2}{n} \iff 2 < 2(\sigma+1) < \frac{2n}{n-2},$$

we have that $u_{k_j} \to 0$ in $L^{2(\sigma+1)}(\mathbb{R}^n)$, which contradicts inequality (3.3.8). Thus, no subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ can satisfy the vanishing criterion of concentration compactness.

Again, by way of contradiction, suppose there exists some subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ satisfying the splitting criterion of concentration compactness. Then there exist bounded sequences $\{v_j\}_{j=1}^{\infty}$ and $\{w_j\}_{j=1}^{\infty}$ in $H^1(\mathbb{R}^n)$ and a real number $\nu \in (0, \mu)$ so that for every number $p$ with $2 \leq p < \frac{2n}{n-2}$,

$$\|v_j\|_{L^2(\mathbb{R}^n)}^2 \to \nu, \qquad \|w_j\|_{L^2(\mathbb{R}^n)}^2 \to \mu - \nu, \qquad \text{and} \qquad \int_{\mathbb{R}^n} \left( |u_{k_j}|^p - |v_{k_j}|^p - |w_{k_j}|^p \right) dx \to 0,$$

as $j \to \infty$. In addition

$$\liminf_{j\to\infty} \int_{\mathbb{R}^n} \left( |Du_{k_j}|^p - |Dv_{k_j}|^p - |Dw_{k_j}|^p \right) dx \geq 0.$$

As such, for all $\varepsilon > 0$ and $j$ sufficiently large, we have that

$$\begin{aligned} I_\mu + 3\varepsilon &\geq \mathcal{H}(u_{k_j}) + 2\varepsilon \\ &= \frac{1}{2}\|Du_{k_j}\|_{L^2(\mathbb{R}^n)}^2 - \frac{1}{2(\sigma+1)}\|u_{k_j}\|_{L^{2(\sigma+1)}(\mathbb{R}^n)}^{2(\sigma+1)} + 2\varepsilon \\ &\geq \frac{1}{2}\|Dv_j\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2}\|Dw_j\|_{L^2(\mathbb{R}^n)}^2 - \frac{1}{2(\sigma+1)}\left(\|v_j\|_{L^{2(\sigma+1)}(\mathbb{R}^n)}^{2(\sigma+1)} + \|w_j\|_{L^{2(\sigma+1)}(\mathbb{R}^n)}^{2(\sigma+1)}\right) + \varepsilon \\ &= \mathcal{H}(v_j) + \mathcal{H}(w_j) + \varepsilon \end{aligned} \tag{3.3.9}$$

Now, observe that there exists strictly positive sequences $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ in $\mathbb{R}^{>0}$ so that $a_j,\ b_j \to 1$ and

$$\|a_j v_j\|_{L^2(\mathbb{R}^n)}^2 = \nu, \qquad \text{and} \qquad \|b_j w_j\|_{L^2(\mathbb{R}^n)}^2 = \mu - \nu$$

for every $j \in \mathbb{N}$. Thus

$$\mathcal{H}(a_j v_j) \geq I_\nu, \qquad \text{and} \qquad \mathcal{H}(a_j v_j) \geq I_{\mu-\nu}, \qquad \forall j \in \mathbb{N}.$$

Further, for $j \gg 1$, we also have

$$\mathcal{H}(v_j) \geq \mathcal{H}(a_j v_j) - \frac{\varepsilon}{2} \qquad \text{and} \qquad \mathcal{H}(w_j) \geq \mathcal{H}(b_j w_j) - \frac{\varepsilon}{2}. \tag{3.3.10}$$

Thus, we can replace $\mathcal{H}(v_j)$ and $\mathcal{H}(w_j)$ in (3.3.9) by their respective rescaled versions in (3.3.10) to obtain

$$I_\mu + 3\varepsilon \geq \mathcal{H}(a_j v_j) + \mathcal{H}(b_j w_j) \geq I_\nu + I_{\mu-\nu}. \tag{3.3.11}$$

Hence, taking $\varepsilon \to 0^+$ in (3.3.11) yelds

$$I_\mu \geq I_\nu + I_{\mu-\nu}, \tag{3.3.12}$$

which contradicts the fact that the map $\mu \mapsto I_\mu$ is strictly subadditive, as shown in Lemma 3.3.3. Thus, splitting cannot occur, and so, concentration compactness therefore implies that there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ and a sequence $\{y_j\}_{j=1}^{\infty}$ in $\mathbb{R}^n$ so that the sequence $\left\{u_{k_j}(\cdot - y_j)\right\}_{j=1}^{\infty}$ converges in $L^p$ to some $\phi \in H^1(\mathbb{R}^n)$ for $p$ satisfying $2 \leq p < \frac{2n}{n-2}$.

In particular, for $p = 2$

$$\mathcal{N}(\phi) = \mu,$$

and for $p = 2(\sigma+1)$,

$$\left\| u_{k_j}(\cdot - y_j) \right\|_{L^{2(\sigma+1)}(\mathbb{R}^n)} \to \|\phi\|_{L^{2(\sigma+1)}(\mathbb{R}^n)}.$$

As such, applying the Banach-Alaoglu theorem and uniqueness of weak limits yields

$$u_{k_j}(\cdot - y_j) \to \phi$$

in $H^1(\mathbb{R}^n)$. Thus, the lower-weak semi-continuity of the map

$$H^1(\mathbb{R}^n) \ni u \mapsto \int_{\mathbb{R}^n} |Du|^2 \, dx \in \mathbb{R}$$

implies that

$$\begin{aligned} I_\mu &\le \mathcal{H}(\phi) \\ &\le \liminf_{j\to\infty} \mathcal{H}\left(u_{k_j}(\cdot - y_j)\right) \\ &= \liminf_{j\to\infty} \mathcal{H}\left(u_{k_j}\right) \\ &= I_\mu, \end{aligned}$$

verifying that $\phi$ solves problem (3.3.7). ❑

Thus, Theorems 3.3.1 and 3.3.4 tell us that an NLS equation is guaranteed to have a bound state solution whenever its nonlinearity exponent $\sigma$ satisfies $0 < \sigma < \frac{2}{n}$. While (3.0.3) is known to have a large number of solutions, the solutions which minimize $I_\mu$ have some very special properties, and are therefore given the name "ground state" solutions. First, notice that $\mathcal{H}(|u|) \le \mathcal{H}(u)$ for all $u \in H^1(\mathbb{R}^n)$. Thus, since the ground state solution $\phi$ is minimizes $\mathcal{H}$ with respect to the constraint $\mathcal{N}(\cdot) = \mu$, it follows that $|\phi| = \phi$. In other words, ground state solutions are positive. Moreover, it can also be shown using symmetric decreasing rearrangement inequalities that ground state solutions are radially symmetric, and experience exponential decay at infinity[2]. Further, Man Kam Kwong showed in his 1989 paper Kwong (1989) that ground state solutions are unique up to translations and complex rotations.

[2]See Theorem 4.5 from Sulem & Sulem (1999).

# Chapter 4

# Heuristic Approach to Stability of Standing Waves

In ODE theory, one often studies the ODE

$$x_t = Ax, \tag{4.0.1}$$

where $A$ is some linear operator. Equation (4.0.1) has a commonly known solution $x(t) = e^{tA}x(0)$. As such, it is natural to assume that the stability of an equilibrium solution $x^*$ to (4.0.1) is complete determined by the spectrum of the linear operator $A$. Indeed, as discussed in Teschl (2012), if the real component of each eigenvalue of $A$ is strictly positive, then $x^*$ is stable. So, by studying the spectrum of the matrix $A$, we can completely determine how solutions near $x^*$ behave.

Moreover, there is powerful theorem from nonlinear ODE theory called the Center Manifold Theorem which tells us that if (4.0.1) is a linearization of the nonlinear ODE $x_t = f(x)$, where $f$ is a nonlinear operator, then the spectrum of $A$ can often still be used to determine the behavior of solutions near $x^*$. In particular, if $M_+$ is the manifold spanned by eigenvectors of $A$ whose corresponding eigenvalues have strictly positive real part, $M_-$ is the manifold spanned by the eigenvectors $A$ whose corresponding eigenvalues have strictly negative real part, and $M_0$ is the manifold spanned by the eigenvectors $A$ whose corresponding eigenvalues which are strictly imaginary, then $M_+$ is what is called a "stable" manifold, $M_-$ is an "unstable" manifold, and $M_0$ is a center manifold. As the name of $M_+$ indicates, during the time period where a solution $x(t)$ of $x_t = f(x)$ lies inside the stable manifold $M_+$, it is attracted to the equilibrium $x^*$. Similarly, while $x(t)$ lies in $M_-$ is repulsed from $x^*$. However, during the time $x(t)$ lies in $M_0$, it is really neither attracted to, nor repulsed from $x^*$.

Unfortunately, the Central Manifold Theorem cannot always be applied to non-linear partial differential equations, such as the NLS. However, a general "rule of thumb" in PDE theory holds that the unstable eigenvalues of a linearized form of a PDE correspond to the unstable spectrum of the original PDE, which makes studying the spectrum of the linearized NLS a natural first step in determining the stability/instability conditions on ground state solutions.

## 4.1 Linearizing the NLS

In order to linearize the NLS equation

$$i\psi_t = -\Delta\psi - |\psi|^{2\sigma}\psi, \tag{4.1.1}$$

we consider a solution to the NLS of the form

$$\psi = e^{i\lambda^2 t}(\phi + \varepsilon\nu), \tag{4.1.2}$$

where $\phi$ is a bound state solution, $|\varepsilon| \ll 1$ and $\phi = \nu_r + i\nu_i$, for real valued functions $\nu_r,\ \nu_i$. Then, since $\phi(x) \geq 0$ for all $x \in \mathbb{R}$ (as we saw in Section 3), by substituting (4.1.2) into (4.1.1) we obtain

$$e^{i\lambda^2 t}\Big(-\lambda^2(\phi+\varepsilon\nu) + i\varepsilon\frac{\partial}{\partial t}\nu\Big) = -e^{i\lambda^2 t}(\Delta\phi + \varepsilon\Delta\nu) - e^{i\lambda^2 t}|\phi+\varepsilon\nu|^{2\sigma}(\phi+\varepsilon\nu). \tag{4.1.3}$$

Now,

$$
\begin{aligned}
|\phi+\varepsilon\nu|^2 &= (\phi+\varepsilon\nu)(\phi+\varepsilon\overline{\nu}) \\
&= \phi^2 + 2\overline{\nu}\phi + \varepsilon\nu\phi + \varepsilon^2\nu\overline{\nu} \\
&= \phi^2 + 2\varepsilon\phi \operatorname{Re}\nu + O\left(\varepsilon^2\right) \qquad (4.1.4)
\end{aligned}
$$

Since the Taylor expansion for the function $x^\sigma$ is

$$
x^\sigma = a^\sigma + \sigma a^{\sigma-1}(x-a) + \frac{1}{2}\sigma(\sigma-1)a^{\sigma-2}(x-a)^2 + \cdots ,
$$

by Taylor expanding the function $x^\sigma$ about $a=\phi^2$ and setting $x = |\phi+\varepsilon\nu|^2 = \phi^2+2\varepsilon\phi\operatorname{Re}\nu + O(\varepsilon^2)$, we obtain

$$
|\phi+\varepsilon\nu|^{2\sigma} = \phi^{2\sigma} + 2\varepsilon\sigma\phi^{2\sigma-1}\operatorname{Re}\nu + O(\varepsilon^2) \qquad (4.1.5)
$$

So, by simplifying (4.1.3) and combing the result with (4.1.5), we find

$$
\begin{aligned}
-\lambda^2(\phi+\varepsilon\nu) + i\varepsilon\frac{\partial}{\partial t}\nu &= -\Delta\phi - \varepsilon\Delta\nu - |\phi+\varepsilon\nu|^{2\sigma}(\phi+\varepsilon\nu) \\
&= -\Delta\phi - \varepsilon\Delta\nu - \left(\phi^{2\sigma} + 2\varepsilon\sigma\phi^{2\sigma-1}\operatorname{Re}\nu + O\left(\varepsilon^2\right)\right)(\phi+\varepsilon\nu) \\
&= -\Delta\phi - \varepsilon\Delta\nu - \phi^{2\sigma+1} - \varepsilon\phi^{2\sigma}\nu - 2\sigma\varepsilon\phi^{2\sigma}\operatorname{Re}(\nu) + O\left(\varepsilon^2\right) \\
&\approx -\Delta\phi - \varepsilon\Delta\nu - \phi^{2\sigma+1} - \varepsilon\phi^{2\sigma}\nu - 2\sigma\varepsilon\phi^{2\sigma}\operatorname{Re}(\nu) \qquad (4.1.6)
\end{aligned}
$$

Replacing $\nu$ in (4.1.6) by $\nu_r + i\nu_i$ therefore gives us

$$-\lambda^2(\phi + \varepsilon\nu_r + \varepsilon i\nu_i) + i\varepsilon\frac{\partial}{\partial t}(\nu_r + i\nu_i) = -\Delta\phi - \varepsilon\Delta(\nu_r + i\nu_i) - \phi^{2\sigma+1} - \varepsilon\phi^{2\sigma}(\nu_r + i\nu_i) - 2\sigma\varepsilon\phi^{2\sigma}\nu_r$$

$$= \Big( - \Delta\phi - \phi^{2\sigma+1}\Big) + \varepsilon\Big( - \Delta\nu_r - \phi^{2\sigma}\nu_r - 2\sigma\phi^{2\sigma}\nu_r\Big) \tag{4.1.7}$$

$$+ i\varepsilon\Big( - \Delta\nu_i - \phi^{2\sigma}\nu_i\Big)$$

Since $-\lambda^2\phi = -\Delta\phi - \phi^{2\sigma+1}$ (as $\phi$ is a bound state solution), the above equality (4.1.7) becomes

$$-\lambda^2(\nu_r + i\nu_i) + i\frac{\partial}{\partial t}(\nu_r + i\nu_i) = \Big( - \Delta - (2\sigma+1)\phi^{2\sigma}\Big)\nu_r + i\Big( - \Delta - \phi^{2\sigma}\Big)\nu_i \tag{4.1.8}$$

Thus

$$i\frac{\partial}{\partial t}(\nu_r + i\nu_i) = \Big( - \Delta + \lambda^2 - (2\sigma+1)\phi^{2\sigma}\Big)\nu_r + i\Big( - \Delta + \lambda^2 - \phi^{2\sigma}\Big)\nu_i$$
$$\frac{\partial}{\partial t}(\nu_r + i\nu_i) = \Big( - \Delta + \lambda^2 - \phi^{2\sigma}\Big)\nu_i - i\Big( - \Delta + \lambda^2 - 2(\sigma+1)\phi^{2\sigma}\Big)\nu_r$$
$$= L_-\nu_i - iL_+v_r, \tag{4.1.9}$$

where

$$L_- := -\Delta + \lambda^2 - \phi^{2\sigma} \qquad \text{and} \qquad L_+ := -\Delta + \lambda^2 - (2\sigma+1)\phi^{2\sigma}.$$

Hence, if we think of $\nu$ as a column vector in $\mathbb{R}^2$, then linearized NLS equation can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix}\nu_r \\ \nu_i\end{pmatrix} = \mathbf{N}\begin{pmatrix}\nu_r \\ \nu_i\end{pmatrix}, \tag{4.1.10}$$

where

$$\mathbf{N} := \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}.$$

While much of the remaining heuristic analysis focuses heavily on determining the spectrum of the matrix operator $\mathbf{N}$, the matrix operator

$$\mathcal{E}''(\phi) := \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$$

actually plays a pivotal role in the rigorous stability analysis used later on.

## 4.2 Spectral Analysis

We start by supposing that $\mu$ is an eigenvalue of the matrix $\mathbf{N}$ and let $(u,\ v)^T$ be an eigenvector of $\mathbf{N}$ corresponding to the eigenvalue $\mu$. Then

$$\mu \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} L_- v \\ -L + u \end{pmatrix},$$

which implies that $\mu u = L_- v$ and $\mu v = -L_+ u$. So, applying the operator $L_-$ to both sides of $\mu v = -L_+ u$ yields

$$L_- L_+ u = -\mu^2 u. \tag{4.2.1}$$

As we discuss in Section 6.2 below, $\ker L_- = \operatorname{span}\{\phi\}$, which means that $L_-$ is invertible on $\{\phi\}^\perp$. Thus, (4.2.1) implies

$$L_+ u = -\mu^2 L_-^{-1} u, \tag{4.2.2}$$

provided $u \perp \phi$. In particular, by taking the inner product of both sides of (4.2.2) and solving for $-\mu^2$ we obtain

$$-\mu^2 = \frac{\langle u, L_+ u\rangle_{L^2(\mathbb{R}^n)}}{\langle u, L_-^{-1} u\rangle_{L^2(\mathbb{R}^n)}}. \tag{4.2.3}$$

At this point, we claim that $-\mu^2$ is a real number. In particularly, either $\mu \in i\mathbb{R}$, or $\mu \in \mathbb{R}$. If the former is true, then the eigenvectors of $\mathbf{N}$ corresponding to $\mu$ lie in the center manifold and we can say nothing about stability in the eigenspace of $\mu$. However, if the later is true, then, since $|\mu|$ and $-|\mu|$ are both eigenvalues of $\mathbf{N}$, we are guaranteed at least one unstable "direction" (corresponding to $-|\mu|$) which causes the entire linearized system to be unstable. In such a case, our ground state solutions to the NLS should also be unstable.

To verify the claim in the preceding paragraph, set

$$\mathbf{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{L} := \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}.$$

Since $\mathbf{N} = \mathbf{JL}$ is Hamiltonian, the fact that $\mu$ is an eigenvalue of $\mathbf{N}$ means that $-\mu$ and $\pm\overline{\mu}$ are also eigenvalues of $\mathbf{N}$. Pego & Weinstein (1992) proved that the number of eigenvalues of $\mathbf{JL}$ with positive real part is less than or equal to the number of eigenvalues of the matrix $\mathbf{L}$ with negative real part. As we discuss in Section 6.2, the operator $L_+$ has exactly one negative eigenvalue—which happens to be simple—while the operator $L_-$ has no negative eigenvalues. Hence $\mathbf{L}$ has only one eigenvalue with negative real part. From which, it follows that $\mathbf{N} = \mathbf{JL}$ can have only one eigenvalue with positive real part. Consequently, if $\text{Re}(\mu) \neq 0$, then $\text{Im}(\mu) = 0$. Otherwise, either $\overline{\mu}$ or $-\overline{\mu}$ would be a distinct second eigenvalue of $\mathbf{N}$ with positive real part (with either $\mu$ or $-\mu$ being the first)—contradicting the restriction on the number of eigenvalues $\mathbf{N}$ can have which posses positive real parts.

Observe that (4.2.3) allows us to determine whether $\mu \in i\,\mathbb{R}$ or $\mu \in \mathbb{R}$. Specifically, if $-\mu^2 > 0$, then $\mu \in i\,\mathbb{R}$, and if $-\mu^2 < 0$ then $\mu \in \mathbb{R}$. So, our goal at this point is to determine the sign of the smallest possible value of the quantity (4.2.3). If the minimum of (4.2.3) is negative, we should expect the ground state solutions of the NLS to be unstable. Alternatively, if the minimum of (4.2.3) is positive, then we may expect ground state solutions could be stable—in which case, further stability analysis is warranted.

Since the spectrum of $L_-$ is non-negative—as we discuss in Section 6.2—and hence

$$\left\langle u, L_-^{-1} u \right\rangle_{L^2(\mathbb{R}^n)} \geq 0,$$

the sign of $-\mu^2$ is completely determined by the sign of $\langle u, L_+ u\rangle_{L^2(\mathbb{R}^n)}$. As such, in order to determine the sign of the minimum of $-\mu^2$, it suffices to minimize the quantity $\langle u, L_+ u\rangle_{L^2(\mathbb{R}^n)}$ subject to the constraint $u \in \ker(L_-)^\perp = \{\phi\}^\perp$. Define

$$\mu_0 := \min_{u \perp \phi} \langle u, L_+ u\rangle_{L^2(\mathbb{R}^n)} = \min \operatorname{spec}\left(\Pi L_+\right),$$

where $\Pi$ is the complementary projection operator defined by

$$\begin{aligned} \Pi : L^2\left(\mathbb{R}^n\right) &\to \{\phi\}^\perp \\ u &\mapsto u - \frac{\langle u, \phi\rangle_{L^2(\mathbb{R}^n)}}{\langle \phi, \phi\rangle_{L^2(\mathbb{R}^n)}} \phi. \end{aligned}$$

Ultimately, we show in Subsection 6.3.2 that the sign of $\mu_0$ happens to be the same as the sign of the quantity

$$\frac{\partial}{\partial \lambda^2} \mathcal{N}(\phi),$$

where $\mathcal{N}(u) := \frac{1}{2} \int_{\mathbb{R}^n} \phi^2 \, dx$.

To determine the sign of $\frac{\partial}{\partial \lambda^2}\mathcal{N}(\phi)$, we use a rescaling argument. Let $\phi_1$ denote a solution to the profile equation (3.0.3) for which $\lambda = 1$—that is,

$$\Delta \phi_1 + \phi_1 - |\phi_1|^{2\sigma} \phi_1.$$

Suppose $\phi$ is a solution to (3.0.3) for arbitrary $\lambda$ and suppose that $\phi(x) = A\phi_1(\beta x)$ for some constants $A$ and $\beta$. Then, by substituting $A\phi_1(\beta x)$ into (3.0.3), we find that

$$A\beta^2\Delta\phi_1 - \lambda^2 a\phi_1 + A^{2\sigma+1}|\phi_1|^{2\sigma}\phi_1 = 0. \tag{4.2.4}$$

Thus, in order for (4.2.4) to be true, we must have $A\beta^2 = A\lambda^2 = A^{2\sigma+1}$, or, equivalently, $\beta^2 = \lambda^2 = A^{2\sigma}$ Hence, $A = \lambda^{1/\sigma}$ and $\beta = \lambda$, and so, $\phi(x) = \lambda^{1/\sigma}\phi_1(\lambda x)$. Using the fact that $\lambda^2 > 0$, we can therefore rewrite $\mathcal{N}(\phi)$ as

$$\begin{aligned}
\mathcal{N}(\phi) &= \frac{1}{2}\int_{\mathbb{R}^n} |\phi|^2 \; dx \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\lambda^{1/\sigma}\phi_1(\lambda x)|^2 \; dx \\
&= |\lambda|^{2/\sigma}|\lambda|^{-n}\frac{1}{2}\int_{\mathbb{R}^n} |\phi_1(x)|^2 \; dx \\
&= \left(\lambda^2\right)^{1/\sigma - n/2}\mathcal{N}(\phi_1),
\end{aligned}$$

which allows us to easily compute $\frac{\partial}{\partial\lambda^2}\mathcal{N}(\phi)$:

$$\begin{aligned}
\frac{\partial}{\partial\lambda^2}\mathcal{N}(\phi) &= \frac{\partial}{\partial\lambda^2}\left(\lambda^2\right)^{1/\sigma - n/2}\mathcal{N}(\phi_1) \\
&= \left(\sigma - \frac{n}{2}\right)\left(\lambda^2\right)^{1/\sigma - n/2 - 1}\mathcal{N}(\phi_1).
\end{aligned} \tag{4.2.5}$$

Observe that (4.2.5) implies $\frac{\partial}{\partial\lambda^2}\mathcal{N}(\phi) > 0$ for $\sigma > 2/n$ and $\frac{\partial}{\partial\lambda^2}\mathcal{N}(\phi) < 0$ for $\sigma < 2/n$. Thus, when $\sigma < 2/n$, the ground state solutions to the NLS could be stable. However, if $\sigma > 2/n$, then we should expect ground state solutions to be unstable.

# Chapter 5

# General Discussion on Tools Required for Stability Theory

We are nearly ready to begin our stability analysis of the NLS. However, before doing so, we take a brief repose from our examination of the NLS to discuss some of the tools we use in the following section:

## 5.1 Functional Derivatives

We can treat $X := H^1(\mathbb{R}^n;\ \mathbb{C})$ as a two dimensional vector space over $\mathbb{R}$ by equipping it with the inner product

$$\langle u,\ v\rangle_X := \mathrm{Re}\left(\int_{\mathbb{R}^n} (u\bar{u} + \nabla u \cdot \nabla \bar{v})\ dx\right).$$

Moreover, we can also identify the dual space $X^*$ of $X$ through the bilinear form

$$\langle u,\ v\rangle_{X^*} := \mathrm{Re}\left(\int_{\mathbb{R}^n} u\bar{v}\ dx\right).$$

As such, given the functional $\mathcal{F} : X \to \mathbb{R}$, the variational (or functional) derivative $\frac{\delta}{\delta u}\mathcal{F}$ (or $\mathcal{F}'$, for short) of $\mathcal{F}$ really is just a mapping from $X$ to the space of bounded linear operators.

To compute $\mathcal{F}'$, take $\psi,\ \eta \in X$ and $\varepsilon > 0$. T he MacLauren expansion for $\mathcal{F}(\psi + \varepsilon\eta)$ is

$$\mathcal{F}(\psi + \varepsilon\eta) = \mathcal{F}(\psi) + \varepsilon \langle \mathcal{F}'(\psi), \eta \rangle + O(\varepsilon^2)$$

In which case,

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}(\psi + \varepsilon\eta) - E(\eta)}{\varepsilon} = \lim_{\varepsilon \to 0} \langle \mathcal{F}'(\psi), \eta \rangle + O(\varepsilon) = \langle \mathcal{F}'(\psi),\ \eta \rangle_{X^*} = \mathrm{Re}\left( \int_{\mathbb{R}^n} \mathcal{F}'(\psi)\bar{\eta}\ dx \right).$$

Thus, the variational derivative $\mathcal{F}'$ of $\mathcal{F}$ is a functional on $X^*$ which satisfies

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}(\psi + \varepsilon\eta) - \mathcal{F}(\eta)}{\varepsilon} = \langle \mathcal{F}'(\psi),\ \eta \rangle_{X^*}.$$

Moreover, to see that the variational derivative is linear, let $\mathcal{F},\ \mathcal{G} \in X^*$ be functionals with corresponding derivatives $\mathcal{F}'$ and $\mathcal{G}'$, and $\alpha \in X$. Then

$$\begin{aligned}
\Big\langle \big(\alpha\mathcal{F} + \mathcal{G}\big)'(\psi),\ \eta \Big\rangle_{X^*} &= \lim_{\varepsilon \to 0} \frac{\big(\alpha\mathcal{F} + \mathcal{G}\big)(\psi + \varepsilon\eta) - \big(\alpha\mathcal{F} + \mathcal{G}\big)(\eta)}{\varepsilon} \\
&= \alpha \lim_{\varepsilon \to 0} \frac{\mathcal{F}(\psi + \varepsilon\eta) - \mathcal{F}(\eta)}{\varepsilon} + \lim_{\varepsilon \to 0} \frac{\mathcal{G}(\psi + \varepsilon\eta) - \mathcal{G}(\eta)}{\varepsilon} \\
&= \alpha \langle \mathcal{F}'(\psi),\ \eta \rangle_{X^*} + \langle \mathcal{G}'(\psi),\ \eta \rangle_{X^*}.
\end{aligned}$$

Unfortunately, computing second order variational derivatives can be slightly trickier. In order to find the second variational derivative $\mathcal{F}''$ of a functional $\mathcal{F} \in X^*$. For $\psi,\ \eta \in X$ and $\varepsilon > 0$, we can Taylor expand $\mathcal{F}(\psi + \varepsilon\eta)$ to find

$$\mathcal{F}(\psi + \varepsilon\eta) = \mathcal{F}(\psi) + \varepsilon \langle \mathcal{F}'(\psi),\ \eta \rangle_{X^*} + \frac{\varepsilon^2}{2} \langle \mathcal{F}''(\psi)\eta,\ \eta \rangle_{X^*} + O(\varepsilon^3).$$

Hence,

$$\left\langle \mathcal{F}''(\psi)\eta,\ \eta \right\rangle_{X^*} = \frac{2}{\varepsilon^2}\Big(\mathcal{F}(\psi+\varepsilon\eta) - \mathcal{F}(\psi) - \varepsilon \left\langle \mathcal{F}'(\psi), \eta \right\rangle_{X^*}\Big) + O(\varepsilon). \tag{5.1.1}$$

So, to find $\mathcal{F}''$ for a specific functional $\mathcal{F}$, we need to compute $\mathcal{F}(\psi+\varepsilon\eta) - \mathcal{F}(\psi) - \varepsilon \left\langle \mathcal{F}'(\psi), \eta \right\rangle_{X^*}$ and hope we can factor an $\varepsilon^2$ from the result. If so, then we can let $\varepsilon \to 0$, and through some voodoo-magical-algebra, eventually find $\mathcal{F}''$, as we do in Section 6.2.

## 5.2 Spectral Theory

Throughout this section, let $B$ denote an arbitrary Banach spaces whose field of scalars is $\mathbb{F}$, and let $D(\mathcal{L}) \subseteq B$ denote the domain of a linear operator $\mathcal{L} : D(\mathcal{L}) \subseteq X \to B$.

**Definition 5.2.1** (Closed Operator)**.** A linear operator $\mathcal{L} : D(\mathcal{L}) \subseteq B \to B$ is said to be *closed* if the set

$$\big\{(x, \mathcal{L}x) \;\big|\; x \in B\big\},$$

called the *graph* of $\mathcal{L}$, is closed under the product topology $B \times B$.

**Definition 5.2.2** (Fredholm Operator)**.** Let $X$ and $Y$ be Banach spaces and $\mathcal{F} : D(\mathcal{F}) \subseteq X \to Y$ be a bounded linear operator. Then $\mathcal{F}$ is said to be *Fredholm* if both the kernel and cokernel of $\mathcal{F}$ are finite dimensional. The *index* of a Fredholm operator $\mathcal{F}$, denoted as ind $\mathcal{F}$, is defined to be

$$\operatorname{ind} \mathcal{F} := \dim(\ker \mathcal{F}) - \dim(\operatorname{coker} \mathcal{F}),$$

where the co-kernel coker$\mathcal{F}$ of $\mathcal{F}$ is the topological quotient space $Y/\mathcal{F}\big(D(\mathcal{F})\big)$.

**Definition 5.2.3** (Densely Defined Operator)**.** A linear operator $\mathcal{L} : D(\mathcal{L}) \subseteq B \to B$ is said to be *densely defined* if its domain $D(\mathcal{L}) \subseteq X$ is dense in $B$.

**Definition 5.2.4** (Resolvent)**.** The *resolvent* $\rho(\mathcal{L})$ of a closed, densely defined linear operator $\mathcal{L} : D(\mathcal{L}) \subseteq B \to B$ is defined to be the set given by

$$\rho(\mathcal{L}) := \left\{ \lambda \in \mathbb{F} \;\middle|\; (\mathcal{L} - \lambda I) \text{ is bijective.} \right\}.$$

**Definition 5.2.5** (Spectrum)**.** Let $\mathcal{L} : D(\mathcal{L}) \subseteq B \to B$ be a closed, densely defined linear operator. Then the *spectrum* $\text{spec}(\mathcal{L})$ of $\mathcal{L}$ is defined to be the set

$$\text{spec}(\mathcal{L}) := \mathbb{F} \backslash \rho(\mathcal{L}),$$

The *point spectrum* of $L$, denoted $\text{spec}_p(\mathcal{L})$ is the set of all isolated eigenvalues of $L$ with finite multiplicity. Specifically, $\lambda \in \text{spec}_p(\mathcal{L})$ if and only if $\ker(\mathcal{L} - \lambda I) \neq \{0\}$, and $\text{ind}(\mathcal{L} - \lambda I) = 0$. Lastly, we define the *essential spectrum* $\text{spec}_{ess}(\mathcal{L})$ of $\mathcal{L}$ as the set of all $\lambda \in \mathbb{F}$ so that either the operator $\mathcal{L} - \lambda I$ is not Fredholm or $\text{ind}(\mathcal{L} - \lambda I) \neq 0$.

*Remark.* It is a simple exercise to show that $\text{spec}(\mathcal{L})$ is the disjoint union of $\text{spec}_{ess}(\mathcal{L})$ and $\text{spec}_p(\mathcal{L})$. As such, for the sake of this paper, we treat the essential spectrum of $L$ as simply $\text{spec}(\mathcal{L}) \backslash \text{spec}_p(\mathcal{L})$.

**Definition 5.2.6** (Compact Operator)**.** A linear operator $\mathcal{L} : B_1 \to B_2$ from a Banach space $B_1$ to a second Banach space $B_2$ is call *compact* if $\mathcal{L}$ maps bounded sets to pre-compact sets. That is, $\mathcal{L}$ is a compact operator if and only if for every bounded subset $U$ of $B_1$, the closure of $\mathcal{L}(U_1)$ is compact.

**Theorem 5.2.7** (Weyl Essential Spectrum)**.** *Suppose $\mathcal{L}$ and $K$ are bounded, self-adjoint linear operators acting on a Hilbert space $H$. If $K$ is also a compact operator and $\mathcal{L}$ and $K$ are closed, then $\text{spec}_{ess}(\mathcal{L}) = \text{spec}_{ess}(\mathcal{L} + K)$.*

The following two theorems are the textbook Spectral and *Dynamical Stability of Nonlinear Waves* by Todd Kapitula and Keith Promislow (see Kapitula & Promislow (2013)) and will be used in analyzing the spectrum of the $L_+$ and $L_-$ given in Subsection 6.2:

**Theorem 5.2.8.** *Consider an operator $\mathcal{L} : H^n(\mathbb{R}) \to L^2(\mathbb{R})$ of the form*

$$\mathcal{L} := \partial_x^n + a_{n-1}(x)\partial_x^{n-1} + \cdots + a_1(x)\partial_x + a_0(x),$$

*where $a_0, \ldots, a_{n-1}$ are Lebesque measurable functions. If $\{a_i\}_{i=0}^{n-1} \subset W^{1,\infty}(\mathbb{R})$ (i.e. each $a_i$ is differentiable and $\|a_i\|_\infty,\ \|\partial_x a_i\|_\infty < \infty$, where $\|\cdot\|_\infty$ denotes the essential supremum norm) then $\mathcal{L}$ is closed.*

**Theorem 5.2.9.** *Let $\mathcal{L} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ be an operator of the form*

$$\mathcal{L} := \partial_x^2 + a_1(x)\partial_x + a_0(x),$$

*where $a_0,\ a_0 \in W^{1,\infty}(\mathbb{R})$. If there exist four (not necessarily distinct) real constants $a_0^\pm$ and $a_1^\pm$ so that*

$$\lim_{x\to\pm\infty} e^{v|x|}\left|a_0(x) - a_0^\pm\right| = 0, \qquad \textit{and} \qquad \lim_{x\to\pm\infty} e^{v|x|}\left|a_1(x) - a_1^\pm\right| = 0$$

*for some $v > 0$, then the point spectrum $spec_p(\mathcal{L})$ of $\mathcal{L}$ consists of a finite number of eigenvalues of algebraic multiplicity one.*

*Moreover, if we define the* asymptotic limit *operator $\mathcal{L}_\infty : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ of $\mathcal{L}$ by*

$$\mathcal{L}_\infty := \partial_x^2 + a_1^\infty(x)\partial_x + a_0^\infty(x),$$

*where*

$$a_i^\infty(x) := \begin{cases} a_i^- & \text{for } x < 0 \\ a_i^+ & \text{for } x \geq 0 \end{cases},$$

*then $spec_{ess}(\mathcal{L}) = spec_{ess}(\mathcal{L}_\infty)$.*

## 5.3 Orbital Stability

As we discuss in Subsection 6.2, one of the difficulties impeding a conventional Lyapunov method to stability analysis is that there are technically an uncountable number of ground state solutions to the NLS. To see this, note that if $\phi$ is a ground state solution, then $e^{i\alpha}\phi(\cdot - x_0)$ also solves the NLS for each $\alpha,\ x_0 \in \mathbb{R}$. Moreover, since $|e^{i\alpha}| = 1$ (for all $\alpha \in \mathbb{R}$) and the Lebesgue integral is invariant under translations,

$$\mathcal{H}(\phi) = \mathcal{H}\big(e^{i\alpha}\phi(\cdot - x_0)\big).$$

Hence, $e^{i\alpha}\phi(\cdot - x_0)$ is also a ground state solution. Nevertheless, we can still partially kludge our method to compensate for this issue by treating all ground states of the NLS as equivalent, which we can do since ground state solutions of the NLS are unique up to translations and complex rotations.

More precisely, the observation that for every $\alpha,\ x_0 \in \mathbb{R}$, the function $u(x,t)$ is a solution to the NLS if and only if $e^{i\alpha}u(\cdot - x_0)$ is also a solution, means that there exists a two parameter symmetry group $G$, which can be identified with $\mathbb{R} \times \mathbb{R}^n$ acting on $X$, where $X$ is as defined in the previous section, through unitary representation

$$\left(R_{\alpha,\xi}u\right)(x) = e^{i\alpha}u(x+\xi), \qquad \forall u \in X,\ (\alpha,\xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{5.3.1}$$

The $G$-orbit generated by a solution $u$, is defined to be

$$\mathcal{O}_u = \big\{R_{\alpha,\xi}u \;\big|\; (\alpha,\xi) \in G\big\}.$$

Now, for a ground state solution $\phi$, the $G$-orbit (or orbit, for short) $\mathcal{O}_\phi$ of $\phi$ is simply the collection of *all* ground state solutions. So, in our stability analysis, we consider the orbit $\mathcal{O}_\phi$ of a ground state $\phi$ *in lieu* of the actual ground state. In doing so, we analyze what is called the *orbital stability* of $\phi$.

To put the concept of orbital stability in precise mathematical language, consider some initial data $u_0 \in X$ which is “close” to an orbit $\mathcal{O}_\phi$. With orbital stability, we want to see if the solution $u(\cdot,t) \in X$ with $u(\cdot,0) = u_0$ stays “close” to $\mathcal{O}_\phi$ for all $t > 0$. In order to quantify what we mean by “close”, define the semi-distance $\rho : X \to \mathbb{R}$ by

$$\rho(g,h) = \inf_{\alpha,\xi \in G} \|g - R_{(\alpha,\xi)} h\|_{X^*}.$$

Since $\rho$ is essentially just the standard semi-norm which gives the distance between a point $g$ and the set $\mathcal{O}_h$, to show that the orbit $\mathcal{O}_\phi$ of the ground state $\phi$ is stable, we need to show that given any $\varepsilon > 0$, we can find a $\delta > 0$ so that

$$\rho\big(u(\cdot,t), \phi\big) < \varepsilon \qquad \forall t > 0,$$

whenever $\rho(u_0, \phi) < \delta$.

# Chapter 6

# Stability Theory for Ground State Solutions

As mentioned in the Introduction, in studying the stability of a solution to a PDE, mathematicians have two major tools: studying the spectrum of the linearized version of the PDE, or the Lyapunov functional method. Since the first approach failed to provide us with any tangible information, as the spectrum of the matrix **N** is almost entirely purely complex, we now turn our attention to applying Lyapunov stability theory to characterizing the stability of ground state solutions.

For a PDE with equilibrium solution $u^*$, with a general Lyapunov functional method, we look for a functional $\mathcal{F} : H^1(\mathbb{C}^n) \to \mathbb{R}$ for which

$$\mathcal{F}\big(u(t)\big) = \mathcal{F}\big(u(0)\big), \qquad \forall t$$

and

$$\mathcal{F}'(u) = 0.$$

Such a functional is called a Lyapunov candidate functional. If a candidate functional exists, then Lyapunov stability theory says that if

$$\frac{\partial}{\partial t}\mathcal{F}(u) \leq 0$$

for every $u$ in some neighborhood $N$ of $u^*$, then $u^*$ is stable.

## 6.1 Meet the Functionals

The first order of business is to construct a candidate Lyapunov functional for the NLS. To do so, we begin by considering the Hamiltonian $\mathcal{H}$ of the NLS, which is given by

$$\mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left( |\nabla u|^2 - \frac{1}{\sigma+1}|u|^{2\sigma+2} \right) \, dx. \tag{6.1.1}$$

In order for a functional to be a candidate for a Lyapunov functional for the NLS, it needs to be both conserved (*i.e.* time invariant) on solutions of the NLS, and have bound states solutions as critical points. Unfortunately, as we show in the following two propositions, while $\mathcal{H}$ is conserved for solutions to the NLS, bound state solutions to the NLS are not critical points of $\mathcal{H}$.

**Proposition 6.1.1.** *The Hamiltonian $\mathcal{H}$ given in* (6.1.1) *is conserved.*

*Proof.* Let $u \in C\big(\mathbb{R};\ H^1(\mathbb{R}^n)\big)$ be a continuous solution to the NLS whose first derivative in time is also continuous. That is

$$iu_t = -\Delta u - |u|^{2\sigma}u. \tag{6.1.2}$$

We take the $X^*$ inner product of (6.1.2) with $\overline{u}_t$ to obtain

$$\operatorname{Re}\int_{\mathbb{R}^n} iu_t\overline{u_t}\, dx = -\operatorname{Re}\int_{\mathbb{R}^n} \Delta u\overline{u}_t\, dx - \operatorname{Re}\int_{\mathbb{R}^n} u|u|^{2\sigma}\overline{u}_t\, dx \tag{6.1.3}$$

Clearly, $\operatorname{Re}\int_{\mathbb{R}^n} iu_t\overline{u}_t\ dx = 0$, as $\int_{\mathbb{R}^n} iu_t\overline{u}_t\ dx \in \mathbb{C}$. Further, since

$$\int_{\mathbb{R}^n} \Delta u\overline{u}_t\ dx = -\int_{\mathbb{R}^n} \left(\overline{\nabla u}\right)_t \cdot \nabla u\ dx$$

and

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\ dx = \operatorname{Re}\int_{\mathbb{R}^n} \nabla\overline{u}_t \cdot \nabla u\ dx,$$

it follows that

$$\operatorname{Re}\int_{\mathbb{R}^n} \Delta u\overline{u}_t\ dx = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2\ dx. \tag{6.1.4}$$

Similarly,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^n} |u|^{2\sigma+2}\ dx &= \frac{d}{dt}\int_{\mathbb{R}^n} (\overline{u})^{\sigma+1}\, u^{\sigma+1}\ dx \\
&= (\sigma+1)\int_{\mathbb{R}^n} \left( \left(u^{\sigma+1}\right)(\overline{u}^{\sigma})\,\overline{u}_t + \left(\overline{u}^{\sigma+1}\right)(u^{\sigma})\,u_t \right) dx \\
&= 2(\sigma+1)\operatorname{Re}\int_{\mathbb{R}^n} (\overline{u}_t)\left(u^{\sigma+1}\right)(\overline{u}^{\sigma})\ dx \\
&= 2(\sigma+1)\operatorname{Re}\int_{\mathbb{R}^n} \overline{u}_t|u|^{2\sigma}u\ dx.
\end{aligned}$$

Hence

$$\operatorname{Re}\int_{\mathbb{R}^n} u|u|^{2\sigma}\overline{u}_t\ dx = \frac{1}{2(\sigma+1)}\frac{d}{dt}\int_{\mathbb{R}^n} |u|^{2\sigma+2}dx. \tag{6.1.5}$$

By combining equations (6.1.4) and (6.1.5) with (6.1.3) yeilds

$$\mathcal{H}(u) = \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\ dx + \frac{1}{2(\sigma+1)}\int_{\mathbb{R}^n} |u|^{2\sigma+2}\ dx\right) = 0,$$

which verifies the claim. ❏

**Proposition 6.1.2.** *The variational derivative* $\mathcal{H}' := \frac{\delta \mathcal{H}}{\delta u}$ *of* $\mathcal{H}$ *is* $\mathcal{H}'(u) = -\Delta u - |u|^{2\sigma}u$.

*Proof.* Assume $u \in H^1(\mathbb{R}^n)$ is a solution of the NLS, $\varepsilon > 0$ is some small number, and $\eta$ is an arbitrary function in $H^1(\mathbb{R}^n)$. Observing that

$$\begin{aligned} |\nabla(u+\varepsilon\eta)|^2 &= (\nabla u + \varepsilon\nabla\eta)\cdot(\nabla\bar{u}+\varepsilon\nabla\bar{\eta}) \\ &= |\nabla u|^2 + \varepsilon\nabla\bar{u}\cdot\nabla\eta + \varepsilon\nabla\bar{\eta}\nabla u + O(\varepsilon^2) \\ &= |\nabla u|^2 + 2\varepsilon\,\mathrm{Re}\,(\nabla u\cdot\nabla\bar{\eta}) + O(\varepsilon^2). \end{aligned} \tag{6.1.6}$$

Additionally, by Taylor expanding the function $x^{\sigma+1}$ about $|u|^2$ and setting $x = |u+\varepsilon\eta|^2$, we obtain

$$\begin{aligned} \left(|u+\varepsilon\eta|^2\right)^{\sigma+1} &= \left(|u|^2 + 2\varepsilon\,\mathrm{Re}\,(u\bar{\eta}) + O(\varepsilon^2)\right)^{\sigma+1} \\ &= \left(|u|^2\right)^{\sigma+1} + (\sigma+1)\left(|u|^2\right)^{\sigma}\cdot 2\varepsilon\,\mathrm{Re}\,(u\bar{\eta}) + O(\varepsilon^2) \end{aligned} \tag{6.1.7}$$

Hence, Equations (6.1.6) and (6.1.7) imply

$$\begin{aligned} 2\mathcal{H}(u+\varepsilon\eta) &= \int_{\mathbb{R}^n}\left(|\nabla\left(u+\varepsilon\eta\right)|^2 - \frac{1}{\sigma+1}|u+\varepsilon\eta|^{2\sigma+2}\right)\,dx \\ &= \int_{\mathbb{R}^n}\left(|\nabla u|^2 + 2\varepsilon\,\mathrm{Re}\,(\nabla u\cdot\nabla\bar{\eta})\right)\,dx \\ &\quad - \frac{1}{\sigma+1}\int_{\mathbb{R}^n}\left(|u|^{2\sigma+2} + (\sigma+1)|u|^{2\sigma}\cdot 2\varepsilon\,\mathrm{Re}\,(u\bar{\eta})\right)\,dx + O(\varepsilon^2) \\ &= \int_{\mathbb{R}^n}\left(|\nabla u|^2 + \frac{1}{\sigma+1}|u|^{2\sigma+2}\right)\,dx \\ &\quad + 2\varepsilon\int_{\mathbb{R}^n}\left(\mathrm{Re}\,(\nabla u\cdot\nabla\bar{\eta}) - |u|^{2\sigma}\,\mathrm{Re}\,(u\bar{\eta})\right)\,dx + O(\varepsilon^2) \\ &= 2\mathcal{H}(u) + 2\varepsilon\,\mathrm{Re}\int_{\mathbb{R}^n}\left(\nabla u\cdot\nabla\bar{\eta} - |u|^{2\sigma}u\bar{\eta}\right)\,dx + O(\varepsilon^2) \\ &= 2\mathcal{H}(u) + 2\varepsilon\,\mathrm{Re}\int_{\mathbb{R}^n}\left(-\Delta u - |u|^{2\sigma}u\right)\bar{\eta}\,dx + O(\varepsilon^2), \end{aligned} \tag{6.1.8}$$

by integration by parts. Combing Equations (6.1.6) through (6.1.8) therefore yields

$$\lim_{\varepsilon\to 0}\frac{\mathcal{H}\left(u+\varepsilon\eta\right)-\mathcal{H}(u)}{\varepsilon}=\lim_{\varepsilon\to 0}\left(\mathrm{Re}\int_{\mathbb{R}^n}\Big(-\Delta u-|u|^{2\sigma}u\Big)\bar{\eta}\;dx+O(\varepsilon)\right)$$
$$=\int_{\mathbb{R}^n}\mathrm{Re}\left(\left(-\Delta u-|u|^{2\sigma}u\right)\bar{\eta}\right)\;dx. \tag{6.1.9}$$

Consequently, by letting $X^*$ denote the dual space of $H^1\left(\mathbb{R}^n;\ \mathbb{C}\right)$ as in the discussion on Functional Derivatives in Section 5.1, we obtain

$$\lim_{\varepsilon\to 0}\frac{\mathcal{H}\left(u+\varepsilon\eta\right)-\mathcal{H}(u)}{\varepsilon}=\big\langle\big(-\Delta u-|u|^{2\sigma}u\big),\ \eta\big\rangle_{X^*}.$$

Hence,

$$\mathcal{H}'(u)=-\Delta u-|u|^{2\sigma}u, \tag{6.1.10}$$

as claimed ❏

Unfortunately, Proposition 6.1.2 implies that for ground state solutions $\phi$,

$$\mathcal{H}'(\phi)=-\Delta\phi-|\phi|^{2\sigma}u=-\lambda^2\phi\neq 0.$$

In other words, ground states solutions are not critical points of $\mathcal{H}$, which means that $\mathcal{H}$ is not the candidate Lyapunov functional we seek. However, all hope is not lost, as we now have significant insight into how to use the Hamiltonian $\mathcal{H}$ to construct a Lyapunov functional candidate. Specifically, if we can find a functional $\mathcal{N}$ which is also conserved on solutions to the NLS and whose variational derivative is $u$, then the linearity of functional derivatives tells us that the functional

$$\mathcal{E}(u):=\mathcal{H}(u)+\lambda^2\mathcal{N}(u)$$

will be a candidate Lyapunov functional. In particular, we claim that

$$\mathcal{N}(u) = \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 \; dx,$$

is the "right choice" for such a functional. As before, we start by showing that $\mathcal{N}$ is conserved.

**Proposition 6.1.3.** *The operator $\mathcal{N}$ is conserved in time.*

*Proof.* Again, let $u \in C\big(\mathbb{R};\ H^1(\mathbb{R}^n)\big)$ be a continuous solution to the NLS whose first derivative in time is also continuous. Through one application of integration by parts, we obtain

$$\int_{\mathbb{R}^n} (\overline{u}\Delta u - u\Delta\overline{u}) \; dx = -\int_{\mathbb{R}^n} (\nabla\overline{u}\cdot\nabla u - \nabla u\cdot\nabla\overline{u}) \; dx = 0. \tag{6.1.11}$$

As such, we see using Leibniz' rule that

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^n} |u|^2 \; dx &= \int_{\mathbb{R}^n} \frac{\partial}{\partial t}(u\overline{u}) \; dx \\
&= \int_{\mathbb{R}^n} (u_t\overline{u} + u\overline{u}_t) \; dx \\
&= \int_{\mathbb{R}^n} \Big( (i\Delta u + i(u\overline{u})^\sigma u)\,\overline{u} + u\big( - i\ \Delta\overline{u} + i(\overline{u}u)^\sigma\overline{u}\big)u \Big) \; dx \\
&= i\int_{\mathbb{R}^n} \big(\overline{u}\Delta u + |u|^{2\sigma+2} - u\Delta\overline{u} - |u|^{2\sigma+2}\big) \; dx \\
&= i\int_{\mathbb{R}^n} (\overline{u}\Delta u - u\Delta\overline{u}) \; dx \\
&= 0,
\end{aligned} \tag{6.1.12}$$

by (6.1.11), as claimed. ❑

**Proposition 6.1.4.** *The variational derivative of the functional $\mathcal{N}(u) = \frac{1}{2}\int_{\mathbb{R}^n} |u|^2 \; dx$ is $\mathcal{N}'(u) = u$.*

*Proof.* As before, we begin by computing $|u + \varepsilon\eta|^2$, where $u$ is an $H^1(\mathbb{R}^n)$ solution of the

NLS, $\varepsilon > 0$ is a small real number, and $\eta \in H^1(\mathbb{R}^n)$ is arbitrary.

$$|u + \varepsilon\eta|^2 = |u|^2 + 2\varepsilon \operatorname{Re}(u\bar{\eta}) + O\left(\varepsilon^2\right).$$

So,

$$\mathcal{N}(u + \varepsilon\eta) = \frac{1}{2}\int_{\mathbb{R}^n} |u + \varepsilon\eta|^2 \, dx = \frac{1}{2}\int_{\mathbb{R}^n} \left(|u|^2 + 2\varepsilon \operatorname{Re}(u\bar{\eta}) + O\left(\varepsilon^2\right)\right) \, dx,$$

which implies that

$$\mathcal{N}(u + \varepsilon\eta) - \mathcal{N}(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(2\varepsilon \operatorname{Re}(u\bar{\eta}) + O\left(\varepsilon^2\right)\right) \, dx = \varepsilon\lambda^2 \operatorname{Re}\left(\int_{\mathbb{R}^n} u\bar{\eta} \, dx\right) + O\left(\varepsilon^2\right). \tag{6.1.13}$$

Hence

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\mathcal{N}(u + \varepsilon\eta) - \mathcal{N}(u)}{\varepsilon} &= \lim_{\varepsilon\to 0}\left(\operatorname{Re}\left(\int_{\mathbb{R}^n} u\bar{\eta} \, dx\right) + O(\varepsilon)\right) \\
&= \operatorname{Re}\left(\int_{\mathbb{R}^n} \lambda^2 u\bar{\eta} \, dx\right) \\
&= \left\langle \lambda^2 u, \ \eta\right\rangle_{X^*}.
\end{aligned}$$

Thus, $\mathcal{N}'(u) = u$. ❑

It therefore follows that solutions of (3.0.3) are critical points of the quantity $\mathcal{E}(u) = \mathcal{H}(u) + \lambda^2\mathcal{N}(u)$. Moreover, we see from our previous work that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\mathcal{E}(u + \varepsilon\eta) - \mathcal{E}(u)}{\varepsilon} &= \operatorname{Re}\int_{\mathbb{R}^n} 2\Big( \left(-\Delta u - |u|^{2\sigma}u - \lambda^2 u\right)\bar{\eta}\Big) \, dx \\
&= -2\left\langle \Delta u + |u|^{2\sigma}u + \lambda^2 u, \ \eta\right\rangle_{X^*},
\end{aligned} \tag{6.1.14}$$

for each $u \in H^1(\mathbb{R}^n)$. Hence, $\mathcal{E}'(u) = \Delta u + |u|^{2\sigma}u + \lambda^2 u$, which implies that $u$ is a critical point of $\mathcal{E}$ if and only if $u$ is a bound state solution to the NLS.

Before we proceed to the next section, let's take stock of exactly what we've shown so far. Note that the NLS can be written as $iu_t = \mathcal{H}'(u)$. By writing the NLS in this way, we can readily see that bound state solutions are not even solutions to the NLS, let alone equilibria (not that we should expect them to be so). However, bound state solutions are equilibria of the PDE $iu_t = \mathcal{H}'(u) + \lambda^2\mathcal{N}'(u)$ So, when viewed in terms of the method of Lagrange multipliers, every bound state solution is a candidate extrema of the functional $\mathcal{H}$ under the constraint $\mathcal{N}(u) = \mu$, for some $\mu > 0$, as

$$\mathcal{H}'(u) + \lambda^2\mathcal{N}'(u) = \frac{\delta}{\delta u}\big(\mathcal{H}(u) + \lambda^2(\mathcal{N}(u) - \mu)\big).$$

This is precisely what we might expect, given that the ground state solutions (a special class of bound state solutions) of the NLS are the global minimizers of $\mathcal{H}$ constrained to $\mathcal{N}(u) = \mu$, since a ground state solution $\phi$ satisfies $\mathcal{H}(\phi) = \inf_{u\in\mathcal{A}} \mathcal{H}(u)$, where $\mathcal{A} = \big\{u \in H^1(\mathbb{R}^n) \;\big|\; \mathcal{N}(u) = \mu\big\}$.

## 6.2 Lyapunov gone sideways

The discussion at the end of the preceding section implies that our choice of $\mathcal{E}$ as the candidate Lyapunov functional really is right one, as the stability of a ground state solution $\phi$ is determined completely by the geometry of the manifold $M_{\mathcal{E}}$ generated by $\mathcal{E}$ near $\phi$. Unfortunately, $\phi$ happens to be a saddle point of the aforementioned manifold. Informally speaking, while for initial data near $\phi$ taken along one "direction" on $M_{\mathcal{E}}$, the corresponding solution may stay near the orbit of $\phi$ (in the sense discussed in Subsection 5.3), the solutions to the NLS corresponding to initial data taken along a different "direction" on $M_{\mathcal{E}}$ may diverge from the orbit of $\phi$. So, our job in this section is to determine the "direction" on $M_{\mathcal{E}}$ along which we have orbital stability. In other words, we want to find the submanifold $\Sigma_0$ of $M_{\mathcal{E}}$ for which solutions $u$ corresponding to initial data $u_0 \in \Sigma_0$ stay close to the orbit $\mathcal{O}_\phi$, provided that $u_0$ is close to $\mathcal{O}_\phi$. Ultimately, we show

$$\Sigma_0 = \big\{ u \in X \;\big|\; \mathcal{N}(u) = \mathcal{N}(\phi) \big\}.$$

Unfortunately, there is no great way to motivate some of the techniques we use to establish the stability criteria for ground state solutions, and it can be easy to get "lost in the details" without understanding how the following techniques work. As such, a brief overview of our stability analysis is warranted.

First of all, observe that if a function $u(\cdot, t)$ is in $\Sigma_0$ for some specific value of $t$, then $u$ will stay in $\Sigma_0$ for all $t$ on which $u$ exists, since $\mathcal{N}$ is conserved in time. This means, that if we can find a solution $u$ to the NLS for which $\rho\big(u(\cdot, 0),\ \phi\big) \ll 1$ and $u(x, 0) \in \Sigma_0$, then we are at least guaranteed that $u \in \Sigma_0$ for as long as $u$ exists. Further, since the NLS is locally well posed, we are also given that $\rho\big(u(\cdot, t),\ \phi\big) \ll 1$ for a small window of time.

Now, for any solution $u$ to the NLS satisfying $\rho\big(u_0,\ \phi\big) \ll 1$, where $u_0 := u(\cdot, 0)$ we can always find $\nu \in C(\mathbb{R};\ X)$ so that for fixed $\alpha,\ x_0 \in \mathbb{R}$, we have $u(\cdot, t) = e^{i\alpha}\phi(\cdot - x_0) + \nu(\cdot, t)$. If we can show that

$$\mathcal{E}(u) - \mathcal{E}(\phi) \geq C\rho\big(u,\ \phi\big)^2,$$

then, using the fact that $\mathcal{E}$ is conserved in time, it follows that

$$\mathcal{E}(u_0) - \mathcal{E}(\phi) = \mathcal{E}(u) - \mathcal{E}(\phi) \geq C\rho\big(u,\ \phi\big)^2$$

for all relevant time. In other words, if $\mathcal{E}$ happens to be coercive with respect to $\rho$ for a certain class of functions, then we can use the quantity $\mathcal{E}(u_0) - \mathcal{E}(\phi)$ to bound $\rho\big(u,\ \phi\big)$.

Along these lines, by Taylor expanding $\mathcal{E}(u)$, we obtain

$$\begin{aligned}\mathcal{E}(u) &= \mathcal{E}\left(e^{i\alpha}\phi(\cdot - x_0) + \nu\right) \\ &= \mathcal{E}\left(e^{i\alpha}\phi(\cdot - x_0)\right) + \big\langle \mathcal{E}'\left(e^{i\alpha}\phi(\cdot - x_0)\right)\nu,\ \nu\big\rangle_{X^*} + \frac{1}{2}\big\langle \mathcal{E}''\left(e^{i\alpha}\phi(\cdot - x_0\right)\nu,\ \nu\big\rangle_{X^*} + O\left(\|\nu\|_X^3\right)\end{aligned}$$

Recall that since $e^{i\alpha}\phi(\cdot - x_0)$ is a ground state solution, $\mathcal{E}'\big(e^{i\alpha}\phi(\cdot - x_0)\big) = 0$. Thus

$$\mathcal{E}(u) - \mathcal{E}(\phi) = \frac{1}{2}\left\langle \mathcal{E}''\left(\phi\right)\nu,\ \nu\right\rangle_{X^*} + O\left(\|\nu\|_X^3\right) \tag{6.2.1}$$

By analyzing the spectrum of $\mathcal{E}''$, we are ultimately able to determine that $\left\langle \mathcal{E}''\left(\phi\right)\nu,\ \nu\right\rangle_{X^*} \geq C\rho(u,\phi)$ provided $\nu$ satisfies the conditions

$$\left\langle \phi,\ \nu\right\rangle_{X^*} = \left\langle \phi', \nu\right\rangle_{X^*} = 0. \tag{6.2.2}$$

As such, (6.2.1) can be written as

$$\mathcal{E}(u) - \mathcal{E}(\phi) \geq \frac{1}{4}\left\langle \mathcal{E}''\left(\phi\right)\nu,\ \nu\right\rangle_{X^*}, \tag{6.2.3}$$

for $\nu$ satisfying (6.2.2).

Using the Implicit Function Theorem, we can show that for each $u \in \Sigma_0$, we can find $\alpha,\ x_0 \in \mathbb{R}$ for which the corresponding function $\nu = u - e^{i\alpha}\phi(\cdot - x_0)$ happens to satisfy the conditions given in (6.2.2).

So, based on the preceding discussion, we now need to determine $\mathcal{E}''(\phi)$ in order to continue with our stability analysis. We do so by first finding both $\mathcal{N}''$ and $\mathcal{H}''$. Take any $\eta \in X := H^n\left(R^n;\ \mathbb{C}\right)$ and $\varepsilon > 0$. Since

$$|\phi + \varepsilon\eta|^2 = |\phi|^2 + 2\varepsilon\,\mathrm{Re}(\phi\bar{\eta}) + \varepsilon^2|\eta|^2,$$

it follows that

$$\mathcal{N}(\phi + \varepsilon\eta) - \mathcal{N}(\phi) = \varepsilon\,\mathrm{Re}\int_{\mathbb{R}^n}\left(\psi\bar{\eta} + \frac{1}{2}\varepsilon|\eta|^2\right)\,dx.$$

Thus, given $\mathcal{N}'(\phi) = \phi$, equation (5.1.1) implies that

$$\begin{aligned}
\langle \mathcal{N}''(\phi)\eta, \eta \rangle_{X^*} &= \frac{2}{\varepsilon^2}\Big(\mathcal{N}(\phi + \varepsilon\eta) - \mathcal{N}(\phi) - \varepsilon \langle \phi, \eta \rangle_{X^*}\Big) + O(\varepsilon). \\
&= \frac{2}{\varepsilon^2}\left(\varepsilon \operatorname{Re} \int_{\mathbb{R}^n} \left(\phi\eta + \frac{1}{2}\varepsilon|\eta|^2\right) dx - \varepsilon \operatorname{Re} \int_{\mathbb{R}^n} \phi\bar{\eta}\; dx\right) + O(\varepsilon) \\
&= \langle \eta, \eta \rangle_{X^*} + O(\varepsilon)
\end{aligned}$$

By taking the limit as $\varepsilon \to 0$, we obtain $\langle \mathcal{N}''(\phi)\eta, \eta \rangle_{X^*} = \langle \eta, \eta \rangle_{X^*}$. Hence, $\mathcal{N}''(\phi) = \mathbb{1}_{X^*}$.

Now, to find $\mathcal{E}''$, we need to explicitly find the $\varepsilon^2$ terms in equations (6.1.7) and (6.1.8). Thus,

$$|\nabla(\phi + \varepsilon\eta)|^2 = |\nabla\phi|^2 + 2\varepsilon \operatorname{Re}(\nabla\phi \cdot \nabla\bar{\eta}) + \varepsilon^2|\nabla\eta|^2,$$

and

$$\begin{aligned}
\left(|\phi + \varepsilon\eta|^2\right)^{\sigma+1} &= \left(|\phi|^2 + 2\varepsilon \operatorname{Re}(\phi\bar{\eta}) + \varepsilon^2|\eta|^2\right)^{\sigma+1} \\
&= \left(|\phi|^2\right)^{\sigma+1} + (\sigma+1)\left(|\phi|^2\right)^{\sigma}\left(2\varepsilon \operatorname{Re}(\phi\bar{\eta}) + \varepsilon^2|\eta|^2\right) \\
&\quad + \frac{1}{2}(\sigma+1)\sigma\left(|\phi|^2\right)^{\sigma-1}\left(2\varepsilon \operatorname{Re}(\phi\bar{\eta}) + \varepsilon^2|\eta|^2\right)^2 + O(\varepsilon^3) \\
&= |\phi|^{2\sigma+2} + (\sigma+1)|\phi|^{2\sigma}\left(2\varepsilon \operatorname{Re}(\phi\bar{\eta}) + \varepsilon^2|\eta|^2\right) \\
&\quad + 2\varepsilon^2(\sigma+1)\sigma|\phi|^{2(\sigma-1)} \operatorname{Re}(\phi\bar{\nu})^2 + O(\varepsilon^3).
\end{aligned}$$

In order to simplify the computation of $\mathcal{H}(\phi - \varepsilon\eta)$ we compute terms of $\mathcal{H}(\phi - \varepsilon\eta)$ based on factors of $\varepsilon$. Let's start with the $O(\varepsilon^0)$ terms:

$$\frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla\phi|^2 - \frac{1}{\sigma+1}|\phi|^2\right)\; dx = H(\phi). \tag{6.2.4}$$

Now for the $O(\varepsilon^1)$ terms:

$$\begin{aligned}
\frac{1}{2}\varepsilon \int_{\mathbb{R}^n} &\left(2\operatorname{Re}(\nabla\phi\nabla\bar{\eta}) - \frac{1}{\sigma+1}\left((\sigma+1)|\phi|^{2\sigma}2\operatorname{Re}(\phi\bar{\eta})\right)\right)\, dx \\
&= \varepsilon \operatorname{Re}\int_{\mathbb{R}^n}\left((-\Delta\phi) - |\phi|^{2\sigma}\phi\right)\bar{\eta}\, dx \\
&= \varepsilon \left\langle \mathcal{H}'(\phi), \eta\right\rangle_{X^*}. \qquad (6.2.5)
\end{aligned}$$

The first equality was obtained by applying integration by parts to $\int_{\mathbb{R}^n} \nabla\phi\nabla\bar{\eta}\, dx$ and factoring $\bar{\eta}$ from the result.

Lastly, the $O(\varepsilon^2)$ terms are

$$\begin{aligned}
\frac{1}{2}\varepsilon^2 \int_{\mathbb{R}^n} &\left(|\nabla\eta|^2 - \frac{1}{\sigma+1}\left((\sigma+1)|\phi|^{2\sigma}|\eta|^2 + 2(\sigma+1)\sigma|\phi|^{2(\sigma-1)}\operatorname{Re}(\phi\bar{\eta})^2\right)\right) dx \\
&= \frac{1}{2}\varepsilon^2 \int_{\mathbb{R}^n}\left(-\Delta\eta\bar{\eta} - |\phi|^{2\sigma}\eta\bar{\eta} - 2\sigma|\phi|^{2(\sigma-1)}\operatorname{Re}(\phi\bar{\eta})\operatorname{Re}(\phi\bar{\eta})\right) dx \\
&\frac{\varepsilon^2}{2}\left\langle\left(-\Delta - |\phi|^{2\sigma} - 2\sigma|\phi|^{2(\sigma-1)}(\phi\otimes\phi)\right)\eta,\ \eta\right\rangle_{X^*}, \qquad (6.2.6)
\end{aligned}$$

where the tensor product $(\cdot\otimes\cdot)$ satisfies

$$\langle(\xi\otimes\xi)u, v\rangle = \int_{\mathbb{R}}(\operatorname{Re}\xi\bar{u})(\operatorname{Re}\xi\bar{v})dx.$$

Combining equations (6.2.4) through (6.2.6) yeilds

$$\begin{aligned}
\langle\mathcal{H}''(\phi)\eta,\eta\rangle &= \frac{2}{\varepsilon^2}\left(\mathcal{H}(\phi-\varepsilon\eta) - \mathcal{H}(\phi) - \varepsilon\left\langle\mathcal{H}'(\phi),\eta\right\rangle_{X^*}\right) \\
&= \left\langle\left(-\Delta - |\phi|^{2\sigma} - 2\sigma|\phi|^{2(\sigma-1)}(\phi\otimes\phi)\right)\eta,\ \eta\right\rangle_{X^*} + O(\varepsilon),
\end{aligned}$$

which implies that

$$\mathcal{H}''(\phi) = -\Delta - |\phi|^{2\sigma} - 2\sigma|\phi|^{2(\sigma-1)}(\phi\otimes\phi), \qquad (6.2.7)$$

and

$$\mathcal{E}''(\phi) = -\Delta - |\phi|^{2\sigma} - 2\sigma|\phi|^{2(\sigma-1)}(\phi \otimes \phi) + \lambda^2 \mathbb{1}_X, \tag{6.2.8}$$

for every $\phi \in X$.

Since $X$ is a real Hilbert space, we can decompose $\mathcal{E}''(\phi)$ into real and imaginary parts giving us a "matrix" representation of $\mathcal{E}''(\phi)$ acting on $X$. To do so, we consider the action of $\mathcal{E}''(\phi)$ on $u + iv \in X$, where both $u$ and $v$ are real-valued functions:

$$\begin{aligned} \langle \mathcal{E}''(\phi)(u+iv),\ u+iv\rangle_{X^*} &= \langle \mathcal{E}''(\phi)u,\ u\rangle_{X^*} + \langle \mathcal{E}''(\phi)u,\ iv\rangle_{X^*} \\ &\quad + \langle \mathcal{E}''(\phi)iv,\ u\rangle_{X^*} + \langle \mathcal{E}''(\phi)iv,\ iv\rangle_{X^*} \end{aligned} \tag{6.2.9}$$

Given $u$, $v$, and $\phi$ are all real valued, $\mathrm{Re}(\phi\overline{iv}) = 0$ (since $\phi\overline{iv} = -i\phi v$ is purely complex), and

$$\begin{aligned} \langle \mathcal{E}''(\phi)u,\ iv\rangle_{X^*} &= \left\langle -\Delta u - |\phi|^{2\sigma}u - 2\sigma|\phi|^{2(\sigma-1)}(\phi\otimes\phi)u + \lambda^2 u,\ iv\right\rangle_{X^*} \\ &= \mathrm{Re}\int_{\mathbb{R}^n} \left(-\Delta u\overline{iv} - |\phi|^{2\sigma}u\overline{iv} - 2\sigma|\phi|^{2(\sigma-1)}\,\mathrm{Re}(\phi\bar{u})\,\mathrm{Re}(\phi\overline{iv}) + u\overline{iv}\right)\ dx \\ &= \mathrm{Re}\left(i\int_{\mathbb{R}^n}\left(\Delta uv + |\phi|^{2\sigma}uv + 0 - uv\right)\ dx\right) \\ &= 0. \end{aligned}$$

Similarly, by expanding the third term in the left hand side of (6.2.9), it is easy to see that $\langle \mathcal{E}''(\phi)iv,\ u\rangle_{X^*}$ is also identically zero.

On the other hand,

$$\begin{aligned}
\langle \mathcal{E}''(\phi)u,\ u\rangle_{X^*} &= \left\langle -\Delta u - |\phi|^{2\sigma}u - 2\sigma|\phi|^{2(\sigma-1)}(\phi\otimes\phi)u + \lambda^2 u,\ u\right\rangle_{X^*} \\
&= \left\langle -\Delta u - |\phi|^{2\sigma}u + \lambda^2 u,\ u\right\rangle_{X^*} - 2\sigma \int_{\mathbb{R}^n} |\phi|^{2(\sigma-1)} \operatorname{Re}(\phi u)\operatorname{Re}(\phi u)\ dx \\
&= \left\langle -\Delta u - |\phi|^{2\sigma}u + \lambda^2 u,\ u\right\rangle_{X^*} - 2\sigma \operatorname{Re} \int_{\mathbb{R}^n} |\phi|^{2(\sigma-1)}|\phi|^2 u\bar{u}\ dx \\
&= \left\langle -\Delta u - |\phi|^{2\sigma}u + \lambda^2 u - 2\sigma|\phi|^{2\sigma}u,\ u\right\rangle_{X^*} \\
&= \left\langle \left(-\Delta - (2\sigma+1)|\phi|^{2\sigma} + \lambda^2\right)u,\ u\right\rangle_{X^*},
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathcal{E}''(\phi)iv,\ iv\rangle_{X^*} &= \left\langle -\Delta iv - |\phi|^{2\sigma}iv - 2\sigma|\phi|^{2(\sigma-1)}(\phi\otimes\phi)iv + \lambda^2 iv,\ iv\right\rangle_{X^*} \\
&= \left\langle \left(-\Delta - |\phi|^{2\sigma} + \lambda^2\right) iv,\ iv\right\rangle,
\end{aligned}$$

as $\langle(\phi\otimes\phi)iv\rangle_{X^*} = \int_{\mathbb{R}^n} \operatorname{Re}\left(\phi\overline{iv}\right)^2\ dx = 0$. As such, the matrix representation $\widehat{\mathcal{H}''(\phi)}$ of $\mathcal{H}''(\phi)$ is

$$\widehat{\mathcal{E}''(\phi)} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix},$$

where

$$L_+ := -\Delta + \lambda^2 - (2\sigma+1)|\phi|^{2\sigma} \qquad \text{and} \qquad L_- := -\Delta + \lambda^2 - |\phi|^{2\sigma}$$

are the same operators studdied in Section 4.1.

Hence, in order to determine the conditions under which the bilinear form

$$\left\langle \widehat{\mathcal{E}''(\phi)}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right\rangle = \langle L_+u, u\rangle + \langle L_-v, v\rangle. \tag{6.2.10}$$

is coercive, it suffices to find the conditions under which both of the operators $L_+$ and $L_-$ are coercive. To so so, we first consider the spectral theory for $L_+$ and $L_-$. Theorem 5.2.8 tells us that $L_+$ and $L_-$ are closed operators, so we can consider the essential spectrum of both operators. Moreover, since the asymptotic limit operator of both $L_-$ and $L_+$ is $-\Delta+\lambda^2$, we see from Theorem 5.2.9 that to find the essential spectra $\text{spec}_{ess}(L_+)$ and $\text{spec}_{ess}(L_-)$ of $L_+$ and $L_-$, respectively, we need only determine the essential spectrum $\text{spec}_{ess}(-\Delta+\lambda^2)$ for the operator $-\Delta+\lambda^2$. To do so, set

$$p(z) := -z \cdot z + \lambda^2,$$

and note that $-\Delta+\lambda^2 = p(\nabla)$. Since $p(i\mathbb{R}) = [\lambda^2, \infty)$, it follows that $\text{spec}_{ess}(-\Delta+\lambda^2) = [\lambda^2, \infty)$. Therefore, the Weyl Essential Spectrum Theorem implies

$$\text{spec}_{ess}(L_+) = \text{spec}_{ess}(L_-) = \text{spec}_{ess}\left(-\Delta+\lambda^2\right) = [\lambda^2, \infty).$$

In continuing our analysis of the spectrum of $L_+$ and $L_-$ it helpful to first find the kernels of both operators. To do so, we restrict our attention to one spatial attention, as the general computation of $\ker L_+$ and $\ker L_-$ for $n > 1$ spatial dimensions given in Kwong (1989) far exceeds the scope of this project. Now, observe by differentiating the bound state PDE—equation (3.0.3)—with respect to $x \in \mathbb{R}$, we obtain

$$0 = -\Delta\phi' - \lambda^2\phi' - (2\sigma+1)\phi^{2\sigma}\phi' = L_+\phi'.$$

Hence $\phi'$ is in the kernel of $L_+$. Moreover, (3.0.3) is equivalent to $L_-\phi = 0$. Thus,

$$\text{span}\,\{\phi'\} \subseteq \ker L_+ \qquad \text{and} \qquad \text{span}\,\{\phi\} \subseteq \ker L_-.$$

Now, since zero is an eigenvalue of both $L_+$ and $L_-$ and $\lambda^2 > 0$, zero cannot be in the essential spectrum for either operator and is therefore in the point spectrum of both operators. Theorem 5.2.9 therefore implies that zero is a simple eigenvalue of $L_+$ and $L_-$, which allows us to conclude that

$$\text{span}\,\{\phi'\} = \ker L_+ \qquad \text{and} \qquad \text{span}\,\{\phi\} = \ker L_-. \tag{6.2.11}$$

Thus

$$\text{span}\left\{\begin{pmatrix}\phi'\\0\end{pmatrix}, \begin{pmatrix}0\\\phi\end{pmatrix}\right\} = \ker\left(\widehat{\mathcal{E}''(\phi)}\right).$$

Further, by Sturm-Liouville Theory, zero is the smallest eigenvalue of $L_-$ and the second smallest eigenvalue of $L_+$, as $\phi$ has no roots, while $\phi'$ has one root. Thus, the solution $\phi$ is a degenerate saddle point of $\mathcal{E}$ with exactly one "negative" direction–precisely as claimed in the beginning of this section.

## 6.3 Strip Tease

Now that we understand the full extent of the problem for why the bilinear form (6.2.10) is not necessarily coercive for all solutions to the which start close to the ground state orbit, let's turn our attention to determining when $L_-$ and $L_+$–and hence (6.2.10)–are coercive. In essence, in the discussion which follows, we use several projection-like operators to "strip off" the parts of the domains for $L_-$ and $L_+$ which cause $L_-$ and $L_+$ to not be coercive, and then show that the two operators are coercive on the remaining domains.

### 6.3.1 Coercivity of $L_-$

To establish the coercivity of $L_-$, first recall that $\ker L_- = \text{span}\{\phi\}$, where $\phi$ is a ground state solution. Consequently, if we define the projection operation

$$P_- : L^2(\mathbb{R};\mathbb{R}) \to \ker L_-$$
$$u \mapsto \frac{\langle u, \phi\rangle_{L^2(\mathbb{R})}}{\langle \phi, \phi\rangle_{L^2(\mathbb{R})}}\phi$$

(that is, $P_-$ projects onto the span of $\phi$), then its complementary projection

$$\Pi_- := I - P_-$$

has range $(\ker L_-)^\perp$. Moreover, when we consider the constrained operator

$$\Pi_- L_- : (\ker L_-)^\perp \to (\ker L_-)^\perp ,$$

then we have that

$$\operatorname{spec}(\Pi_- L_-) = \operatorname{spec}(L_-) \setminus \{0\},$$

as $\Pi_-$ is a spectral projection for $L_-$. Hence, $\Pi_- L_-$ does not have zero as an eigenvalue, and zero is not a accumulation point for the spectrum of $\Pi_- L_-$ (as zero was an isolated point of the spectrum of $L_-$). Thus, $\Pi_- L_-$ is coercive, which means that there exists $k > 0$ so that

$$\langle \Pi_- L_- v, v \rangle_{L^2(\mathbb{R})} \geq k \|v\|^2_{L^2(\mathbb{R})}$$

for all $v \in (\ker L_-)^\perp = \operatorname{ran}(\Pi_-)$. Consequently, if $v \in (\ker L_-)^\perp$ (*i.e.* $\langle \phi, u \rangle = 0$), then

$$\langle L_- v, v \rangle \geq k \|v\|^2_{L^2(\mathbb{R})},$$

as desired. Hence, $L_-$ is coercive for those $v \in H^1(\mathbb{R};\ \mathbb{R})$ which satisfy $\langle \phi, v \rangle_{L^2(\mathbb{R})} = 0$.

### 6.3.2 Coercivity of $L_+$

Similarly, since $\ker L_+ = \operatorname{span}\{\phi'\}$, the projection $P_+$ onto $\ker L_+$ is given by

$$\begin{aligned} P_+ : L^2(\mathbb{R};\mathbb{R}) &\to \ker L_+ \\ u &\mapsto \frac{\langle u, \phi' \rangle}{\langle \phi', \phi' \rangle} \phi'. \end{aligned}$$

Again, note that the complementary projection

$$\Pi_+ := I - P_+$$

has range $(\ker L_+)^\perp$. Specifically, when we consider the operator $\Pi_+ L_+$ as the constrained operator

$$\Pi_+ L_+ : (\ker L_+)^\perp \to (\ker L_+)^\perp\,,$$

then since $\Pi_+$ is a spectral projection for $L_+$, we have

$$\operatorname{spec}(\Pi_+ L_+) = \operatorname{spec}(L_+) \setminus \{0\}.$$

Hence, $\Pi_+ L_+$ is a boundedly invertible operator. Moreover, Sturm-Liouville theory implies that since $\phi'$ has exactly one root, and $\phi'$ is an eigenvector of $L_+$ corresponding to the eigenvalue 0, 0 is the second smallest eigenvalue of $L_+$. Hence exactly one of the eigenvalues of $L_+$ must be negative, and the rest (not counting the zero eigenvalue) must be positive. Since the operator $\Pi_+$ removes the zero eigenvalue from the spectrum of $L_+$, but leaves the remaining spectrum untouched, only one of the eigenvalues of $\Pi_+ L_+$ is negative, and the remaining eigenvalues of $\Pi_+ L_+$ must be strictly positive.

Unfortunately, the one negative eigenvalue of $L_+$ can still cause $L_+$ to be non-coercive, which means we need to find a way of removing it as well. To that end, we introduce the self-adjoint complementary projection

$$\Pi u = u - \frac{\langle u, \phi \rangle_{L^2(\mathbb{R})}}{\langle \phi, \phi \rangle_{L^2(\mathbb{R})}} \phi.$$

Note that the operator $\Pi$ effectively "strips away" any part of a vector $u$ which is parallel to $\phi$. Unfortunately, $\Pi$ is not a spectral projection for either $L_+ P$ or $\Pi_+ L_+$. However, since $\Pi\Pi_+$ projects onto $\{\phi, \phi'\}^\perp$ and $\phi \perp \phi'$, it is easy to see that $\Pi$ still commutes with $\Pi_+$.

Consequently, as we consider the constrained operator

$$\Pi\,\Pi_+ L_+ : \{\phi, \phi'\}^\perp \cap D(L_+) \subset (\ker L_+)^\perp \to \Pi\,(\ker L_+)^\perp\,,$$

we are still able to obtain useful information.

Now, a simple direct computation shows that the complementary projection operators $\Pi$ and $\Pi_+$ are both self-adjoint. Similarly, through two applications of integration by parts it is easy to see that

$$\langle L_+ u, v\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (L_+ u)\, v\; dx = \int_{\mathbb{R}} u\,(L_+ v)\;\; dx = \langle u, L_+ v\rangle_{L^2(\mathbb{R})}$$

for each $u,\ v \in \{\phi, \phi'\}^\perp \cap D(L_+)$. Thus, the constrained operator $\Pi\,\Pi_+ L_+$ is also self-adjoint and therefore has a real valued spectrum. Moreover, since $I - \Pi$ acting on $(\ker L_+)^\perp$ is rank one, we see that

$$\Pi_+ L_+ - \Pi\Pi_+ L_+ = (I - \Pi)(\Pi_+ L_+)$$

is compact. Therefore, the Weyl Essential Spectral Theorem implies that

$$\begin{aligned}
\text{spec}_{ess}\,(\Pi\,\Pi_+ L_+) &= \text{spec}_{ess}\Big(\Pi\,\Pi_+ L_+ + (\Pi_+ L_+ - \Pi\Pi_+ L_+)\Big) \\
&= \text{spec}_{ess}\,(\Pi_+ L_+) \\
&= \text{spec}_{ess}(L_+) = \left[\lambda^2, \infty\right)\,,
\end{aligned}$$

as $\Pi_+$ is a spectral operator and therefore preserves essential spectrum of $L_+$. As such, if we can find the conditions under which the point spectrum of $\Pi\Pi_+ L_+$ is strictly positive, then such conditions will also guarantee the bound

$$\langle \Pi\,\Pi_+ L_+ v, v\rangle \geq c\|v\|_X^2$$

is valid for some $c > 0$ and every $v \in \{\phi, \phi'\}^\perp \cap D(L_+)$. To determine these conditions, we use the Vakhitov-Kolokolov projection method.

Set $L_\Pi := \Pi\, \Pi_+ L_+$ and let $\alpha_0 \in \mathbb{R}$ denote the ground state (*i.e.* lowest) eigenvalue for $L_\Pi$. Then

$$\alpha_0 = \inf_{v \in \tau_0} \frac{\langle L_+ v, v\rangle}{\langle v, v\rangle},$$

where $\tau_0 := \{\phi, \phi'\}^\perp \cap D(L_+)$. Further, if $-\chi^2 < 0$ is the ground state eigenvalue for $L_+$, then

$$-\chi^2 = \inf_{v \in H^1(\mathbb{R}^1)} \frac{\langle L_+ v, v\rangle}{\langle v, v\rangle}.$$

Clearly, $-\chi^2 \le \alpha_0$. Moreover, since the ground state eigenfunction $\psi_0$ for $L_+$ has a fixed sign (since it has no roots, by Sturm-Liouville theory), it follows that $\langle \psi_0, \phi\rangle \neq 0$, given $\phi$ also has fixed sign. Hence $\psi_0 \notin \tau_0$, which implies that

$$-\chi^2 < \alpha_0.$$

Now suppose $\alpha > -\chi^2$ is an eigenvalue of $L_\Pi$. Then there exists some $\psi_\alpha \in \tau_0$ so that $L_\Pi \psi_\alpha = \alpha\psi_\alpha$. In which case, since $\Pi\phi = 0$ and $\Pi\psi_\alpha = \psi_\alpha$, we can find some $\mu \in \mathbb{R}$ so that

$$L_0 \psi_\alpha = \alpha\psi_\alpha + \mu\phi,$$

where $L_0 := \Pi_+ L_+$. If $\alpha \notin \operatorname{spec}(L_+) = \operatorname{spec}(L_0) \cup \{0\}$, then $\mu \neq 0$ and

$$\psi_\alpha = \mu(L_0 - \alpha)^{-1}\phi,$$

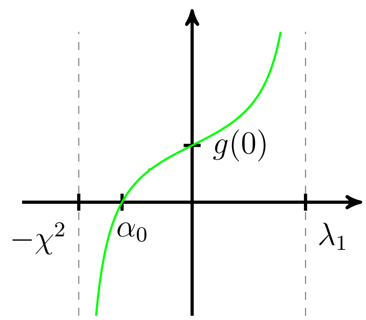
as $(L_0 - \alpha)$ is invertible. In particular, given the function,

$$g(\beta) := \left\langle (L_0 - \beta)^{-1}\phi, \phi \right\rangle,$$

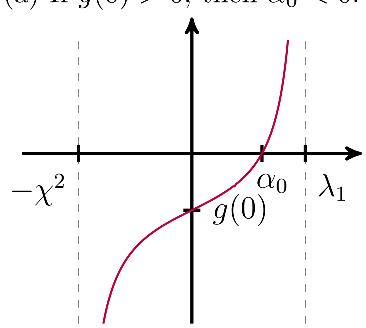
it is easy to show that $\alpha$ is a root of $g$ given that $\psi_\alpha \perp \phi$:

$$g(\alpha) = \left\langle (L_0 - \alpha)^{-1}\phi, \phi \right\rangle = \mu^{-1} \left\langle \psi_\alpha, \phi \right\rangle = 0.$$

Conversely, to see that $\alpha \in \mathbb{R}$ is in $\operatorname{spec}(L_\Pi) \backslash \operatorname{spec}(L_0)$ if and only if $g(\alpha) = 0$, suppose $g(\alpha) = 0$ for some $\alpha \in \mathbb{R}$. Then we can find a $\psi_\alpha \in \tau_0$ so that $\psi_\alpha = (L_0 - \alpha)^{-1}\phi$, which means that $\phi = L_0\psi_\alpha - \alpha\psi_\alpha$. Therefore, applying the operator $\Pi$ to both sides of $\phi = L_0\psi_\alpha - \alpha\psi_\alpha$ yields $L_0\psi_\alpha = \alpha\psi_a$. So, given that $g$ is not defined on spec$(L_0)$—as $L_0 - \alpha$ is not invertible for $\alpha$ in the spectrum of $L_0$—it follows that $\alpha \in \operatorname{spec}(L_\Pi)\backslash\operatorname{spec}(L_0)$.



(a) If $g(0) > 0$, then $\alpha_0 < 0$.



(b) If $g(0) < 0$, then $\alpha_0 > 0$.

Figure 6.1: Sketches of the function $g(x)$ for (a) $g(0) > 0$ and (b) for $g(0) < 0$.

From above, $g(\alpha) \neq 0$ for every $\alpha \leq -\chi^2$, as this requirement forces $\alpha$ not to be an eigenvalue of $L_\Pi$. So, if we can show $g(\alpha) \neq 0$ for every $\alpha \leq 0$, then the ground state eigenvalue $\alpha_0$ of $L_\Pi$ must be strictly positive.

Note that $g$ is (real) analytic for $a \notin \operatorname{spec}(L_0)$ with pole singularities on spec$(L_0)$. Further, since

$$g'(\alpha) = \left\langle (L - \alpha)^{-2}\phi, \phi \right\rangle = \left\| (L - \alpha)^{-1}\phi \right\|_{L^2}^2 > 0,$$

we have that $g$ is a strictly increasing function on spec$(L_0)^c$. Further, if $\lambda_1 := \min\left(\operatorname{spec}(L_0)\backslash\{-\chi^2\}\right)$, then $\lambda_1 > 0$ and $g$ is smooth and strictly increasing on $(-\chi^2, \lambda_1)$.

Since $\langle \phi, \psi_0 \rangle \neq 0$, we have

$$\lim_{\alpha \to (-\chi^2)-} g(\alpha) = -\infty.$$

So, as we illustrate in Figure 6.1, $\alpha_0 > 0$ if $g(0) < 0$ and $\alpha_0 < 0$ if $g(0) > 0$. Now

$$g(0) = \left\langle L_0^{-1}\phi, \phi \right\rangle . \tag{6.3.1}$$

Recall that Equation (3.0.3) implies that

$$\Delta\phi - \lambda^2\phi + \phi^{2\sigma+1} = 0.$$

Thus, by differentiating Equation (3.0.3) with respect to $\lambda^2$ we find

$$\Delta\frac{\partial}{\partial\lambda^2}\phi - \lambda^2\frac{\partial}{\partial\lambda^2}\phi - \phi + (2\sigma+1)\phi^{2\sigma} = 0,$$

which implies

$$\phi = \left(-\Delta\lambda^2 + (2\sigma+1)\phi^{2\sigma}\right)\left(-\frac{\partial}{\partial\lambda^2}\phi\right) = L_+\left(-\frac{\partial}{\partial\lambda^2}\phi\right).$$

Hence

$$\Pi_+\phi = L_0\left(-\frac{\partial}{\partial\lambda^2}\phi\right),$$

which means that

$$L_0^{-1}\phi = -\frac{\partial}{\partial\lambda^2}\phi.$$

As such, Equation (6.3.1) becomes

$$g(0) = \left\langle -\frac{\partial}{\partial\lambda^2}\phi, \phi \right\rangle = -\frac{1}{2}\frac{\partial}{\partial\lambda^2}\mathcal{N}(\phi).$$

Hence, provided $\frac{\partial}{\partial\lambda^2}\mathcal{N}(\phi) > 0$, the operator $L_\Pi$ is coercive. Through direct computation of $\frac{\partial}{\partial\lambda^2}\mathcal{N}(\phi)$, we see that $\frac{\partial}{\partial\lambda^2}\mathcal{N}(\phi)$ is positive for $\sigma < 2/n$. Thus, since by construction, both $L_\Pi$ and $L_+$ agree on $\tau_0$, we see that $L_+$ is coercive on $\tau_0$ whenever $0 < \sigma < 2/n$. Given $D(L_+) = D(L_-)$, $\{\phi\}^\perp \subset \tau_0$ and $L_-$ is coercive on $\{\phi\}^\perp \cap D(L_-)$, it follows that the bilinear

(6.2.10) is coercive on $\{\phi, \phi'\}^{\perp}$ for $\sigma < 2/n$.

As we mention at the beginning of Section 6.2, elements $u$ of the manifold $\Sigma_0$ have the property that they can be written as $u = e^{i\alpha}\phi(\cdot - x_0) + \nu$, where $\nu \in \{\phi, \phi'\}^{\perp}$. Equation (6.2.3) consequently implies that

$$\rho(u, \phi)^2 \leq C\big(\mathcal{E}(u) - \mathcal{E}(\phi)\big) = C\big(\mathcal{E}(u_0) - \mathcal{E}(\phi)\big),$$

for $u \in \Sigma_0$ and some constant $C$. In other words, ground state solutions to the nonlinear Schrödinger equation are orbitally stable on the manifold $\Sigma_0$ whenever $\sigma < 2/n$.

# Bibliography


Balakrishnan, R. (1985). Soliton propagation in nonuniform media. *Phys. Rev. A*, 32, 1144–1149.

Conway, J. B. (1997). *A Course in Functional Analysis.* New York: Springer-Verlag.

Johnson, M. (2014). Advanced partial differential equations class lecture notes.

Kapitula, T. & Promislow, K. (2013). *Spectral and Dynamical Stability of Nonlinear Waves.* New York: Springer-Verlag.

Kwong, M. K. (1989). Uniqueness of positive solutions of $\triangle u - u + u^p = 0$ in $\mathbb{R}^n$. *Archive for Rational Mechanics and Analysis*, 105(3), 243–266.

Malomed, B. (2005). Nonlinear schrödinger equations. *Encyclopedia of Nonlinear Science*, (pp. 639–643).

Pego, R. L. & Weinstein, M. I. (1992). Eigenvalues, and instabilities of solitary waves. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 340(1656), 47–94.

Sulem, C. & Sulem, P.-L. (1999). *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse.* New York: Springer-Verlag.

Teschl, G. (2012). *Ordinary Differential Equations and Dynamical Systems.* United States of America: American Mathematical Society.