Categorical and homological aspects of module theory over commutative rings

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Abstract

The purpose of this work is to understand the structure of the subcategories of mod(R) and the derived category $D^b(R)$ for a commutative Noetherian ring R. Special focus is given to categories involving duality. We use these results to study homological dimension, maximal Cohen-Macaulay modules, and the singularities of a ring. Specifically, we classify certain resolving subcategories using semidualizing modules and also explore the relationship between these resolving subcategories and homological dimension. We also investigate the connections between semidualizing modules and rational singularities. Furthermore, using the theory of semidualizing modules and relative homological algebra, we prove a result on the depth formula. In order to construct Gersten-like complexes for singular schemes, we give an equivalence of derived categories. We also use this equivalence to study the Witt groups of categories associated to semidualizing modules. Lastly, we study the geometry of cohomological supports, a tool for understanding the thick subcategories over complete intersection rings. In particular, we show that when the Tor modules vanish, the cohomological support of the tensor product of two modules is the geometric join of the cohomological support of the original modules.

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Chapter 1

Introduction

Classifying objects is a natural and fundamental question in mathematics. A classical example is the classification of modules over a ring R, particularly when R is a group algebra. However, unless there are a finite number of isomorphism classes of indecomposable R-modules, i.e. R has finite representation type, the problem is generally intractable. In recent years, mathematicians have begun considering the question of classifying certain types of subcategories. From a representation theoretic viewpoint, this question is easier since there are fewer nice categories than modules. Benson, Iyengar, and Krause demonstrated the success of this approach by classifying the thick subcategories of the stable module category of a modular group algebra, Benson et al. (2011b). Mathematicians are actively classifying subcategories in a variety of fields including algebraic geometry, commutative algebra, stable homotopy theory, modular representation theory, K-theory, motivic theory, and symplectic geometry. See Balmer (2010); Kontsevich (1995); Hopkins & Smith (1998); Voevodsky et al. (2000); Benson & Carlson (2008); Verdier (1996); Thomason & Trobaugh (1990).

Commutative algebraists consider the question of classifying the thick subcategories of the category of finitely generated modules, $\operatorname{mod}(R)$, over a commutative Noetherian ring R. A subcategory $\mathcal{C} \subseteq \operatorname{mod}(R)$ is *thick* if it is closed under direct summands and has the following two-out-of-three property: if two terms of a short exact sequence are in $\mathcal{C}$, the third is too. Thick subcategories arise in many homological contexts such as homological dimension and the vanishing of Ext and Tor. The structure of the thick subcategories of $\operatorname{mod}(R)$ detects important homological information about the ring R.

For a commutative ring R, the thick subcategories of $\operatorname{mod}(R)$ are related to the thick subcategories of $D^b(R)$, the bounded derived category. The category $D^b(R)$ is triangulated, and in this context a subcategory $\mathcal{C} \subseteq D^b(R)$ is *thick* if it is a full triangulated subcategory closed under direct summands. Since a thick subcategory $\mathcal{C}$ is closed under mapping cones, it has the following two-out-of-three property: if two terms of an exact triangle are in $\mathcal{C}$, the third is as well. The classification of thick subcategories is an active research of research which draws ideas from many corners of mathematics. For example, Hopkins and Neeman classified the thick subcategories of the perfect complexes using intuition from stable homotopy theory, Hopkins (1987).

Surprisingly, the classification of thick subcategories is actually a geometric question. In their classification, Hopkins and Neeman give a bijection between the thick subcategories of the perfect complexes and the specialization closed subsets of $\operatorname{spec} R$, Hopkins (1987); Neeman (1992). In attempting to generalize this framework, the two current classification theories assign objects of a triangulated category (with additional structure) supports lying in some geometric object, Balmer (2005); Stevenson (2013b); Benson et al. (2008, 2011a). Due in part to this geometric connection, the singularities of R are intrinsically related to the structure of the thick subcategories of $\operatorname{mod}(R)$ and $D^b(R)$.

There are a plethora of different types of subcategories of $\operatorname{mod}(R)$ related to thick subcategories whose classification has received much attention from researchers. As with the thick subcategories, the majority of these classification results are also intrinsically related to $\operatorname{spec} R$ or some other topological space. For instance Gabriel's Theorem in Gabriel (1962) gives a bijection between the Serre subcategories of $\operatorname{mod}(R)$ and the specialization closed subsets of $\operatorname{spec} R$ (and hence also the thick subcategories of the perfect complexes). Another example is the work regarding resolving subcategories, for example in Dao & Takahashi

(2013).

Duality provides an interesting context to approach these classification questions. Duality is a property present in mathematical structures, the most elementary of which are finite dimensional vector spaces. Modern theories involving duality include Calabi-Yau manifolds, string theory (see Aganagic (2015)), and quadratic forms (see Lam (2005)). In commutative algebra and algebraic geometry, dualizing modules play a pivotal role in the celebrated local duality theorem (Bruns & Herzog, 1993, Theorem 3.5.8). In fact, Serre duality and Grothendieck's work on dualizing complexes were impetuses for Grothendieck's student Verdier to discover the derived category and to pioneer the notion of a triangulated category, see Marquis (2014).

Semidualizing modules are objects intrinsically related to duality. They are similar to dualizing modules, except they do not necessarily have finite injective dimension. As a result, semidualizing modules give a duality on only a subcategory of maximal Cohen-Macaulay modules (when R is Cohen-Macaulay) whereas dualizing modules give a duality on all maximal Cohen-Macaulay modules. Just as dualizing complexes are analogues in $D^b(R)$ of dualizing modules, we may also define semidualizing complexes. These objects yield interesting thick subcategories rich in structure, Gerko (2001, 2005); Christensen (2001).

1.1 Overview

In this section, I briefly present the main results of this document. The principal theme of my research is understanding the structure of the subcategories of $\text{mod}(R)$ and $D^b(R)$ for a commutative Noetherian ring R and their relation to duality, singularities, maximal Cohen-Macaulay modules, and homological dimension. Most results in this document may also be found in the following papers written in whole or in part by the author, Sanders (2014a,b); Sanders & Sane (2014); Dao & Sanders (2014).

In Chapter 3, I classify certain resolving subcategories using grade consistent functions.

A function $f : \operatorname{spec} R \to \mathbb{N}$ is grade consistent if it is weakly increasing as a map of posets and $f(p) \leq \operatorname{grade}(p)$ for all $p \in \operatorname{spec} R$. A resolving subcategory is weaker than a thick subcategory. It is particularly useful in understanding homological dimension. A special case of the main theorem of this chapter is the following.

Theorem 1.1.1. *Let R be a Cohen-Macaulay ring with a dualizing module D. Suppose $\mathcal{X}$ is a thick subcategory of the maximal Cohen-Macaulay modules such that $D \in \mathcal{X}$. Then there is a bijection between the grade consistent functions and the class of resolving subcategories*

$$\{\mathcal{Y} \subseteq \operatorname{mod}(R) \mid \mathcal{Y} \text{ is resolving}, \quad \mathcal{X} \subseteq \mathcal{Y}, \quad \mathcal{X}\text{-}\dim Y < \infty \; \forall Y \in \mathcal{Y}\}.$$

The full statement, Theorem 6.8, generalizes the main result of Dao & Takahashi (2013), which shows a similar bijection for the resolving subcategories of the category of modules of finite projective dimension. A corollary of this result is that when R is Cohen-Macaulay and contains a dualizing module, the dominant resolving subcategories of $\operatorname{mod}(R)$ are in bijection with the grade consistent functions. This bijection also shows that these resolving subcategories can be described using dimension types, a concept from Auslander's 1962 ICM address, Auslander (1962). The key to the proof of Theorem 1.1 is duality, and actually Theorem 6.8 is stated using a class of modules heavily associated with duality. This class of modules is called semidualizing.

In Chapter 4, we discuss the existence of semidualizing modules and their relation to the singularities of R. Since singularities are related to thick subcategories and since semidualizing modules define thick subcategories, the two subjects are naturally linked. However, in this chapter we investigate a more direct connection. In particular, we investigate the following question.

Question 1. *Suppose R has rational singularities. Does R have only trivial semidualizing modules?*

As discussed in Section 3, there is much evidence suggesting the answer to this question is yes. I add to this evidence with the following theorem which appears later as Theorem 4.5.

Theorem 1.1.2. *If S is a power series ring over a field k in finitely many variables and G is a cyclic group of order p^l acting on S with $\operatorname{Char} k \neq p$ and p a prime, then the only semidualizing modules over S^G are itself and the dualizing module.*

In Chapter 5, for a semidualizing module C, we define the relative Tate homology and cohomology functors $\widehat{\operatorname{Tor}}_i^C(X_\bullet, Y_\bullet)$ and $\widehat{\operatorname{Ext}}_C^i(X_\bullet, Y_\bullet)$ for any bounded complexes $X_\bullet$ and $Y_\bullet$. We discuss the relationship between these functors and the previous work on Tate (co)homology functors in Veliche (2006); Christensen & Jorgensen (2014); Sather-Wagstaff et al. (2010b); Di et al. (2014). We use these functors to prove the following result. Lastly, we also discuss in this chapter the relation between AB rings and the depth formula. In particular, we show that nonartinian Gorenstein isolated singularities are AB if and only if every pair of modules satisfies the depth formula.

Theorem 1.1.3. *Let C be a semidualizing module. Suppose $X_\bullet, Y_\bullet$ are complexes with $X_\bullet$ totally C-reflexive and $Y_\bullet$ in $D^b(\mathcal{A}_C)$ where $\mathcal{A}_C$ is the Auslander category (see Definition 2.1). If $\widehat{\operatorname{Tor}}_i^C(M, N) = 0$ for all $i \in \mathbb{Z}$, then the derived depth formula holds, i.e.*

$$\operatorname{depth} X_\bullet \otimes^{\mathbf{L}} Y_\bullet + \operatorname{depth} R = \operatorname{depth} X_\bullet + \operatorname{depth} Y_\bullet$$

This result recovers the main theorem in Christensen & Jorgensen (2015), but the techniques used are very different.

In Chapter 6, we discuss an equivalence of derived categories. The work in this chapter is joint with Sarang Sane. The following is a special case of Theorem 5.5, the main result of this chapter.

Theorem 1.1.4. *Set*

$$\mathrm{Fpd}_{\mathrm{fl}} = \{M \in \operatorname{mod}(R) \mid \operatorname{pd}(M) < \infty \quad \operatorname{length}(M) < \infty\}$$

$$\mathrm{Perf}_{\mathrm{fl}} = \{X_\bullet \in D^b(R) \mid X_\bullet \textit{ is perfect}, \quad \mathrm{length}(H_i(X_\bullet)) < \infty \ \forall i\}.$$

If R is a local Cohen-Macaulay ring, there is an equivalence of derived categories

$$\rho : D^b(\mathrm{Fpd}_{\mathrm{fl}}) \xrightarrow{\sim} \mathrm{Perf}_{\mathrm{fl}}$$

where ρ is the projective resolution functor.

We apply this result to the study of K and Witt groups. In particular, the nonconnective K-theory spectra of these categories are homotopy equivalent (see Schlichting (2011); Toën (2011) for definitions), and hence all the K-groups coincide, generalizing a result in Roberts & Srinivas (2003). Furthermore, this result gives a more concrete description of the terms in the weak Gersten complexes constructed in Balmer (2005). Another corollary is that the Witt groups $W(\mathrm{Fpd}_{\mathrm{fl}})$ and $W(\mathrm{Perf}_{\mathrm{fl}})$ are isomorphic.

The thick subcategories of a complete intersection ring are classified. A main tool in this classification is the theory of cohomological supports. The cohomological support, also called the support variety, of a module is a particular subscheme of projective space which contains homological information regarding the module. Approaching this subject with a new perspective, in Chapter 7 we examine the geometry of cohomological supports. Note that this work is joint with my advisor, Hailong Dao. The following is Theorem 3.4, the main result of the chapter.

Theorem 1.1.5. *If R is a complete intersection ring, and $\mathrm{Tor}_{>0}(M,N) = 0$, then*

$$V^*(M \otimes N) = \mathrm{Join}(V^*(M), V^*(N)).$$

We will now present an in-depth summary of each chapter.

1.2 Homological dimension and resolving subcategories

Resolving subcategories are the appropriate categories over which to define homological dimension. During their work on totally reflexive modules and Gorenstein dimension, Auslander and Bridger in Auslander & Bridger (1969) define a subcategory $\mathcal{C} \subseteq \operatorname{mod}(R)$ to be resolving if

1. R is in $\mathcal{C}$;
2. $X \oplus Y$ is in $\mathcal{C}$ if and only if X and Y are in $\mathcal{C}$; and
3. If $0 \to X \to Y \to Z \to 0$ is exact, then $Z \in \mathcal{C}$ implies that $Y \in \mathcal{C}$ if and only if $X \in \mathcal{C}$.

Natural examples of resolving subcategories include the category of projective modules, the category of totally reflexive modules, and the subcategory

$$\mathcal{C}_N = \{M \in \operatorname{mod}(R) \mid \operatorname{Ext}^{>0}(M, N) = 0\}$$

for any fixed $N \in \operatorname{mod}(R)$. For a resolving subcategory $\mathcal{C} \subseteq \operatorname{mod}(R)$ and a module $M \in \operatorname{mod}(R)$, we say that $\mathcal{C}\text{-}\dim M = n$ if $n \in \mathbb{N}$ is the smallest number n such that there is an exact sequence

$$0 \to C_n \to \cdots \to C_0 \to M \to 0$$

with $C_0, \ldots, C_n \in \mathcal{C}$.

Resolving subcategories are intimately connected with depth. In his 1962 ICM address, Auslander (1963), Auslander defines two modules M and N to have the same *dimension type* if $\operatorname{pd}_{R_p} M_p = \operatorname{pd}_{R_p} N_p$ for all $p \in \operatorname{spec} R$. Auslander was motivated by the regular case, but we can extend this definition to all modules over Cohen-Macaulay rings by saying that two modules M, N have the same dimension type if $\operatorname{MCM}(R_p)\text{-}\dim M_p = \operatorname{MCM}(R_p)\text{-}\dim N_p$ for all $p \in \operatorname{spec} R$, where $\operatorname{MCM}(R_p)$ denotes the category of maximal Cohen-Macaulay R_p-modules. Both definitions are equivalent to saying that $\operatorname{depth}_{R_p} M_p = \operatorname{depth}_{R_p} N_p$ for all

$p \in \operatorname{spec} R$. The set of dimension types has a natural poset structure. Assume, for now, that R is a Cohen-Macaulay ring. Letting $[M]$ be the dimension type of M, we may view $[M]$ as a function $[M] : \operatorname{spec} R \to \mathbb{N}$ given by $p \mapsto \mathrm{MCM}(R_p)\text{-}\dim M_p$. With this viewpoint, for a fixed $f : \operatorname{spec} R \to \mathbb{N}$, the subcategory $\{N \in \operatorname{mod}(R) | [N] \leq f\}$ is resolving. An example of such a resolving subcategory is the collection of modules satisfying Serre's condition S_n for some fixed n. A module M satisfies S_n if $\operatorname{depth} M_p \geq \min\{n, \operatorname{ht} p\}$ for all $p \in \operatorname{spec} R$. This is equivalent to saying that the dimension type of M is bounded by $\max\{n - \operatorname{ht} p, 0\}$.

Note that for each dimension type, the function $[M]$ is an increasing function from the poset $\operatorname{spec} R$ to the poset $\mathbb{N}$ and is bounded by the function $p \mapsto \operatorname{grade} p$. We call such a function *grade consistent*. Dao and Takahashi in Dao & Takahashi (2013) give a bijection between grade consistent functions and the resolving subcategories of modules with finite projective dimension. We extend this result in Theorem 6.8, of which the following is a special case.

Theorem 1.2.1. *If R has a dualizing module D, and $\mathcal{X}$ is a thick subcategory of MCM(R) containing R and D, then there is a bijection between grade consistent functions and resolving subcategories $\mathcal{Y}$ such that $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X}\text{-}\dim M < \infty$ for all $M \in \mathcal{Y}$. This bijection is given by*

$$f \mapsto \{M \in \operatorname{mod}(R) \mid \operatorname{add} \mathcal{X}_p\text{-}\dim M_p \leq f(p)\}$$

In particular, there is a bijection between grade consistent functions and resolving subcategories containing MCM(R), the maximal Cohen-Macaulay R-modules.

where $\operatorname{add} \mathcal{X}_p$ is the additive closure of $\mathcal{X}_p$. In particular, when R has a dualizing module, the resolving subcategory $\operatorname{res}(\mathrm{MCM}(R), M)$ is determined by the dimension type of M. Therefore, this theorem demonstrates a deep relationship between the depth of a module and the resolving subcategories containing it.

In Hügel et al. (2014), the authors also classify the resolving subcategories of modules with finite projective dimension by solving the equivalent problem of classifying the tilting and cotilting classes. In Hügel & Saorín (2014), the authors show that these classes are also in bijection with the compactly generated t-structures $(\mathcal{U}, \mathcal{V})$ of $D(R)$ with $R[1]$ in their heart such that the homology of the objects in $\mathcal{U}$ are concentrated in positive degrees. These objects are thus in bijection with grade consistent functions. These results suggest that resolving subcategories can be viewed as the module category analog of t-structures.

1.3 Semidualizing modules and birational geometry

Consider a ring homomorphism $R \to S$ with S a finitely generated R-module. The induced functor $\theta : D^b(S) \to D^b(R)$ is poorly behaved, and relating the thick subcategories of $D^b(S)$ and $D^b(R)$ is difficult. However, duality allows us to compare certain thick subcategories. For any complex $C_\bullet \in D^b(R)$, there is a thick subcategory $\mathcal{G}_{C_\bullet} \subseteq D^b(R)$ over which $\mathbf{R}\mathrm{Hom}(-, C_\bullet)$ is dualizing. We call $C_\bullet$ a semidualizing complex when $R \in \mathcal{G}_{C_\bullet}$. If $R, S \in \mathcal{G}_{C_\bullet} \subseteq D^b(R)$, it turns out that $\mathcal{G}_{B_\bullet} = \theta^{-1}\mathcal{G}_{C_\bullet}$ where $B_\bullet = \mathbf{R}\mathrm{Hom}_R(S, C_\bullet)$ is also semidualizing Gerko (2005). Therefore we can relate thick subcategories of $D^b(R)$ and $D^b(S)$ of the form $\mathcal{G}_{C_\bullet}$ where $C_\bullet$ is a semidualizing complex. Examples of semidualizing complexes always include dualizing complexes and R.

We can see already that semidualizing complexes are intrinsically linked to the study of thick subcategories, but the relationship goes deeper. For example, for a semidualizing complex, $C_\bullet$, there is an equivalence of categories between perfect complexes and $\mathcal{T}_{C_\bullet}$, the smallest thick subcategory of $D^b(R)$ containing $C_\bullet$. Thus, the Hopkins-Neeman theorem immediately implies that the thick subcategories of $\mathcal{T}_{C_\bullet}$ are in bijection with the specialization closed subsets of $\operatorname{spec} R$. Furthermore, we have the following conjecture regarding thick subcategories which is central in the study of semidualizing complexes, Gerko (2001, 2005).

Conjecture 1.3.1. *Let $B_\bullet$ and $C_\bullet$ be semidualizing complexes. Then $B_\bullet \in \mathcal{G}_{C_\bullet}$ implies $\mathcal{G}_{B_\bullet} \subseteq \mathcal{G}_{C_\bullet}$.*

Besides showing that semidualizing complexes behave intuitively, an affirmative answer would imply the transitivity of Gorenstein dimension.

When $C_\bullet = C$ is a module, C is semidualizing if and only if $\operatorname{Hom}(C, C) \cong R$ and $\operatorname{Ext}^{>0}(C, C) = 0$. Semidualizing modules have been studied extensively under several names in Araya & Takahashi (2009); Vasconcelos (1974); Nasseh & Sather-Wagstaff (2012); Jorgensen et al. (2012); Sather-Wagstaff (2009a,b). We call R and any dualizing modules *trivial semidualizing modules*. In Chapter 4, we study the following question.

Question 2. *Suppose R has rational singularities. Does R have only trivial semidualizing modules?*

This question is important because an understanding of the relationship between semidualizing modules and their singularities in conjunction with the earlier discussions will give new insights into the behavior of singularities under a ring homomorphism.

The evidence suggests the answer to Question 2 is yes. In Celikbas & Dao (2014), the authors show that only trivial semidualizing modules exist over Veronese subrings, which have a quotient singularity and hence a rational singularity. Furthermore, Sather-Wagstaff shows in Sather-Wagstaff (2007) that only trivial semidualizing modules exist for determinantal rings, which also have rational singularities. By (Sather-Wagstaff, 2009b, Example 4.2.14), all Cohen-Macaulay rings with minimal multiplicity have only trivial semidualizing modules. Since rational singularities with dimension 2 have minimal multiplicity, such rings have only trivial semidualizing modules. In Chapter 4, we add to the evidence with the following theorem.

Theorem 1.3.2 (Theorem 4.5)**.** *If S is a power series ring over a field k in finitely many variables and G is a cyclic group of order p^l acting on S with $\operatorname{Char} k \neq p$, then S^G has only trivial semidualizing modules.*

The proofs of Theorem 3.2 and the aforementioned results use ad hoc methods and not the powerful tools of birational geometry.

1.4 Relative functors and the depth formula

In Auslander (1961), Auslander proved that any R-modules M and N with $\operatorname{pd} M < \infty$ satisfy the following condition.

Definition 1.4.1. The *depth formula* holds for modules M and N if $\operatorname{Tor}_{>0}(M, N) = 0$ implies

$$\operatorname{depth} M \otimes N + \operatorname{depth} R = \operatorname{depth} M + \operatorname{depth} N.$$

Following in Auslander's footsteps, there have been many results on sufficient conditions for the depth formula to hold for modules M and N. For instance, Huneke and Wiegand show in Huneke & Wiegand (1994) that the depth formula holds for any pair of modules over complete intersection rings. Later it was shown independently in Araya & Yoshino (1998) and Iyengar (1999) that the depth formula holds if one of the modules has finite complete intersection dimension. Christensen and Jorgensen proceeded to show that any pair of modules over an AB ring satisfy the depth formula in Christensen & Jorgensen (2015). We observe in Corollary 5.6 that the proof of this result holds if one of the modules has finite AB-dimension, a concept which postdates the work of Christensen and Jorgensen. Taking this line of investigation into a different direction, Celikbas and Dao study necessary conditions for a pair of modules to satisfy the equality

$$\operatorname{depth} M \otimes N + \operatorname{depth} R = \operatorname{depth} M + \operatorname{depth} N$$

in Celikbas & Dao (2014).

Iyengar and Foxby in Iyengar (1999) and Foxby & Iyengar (2003) extensively study the depth of a complex. Using this notion, we can state the derived version of the depth formula.

Definition 1.4.2. Bounded complexes $X_\bullet$ and $Y_\bullet$ satisfy the *derived depth formula* if

$$H_{\gg 0}(X_\bullet \otimes^{\mathbf{L}} Y_\bullet) = 0$$

implies

$$\operatorname{depth} X_\bullet \otimes^{\mathbf{L}} Y_\bullet + \operatorname{depth} R = \operatorname{depth} X_\bullet + \operatorname{depth} Y_\bullet.$$

If the derived depth formula holds for modules M and N (viewed as complexes) then so does the depth formula, because $\operatorname{Tor}_{>0}(M, N) = 0$ implies $M \otimes^{\mathbf{L}} N \cong M \otimes N$. Foxby and Iyengar show in Foxby (1980) and Iyengar (1999) that the derived depth formula holds if one of the complexes is perfect. Christensen and Jorgensen generalize this result in Christensen & Jorgensen (2015) with the following theorem.

Theorem 1.4.3. *If $\widehat{\operatorname{Tor}}_i(X_\bullet, Y_\bullet) = 0$ for all $i \in \mathbb{Z}$, then $X_\bullet$ and $Y_\bullet$ satisfy the derived depth formula.*

In Chapter 5, we generalize this result to the semidualizing setting using relative Tate homology functors.

In Avramov & Martsinkovsky (2002), the authors develop the functors $\widehat{\operatorname{Ext}}^i(M, N)$ called the Tate cohomology functors. This idea can be traced back to Buchweitz (1989) and was further developed by Jørgensen (2007) and Veliche (2006). This functor is defined when M has finite Gorenstein-dimension. Iacob developed the Tate homology functor $\widehat{\operatorname{Tor}}_i(M, N)$ when M has finite Gorenstein dimension in Iacob (2007). This is the functor used in Theorem 4.3. The theories of Tate homology and cohomology for complexes were developed in Christensen & Jorgensen (2014) and Veliche (2006). This area of research is sometimes referred to as Gorenstein homological algebra.

First formulated in Eilenberg & Moore (1965), relative homological algebra defines versions of derived functors by resolving modules with categories other than projective modules. Relative homological algebra was utilized in Enochs & Jenda (1995) to define and study Gorenstein projective modules. This theory was crucial in developing the relative cohomology functor of Avramov & Martsinkovsky (2002) and the Gorenstein homology functor of Iacob (2007).

Relative homological algebra was also used to generalize Gorenstein homological algebra to the semidualizing case. Indeed Gorenstein C-projective modules were defined and developed in Holm & Jørgensen (2006) and White (2010). In Sather-Wagstaff et al. (2010a), the authors define relative cohomology functors involving semidualizing modules, and analogous constructions for homology are discussed in Salimi et al. (2014). Generalizing the work of Avramov and Martsinkovsky, a relative version of Tate cohomology for semidualizing modules is studied in Sather-Wagstaff et al. (2010b). Similarly, in Di et al. (2014) the authors give a semidualizing analog of the Tate homology defined by Iacob.

In Chapter 5,we do three things. First, we construct a functor $T_C : D^b(R) \to K(R)$ which allows us to define for complexes relative Tate homology and cohomology functors $\widehat{\mathrm{Tor}}^C$ and $\widehat{\mathrm{Ext}}_C$ for a semidualizing module C. Second, we prove the following result which is Theorem 5.1.

Theorem 1.4.4. *Let C be a semidualizing module. Suppose $X_\bullet, Y_\bullet$ are complexes with $X_\bullet$ totally C-reflexive and $Y_\bullet$ in $D^b(\mathcal{A}_C)$ where $\mathcal{A}_C$ is the Auslander category (see Definition 2.1). If $\widehat{\mathrm{Tor}}_i^C(M,N) = 0$ for all $i \in \mathbb{Z}$, then the derived depth formula holds, i.e.*

$$\operatorname{depth} X_\bullet \otimes^{\mathbf{L}} Y_\bullet + \operatorname{depth} R = \operatorname{depth} X_\bullet + \operatorname{depth} Y_\bullet$$

Thirdly, we also explore the relationship between AB and UAC rings and the depth formula. In particular, we show that for nonartinian Gorenstein isolated singularities, the depth formula holds if and only if the ring is AB.

Taking $C = R$, Theorem 4.4 recovers Theorem 4.3 for bounded complexes of finitely generated modules. Furthermore, our techniques differ from those in Christensen & Jorgensen (2015). Furthermore, our construction of $\widehat{\mathrm{Tor}}^C$ and $\widehat{\mathrm{Ext}}_C$ differs from Veliche (2006) and Christensen & Jorgensen (2014).

Our initial interest in $\widehat{\mathrm{Tor}}^C$ was to state the above theorem. However, $\widehat{\mathrm{Tor}}^C$ and $\widehat{\mathrm{Ext}}_C$ seem to warrant independent interest, and have both advantages and disadvantages over the presently defined functors. On the one hand, we may define $\widehat{\mathrm{Tor}}^C(X_\bullet, Y_\bullet)$ and $\widehat{\mathrm{Ext}}_C(X_\bullet, Y_\bullet)$

for any pair of bounded complexes. The theories of Sather-Wagstaff et al. (2010b) and Di et al. (2014) only hold for modules admitting certain resolutions. Furthermore, $\widehat{\mathrm{Tor}}$ and $\widehat{\mathrm{Ext}}$ are only defined when one of the complexes admits a certain type of resolution. That being said, $X_\bullet$ and $Y_\bullet$, $\widehat{\mathrm{Tor}}^C(X_\bullet, Y_\bullet)$ and $\widehat{\mathrm{Ext}}_C(X_\bullet, Y_\bullet)$ may not have certain nice properties for arbitrary complexes. For example, $\widehat{\mathrm{Tor}}^C(X_\bullet, R)$ does not necessarily vanish unless we put some restrictions on $X_\bullet$.

The functors $\widehat{\mathrm{Tor}}^C$ and $\widehat{\mathrm{Ext}}_C$ have other limitations as well. Avramov and Martsinkovsky also define the relative cohomology functor $\mathrm{Ext}_{\mathcal{G}}$ which fits into long exact sequence

$$\cdots \to \mathrm{Ext}^i_{\mathcal{G}}(M, N) \to \mathrm{Ext}^i(M, N) \to \widehat{\mathrm{Ext}}^i(M, N) \to \mathrm{Ext}^{i+1}_{\mathcal{G}}(M, N) \to \cdots$$

and the works of Veliche, Iacob, Wagstaff et. al., etc all generalize this construction. Our functors $\widehat{\mathrm{Tor}}^C$ and $\widehat{\mathrm{Ext}}_C$ do not currently fit into such an exact sequence. Furthermore, we have no balance results analogous to those in Sather-Wagstaff et al. (2010b), Di et al. (2014), and Christensen & Jorgensen (2014). Lastly, the work Veliche (2006) and Christensen & Jorgensen (2014) allows us to define $\widehat{\mathrm{Tor}}(X_\bullet, Y_\bullet)$ and $\widehat{\mathrm{Ext}}(X_\bullet, Y_\bullet)$ when for certain cases where $X_\bullet$ is unbounded, where as the constructions in this work always requires $X_\bullet$ to be bounded.

The main difference between this work and the work of Veliche (2006),Sather-Wagstaff et al. (2010b), etc. is our treatment of of resolutions of of an object $X_\bullet$. Loosely speaking, in Veliche (2006), Sather-Wagstaff et al. (2010b), the authors define a certain type of resolution for a module or a complex, and then show by hand that these resolutions lift and also prove some generalization of the Horseshoe Lemma. In our work, we use the projective resolution functor and $\hom(-, C)$ to construct a functor $T_C : D^b(R) \to K(R)$. The machinery of triangulated categories then gives the desired properties.

1.5 A derived equivalence, K-theory, and the New Intersection Theorem

Much attention has been given to the category, $\mathrm{Fpd}_{\mathrm{fl}}$, of finite length modules with finite projective dimension and the derived category, $\mathrm{Perf}_{\mathrm{fl}}$, of perfect complexes whose homologies have finite length. Bass essentially conjectured that nonzero modules of finite projective dimension and finite length exist if and only if R is Cohen-Macaulay. This conjecture was resolved in the affirmative by the celebrated New Intersection Theorem, Roberts (1987); Peskine & Szpiro (1973); Hochster (1975). In Roberts & Srinivas (2003), Roberts and Srinivas show that when R is Cohen-Macaulay, $K_0(\mathrm{Fpd}_{\mathrm{fl}}) \cong K_0(\mathrm{Perf}_{\mathrm{fl}})$. Sarang Sane and the author vastly generalized this result in Sanders & Sane (2014). Here is a special case of our main result which we state in Chapter 6 as Theorem 5.5.

Theorem 1.5.1 (Sanders, Sane)**.** *If R is a local Cohen-Macaulay ring, there is an equivalence of derived categories*

$$\rho : D^b(\mathrm{Fpd}_{\mathrm{fl}}) \xrightarrow{\sim} \mathrm{Perf}_{\mathrm{fl}}$$

where ρ is the projective resolution functor.

In particular, there is a homotopy equivalence of nonconnective K-theory spectra of the category of finite length modules with finite projective dimension and the derived category of perfect complexes with finite length homologies, and hence these categories have the same K-groups (see Schlichting (2011); Toën (2011)). Balmer constructs in Balmer (2009) a niveau spectral sequence abutting to $K_i(R)$ which yields a weak Gersten complex whose terms have direct summands of the form $K_j(\mathrm{Perf}_{\mathrm{fl}})$. Hence when R is Cohen-Macaulay, the theorem allows us to rewrite the direct summands of this complex as $K_j(\mathrm{Fpd}_{\mathrm{fl}})$, offering an alternative to dévissage in the nonregular case.

The Hopkins-Neeman theorem of Hopkins (1987) and Neeman (1992) shows that $\mathrm{Perf}_{\mathrm{fl}}$ is never empty. But when R is not Cohen-Macaulay, $\mathrm{Fpd}_{\mathrm{fl}}$ is always zero by the New

Intersection Theorem of Roberts (1987); Peskine & Szpiro (1973); Hochster (1975), and thus Theorem 5.1 holds precisely when R is Cohen-Macaulay, suggesting a deep relation between Theorem 5.1 and the New Intersection Theorem. Interestingly, the proof of Theorem 5.1 does not use the New Intersection Theorem.

We actually prove a much more general result in Theorem 5.5, which shows that Theorem 5.1 holds for resolving subcategories, a generalization of the subcategory of projective modules discussed in Section 2 and Chapter 3. Furthermore, this more general result holds for other Serre subcategories besides finite length modules and does not involve the Cohen-Macaulay assumption in its statement or proof. Because of this more general result, any relations between Theorem 5.1 and the New Intersection Theorem may hold for other resolving subcategories. Also, since Theorem 5.1 holds for other resolving subcategories, we can use Balmer (2009) to construct weak Gersten complexes for those categories in the same manner as previously discussed.

1.6 Cohomological supports and tensor products

Developed by Avramov and Buchweitz in Avramov & Buchweitz (2000b), the theory of cohomological supports over a complete intersection ring (which are also known as support varieties) encodes important homological information of a module into a geometric object. In this section, let R be a local complete intersection ring of codimension $c + 1$ with an algebraically closed residue field. Then for each $M \in \operatorname{mod}(R)$, there is an algebraic set $V^*(M) \subseteq \mathbb{P}^c_k$ with the property that $V^*(M) \cap V^*(N) = \emptyset$ if and only if $\operatorname{Tor}_{\gg 0}(M, N) = 0$. Furthermore, if M is in the thick subcategory generated by N, then $V^*(M) \subseteq V^*(N)$. In fact, if a ring has an isolated singularity, then the cohomological supports classify all of the thick subcategories containing R. See Stevenson (2013a) and Carlson & Iyengar (2012).

In Chapter 7, we embark on a geometric study of cohomological supports. The work in this chapter is joint with Hailong Dao. The main result of the chapter is the following.

Theorem 1.6.1 (Theorem 3.4). *If* $\operatorname{Tor}_{>0}(M,N)=0$*, then*

$$V^*(M\otimes N)=\operatorname{Join}(V^*(M),V^*(N)).$$

For any closed sets $V,U\subseteq\mathbb{P}^c_k$ with $U\cap V=\emptyset$ (as is the case in this theorem), the join of U and V is the set $\bigcup_{v\in V,u\in U}l(u,v)$ where $l(u,v)$ is the projective line containing u and v. Consequently, if R is an isolated singularity, then M is in the thick closure of $M\otimes N$ when $\operatorname{Tor}_{>0}(M,N)=0$. Augmenting the potential utility of this theorem, we also show that if M and N are Cohen-Macaulay (not necessarily maximal), then $\operatorname{Tor}_{\gg 0}(M,N)=0$ implies that $\operatorname{Tor}_{>0}(M,N)=0$.

The geometric join is an active topic of research. Of particular interest is the understanding generating set of the defining ideal. This result provides another avenue to study this question. Much attention is also being given to understanding the dimension of the join when the sets do not intersect. At the end of the chapter, we provide interesting questions involving Tor whose answers may shed light on this topic.

Chapter 2

Background

2.1 Historical note

Commutative algebra arose in the early twentieth century as a synthesis of tools and concepts arising from algebraic geometry and number theory. Mathematicians are naturally interested in the vanishing loci of polynomials. For an ideal $I \subseteq k[x_1, \ldots, x_n]$ where k is a field, let $V(I)$ denote the vanishing locus of I. We call a set of this form *algebraic*. Proved in 1893, Hilbert's Nullstellensatz demonstrated how to use algebraic methods to study algebraic sets and was a major impetus in the development of commutative algebra. The theorem shows that when k is an algebraically closed field, the points of an algebraic set $V(I)$ are in bijection with the maximal ideals containing I. Hence, to study the algebraic set V, one can study the coordinate ring $k[x_1, \ldots, x_n]/I$. In this vein, algebraists began translating geometric notions, such as dimension, into algebraic terms. Originally, much of the focus was on quotients of polynomials, but the true birth of commutative algebra came when mathematicians realized that this work could be generalized to all commutative Noetherian rings including such as $\mathbb{Z}$ and $\mathbb{Z}[\sqrt{-5}]$. For example, Emmy Noether related concurrent ideas on primary decomposition in algebraic geometry and number theory, Noether (1921). Later it was discovered that one can view $\operatorname{spec} R$, the set of prime ideals of R, as a geometric object.

See (Bourbaki, 1989, Historical Note) for a fascinating and more comprehensive discussion.

From the late 1920s to the late 1940s, the important tool of localization was slowly introduced into commutative algebra by authors including Grell, Krull, Chevalley, and Uzkov in Grell (1927); Chevalley (1944); Uzkov (1948). Krull's 1938 memoir Krull (1938) contained the first study of local rings. His chief motivation was to develop dimension theory. Krull was one of the first to use the local-global principle, see Krull (1935). He improved upon the work of Hensel on completions of a ring with respect to an ideal, another natural example of a local ring. Through these works, local Noetherian rings became a central component of modern commutative algebra. Again, consult (Bourbaki, 1989, Historical Note) for a more detailed account.

The work of Emmy Noether and Krull introduced another paradigm shift in commutative algebra and ring theory by using tools from linear algebra to study ideals. They took quotients of ideals and then applied other operations rooted in linear algebra. The resulting objects were not ideals but modules. Thus algebraists began to use module theory to understand commutative rings. The advent of homological algebra further established this viewpoint.

In the mid-twentieth century, topology was revolutionized by the development of the modern formulations of homotopy theory and homology. In the 1950s, the field of homological algebra emerged as mathematicians realized that these new tools also yielded invariants of algebraic objects. In particular, this decade saw the modern construction of the derived functors Tor and Ext. The functor Ext first gained attention when it was realized that the extensions of Abelian groups A and B were in bijection with the modern definition of $\mathrm{Ext}^1(A, B)$. These functors gained further notoriety with the universal coefficient theorem. It was quickly realized that the vanishing of these functors detected interesting information about rings and modules. See Weibel (1999) for a fascinating account of this historical development.

The power of homological methods in commutative algebra was demonstrated by Serre's landmark theorem Serre (1956): a local ring R is regular if and only if every R-module has finite projective dimension. Motivated by this result, algebraists such as Auslander and Buchsbaum began studying the interplay between homological and commutative algebraic invariants . See Auslander & Buchsbaum (1956, 1957, 1958b, 1959, 1958a); Auslander (1962). A celebrated result of these inquiries is the Auslander-Buchsbaum formula: for a module M over a local ring R with projective dimension $\operatorname{pd} M < \infty$,

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R.$$

Auslander's landmark paper Auslander (1963), motivated by torsion and depth, includes two major results. First, Tor is rigid over unramified regular local rings: if $\operatorname{Tor}_i(M, N) = 0$, then $\operatorname{Tor}_{>i}(M, N) = 0$. Second is the depth formula: if $\operatorname{Tor}_{>0}(M, N) = 0$ and $\operatorname{pd} M < \infty$, then $\operatorname{depth} M + \operatorname{depth} N = \operatorname{depth} R + \operatorname{depth} M \otimes N$. He uses these results to prove over an unramified regular local ring that if $M \otimes N$ is torsion free, then so are M and N.

The same advances in algebraic topology in the mid twentieth century which gave rise to homological algebra also gave birth to category theory. Categorical language first appeared in print in Eilenberg and Mac Lane's paper Eilenberg & MacLane (1945). Their primary aim was to understand the concept of a natural transformation which arose in their earlier work on universal coefficient theorems which laid some of the foundations for the modern notional of Ext in Čech cohomology, Eilenberg & MacLane (1942). For over a decade, category theory was established as a convenient language to talk about homological machinery. In this work, they defined the term *functor*, a term borrowed from the philosopher Carnap, and they defined the term *category*, which they borrowed from Aristotle, Immanuel Kant, and Charles Sanders Peirce (see Marquis (2014)). In particular, the revolutionary books Cartan & Eilenberg (1956) and Eilenberg & Steenrod (1952) were written in the language of category theory. Category theory was elevated from a useful linguistic tool to a mathematical theory by Grothendieck in his landmark paper, Grothendieck (1957). Here he defined Abelian categories and used them to begin doing homological algebra using sheaves, changing the face of algebraic geometry forever. See Marquis (2014) for an interesting exposition on the matter.

After this revolutionary perspective, many new categories were discovered. As an example, the derived category was discovered in the 1960s by Verdier in his thesis Verdier (1996) under the direction of Grothendieck. Their motivation was to extend Serre duality to singular schemes. It was discovered that the derived category had an elegant structure that differed from Abelian categories. Verdier axiomatized this behavior and defined triangulated categories. The later half of the twentieth century saw numerous applications of triangulated categories to a plethora of fields. Notable triumphs of triangulated categories include Buchweitz's theorem on maximal Cohen-Macaulay modules over Gorenstein rings Buchweitz (1989), the use of spectra in higher homotopy theory, Lurie's derived algebraic geometry Lurie (2004), the Brown representation theorem Brown (1962), and the Thomason Trobaugh theorem of K-theory Thomason & Trobaugh (1990).

From these accounts it is apparent that the field of homological algebra has developed alongside category theory and that categorical methods have been indispensable in forging fruitful links between Algebra, Topology and Geometry.

2.2 Resolving subcategories

We proceed with an overview of resolving subcategories. In order to understand the homological properties of modules which characterize Gorenstein rings, Auslander and Bridger defined and studied Gorenstein dimension (see Definition 3.2) in Auslander & Bridger (1969). In the process, they defined a special class of subcategories called resolving subcategories. It turns out, that these categories are the appropriate categories to use in defining homological dimension. Resolving subcategories are also intricately related to tilting classes, see (Göbel & Trlifaj, 2012, Section 13.2.3). Note that all subcategories considered in this document are full and closed under isomorphisms.

Definition 2.2.1. Given a ring R, a full subcategory $\mathcal{C} \subseteq \operatorname{mod}(R)$ is *resolving* if the following hold.

1. R is in $\mathcal{C}$
2. $X \oplus Y$ is in $\mathcal{C}$ if any only if X and Y are in $\mathcal{C}$
3. If $0 \to X \to Y \to Z \to 0$ is exact, then $Z \in \mathcal{C}$ implies that $Y \in \mathcal{C}$ if and only if $X \in \mathcal{C}$.

By (Yoshino, 2005, Lemma 3.2), this is equivalent to saying these conditions hold.

1. All projectives are in $\mathcal{C}$
2. If $X \in \mathcal{C}$, then $\operatorname{add} X \subseteq \mathcal{C}$ where $\operatorname{add} X$ is the smallest category containing X closed under direct sums and summands
3. $\mathcal{C}$ is closed under extensions
4. $\mathcal{C}$ is closed under syzygies

For a subset $\mathcal{X} \subseteq \operatorname{mod}(R)$, let $\operatorname{res}(\mathcal{X})$ be the smallest resolving subcategory containing $\mathcal{X}$. Let $\mathcal{P}$ and $\mathcal{P}(R)$ be the category of finitely generated projective R-modules.

Example 2.2.2. The following categories are easily seen to be resolving.

1. $\mathcal{P}$
2. $\operatorname{mod}(R)$
3. The set of Gorenstein dimension zero modules (see Definition 3.2)
4. For any $\mathcal{X} \subseteq \operatorname{Mod}(R)$ and any $n \geq 0$, $\{Y \mid \operatorname{Ext}^{>n}(Y, X) = 0 \ \forall X \in \mathcal{X}\}$
5. For any $\mathcal{X} \subseteq \operatorname{Mod}(R)$ and any $n \geq 0$, $\{Y \mid \operatorname{Tor}_{>n}(Y, X) = 0 \ \forall X \in \mathcal{X}\}$
6. $MCM(R)$ when R is Cohen-Macaulay and local

Resolving subcategories are studied in part because dimension with respect to a resolving subcategory has nice properties. For a subset $\mathcal{C} \subseteq \text{mod}(R)$, and a module $M \in \text{mod}(R)$ we say that $\mathcal{C}\text{-dim}\, M = n$ if $n \in \mathbb{N}$ is the smallest number such that there is an exact sequence

$$0 \to C_n \to \cdots \to C_0 \to M \to 0$$

with $C_0, \ldots, C_n \in \mathcal{C}$. Projective dimension and Gorenstein dimension are dimensions with respect to resolving subcategories of projective modules and Gorenstein dimension zero modules respectively. The following proposition from (Auslander & Buchweitz, 1989, Proposition 3.3) causes nice properties to hold for dimension with respect to a resolving subcategory.

Proposition 2.2.3. *If $\mathcal{C}$ is resolving and $\mathcal{C}\text{-dim}(X) \leq n$, then for any exact sequence*

$$0 \to U \to C_{n-1} \to \cdots \to C_0 \to X \to 0$$

with each $C_i \in \mathcal{C}$, we have $U \in \mathcal{C}$.

This proposition allows us to prove the following results.

Corollary 2.2.4. *If $\mathcal{C}$ is resolving, then $\mathcal{C}\text{-dim}(X) = \inf\{n \mid \Omega^n M \in \mathcal{C}\}$.*

Proof. If $\Omega^n M \in \mathcal{C}$, then we have

$$0 \to \Omega^n M \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

with each F_i projective. This shows that $\mathcal{C}\text{-dim}\, M \leq n$. If $n \leq \mathcal{C}\text{-dim}\, M$, the same sequence and Proposition 2.4 show that $\Omega^n M$ is in $\mathcal{C}$. □

Lemma 2.2.5. *If $\mathcal{C}$ is resolving, then $\mathcal{C}\text{-dim}\, X \oplus Y = \max\{\mathcal{C}\text{-dim}\, X, \mathcal{C}\text{-dim}\, Y\}$.*

Proof. We have $\Omega^n(X \oplus Y) = \Omega^n X \oplus \Omega^n Y$ for a suitable choice of syzygies. Since $\Omega^n(X \oplus Y)$ is in $\mathcal{C}$ if and only if $\Omega^n X$ and $\Omega^n Y$ are in $\mathcal{C}$, the result follows from Corollary 2.4. Parts (1) and (2) are essentially proved in Masek (1999)[Theorem 18]. □

Lemma 2.2.6. *If $\mathcal{C}$ is a resolving category, and $0 \to K \to L \to M \to 0$ is exact, then the following inequalities hold.*

1. $\mathcal{C}\text{-}\dim K \leq \max\{\mathcal{C}\text{-}\dim L, \mathcal{C}\text{-}\dim M - 1\}$

2. $\mathcal{C}\text{-}\dim L \leq \max\{\mathcal{C}\text{-}\dim K, \mathcal{C}\text{-}\dim M\}$

3. $\mathcal{C}\text{-}\dim M \leq \max\{\mathcal{C}\text{-}\dim K, \mathcal{C}\text{-}\dim L\} + 1$

Proof. For suitable choices of syzygies, we have the following.

$$0 \to \Omega^k K \to \Omega^k L \to \Omega^k M \to 0$$

If $k = \max\{\mathcal{C}\text{-}\dim K, \mathcal{C}\text{-}\dim M\}$, then, by Corollary 2.4, $\Omega^k K$ and $\Omega^k M$ are in $\mathcal{C}$, and thus, so is $\Omega^k L$, giving us (2). If $k = \max\{\mathcal{C}\text{-}\dim K, \mathcal{C}\text{-}\dim L\}$, then, again by Corollary 2.4, $\Omega^k K$ and $\Omega^k L$ will be in $\mathcal{C}$. Therefore $\mathcal{C}\text{-}\dim \Omega^k M \leq 1$, and so $\Omega^{k+1} M$ will be in $\mathcal{C}$. Thus by Corollary 2.4, $\mathcal{C}\text{-}\dim M \leq k+1$, proving (3).

Now take $k = \max\{\mathcal{C}\text{-}\dim L, \mathcal{C}\text{-}\dim M - 1\}$. Then L_k and M_{k+1} are in $\mathcal{C}$. We take the pushout diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega^{k+1} M & = & \Omega^{k+1} M & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^k K & \longrightarrow & Z & \longrightarrow & F & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^k K & \longrightarrow & \Omega^k L & \longrightarrow & \Omega^k M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with F free and hence in $\mathcal{C}$. Since, by Corollary 2.4, $\Omega^{k+1} M$ and $\Omega^k L$ are in $\mathcal{C}$, so is Z. Since $F \in \mathcal{C}$, $\Omega^k K$ has to also be in $\mathcal{C}$. Hence $\mathcal{C}\text{-}\dim K \leq k$, and we have (1). $\square$

We would like to discuss the categories of modules with finite $\mathcal{C}$-dimension. From the lemma, we can see that this category is resolving, but it is actually stronger. A special class of resolving subcategories are thick subcategories.

Definition 2.2.7. Let $\mathcal{X} \subseteq \operatorname{mod}(R)$. A full subcategory $\mathcal{C} \subseteq \mathcal{X}$ is a *thick subcategory* of $\mathcal{X}$ (or $\mathcal{C}$ is thick in $\mathcal{X}$) if it is resolving and for any exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Y, Z \in \mathcal{X}$, if X and Y are in $\mathcal{C}$, then Z is in $\mathcal{C}$ too. We let $\operatorname{Thick}_{\mathcal{X}}(\mathcal{C})$ be the smallest thick subcategory of $\mathcal{X}$ containing $\mathcal{C}$. If $\mathcal{X}$ is not specified, we understand then that $\mathcal{X} = \operatorname{mod}(R)$. In this situation, let $\operatorname{Thick}(\mathcal{C})$ and $\bar{\mathcal{C}}$ both denote the smallest thick subcategories containing $\mathcal{C}$.

The following is immediate from the previous lemma.

Lemma 2.2.8. *Let $\mathcal{C}$ be resolving. For any n, the class $\{M \in \operatorname{mod}(R) \mid \mathcal{C}\text{-}\dim M \leq n\}$ is is equal to* $\operatorname{Thick}(\mathcal{C}) = \bar{\mathcal{C}}$.

Example 2.2.9. The following categories are obtained as the thick closures in $\operatorname{mod}(R)$ of the resolving categories in Example 2.2. Equivalently, each of these categories are the class of modules with finite dimensions with respect to the categories in Example 2.2.

1. The set of modules with finite projective dimension
2. $\operatorname{mod}(R)$
3. The set of modules with finite Gorenstein dimension
4. For any $\mathcal{X} \subseteq \operatorname{Mod}(R)$, $\{Y \mid \operatorname{Ext}^{\gg 0}(Y, X) = 0 \ \forall X \in \mathcal{X}\}$
5. For any $\mathcal{X} \subseteq \operatorname{Mod}(R)$, $\{Y \mid \operatorname{Tor}_{\gg 0}(Y, X) = 0 \ \forall X \in \mathcal{X}\}$
6. When R is Cohen-Macaulay and local, $\overline{MCM(R)} = \operatorname{mod}(R)$.

Through these results, we may construct many resolving and thick subcategories. It is easy to show that the intersection of a collection of resolving subcategories and the intersection of a collection of thick subcategories are resolving and thick respectively. The following lemma allows us to construct even more resolving subcategories. For $\mathcal{C} \subseteq \operatorname{mod}(R)$, we set $\mathcal{C}_p = \{C_p \mid C \in \mathcal{C}\}$.

Lemma 2.2.10. *Let R and S be rings and $F : \operatorname{mod}(R) \to \operatorname{mod}(S)$ be an exact functor with $F(R) = S$. Then for any resolving subcategory $\mathcal{C} \subseteq \operatorname{mod}(S)$, $F^{-1}(\mathcal{C})$ is a resolving subcategory of $\operatorname{mod}(R)$.*

The proof is elementary. Applying this lemma to the localization functor, for any $V \subseteq \operatorname{spec} R$, the category $\{M \in \operatorname{mod}(R) \mid M_p \text{ free } \forall p \in V\}$ is also resolving. The following lemmas give insight into the behavior of resolving categories under localization. The first lemma is from (Takahashi, 2010, Lemma 4.8) and (Dao & Takahashi, 2012, Lemma 3.2(1)), and the second is from (Dao & Takahashi, 2013, Proposition 3.3).

Lemma 2.2.11. *If $\mathcal{X}$ is a resolving subcategory, then so is $\operatorname{add} \mathcal{X}_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{spec} R$.*

Lemma 2.2.12. *The following is equivalent for a resolving subcategory $\mathcal{X}$ and a module $M \in \operatorname{mod}(R)$.*

1. $M \in \mathcal{X}$
2. $M_p \in \operatorname{add} \mathcal{X}_p$ *for all* $p \in \operatorname{spec} R$
3. $M_\mathfrak{m} \in \operatorname{add} \mathcal{X}_\mathfrak{m}$ *for all maximal ideals* $\mathfrak{m}$.

We can use these results to describe a multitude of resolving subcategories.

Definition 2.2.13. 1. A function $f : \operatorname{spec} R \to \mathbb{N}$ is *grade consistent* if it is increasing as a poset homomorphism and $f(p) \leq \operatorname{grade} p$.

2. Let Γ be the set of grade consistent functions.

3. For a subcategory $\mathcal{C} \subseteq \operatorname{mod}(R)$ and an $f \in \Gamma$, define

$$\Lambda(\mathcal{C})(f) = \{M \in \operatorname{mod}(R) \mid \operatorname{add} \mathcal{C}_p\text{-}\dim M_p \leq f(p) \quad \forall p \in \operatorname{spec} R\}.$$

4. For a resolving subcategory $\mathcal{X} \subseteq \operatorname{mod}(R)$ define the function

$$\Phi_C(\mathcal{X}) : \operatorname{spec} R \to \mathbb{N} \qquad p \to \sup\{\operatorname{add} C_p\text{-}\dim X_p | X \in \mathcal{X}\}.$$

This result follows from our discussions.

Lemma 2.2.14. *If $\mathcal{C}$ is resolving, then for all $f \in \Gamma$, $\Lambda(\mathcal{C})(f)$ is a resolving subcategory.*

An important example of a resolving subcategory is the category of maximal Cohen-Macaulay modules.

Definition 2.2.15. A commutative Noetherian local ring is said to be *Cohen-Macaulay* if its depth equals its dimension. A commutative Noetherian ring R is Cohen-Macaulay if each localization $R_{\mathfrak{p}}$ is Cohen-Macaulay local. For a local ring we denote the subcategory of maximal Cohen-Macaulay modules (i.e. modules M which satisfy $\operatorname{depth}(M) = \dim R$) by MCM or MCM($R$). When R is not local, MCM will be the category of modules M such that M_p is maximal Cohen-Macaulay for every $p \in \operatorname{spec} R$.

As noted earlier, when R is Cohen-Macaulay, MCM is resolving. Furthermore, letting $d = \dim R$, $\Omega^d M$ is in MCM for every $M \in \operatorname{mod}(R)$. Hence, $\overline{\mathrm{MCM}} = \operatorname{mod}(R)$. Dimension with respect to MCM is very computable.

Lemma 2.2.16. *Suppose $\mathcal{C}$ is a thick subcategory of a resolving subcategory $\mathcal{D}$. Then for any module $M \in \overline{\mathcal{C}}$, we have $\mathcal{C}\text{-}\dim M = \mathcal{D}\text{-}\dim M$. Furthermore, if R is Cohen-Macaulay, $\mathcal{C}$ is a thick subcategory of MCM if and only if dimension with respect to $\mathcal{C}$ satisfies the Auslander Buchsbaum Formula, i.e. for all $M \in \overline{\mathcal{C}}$ we have*

$$\mathcal{C}\text{-}\dim M + \operatorname{depth} M = \operatorname{depth} R.$$

Proof. Suppose $M \in \overline{\mathcal{C}}$. Then we may write $0 \to C_d \to \cdots \to C_0 \to M \to 0$ with $C_i \in \mathcal{C}$ and $d = \mathcal{C}\text{-}\dim M$. Since each C_i is also in $\mathcal{D}$, we have $\mathcal{D}\text{-}\dim M \leq d$. Setting $e = \mathcal{D}\text{-}\dim M \leq d$, by Corollary 2.4, there exists a $D \in \mathcal{D}$ such that

$$0 \to D \to C_{e-1} \to \cdots \to C_0 \to M \to 0 \qquad 0 \to C_d \to \cdots \to C_e \to D \to 0$$

are exact. However, since $\mathcal{C}$ is thick in $\mathcal{D}$, D is also in $\mathcal{C}$, which implies that $e = d$, proving the second statement.

Assume R is Cohen-Macaulay. Let $\mathcal{C}$ be a resolving subcategory whose dimension satisfies the Auslander Buchsbaum formula. Then for any module $M \in \overline{\mathcal{C}} \cap \mathrm{MCM}$, we have

$$\operatorname{depth} R = \mathcal{C}\text{-}\dim M + \operatorname{depth} M = \mathcal{C}\text{-}\dim M + \operatorname{depth} R.$$

Thus $\mathcal{C}\text{-}\dim M = 0$ forcing M to be in $\mathcal{C}$. Hence $\mathcal{C}$ is thick in MCM.

Now we prove the converse. By what we have proved so far, it suffices to show that dimension with respect to MCM satisfies the Auslander Buchsbaum formula. Take an M in $\overline{\mathrm{MCM}}$. We will show that $\mathcal{C}\text{-}\dim M = d - \operatorname{depth} M$ by induction on $d - \operatorname{depth} M$. Suppose $d - \operatorname{depth} M = 0$. Then $M \in \mathrm{MCM}$. Now suppose $d - \operatorname{depth} M = n > 0$. Then $\operatorname{depth} M < d$ and so $\operatorname{depth} \Omega M = \operatorname{depth} M + 1$. Therefore $d - \operatorname{depth} M > d - \operatorname{depth} \Omega M$. So by induction, we have

$$\mathcal{C}\text{-}\dim M = \mathcal{C}\text{-}\dim \Omega M + 1 = d - \operatorname{depth} \Omega M + 1 = d - \operatorname{depth} M.$$

□

If dimension with respect to $\operatorname{add} \mathcal{C}_p$ satisfies the Auslander Buchsbaum formula for all $p \in \operatorname{spec} R$, then for all $\mathcal{X} \subseteq \overline{\mathcal{C}}$, $\Phi_{\mathcal{C}}(\mathcal{X})$ is in Γ. Before proceeding, we need one more definition and a result.

Definition 2.2.17. Let $\mathcal{A} \subseteq \mathcal{C}$. We say $\mathcal{A}$ *cogenerates* $\mathcal{C}$, if for every $C \in \mathcal{C}$, there exists an exact sequence $0 \to C \to A \to C' \to 0$ with $C' \in \mathcal{C}$ and $A \in \mathcal{A}$.

The following is an important theorem from (Auslander & Buchweitz, 1989, Theorem 1.1).

Theorem 2.2.18. *Suppose $\mathcal{X}$ and $\mathcal{A}$ are resolving with $\mathcal{A} \subseteq \mathcal{X}$. If $\mathcal{A}$ cogenerates $\mathcal{X}$, then for every $X \in \overline{\mathcal{X}}$ with $\mathcal{X}$-dim $X = n$, there exists a $Y \in \overline{\mathcal{A}}$ with $\mathcal{A}$-dim $Y = n$ and $Z \in \mathcal{X}$ such that $0 \to X \to Y \to Z \to 0$ is exact.*

Such an exact sequence is called an Auslander Buchweitz approximation.

2.3 Semidualizing modules

We fix a module $C \in \mathrm{mod}(R)$ and write $M^\dagger = \mathrm{Hom}(M, C)$.

Definition 2.3.1. A finitely generated module M is *totally C-reflexive* if it satisfies the following.

1. $\mathrm{Ext}^{>0}(M, C) = 0$
2. $\mathrm{Ext}^{>0}(M^\dagger, C) = 0$
3. The natural homothety map $\eta_M : M \to M^{\dagger\dagger}$ defined by $\mu \mapsto (\varphi \mapsto \varphi(\mu))$ is an isomorphism.

Let $\mathcal{G}_C$ denote the category of totally C-reflexive modules.

The category $\mathcal{G}_C$ is essentially the subcategory over which $\dagger$ is a dualizing functor. The notion of totally C-reflexivity generalizes the notion of Gorenstein dimension zero.

Definition 2.3.2. The category of $\mathcal{G}_R$ is the *category of Gorenstein dimension zero modules*. It is also known as the category of totally reflexive modules. Gorenstein dimension is the dimension with respect to the category of Gorenstein dimension zero modules.

See Masek (1999) for further information on Gorenstein dimension. The following proposition shows us that $\mathcal{G}_C$ is almost resolving.

Lemma 2.3.3. *The set $\mathcal{G}_C$ is closed under direct sums, summands, and extensions.*

Proof. It is easy to show that $\mathcal{G}_C$ is closed under direct sums and direct summands. Suppose we have

$$0 \to X \to Y \to Z \to 0$$

with $X, Z \in \mathcal{G}_C$. It is easy to check that Y satisfies condition (1) of Definition 3.1. We have

$$0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0 \qquad 0 \to X^{\dagger\dagger} \to Y^{\dagger\dagger} \to Z^{\dagger\dagger} \to 0.$$

From the first exact sequence, it is easy to see that Y satisfies condition (2) of Definition 3.1. We can then use the five lemma to show that Y satisfies condition (3) of Definition 3.1. □

In general, $\mathcal{G}_C$ will not be resolving. For example, if $C = R/xR$ for a regular element $x \in R$, then we have $\mathrm{Ext}^1(R/xR, R/xR) = R/xR \neq 0$. So R cannot be in $\mathcal{G}_{R/xR}$, and thus $\mathcal{G}_{R/xR}$ cannot be resolving. It is clear from the definition that $R \in \mathcal{G}_C$ is a necessary condition for $\mathcal{G}_C$ to be resolving. In fact, this condition is sufficient.

Proposition 2.3.4. *The category $\mathcal{G}_C$ is resolving if and only if $\mathcal{G}_C$ contains R.*

Proof. If $\mathcal{G}_C$ is resolving, by definition it contains R, so we prove the converse. So suppose R is in $\mathcal{G}_C$. In light of the last lemma, we need only to prove that if $0 \to X \to Y \to Z \to 0$ is exact with $Y, Z \in \mathcal{G}_C$, then X is in $\mathcal{G}_C$ as well. Since Y and Z satisfy condition (1) of Definition 3.1, it is easy to show that X does too. Also, since $\mathrm{Ext}^1(Z, C) = 0$, we have

$$0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0.$$

Hence, we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} & & \downarrow{\scriptstyle \eta_Z} & & \downarrow & & \\
0 & \longrightarrow & X^{\dagger\dagger} & \longrightarrow & Y^{\dagger\dagger} & \longrightarrow & Z^{\dagger\dagger} & \longrightarrow & \mathrm{Ext}^1(X^\dagger, C) & \longrightarrow & 0
\end{array}$$

Since η_Y and η_Z are isomorphisms, the five lemma shows that η_X is too, and that $\mathrm{Ext}^1(X^\dagger, C) =$

0. Thus X satisfies condition (3) of Definition 3.1. It is easy to check using the first exact sequence that $\operatorname{Ext}^{>1}(X^\dagger, C) = 0$, showing that X satisfies condition (2) of Definition 3.1. $\square$

Motivated by this proposition, we say a module C is semidualizing if R is in $\mathcal{G}_C$. This is easily seen to be equivalent to the following definition which is standard in the literature.

Definition 2.3.5. A module C is *semidualizing* if $\operatorname{Ext}^{>0}(C, C) = 0$ and $R \cong \operatorname{Hom}(C, C)$ via the map $r \mapsto (c \mapsto rc)$.

For the remainder of this section, we will let C denote a semidualizing module. Semidualizing modules were first discovered by Foxby in Foxby (1972) and were later rediscovered in different guises by various authors, including Vasconcelos in Vasconcelos (1974), who called them spherical modules, and Golod, who called them suitable modules. For an excellent treatment of the general theory of semidualizing modules, see Sather-Wagstaff (2009b). Since their discovery, semidualizing modules have been the subject of much research. See for example Araya & Takahashi (2009); Gerko (2005); Vasconcelos (1974); Nasseh & Sather-Wagstaff (2012); Jorgensen et al. (2012); Sather-Wagstaff (2009a).

Examples of semidualizing modules include R and dualizing modules. If R is Cohen-Macaulay and D is a dualizing module, then $\mathcal{G}_D$ is simply MCM. Dimension with respect to $\mathcal{G}_C$ is often called Gorenstein C-dimension, or G_C-dimension for short, since it is a generalization of Gorenstein dimension. We would expect G_C and Gorenstein dimension to have similar properties. Thus we have the following lemma and proposition, which the first of which is easy to see and the second from (Gerko, 2001, Theorem 1.22) respectively.

Lemma 2.3.6. *If $M \in \bar{\mathcal{G}_C}$, then $\mathcal{G}_C\text{-}\dim M = \min\{n \mid \operatorname{Ext}^{>n}(M, C)\} = 0$.*

Proposition 2.3.7. *For any semidualizing module C, G_C-dimension satisfies the Auslander Buchsbaum formula, i.e. for any module $M \in \bar{\mathcal{G}_C}$, we have*

$$\mathcal{G}_C\text{-}\dim M + \operatorname{depth} M = \operatorname{depth} R.$$

In light of Lemma 2.16, when R is Cohen-Macaulay this means that $\mathcal{G}_C$ is a thick subcategory of MCM. Interest in understanding G_C-dimension and the structure of $\mathcal{G}_C$ is not new. The following conjecture by Gerko from (Gerko, 2001, Conjecture 1.23) is equivalent to saying that $\mathcal{G}_R$ is a thick subcategory of $\mathcal{G}_C$.

Conjecture 2.3.8. *If C is semidualizing, then for any module M, $\mathcal{G}_C\text{-}\dim M \leq \mathcal{G}_R\text{-}\dim M$, and equality holds when both are finite.*

The category $\mathcal{G}_C$ has another interesting categorical property, namely that it has enough relative injectives. To see this, first note that any module in $\operatorname{add} C$ is an injective object in $\mathcal{G}_C$. Take any $M \in \mathcal{G}_C$. Then we have $0 \to \Omega M^\dagger \to R^n \to M^\dagger \to 0$ is exact. Since $R^\dagger \cong C$, applying $\dagger$ yields the exact sequence

$$0 \to M \to C^n \to (\Omega M^\dagger)^\dagger \to 0.$$

Using the language of Auslander & Buchweitz (1989), we can say that the category $\operatorname{add} C$ is an injective cogenerator of $\mathcal{G}_C$. Furthermore, if $F_\bullet$ is a projective resolution of $M^\dagger$ with $M \in \mathcal{G}_C$, then $F_\bullet^\dagger$ is an $\operatorname{add} C$ coresolution of M. Splicing this together with a free resolution $G_\bullet$ of M, we get what is called a complete PP_C or a complete P_C resolution of M. See White (2010) or Sather-Wagstaff (2009b) for more on the matter.

Semidualizing modules have interesting properties when we begin considering the geometry of the ring. Take, for example, the following lemma from (Sather-Wagstaff, 2007, Fact 2.4) and (Gerko, 2004, Theorem 3.1).

Lemma 2.3.9. *If C is a semidualizing R-module and R is a normal domain, then C is reflexive and hence an element of the class group.*

This is particularly interesting given the following result from Jorgensen et al. (2012).

Lemma 2.3.10. *If C is a semidualizing R-module and D is a dualizing module for R, then the homomorphism $\eta : C \otimes \operatorname{Hom}_R(C, D) \to D$ given by $x \otimes \varphi \mapsto \varphi(x)$ is an isomorphism.*

The map η being an isomorphism is a strong condition since D is torsionless and since tensor products often have torsion. Therefore, when R is normal, $\mathrm{Hom}(C, D)$ is the element of the class group associated with $C^{-1} \circ D$, and all three modules involved in Lemma 3.10 are elements of the class group.

We close this section by noting that many arguments involving semidualizing modules can be reduced to the complete case using the following result in (Sather-Wagstaff, 2009b, Proposition 2.2.1).

Lemma 2.3.11. *If $R \to S$ is a faithfully flat extension, then C is a semidualizing R-module if and only if $C \otimes S$ is a semidualizing S-module.*

2.4 The derived category

In homological algebra, in order to consider the derived functors of a module, one must consider its resolution. Therefore, to consider the derived functors over a complex, one must understand what a resolution of a complex should be. Moreover, if one's interest in a module is only in its derived functors, then the complex contains all the necessary information regarding the module, rendering the distinction between the two superfluous. Therefore it is reasonable to identify a module with its resolution. Such considerations are motivations for the derived category. The resulting category is triangulated.

The homotopy category

We begin with $\mathrm{Ch}(R)$, the category of complexes of (not necessarily finitely generated) R-modules. The morphisms of this category are chain maps.

Definition 2.4.1. For any complexes $A_\bullet$ and $B_\bullet$, a *chain map* $\varphi : A_\bullet \to B_\bullet$ is a collection of homomorphisms $\{\varphi_n : A_n \to C_n\}$ such that the following diagram commutes.

$$\begin{array}{ccc} A_n & \xrightarrow{\partial^A} & A_{n-1} \\ \Big\downarrow{\scriptstyle \varphi_n} & & \Big\downarrow{\scriptstyle \varphi_{n-1}} \\ B_n & \xrightarrow{\partial^B} & B_{n-1} \end{array}$$

The category $\mathrm{Ch}(R)$ is Abelian. It is well known that for a chain map $\varphi : A_\bullet \to B_\bullet$, there is a homomorphism $H_n(\varphi) : H_n(A_\bullet) \to H_n(B_\bullet)$ for all $n \in \mathbb{Z}$. Furthermore, one can show that $H_i : \mathrm{Ch}(R) \to \mathrm{Mod}(R)$ is a functor.

Definition 2.4.2. A chain map $\varphi : A_\bullet \to B_\bullet$ with $A_\bullet, B_\bullet \in \mathrm{Ch}(R)$ is a *quasi-isomorphism* if the induced homomoprhism $H_n(\varphi) : H_n(A_\bullet) \to H_n(B_\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$.

A *projective resolution* of a complex $A_\bullet$ is a quasi-isomorphism $\pi : P_\bullet \to A_\bullet$ where $P_\bullet$ is a complex of projectives.

Example 2.4.3. Let M be an R-module. Let $\cdots P_1 \to P_0 \to M \to 0$ be a projective resolution. The following diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

gives a chain map $P_\bullet \to M$ where we view M as a complex with only one nonzero term. This chain map is actually a quasi-isomorphism because then $H_0(P_\bullet) = M$ and $H_i(P_\bullet) = 0$ for all $i \neq 0$.

It easy to see that a projective resolution of a complex $A_\bullet$ exists if $A_n = 0$ for $n \ll 0$. Thus we define $\mathrm{Ch}^+(R)$ to be the complexes satisfying this condition. Now chain complexes are only unique up to homotopy, a definition we recall below.

Definition 2.4.4. Two chain maps φ, ψ are *homotopic*, written $\varphi \sim \psi$, if there exists a set of homomorphisms $\sigma = \{\sigma_n : A_n \to B_{n+1}\}$ such that:

$$\varphi_n - \psi_n = \partial^B_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial^A_n$$

Note that we call σ a homotopy from φ to ψ.

Two complexes $A_\bullet$ and $B_\bullet$ are homotopic if there exist chain maps $\varphi : A_\bullet \to B_\bullet$ and $\psi : B_\bullet \to A_\bullet$ such that $\psi\varphi \sim \mathbf{id}_{A_\bullet}$ and $\varphi\psi \sim \mathbf{id}_{B_\bullet}$.

Let $K(R)$ denote the homotopy category, the category whose objects are complexes of R-modules and whose morphisms are chain maps modulo homotopy equivalence. One must check that compositions in this category are well defined, but such an exercise is routine. We let $K^+(\mathcal{P}(R))$ denote the subcategory of $K(R)$ consisting of complexes of projective modules. We can now give a categorical description of projective resolutions.

Proposition 2.4.5. *The mapping $\rho : \mathrm{Ch}^+(R) \to \mathcal{K}^+(\mathcal{P}(R))$ where $\rho(A_\bullet)$ is a projective resolution defines a functor.*

To prove this proposition one must show that any chain map $\varphi : A_\bullet \to B_\bullet$ lifts to a chain map $\rho(\varphi) : \rho(A_\bullet) \to \rho(B_\bullet)$ and that such a lift is unique up to homotopy. This category has two interesting features.

Definition 2.4.6.

1. The *shift functor* $T : K(R) \to K(R)$ is given by $(TA)_n = A_{n-1}$ and $\partial^{TA}_n = \partial^A_{n-1}$.

2. Given a chain map $\varphi : A_\bullet \to B_\bullet$ we define the complex $\mathrm{cone}(\varphi)_n = A_{n-1} \oplus B_n$ and

$$\partial^{\mathrm{cone}(\varphi)}_n(a, b) = (-\partial^A_{n-1}(a), \partial^B_n(b) - \varphi(a)).$$

Note that we have an exact sequence $0 \to B_\bullet \to \mathrm{cone}(\varphi) \to TA_\bullet \to 0$ of complexes, i.e we have an short exact sequence of modules $0 \to B_n \to \mathrm{cone}(\varphi)_n \to (TA)_n \to 0$ for all $n \in \mathbb{Z}$. This proves the following.

Lemma 2.4.7. *If $\varphi : A_\bullet \to B_\bullet$ is a chain map, we have the long exact sequence*

$$\cdots \to H_{n-1}(\mathrm{cone}(\varphi)) \to H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(\mathrm{cone}(\varphi)) \to H_{n+1}(A_\bullet) \to \cdots$$

Also note that we have a sequence of chain of maps

$$\cdots \to T^{-1}\,\mathrm{cone}(\varphi)_\bullet \to A_\bullet \xrightarrow{\varphi} B_\bullet \to \mathrm{cone}(\varphi)_\bullet \to TA_\bullet \xrightarrow{T\varphi} TB_\bullet \to T\,\mathrm{cone}(\varphi)_\bullet \to T^2A \to \cdots$$

which is almost never an exact sequence.

Example 2.4.8. If $f : M \to N$ is a module homomorphism, then considering M and N as complexes gives

$$\mathrm{cone}(f) = \cdots 0 \to 0 \to M \xrightarrow{f} N \to 0 \to 0 \to \cdots .$$

Note that $H_1(f) = \ker f$ and $H_0(f) = \operatorname{coker} f$.

The category $\mathrm{Ch}(R)$ is Abelian. However, the homotopy category is not Abelian because kernels and cokernels are not well defined up to homotopy. To remedy this, we replace these notions with the mapping cone in Definition 4.6. Example 4.8 provides a philosophical justification for this viewpoint because here the mapping cone contains the information of both the kernel and cokernel. In order to do this, we must also develop a notion that replaces that of a short exact sequence.

Definition 2.4.9. A *triangle* is a sequence of complexes and maps $A_\bullet \to B_\bullet \to C_\bullet \to TA_\bullet$. Two triangles $A_\bullet \to B_\bullet \to C_\bullet \to TA_\bullet$ and $A'_\bullet \to B'_\bullet \to C'_\bullet \to TA'_\bullet$ are isomorphic in $K(R)$ if there exists a commutative diagram in $K(R)$

$$\begin{array}{ccccccc} A_\bullet & \longrightarrow & B_\bullet & \longrightarrow & C_\bullet & \longrightarrow & TA_\bullet \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ A'_\bullet & \longrightarrow & B'_\bullet & \longrightarrow & C'_\bullet & \longrightarrow & TA'_\bullet \end{array}$$

where the vertical arrows are isomorphisms in $K(R)$ (i.e. homotopy equivalences). We may abbreviate a triangle by writing $A_\bullet \to B_\bullet \to C_\bullet \to$ instead of $A_\bullet \to B_\bullet \to C_\bullet \to TA_\bullet$.

Proposition 2.4.10. *If $\varphi, \psi : A_\bullet \to B_\bullet$ are homotopic, then the triangles $A_\bullet \xrightarrow{\varphi} B_\bullet \to \operatorname{cone}(\varphi) \to$ and $A_\bullet \xrightarrow{\psi} B_\bullet \to \operatorname{cone}(\psi) \to$ are isomorphic in $K(R)$. In particular, mapping cones are well defined in $K(R)$.*

Therefore, the mapping cone is an invariant of morphisms in the homotopy category.

Definition 2.4.11. A triangle is *exact* if it is isomorphic in $K(R)$ to a triangle of the form $A_\bullet \xrightarrow{\varphi} B_\bullet \to \operatorname{cone}(\varphi) \to$.

The reader should be cautioned that these triangles are not exact in any traditional homological sense. The term exact is used as an analogy because, as we will see, exact triangles behave similarly to short exact sequences.

Triangulated categories

We would like to perform some sort of homological algebra in the homotopy category. Since it is not Abelian, we need to recognize a different type of structure on the category. Thus we must begin looking at triangulated categories.

Definition 2.4.12. Let $\mathcal{T}$ be an additive category equipped with a functor $T : \mathcal{T} \to \mathcal{T}$ which is an equivalence of categories. Let $\mathbb{T}$ be a class of triangles closed under isomorphisms (see Definition 4.9). A triangle in $\mathbb{T}$ will be referred to as exact. The tuple $(\mathcal{T}, T, \mathbb{T})$ is a *triangulated category* if the following axioms hold.

1. Every morphism $\varphi : A \to B$ can be completed to an exact triangle $A \xrightarrow{\varphi} B \to C \to$. Furthermore, $A \xrightarrow{\mathbf{id}_A} A \to 0 \to$ is exact.

2. A triangle $A \to B \to C \to$ is exact if and only if $B \to C \to TA \to$ is exact.

3. Given exact triangles $A \to B \to C \to$ and $A' \to B' \to C' \to$ and a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & & & \downarrow{\scriptstyle T\alpha} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

there exists a morphism $\gamma : C \to C$ making the diagram commute.

4. Given the commutative diagram

$$\begin{array}{ccccccc} A & = & A & & & & \\ \downarrow & & \downarrow & & & & \\ B & \longrightarrow & D & \longrightarrow & E & \longrightarrow & TB \\ \downarrow & & \downarrow & & & & \downarrow \\ C & \longrightarrow & F & \longrightarrow & G & \longrightarrow & TC \\ \downarrow & & \downarrow & & & & \\ TA & = & TA & & & & \end{array}$$

where the first two columns and second two rows exact, there exists an isomorphism $E \to G$ which makes the diagram commute.

Theorem 2.4.13. *The category $K(R)$ is triangulated.*

Most of the proof is routine. See (Weibel, 1994, Proposition 10.2.4) for a proof. We now state basic results on triangulated categories.

Lemma 2.4.14. *For any exact triangle $A \xrightarrow{\alpha} B \to C \to$, α is an isomorphism if and only if $C \cong 0$.*

Proof. Suppose that $\alpha : A \to B$ is an isomorphism. Then we have the following isomorphism of triangles

$$\begin{array}{ccccccc} A & = & A & \longrightarrow & 0 & \longrightarrow & TA \\ \| & & \downarrow{\scriptstyle \alpha} & & \| & & \| \\ A & \xrightarrow[\alpha]{} & B & \longrightarrow & 0 & \longrightarrow & TA \end{array} .$$

Since exact triangles are closed under isomorphism, the triangle $A \xrightarrow{\alpha} B \to 0 \to$ is exact.

Conversely, uppose that $A \xrightarrow{\alpha} B \to 0 \to$ is a triangle. By Axiom 3, we have a commutative diagram

$$\begin{CD}
0 @>>> A @>>{\alpha}> B @>>> 0 \\
@VVV @| @VV{\beta}V @VVV \\
0 @>>> A @= A @>>> 0
\end{CD}$$

and we can see that $\beta\alpha = \mathbf{id}_A$. Axiom 4 yields the commutative diagram with exact rows and columns.

$$\begin{CD}
0 @= 0 @. @. \\
@VVV @VVV @. @. \\
A @= A @>>> 0 @>>> TA \\
@VV{\alpha}V @| @| @VV{T\alpha}V \\
B @>>{\beta}> A @>>> 0 @>>> TB \\
@VVV @VVV @. @. \\
@. @. @. @.
\end{CD}$$

Therefore $B \xrightarrow{\beta} A \to 0 \to$ is exact. Now we have the commutative diagram

$$\begin{CD}
0 @= 0 @. @. \\
@VVV @VVV @. @. \\
0 @>>> B @= B @>>> 0 \\
@VVV @VV{\beta}V @. @VVV \\
0 @>>> A @>>{\alpha}> B @>>> 0 \\
@VVV @VVV @. @. \\
@. @. @. @.
\end{CD}$$

and Axiom 4 thus implies that there is an isomorphism $\zeta : B \to B$ which equals $\alpha\beta$. It follows that α is an isomorphism as desired.

□

Lemma 2.4.15.

Given exact triangles $A \to B \to C \to TA$ *and* $A' \to B' \to C' \to TA'$ *and a commutative* *diagram*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\wr} & & {\scriptstyle\beta}\downarrow{\scriptstyle\wr} & & & & {\scriptstyle T\alpha}\downarrow{\scriptstyle\wr} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

with the vertical arrows isomorphisms, there exists an isomorphism $\gamma : C \to C$ *making the diagram commute. In particular, for a* $\varphi : A \to B$*, the object* C *guaranteed by Axiom 3 is unique up to isomorphism, which we denote* $\mathrm{cone}(\varphi) = C$.

Proof. By the previous lemma, we have the commutative diagram

$$\begin{array}{ccccccc} 0 & = & 0 & & & & \\ \downarrow & & \downarrow & & & & \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\wr} & & {\scriptstyle\beta}\downarrow{\scriptstyle\wr} & & & & {\scriptstyle T\alpha}\downarrow{\scriptstyle\wr} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

with the first two columns and second two rows exact. Axiom 4 implies the existence of the desired isomorphism.

□

Unfortunately, the isomorphism in this lemma is not unique. This is because the morphism guaranteed is not unique, see (Kashiwara & Schapira, 2006, Proposition 10.1.17). This can be the cause of great consternation, as we will see in Chapter 5 and Chapter 6.

Axiom 4 seems opaque, but it is the mechanism behind most proofs involving triangulated categories. One can think of it as the triangulated version of the Snake lemma. The following result demonstrates the utility of the axiom.

Lemma 2.4.16. *Suppose a category* $\mathcal{T}$ *satisfies Axioms 1,2 and 3. Then Axiom 4 is equivalent to the following Axiom.*

4′. *Suppose we have the commutative diagram*

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A'' & \underset{\alpha}{\longrightarrow} & B'' & & C'' & & TA'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
TA & \longrightarrow & TB & \longrightarrow & TC & \longrightarrow & T^2A
\end{array} \tag{2.1}$$

where each column is an exact triangle and the first, second and fourth rows are exact triangles. Then there exists map β *and* γ *such that* $A'' \xrightarrow{\alpha} B'' \xrightarrow{\beta} C'' \xrightarrow{\gamma} TA''$ *is exact and which fits into the commutative diagram 1.*

Proof. First assume Axiom 4′. Note that one can prove Lemma 4.14 using Axiom 4′, and so we will use it freely. Consider the following commutative diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\
\downarrow & & \downarrow & & \| & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & TA' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A'' & & B'' & & 0 & & TA'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
TA & \longrightarrow & TB & \longrightarrow & TC & \longrightarrow & T^2
\end{array}$$

where each column is an exact triangle and the first, second and fourth rows are exact triangles. By Axiom 3, there exists an $\alpha : A'' \to B''$ that makes the diagram commute. By the above statement, $A'' \xrightarrow{\alpha} B'' \to 0 \to$ is exact. Therefore, α is an isomorphism proving that Axiom 4 holds.

Now we show the converse. Assume that $\mathcal{T}$ is triangulated and that we have diagram 1. Set $X = \text{cone}(\alpha)$. We have the triangle, $A'' \xrightarrow{\alpha} B'' \xrightarrow{\beta} X \to$. To show that Axiom 4′

holds, we need an isomorphism $\nu : X \to C''$ the exact triangle $A'' \xrightarrow{\alpha} B'' \xrightarrow{\nu\beta} C'' \to$ fits into commutative diagram 1.

Composing the maps $B' \to C'$ and $C' \to C''$ gives us a morphism $\lambda : B' \to C''$. Set $M = \operatorname{cone} \lambda$. Consider the commutative diagram

$$\begin{array}{ccccccc}
B' & = & B' & & & & \\
\downarrow & & \downarrow{\scriptstyle \lambda} & & & & \\
C' & \longrightarrow & C'' & \longrightarrow & TC & \longrightarrow & TC' \\
\downarrow & & \downarrow & & & & \downarrow \\
TA' & \underset{\kappa}{\longrightarrow} & M & \longrightarrow & D & \longrightarrow & T^2A' \\
\downarrow & & \downarrow & & & &
\end{array}$$

where κ is the map induced by Axiom 3 and $D = \operatorname{cone}(\kappa)$. By Axiom 4, there is a isomorphism between TC and D making the diagram commute. Therefore, $C \to TA' \xrightarrow{\kappa} M \to$ is an exact triangle. Now consider the commutative diagram

$$\begin{array}{ccccccc}
C & = & C & & & & \\
\downarrow & & \downarrow & & & & \\
TA & \longrightarrow & TA' & \longrightarrow & TA'' & \longrightarrow & T^2A \\
\downarrow & & \downarrow{\scriptstyle \kappa} & & & & \downarrow \\
TB & \underset{\mu}{\longrightarrow} & M & \longrightarrow & E & \longrightarrow & T^2B \\
\downarrow & & \downarrow & & & &
\end{array}$$

where μ is the map induced by Axiom 3 and $E = \operatorname{cone}(\mu)$. By Axiom 4, there is a isomorphism between TA'' and E making the diagram commute. Therefore,

$$B \xrightarrow{T^{-1}\mu} T^{-1}M \to A'' \to$$

is an exact triangle. Given the commutative diagram

$$\begin{array}{ccccccc}
B & = & B & & & & \\
\downarrow{\scriptstyle T^{-1}\mu} & & \downarrow & & & & \\
T^{-1}M & \longrightarrow & B' & \xrightarrow[\lambda]{} & C'' & \longrightarrow & M \\
\downarrow & & \downarrow & & & & \downarrow \\
A'' & \longrightarrow & B'' & \longrightarrow & X & \longrightarrow & T^2B \\
\downarrow & & \downarrow & & & &
\end{array}$$

Axiom 4 guarantees the existence of an isomorphism ν from C'' to X such that the diagram commutes. By factoring λ into the original maps $B' \to C'$ and $C' \to C''$ and examining the above diagrams, one can see that ν is the desired isomorphism.

□

We thus can give an alternate characterization of triangulated categories.

Theorem 2.4.17. *Let $\mathcal{T}$ be an additive category equipped with a functor $T : \mathcal{T} \to \mathcal{T}$ which is an equivalence of categories. Let $\mathbb{T}$ be a class of triangles closed under isomorphisms. A triangle in $\mathbb{T}$ will be referred to as exact. The tuple $(\mathcal{T}, T, \mathbb{T})$ is a triangulated category if and only if the following axioms hold.*

1. *The triangle $A \xrightarrow{id_A} A \to 0 \to$ is exact.*

2. *A triangle $A \to B \to C \to$ is exact if and only if $B \to C \to TA \to$ is exact.*

3. *Every commutative square*

$$\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & B'
\end{array}$$

can be extended to a commutative diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & TA'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
TA & \longrightarrow & TB & \longrightarrow & TC & \longrightarrow & T^2A
\end{array}$$

where all the rows and columns are exact.

Verdier quotients

Triangulated categories have interesting subcategories.

Definition 2.4.18. A full subcategory $\mathcal{X}$ of a triangulated category $\mathcal{T}$ is *thick* if

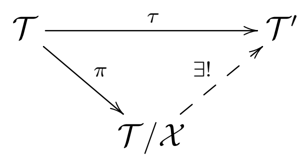
1. If $A \to B \to C \to$ is exact and $A, B \in \mathcal{X}$, then $C \in \mathcal{C}$. Hence $\mathcal{X}$ is closed under cones.
2. We have $A \coprod B \in \mathcal{X}$ if and only if $A, B \in \mathcal{X}$.

In other words, thick subcategories are idempotent complete subtriangulated categories.

We can define the quotient of triangulated categories by a thick subcategory, which is a unique categorical feature.

Definition 2.4.19. An additive functor $\tau : \mathcal{T} \to \mathcal{T}'$ is *triangulated* if $\tau \circ T_{\mathcal{T}} = T_{\mathcal{T}'} \circ \tau$ and for every $\varphi : A \to B$, $\operatorname{cone}(\tau\varphi) = \tau(\operatorname{cone}\varphi) : \tau(A) \to \tau(B)$.

Definition 2.4.20. Let $\mathcal{X}$ be a thick subcategory of a triangulated category $\mathcal{T}$. The *Verdier quotient* of $\mathcal{T}$ by $\mathcal{X}$ is a triangulated category $\mathcal{T}/\mathcal{X}$ and a triangulated functor $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{X}$ such that $\pi(\mathcal{X}) = 0$ and any triangulated functor $\tau : \mathcal{T} \to \mathcal{T}'$ such that $\tau(\mathcal{X}) = 0$ factors uniquely through π.



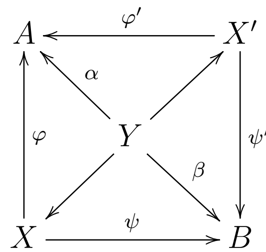
It is far from clear that Verdier quotients exist. In light of Lemma 4.14, the condition $\tau(\mathcal{X}) = 0$ is equivalent to saying that $\tau(\varphi)$ is invertible whenever $\operatorname{cone}(\varphi) \in \mathcal{X}$. Because of this, we can actually construct Verdier quotients using categorical localization.

Localizing a category is an extension of the classical theory of localization in commutative algebra. Let $\mathcal{C}$ be a category and S a collection of morphisms in $\mathcal{C}$. In certain circumstances, one can construct $S^{-1}\mathcal{C}$ by formally introducing an inverse s^{-1} for each element $s \in S$. Specifically, $S^{-1}\mathcal{C}$ is the category whose objects are the same as $\mathcal{C}$ and whose morphisms are formal sequence $r_1 s_1^{-1} r_2 s_2^{-1} \cdots r_n s_n^{-1} r_{n+1}$ where $s_i \in S$ and $r_i \in R$ modulo an equivalence relation which forces $ss^{-1} = \mathbf{id}$ for each $s \in S$. See Gabriel & Zisman (1967) or (Weibel, 1994, Section 10.3) for a rigorous construction. Unfortunately, there are foundational considerations which sometimes prevents $S^{-1}\mathcal{C}$ from being a category. Namely, the class $\operatorname{Hom}_{S^{-1}\mathcal{C}}(A, B)$ might not be a set, a necessary condition for a category. Luckily, in the case of thick categories we can perform this construction.

Theorem 2.4.21. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{X} \subseteq \mathcal{T}$ a thick subcategory. Let S be the collection of morphisms of T whose cone lies in $\mathcal{X}$. The Verdier quotient $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{X}$ and the localization $S^{-1}\mathcal{T}$ both exist and $\mathcal{T}/\mathcal{X} \cong S^{-1}\mathcal{T}$. Also, $\pi(A) = 0$ if and only if $A \in \mathcal{T}$. Moreover, for any two objects $A, B \in \mathcal{T}$, $\operatorname{Hom}_{\mathcal{T}/\mathcal{X}}(A, B)$ is the set of equivalent roof diagrams (see the following definition).*

See (Neeman, 2001, Theorem 2.1.8) or (Weibel, 1994, Proposition 10.4.1) for a proof.

Definition 2.4.22. In the set up of the previous theorem, a *roof diagram* from A to B is pair of morphisms $A \xleftarrow{\varphi} X \xrightarrow{\psi} B$ with $\varphi \in S$. Two roof diagrams $A \xleftarrow{\varphi} X \xrightarrow{\psi} B$ and $A \xleftarrow{\varphi'} X' \xrightarrow{\psi'} B$ are equivalent if there is a roof diagram $A \xleftarrow{\alpha} Y \xrightarrow{\beta} B$ fitting into the following commutative diagram.



Roof diagrams are really just fractions. Indeed, the roof diagram $A \overset{\varphi}{\leftarrow} X \overset{\psi}{\rightarrow} B$ is the fraction $\psi\varphi^{-1}$. Furthermore, Theorem 4.21 tells us that in $S^{-1}\mathcal{T}$, the formal sequence $r_1 s_1^{-1} r_2 s_2^{-1} \cdots r_n s_n^{-1} r_{n+1}$ where $s_i \in S$ and $r_i \in R$ is equivalent to a sequence, or fraction, $r_1 s_1^{-1}$. It is here that one can see the parallels between categorical localization and localization in the commutative case.

The Derived Category

Our original goal was to identify complexes with their projective resolutions. We will do this using the Verdier quotients in the previous subsection. A complex is acyclic if all of its homologies are trivial.

Definition 2.4.23. Let $\mathcal{X} \subseteq K(R)$ be the subcategory of acyclic complexes. By Lemma 4.7, this subcategory is thick. Let $D(R) = K(R)/\mathcal{X}$, be the Verdier quotient. We call this the *derived category of R.*

From Theorem 4.21, the derived category exists and is triangulated.

Remark 2.4.24. A morphism φ is invertible in $D(R)$ if and only if $\operatorname{cone}(\varphi)$ is acyclic. Thus by Lemma 4.7, a morphism φ is invertible in $D(R)$ if and only if it is a quasi-isomorphism. Therefore, the derived category can be constructed by inverting all the quasi-isomorphisms in $K(R)$. However, we can now see a short cut in constructing the derived category. Indeed $D(R) = S^{-1}\operatorname{Ch}(R)$ where S is the collection of quasi-isomorphisms.

Unfortunately, since the morphisms of the derived category are roof diagrams and not chain maps, the derived category can sometimes be difficult to work with. Thus the following result can be very useful.

Proposition 2.4.25 ((Weibel, 1994, Corollary 10.4.7))**.** *There natural functor* $K^+(R) \to D(R)$ *is a full and faithful functor. In particular, if* $A_\bullet$ *and* $B_\bullet$ *are two complexes in* $\mathrm{Ch}^+(R)$ *and* $P_\bullet$ *and* $Q_\bullet$ *are their projective resolutions, then* $\mathrm{Hom}_{D(R)}(A_\bullet, B_\bullet) \cong \mathrm{Hom}_{K(\mathcal{P}(R))}(P_\bullet, Q_\bullet)$.

From this proposition, we can see the following result.

Proposition 2.4.26. *If* M, N *are* R*-modules, then* $\mathrm{Hom}_{D(R)}(M, T^n N) \cong \mathrm{Ext}^n(M, N)$.

One can perform the this same construction using various thick subcategories of $K(R)$. For example, $D^+(R)$ is the quotient of $K^+(R)/(\mathcal{X} \cap K^+(R))$. Also we may let $\mathrm{Ch}^b(R)$ be the category of bounded chain complexes of finitely generated modules and then define $K^b(R)$ and $D^b(R)$ similarly. The category $D^b(R)$ is called the *bounded derived category of* R. It turns out that $D^+(R)$ and $D^b(R)$ are equivalent to thick subcategories of $D(R)$.

Furthermore, given any exact category $\mathcal{E}$, i.e. an extension closed full subcategory of an Abelian category, one can define $\mathrm{Ch}(\mathcal{E})$ and $\mathrm{Ch}^b(\mathcal{E})$ thus also $K(\mathcal{E})$ and $K^b(R)$. Therefore, one can also construct the derived category $D(\mathcal{E})$ and bounded derived category $D^b(\mathcal{E})$ for any exact category. For example, $D^b(\mathcal{P})$ is the category of perfect complexes.

2.5 Cohomological supports

The study of cohomological supports over complete intersection rings was initiated by Avramov and Buchweitz in Avramov & Buchweitz (2000b). For the entirety of this section, $(R, \mathfrak{m}, k)$ will be a local complete intersection of codimension c such that $\hat{R} = Q/(f_1, \ldots, f_c)$ for a regular local ring Q and a regular sequence $\underline{f} = f_1, \ldots, f_c$ not contained in the square of the maximal ideal of Q. Let $\tilde{k}$ be the algebraic closure of k and set $T = \hat{R}$. The cohomological support of a finitely generated T-module X is essentially the support of $\mathrm{Ext}(X, k)$ as a module over the polynomial ring $S = k[\chi_1, \ldots, \chi_c]$.

We recall a construction from Eisenbud (1980), which gives the action of S on $\operatorname{Ext}(X,k)$. Eisenbud's construction was motivated by Gulliksen (1974). Let $(F_\bullet, \partial)$ be a free resolution of X over T. Each $F_n = T^{i_n}$ and we may view ∂ is a sequence of matrices with entries in T. Let $\tilde{F}_n = Q^{i_n}$ and $\tilde{\partial}$ be the lift of ∂ to $\tilde{F}_\bullet$. Since $\partial^2 = 0$, we know that $\tilde{\partial}^2$ is a sequence of matrices whose entries are in the ideal $(f_1, \dots, f_c)$. Thus we may write

$$\tilde{\partial}^2 = \sum_{i=1}^{c} f_i \tilde{\Phi}_i$$

where $\tilde{\Phi}_i$ is a sequence of matrices with entries in Q. Set $\Phi_i = \tilde{\Phi}_i \otimes T$. We may now view $\oplus_{n=0}^{\infty} F_n$ as a module over $T[\chi_1, \dots, \chi_c]$ where $\chi_i r = \Phi_i(r)$ for every $r \in F_n$. Furthermore, this defines an action of $T[\chi_1, \dots, \chi_c]$ on $\operatorname{Ext}(X,k) = \bigoplus_{i=0}^{\infty} \operatorname{Ext}^i(X,k)$. This action actually turns $\operatorname{Ext}(X,k)$ into a graded S-module, where each χ_i is degree 2. It is shown in Eisenbud (1980) that the operators Φ_i commute and that this action is independent of our choices of $F_\bullet$ and $\tilde{\Phi}_i$. Furthermore, in the same work, Eisenbud show that $\operatorname{Ext}(X,k)$ is a finitely generated over S. The ring S is known as the ring of cohomological operators and has been studied in many works, including Avramov (1989); Avramov & Buchweitz (2000a); Avramov et al. (1997); Eisenbud (1980); Mehta (1976). There are alternative ways one can construct this action. See Avramov & Sun (1998) for a detailed discussion. In this notation, we define

$$V(Q,T;X) = \{\bar{a} \in \mathbb{A}^c_{\tilde{k}} \mid g(\bar{a}) = 0 \quad \forall g \in \operatorname{ann}_S \operatorname{Ext}(X,k)\}$$

where $\tilde{k}$ is the algebraic closure of k. This set is a cone in $\mathbb{A}^c_{\tilde{k}}$.

Definition 2.5.1. Let R be a complete intersection ring. For a finitely generated R-module M, the *cohomological support*, denoted $V^*(M)$, is the projectivization in $\mathbb{P}^{c-1}_{\tilde{k}}$ of the cone $V(T,Q;\hat{M})$.

Remark 2.5.2. What we call the cohomological support is referred to as the support variety in Avramov & Buchweitz (2000b). In Avramov & Iyengar (2007), the terminology cohomological support and cohomological variety are both used. Since geometers generally require varieties to be irreducible closed subsets and since $V^*(M)$ is generally not irreducible, we have decided to use the term cohomological support.

Remark 2.5.3. In Avramov & Buchweitz (2000b) and in other works, the authors consider $V^*(M)$ as a cone in $\mathbb{A}^c_{\tilde{k}}$. To facilitate the statement of certain results, we have found it easiest to work in projective space.

The following is a combination of the results (Avramov & Buchweitz, 2000b, Theorem 5.6,Theorem 6.1).

Theorem 2.5.4. *For finitely generated R-modules M and N, the following are equivalent.*

1. $V^*(M) \cap V^*(N) = \emptyset$
2. $\operatorname{Tor}_{\gg 0}(M, N) = 0$
3. $\operatorname{Ext}^{\gg 0}(M, N) = 0$
4. $\operatorname{Ext}^{\gg 0}(N, M) = 0$

Hence cohomological supports encode homological information about a module. The following result gives another description of cohomological supports.

Theorem 2.5.5 ((Avramov & Buchweitz, 2000b, Theorem 5.2),(Avramov, 1989, Corollary 3.11))**.** *Suppose that the residue field k is algebraically closed. For any module $M \in \operatorname{mod} R$, we have*

$$V^*(M) = \{(a_1, \ldots, a_c) \in \mathbb{P}^{c-1}_k \mid \operatorname{pd}_{Q/(\tilde{a}_1 f_1 + \cdots + \tilde{a}_c f_c)} \hat{M} = \infty\}$$

where $\tilde{a}_i$ is a lift in Q of a_i.

From this result and Lemma 5.19, can easily deduce these corollaries.

Corollary 2.5.6. *For a finitely generated R-module M, $V^*(M) = \emptyset$ if and only if* $\operatorname{pd} M < \infty$*. Also $V^*(k) = \mathbb{P}^{c-1}_{\tilde{k}}$.*

Corollary 2.5.7. *Let H be the hyperplane defined by $\chi_1 = 0$. For any module M over T, the cohomological support $V^*_{Q/(f_2,\dots,f_c)}(M)$ computed over $Q/(f_2,\dots,f_c)$ is equal to $V^*_T(M) \cap H$.*

A generalization of Corollary 5.6 exists involving complexity.

Definition 2.5.8. For a sequence $(a_n)_{n\geq 0}$ of nonnegative integers, we can define the *complexity*

$$\operatorname{cx}(a_n)_{n\geq 0} = \min\{\deg f \mid f \in \mathbb{Q}[t] \quad a_n \leq f(n) \quad \forall n \gg 0\} + 1.$$

For a module M, we set $\operatorname{cx} M = \operatorname{cx} \beta_n(M)$.

A module has finite projective dimension if and only if $\operatorname{cx} M = 0$. Since R is a complete intersection of codimension c, $\operatorname{cx} k = c$.

Proposition 2.5.9 ((Avramov & Buchweitz, 2000b, Theorem 5.6))**.** *For any R-module, we have* $\dim V^*(M) = \operatorname{cx} M - 1$

Remark 2.5.10. Note that the previous result considers $V^*(M)$ as a closed set of projective space instead of a cone in affine space.

An obvious question regarding our choice of cohomological supports is the affect of changing the regular sequence $f_1, \dots, f_c$. To that end, we have the following result due to Eisenbud.

Theorem 2.5.11 ((Eisenbud, 1980, Proposition 1.7), cf. (Avramov & Sun, 1998, (3.11)))**.** *Let $f_1, \dots, f_c$ and $f'_1, \dots, f'_c$ be two regular sequences of Q which generate the same ideal. Write*

$$f_i = \sum_{j=1}^{c} q_{i,j} f'_j$$

with each $q_{i,j} \in Q$. Letting $\chi_1, \dots, \chi_c$ and $\chi'_1, \dots, \chi'_c$ be the cohomological operators associated to $f_1, \dots, f_c$ and $f'_1, \dots, f'_c$ respectively, we have

$$\chi'_j = \sum_{i=1}^{c} q_{i,j} \chi_i$$

Thus the matrix $(q_{i,j})$ acts as a change of basis matrix, changing the coordinates of $\mathbb{P}^c_{\tilde{k}}$. When $k = \tilde{k}$, any change of coordinates of $\mathbb{P}^{c-1}_{\tilde{k}}$ corresponds to choosing a different regular sequence which generates the ideal $(f_1, \ldots, f_c)$. This important fact is critical to several proofs in this document, thus we state it precisely.

Proposition 2.5.12. *Assume that k is algebraically closed and set $I = (f_1, \ldots, f_c)$. Let $\varphi : \mathbb{P}^{c-1}_k \to \mathbb{P}^{c-1}_k$ be an automorphism, i.e. a change of coordinates. Then there exists a regular sequence $f_1', \ldots, f_c'$ generating I such that $\varphi_*(\chi_i) = \chi_i'$ where $\chi_1, \ldots, \chi_c$ and $\chi_1', \ldots, \chi_c'$ are the cohomological operators associated to $f_1, \ldots, f_c$ and $f_1', \ldots, f_c'$ respectively.*

Proof. Set $\psi = \varphi^{-1}$, and let $\tilde{\varphi}$ and $\tilde{\psi}$ be the lifts of φ and ψ in Q such that $\tilde{\phi} = \tilde{\varphi}^{-1}$. We can regard $\tilde{\varphi}$ and and $\tilde{\psi}$ as a matrices whose entries are $q_{i,j} \in Q$ and $p_{i,j} \in Q$ respectively. Set

$$f_i' = \sum_{i=1}^{c} p_{i,j} f_j.$$

By Nakayama's lemma, $f_1', \ldots, f_c'$ generates I, and since there are c elements, $f_1', \ldots, f_c'$ is necessarily a regular sequence. However since $\tilde{\varphi}\tilde{\psi}$ is the identity, we also have

$$f_i = \sum_{j=1}^{c} q_{i,j} f_j'$$

It follows from Theorem 5.11 that

$$\chi_j' = \sum_{i=1}^{c} q_{i,j} \chi_i = \varphi_*(\chi_j)$$

□

There is a deep connection between cohomological supports and the thick subcategories of mod R. This connection begins with the following result.

Proposition 2.5.13 ((Avramov & Buchweitz, 2000b, Theorem 5.6))**.** *If*

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

is exact, then

$$V^*(X_i) \subseteq V^*(X_j) \cap V^*(X_l)$$

with $\{i,j,l\} = \{1,2,3\}$*. In particular,* $V^*(M) = V^*(\Omega m)$*. Furthermore*

$$V^*(X \oplus Y) = V^*(X) \cup V^*(Y)$$

Definition 2.5.14. A subcategory $\mathcal{C} \subseteq \operatorname{mod} R$ is thick if

1. $R \in \mathcal{C}$
2. $\mathcal{C}$ is closed under direct summands, that is if $X \oplus Y \in C$ then $X, Y \in \mathcal{C}$
3. $\mathcal{C}$ has the two out of three property, that is if $0 \to X_1 \to X_2 \to X_3 \to 0$ and $X_i, X_j \in \mathcal{C}$, then $X_l \in \mathcal{C}$ with $\{i,j,l\} = \{1,2,3\}$.

Let $\operatorname{Thick} M$ denote the smallest thick category containing M.

By Proposition 5.13, for any $V \subseteq \mathbb{P}^{c-1}_{\tilde{k}}$, the category

$$\{M \in \operatorname{mod} R \mid V^*(M) \subseteq V\}$$

is thick. In fact, we can use the cohomological supports to classify all the thick subcategories of $\operatorname{mod} R$. The following result is due to in Stevenson (2012), and in the zero dimensional case to Carlson and Iyengar in (Carlson & Iyengar, 2012, Remark 5.12). Since

$$V^*(M) \subseteq \mathbb{P}^{c-1}_{\tilde{k}} = \operatorname{MaxSpec} \operatorname{Proj} \tilde{k}[\chi_1, \ldots, \chi_c]$$

we may let $\overline{V^*(M)}$ be the closure of $V^*(M)$ in $\operatorname{Proj} \tilde{k}[\chi_1, \ldots, \chi_c]$.

Theorem 2.5.15. *If* R *is an isolated singularity and a complete intersection, then the thick subcategories of* $\operatorname{mod}(R)$ *are in bijection with the specialization closed subsets of the scheme* $\operatorname{Proj} k[\chi_1, \ldots, \chi_c]$*. Furthermore, the cyclic thick subcategories, i.e. the thick subcategories of* *the form* Thick M*, are in bijection with the closed sets of the scheme* Proj $k[\chi_1, \ldots, \chi_c]$*.This bijection is given by* Thick $M \mapsto \overline{V^*(M)} \cap \operatorname{Proj} k[\chi_1, \ldots, \chi_c]$*. When k is algebraically closed, the cyclic thick subcategories are in bijection with the closed subsets of* $\mathbb{P}_k^{c-1}$.

Remark 2.5.16. In Stevenson (2012), Stevenson actually classifies the thick subcategories over an arbitrary complete intersection ring in terms of the specialization closed subsets of a scheme X. He does this by assigning $M \in \operatorname{mod}(R)$ a closed set in X which we will denote by $\mathfrak{S}(M)$. On a set theoretic level, we have

$$X = \coprod_{p \in \operatorname{Sing} R} \operatorname{Proj} k(p)[\chi_1, \ldots, \chi_{c(p)}]$$

where $c(p)$ is the codimension of R_p and $k(p)$ is the residue field of R_p. Furthermore, we have

$$\mathfrak{S}(M) \cap \operatorname{Proj} k(p)[\chi_1, \ldots, \chi_{c(p)}] = \overline{V^*_{R_p}(M_p)} \cap \operatorname{Proj} k(p)[\chi_1, \ldots, \chi_{c(p)}].$$

Suppose we have two modules $M, N \in \operatorname{mod}(R)$ such that $V^*_{R_p}(M_p) \subseteq V^*_{R_p}(N_p)$ for every $p \in \operatorname{Sing} R$ (and hence $\operatorname{spec} R$). It follows that

$$\mathfrak{S}(M) \cap \operatorname{Proj} k(p)[\chi_1, \ldots, \chi_{c(p)}] \subseteq \mathfrak{S}(N) \cap \operatorname{Proj} k(p)[\chi_1, \ldots, \chi_{c(p)}].$$

We thus have $\mathfrak{S}(M) \subseteq \mathfrak{S}(N)$. By Stevenson's classification theorem, we may conclude that Thick $M \subseteq$ Thick N.

The following is an important theorem helping prove the classification result of (Carlson & Iyengar, 2012, Remark 5.12).

Theorem 2.5.17 ((Bergh, 2007, Corollary 2.3),(Avramov & Iyengar, 2007, Theorem 7.8))**.** *If k is algebraically closed, for each closed set $V \subseteq \mathbb{P}_k^{c-1}$, there is a maximal Cohen-Macaulay module M such that $V^*(M) = V$.*

When working with cohomological supports, it is important to be able to reduce to the case where R is complete and k is algebraically closed. We give two results which allow us to do this.

Lemma 2.5.18. *For any R-module M, we have $V_R^*(M) = V_{\hat{R}}^*(\hat{M})$.*

Lemma 2.5.19 ((Avramov & Buchweitz, 2000b, Lemma 2.2),(Bourbaki, 1983, App., Théorèm 1, Corollaire))). *There exists a local complete intersection ring $(\tilde{R}, \tilde{\mathfrak{m}}, \tilde{k})$ of codimension c such that $\tilde{R}$ is a flat extension of R, $\mathfrak{m}\tilde{R} = \tilde{m}$, and the induced map $k \to \tilde{k}$ is the inclusion of k into its algebraic closure. Furthermore, we have $V_R^*(M) = V_{\tilde{R}}^*(M \otimes \tilde{R})$.*

Chapter 3

Classifying resolving subcategories

3.1 Introduction

Recently, there has been research in classifying the resolving subcategories of $\operatorname{mod}(R)$. The study of resolving subcategories began with Auslander and Bridger's influential work in Auslander & Bridger (1969) where they define the category of Gorenstein dimension zero modules, which we will denote by $\mathcal{G}_R$. Also, they generalize the notion of projective dimension by defining Gorenstein dimension through approximations of Gorenstein dimension zero modules. In their paper, they also prove that $\mathcal{G}_R$ has certain homological closure properties which cause Gorenstein dimension to behave similarly to projective dimension. They then take these homological closure properties of $\mathcal{G}_R$ as the definition of resolving subcategories. We can take dimension with respect to a resolving subcategory, and, as in the case of $\mathcal{G}_R$, these homological closure properties force this dimension function to also behave similarly to projective dimension. See Section 2 for further exposition.

The classification of resolving subcategories was significantly advanced by Dao and Takahashi in Dao & Takahashi (2013), where they give a bijection between the set of resolving subcategories of the category of finite projective dimension modules and the set of grade consistent functions. A function $f : \operatorname{spec} R \to \mathbb{N}$ is called grade consistent if it is increasing (as a morphism of posets) and $f(p) \leq \text{grade}(p)$ for all $p \in \text{spec}(R)$. This result motivated the author to find other situations where a similar bijection exists, furthering the use of grade consistent functions in classifying resolving subcategories. Before the work of Dao and Takahashi, Takahashi in Takahashi (2013) classifies the resolving subcategories closed under tensor products and Auslander transposes in the Cohen-Macaulay case. In Takahashi (2011) he classifies the contravariantly finite resolving subcategories of a Henselian local Gorenstein ring. In Takahashi (2009), Takahashi also studies resolving subcategories which are free on the punctured spectrum. In Auslander & Reiten (1991), Auslander and Reiten discover a connection between resolving subcategories and tilting theory, and they classify all the contravariantly finite resolving subcategories using cotilting bundles. Also, the result in Dao & Takahashi (2013) was later reproved in Hügel et al. (2014) by classifying all the tilting classes, an approach which is very different from Dao and Takahashi's.

Let Γ be the set of grade consistent functions. For categories $\mathcal{C}, \mathcal{X} \subseteq \text{mod}(R)$ and $f \in \Gamma$, we define

$$\Lambda(\mathcal{C})(f) = \{M \in \text{mod}(R) \mid \text{add}\, \mathcal{C}_p\text{-dim}\, M_p \leq f(p) \quad \forall p \in \text{spec}\, R\}$$

$$\Phi_{\mathcal{C}}(\mathcal{X}) : \text{spec}\, R \to \mathbb{N} \qquad p \mapsto \sup\{\text{add}\, \mathcal{C}_p\text{-dim}\, X_p \mid X \in \mathcal{X}\}.$$

Let $\mathfrak{R}$ denote all the resolving subcategories of $\text{mod}(R)$, and for any $\mathcal{C} \subseteq \text{mod}(R)$ let $\mathfrak{R}(\mathcal{C})$ be all the resolving subcategories that contain $\mathcal{C}$ and whose objects all have finite dimension with respect to $\mathcal{C}$. Using our new notation, we can restate Dao and Takahashi's result from Dao & Takahashi (2013).

Theorem 3.1.1. *When R is Noetherian, the following is a bijection*

$$\mathfrak{R}(\mathcal{P}) \leftrightarrows^{\Lambda(\mathcal{P})}_{\Phi_{\mathcal{P}}} \Gamma$$

where $\Lambda(\mathcal{P})$ and $\Phi_{\mathcal{P}}$ are inverses of each other.

The first main result of this chapter is Theorem 2.2, which is the following.

Theorem (A). *Let Ψ be a set of increasing functions from* $\operatorname{spec} R$ *to* $\mathbb{N}$. *Suppose* $\mathcal{C} \subseteq \mathcal{D}$ *such that* $\mathcal{C}$ *cogenerates* $\mathcal{D}$ *and* $\operatorname{add} \mathcal{C}_p$ *is thick in* $\operatorname{add} \mathcal{D}_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{spec} R$. *Define* $\eta_{\mathcal{C}}^{\mathcal{D}} : \mathfrak{R}(\mathcal{C}) \to \mathfrak{R}(\mathcal{D})$ *by* $\eta_{\mathcal{C}}^{\mathcal{D}}(\mathcal{X}) = \operatorname{res}(\mathcal{X} \cup \mathcal{D})$ *and* $\rho_{\mathcal{C}}^{\mathcal{D}} : \mathfrak{R}(\mathcal{D}) \to \mathfrak{R}(\mathcal{C})$ *by letting* $\rho_{\mathcal{C}}^{\mathcal{D}}(\mathcal{X})$ *be the subcategory of modules in* $\mathcal{X}$ *of finite dimension with respect to* $\mathcal{C}$. *If* $\Phi_{\mathcal{C}}$ *and* $\Lambda(\mathcal{C})$ *are inverses of each other giving a bijection between* $\mathfrak{R}(\mathcal{C})$ *and* Ψ, *then we have the following commutative diagram.*

$$\begin{array}{ccc} \mathfrak{R}(\mathcal{D}) & & \\ & \overset{\Phi_{\mathcal{D}}}{\searrow} & \\ \eta_{\mathcal{C}}^{\mathcal{D}} \uparrow & & \Psi \\ & \underset{\Phi_{\mathcal{C}}}{\nearrow} & \\ \mathfrak{R}(\mathcal{C}) & & \end{array}$$

Furthermore, $\Phi_{\mathcal{D}}$ *and* $\Lambda(\mathcal{D})$, *and also* $\eta_{\mathcal{C}}^{\mathcal{D}}$ *and* $\rho_{\mathcal{C}}^{\mathcal{D}}$, *are pairs of inverse functions.*

This result allows us to extend the bijection from Dao & Takahashi (2013) to a plethora of categories. We use it to prove the following results which are Theorem 6.8 and essentially Corollary 6.6.

Theorem (B). *Let C be a semidualizing module. For any thick subcategory* $\mathcal{C}$ *of* $\mathcal{G}_C$ *containing* C, $\Lambda(\mathcal{C})$ *and* $\Phi_{\mathcal{C}}$ *give a bijection between* $\mathfrak{R}(\mathcal{C})$ *and* Γ.

Theorem (C). *Let C be a semidualizing module, and let* $\mathfrak{S}(C)$ *be the collection of thick subcategories of totally C-reflexive modules containing C. Then the following is a bijection.*

$$\Lambda : \mathfrak{S}(C) \times \Gamma \to \bigcup_{\mathcal{C} \in \mathfrak{S}(C)} \mathfrak{R}(\mathcal{C}) \subseteq \mathfrak{R}$$

Theorem C is really just the bijections of Theorem B patched together. These theorems show that the classification of resolving subcategories is intrinsically linked to the classification of thick subcategories of totally C-reflexive modules and hence to the classification of thick subcategories of $\mathrm{mod}(R)$. Applying these results in the Gorenstein case yields Theorem 7.1 which, letting MCM denote the category of maximal Cohen-Macaulay modules, states

Theorem (D)**.** *If R is Gorenstein, then we have the following bijections which commute*

$$\begin{array}{ccc} \{\textit{Thick subcategories of MCM}\} \times \Gamma & \xrightarrow{\Lambda} & \\ \Big\downarrow \Lambda(\mathcal{P}) & & \{\mathcal{C} \in \mathfrak{R} \mid \mathcal{C} \cap \textit{MCM is thick in MCM}\} \\ \{\textit{Thick subcategories of MCM}\} \times \mathfrak{R}(\mathcal{P}) & \xrightarrow{\Xi} & \end{array}$$

where $\Xi(\mathcal{X}, \mathcal{Y}) = \mathrm{res}(\mathcal{X} \cup \mathcal{Y})$.

Of independent interest, using semidualizing modules, we generalize the famed Auslander transpose. This generalization is similar, but different, to the generalizations of Geng and Huang in Geng (2013) and Huang (1999).

This chapter is organized as follows: We prove Theorem A in Section 2. In Section 3, we give the generalization of the Auslander transpose, which we use in Section 4 to prove a theorem about resolving subcategories that are locally Maximal Cohen-Macaulay. This result is used in Section 5 to prove that Theorem B holds for certain thick subcategories containing a semidualizing module C. Section 6 then proves Theorems B and C by examining thick subcategories of maximal Cohen-Macaulay modules that contain C, and then by applying Theorem A. In the last section, these results are applied to the Gorenstein case, and Theorem D and several other results are proven.

3.2 Comparing resolving subcategories

For the entirety of this section, let $\mathcal{C}$ and $\mathcal{D}$ be resolving subcategories. Recall that $\mathfrak{R}(\mathcal{C})$ is the collection of resolving subcategories $\mathcal{X}$ such that $\mathcal{C} \subseteq \mathcal{X} \subseteq \overline{\mathcal{C}}$. In this section, we compare $\mathfrak{R}(\mathcal{C})$ and $\mathfrak{R}(\mathcal{D})$ when $\mathcal{C}$ is contained in $\mathcal{D}$. If $\mathcal{C} \subseteq \mathcal{D}$, we may define $\eta_{\mathcal{C}}^{\mathcal{D}} : \mathfrak{R}(\mathcal{C}) \to \mathfrak{R}(\mathcal{D})$ by

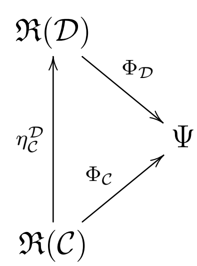
$\mathcal{X} \mapsto \operatorname{res}(\mathcal{X} \cup \mathcal{D})$ and $\rho_{\mathcal{C}}^{\mathcal{D}} : \mathfrak{R}(\mathcal{D}) \to \mathfrak{R}(\mathcal{C})$ by $\mathcal{X} \mapsto \mathcal{X} \cap \overline{\mathcal{C}}$. Note that if $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{E}$, then $\eta_{\mathcal{C}}^{\mathcal{E}} = \eta_{\mathcal{D}}^{\mathcal{E}} \eta_{\mathcal{C}}^{\mathcal{D}}$ and $\rho_{\mathcal{C}}^{\mathcal{E}} = \rho_{\mathcal{C}}^{\mathcal{D}} \rho_{\mathcal{D}}^{\mathcal{E}}$.

Proposition 3.2.1. *If $\mathcal{C}$ cogenerates $\mathcal{D}$, then the map $\rho_{\mathcal{C}}^{\mathcal{D}}$ is injective.*

Proof. Suppose that for $\mathcal{X}, \mathcal{Y} \in \mathfrak{R}(\mathcal{C})$, we have $\rho_{\mathcal{C}}^{\mathcal{D}}(\mathcal{X}) = \rho_{\mathcal{C}}^{\mathcal{D}}(\mathcal{Y})$, i.e. $\mathcal{X} \cap \overline{\mathcal{C}} = \mathcal{Y} \cap \overline{\mathcal{C}}$. Take any $X \in \mathcal{X}$. Since $X \in \overline{\mathcal{D}}$ and $\mathcal{C}$ cogenerates $\mathcal{D}$, by Theorem 2.18, there exists $M \in \overline{\mathcal{C}}$ and $D \in \mathcal{D}$ such that $0 \to X \to M \to D \to 0$ is exact. Since $D \in \mathcal{D} \subseteq \mathcal{X}$ and $X \in \mathcal{X}$, we know that M is also in $\mathcal{X}$. But then M is in $\mathcal{X} \cap \overline{\mathcal{C}} = \mathcal{Y} \cap \overline{\mathcal{C}}$ and thus also in $\mathcal{Y}$. Since also $D \in \mathcal{D} \subseteq \mathcal{Y}$, we know that X is must also be in $\mathcal{Y}$. Hence $\mathcal{X} \subseteq \mathcal{Y}$, and, by symmetry, we have equality. Therefore, $\rho_{\mathcal{C}}^{\mathcal{D}}$ is injective. □

In certain circumstances, this map is a bijection. The following is Theorem A from the introduction.

Theorem 3.2.2. *Let Ψ be a set of increasing functions from* $\operatorname{spec} R$ *to* $\mathbb{N}$. *Suppose, $\mathcal{C} \subseteq \mathcal{D}$ such that $\mathcal{C}$ cogenerates $\mathcal{D}$ and* $\operatorname{add} \mathcal{C}_p$ *is thick in* $\operatorname{add} \mathcal{D}_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{spec} R$. *If $\Phi_{\mathcal{C}}$ and $\Lambda(\mathcal{C})$ are inverse functions giving a bijection between $\mathfrak{R}(\mathcal{C})$ and Ψ, then the following diagram commutes.*



Furthermore, $\Phi_{\mathcal{D}}$ and $\Lambda(\mathcal{D})$ and also $\eta_{\mathcal{C}}^{\mathcal{D}}$ and $\rho_{\mathcal{C}}^{\mathcal{D}}$ are pairs of inverse functions.

The proof of this proposition will be given after this brief lemma.

Lemma 3.2.3. *If $\mathcal{X}$ and $\mathcal{Y}$ are subcategories and $\mathcal{C}$ is resolving, then* $\Phi_{\mathcal{C}}(\operatorname{res}(\mathcal{X} \cup \mathcal{Y})) = \Phi_{\mathcal{C}}(\mathcal{X}) \vee \Phi_{\mathcal{C}}(\mathcal{Y})$.

Proof. Since every element in $\operatorname{res}(\mathcal{X} \cup \mathcal{Y})$ is obtained by taking extensions, syzygies, and direct summands a finite number of times, and since these operations never increase the $\mathcal{C}$

dimension, we have $\Phi_{\mathcal{C}}(\operatorname{res}(\mathcal{X} \cup \mathcal{Y})) \leq \Phi_{\mathcal{C}}(\mathcal{X}) \vee \Phi_{\mathcal{C}}(\mathcal{Y})$. However, since $\mathcal{X}, \mathcal{Y} \subseteq \operatorname{res}(\mathcal{X} \cup \mathcal{Y})$, we actually have equality. □

Proof of Theorem 2.2. First, we will show that $\rho_{\mathcal{C}}^{\mathcal{D}}$ and $\eta_{\mathcal{C}}^{\mathcal{D}}$ are inverse functions and are thus both bijections. Proposition 2.1 shows that $\rho_{\mathcal{C}}^{\mathcal{D}}$ is injective. Let $\mathcal{Z} = \rho_{\mathcal{C}}^{\mathcal{D}} \eta_{\mathcal{C}}^{\mathcal{D}}(\mathcal{X}) = \operatorname{res}(\mathcal{X} \cup \mathcal{D}) \cap \overline{\mathcal{C}}$. It suffices to show that $\mathcal{Z} = \mathcal{X}$. Setting $f = \Phi_{\mathcal{C}}(\mathcal{X})$, since $\Phi_{\mathcal{C}}$ and $\Lambda(\mathcal{C})$ are inverse functions, this is equivalent to showing that $\Phi_{\mathcal{C}}(\mathcal{Z}) = f$. Since $\mathcal{X} \subseteq \mathcal{Z}$, we know that $\Phi_{\mathcal{C}}(\mathcal{Z}) \geq f$. From Lemma 2.3, we have

$$\Phi_{\mathcal{D}}(\operatorname{res} \mathcal{X} \cup \mathcal{D}) = \Phi_{\mathcal{D}}(\mathcal{X}) \vee \Phi_{\mathcal{D}}(\mathcal{D}) = \Phi_{\mathcal{D}}(\mathcal{X}).$$

Furthermore, since $\operatorname{add} \mathcal{C}_p$ is thick in $\operatorname{add} \mathcal{D}_{\mathfrak{p}}$ for all $p \in \operatorname{spec} R$, by Lemma 2.16, $\operatorname{add} \mathcal{C}_p$-dim M and $\operatorname{add} \mathcal{D}_p$-dim M are the same for all $p \in \operatorname{spec} R$ and $M \in \overline{\mathcal{C}}$. Hence $\Phi_{\mathcal{C}}(\mathcal{W})$ equals $\Phi_{\mathcal{D}}(\mathcal{W})$ for all $\mathcal{W} \subseteq \overline{\mathcal{C}}$. Therefore,

$$f \leq \Phi_{\mathcal{C}}(\mathcal{Z}) = \Phi_{\mathcal{D}}(\mathcal{Z}) \leq \Phi_{\mathcal{D}}(\operatorname{res}(\mathcal{X} \cup \mathcal{D})) = \Phi_{\mathcal{D}}(\mathcal{X}) = \Phi_{\mathcal{C}}(\mathcal{X}) = f$$

and so, $\Phi_{\mathcal{C}}(\mathcal{Z}) = f$. Hence, $\rho_{\mathcal{C}}^{\mathcal{D}}$ and $\eta_{\mathcal{C}}^{\mathcal{D}}$ are inverse functions. Also, this argument shows that $\Phi_{\mathcal{C}}(\mathcal{X}) = \Phi_{\mathcal{D}}(\operatorname{res}(\mathcal{X} \cup \mathcal{D})) = \Phi_{\mathcal{D}}(\eta_{\mathcal{C}}^{\mathcal{D}}(\mathcal{X}))$, showing that the diagram commutes and hence $\Phi_{\mathcal{D}}$ also gives a bijection.

It remains to show that $\Lambda(\mathcal{D}) = \Phi_{\mathcal{D}}{}^{-1}$. For $f \in \Psi$, we have $\eta_{\mathcal{C}}^{\mathcal{D}}(\Lambda(\mathcal{C})(f))$ contained in $\Lambda(\mathcal{D})(f)$. Because $\Phi_{\mathcal{D}}$ is an increasing function and both $\Phi_{\mathcal{C}}$ and $\Lambda(\mathcal{C})$ are inverse functions, we have

$$f = \Phi_{\mathcal{C}} \Lambda(\mathcal{C})(f) = \Phi_{\mathcal{D}}(\eta_{\mathcal{C}}^{\mathcal{D}}(\Lambda(\mathcal{C})(f)) \leq \Phi_{\mathcal{D}} \Lambda(\mathcal{D})(f) \leq f.$$

Thus we have $\Phi_{\mathcal{D}} \Lambda(\mathcal{D})(f) = f$, and we are done. □

For a resolving subcategory $\mathcal{A}$, let $\mathfrak{S}(\mathcal{A})$ be the collection of resolving subcategories $\mathcal{C}$ such that $\mathcal{C}$ and $\mathcal{A}$ satisfy the hypotheses of Theorem 2.2, i.e. $\mathcal{A}$ cogenerates $\mathcal{C}$ and $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$ is thick in $\operatorname{add} \mathcal{C}_{\mathfrak{p}}$ for all $p \in \operatorname{spec} R$. The following theorem shows that we can patch together the bijections in Theorem 2.2.

Theorem 3.2.4. *Let Ψ be a set of increasing functions from* $\operatorname{spec} R$ *to* $\mathbb{N}$. *If $\Phi_{\mathcal{A}}$ and $\Lambda(\mathcal{A})$ are inverse functions giving a bijection between $\mathfrak{R}(\mathcal{A})$ and Ψ, then the following is a bijection*

$$\Lambda : \mathfrak{S}(\mathcal{A}) \times \Psi \to \bigcup_{\mathcal{C} \in \mathfrak{S}} \mathfrak{R}(\mathcal{C}) \subseteq \mathfrak{R}.$$

Furthermore, for any $\mathcal{C}, \mathcal{D} \in \mathfrak{S}(\mathcal{A})$ the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{R}(\mathcal{D}) & \xrightarrow{\Phi_{\mathcal{D}}} & \Psi \\ {\scriptstyle \eta_{\mathcal{C}}^{\mathcal{D}}}\uparrow & & \| \\ \mathfrak{R}(\mathcal{C}) & \xrightarrow{\Phi_{\mathcal{C}}} & \Psi \\ {\scriptstyle \eta_{\mathcal{A}}^{\mathcal{C}}}\uparrow & & \| \\ \mathfrak{R}(\mathcal{A}) & \xrightarrow{\Phi_{\mathcal{A}}} & \Psi \end{array} \tag{3.1}$$

Furthermore, $\rho_{\mathcal{C}}^{\mathcal{D}}$ and $\eta_{\mathcal{C}}^{\mathcal{D}}$ are inverse functions.

The second part of this result is interesting because there is no reason a priori that $\rho_{\mathcal{C}}^{\mathcal{D}}$ and $\eta_{\mathcal{C}}^{\mathcal{D}}$ should be bijections. Before we proceed with the proof of Theorem 2.4, we need a lemma.

Lemma 3.2.5. *The set $\mathfrak{S}(\mathcal{A})$ is closed under intersections.*

Proof. Let $\mathcal{C}, \mathcal{D} \in \mathfrak{S}(\mathcal{A})$. Take any $p \in \operatorname{spec} R$. Suppose $0 \to L \to M \to N \to 0$ is an exact sequence of R_p-modules with $L, M \in \operatorname{add} \mathcal{A}_p$ and $N \in \operatorname{add}(\mathcal{C} \cap \mathcal{D})_p$. Then N is in $\operatorname{add} \mathcal{C}_p$. Therefore, since $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$ is thick in $\operatorname{add} \mathcal{C}_p$ by assumption, N is in $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$. Since $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$ is resolving and contained in $\operatorname{add}(\mathcal{C} \cap \mathcal{D})_p$, $\operatorname{add} \mathcal{A}_{\mathfrak{p}}$ is thick in $\operatorname{add}(\mathcal{C} \cap \mathcal{D})_p$.

It remains to show that $\mathcal{A}$ cogenerates $\mathcal{C} \cap \mathcal{D}$. Take $X \in \mathcal{C} \cap \mathcal{D}$. We have

$$0 \to X \to A \to C \to 0 \qquad 0 \to X \to A' \to D \to 0$$

with $C \in \mathcal{C}$, $D \in \mathcal{D}$, and $A, A' \in \mathcal{A}$. Consider the following pushout diagram.

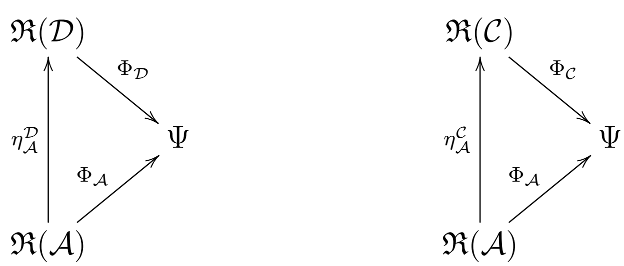
$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & X & \longrightarrow & A & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & T & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & D & = & D & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

It is easy to see $T \in \mathcal{C} \cap \mathcal{D}$. We also have the exact sequence

$$0 \to X \to A \oplus A' \to T \to 0$$

Since $A \oplus A' \in \mathcal{A}$, this completes the proof. $\square$

Proof of Theorem 2.4. Suppose $\mathcal{C}, \mathcal{D} \in \mathfrak{S}$ with $\mathcal{C} \subseteq \mathcal{D}$. From Theorem 2.2, the following diagrams commute.



From here, it is easy to show that (2) commutes and $\Phi_{\mathcal{D}}$ and $\eta^{\mathcal{D}}_{\mathcal{C}}$ are bijections with $(\eta^{\mathcal{D}}_{\mathcal{C}})^{-1} = \rho^{\mathcal{D}}_{\mathcal{C}}$.

Also, Theorem 2.2 shows that $\mathrm{Im}(\Lambda) = \bigcup_{\mathcal{C} \in \mathfrak{S}} \mathfrak{R}(\mathcal{C})$. It remains to show that Λ is injective. Suppose $\mathcal{Z} = \Lambda(\mathcal{C})(f) = \Lambda(\mathcal{D})(g)$. Then we have $\mathcal{C} \subseteq \mathcal{Z}$ and $\mathcal{D} \subseteq \mathcal{Z}$, and hence $\mathcal{C} \cap \mathcal{D} \subseteq \mathcal{Z}$. For any Z and any n greater than $\mathcal{C}$-dim Z and $\mathcal{D}$-dim Z, $\Omega^n Z$ is in $\mathcal{C} \cap \mathcal{D}$ by Corollary 2.4.

Therefore, $\mathcal{Z}$ is contained in $\overline{\mathcal{C} \cap \mathcal{D}}$ and thus $\mathcal{Z} \in \mathfrak{R}(\mathcal{C} \cap \mathcal{D})$. Since by the last lemma $\mathcal{C} \cap \mathcal{D}$ is in $\mathfrak{S}(\mathcal{A})$, by Theorem 2.2, $\Lambda(\mathcal{C} \cap \mathcal{D}) : \Psi \to \mathfrak{R}(\mathcal{C} \cap \mathcal{D})$ is a bijection. So there exists an h such that $\Lambda(\mathcal{C} \cap \mathcal{D})(h) = \mathcal{Z} = \Lambda(\mathcal{C})(f) = \Lambda(\mathcal{D})(g)$. Therefore, we may assume that $\mathcal{C}$ is contained in $\mathcal{D}$.

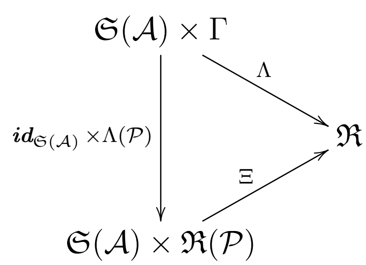
Then, by assumption, we have $\mathcal{D} \subseteq \mathcal{Z} \subseteq \overline{\mathcal{C}}$. Thus, because $\mathcal{A} \subseteq \mathcal{D}$, we have the following.

$$\mathcal{D} = \mathcal{D} \cap \overline{\mathcal{C}} = \rho^{\mathcal{D}}_{\mathcal{C}}(\mathcal{D}) = \eta^{\mathcal{C}}_{\mathcal{A}} \rho^{\mathcal{C}}_{\mathcal{A}} \rho^{\mathcal{D}}_{\mathcal{C}}(\mathcal{D}) = \eta^{\mathcal{C}}_{\mathcal{A}} \rho^{\mathcal{D}}_{\mathcal{A}}(\mathcal{D}) = \eta^{\mathcal{C}}_{\mathcal{A}}(\mathcal{A}) = \mathcal{C}$$

Since $\Lambda(\mathcal{C})$ is injective, we then also have $f = g$. □

As mentioned earlier, in Dao & Takahashi (2013), we have $\Lambda(\mathcal{P})$ is a bijection from Γ to $\mathfrak{R}(\mathcal{P})$. In Section 6 and Section 7 we apply Theorem 2.4 when $\mathcal{A} = \mathcal{P}$, and show that $\mathfrak{S}(\mathcal{P})$ is simply the collection of thick subcategories of $\mathcal{G}_R$. The following results gives an alternative way of viewing Theorem 2.4.

Proposition 3.2.6. *In the situation of Theorem 2.4, if* $\Psi = \Gamma$ *and* $\mathcal{P}$ *is thick in* $\mathcal{A}$*, then the following diagram commutes.*



where $\Xi(\mathcal{C}, \mathcal{X}) = \operatorname{res}(\mathcal{C} \cup \mathcal{X})$*. Furthermore,* $\mathbf{id}_{\mathfrak{S}(\mathcal{A})} \times \Lambda(\mathcal{P})$ *is bijective.*

Proof. Since $\Lambda(\mathcal{P})$ is bijective, $\mathbf{id}_{\mathfrak{S}(\mathcal{A})} \times \Lambda(\mathcal{P})$ is too. It suffices to show that for any $(\mathcal{C}, f) \in \mathfrak{S} \times \Gamma$ we have $\Xi(\mathcal{C}, \Lambda(\mathcal{P})(f)) = \Lambda(\mathcal{C})(f)$. Set $\mathcal{X} = \Xi(\mathcal{C}, \Lambda(\mathcal{P})(f))$. First note that $\mathcal{X}$ is in $\mathfrak{R}(\mathcal{C})$. Since $\mathcal{P}$ is thick in $\mathcal{A}$ and hence in $\mathcal{C}$, by Lemma 2.3, we have

$$\Phi_{\mathcal{C}}(\mathcal{X}) = \Phi_{\mathcal{C}}(\operatorname{res}(\mathcal{C} \cup \Lambda(\mathcal{P})(f))) = \Phi_{\mathcal{C}}(\mathcal{C}) \vee \Phi_{\mathcal{C}}(\Lambda(\mathcal{P})(f)) = \Phi_{\mathcal{P}}(\lambda(\mathcal{P})(f)) = f$$

and thus $\Lambda(\mathcal{C})(f) = \mathcal{X}$, proving the claim. $\square$

3.3 A generalization of the Auslander transpose

Let C be a semidualizing module, and set $-^\dagger = \mathrm{Hom}(-, C)$. For the entirety of this section, $\mathcal{A}$ denotes a thick subcategory of $\mathcal{G}_C$ that is closed under $\dagger$. Recalling Proposition 3.7, $\mathcal{A}$-dim satisfies the Auslander Buchsbaum formula.

The Auslander transpose has been an invaluable tool in both representation theory and commutative algebra. In this section we generalize the notion of the Auslander transpose using semidualizing modules and list some properties which we will use. The Auslander transpose has previously been generalized in Geng (2013) and Huang (1999), but the construction here is different.

Definition 3.3.1. An *$\mathcal{A}$-presentation* of M is an exact sequence $G \xrightarrow{\varphi} F \to M \to 0$ with $F, G \in \mathcal{A}$. We set $\mathrm{Tr}_{\mathcal{A}} M = \mathrm{coker}\, \varphi^\dagger$ and $\tilde{\Omega}\, \mathrm{Tr}_{\mathcal{A}} M = \mathrm{Im}\, \varphi^\dagger$.

These "functors" are not well defined up to isomorphism, motivating a new equivalence relation. For modules A and B, we write $A \sim' B$ and $B \sim' A$ if there exists a $K \in \mathcal{A}$ such that $0 \to A \to B \to K \to 0$ is exact. Let $\mathcal{A}$-equivalence, denoted by $\sim$, be the transitive closure of the relation $\sim'$. Since $\sim'$ is symmetric and reflexive, $\sim$ is an equivalence relation. Stable equivalence implies $\mathcal{A}$ equivalence, and when $\mathcal{A} = \mathcal{P}$, they are the same.

Proposition 3.3.2. *The functors* $\mathrm{Tr}_{\mathcal{A}}$ *and* $\tilde{\Omega}\, \mathrm{Tr}_{\mathcal{A}}$ *are unique up to $\mathcal{A}$-equivalence.*

Proof. We say that an $\mathcal{A}$-presentation of M, π, dominates another $\mathcal{A}$-presentation, ρ, if there is an epimorphism from π to ρ. Suppose that π is the projective presentation

$$P_1 \to P_0 \to M \to 0$$

and ρ is the $\mathcal{A}$-presentation

$$G \to F \to M \to 0.$$

Furthermore, suppose that π dominates ρ. Then we have the following exact commutative diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \| & & \\
 & & & & G & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.2}$$

The map $K_1 \to K_0$ is surjective by the snake lemma. Note that K, K_1, and K_0 are in $\mathcal{A}$. Applying $\dagger$ to the diagram yields the following.

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^\dagger & \longrightarrow & F^\dagger & \longrightarrow & G^\dagger & \longrightarrow & \operatorname{Tr}^\rho_{\mathcal{A}} M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^\dagger & \longrightarrow & P_0^\dagger & \longrightarrow & P_1^\dagger & \longrightarrow & \operatorname{Tr}^\pi_{\mathcal{A}} M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & K_0^\dagger & \longrightarrow & K_1^\dagger & \longrightarrow & K^\dagger & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

Since $\mathcal{A}$ is closed under $\dagger$, $K^\dagger$ is in $\mathcal{A}$, and so $\operatorname{Tr}^\rho_{\mathcal{A}} M \sim \operatorname{Tr}^\pi_{\mathcal{A}} M$. Applying the snake lemma to the second two columns gives us

$$0 \to \tilde{\Omega} \operatorname{Tr}^\rho_{\mathcal{A}} M \to \tilde{\Omega} \operatorname{Tr}^\pi_{\mathcal{A}} M \to K_0^\dagger \to 0.$$

Since $K^\dagger \in \mathcal{A}$, we have $\tilde{\Omega} \operatorname{Tr}^\rho_{\mathcal{A}} M \sim \tilde{\Omega} \operatorname{Tr}^\pi_{\mathcal{A}} M$.

It suffices to show that for any two $\mathcal{A}$-presentations, ρ and ρ', there exists a projective presentation that dominates both of them. It is easy to construct projective presentations ψ and ψ' which dominate ρ and ρ' respectively, then the proof of Masek (1999)[Proposition 4] shows that there is a projective presentation of π which dominates both ψ and ψ'. But then that π will also dominate ρ and ρ'. $\square$

One can easily use this lemma to show that $\operatorname{res}_{\mathcal{A}}(\operatorname{Tr}_{\mathcal{A}} X) = \mathcal{A}$ for any $X \in A$. The following will show that $\mathcal{A}$-equivalence is well behaved under many important operations that will be used in the remainder of this paper.

Lemma 3.3.3. *For any $A, B \in \operatorname{mod}(R)$ such that $A \sim B$, the following are true.*

1. $\operatorname{res}_{\mathcal{A}} A = \operatorname{res}_{\mathcal{A}} B$
2. $\Omega A \sim \Omega B$
3. $\operatorname{Tr}_{\mathcal{A}} A \sim \operatorname{Tr}_{\mathcal{A}} B$
4. $\tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} A \sim \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} B$
5. $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} A \sim A$

Proof. For statements (1)-(4), it suffices to assume that $0 \to A \to B \to K \to 0$ with $K \in \mathcal{A}$. Proving (1) is trivial. For suitable choices of syzygies, we have $0 \to \Omega A \to \Omega B \oplus P \to \Omega K \to 0$. Since ΩK is in $\mathcal{A}$, and since syzygies are unique up to stable, and hence $\mathcal{A}$-equivalence, this implies (2).

Now we wish to show (3). There exist projective modules P, Q_K, Q_B such that we may write the following.

$$\begin{array}{ccccccccc} & & P_B & \longrightarrow & Q_B & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega K & \longrightarrow & Q_K & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

where the rows are exact and the vertical maps surjective. The snake lemma yields the following diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & P_A & \longrightarrow & Q_A & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & P_B & \longrightarrow & Q_B & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega K & \longrightarrow & Q_K & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since K is in $\mathcal{A}$, so are ΩK, P_A, and Q_A . Thus, for suitable choices of $\operatorname{Tr}_{\mathcal{A}} A$ and $\operatorname{Tr}_{\mathcal{A}} B$, applying $\dagger$ to this diagram gives the following.

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K^\dagger & \longrightarrow & (Q_K)^\dagger & \longrightarrow & (\Omega K)^\dagger & \longrightarrow & \operatorname{Ext}^1(K, C) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B^\dagger & \longrightarrow & (Q_B)^\dagger & \longrightarrow & P_B^\dagger & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C^\dagger & \longrightarrow & (Q_A)^\dagger & \longrightarrow & P_A^\dagger & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} A & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Ext}^1(K^\dagger, C) & & 0 & & 0 & & & &
\end{array}
$$

Since $\operatorname{Ext}^1(K^\dagger, C) = \operatorname{Ext}^1(K, C) = 0$, we have $\operatorname{Tr}_{\mathcal{A}} A \cong \operatorname{Tr}_{\mathcal{A}} B$ by applying the snake lemma to the middle two columns. This shows (3). Applying the snake lemma to the first two columns gives us (4).

Because of (3), we know $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} M$ is well defined up to $\mathcal{A}$-equivalence. With $F, G \in \mathcal{A}$, consider the sequence $G \xrightarrow{f} F \to M \to 0$. Then we have

$$F^\dagger \xrightarrow{f^\dagger} G^\dagger \to \operatorname{Tr}_{\mathcal{A}} M \to 0$$

Since F and G are totally C reflexive, we have $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} M \sim \operatorname{coker} f^{\dagger\dagger} \cong \operatorname{coker} f = M$. Thus we have shown (5). $\square$

Because syzygies are unique up to stable equivalence, and hence $\mathcal{A}$-equivalence, this lemma shows us that the main characters of our proofs, $\operatorname{Tr}_{\mathcal{A}} \Omega^i M$, $\operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Tr}_{\mathcal{A}} \Omega^i M$, and $\operatorname{Tr}_{\mathcal{A}} \Omega^i \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} M$ are all well defined up to $\mathcal{A}$ equivalence. It also shows that

$$\operatorname{Tr}_{\mathcal{A}} \Omega^i \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} M \sim \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} \operatorname{Tr}_{\mathcal{A}} M.$$

We close this section with an example of a property shared by $\operatorname{Tr}_{\mathcal{A}}$ and Tr.

Lemma 3.3.4. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in* $\operatorname{mod}(R)$. *For suitable choices of* $\operatorname{Tr}_{\mathcal{A}}$, *we have the exact sequence*

$$0 \to N^\dagger \to M^\dagger \to L^\dagger \to \operatorname{Tr}_{\mathcal{A}} N \to \operatorname{Tr}_{\mathcal{A}} M \to \operatorname{Tr}_{\mathcal{A}} L \to 0.$$

Furthermore, if $\operatorname{Ext}^i(L, C) = 0$, *then we have*

$$0 \to \operatorname{Tr}_{\mathcal{A}} \Omega^i N \to \operatorname{Tr}_{\mathcal{A}} \Omega^i n M \to \operatorname{Tr}_{\mathcal{A}} \Omega^i L \to 0.$$

Proof. Let θ denote the map from M to N. We have the short exact sequence

$$0 \to \Omega^i L \to \Omega^i M \to \Omega^i N \to 0.$$

Consider the following diagram where each Q_j^i projective.

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Q_1^0 & \longrightarrow & Q_0^0 & \longrightarrow & \Omega^i L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Q_1^1 & \longrightarrow & Q_0^1 & \longrightarrow & \Omega^i M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega^i\theta} \\
 & & Q_1^2 & \longrightarrow & Q_0^2 & \longrightarrow & \Omega^i N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
$$

Applying † gives the following.

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (\Omega^i N)^\dagger & \longrightarrow & (Q_0^2)^\dagger & \longrightarrow & (Q_1^2)^\dagger & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} \Omega^i N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\Omega^i\theta)^\dagger} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\Omega^i M)^\dagger & \longrightarrow & (Q_0^1)^\dagger & \longrightarrow & (Q_1^1)^\dagger & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} \Omega^i M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^i L^\dagger & \longrightarrow & (Q_0^0)^\dagger & \longrightarrow & (Q_1^0)^\dagger & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} \Omega^i L & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

The snake lemma yields

$$0 \to (\Omega^i N)^\dagger \xrightarrow{(\Omega^i\theta)^\dagger} \Omega^i M \xrightarrow{\lambda} \Omega^i L^\dagger \xrightarrow{\varepsilon} \operatorname{Tr}_{\mathcal{A}} \Omega^i N \xrightarrow{\eta} \operatorname{Tr}_{\mathcal{A}} \Omega^i M \to \operatorname{Tr}_{\mathcal{A}} \Omega^i L \to 0.$$

Setting $i = 0$ at this stage gives us the first claim. The short exact sequence

$$0 \to \Omega^i L \to \Omega^i M \to \Omega^i N \to 0$$

gives the following long exact sequence of Ext modules.

$$0 \to (\Omega^i M)^\dagger \xrightarrow{(\Omega^i\theta)^\dagger} \Omega^i M \xrightarrow{\lambda} \Omega^i L^\dagger \xrightarrow{\delta} \operatorname{Ext}^1(\Omega^i N, C) \xrightarrow{\operatorname{Ext}^1(\Omega^i\theta, C)} \operatorname{Ext}^1(\Omega^i M, C) \to \cdots$$

We also have

$$\cdots \to \operatorname{Ext}^i(L, C) \to \operatorname{Ext}^{i+1}(N, C) \xrightarrow{\operatorname{Ext}^{i+1}(\theta, C)} \operatorname{Ext}^{i+1}(M, C) \to \cdots .$$

Since, by assumption, $\operatorname{Ext}^i(L, C) = 0$, $\operatorname{Ext}^{i+1}(\theta, C)$ and thus $\operatorname{Ext}^1(\Omega^i\theta, C)$ are injective, forcing δ to be zero. Hence λ is surjective. Then the first long exact sequence shows that ε is zero, and so η is injective, giving the desired result. □

3.4 Modules which are maximal Cohen-Macaulay on the punctured spectrum

For the entirety of this section $(R, \mathfrak{m}, k)$ is a Noetherian local ring, and $\mathcal{A}$ denotes a thick subcategory of $\mathcal{G}_C$ that is closed under †. Recall that according to Proposition 3.7, dimension with respect to $\mathcal{A}$ satisfies the Auslander Buchsbaum formula. Set

$$\operatorname{res}_{\mathcal{A}} M = \operatorname{res}(M \cup \mathcal{A}) \qquad \Delta(\mathcal{A})_0 = \{M \in \Delta(\mathcal{A}) \mid M_p \in \operatorname{add} \mathcal{A}_{\mathfrak{p}} \ \forall p \in \operatorname{spec} R \backslash \mathfrak{m}\}$$

$$\Delta(\mathcal{A})_0^i = \{M \in \Delta(\mathcal{A})_0 \mid \mathcal{A}\text{-}\dim M \le i\}.$$

This section is devoted to proving the following.

Theorem 3.4.1. *If $(R, \mathfrak{m}, k)$ is a local ring with $\dim R = d$, the filtration*

$$\mathcal{A} = \Delta(\mathcal{A})_0^0 \subsetneq \Delta(\mathcal{A})_0^1 \subsetneq \cdots \subsetneq \Delta(\mathcal{A})_0^d = \Delta(\mathcal{A})_0$$

gives all the resolving subcategories of $\Delta(\mathcal{A})_0$ containing $\mathcal{A}$.

This theorem and its proof are generalizations of (Dao & Takahashi, 2013, Theorem 2.1). We now use our new "functors" from the previous section to make the building blocks of the proof of Theorem 4.1.

Lemma 3.4.2. *For any module* $M \in \operatorname{mod}(R)$*, for suitable choices of* $\operatorname{Tr}_{\mathcal{A}} M$ *and* $\tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} M$*, we have*

$$0 \to \operatorname{Ext}^1(M, C) \to \operatorname{Tr}_{\mathcal{A}} M \to \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega M \to 0.$$

Proof. With F_0, F_1, F_2 projective, consider the sequence

$$F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0 \to M \to 0.$$

We have $\operatorname{coker} g^\dagger = \operatorname{Tr}_{\mathcal{A}} M$ and $\operatorname{Im} f^\dagger = \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega M$. By the universal property of kernel and cokernel, we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} g^\dagger & \longrightarrow & F^\dagger & \longrightarrow & \operatorname{coker} g^\dagger & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \| & & \downarrow{\scriptstyle \varepsilon} & & \\ 0 & \longrightarrow & \ker f^\dagger & \longrightarrow & F^\dagger & \longrightarrow & \operatorname{Im} f^\dagger & \longrightarrow & 0 \end{array}$$

The snake lemma yields the exact sequence

$$0 \to \ker \iota \to 0 \to \ker \varepsilon \to \operatorname{Ext}^1(M, C) \to 0 \to \operatorname{coker} \varepsilon \to 0.$$

Thus ε is surjective and $\ker \varepsilon \cong \operatorname{Ext}^1(M, C)$. The result follows. □

Proposition 3.4.3. *If* $M \in \Delta(\mathcal{A})_0$*, for all* $0 \leq i < \operatorname{depth} C$*, for suitable choices of* $\operatorname{Tr}_{\mathcal{A}}$ *and* $\tilde{\Omega} \operatorname{Tr}_{\mathcal{A}}$*, the following is exact.*

$$0 \to \operatorname{Tr}_{\mathcal{A}} \Omega^i \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} M \to \operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Tr}_{\mathcal{A}} \Omega^i M \to \operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Ext}^{i+1}(M, C) \to 0$$

Proof. Using Lemma 4.2, we have

$$0 \to \operatorname{Ext}^{i+1}(M, C) \to \operatorname{Tr}_{\mathcal{A}} \Omega^i M \to \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} M \to 0.$$

Since $M \in \Delta(\mathcal{A})_0$, $\operatorname{Ext}^{i+1}(M, C)_p = 0$ for every nonmaximal prime p. Thus $\operatorname{Ext}^{i+1}(M, C)$ has finite length, and so $\operatorname{Ext}^i(\operatorname{Ext}^{i+1}(M, C), C) = 0$ for all $0 \le i < \operatorname{depth} C$. Thus, we can apply Lemma 3.4. □

Lemma 3.4.4. *Let $0 \le n < \operatorname{depth} R$ and L be a finite length module. There exists an $\mathcal{A}$-resolution $(G_\bullet, \partial^{L,n})$ of $\operatorname{Tr}_{\mathcal{A}} \Omega^n L$ such that $G_i = 0$ for all $i > n+1$ and*

$$\ker \partial_i^{L,n} = \operatorname{Tr}_{\mathcal{A}} \Omega^{n-i} L$$

for all $1 \le i \le n$. In particular, $\operatorname{Tr}_{\mathcal{A}} \Omega^i L \in \operatorname{res}_{\mathcal{A}}(\operatorname{Tr}_{\mathcal{A}} \Omega^n L)$ for all $0 \le i \le n$ and $\mathcal{A}\text{-}\dim(\operatorname{Tr}_{\mathcal{A}} \Omega^n L) = n+1$.

Proof. Let $(F_\bullet, \partial)$ be a free resolution of L. Then we have

$$F_{n+1} \to F_n \to \Omega^n L \to 0 \qquad 0 \to \Omega^n L \to F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to L \to 0.$$

Because L has finite length, and since $\operatorname{depth} C = \operatorname{depth} R$ by Proposition 3.7, we have $\operatorname{Ext}^i(L, C) = 0$ for all $0 \le i \le n$, and so we have the exact sequence

$$0 \to L^\dagger \to F_0^\dagger \xrightarrow{\partial_1^\dagger} F_1^\dagger \xrightarrow{\partial_2^\dagger} \cdots \xrightarrow{\partial_{n-1}^\dagger} F_{n-1}^\dagger \to (\Omega^n L)^\dagger \to 0.$$

Now $L^\dagger = 0$ since L has finite length. Thus, splicing this with

$$0 \to (\Omega^n L)^\dagger \to F_n^\dagger \xrightarrow{\partial_{n+1}^\dagger} F_{n+1}^\dagger \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L \to 0$$

we create an $\mathcal{A}$-resolution of $\operatorname{Tr}_{\mathcal{A}} \Omega^n L$. So we set $G_i = F_{n+1-i}^\dagger$ for $0 \le i \le n+1$ and $G_i = 0$

for $i > n+1$. Set $\partial_i^{L,n} = \partial_{n+2-i}^{\dagger}$ for $1 \leq i \leq n+1$ and $\partial_i^{L,n} = 0$ for all $i > n+1$. Using our previous arguments for values less that n, we see that $\ker \partial_i^{L,n} = \operatorname{Tr}_{\mathcal{A}} \Omega^{n-i} L$ for $0 \leq i \leq n$. Showing the first two claims.

It is now apparent that $\mathcal{A}\text{-}\dim \operatorname{Tr}_{\mathcal{A}} \Omega^n \leq n+1$. If $\ker \partial_n^{L,n} = \operatorname{Tr}_{\mathcal{A}} L$ is in $\mathcal{A}$, then so is L since $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} L \sim L$. In order for an n satisfying our assumption to exist, $\operatorname{depth} R$ must not be zero. So since $\mathcal{A}$ is thick in $\mathcal{G}_C$, $\mathcal{A}\text{-}\dim L = \operatorname{depth} R - \operatorname{depth} L = \operatorname{depth} R > 0$, and thus L cannot be in $\mathcal{A}$. We thus have $\mathcal{A}\text{-}\dim \operatorname{Tr}_{\mathcal{A}} \Omega^n L = n+1$ as desired. □

Lemma 3.4.5. *For all* $0 \leq n < \operatorname{depth} R$ *and all nonzero finite length modules* L*,*

$$\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L = \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n k.$$

Proof. Let λ denote the length function. If $L \neq 0$, then we can write $0 \to L' \to L \to k \to 0$ with $\lambda(L') < \lambda(L)$. Since by Proposition 3.7 $n < \operatorname{depth} R = \operatorname{depth} C$, we have $\operatorname{Ext}^n(L', C) = 0$, and so from Lemma 3.4, we have

$$0 \to \operatorname{Tr}_{\mathcal{A}} \Omega^n k \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L' \to 0.$$

Thus, by induction we have $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n k$.

Now we wish to show $\operatorname{Tr}_{\mathcal{A}} \Omega^n k \in \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L$. We proceed by double induction first on $\lambda(L)$ and then on n. The case $L = k$ is trivial, so suppose $\lambda(L) > 1$. Write

$$0 \to L' \to L \to k \to 0$$

again. Since L' has depth zero, we can use Lemma 4.4 to get the resolution $(G_\bullet, \partial^{L'})$. Thus we have the exact sequence

$$0 \to \ker \partial_1^{L',n} \to G_0 \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L \to 0.$$

Taking the pullback diagram with our last exact sequence yields the following.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker \partial_1^{L',n} & = & \ker \partial_1^{L',n} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} \Omega^n k & \longrightarrow & T & \longrightarrow & G_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} \Omega^n k & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} \Omega^n L & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} \Omega^n L' & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It is now easy to see that it suffices to show that $\ker \partial_1^{L',n}$ is in $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L$. When $n = 0$, $(G_\bullet, \partial^{L',n})$ is the resolution

$$0 \to G_1 \xrightarrow{\partial_1^{L',0}} G_0 \to \operatorname{Tr}_{\mathcal{A}} L' \to 0,$$

and we are done since $\ker \partial_1^{L',0} = G_1 \in \mathcal{A} \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} L$. So suppose $n > 0$. We have $\ker \partial_1^{L',n} = \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L'$, by Lemma 4.4. By induction, $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L$ and $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L'$ are the same as $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} k$. So we have $\ker \partial_1^{L',n} \in \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L$, where the inclusion follows from Lemma 4.4, and we are done. Note that this argument works for any choice of $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L$ or $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^n L'$. □

These next proofs are essentially identical to the proofs in Dao & Takahashi (2013) with the appropriate changes. They are included here for the sake of completeness.

Proposition 3.4.6. *For every* $1 \leq n \leq \operatorname{depth} R$, *we have* $\Delta(\mathcal{A})_0^n = \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L$ *for every nonzero finite length module* L.

Proof. By Lemma 4.5, we may assume that $L = k$. By Lemma 4.4, we know that $\mathcal{A}\text{-}\dim(\operatorname{Tr} \Omega^{n-1} k) = n$. Since localization commutes with cokernels, duals and syzygies, we have $\operatorname{Tr} \Omega^n k$ is in $\Delta(\mathcal{A})$

and hence in $\Delta(\mathcal{A})_0^n$. Suppose $M \in \Delta(\mathcal{A})_0^n$. Proposition 4.3 tells us that for each $0 \leq i < n$, we have

$$\operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Tr}_{\mathcal{A}} \Omega^i M \in \operatorname{res}_{\mathcal{A}}(\operatorname{Tr}_{\mathcal{A}} \Omega^i \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} M, \operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Ext}^{i+1}(M, C)).$$

Lemma 3.3 says that $\operatorname{Tr}_{\mathcal{A}} \Omega^i \tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} M \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} M$. Furthermore, because $\operatorname{Ext}^{i+1}(M, C)$ has finite length, Lemma 4.5 implies that $\operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Ext}^{i+1}(M, C)$ is in $\operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^i k \subseteq \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} k$, where the inclusion follows from Lemma 4.4. Hence we have

$$\operatorname{Tr}_{\mathcal{A}} \Omega^i \operatorname{Tr}_{\mathcal{A}} \Omega^i M \in \operatorname{res}_{\mathcal{A}}(\operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} \operatorname{Tr}_{\mathcal{A}} \Omega^{i+1} M, \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} k).$$

It follows by induction that $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} M \sim M$ is in

$$\operatorname{res}_{\mathcal{A}}(\operatorname{Tr}_{\mathcal{A}} \Omega^n \operatorname{Tr}_{\mathcal{A}} \Omega^n M, \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} k).$$

However, $\Omega^n M \in \mathcal{A}$ and thus so is $\operatorname{Tr}_{\mathcal{A}} \Omega^n \operatorname{Tr}_{\mathcal{A}} \Omega^n M$, and we have $M \in \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} k$, which completes the proof. $\square$

We now prove the main result of this section.

Proof of Theorem 4.1. We clearly have the chain

$$\mathcal{A} = \Delta(\mathcal{A})_0^0 \subsetneq \Delta(\mathcal{A})_0^1 \subsetneq \cdots \subsetneq \Delta(\mathcal{A})_0^d = \Delta(\mathcal{A})_0.$$

Take $X \in \Delta(\mathcal{A})_0^n \setminus \Delta(\mathcal{A})_0^{n-1}$ for $d \geq n \geq 1$. We need to show that $\operatorname{res}_{\mathcal{A}} X = \Delta(\mathcal{A})_0^n$, and we have $\operatorname{res}_{\mathcal{A}} X \subseteq \Delta(\mathcal{A})_0^n$. For the reverse inclusion, by Proposition 4.6, it suffices to show $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L \in \operatorname{res}_{\mathcal{A}} X$ for some finite length module L.

Since $\operatorname{Ext}^n(X, C)$ is not zero and its localization is zero at every prime not equal to $\mathfrak{m}$, $\operatorname{Ext}^n(X, C)$ has finite length, and for every finite length module L,

$$\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} L \in \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C).$$

Using the $\mathcal{A}$-resolution of X

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to X \to 0$$

to compute $\operatorname{Ext}^i(X, C)$ and the $\mathcal{A}$-presentation $0 \to F_n \to F_{n-1} \to \Omega^{n-1}X \to 0$ to compute $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}X$, we have $\operatorname{Ext}^n(X, C) \sim \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}X$ (this is why the generalization of the Auslander transpose in Geng (2013) and Huang (1999) was insufficient). So, for any finite length module L, we have

$$\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}L \in \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C) = \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}X. \tag{3.3}$$

Therefore it suffices to show that $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}X$ is in $\operatorname{res}_{\mathcal{A}} X$ for some (and hence every) choice of $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}X$. We will show this by induction on n. When $n = 1$, we are done by $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} X \sim X$. Now assume $n > 1$. By induction, (4), and Lemma 4.5, for all finite length L we have

$$\operatorname{Tr}_{\mathcal{A}} \Omega^{n-2}L \in \operatorname{res}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-2} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-2}(\Omega X) \subseteq \operatorname{res}_{\mathcal{A}} \Omega X \subseteq \operatorname{res}_{\mathcal{A}} X. \tag{3.4}$$

Then Proposition 4.3 gives us the exact sequence

$$0 \to \operatorname{Tr}_{\mathcal{A}} \Omega^{n-2}\tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}X \to \operatorname{Tr}_{\mathcal{A}} \Omega^{n-2} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-2}X \to \operatorname{Tr}_{\mathcal{A}} \Omega^{n-2} \operatorname{Ext}^{n-1}(X, C) \to 0.$$

But since $\operatorname{Ext}^{n-1}(X, C)$ has finite length, (5) tells us that

$$\operatorname{Tr}_{\mathcal{A}} \Omega^{n-2} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-2}X, \operatorname{Tr}_{\mathcal{A}} \Omega^{n-2} \operatorname{Ext}^{n-1}(X, C) \in \operatorname{res}_{\mathcal{A}} X.$$

Therefore $\operatorname{Tr}_{\mathcal{A}} \Omega^{n-2}\tilde{\Omega} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}X$ is in $\operatorname{res}_{\mathcal{A}} X$, and so, by Lemma 3.3,

$$\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1}X \in \operatorname{res}_{\mathcal{A}} X.$$

Thus, we are done. $\square$

The following corollary is immediate from Theorem 4.1

Corollary 3.4.7. *If* $M \in \Delta(\mathcal{A})_0^t \setminus \Delta(\mathcal{A})_0^{t-1}$, *then* $\operatorname{res}_{\mathcal{A}} M = \Delta(\mathcal{A})_0^t$.

3.5 Resolving subcategories and semidualizing modules

In this section, we keep the same notations and conventions as the previous section, except we will not assume that R is local. In this section, we prove Theorem 5.4, which is a critical step towards proving Corollary 6.6 and Theorem 6.8. Note that it is easy to check that C_p is a semidualizing R_p-module for all $p \in \operatorname{spec} R$. Using Lemma 6.2, it is also easy to show that for all $p \in \operatorname{spec} R$, $\operatorname{add} \mathcal{A}_p$ is a thick subcategory of $\mathcal{G}_{C_p}$ closed under duals and contains C_p. The following is a modified version of Dao & Takahashi (2012)[Lemma 4.6], which is a generalization of Takahashi (2009)[Proposition 4.2]. For a module X, let $\mathrm{NA}(X) = \{p \in \operatorname{spec} R \mid X_p \notin \operatorname{add} \mathcal{A}_p\}$.

Proposition 3.5.1. *Suppose* $X \in \Delta(\mathcal{A})$. *For every* $p \in \mathrm{NA}(X)$, *there is a* $Y \in \operatorname{res}_{\mathcal{A}} X$ *such that* $\mathrm{NA}(Y) = \mathrm{V}(p)$ *and* $\operatorname{add} \mathcal{A}_\pi \text{-}\dim Y_\pi = \operatorname{add} \mathcal{A}_\pi \text{-}\dim X_\pi$ *for all* $\pi \in V(p)$.

Proof. If $\mathrm{NA}(X) = \mathrm{V}(p)$ we are done. So fix a $q \in \mathrm{NA}(X) \setminus \mathrm{V}(p)$. As in the proof of Dao & Takahashi (2012)[Lemma 4.6], choose an $x \in p \setminus q$ and consider the following pushout diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega X & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle x}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega X & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

with F projective. Immediately, we have $Y \in \operatorname{res}_{\mathcal{A}} X$. Therefore, for all $\pi \in \operatorname{spec} R$, if X_π is in a resolving subcategory, then so is Y_π, and thus we have $\mathrm{NA}(Y) \subseteq \mathrm{NA}(X)$. The proof of Dao & Takahashi (2012)[Lemma 4.6] tells us that

$$\operatorname{depth}(Y_\pi) = \min\{\operatorname{depth}(X_\pi), \operatorname{depth}(R_\pi)\}$$

for all $\pi \in V(p)$. Thus, by Proposition 3.7, $\operatorname{add}\mathcal{A}_\pi\text{-dim}\, Y_\pi = \operatorname{add}\mathcal{A}_\pi\text{-dim}\, X_\pi$, for all $\pi \in V(p)$. In particular, this shows that $V(p)$ is contained in $\mathrm{NA}(Y)$.

Localizing at q, yields the following.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega X_q & \longrightarrow & F_q & \longrightarrow & X_q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega X_q & \longrightarrow & Y_q & \longrightarrow & X_q & \longrightarrow & 0 \end{array}$$

Note x is a unit in R_q. Thus, by the five lemma, Y_q is isomorphic to F_p and therefore is projective. So we have $q \notin \mathrm{NA}(Y)$ and hence $\mathrm{NA}(Y) \subsetneq \mathrm{NA}(X)$.

If $\mathrm{NA}(Y) \neq \mathrm{V}(p)$, then we may repeat this process and construct a Y' that, like Y, satisfies all the desired properties except $\mathrm{V}(p) \subseteq \mathrm{NA}(Y') \subsetneq \mathrm{NA}(Y) \subsetneq \mathrm{NA}(X)$. Since $\operatorname{spec} R$ is Noetherian, this process must stabilize after some iteration, producing the desired module.

□

Lemma 3.5.2. *Let Z be a nonempty finite subset of $\operatorname{spec} R$. Let M be a module and $\mathcal{X}$ a resolving subcategory such that $M_p \in \operatorname{add}\mathcal{X}_p$ for some $p \in \operatorname{spec} R$. Then there exist exact sequences*

$$0 \to K \to X \to M \to 0 \qquad 0 \to L \to M \oplus K \oplus R^t \to X \to 0$$

with $X \in \mathcal{X}$ and $\mathrm{NA}(L) \subseteq \mathrm{NA}(M)$ and $\mathrm{NA}(L) \cap Z = \emptyset$.

Proof. The result is essentially contained in the proof of Takahashi (2010)[Proposition 4.7]. It shows the existence of the exact sequences and shows that Z is contained in the free locus of L and thus $\mathrm{NA}(L) \cap Z = \emptyset$. Furthermore, the last exact sequence in that proof shows that for any $p \in \operatorname{spec} R$, L_p is in $\operatorname{res} M_p$. Hence, if L_p is not in a resolving subcategory, then M_p cannot be in that category giving us $\mathrm{NA}(L) \subseteq \mathrm{NA}(M)$. □

These lemmas help prove the following proposition which is a key component of the proof of Theorem 5.4. This next result is also where we use Corollary 4.7 of the last section.

Proposition 3.5.3. *For a module $M \in \operatorname{mod}(R)$ and a category $\mathcal{X} \in \mathfrak{R}(\mathcal{A})$, if for every $p \in \operatorname{spec} R$, there exists an $X \in \mathcal{X}$ such that $\operatorname{add} \mathcal{A}_\mathfrak{p}\text{-dim}\, M_p \leq \operatorname{add} \mathcal{A}_\mathfrak{p}\text{-dim}\, X_p$, then M is in $\mathcal{X}$.*

Proof. Because of Lemma 2.12, we may assume $(R, \mathfrak{m}, k)$ is local. We proceed by induction on $\dim(\operatorname{NA} M)$. If $\dim \operatorname{NA} M = -\infty$, then M is in $\mathcal{A}$ and we are done. Suppose $\dim \operatorname{NA}(M) = 0$. Then M is in $\Delta(\mathcal{A})_0^t$ where $t = \mathcal{A}\text{-dim}\, X$. By Proposition 5.1, there exists a $Y \in \operatorname{res}_\mathcal{A} X \subseteq \mathcal{X}$ with $\mathcal{A}\text{-dim}\, Y = t$ and $Y \in \Delta(\mathcal{A})_0$, and thus $Y \in \Delta(\mathcal{A})_0^t \setminus \Delta(\mathcal{A})_0^{t-1}$. By Corollary 4.7, $\operatorname{res}_\mathcal{A} Y = \Delta(\mathcal{A})_0^t$, and thus $M \in \operatorname{res}_\mathcal{A}(Y) \subseteq \mathcal{X}$.

The rest of the proof uses Lemma 5.2 and is identical to Dao & Takahashi (2013)[Theorem 3.5], except one replaces the nonfree locus of M with $\operatorname{NA}(M)$ and replaces projective dimension with $\mathcal{A}$-dim. □

We come to the main theorem of this section. Recall that Γ is the set of grade consistent functions.

Theorem 3.5.4. *Assume R is Noetherian. If $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$ containing C and is closed under duals, then $\Lambda(\mathcal{A})$ and $\Phi_\mathcal{A}$ are inverse functions and give a bijection between Γ and $\mathfrak{R}(\mathcal{A})$.*

Proof. The previous proposition shows that $\Lambda(\mathcal{A})\Phi_\mathcal{A}$ is the identity on $\mathfrak{R}(\mathcal{A})$. Let $f \in \Gamma$ and $p \in \operatorname{spec} R$. Since $\operatorname{add} \mathcal{A}_p\text{-dim}\, X_p \leq f(p)$ for every $X \in \Lambda(\mathcal{A})(f)$, we have

$$\Phi_\mathcal{A}(\Lambda(\mathcal{A})(f))(p) \leq f(p).$$

However, by Dao & Takahashi (2013)[Lemma 5.1] there is an $M \in \overline{\mathcal{P}} \subseteq \overline{\mathcal{A}}$ such that $\operatorname{pd}_{R_p} M_p = f(p)$ and $\operatorname{pd}_{R_q} M_q \leq f(q)$ for all $q \in \operatorname{spec} R$. Since for all $q \in \operatorname{spec} R$ $\operatorname{pd}_q M_q = \operatorname{add} \mathcal{A}_q\text{-dim}\, M_q$, M is in $\Lambda(\mathcal{A})(f)$, and we have $\Phi_\mathcal{A}(\Lambda(\mathcal{A})(f))(p) = f(p)$. Thus $\Phi_\mathcal{A}\Lambda(\mathcal{A})$ is the identity on Γ. □

3.6 Resolving subcategories that are closed under duals

We wish to expand upon Theorem 5.4 using the results in Section 2. However, to use Theorem 5.4, we need to understand which thick subcategories of $\mathcal{G}_C$ are closed under duals and contain C. In this section, C will be a semidualizing module. Since $\mathcal{G}_C$ is cogenerated by $\operatorname{add} C$, as seen at the end of Section 3, it stands to reason that the results of Section 2 are applicable.

Lemma 3.6.1. *Suppose $\mathcal{X} \subseteq \mathcal{G}_C$ is resolving with $C \in \mathcal{X}$. Then $\mathcal{X}$ is thick in $\mathcal{G}_C$ if and only if for every $X \in \mathcal{X}$, $(\Omega X^\dagger)^\dagger$ is in $\mathcal{X}$. In particular, $\mathcal{X}$ is thick in $\mathcal{G}_C$ if and only if it is cogenerated by* $\operatorname{add} C$.

When $R = C$, this is equivalent to saying that a resolving subcategory $\mathcal{X}$ of $\mathcal{G}_R$ is thick if and only if it is closed under cosyzygies. Also, since syzygies are unique up to projective summands, $(\Omega X^\dagger)^\dagger$ is unique up to $\operatorname{add} C$ summands. Thus, for our purposes, our choice of syzygy is inconsequential.

Proof. We have the following exact sequence.

$$0 \to \Omega X^\dagger \to R^n \to X^\dagger \to 0$$

Applying $\dagger$ yields

$$0 \to X \to C^n \to (\Omega X^\dagger)^\dagger \to 0.$$

Since $C \in \mathcal{X}$, if $\mathcal{X}$ is thick in $\mathcal{G}_C$, $(\Omega X^\dagger)^\dagger$ is in $\mathcal{X}$. With regards to the second statement, $\operatorname{add} C$ cogenerates $\mathcal{X}$ in this case.

Conversely, suppose for every $X \in \mathcal{X}$, $(\Omega X^\dagger)^\dagger$ is in $\mathcal{X}$. First we show that $\mathcal{X}^\dagger$ is resolving. Since $C \in \mathcal{X}$, R is in $\mathcal{X}^\dagger$. For every $X^\dagger \in \mathcal{X}^\dagger$, we have $(\Omega X^\dagger)^{\dagger\dagger} \cong \Omega X^\dagger \in \mathcal{X}^\dagger$, and so $\mathcal{X}^\dagger$ is closed under syzygies. Given a short exact sequence $0 \to X^\dagger \to Y \to Z^\dagger \to 0$. With $X^\dagger, Z^\dagger \in \mathcal{X}^\dagger$, we have $0 \to Z \to Y^\dagger \to X \to 0$. Thus $Y^\dagger \in \mathcal{X}$, and so Y is in $\mathcal{X}^\dagger$. Showing $\mathcal{X}^\dagger$ is closed under direct summands is easy, and so $\mathcal{X}^\dagger$ is resolving.

Given a short exact sequence $0 \to M \to N \to L \to 0$ with $M, N \in \mathcal{X}$, then we have $0 \to L^\dagger \to N^\dagger \to M^\dagger \to 0$ with $N^\dagger, M^\dagger \in \mathcal{X}^\dagger$. So $L^\dagger$ is in $\mathcal{X}^\dagger$ since $\mathcal{X}^\dagger$ is resolving, and $L \cong L^{\dagger\dagger}$ is in $\mathcal{X} = \mathcal{X}^{\dagger\dagger}$. Since $\mathcal{X}$ is resolving, this completes the proof of the first statement.

To prove the second statement, suppose $\mathcal{X}$ is cogenerated by $\operatorname{add} C$. Then we have $0 \to X \to C^n \to Y \to 0$ for any $X \in \mathcal{X}$ which yields $0 \to Y^\dagger \to R^n \to X^\dagger \to 0$. So $\Omega X^\dagger$ is stably equivalent to Y, and thus $(\Omega X^\dagger)^\dagger \sim Y$. So $(\Omega X^\dagger)^\dagger$ is in $\mathcal{X}$.

□

The following corollary, although intuitive, is not obvious, and it is not clear if it holds for other subcategories besides $\mathcal{G}_C$.

Corollary 3.6.2. *If $\mathcal{X}$ is thick in $\mathcal{G}_C$, then $\operatorname{add} \mathcal{X}_p$ is thick in $\mathcal{G}_{C_p}$ for all $p \in \operatorname{spec} R$.*

Proof. Take $p \in \operatorname{spec} R$. From Lemma 2.11, we know that $\operatorname{add} \mathcal{X}_p$ is resolving. By the previous lemma, it suffices to show that for all $X \in \operatorname{add} \mathcal{X}_p$, $(\Omega_{R_p} X^\dagger)^\dagger = \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(X, C_p), C_p)$ is in $\operatorname{add} \mathcal{X}_p$. For every $X \in \operatorname{add} \mathcal{X}_p$, there exists a Y such that $X \oplus Y = Z_p$ for some $Z \in \mathcal{X}$. Consider the following.

$$\begin{aligned}
{(\Omega Z^\dagger)^\dagger}_p &= \operatorname{Hom}(\Omega_R \operatorname{Hom}(Z, C), C)_p \\
&= \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(Z_p, C_p), C_p) \\
&= \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(X \oplus Y, C_p), C_p) \\
&= \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(X, C_p), C_p) \oplus \operatorname{Hom}(\Omega_{R_p} \operatorname{Hom}(Y, C_p), C_p)
\end{aligned}$$

By the previous lemma, $(\Omega Z^\dagger)^\dagger$ is in $\mathcal{X}$, and so $(\Omega_{R_p} X^\dagger)^\dagger$ is in $\operatorname{add} \mathcal{X}_p$. □

The following lemmas show how to construct thick subcategories of $\mathcal{G}_C$ closed under duality.

Lemma 3.6.3. *For a subset $\mathcal{W} \subseteq \mathcal{G}_C$ containing C, if for every $X \in \mathcal{W}$, $(\Omega X^\dagger)^\dagger$ is in $\mathcal{W}$ and $(\Omega C)^\dagger$ is in $\mathcal{W}$, then $\operatorname{res} \mathcal{W}$ is thick in $\mathcal{G}_C$.*

Proof. In light of the last lemma, we only need to show that for every $X \in \operatorname{res} \mathcal{W}$, $(\Omega X^\dagger)^\dagger$ is in $\operatorname{res} \mathcal{W}$. We proceed by induction on the number of steps it takes to construct an element in $\operatorname{res} \mathcal{X}$. See Takahashi (2009) for a precise definition of the notion of steps with regards to a resolving subcategory. The elements in $\operatorname{res} \mathcal{X}$ that take zero steps are $R \cup \mathcal{W}$, and these satisfy our claim by assumption. Suppose X is constructed in $n > 0$ steps. Then there exists a Y and a Z that can be constructed in $n-1$ steps that satisfy one of the following situations.

1. $0 \to X \to Y \to Z \to 0$
2. $0 \to Y \to X \to Z \to 0$
3. $Z = X \oplus W$

Therefore one of the following is true.

1. $0 \to (\Omega X^\dagger)^\dagger \to (\Omega Y^\dagger)^\dagger \to (\Omega Z^\dagger)^\dagger \to 0$
2. $0 \to (\Omega Y^\dagger)^\dagger \to (\Omega X^\dagger)^\dagger \to (\Omega Z^\dagger)^\dagger \to 0$
3. $(\Omega Z^\dagger)^\dagger = (\Omega X^\dagger)^\dagger \oplus (\Omega W^\dagger)^\dagger$

By induction, $(\Omega Y^\dagger)^\dagger$ and $(\Omega Z^\dagger)^\dagger$ are in $\operatorname{res} \mathcal{W}$, and the result follows. $\square$

Lemma 3.6.4. *Suppose $\mathcal{W} \subseteq \mathcal{G}_C$ is a subset containing C and $(\Omega C)^\dagger$. If for every $X \in \mathcal{W}$, $X^\dagger$ and $(\Omega X^\dagger)^\dagger$ are in $\mathcal{W}$, then* $\operatorname{res} \mathcal{W}$ *is a thick subcategory of $\mathcal{G}_C$ that is closed under $\dagger$.*

Proof. The previous lemma shows that $\operatorname{res} \mathcal{W}$ is thick in $\mathcal{G}_C$. We will show that for every $X \in \operatorname{res} \mathcal{W}$, $X^\dagger$ is in $\operatorname{res} \mathcal{W}$ by inducting on the number of steps it takes to construct an element in the thick subcategory $\operatorname{res} \mathcal{W}$. The step zero modules are $\mathcal{W}$ and hence the claim is true. Now assume that the statement is true for step n. Given $0 \to X \to Y \to Z \to 0$, we have $0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0$. Hence if any two of X, Y, Z are in step n, then two of $X^\dagger, Y^\dagger, Z^\dagger$ are in $\operatorname{res} \mathcal{W}$, and so the third is too. Since $\dagger$ splits across direct sums, the result follows. $\square$

Let $\mathcal{W}$ be the set of of modules obtained by applying † and Ω to C successive times. The following motivates us to set $\mathcal{A} = \text{res}\{\mathcal{W}\}$.

Proposition 3.6.5. *The category $\mathcal{A}$ is the smallest thick category in $\mathcal{G}_C$ containing C which is closed under †.*

Proof. The previous lemma shows that $\mathcal{A}$ is thick and closed under duals. It is also easy to see that any thick subcategory containing C must contain $\mathcal{W}$. □

In the notation of Section 2, set $\mathfrak{S}(C) = \mathfrak{S}(\mathcal{A})$. We may now apply our results from the beginning of the paper.

Corollary 3.6.6. *The following is a bijection.*

$$\Lambda : \mathfrak{S}(C) \times \Gamma \to \bigcup_{\mathcal{C} \in \mathfrak{S}} \mathfrak{R}(\mathcal{C}) \subseteq \mathfrak{R}$$

Furthermore, for any $\mathcal{C}, \mathcal{D} \in \mathfrak{S}(C)$, then the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{R}(\mathcal{D}) & \xrightarrow{\Phi_{\mathcal{D}}} & \Gamma \\ {\scriptstyle \eta^{\mathcal{D}}_{\mathcal{C}}}\uparrow & & \| \\ \mathfrak{R}(\mathcal{C}) & \xrightarrow{\Phi_{\mathcal{C}}} & \Gamma \\ {\scriptstyle \eta^{\mathcal{C}}_{\mathcal{A}}}\uparrow & & \| \\ \mathfrak{R}(\mathcal{A}) & \xrightarrow{\Phi_{\mathcal{A}}} & \Gamma \end{array}$$

In particular, $\rho^{\mathcal{D}}_{\mathcal{C}}$ and $\eta^{\mathcal{D}}_{\mathcal{C}}$ are inverse functions.

Proof. The previous proposition tells us that $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$ which contains C and is closed under †. Therefore, by Theorem 5.4, $\Lambda(\mathcal{A})$ and $\Phi_{\mathcal{A}}$ give a bijection between $\mathfrak{R}(\mathcal{A})$ and Γ. The result then follows immediately from Theorem 2.4. □

Lemma 3.6.7. *The collection of all the thick subcategories of $\mathcal{G}_C$ containing C is contained in $\mathfrak{S}(C)$. Furthermore, when R is Cohen-Macaulay, every element in $\mathfrak{S}(C)$ is contained in*

MCM. In particular, when $C = D$ is a dualizing module, $\mathfrak{S}(D)$ is the collection of thick subcategories of MCM containing D.

Proof. Suppose $\mathcal{X}$ is thick in $\mathcal{G}_C$ and contains C. Then $\mathcal{X}$ must contain $\mathcal{W}$ and hence contains $\mathcal{A}$. By Lemma 6.1, $\mathcal{X}$ is cogenerated by $\operatorname{add} C$ and hence, by $\mathcal{A}$. Since $\mathcal{A}$ is thick in $\mathcal{G}_C$, $\mathcal{A}$ is thick in $\mathcal{X}$ as well, since any short exact sequence in $\mathcal{X}$ is a short exact sequence in $\mathcal{G}_C$. By Corollary 6.2, $\operatorname{add} \mathcal{A}_\mathfrak{p}$ is thick in $\mathcal{G}_{C_p}$ and thus thick in $\operatorname{add} \mathcal{X}_p$ for all $p \in \operatorname{spec} R$. Hence $\mathcal{X}$ is in $\mathfrak{S}(C)$.

Now suppose that R is Cohen-Macaulay and $\mathcal{X} \in \mathfrak{S}(C)$. Since $\mathcal{A}$ cogenerates $\mathcal{X}$, for any $X \in \mathcal{X}$ there exists $0 \to X \to A_0 \to \cdots \to A_d \to X' \to 0$ with $A \in \mathcal{A}$ and $d = \operatorname{depth} R$. Since $\mathcal{A} \subseteq$ MCM, X is in MCM. The last statement is now clear, since in that case $\mathcal{G}_D =$ MCM. ☐

We now come to one of the main results of the paper.

Theorem 3.6.8. *For any thick subcategory $\mathcal{C}$ of $\mathcal{G}_C$ containing C, $\Lambda(\mathcal{C})$ and $\Phi_\mathcal{C}$ give a bijection between $\mathfrak{R}(\mathcal{C})$ and Γ.*

Proof. The previous lemma shows that $\mathcal{C}$ is in $\mathfrak{S}(C)$. The rest follows from Theorem 2.2 and Theorem 5.4. ☐

A resolving subcategory $\mathcal{X}$ is dominant if for every $p \in \operatorname{spec} R$, there is an $n \in \mathbb{N}$ such that $\Omega^n_{R_p} R_p/pR_p \in \operatorname{add} \mathcal{X}_p$.

Corollary 3.6.9. *Suppose R is Cohen-Macaulay and has a canonical module. Then there is a bijection between resolving subcategories containing MCM and grade consistent functions. Furthermore, the following are equivalent for a resolving subcategory $\mathcal{X}$.*

1. *$\mathcal{X}$ is dominant*

2. *MCM $\subseteq \mathcal{X}$*

3. Thick $\mathcal{X} = \operatorname{mod}(R)$

Proof. Letting D be the dualizing modules of R, MCM is the same as $\mathcal{G}_D$. Hence, by the previous theorem, $\Lambda(\mathrm{MCM}) : \Gamma \to \mathfrak{R}(\mathrm{MCM})$ is a bijection, showing the first statement. From (Dao & Takahashi, 2013, Theorem 1.3), the following is a bijection.

$$\xi : \Gamma \to \{\text{Dominant Resolving subcategories of } \operatorname{mod}(R)\}$$

$$\xi(f) = \{M \in \operatorname{mod}(R) \mid \operatorname{depth} M_{\mathfrak{p}} \geq \operatorname{ht} \mathfrak{p} - f(\mathfrak{p})\}$$

It is clear that $\xi(0) = \mathrm{MCM}$, hence every dominant subcategory contains MCM. Furthermore, we have $\operatorname{mod}(R) = \overline{\mathrm{MCM}}$, and hence every dominant resolving subcategory is an element of $\mathfrak{R}(\mathrm{MCM})$. Then for any $f \in \Gamma$, we have

$$\begin{aligned}
\xi(f) &= \{M \in \operatorname{mod}(R) \mid \operatorname{depth} M_{\mathfrak{p}} \geq \operatorname{ht} \mathfrak{p} - f(\mathfrak{p})\} \\
&= \{M \in \operatorname{mod}(R) \mid \operatorname{add} \mathrm{MCM}_{\mathfrak{p}}\text{-}\dim M_{\mathfrak{p}} \leq f(\mathfrak{p})\} \\
&= \Lambda(\mathrm{MCM})(f)
\end{aligned}$$

Thus ξ equals $\Lambda(\mathrm{MCM})$, showing the equivalence of 1 and 2.

It is clear that 2 implies 3. Assume 3. Take a $p \in \operatorname{spec} R$. Then we have

$$\mathcal{X}\text{-}\dim R/p < \infty.$$

This implies that $\Omega^n R/p \in \mathcal{X}$ for some n. Hence $\Omega^n_{R_p} R_p/pR_p \in \operatorname{add} \mathcal{X}_p$, and so $\mathcal{X}$ is dominant.

□

3.7 Gorenstein rings and vanishing of ext

In this section, $(R, \mathfrak{m}, k)$ is a local Gorenstein ring. In this case, MCM is the same as $\mathcal{G}_R$, and Lemma 6.7 implies that $\mathfrak{S}(R)$ is merely the collection of thick subcategories of MCM.

This gives us the following which recovers Dao & Takahashi (2013)[Theorem 7.4].

Theorem 3.7.1. *If R is Gorenstein, then we have the following commutative diagram of bijections.*

$$\begin{array}{ccc}
\{\textit{Thick subcategories of MCM}\} \times \Gamma & & \\
& \xrightarrow{\Lambda} & \\
\Big\downarrow \Lambda(\mathcal{P}) & & \{\mathcal{C} \in \mathfrak{R} \mid \mathcal{C} \cap \textit{MCM is thick in MCM}\} \\
& \xrightarrow{\Xi} & \\
\{\textit{Thick subcategories of MCM}\} \times \mathfrak{R}(\mathcal{P}) & &
\end{array}$$

Proof. Let $\mathcal{T}$ be the collection of resolving subcategories whose intersection with MCM is thick in MCM. As observed before the Theorem, $\mathfrak{S}(R)$ is simply the thick subcategories of MCM. Since for any $\mathcal{C} \in \mathfrak{S}(R)$, $\overline{\mathcal{C}} \cap \mathrm{MCM}$ is $\mathcal{C}$, the image of Λ lies in $\mathcal{T}$. Furthermore, for any $\mathcal{X} \in \mathcal{T}$, $\mathcal{X}$ is in $\mathfrak{R}(\mathcal{X} \cap \mathrm{MCM})$, thus the result follows from Proposition 2.6 and Theorem 6.8. □

It is natural to ask when the image Λ is all of $\mathfrak{R}(R)$. This will happen precisely when every resolving subcategory of MCM is thick. This occurs, by Dao & Takahashi (2013)[Theorem 6.4], when R is a complete intersection. We will give a necessary condition for $\operatorname{Im}\Lambda = \mathfrak{R}(R)$ by examining the resolving subcategories of the form

$$\mathcal{C}_{\mathcal{B}} = \{M \in \mathrm{mod}(R) \mid \mathrm{Ext}^{i>0}(M,N) = 0 \quad \forall N \in \mathcal{B}\}$$

where $\mathcal{B} \subseteq \mathrm{mod}(R)$. Dimension with respect to this category can be calculated in the following manner.

Lemma 3.7.2. *For all $\mathcal{B} \subseteq \mathrm{mod}(R)$, we have the following.*

$$\mathcal{C}_{\mathcal{B}}\text{-}\dim M = \inf\{n \mid \mathrm{Ext}^{i>n}(M,N) = 0 \quad \forall N \in \mathcal{B}\}$$

Proof. Let $M \in \text{mod}(R)$. For all $i > 0$ and $j \geq 0$ and each $N \in \mathcal{B}$, we have

$$\text{Ext}^{i+j}(M, N) = \text{Ext}^i(\Omega^j M, N).$$

So $\text{Ext}^{i+n}(M, N) = 0$ for all $i \geq 0$ if and only if $\Omega^n M$ is in $\mathcal{C}_\mathcal{B}$. □

Lemma 3.7.3. *For any $\mathcal{B} \subseteq \text{mod}(R)$, we have $\mathcal{C}_\mathcal{B} \cap D(\mathcal{P}) = \mathcal{P}$.*

Proof. To prove this, it suffices to show that if $\text{pd}(X) = n > 0$, then $\text{Ext}^n(X, M) \neq 0$. Take a minimal free resolution

$$0 \to F_n \xrightarrow{d} F_{n-1} \to \cdots \to F_0 \to X \to 0.$$

Note that $\text{Im}(d) \subseteq \mathfrak{m}F_{n-1}$. We then get the complex

$$0 \to \text{Hom}(X, M) \to \text{Hom}(F_0, M) \to \cdots \to \text{Hom}(F_{n-1}, M) \xrightarrow{d^*} \text{Hom}(F_n, M) \to 0.$$

Now $\text{Im}(d^*)$ will still lie in $\mathfrak{m}\,\text{Hom}(F_n, M)$, and thus by Nakayama, d^* cannot be surjective. Hence we have $\text{Ext}^n(X, M) = \text{coker}\, d^* \neq 0$. □

Araya in Araya (2012a) defined AB dimension by $\text{AB-dim}\, M = \max\{b_m, \mathcal{G}_R\text{-dim}\, M\}$ where

$$b_M = \min\{i \mid \text{Ext}^{\gg 0}(M, X) = 0 \Rightarrow \text{Ext}^{>i}(M, X) = 0\}.$$

Note that AB dimension satisfies the Auslander Buchsbaum formula. Also, a ring is AB if and only if every module has finite AB dimension.

Lemma 3.7.4. *Taking $\mathcal{B} \subseteq \text{mod}(R)$, if $\text{AB-dim}\, M < \infty$ for all $M \in \overline{\mathcal{C}_\mathcal{B}}$, then $\mathcal{C}_\mathcal{B}$ is a thick subcategory of MCM.*

Proof. Suppose $\text{AB-dim}\, \overline{\mathcal{C}} < \infty$. First, we show that $\mathcal{C}_\mathcal{B}$ is contained in MCM. Take any $M \in \mathcal{C}_\mathcal{B}$. There is an exact sequence $0 \to M \to Y \to X \to 0$ with $\text{pd}(Y) < \infty$ and $X \in \text{MCM}$.

We claim that X has AB dimension zero. Suppose $\operatorname{Ext}^{\gg 0}(X,Z)=0$. Then $\operatorname{Ext}^{\gg 0}(Y,Z)=0$ and since $\operatorname{pd} Y = \text{AB-dim}\, Y$, $\operatorname{Ext}^{>\operatorname{pd} Y}(Y,Z)$ is zero. Then we have $\operatorname{Ext}^{\gg 0}(M,Z)=0$ and thus $\operatorname{Ext}^{>b_M}(M,Z)=0$. Therefore for all $i > \max\{\operatorname{pd}(Y), b_M\}+1$ we have $\operatorname{Ext}^i(X,Z)=0$. Since R is Gorenstein, that means that X has finite $\mathcal{G}_R$ dimension, and thus X has finite AB dimension. But since AB dimension satisfies the Auslander Buchsbaum formula, AB-dim X must be zero.

Since $Y \in \overline{\mathcal{C}_\mathcal{B}}$, we have $X \in \overline{\mathcal{C}_\mathcal{B}}$. So $\operatorname{Ext}^{\gg 0}(X,N)=0$ for all $N \in \mathcal{B}$, and we have $\operatorname{Ext}^{>0}(X,N)=0$ for all $N \in \mathcal{B}$. Hence X is in $\mathcal{C}_\mathcal{B}$. Therefore, Y is also in $\mathcal{C}_\mathcal{B}$, which, by Lemma 7.3, means that Y is projective and hence in MCM, forcing M to be in MCM as well.

Now to show that $\mathcal{C}_\mathcal{B}$ is thick in MCM, it suffices to show that $\mathcal{C}_\mathcal{B}$ is closed under cokernels of surjections in MCM. So take $0 \to A \to B \to C \to 0$ with $A, B, C \in$ MCM and $A, B \in \mathcal{C}_\mathcal{B}$. Then $C \in \overline{\mathcal{C}_\mathcal{B}}$ and so $\operatorname{Ext}^{\gg 0}(C,N)=0$ for all $N \in \mathcal{B}$. But then C has finite AB dimension by assumption. Since AB dimension satisfies the Auslander Buchsbaum formula, AB-dim C is zero. So we have $\operatorname{Ext}^{>0}(C,N)=0$ for all $N \in \mathcal{B}$, and hence, C is in $\mathcal{C}_\mathcal{B}$. □

Now let $d = \dim R$.

Theorem 3.7.5. *If R is Gorenstein, then the following are equivalent.*

1. *R is AB*

2. *$\mathcal{C}_\mathcal{B}$ is a thick subcategory of MCM for all $\mathcal{B} \subseteq \operatorname{mod}(R)$*

3. *MCM $\cap\, \mathcal{C}_\mathcal{B}$ is thick in MCM for every $\mathcal{B} \subseteq \operatorname{mod}(R)$*

4. *$\Lambda(\mathcal{C}_\mathcal{B})$ gives a bijection between $\mathfrak{R}(\mathcal{C}_\mathcal{B})$ and Γ*

5. *For all $\mathcal{B} \subseteq \operatorname{mod}(R)$ and $M \in \mathcal{C}_\mathcal{B}$, Γ contains the function $f : \operatorname{spec} R \to \mathbb{N}$ defined by the following:*

$$f(p) = \min\{i \mid \operatorname{Ext}^{i>0}(M,\mathcal{B})=0\}$$

Proof. The previous lemma shows that 1 implies 2, and 2 implies 3 is trivial. Assuming 3, we will show 1. Suppose $\mathrm{Ext}^{\gg 0}(M,N)=0$. Then M is in $\overline{\mathcal{C}_N}$. Letting $\dim R = d$, we have $\Omega^d M \in \overline{\mathcal{C}_N} \cap \mathrm{MCM}$. For some $n \geq d$ we have $\Omega^n M \in \mathcal{C}_{\mathcal{B}} \cap \mathrm{MCM}$. But then we have

$$0 \to \Omega^n M \to F_{n-1} \to \cdots \to F_d \to \Omega^d M \to 0$$

where each F_i is projective. By 3, $\Omega^d M$ is in $\mathcal{C}_{\mathcal{B}}$. So we have -$\dim_{C_N} M \leq d$, and so $\mathrm{Ext}^{>d}(M,N)=0$.

Theorem 2.2 shows that 2 implies 4. Lemma 7.2 shows that 4 implies 5. Since R is local, evaluating f at the maximal ideal shows that 5 implies 1. □

Corollary 3.7.6. *Set $r = d - depth M$. If R is AB and $\mathrm{Ext}^{\gg 0}(M,N)=0$, then we have $\mathrm{Ext}^r(M,N) \neq 0$. Furthermore, if $\mathrm{Ext}^r(M,N)=0$ or $\mathrm{Ext}^i(M,N) \neq 0$ for $i > r$, then $\mathrm{Ext}^j(M,N) \neq 0$ for arbitrarily large j.*

Proof. Suppose R is AB. Then 2 holds and so $\mathcal{C}_N$-dim satisfies the Auslander Buchsbaum formula. If $\mathrm{Ext}^{\gg 0}(M,N)=0$ then $r = \mathcal{C}_N\text{-}\dim M = \max\{i \mid \mathrm{Ext}^i(M,N) \neq 0\}$. The second statement is just the contrapositive of the first statement. □

Corollary 3.7.7. *If R is Gorenstein and every resolving subcategory of MCM is thick, then R is AB.*

Proof. The assumption implies 2 in Theorem 7.5. □

Thus if Λ in Theorem 6.8 is a bijection from $\mathfrak{S}(R) \times \Gamma$ to $\mathfrak{R}(R)$, then R is AB. In Stevenson (2013a), Stevenson shows that when R is a complete intersection, every resolving subcategory of MCM is closed under duals. The following gives a necessary condition for this property.

Corollary 3.7.8. *If R is Gorenstein and every resolving subcategory of MCM is closed under duals, then R is AB.*

Proof. Suppose every resolving subcategory of MCM is closed under duals. Let $\mathcal{X} \subseteq$ MCM be resolving. Then for every $X \in \mathcal{X}$, $(\Omega X^*)^*$ is in $\mathcal{X}$. By Lemma 6.1, $\mathcal{X}$ is thick. The result follows from the previous corollary. $\square$

Chapter 4

Semidualizing modules and rational singularities

4.1 Introduction

This chapter is concerned with the existence of nontrivial semidualizing modules. See Section 3 for the definition. In Vasconcelos (1974), Vasconcelos asks if there exists only a finite number of nonisomorphic semidualizing modules. This question is answered in the affirmative in Christensen & Sather-Wagstaff (2008) for equicharacteristic Cohen-Macaulay algebras, and in Nasseh & Sather-Wagstaff (2012) for the semilocal case. It is natural to ask which rings have only trivial semidualizing modules. In Jorgensen et al. (2012), Jorgensen, Leuschke and Sather-Wagstaff give a very nice characterization of rings with a dualizing module and only trivial semidualizing modules. However, this characterization is somewhat abstract and it is difficult to tell whether the conditions hold for a particular ring. Also in Sather-Wagstaff (2009a), Sather-Wagstaff proves results relating the existence of nontrivial semidualizing modules to Bass numbers. In this chapter, we investigate the following question:

Question 3. *If a ring R has a nice (e.g. rational) singularity, then does R have only trivial semidualizing modules?*

The evidence suggests the answer is yes. In Celikbas & Dao (2014), Celikbas and Dao show that only trivial semidualizing modules exist over Veronese subrings, which have a quotient singularity and hence a rational singularity. Furthermore, Sather-Wagstaff shows in Sather-Wagstaff (2007) that only trivial semidualizing modules exist for determinantal rings, which also have a rational singularity. It is proven in Sather-Wagstaff (2009b)[Example 4.2.14] that all Cohen-Macaulay rings with minimal multiplicity have no nontrivial semidualizing modules. Since rational singularity and dimension 2 imply minimal multiplicity, all rings with rational singularity and dimension 2 have no nontrivial semidualizing modules. The following example shows that there are dimension 3 rings with rational singularity that do not have minimal multiplicity.

Example 4.1.1. Let

$$R = k[[x, y, z]]^{(3)} = k[[x^3, y^3, z^3, x^2y, x^2z, y^2x, y^2z, z^2x, z^2y, xyz]]$$

which is the third Veronese subring in three variables. For the multiplicity of R to be minimal, it must equal $\operatorname{edim} R - \dim R + 1 = 10 - 3 + 1 = 8$. However, setting $\bar{R} = R/(x^3, y^3, z^3)R$, $e(R) = e(\bar{R}) = \lambda(\bar{R})$ where λ is length. Since

$$\bar{R} = k \oplus kx^2y \oplus kx^2z \oplus ky^2x \oplus ky^2z \oplus kz^2x \oplus kz^2y \oplus kxyz \oplus kx^2y^2z^2$$

we thus have $e(R) = 9$.

In this chapter, we add to the evidence that suggests that the answer to Question 1 is yes by investigating the case where R is a ring of invariants, a large class of rings with rational singularity. The following theorem is the main result of this chapter.

Theorem. *If R is a power series ring over a field k in finitely many variables and G is a cyclic group of order p^l acting on R with $\operatorname{Char} k \neq p$, then R^G has only trivial semidualizing modules.*

Our approach to the proof of this result, relying on Lemma 3.10, is different than those of the results in Celikbas & Dao (2014) and Sather-Wagstaff (2007). In each of those papers, the key technique involves counting the number of generators, whereas we use Lemma 3.10. See Section 2 for a further explanation.

Section 2 gives preliminary results concerning rings of invariants and also gives a sketch of the proof. Section 3 proves a key technical theorem about when a ring has only trivial semidualizing modules, and then Section 4 uses this result to prove our main theorem.

4.2 Preliminaries

In this section, we sketch the proof the main result of this chapter. First let us recall Lemma 3.10.

Lemma 4.2.1. *If C is a semidualizing R-module and D is a dualizing module for R, then the homomorphism $\eta : C \otimes \operatorname{Hom}_R(C, D) \to D$ given by $x \otimes \varphi \mapsto \varphi(x)$ is an isomorphism.*

Note that by Lemma 3.9, C and D are in the class group of R. In Theorem 3.2, with strong assumptions on R, we show that $A \otimes B$ has torsion for any elements A and B in the class group of R which are not isomorphic to R. The construction of a torsion element is easy, however, it requires considerable work to show that this element is not zero in the tensor product. With this setup, because of Lemma 3.10 and because D does not have torsion, nontrivial semidualizing modules cannot exist.

For the remainder of this chapter, let S be a polynomial ring in finitely many variables over an algebraically closed field k, and let G be a finite group acting linearly on S. We shall assume that the characteristic of k does not divide the order of the group. To prove the main result, Section 4 shows that when $|G| = p^l$ for some prime, S^G satisfies the assumptions of Theorem 3.2. In order to do this, we need the following definition and lemma.

Definition 4.2.2. Given a character $\chi : G \to k^\times$, we denote by S_χ the set of *relative invariants*, namely, the polynomials $f \in S$ such that $gf = \chi(g)f$.

Note that S_χ is an S^G-module. The following lemma is from Benson (1993)[Theorem 3.9.2].

Lemma 4.2.3. *The ring S^G is a normal domain whose class group is the subgroup $H \subseteq \operatorname{Hom}(G, k^\times)$ which consists of the characters that contain all the pseudoreflections in their kernel. Furthermore, for any $\chi \in H$, the relative invariants $S_{\chi^{-1}}$ form the reflexive module corresponding to the element χ.*

4.3 Class Groups

In this section, let R be a Noetherian ring. We say that an element μ in an R-module M is indivisible if there exists no nonunit $a \in R$ and $\nu \in M$ such that $\mu = a\nu$.

Lemma 4.3.1. *Suppose R is a k-algebra, with k a field and M and N are R-modules. Furthermore, suppose $f \in M$ and $g \in N$ are indivisible, and $\gamma \in M$ and $\rho \in N$ are not unit multiples of f and g respectively. If there exists k-bases E, F, X of M, N, R respectively with $f, \gamma \in E$ and $g, \rho \in F$ such that for every $\xi \in X$ and $\varepsilon \in E$ and $\eta \in F$, $\xi\varepsilon$ is a k-linear multiple of an element in E and $\xi\eta$ is a k-linear multiple of an element of F, then $f \otimes g - \gamma \otimes \rho$ is not zero in $M \otimes_R N$.*

Proof. Suppose that such bases E, F, X exist. Let $\mathcal{F}$ denote the free abelian group functor. Recall that for any modules U and V over a ring S, we construct $U \otimes_S V$ by quotienting $\mathcal{F}(U \cup V)$ by the submodule, which we will call $K_{U,V}(S)$, generated by the relations

$$(v_1, u_1+u_2)-(v_1, u_1)-(v_1, u_2) \qquad (v_1+v_2, u_1)-(v_1, u_1)-(v_2, u_1) \qquad (\lambda v_1, u_1)-(v_1, \lambda u_1)$$

with $v_i \in U$, $u_i \in V$ and $\lambda \in S$. Hence we have

$$M \otimes_R N \cong \mathcal{F}(M \cup N)/K_{M,N}(R) \qquad M \otimes_k N \cong \mathcal{F}(M \cup N)/K_{M,N}(k).$$

Notice that, since $k \subseteq R$, $K_{M,N}(k) \subseteq K_{M,N}(R)$. So $M \otimes_R N$ is a quotient of $M \otimes_k N$. Specifically, we have the following isomorphism

$$\frac{M \otimes_k N}{K_{M,N}(R)/K_{M,N}(k)} \cong \frac{\mathcal{F}(M \cup N)/K_{M,N}(k)}{K_{M,N}(R)/K_{M,N}(k)} \cong \frac{\mathcal{F}(M \cup N)}{K_{M,N}(R)} \cong M \otimes_R N$$

We claim that every element of $K_{M,N}(R)/K_{M,N}(k) \subseteq M \otimes_k N$ is of the form

$$\sum_{s=1}^{r} \lambda_s(\mu_s \tau_s \otimes \nu_s) - \lambda_s(\mu_s \otimes \tau_s \nu_s)$$

with $\lambda_i \in k$, and $\mu_i \in E$, $\nu_i \in F$, $\tau_i \in X \backslash k$ and $\lambda_i \in k$. Take $z \in K(R)/K(k)$. Since the generators of $K(R)$ of the form $(v_1, u_1 + u_2) - (v_1, u_1) - (v_1, u_2)$ and $(v_1 + v_2, u_1) - (v_1, u_1) - (v_2, u_1)$ are in $K(k)$, we may write

$$z = \sum_i (m_i t_i \otimes n_i - m_i \otimes t_i n_i)$$

with $m_i \in M$, $n_i \in N$, and $t_i \in R$. However, since E, F, X are bases of M, N, X respectively, we may also write

$$m_i = \sum_j \alpha_{i,j} \mu_{i,j} \qquad n_i = \sum_l \beta_{i,l} \nu_{i,l} \qquad t_i = \sum_k \kappa_{i,k} \tau_{i,k}$$

with each $\lambda_s \in k$, and $\mu_s \in E$, $\nu_s \in F$, $\tau_s \in X \backslash k$ and $\lambda_s \in k$. So we have

$$\begin{aligned} z &= \sum_i (m_i t_i \otimes n_i - m_i \otimes t_i n_i) \\ &= \sum_i \left(\left(\sum_j \alpha_{i,j} \mu_{i,j} \right) \left(\sum_k \kappa_{i,k} \tau_{i,k} \right) \otimes \left(\sum_l \beta_{i,l} \nu_{i,l} \right) - \left(\sum_j \alpha_{i,j} \mu_{i,j} \right) \otimes \left(\sum_k \kappa_{i,k} \tau_{i,k} \right) \left(\sum_l \beta_{i,l} \nu_{i,l} \right) \right) \\ &= \sum_{i,j,k,l} (\alpha_{i,j} \mu_{i,j} \kappa_{i,k} \tau_{i,k} \otimes \beta_{i,l} \nu_{i,l} - \alpha_{i,j} \mu_{i,j} \otimes \kappa_{i,k} \tau_{i,k} \beta_{i,l} \nu_{i,l}) \end{aligned}$$

$$= \sum_{i,j,k,l} \alpha_{i,j}\beta_{i,l}\kappa_{i,k}\left(\mu_{i,j}\tau_{i,k} \otimes \nu_{i,l} - \mu_{i,j} \otimes \tau_{i,k}\nu_{i,l}\right)$$

$$= \sum_{i,j,k,l} \alpha_{i,j}\beta_{i,l}\kappa_{i,k}(\mu_{i,j}\tau_{i,k} \otimes \nu_{i,l}) - \alpha_{i,j}\beta_{i,l}\kappa_{i,k}(\mu_{i,j} \otimes \tau_{i,k}\nu_{i,l})$$

Lastly, if $\tau_{i,k}$ is in k, then $\mu_{i,j}\tau_{i,k} \otimes \nu_{i,l} - \mu_{i,j} \otimes \tau_{i,k}\nu_{i,l}$ is already zero in $M \otimes_k N$. Therefore, setting $\lambda_{i,j,k,l} = \alpha_{i,j}\beta_{i,l}\kappa_{i,k} \in k$, the claim is shown.

Now suppose $f \otimes g - \gamma \otimes \rho$ is zero in $M \otimes_R N$. Then in $M \otimes_k N$, we may write

$$f \otimes g - \gamma \otimes \rho = \sum_{s=1}^{r} \lambda_s(\mu_s\tau_s \otimes \nu_s) - \lambda_s(\mu_s \otimes \tau_s\nu_s)$$

with $\lambda_s \in k$, and $\mu_s \in E$, $\nu_s \in F$, $\tau_s \in X \backslash k$ and $\lambda_s \in k$. Now $Z = \{a \otimes b \mid a \in E, b \in F\}$ is a k-basis of $M \otimes_k N$. Since $f, \gamma \in E$ and $g, \rho \in F$, $f \otimes g$ and $\gamma \otimes \rho$ are in Z. By assumption, each $\mu_s\tau_s \otimes \nu_s$ and $\mu_s \otimes \tau_s\nu_s$ is a linear multiple of an element in Z. Thus, $f \otimes g$ must be a linear multiple of either $\mu_s\tau_s \otimes \nu_s$ or $\mu_s \otimes \tau_s\nu_s$ for some s. But, since f and g are indivisible and for all s, neither $\mu_s\tau_s$ nor $\tau_s\nu_s$ is indivisible, this is a contradiction. Therefore, $f \otimes g - \gamma \otimes \rho$ cannot be zero in $M \otimes_R N$.

□

Take a ring R with class group L with operation $\circ$. Let $T = \bigoplus_{A \in L} A$. We can give this R-module an L-graded R-algebra structure. For any $A, B \in L$, recall that

$$A \circ B = \text{Hom}(\text{Hom}(A \otimes_R B, R), R) \in L.$$

We will define the multiplication on the homogenous elements of T with the natural map $\varphi_{A,B} : A \otimes_R B \to \text{Hom}(\text{Hom}(A \otimes_R B, R), R)$ by setting $ab = \varphi_{A,B}(a \otimes b)$, for any $a \in A$ and $b \in B$. We can extend this multiplication linearly to the nonhomogenous elements of T.

Since R is contained in T, this algebra is unital, and, because

$$\operatorname{Hom}(\operatorname{Hom}(A \otimes_R B, R), R) \cong \operatorname{Hom}(\operatorname{Hom}(B \otimes_R A, R), R)$$

it is commutative as well. This construction is similar to an algebra considered in Tomari & Watanabe (1992).

Theorem 4.3.2. *Let R be a Noetherian k-algebra, with k a field. Suppose L is finite and cyclic with generator Λ. Also suppose that the L-grading on T can be refined to a grading Γ such that every Γ-homogenous component is one dimensional. If there exists a Γ-homogenous element $x \in \Lambda \subseteq T$ such that $x^n \in \Lambda^n \subseteq T$ is indivisible (as an element of an R-module) for all $n \in \mathbb{N}$ strictly less than $|\Lambda|$, then for any $A, B \in L$ where neither A nor B is isomorphic to R, the module $A \otimes_R B$ has torsion.*

Proof. Since Λ generates L, there exists a and b such that $\Lambda^a = A$ and $\Lambda^b = B$. Then there exists $a, b \in \mathbb{N}$ such that $x^a \in A$ and $x^b \in B$. Since neither A nor B is isomorphic to R, a and b are both strictly less than $|L|$ and so x^a and x^b are indivisible. We may assume without loss of generality that $a \geq b$.

Let Q a minimal homogenous generating set of B which contains x^b. We may assume every element in Q is indivisible, since, by the Noetherian condition, we can replace any divisible element by an indivisible one. Since B is not isomorphic to R and is torsionless, we know that Q has another element y besides x^b. Besides being indivisible and homogeneous, y is also not a unit multiple of x^b.

Set $z = x^a \otimes y - yx^{a-b} \otimes x^b$. We show that z is a torsion element. Since x^{a-b} is in Λ^{a-b} and y is in $B = \Lambda^b$, yx^{a-b} is Λ^a which is A. Thus z is in $A \otimes_R B$. Furthermore, for any $f \in (A \circ B)^{-1}$ we have $x^a yf, x^{a+b} f \in R$. Thus we have,

$$(x^a yf)z = x^{2a} yf \otimes y - yx^{a-b} \otimes x^{a+b} yf = x^{2a} yf \otimes y - x^{2a} yf \otimes y = 0$$

Thus to show that z is a torsion element, it suffices to show that z is not zero in $A \otimes_R B$.

Note that, by construction, x^a and y are indivisible, and since y and x^b are not unit multiples of each other, neither are x^a and yx^{a-b}. Also yx^{a-b} is homogenous since x^{a-b} is. We can choose Γ-homogenous bases E and F of A and B respectively such that $x^a, y \in E$ and $x^a, yx^{a-b} \in F$. Similarly we can choose a Γ-homogenous basis X of R. Since every Γ-homogenous component of T is one dimensional, for every $\xi \in X$ and $\varepsilon \in E$ and $\eta \in F$, $\xi\varepsilon$ is a linear multiple of an element in E and $\xi\eta$ is a linear multiple of an element of F. Thus z meets the hypotheses of the previous proposition. Therefore, z is not zero in $A \otimes_R B$.

□

Corollary 4.3.3. *Assume the set up of the last Theorem and that R has a dualizing module. Then R has no nontrivial semidualizing modules.*

Proof. Let C be a semidualizing module for R. Then $C \otimes \mathrm{Hom}(C, D) \cong D$ where D is a dualizing module. However, $\mathrm{Hom}(C, D) \cong C^{-1} \circ D$ is also an element of the class group. Thus by the previous theorem, since D is torsionless, either C or $\mathrm{Hom}(C, D)$ is isomorphic to R. Therefore, C is isomorphic to R or D.

□

4.4 Semidualizing modules of rings of invariants

Let S be the polynomial ring in d variables over k. We can apply the previous results to the semidualizing modules over rings of invariants for a certain cyclic group, but first we need a lemma.

Lemma 4.4.1. *Assume k is an algebraically closed field. If G is a finite cyclic group acting linearly on S generated by g whose order is not divisible by the characteristic of k, then there exist algebraically independent $x_1, \ldots, x_d \in S$ such that $S = k[x_1, \ldots, x_d]$ and $gx_i = \zeta^{\eta_i} x_i$ with $\zeta \in k$ a primitive $|G|$th root of unity.*

Proof. By putting g in the Jordan canonical form, it is easy to see that g is diagonalizable since $|G|$ and $\operatorname{Char} k$ are coprime. Thus, we may choose an eigenbasis, $x_1, \ldots, x_d$, of S_1. So, $gx_i = \xi_i x_i$ with $\xi_i \in k$. Since $g^{|G|}$ should act as the identity, each ξ_i must be a $|G|$th root of unity, and so we may write $\xi = \zeta^{\eta_i}$ where ζ is some fixed primitive $|G|$th root of unity. Also, S is isomorphic to the symmetric algebra of S_1 which is a polynomial ring in the variables $x_1, \ldots, x_d$. Hence, $x_1, \ldots, x_d$ are algebraically independent.

□

To apply the results of Section 3, we will observe that in this case

$$T = \bigoplus_{\chi \in L} S_{\chi^{-1}} \subseteq S$$

where L is the class group of S^G. The desired grading Γ of T will be the monomial grading with respect to the variables $x_1, \ldots, x_d$ defined in the previous lemma. Before we proceed however, we need to show that this grading is a refinement of L, to which end, the following lemma suffices.

Lemma 4.4.2. *If G consists of diagonal matrices, then for any character $\chi : G \to k^\times$, the set of all monomials in S_χ is a k-basis.*

Proof. Let X be the set of all monomials of S_χ. Since any distinct monomials are linearly independent, X is linearly independent. Take any $g \in G$. Then for each i, $gx_i = \lambda_i x_i$ with $\lambda_i \in k$. So for any $\underline{x}^{\underline{\alpha}} = x_1^{a_1} \cdots x_d^{a_d}$ in S, we have

$$g\underline{x}^{\underline{\alpha}} = gx_1^{a_1} \cdots x_n^{a_d} = (\lambda_1 x_1)^{a_1} \cdots (\lambda_d x_d)^{a_d} = \underline{\lambda} x_1^{a_1} \cdots x_d^{a_d} = \underline{\lambda}\underline{x}^{\underline{\alpha}}$$

with $\underline{\lambda} = \lambda_1^{a_1} \cdots \lambda_d^{a_d}$. Take any $f \in S_\chi$. We may write $f = \kappa_1 \underline{x}^{\underline{\alpha}_1} + \cdots + \kappa_m \underline{x}^{\underline{\alpha}_m}$. On the one hand, we know that

$$gf = g(\kappa_1 \underline{x}^{\underline{\alpha}_1} + \cdots + \kappa_m \underline{x}^{\underline{\alpha}_m}) = g\kappa_1 \underline{x}^{\underline{\alpha}_1} + \cdots + g\kappa_m \underline{x}^{\underline{\alpha}_m} = \kappa_1 \underline{\lambda}_1 \underline{x}^{\underline{\alpha}_1} + \cdots + \kappa_m \underline{\lambda}_m \underline{x}^{\underline{\alpha}_m}$$

with $\underline{\lambda}_i = \lambda_1^{a_{1i}} \cdots \lambda_d^{a_{di}}$. By virtue of f being in S_χ, we also know that

$$gf = \chi(g)f = \chi(g)\kappa_1\underline{x}^{\underline{\alpha}_1} + \cdots + \chi(g)\kappa_m\underline{x}^{\underline{\alpha}_m}$$

However, since monomials are linearly independent, this means that for each i, $\kappa_i\underline{\lambda}_i = \chi(g)\kappa_i$, and so $\underline{\lambda}_i = \chi(g)$. Therefore, for each i, $\underline{x}^{\underline{\alpha}_i}$ is in S_χ and thus also in X. Hence, X spans S_χ and is a basis.

□

Proposition 4.4.3. *Suppose R is a power series ring over a field k in d variables and G is a cyclic group of order n acting on S with* Char k *not dividing n. If g generates G and has a primitive nth root of unity as an eigenvalue, then R^G has only trivial semidualizing modules.*

Proof. By Lemma 3.11, since $\bar{k} \otimes R^G$ is a faithfully flat extension of R^G, C is a semidualizing R^G-module if and only if $\bar{k} \otimes_{R^G} C$ is a semidualizing $\bar{k} \otimes R^G$-module. Thus, if there are no nontrivial semidualizing modules for $\bar{k} \otimes R^G$, then there are none for R^G. So, we may assume that k is algebraically closed.

Since G is cyclic and is generated by g, a character in $\mathrm{Hom}(G, k^\times)$ is completely determined by the image of g. However, g can only be sent to an nth root of unity. Since k is algebraically closed, and since $\mathrm{Char}\, k$ does not divide n, there are n distinct nth roots of unity, which form a cyclic group. Therefore, G is isomorphic to $\mathrm{Hom}(G, k^\times)$. Since class group of S^G is a subgroup of $\mathrm{Hom}(G, k^\times)$, this means the class group must be cyclic.

By the previous lemma, we may write $S = k[x_1, \ldots, x_d]$ where $gx_i = \zeta^{\eta_i} x_i$ with $\zeta \in k$ a primitive $|G|$th root on unity. The assumption tells us that we may assume that $\eta_1 = 0$. Define $\chi : G \to k^\times$ by $g \mapsto \zeta^{-1}$. Since ζ^{-1} is a primitive $|G|$th root of unity, χ generates $\mathrm{Hom}(G, k^\times)$. So, for some $\lambda \in \mathbb{N}$, χ^λ generates the class group L. Assume that λ is as small as possible. Note that $gx_1^\lambda = (\zeta x_1)^\lambda = \zeta^\lambda x_1^\lambda$, and so $x_1^\lambda \in S_{\chi^{-\lambda}}$, the reflexive module corresponding to χ^λ. Since we have chosen λ to be as small as possible, $|\chi^\lambda| = n/\lambda$. Thus, for each $1 \le \nu < |\chi^{-\lambda}| = n/\lambda$, $\lambda\nu$ is strictly less than n. Since the smallest power of x_1 that is invariant is n, this means that $(x_1^\lambda)^\nu$ is indivisible. Therefore, using the monomial grading, the conditions of Corollary 3.3 and Theorem 3.2 are satisfied, and thus S^G has no nontrivial semidualizing modules. Since R^G is the completion of S^G, and completion is faithfully flat, we are done by Lemma 3.11.

□

We can recover the non modular case of (Celikbas & Dao, 2014, Corollary 3.21).

Corollary 4.4.4. *The there exists no semidualizing modules over nonmodular Veronese subrings.*

Proof. Let g be an $d \times d$ diagonal matrix whose entries are all ζ_n, a primitive nth root of unity. Then the n-Veronese subring in d variables is $S = k[[x_1, \ldots, x_d]]^G$ where G is the group generated by g. Since the order of G is n, the result follows from the previous proposition.

□

We now come to our main theorem.

Theorem 4.4.5. *If R is a power series ring over a field k in finitely many variables and G is a cyclic group of order p^l acting on R with* $\operatorname{Char} k \neq p$, *then R^G has only trivial semidualizing modules.*

Proof. By Lemma 3.10, we may write $S = k[x_1, \ldots, x_d]$ where $gx_i = \zeta^{\eta_i} x_i$ with $\zeta \in k$ a primitive $|G|$th root on unity. We may assume that ζ^{η_1} has the greatest order of all the ζ^{η_i} and set $z = |\zeta^{\eta_1}|$. Since $|\zeta^{\eta_i}|$ is a power of p less than z, we have $|\zeta^{\eta_i}|$ divides z for each i, and so $(\zeta^{\eta_i})^z = 1$. Thus, viewing g as a diagonal matrix with entries ζ^{η_i}, g^z is the identity, and so $n \leq z$. But, z has to be less than n, giving us equality. Hence, ζ^{η_1} is a primitive nth root of unity. However, since our choice of ζ is arbitrary, we may assume that $\eta_1 = 1$. In short, we have $gx_1 = \zeta x_1$. The result follows from the previous proposition.

□

The proofs of Theorem 4.5 and Proposition 4.3 show that Theorem 3.2 applies to the class of rings under consideration. Thus we actually have the following result, which resolves in the affirmative a special case of Conjecture 1.3 in Goto et al. (2013).

Corollary 4.4.6. *Assume the set up of the previous theorem, and let D be a dualizing module for R. If M is a reflexive module of rank 1 and $M \otimes_R \mathrm{Hom}_R(M, D)$ is torsionfree, then M is isomorphic to either R or D.*

Proof. Since M and $\mathrm{Hom}_R(M, D)$ are both elements of the class group, and since Theorem 3.2 applies, either M or $\mathrm{Hom}(M, D)$ is isomorphic to S. The latter case implies that $M \cong D$. □

Chapter 5

The depth formula and semidualizing modules

5.1 Introduction

This chapter is concerned with the following property.

Definition 5.1.1. A pair of modules M and N satisfy the *depth formula* if $\operatorname{Tor}_{>0}(M, N) = 0$ implies

$$\operatorname{depth} M \otimes N + \operatorname{depth} R = \operatorname{depth} M + \operatorname{depth} N.$$

Over the past fifty years, there has been research on which pairs of = modules satisfy the depth formula. Auslander originally showed in Auslander (1961) that M and N satisfy the depth formula if one of the modules has finite projective dimension, hence any pair of modules satisfy the depth formula over a regular local ring. In Huneke & Wiegand (1994), the authors show that any pair of modules over complete intersection rings satisfy the depth formula. The complete intersection assumption was weakened in Araya & Yoshino (1998) and Iyengar (1999) where the authors show that the depth formula holds if one of modules has finite complete intersection dimension. Besides this work, the most general result on the depth formula is the following theorem in Christensen & Jorgensen (2015).

Theorem 5.1.2. *If* $\widehat{\operatorname{Tor}}_i(X_\bullet, Y_\bullet) = 0$ *for all* $i \in \mathbb{Z}$, *then*

$$\operatorname{depth} X_\bullet \otimes^{\mathbf{L}} Y_\bullet + \operatorname{depth} R = \operatorname{depth} X_\bullet + \operatorname{depth} Y_\bullet.$$

We call the conclusion of this statement the derived depth formula. Note that if a pair of modules M and N satisfy the derived depth formula, they satisfy the depth formula. This is because if $\operatorname{Tor}_{>0}(M, N) = 0$ implies $M \otimes^{\mathbf{L}} N \cong M \otimes N$. The following result, which appears later as Theorem 5.1, generalizes Christensen and Jorgensen's result to the semidualizing setting

Theorem 5.1.3. *Let* C *be a semidualizing module. Suppose* $X_\bullet, Y_\bullet$ *are complexes with* $X_\bullet$ *totally* C*-reflexive and* $Y_\bullet$ *in the Auslander category of* C*. If* $\widehat{\operatorname{Tor}}_i^C(M, N) = 0$ *for all* $i \in \mathbb{Z}$, *then*

$$\operatorname{depth} X_\bullet \otimes^{\mathbf{L}} Y_\bullet + \operatorname{depth} R = \operatorname{depth} X_\bullet + \operatorname{depth} Y_\bullet$$

Taking $C = R$ recovers the result in Christensen & Jorgensen (2015). Before we can prove this theorem, we need to construct the functor $\widehat{\operatorname{Tor}}^C(X_\bullet, Y_\bullet)$ for complexes $X_\bullet$ and $Y_\bullet$. This functor is new and is a relative version of the functor in Veliche (2006) and Christensen & Jorgensen (2014) used in Theorem 1.2. The construction of $\widehat{\operatorname{Tor}}^C$ easily leads to the construction of $\widehat{\operatorname{Ext}}_C$. These functors are interesting in their own right and are extensions of the theory laid out in Sather-Wagstaff et al. (2010a); Salimi et al. (2014); Sather-Wagstaff et al. (2010b); Di et al. (2014).

The philosophy behind our definition of $\widehat{\operatorname{Tor}}^C$ is the following simple observation. Let $\mathcal{X}$ be any category, and $F : \mathcal{X} \to K(R)$ be any functor. For any $X \in \mathcal{X}$ and $Y_\bullet \in K(R)$, we may define $\widehat{\operatorname{Tor}}_i^F(X, Y_\bullet)$ and $\widehat{\operatorname{Ext}}_F^i(X, Y_\bullet)$ by composing the functors

$$\mathcal{X} \xrightarrow{F} K(R) \xrightarrow{-\otimes Y_\bullet} K(R) \xrightarrow{H^i} \operatorname{Mod}(R)$$

$$\mathcal{X} \xrightarrow{F} K(R) \xrightarrow{\operatorname{Hom}(-, Y_\bullet)} K(R) \xrightarrow{H_i} \operatorname{Mod}(R)$$

respectively. If $\mathcal{X}$ is exact (or triangulated) and F sends exact sequences (or triangles) to exact triangles, then since $-\otimes Y_\bullet$ and $\mathrm{Hom}(-, Y_\bullet)$ are triangulated functors, exact sequences (or triangles) in $\mathcal{X}$ will give rise to the usual long exact sequences of $\widehat{\mathrm{Tor}}_i^F$ and $\widehat{\mathrm{Ext}}_F^i$. Thus to suitably define $\widehat{\mathrm{Tor}}_i^C$ one needs only to define the "correct" functor F. For our purposes, this functor is $T_C : D^b(R) \to K(R)$ and is developed in Section 3.

We begin in Section 2 with a treatment of subcategories of $D^b(R)$ related to semidualizing modules. In particular, we define the derived version of $\mathcal{G}_C$. Also, in this section, we define the depth of a complex. In Section 3, we define the functor $T_C : D^b(R) \to K(R)$. It is with this functor that we define $\widehat{\mathrm{Tor}}^C$ and $\widehat{\mathrm{Ext}}_C$ in Section 4. We also discuss some basic properties of these functors. In Section 5 we prove the main theorem of the chapter. In Section 6 we discuss the the relationship between the UAC condition on a ring and when modules satisfy the depth formula.

5.2 Preliminaries

We begin by defining three subcategories of $\mathrm{mod}(R)$ associated to a semidualizing module C, the first of which was already defined in Chapter 2. These categories were first defined in Christensen (2001); Avramov & Foxby (1997). In the rest of this chapter, C will be a semidualizing module.

Definition 5.2.1. 1. The category of *totally reflexive modules*, $\mathcal{G}_C(R)$, is the category of modules $M \in \mathrm{mod}(R)$ such that

(i) $\mathrm{Ext}^{>0}(M, C) = 0$

(ii) $\mathrm{Ext}^{>0}(M^\dagger, C) = 0$

(iii) The natural map $M \to M^{\dagger\dagger}$ is an isomorphism.

2. The *Auslander class*, $\mathcal{A}_C(R)$, is the category of modules $M \in \mathrm{Mod}(R)$ such that

(i) $\mathrm{Tor}_{>0}(C, M) = 0$

(ii) $\operatorname{Ext}^{>0}(C, C \otimes M) = 0$

(iii) The natural map $M \to \operatorname{Hom}(C, C \otimes M)$ is an isomorphism

3. The *Bass class*, $\mathcal{B}_C(R)$, is the category of modules $M \in \operatorname{Mod}(R)$ such that

(i) $\operatorname{Ext}^{>0}(C, M) = 0$

(ii) $\operatorname{Tor}^{>0}(C, \operatorname{Hom}(C, M)) = 0$

(iii) The natural map $C \otimes \operatorname{Hom}(C, M) \to M$ is an isomorphism

We often denote these categories as $\mathcal{G}_C$, $\mathcal{A}_C$, and $\mathcal{B}_C$ respectively. When $C = R$, we have $\mathcal{A}_R = \operatorname{Mod}(R)$ and $\mathcal{B}_R = \operatorname{Mod}(R)$. In this chapter, we would like to discuss the derived analogues of these categories. Thus we also need a derived version of our duality functor.

Definition 5.2.2. For an object $X_\bullet \in D^b(\operatorname{Mod}(R))$, set $X_\bullet^\# = \mathbf{R}\operatorname{Hom}(X_\bullet, C)$. For a complex $X_\bullet \in K(R)$, we set $X^\dagger = \operatorname{Hom}(X_\bullet, C)$. For a complex $X_\bullet \in K(R)$, we set $X^* = \operatorname{Hom}(X_\bullet, R)$.

The functors $-^\#$ and $-^\dagger$ are very different. For example, for a local ring $(R, \mathfrak{m}, k)$ with depth $R > 0$, $k^\dagger = 0$ but $k^\#$ is an unbounded complex unless C is dualizing. See Christensen (2001) and Yassemi (1995) for an exhaustive study on the following definitions.

Definition 5.2.3. 1. the category of *totally C-reflexive complexes*, $D\mathcal{G}_C(R)$, is the category of complexes $X_\bullet \in \mathcal{D}^b(R)$ such that

(i) $X_\bullet^\dagger \in D^b(R)$

(ii) The natural morphism $M \to M^{\#\#}$ is an isomorphism.

2. The *Auslander class*, $D\mathcal{A}_C(R)$, is the category of complexes $X_\bullet \in \mathcal{D}^b(\operatorname{Mod}(R))$ such that

(i) $C \otimes^{\mathbf{L}} X_\bullet \in D^b(\operatorname{Mod}(R))$

(ii) The natural morphism $X_\bullet \to \mathbf{R}\operatorname{Hom}(C, C \otimes^{\mathbf{L}} M)$ is an isomorphism

3. The *Bass class*, $D\mathcal{B}_C(R)$, is the category of complexes $X_\bullet \in \mathcal{D}^b(\operatorname{Mod}(R))$ such that

(i) $\mathbf{R}\mathrm{Hom}(C, X_\bullet) \in D^b(R)$

(ii) The natural morphism $C \otimes^{\mathbf{L}} \mathbf{R}\mathrm{Hom}(C, X_\bullet) \to M$ is an isomorphism

We often denote these categories as $D\mathcal{G}_C$, $D\mathcal{A}_C$, and $D\mathcal{B}_C$ respectively. The following lemma is an easy to see.

Lemma 5.2.4. *The categories $D\mathcal{G}_C$, $D\mathcal{A}_C$, and $D\mathcal{B}_C$ are all thick.*

Lemma 5.2.5. *The natural functors are faithful embeddings.*

1. $D^b(\mathcal{G}_C) \to D\mathcal{G}_C$
2. $D^b(\mathcal{A}_C) \to D\mathcal{A}_C$
3. $D^b(\mathcal{B}_C) \to D\mathcal{B}_C$

We recall that a category $\mathcal{X}$ has the two-out-of-three property if $0 \to X_1 \to X_2 \to X_3 \to 0$ is exact, then $X_l, X_m \in \mathcal{X}$ implies $X_n \in \mathcal{X}$ whenever $\{l, m, n\} = \{1, 2, 3\}$.

Proof. The categories $\mathcal{A}_C$ and $\mathcal{B}_C$ contain all projective and injective R-modules respectively by (Sather-Wagstaff, 2009b, Proposition 3.1.9) and (Sather-Wagstaff, 2009b, Proposition 3.1.10). First note that $D^b(\mathcal{G}_C) \simeq D^b(\overline{\mathcal{G}_C})$ as explained in Lemma 5.6. The categories $\overline{\mathcal{G}_C}$ and $\mathcal{A}_C$ both have the two-out-of-three property by Lemma 2.8 and (Sather-Wagstaff, 2009b, Proposition 3.1.7). Also, they have enough projective R-modules. Thus statements (1) and (2) follow from the proof of Corollary 5.11. The category $\mathcal{B}_C$ also has the two-out-of-three property, (Sather-Wagstaff, 2009b, Proposition 3.1.7). Since it has enough injective R-modules, we can use a dual argument to prove (3).

□

The following category is a semidualizing analogue of the perfect complexes. It is easy to check that this category is contained in $D\mathcal{G}_C$.

Definition 5.2.6. The subcategory of *C-perfect complexes*, $C \otimes \text{Perf} \subseteq D^b(R)f$, is the subcategory of objects $X_\bullet$ isomorphic in $D^b(R)$ to a complex of the form $C \otimes P_\bullet$ with $P_\bullet \in \text{Perf}$.

An interesting feature of these categories is the following.

Theorem 5.2.7 (Foxby Duality). *The functors*

$$C \otimes^{\mathbf{L}} - : D\mathcal{A}_C \to D\mathcal{B}_C$$

$$\mathbf{R}\text{Hom}(C, -) : D\mathcal{B}_C \to D\mathcal{A}_C$$

give an equivalence of triangulated categories. Furthermore, the functors $C \otimes - : \mathcal{A}_C \to \mathcal{B}_C$ and $\text{Hom}(C, -) : \mathcal{B}_C \to \mathcal{A}_C$ also give an equivalence of categories. Lastly, these functors give a derived equivalence $C \otimes^{\mathbf{L}} - : D^b(\mathcal{A}_C) \to D^b(\mathcal{B}_C)$ and $\mathbf{R}\text{Hom}(C, -) : D^b(\mathcal{B}_C) \to D^b(\mathcal{A}_C)$.

This result was proved in this generality in (Christensen, 2001, Theorem 4.6). When $C = R$, the result was originally proved in Avramov & Foxby (1997).

We close this section with a few requisite definitions.

Definition 5.2.8. Let $P_\bullet$, be a complex of R-modules.

1. $\simeq$ will denote quasi-isomorphic complexes, that is $X_\bullet \simeq Y_\bullet$ if they are isomorphic in the derived category.

2. $\min_c(P_\bullet) = \sup\{n \mid P_i = 0 \quad \forall\ i < n\}$

3. $\max_c(P_\bullet) = \inf\{n \mid P_i = 0 \quad \forall\ i > n\}$

4. $\min(P_\bullet) = \sup\{n \mid H_i(P_\bullet) = 0 \quad \forall i < n\}$

5. $\max(P_\bullet) = \inf\{n \mid H_i(P_\bullet) = 0 \quad \forall i > n\}$

The notion of depth for complexes is subtle. It is studied in Iyengar (1999) and Foxby & Iyengar (2003).

Definition 5.2.9. Let $I = (x_1, \dots, x_n)$ be a (finitely generated) ideal. Let $K_\bullet$ be the Koszul complex on $x_1, \dots, x_n$. For a complex $X_\bullet$ in $\mathrm{Ch}(R)$, set

$$\operatorname{depth}(I, X_\bullet) = n - \max K_\bullet \otimes X_\bullet$$

and when $(R, \mathfrak{m}, k)$ is local and Noetherian, set $\operatorname{depth} X_\bullet = \operatorname{depth}(\mathfrak{m}, X_\bullet)$.

Note that for an arbitrary complex $X_\bullet$, $\operatorname{depth} X_\bullet$ can be any positive or negative integer or even ∞. It is shown in (Iyengar, 1999, 1.3) that this definition is independent of the generating set of I. Furthermore, for bounded complexes if $X_\bullet$ and $X'_\bullet$ are quasi-isomorphic, then so are $K_\bullet \otimes X_\bullet$ and $K_\bullet \otimes X'_\bullet$. Hence $\operatorname{depth} X_\bullet = \operatorname{depth} X'_\bullet$. Thus depth is an invariant of isomorphism classes of objects in the derived category. Also note that this definition specializes to the usual definition of depth for modules.

For the rest of this chapter, we will work over a Noetherian local ring $(R, \mathfrak{m}, k)$. We also set the following convention in this chapter.

Definition 5.2.10. For a complex $X_\bullet$, $\Sigma X_\bullet$ will be the shift functor. Specifically, $(\Sigma X_\bullet)_n = X_{n-1}$ and $\partial_n^{\Sigma X} = \partial_{n-1}^{X}$.

5.3 Functors and resolutions

In this section, we construct a version of relative Tate cohomology. To define Tate (co)homology of a totally reflexive module M, forming what is known as a complete resolution, we spice the complexes $\rho(X)$ and $\rho(X^\dagger)^\dagger$ where ρ denotes the projective resolution functor. We then are able to define $\widehat{\mathrm{Tor}}$ and $\widehat{\mathrm{Ext}}$ using this resolution. Essentially we wish to perform the same construction for totally C-reflexive modules or more generally for complexes is in $D\mathcal{G}_C$ for semidualizing modules. In this case, M is a totally reflexive module, the resulting complex will be called a complete P_C or a complete PP_C resolution in White (2010) and (Sather-Wagstaff, 2009b, Section 5.2). The construction of such resolutions is simple, however we need a functorial construction which we give here.

We begin by concatenating a plethora of natural transformations.

Definition 5.3.1. Let $X_\bullet$ be a complex in $\mathrm{Ch}^b(R)$.

1. For $X_\bullet \in \mathrm{Ch}^b(R)$, let $\rho(X_\bullet)$ denote the projective resolution of $X_\bullet$. We may consider this as either as a full and faithful functor $K^b(R) \to K^+(R)$ or $D^b(R) \to K^+(R)$ since the result is the same in either case.

2. Let $\pi : \rho \to \mathbf{id}_{K^+(R)}$ be the natural transformation arising from the quasi-isomorphism $\rho(X_\bullet) \to X_\bullet$.

3. Let $\eta : \mathbf{id}_{K(R)} \to -^{\dagger\dagger}$ be the natural transformation arising from the natural map $X_\bullet \to X^{\dagger\dagger}$.

4. Let $\dagger\pi\dagger : -^{\dagger\dagger} \to \rho(-^\dagger)^\dagger$ be the natural transformation defined by

$$\dagger\pi\dagger_X = \pi_{X^\dagger}{}^\dagger : X_\bullet^{\dagger\dagger} \to \rho(X_\bullet^\dagger)^\dagger$$

We now have a sequence of natural transformations

$$\rho \xrightarrow{\pi} \mathbf{id}_{K^b(R)} \xrightarrow{\eta} \dagger \circ \dagger \xrightarrow{\dagger\pi\dagger} \dagger \circ \rho \circ \dagger$$

whose composition is a natural transformation of functors from $K^b(R) \to K(R)$. Essentially, we wish to talk about the mapping cone of this sequence of functors. Unfortunately, the axioms of a triangulated category do not show that this definition is well defined, because the morphism guaranteed in Axiom 3 (see Definition 4.12) is not functorial, see (Kashiwara & Schapira, 2006, Proposition 10.1.17). However, an examination of the natural transformations involved allows us to make the following definition by working in the category $\mathrm{Ch}(R)$.

Definition 5.3.2. Let $\sigma : \rho \to \dagger\circ\rho\circ\dagger$ be the natural transformation defined by $\sigma = \dagger\pi\dagger\circ\eta\circ\pi$. We set $S = \mathrm{cone}(\sigma) : \mathrm{Ch}^b(R) \to \mathrm{Ch}(R)$. Specifically, for a complex $X_\bullet \in \mathrm{Ch}^b(R)$, we let

$SX_\bullet$ be the complex cone(σ_X), the cone of the chain map σ_X.

Note that the chain map $\sigma_{X_\bullet}$ is well defined morphism in $K(R)$ for any complex $X_\bullet$ i.e. $\sigma_{X_\bullet}$ is independent, up to homotopy, of the choice of projective resolutions of $X_\bullet$ and $X_\bullet^\dagger$.

Lemma 5.3.3. *The mapping* $S : \mathrm{Ch}^b(R) \to \mathrm{Ch}(R)$ *is well defined up to homotopy, giving a functor* $S : K^b(R) \to K(R)$.

Proof. Set $\psi = \dagger \circ \rho \circ \dagger$. It suffices to show that for a chain map $\varphi : X_\bullet \to Y_\bullet$ that is homotopic to zero, every choice of $S\varphi$ is also homotopic to zero. So suppose that h is a homotopy from φ to zero. When working over $\mathrm{Ch}^b(R)$, the complexes and maps $\rho(X_\bullet), \rho(Y_\bullet), \psi(X_\bullet), \psi(Y_\bullet), \rho\varphi, \psi\varphi$ are all not uniquely defined. Thus we fix these choices for the rest of the proof. We also use these choices to construct $SX_\bullet, SY_\bullet, S\varphi$. Furthermore, given these choices, the maps $\sigma_{X_\bullet}$ and $\sigma_{Y_\bullet}$ are still natural i.e. the following square still commutes in $\mathrm{Ch}^b(R)$.

$$\begin{array}{ccc} \rho(X_\bullet) & \xrightarrow[\sigma_{X_\bullet}]{} & \psi(X_\bullet) \\ \downarrow{\scriptstyle \rho\varphi} & & \downarrow{\scriptstyle \psi\varphi} \\ \rho(Y_\bullet) & \xrightarrow[\sigma_{Y_\bullet}]{} & \psi(Y_\bullet) \end{array}$$

Furthermore, by the proof of (Weibel, 1994, Exercise 5.7.3), there exists a lift ρh of ρ such that ρh is a homotopy from $\rho\varphi$ to zero. Similarly, there exists a choice of ψh of h such that ψh is a homotopy from $\psi\varphi$ to zero. Again, as before, the following commutes

$$\begin{array}{ccc} \rho(X_\bullet) & \xrightarrow[\sigma_{X_\bullet}]{} & \psi(X_\bullet) \\ \downarrow{\scriptstyle \rho h} & & \downarrow{\scriptstyle \psi h} \\ \rho(Y_\bullet) & \xrightarrow[\sigma_{Y_\bullet}]{} & \psi(Y_\bullet) \end{array}$$

and thus we have $\sigma_Y \circ \rho h - \psi h \circ \sigma_X = 0$.

Define the homotopy $h'_n = \rho h_{n-1} \oplus \psi h_n$. The next calculation follows from the previous discussion and completes the proof.

$$\begin{aligned}\partial^{SY}h' + h'\partial^{SX} &= (-\partial^{\rho Y} \oplus (\partial^{\psi Y} - \sigma_Y)) \circ (-\rho h \oplus \psi h) + (-\rho h \oplus \psi h) \circ (-\partial^{\rho X} \oplus (\partial^{\psi X} - \sigma_X)) \\ &= \left(\partial^{\rho Y} \circ \rho h + \rho h \circ \partial^{\rho X}\right) \oplus (\partial^{\psi Y} \circ \psi h + \sigma_Y \circ \rho h + \psi h \circ \partial^{\psi X} - \psi h \circ \sigma_X)) \\ &= \rho\varphi \oplus (\psi\varphi + \sigma_Y \circ \rho h - \psi h \circ \sigma_X) \\ &= \rho\varphi \oplus \psi\varphi\end{aligned}$$

□

Our end goal is to construct a functor from $D^b(R) \to K(R)$. To do this, we need some intermediate steps.

Definition 5.3.4. Let $\tau_{\leq n}(X_\bullet)$ be the complex

$$\cdots \to 0 \to 0 \to \ker \partial_n^X \to X_n \to X_{n-1} \to \cdots$$

known as the truncation of $X_\bullet$. This defines a functor $\tau_{\leq n} : K^+(R) \to K^b(R)$.

In this situation, we have the obvious natural transformations $\tau_{\leq n} \to \tau_{\leq n+1}$ which yields a directed system such that

$$\lim_{\substack{\to \\ n}} \tau_{\leq n} = \mathbf{id}_{K^+(R)} .$$

Note that the functor $\rho : D^b(R) \to K^+(R)$ gives us the directed system of functors $\tau_{\leq n} \circ \rho : D^b(R) \to K^b(R)$.

Definition 5.3.5. Set

$$T_C = \Sigma^{-1} \lim_{\substack{\to \\ n}} S \circ \tau_{\leq n} \circ \rho : D^b(R) \to K(R)$$

This is our desired functor. We justify this claim with the following example. It is important to note that since we are working in the homotopy category, we can are free to work with any choice of projective resolution.

Example 5.3.6. Let M be a totally C-reflexive module. Let $n > 0$ and $P_\bullet$ be a projective resolution of $P_\bullet$. The complex $\tau_n P_\bullet$ is the complex

$$\cdots \to 0 \to \Omega^n M \to P_n \to \cdots \to P_0 \to 0 \to \cdots$$

and the complex $(\tau_n P_\bullet)^\dagger$ is the complex

$$\cdots 0 \to P_0^\dagger \to \cdots P_n^\dagger \to (\Omega_n M)^\dagger \to 0 \to \cdots .$$

Since M is totally C-reflexive, $H_0((\tau_{\leq n} P_\bullet)^\dagger) \cong M^\dagger$ and $H_i((\tau_{\leq n} P_\bullet)^\dagger) = 0$ otherwise. We also have a map $M^\dagger \to (\tau_n P_\bullet)^\dagger$. Let $Q_\bullet \xrightarrow{\pi_{M^\dagger}} M^\dagger$ be a projective resolution. Composing these maps yields the following quasi-isomorphism.

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & P_0^\dagger & \longrightarrow & P_1^\dagger & \longrightarrow & \cdots & \longrightarrow & P_n^\dagger & \longrightarrow & (\Omega_n M)^\dagger & \longrightarrow & 0
\end{array}$$

Furthermore, the map $\eta : \tau_{\leq n} P_\bullet \to (\tau_{\leq n} P_\bullet)^{\dagger\dagger}$ will be an isomorphism and so, for each n, $S \circ \tau_{\leq n} \circ \rho(M)$ will simply be the acyclic complex

$$\cdots P_1 \to P_0 \to Q_0^\dagger \to Q_1^\dagger \to \cdots .$$

where P_0 is in homological degree one. Therefore, $T_C M = \Sigma^{-1} S \circ \tau_{\leq n} \circ \rho$ is simply the shift of this complex. Furthermore, $T_C M$ will be a complete P_C resolution of M.

The following is a crucial property of T_C.

Proposition 5.3.7. *The functor T_C is triangulated.*

Proof. Set $\psi = \dagger \circ \rho \circ \dagger$. Sine ρ and ψ are both triangulated, it follows that $\Sigma T_C = T_C \Sigma$. Furthermore, given a triangle $X_\bullet \to Y_\bullet \to Z_\bullet \to$, we have the commutative diagram

$$\begin{array}{ccccccc}
\rho(X_\bullet) & \longrightarrow & \psi(X_\bullet) & \longrightarrow & T_C(X_\bullet) & \longrightarrow & \Sigma\rho(X_\bullet) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\rho(Y_\bullet) & \longrightarrow & \psi(Y_\bullet) & \longrightarrow & T_C(Y_\bullet) & \longrightarrow & \Sigma\rho(Y_\bullet) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\rho(Z_\bullet) & \longrightarrow & \psi(Z_\bullet) & \longrightarrow & T_C(Z_\bullet) & \longrightarrow & \Sigma\rho(Z_\bullet) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}$$

with exact rows and the first, second, and fourth columns exact. It follows from Lemma 4.16 that the third column is also exact, proving the result.

□

One should note that $\mathcal{T}_C(X_\bullet)$ may not be acyclic. In fact, it may not even be a complex of finitely generated modules.

Remark 5.3.8. One can see by the arguments above that we have actually defined a functor $T_C : D^b(\mathrm{Mod}(R)) \to K(R)$, since we did not use the fact that any of the modules involved were finitely generated. In this setting, we can even remove the Noetherian hypothesis.

It is natural to ask how this construction relates with previous works. We discuss the case where $C = R$. A complex $T_\bullet$ is totally acylic if each T_i is projective and both $T_\bullet$ and $T_\bullet^*$ are acyclic. In Veliche (2006), the author defines a complete resolution of $X_\bullet \in \mathrm{Ch}(R)$ to be a diagram of complexes and chain maps

$$T_\bullet \xrightarrow{\lambda} P_\bullet \xrightarrow{\pi} X_\bullet$$

where $T_\bullet$ is totally acyclic, $P_\bullet$ is a projective resolution of $X_\bullet$, and $\lambda_{\gg 0}$ is an isomorphism. A complex admitting a complete resolution is said to have finite Gorenstein projective dimension. Given a complete resolution of $X_\bullet$, Veliche (2006) defines $\widehat{\mathrm{Ext}}^i(X_\bullet, Y_\bullet) =$

$H_{-i}\operatorname{Hom}(T_\bullet, Y_\bullet)$ and Christensen & Jorgensen (2015) defines $\widehat{\operatorname{Tor}}_i(X_\bullet, Y_\bullet) = H_i(T_\bullet, Y_\bullet)$. When $X_\bullet \in D\mathcal{G}_C$ (and R is commutative and Noetherian), in (Veliche, 2006, Theorem 5.3) it is shown that $X_\bullet$ admits a complete resolution. It is in this case that the two theories coincide.

Proposition 5.3.9. *Suppose $X_\bullet \in D\mathcal{G}_R$. There exists a complete resolution of the form*

$$T_R(X_\bullet) \to P_\bullet \to X_\bullet.$$

Proof. Let $P_\bullet$ be a projective resolution of $X_\bullet$. Since $X_\bullet \in D\mathcal{G}_R$, we may let n be such that $H_{\geq n}(X_\bullet) = H_{\leq -n}(X_\bullet^*) = 0$. Using arguments similar to those in Example 3.6, for any $m \geq n$ the natural map

$$(\tau_{\leq n}P_\bullet)^* \to (\tau_{\leq m}P_\bullet)^*$$

is a quasi-isomorphism. Therefore a projective resolution of $(\tau_{\leq n}P_\bullet)^*$ is also a projective resolution of $(\tau_{\leq m}P_\bullet)^*$ showing that $T_R(X_\bullet) = \Sigma^{-1}S\tau_{\leq n}P_\bullet$. Recall however $S\tau_{\leq n}P_\bullet$ is the cone of the morphism

$$P_\bullet \to (\pi(\tau_{\leq n}P_\bullet)^*)^*$$

which implies there is a morphism

$$S\tau_{\leq n}P_\bullet \xrightarrow{\lambda} \Sigma P_\bullet.$$

Furthermore, since $(\pi(\tau_{\leq n}P_\bullet)^*)^*$ is bounded above, τ_i is an isomorphism for $i \gg 0$. We argue that

$$T_R(X_\bullet) \xrightarrow{\Sigma^{-1}\lambda} P_\bullet \to X_\bullet$$

is a complete resolution. It remains to show that $T_R(X_\bullet)$ is totally acyclic.

Set $M_\bullet = \tau_{\leq n}P_\bullet$. Letting $Q_\bullet$ be a projective resolution of $M_\bullet^*$, $Q_\bullet^*$ is isomorphic in $D(R)$

to $(M_\bullet^*)^\#$. We also have

$$M_\bullet^* \to P_\bullet^*$$

is a quasi-isomorphism. Therefore $(M_\bullet^*)^\#$ is also isomorphic in $D(R)$ to $(P_\bullet^*)^\# \cong M_\bullet^{\#\#} \cong M_\bullet$, since $M_\bullet \in D\mathcal{G}_R$. It follows that $Q_\bullet^*$ is quasi-isomorphic to $M_\bullet$ and $P_\bullet$. A close examination of the quasi-isomorphisms mentioned here show that the map $P_\bullet \xrightarrow{\sigma_M} Q_\bullet^*$ is a quasi-isomorphism. This shows that $T_R(X_\bullet)$ is acyclic. However, we also have the exact triangle

$$Q_\bullet^{**} \xrightarrow{\sigma_M^*} P_\bullet^* \to (\Sigma^{-1}T_R(X_\bullet))^* \to$$

However, $Q_\bullet^{**} \cong Q_\bullet$, and so $(\Sigma^{-1}T_R(X_\bullet))^*$ is also acyclic.

□

Remark 5.3.10. It is not clear that if Proposition 3.9 is true when $X_\bullet$ is only assumed to have finite Gorenstein Projective dimension.

Remark 5.3.11. Let $\mathcal{P}_C(R)$ be the category of modules of the form $C \otimes P$ where P is a (not necessarily finitely generated) projective module. In Sather-Wagstaff et al. (2010b), the authors develop what they define a Tate $\mathcal{P}_C(R)$ resolution of a module M as a diagram

$$T_\bullet \xrightarrow{\kappa} W_\bullet \xrightarrow{\pi} M$$

where κ_n is an isomorphism for $n \gg 0$, $W_\bullet$ is a complex of modules in $\mathcal{P}_C$, π is a quasi-isomorphism, and $T_\bullet$ is a totally $\mathcal{P}_C$-acyclic. They show that a module admits such a resolution if and only if it is in $\mathcal{B}_C(R)$ with finite Gorenstein C-projective dimension. See Sather-Wagstaff et al. (2010b) for the relevant definitions. Given such a Tate $\mathcal{P}_C$ resolution of a module M, it is not clear whether or not $T_\bullet$ is homotopic to $T_C(M)$ when $C \neq R$. When R is Noetherian, a finitely generated module M with finite Gorenstein C-projective dimension is in $\overline{\mathcal{G}_C}$. Using arguments similar to Proposition 3.9 and Example 3.6, one can see that any such module $T_C(M)_n$ is projective for $n \gg 0$. Since $T_n \in \mathcal{P}_C(R)$ for $n \gg 0$, it is clear that $T_\bullet$ cannot equal $T_C(M)$. However, this does not rule out the possibility that the two are homotopic

5.4 Relative Tate (co)homology

Since homology is a functor $H_i : K(R) \to \mathrm{Mod}(R)$ we have the following definitions.

Definition 5.4.1. For any complexes $X_\bullet \in D^b(R)$ and $Y_\bullet \in K(R)$, we set

$$\widehat{\mathrm{Tor}}_i^C(X_\bullet, Y_\bullet) = H_i(T_C(X_\bullet) \otimes Y_\bullet) \qquad \widehat{\mathrm{Ext}}_C^i(X_\bullet, Y_\bullet) = H^i(\mathrm{Hom}(T_C(X_\bullet), Y_\bullet))$$

It is clear that $\widehat{\mathrm{Tor}}_i^C$ and $\widehat{\mathrm{Ext}}_C^i$ are bifunctors $D^b(R) \times K(R) \to \mathrm{Mod}(R)$. This section is devoted to understanding the properties of this functor. One should note that for an arbitrary complex, $\mathcal{T}_C$ may not be exact. Hence $\widehat{\mathrm{Tor}}_i^C(X_\bullet, R)$ could possibly be nonzero. It follows from Proposition 3.9 that when $X_\bullet \in D\mathcal{G}_R$, then $\widehat{\mathrm{Tor}}_i^R(X_\bullet, -)$ and $\widehat{\mathrm{Ext}}_R^i(X_\bullet, -)$ are equivalent to functors $\widehat{\mathrm{Tor}}_i(X_\bullet, -)$ and $\widehat{\mathrm{Ext}}_R^i(X_\bullet, -)$ respectively. Remark 3 shows that it is not clear that the cohomology theory defined in Sather-Wagstaff et al. (2010b) by Tate $\mathcal{P}_C(R)$-resolutions is the same as $\widehat{\mathrm{Ext}}_C$ when $C \neq R$. Similar difficulties impede the comparison between $\widehat{\mathrm{Tor}}^C$ and the homology theory in Di et al. (2014).

Lemma 5.4.2. *For exact triangles*

$$M_1 \to M_2 \to M_3 \to \qquad N_1 \to N_2 \to N_3 \to$$

in $D^b(R)$ and $K(R)$ respectively and $M \in D^b(R)$ and $N \in K(R)$, we have the following long exact sequences.

$$\cdots \to \widehat{\mathrm{Tor}}_{i+1}^C(M_3, N) \to \widehat{\mathrm{Tor}}_i^C(M_1, N) \to \widehat{\mathrm{Tor}}_i^C(M_2, N) \to \widehat{\mathrm{Tor}}_i^C(M_3, N) \to \cdots$$

$$\cdots \to \widehat{\mathrm{Tor}}_{i+1}^C(M, N_3) \to \widehat{\mathrm{Tor}}_i^C(M, N_1) \to \widehat{\mathrm{Tor}}_i^C(M, N_2) \to \widehat{\mathrm{Tor}}_i^C(M, N_3) \to \cdots$$

$$\cdots \to \widehat{\mathrm{Ext}}_C^{i-1}(M_1, N) \to \widehat{\mathrm{Ext}}_C^{i}(M_3, N) \to \widehat{\mathrm{Ext}}_C^{i}(M_2, N) \to \widehat{\mathrm{Ext}}_C^{i}(M_1, N) \to \cdots$$

$$\cdots \to \widehat{\mathrm{Ext}}_C^{i-1}(M, N_3) \to \widehat{\mathrm{Ext}}_C^{i}(M, N_1) \to \widehat{\mathrm{Ext}}_C^{i}(M, N_2) \to \widehat{\mathrm{Ext}}_C^{i}(M, N_3) \to \cdots$$

Proof. Since the functors $- \otimes X_\bullet, X_\bullet \otimes -, \mathrm{Hom}(-, X_\bullet), \mathrm{Hom}(X_\bullet, -) : K(R) \to K(R)$ are triangulated, the result follows immediately from Proposition 3.7.

□

Corollary 5.4.3. *For any modules $M, N \in \mathrm{mod}(R)$, we have $\widehat{\mathrm{Tor}}_i^C(\Omega M, N) = \widehat{\mathrm{Tor}}_{i+1}^C(M, N)$.*

Proof. Consider the exact sequence $0 \to \Omega M \to R^n \to M \to 0$. Since $T_C(R)$ is the complex $0 \to R \to R \to 0$, $\widehat{\mathrm{Tor}}_i^C(R, N) = 0$ for all $i \in \mathbb{Z}$. The result follows from Lemma 4.2.

□

The functor $\widehat{\mathrm{Tor}}^C(X_\bullet, Y_\bullet)$ is best behaved when $X_\bullet \in D\mathcal{G}_C$ and $Y_\bullet \in D^b(\mathcal{A}_C)$.

Lemma 5.4.4. *We have a functor $\widehat{\mathrm{Tor}}^C : D\mathcal{G}_C \times D^b(\mathcal{A}_C) \to \mathrm{Mod}(R)$.*

Proof. Since we already have a functor $\widehat{\mathrm{Tor}}^C : D\mathcal{G}_C \times K^b(\mathcal{A}_C) \to \mathrm{mod}(R)$, it suffices to show that $\widehat{\mathrm{Tor}}_i^C(X_\bullet, Y_\bullet) = 0$ for all $i \in \mathbb{Z}$ if $X_\bullet \in \mathcal{G}_C$ and $Y_\bullet \in K^b(\mathcal{A}_C)$ and $Y_\bullet$ is acyclic. Note that $T_C(X_\bullet)$ is an acyclic complex of modules in $\mathrm{add}\, R \oplus C$. Define the complex $T_\bullet^m$ by letting $T_n^m = T_C(X_\bullet)_n$ for all $n \geq m$ and $T_n^m = 0$ otherwise. Let ${}^mE_0^{p,q} = T_p^m \otimes Y_q$. Since $Y_\bullet$ is in $D^b(\mathcal{A}_C)$ and is acyclic, the complex $T_p^m \otimes Y_\bullet$ is also acyclic for all p. It follows that ${}^mE_1^{p,q} = 0$ for all m, and so the total complex of $T_\bullet^m \otimes Y_\bullet$ is acyclic, since ${}^mE_i^{p,q}$ is a third quadrant spectral sequence. But since $Y_\bullet$ is bounded above and below $H_n(T_C(X_\bullet) \otimes Y_\bullet) = H_i(T_\bullet^m \otimes Y_\bullet) = 0$ for $m \ll 0$. It follows that $T_C(X_\bullet) \otimes Y_\bullet$ is acyclic and so $\widehat{\mathrm{Tor}}_i^C(X_\bullet, Y_\bullet) = 0$ for all $i \in \mathbb{Z}$ as desired.

□

Remark 5.4.5. One may also use the same arguments to show that we have a functor $\widehat{\mathrm{Ext}}_C : D\mathcal{G}_C \times D^b(\mathcal{B}_C) \to \mathrm{mod}\, R$.

The following lemma is a relative version of the fact that $\widehat{\operatorname{Tor}}_i(P_\bullet, -) = 0$ for all $i \in \mathbb{Z}$ and $P_\bullet \in \operatorname{Perf}$, (Christensen & Jorgensen, 2014, Proposition 2.5).

Lemma 5.4.6. *If* $X_\bullet \in C \otimes \operatorname{Perf}$ *and* $Y_\bullet \in D^b(\mathcal{A}_C)$*, then* $\widehat{\operatorname{Tor}}_i^C(X_\bullet, Y_\bullet) = 0$ *for all* $i \in \mathbb{Z}$*.*

Proof. We may assume that $X_\bullet = C \otimes Q_\bullet$ for a perfect complex $Q_\bullet$. Set $P_\bullet = \rho(X_\bullet)$ and set $W_\bullet = \operatorname{cone}(P_\bullet \xrightarrow{\pi} X_\bullet)$. Now $W_\bullet$ is an acyclic complex of modules in $D\mathcal{G}_\mathcal{C}$ which is bounded below. It follows that $W^\dagger$ is also acyclic. Since $X_\bullet^\dagger \to P_\bullet^\dagger \to W_\bullet^\dagger \to$ is an exact triangle, it follows that $X_\bullet^\dagger$ and $P_\bullet^\dagger$ are quasi-isomorphic. Moreover, setting $-^* = \operatorname{Hom}(-, R)$,

$$X_\bullet^\dagger = \operatorname{Hom}(C \otimes Q_\bullet, C) \cong \operatorname{Hom}(Q_\bullet, \operatorname{Hom}(C, C)) \cong Q_\bullet^*$$

is a perfect complex, which is a projective resolution of $(\tau_{\leq n} P_\bullet)^\dagger$ for $n \gg 0$. Furthermore, we have $Q_\bullet^{*\dagger} \cong X^{\dagger\dagger} \cong X_\bullet$. It follows that $W_\bullet \cong T_C X_\bullet$.

It remains to compute the total homology of $W_\bullet \otimes Y_\bullet$. Computing the spectral sequence $E_0^{p,q} = W_p \otimes Y_q$ by filtering horizontally yields $E_1^{p,q} = 0$ because each complex $W_\bullet \otimes Y_q$ is acyclic since $W_\bullet$ is a bounded below complex of modules in $\operatorname{add} R \oplus C$. Since $W_\bullet$ is bounded below, this is a third quadrant spectral sequence, allowing us to conclude that the total homology of $W_\bullet \otimes Y_\bullet$ is zero.

□

5.5 The depth formula

In this section, we prove the main result of this chapter.

Theorem 5.5.1. *Suppose* $X_\bullet \in D\mathcal{G}_C$ *and* $Y_\bullet \in D^b(\mathcal{A}_C)$*. If* $\widehat{\operatorname{Tor}}_i^C(X_\bullet, Y_\bullet) = 0$ *for all* $i \in \mathbb{Z}$*, then* $\operatorname{depth} X_\bullet \otimes^{\mathbf{L}} Y_\bullet + \operatorname{depth} R = \operatorname{depth} X_\bullet + \operatorname{depth} Y_\bullet$.

Note that this result recovers the main theorem of Christensen & Jorgensen (2015) when R is Noetherian and $X_\bullet$ is in $\operatorname{Ch}^b(R)$ and $Y_\bullet \in \operatorname{Ch}^b(\operatorname{Mod}(R))$. Indeed, $\mathcal{A}_R = \operatorname{Mod}(R)$

and also the complexes of finite Gorenstein projective dimension are in $D\mathcal{G}_C$. Furthermore, $\widehat{\operatorname{Tor}}_i^R(X_\bullet, Y_\bullet) = \widehat{\operatorname{Tor}}_i(X_\bullet, Y_\bullet)$ when $X_\bullet \in D\mathcal{G}_R$ by Proposition 3.9.

We now give some results necessary for the proof. The following may be thought of as a type of Auslander Buchweitz approximation for complexes.

Lemma 5.5.2. *For any $X_\bullet \in D\mathcal{G}_C$ and for all $n \ll 0$ there exists an exact triangle*

$$L_\bullet \to X_\bullet \to \Sigma^n N \to$$

with $N \in \mathcal{G}_C$ and $L_\bullet \in C \otimes \operatorname{Perf}$.

Proof. Choose an m such that $H_{\geq m}(X_\bullet) = H_{\leq -m}(X_\bullet^\#) = 0$. Fix a bounded complex in the isomorphism class of $X_\bullet^\#$. Let $P_\bullet$ be a projective resolution of $(X_\bullet)^\#$. It is clear that $\tau_{\leq m} P_\bullet$ and $(X_\bullet)^\#$ are quasi isomorphic. Set $M = \ker \partial_m^P$. The complex $P'_\bullet = \cdots \to P_{m+2} \to P_{m+1} \to 0$ is a projective resolution of $\Sigma^{m+1} M$. By our assumptions on m, the complex

$$0 \to M^\dagger \to P_{n+1}^\dagger \to P_{n+2}^\dagger \to \cdots$$

is also exact. This also shows that $\operatorname{Ext}^{>0}(M, C) = 0$. It also shows that $M^\dagger \cong M^\#$ in $D^b(R)$. Since M is also in $D\mathcal{G}_C$, we know that $M \cong M^{\#\#} \cong M^{\dagger\#}$ in $D^b(R)$. This implies that $\operatorname{Ext}^{>0}(M^\dagger, C) = 0$. Therefore M is actually in $\mathcal{G}_C$.

We have a natural map $\Sigma^{m+1} M \to (\tau_{\leq m} P_\bullet)$ whose cone is a perfect complex which we will denote by $Q_\bullet$. Thus we have the exact triangle in $D^b(R)$

$$\Sigma^{m+1} M \to X_\bullet^\# \to Q_\bullet \to$$

which gives us the exact triangle

$$Q_\bullet^\# \to X_\bullet^{\#\#} \to (\Sigma^{-m-1} M)^\# \to$$

which is isomorphic to

$$Q_\bullet^\# \to X_\bullet \to \Sigma^{-m-1} M^\dagger \to .$$

Set $N = M^\dagger$ and $n = -m-1$ and $L_\bullet = Q_\bullet^\#$. Since we can make m arbitrarily greater than zero, we can make n arbitrarily less than zero.

It remains to show that $L_\bullet$ is in $C \otimes \mathrm{Perf}$. Since $Q_\bullet$ is perfect, both $L_\bullet$ and $Q_\bullet$ are in $D\mathcal{G}_C$. Let $I^\bullet$ be an injective resolution of C. We have

$$\begin{aligned}
L_\bullet &= \mathrm{Hom}(Q_\bullet, C) \\
&\simeq \mathrm{Hom}(Q_\bullet, I^\bullet) \\
&\cong \mathrm{Hom}(Q_\bullet^{**}, I^\bullet) \\
&\cong \mathrm{Hom}(\mathrm{Hom}(Q_\bullet^*, \mathrm{Hom}(C, C)), I^\bullet) \\
&\simeq \mathrm{Hom}(\mathrm{Hom}(Q_\bullet^*, \mathrm{Hom}(C, I^\bullet)), I^\bullet) \\
&\cong \mathrm{Hom}(\mathrm{Hom}(Q_\bullet^* \otimes C, I^\bullet), I^\bullet) \\
&\simeq (Q_\bullet^*) \otimes C
\end{aligned}$$

where $-^* = \mathrm{Hom}(-, *R)$. The last quasi-isomorphism is because the complex $Q_\bullet^*$ is perfect and so $Q_\bullet \otimes C \in C \otimes \mathrm{Perf} \subseteq D\mathcal{G}_C$. This also completes the proof.

□

We now give an interesting fact concerning Foxby duality.

Lemma 5.5.3. *Foxby duality preserves* max. *Specifically, For every $A_\bullet \in D^b(\mathcal{A}_C)$ and $B_\bullet \in D^b(\mathcal{B}_C)$, we have*

$$\max A_\bullet = \max C \otimes^{\mathbf{L}} A_\bullet \qquad \max B_\bullet = \max \mathbf{R}\mathrm{Hom}(C, B_\bullet).$$

Proof. Because of Foxby duality, Theorem 2.7, it suffices to prove the following claims.

1. $\max A_\bullet \geq \max C \otimes^{\mathbf{L}} A_\bullet$

2. $\max B_\bullet \geq \max \mathbf{R}\mathrm{Hom}(C_\bullet, B_\bullet)$

First, since $\mathcal{A}_C$ has the two out of three property by (Sather-Wagstaff, 2009b, Proposition 3.1.7), there exists a complex $A'_\bullet$ such that $A_\bullet \simeq A'_\bullet$ and $\max_c A'_\bullet = \max A'_\bullet$. Since $C \otimes^{\mathbf{L}} A_\bullet$ and $C \otimes^{\mathbf{L}} A'_\bullet$ are quasi-isomorphic, we may replace $A_\bullet$ with $A'_\bullet$. Hence, we may assume that $\max_c A_\bullet = \max A_\bullet$. Let $P_\bullet$ be a projective resolution of C. Set $E_0^{i,j} = P_i \otimes A_j$. Since $A_\bullet$ is bounded, this is a third quadrant spectral sequence. Computing the spectral sequence using a vertical filtration, we have $E_1^{i,j} = H_i(P_\bullet \otimes A_j) = \mathrm{Tor}_i(C, A_i)$. Therefore $E_1^{0,j} = C \otimes A_j$ and $E_1^{i,j} = 0$ when $i \neq 0$ since $A_j \in \mathcal{A}_C$. Hence the spectral sequence collapses after the second page, showing that $C \otimes A_\bullet$ is quasi-isomorphic to $C \otimes^{\mathbf{L}} A_\bullet$. We therefore, have the inequality

$$\max A_\bullet = \max_c A_\bullet = \max_c C \otimes A_\bullet \geq \max C \otimes A_\bullet = \max C \otimes^{\mathbf{L}} A_\bullet$$

Similarly, since $\mathcal{B}_C$ has the two out of three property by (Sather-Wagstaff, 2009b, Proposition 3.1.7), there exists a complex $B'_\bullet$ such that $B_\bullet \simeq B'_\bullet$ and $\max_c B'_\bullet = \max B'_\bullet$. Since $\mathbf{R}\mathrm{Hom}(C, B_\bullet) \simeq \mathbf{R}\mathrm{Hom}(C, B'_\bullet)$, we may replace $B_\bullet$ with $B'_\bullet$. Hence, we may assume that $\max_c B_\bullet = \max B_\bullet$. Set $E_0^{i,j} = \mathrm{Hom}(P_i, B_j)$ where $P_\bullet$ is still a projective resolution of C. Since $B_\bullet$ is bounded, this is a third quadrant spectral sequence. Computing the spectral sequence using a vertical filtration, we have $E_1^{i,j} = H_i(\mathrm{Hom}(P_\bullet, B_j)) = \mathrm{Ext}^{-j}(C, A_i)$. Therefore $E_1^{0,j} = \mathrm{Hom}(C, B_j)$ and $E_1^{i,j} = 0$ when $i \neq 0$ since $B_j \in \mathcal{B}_C$. Hence the spectral sequence collapses after the second page, showing that $\mathrm{Hom}(C, B_\bullet)$ is quasi-isomorphic to $\mathbf{R}\mathrm{Hom}(C, B_\bullet)$. We therefore, have the inequality

$$\max B_\bullet = \max_c B_\bullet = \max_c \mathrm{Hom}(C, B_\bullet) \geq \max \mathrm{Hom}(C, B_\bullet) = \max \mathbf{R}\mathrm{Hom}(C, B_\bullet)$$

□

We now prove the following corollary which is interesting in its own right.

Corollary 5.5.4. *Foxby duality preserves depth. Specifically, for any complex* $A_\bullet \in D^b(\mathcal{A}_C)$

and for any complex $B_\bullet \in D^b(\mathcal{B}_C)$, we have

$$\operatorname{depth} C \otimes^{\mathbf{L}} A_\bullet = \operatorname{depth} A_\bullet$$

$$\operatorname{depth} \mathbf{R}\mathrm{Hom}(C, B_\bullet) = \operatorname{depth} B_\bullet.$$

Proof. Let $K_\bullet$ be the Koszul complex on a generating set of $\mathfrak{m}$. Since $K_\bullet$ is a complex of free modules, $K_\bullet \otimes A_\bullet$ is also in $D^b(\mathcal{A}_C)$. It follows from the previous lemma that

$$\max K_\bullet \otimes A_\bullet = \max C \otimes^{\mathbf{L}} K_\bullet \otimes A_\bullet = \max K_\bullet \otimes (C \otimes^{\mathbf{L}} A_\bullet).$$

The first statement is now clear.

Let $P_\bullet$ be a projective resolution of C. One can see that

$$K_\bullet \otimes \mathrm{Hom}(P_\bullet, B_\bullet) \cong \mathrm{Hom}(P_\bullet, K_\bullet \otimes B_\bullet) = \mathbf{R}\mathrm{Hom}(C, K_\bullet \otimes B_\bullet)$$

since $K_\bullet$ is a complex of free modules. Furthermore, $K_\bullet \otimes B_\bullet$ is also a complex of modules in $\mathcal{B}_C$ since each K_n is free and $B_\bullet \in \mathrm{Ch}^b(\mathcal{B}_C)$. Hence the second statement follows from Lemma 5.3.

□

The following lemma is a semidualizing analogue of a result in Iyengar (1999) which states that the derived depth formula is satisfied by perfect complexes.

Lemma 5.5.5. *For any $L_\bullet \in C \otimes \mathrm{Perf}$ and any $Y_\bullet \in D^b(\mathcal{A}_C)$, the derived depth formula holds, i.e*

$$\operatorname{depth} L_\bullet \otimes^{\mathbf{L}} Y_\bullet + \operatorname{depth} R = \operatorname{depth} L_\bullet + \operatorname{depth} Y_\bullet.$$

Note that this is a special case of Theorem 5.1 since $\widehat{\mathrm{Tor}}_i^C(L_\bullet, Y_\bullet) = 0$ for all $i \in \mathbb{Z}$.

Proof. By assumption, we may write $L_\bullet = C \otimes P_\bullet$ where $P_\bullet$ is a perfect complex. Let $Q_\bullet$

be a projective resolution of $Y_\bullet$. Then we have $Y_\bullet \otimes^L L_\bullet = Q_\bullet \otimes C \otimes P_\bullet = P_\bullet \otimes^{\mathbf{L}} (Y_\bullet \otimes^{\mathbf{L}} C)$ is bounded. Hence $P_\bullet$ is perfect, so by Iyengar (1999), $P_\bullet$ and $Y_\bullet$ satisfy the depth formula. Hence

$$\operatorname{depth} Y_\bullet \otimes^{\mathbf{L}} L_\bullet = \operatorname{depth} P_\bullet \otimes^{\mathbf{L}} (Y_\bullet \otimes^L C) = \operatorname{depth} P_\bullet + \operatorname{depth} Y_\bullet \otimes^L C - \operatorname{depth} R$$

By the previous lemma, $\operatorname{depth} Y_\bullet \otimes^L C = \operatorname{depth} Y_\bullet$. Again by Iyengar (1999), $\operatorname{depth} L_\bullet = \operatorname{depth} P_\bullet + \operatorname{depth} C - \operatorname{depth} R = \operatorname{depth} P_\bullet$. The result now follows.

□

We now proceed with the proof of the main theorem.

Proof of Theorem 5.1. From Lemma 5.2, we have an exact triangle $L_\bullet \to X_\bullet \to \Sigma^n N \to$ with $N \in \mathcal{G}_C$ and $L_\bullet \in C \otimes \operatorname{Perf}$. Because of Lemma 4.6, Lemma 4.2, and our assumptions, we may conclude that $\widehat{\operatorname{Tor}}_i^C(\Sigma^n N, Y_\bullet) = 0$ for all $i \in \mathbb{Z}$. Set $M_\bullet = \rho((\Sigma^n N)^\dagger)^\dagger$. By definition, we have the exact triangle

$$\rho(\Sigma^n N) \to M_\bullet \to T_C(\Sigma^n N) \to .$$

Since $\widehat{\operatorname{Tor}}_i^C(\Sigma^n N, Y_\bullet) = 0$ for all $i \in \mathbb{Z}$, the exact triangle

$$\rho(\Sigma^n N) \otimes Y_\bullet \to M_\bullet \otimes Y_\bullet \to T_C(\Sigma^n N) \otimes Y_\bullet \to$$

shows that the map $\rho(\Sigma^n N) \otimes Y_\bullet \to M_\bullet \otimes Y_\bullet$ is a quasi-isomorphism. Let $K_\bullet$ be the Koszul complex on a generating set of $\mathfrak{m}$. It follows that $K_\bullet \otimes \Sigma^n N \otimes^{\mathbf{L}} Y_\bullet \simeq K_\bullet \otimes \Sigma^n \rho(N) \otimes Y_\bullet$ is quasi-isomorphic to $K_\bullet \otimes M_\bullet \otimes Y_\bullet$. We thus have

$$\max K_\bullet \otimes \Sigma^n N \otimes^{\mathbf{L}} Y_\bullet = \max K_\bullet \otimes M_\bullet \otimes Y_\bullet \leq \max_c K_\bullet + \max_c M_\bullet + \max_c Y_\bullet = \max_c K_\bullet + \max_c Y_\bullet + n.$$

However, by Lemma 5.2, we can choose n small enough such that

$$\max_c K_\bullet + \max_c Y_\bullet + n < \max K_\bullet \otimes X_\bullet \otimes^{\mathbf{L}} Y_\bullet$$

which means that by Lemma 4.2 we can choose n small enough such that

$$\max K_\bullet \otimes X_\bullet \otimes^{\mathbf{L}} Y_\bullet = \max K_\bullet \otimes L_\bullet \otimes^{\mathbf{L}} Y_\bullet$$

$$\max K_\bullet \otimes X_\bullet = \max K_\bullet \otimes L_\bullet.$$

We therefore have $\operatorname{depth} X_\bullet \otimes^{\mathbf{L}} Y_\bullet = \operatorname{depth} L_\bullet \otimes^{\mathbf{L}} Y_\bullet$. We now merely need to apply Lemma 5.5:

$$\operatorname{depth} X_\bullet + \operatorname{depth} Y_\bullet = \operatorname{depth} L_\bullet + \operatorname{depth} Y_\bullet = \operatorname{depth} L_\bullet \otimes^{\mathbf{L}} Y_\bullet + \operatorname{depth} R.$$

□

5.6 Some observations on AB rings

The AB and UAC conditions are intertwined with the depth formula. In order to prevent redundancies, we wish to state these observations in the most general setting. To do this, we like to generalize the notion of finite AB-dimension. We recall that $(R, \mathfrak{m}, k)$ is a local ring.

Definition 5.6.1. 1. A class of modules $\mathcal{X} \subseteq \operatorname{mod}(R)$ satisfies *UAC* if there is an $\eta \in \mathbb{N}$ such that for all $X \in \mathcal{X}$ and $Y \in \operatorname{mod}(R)$, $\operatorname{Ext}^{\gg 0}(X, Y) = 0$ implies $\operatorname{Ext}^{>\eta}(X, Y) = 0$. The UAC index of $\mathcal{X}$ is the smallest such η. A ring is UAC if $\operatorname{mod}(R)$ is UAC.

2. A class of modules $\mathcal{X}$ satisfies *UTAC* if there exists an $\eta \in \mathbb{N}$ such that for all $X \in \mathcal{X}$ and $Y \in \operatorname{mod}(R)$, $\operatorname{Tor}_{\gg 0}(X, Y) = 0$ implies $\operatorname{Tor}_{>\eta}(X, Y) = 0$. The UTAC index of $\mathcal{X}$ is the smallest such η. A ring is UTAC if $\operatorname{mod}(R)$ is UTAC.

Note that for a class of modules $\mathcal{X} \subseteq \overline{\mathcal{G}_C}$, $\mathcal{X}$ is UAC if and only it its elements have finite AB-dimension. We give some simple lemmas regarding these properties.

Lemma 5.6.2. *Let $\mathcal{X} \subseteq \operatorname{mod}(R)$. Suppose that $x \in R$ is regular on R and on every module in $\mathcal{X}$. Setting $\mathcal{X}' = \{X/xX \mid X \in \mathcal{X}\}$, we have the following.*

1. *$\mathcal{X}'$ is UAC if and only if $\mathcal{X}$ is UAC*
2. *$\mathcal{X}'$ is UTAC if and only if $\mathcal{X}$ is UTAC*

Proof. The proof of prove (1) is essentially the same as the proof of (Christensen & Holm, 2012, Lemma 2.1). The proof of (2) is essentially given in (Huneke & Jorgensen, 2003, Proposition 3.2). □

Lemma 5.6.3. *If R is Cohen-Macaulay, then $\mathcal{X} \subseteq \operatorname{mod}(R)$ is UAC if and only if it is UTAC.*

The proof is essentially the same as (Huneke & Jorgensen, 2003, Theorem 3.3).

Proof. We induct on $\dim R$. Suppose $\dim R = 0$. First suppose that $\mathcal{X}$ is UAC with UAC index η and that $\operatorname{Tor}_{\gg 0}(X,Y) = 0$ with $X \in \mathcal{X}$. Letting $\vee$ be the Matlis dual, we have $\operatorname{Tor}_i(X,Y) \cong \operatorname{Ext}^i(X,Y^\vee)^\vee$ and thus $\operatorname{Ext}^{\gg 0}(X,Y^\vee) = 0$. However, $\operatorname{Ext}^{>\eta}(X,Y^\vee) = 0$ and hence $\operatorname{Tor}_{>\eta}(X,Y) = 0$. Showing that UTAC implies UAC is similar, only one uses the isomorphism $\operatorname{Tor}_i(X,Y^\vee) \cong \operatorname{Ext}^i(X,Y)^\vee$.

Now suppose that $\dim R = d > 0$. Since $\operatorname{Tor}_{i+d}(X,Y) \cong \operatorname{Tor}_i(\Omega^d X,Y)$ and $\operatorname{Ext}^{i+d}(X,Y) \cong \operatorname{Ext}^i(X,Y)$, we can see that $\mathcal{X}$ is UAC or UTAC if and only if $\Omega^d\mathcal{X} = \{\Omega^d X \mid X \in \mathcal{X}\}$ is UAC or UTAC respectively. Therefore we may work with $\Omega^d\mathcal{X}$ assume $\mathcal{X} \subseteq$ MCM. Let $x \in R$ be a regular element, and set $\mathcal{X}' = \{X/xX \mid X \in \mathcal{X}\}$. By induction, $\mathcal{X}'$ is UAC if and only if it is UTAC. The result now follows from Lemma 6.2.

□

We now show that these conditions have a relationship with the depth formula.

Proposition 5.6.4. *Let $\mathcal{X} \subseteq \operatorname{mod}(R)$. Suppose we have the following*

1. *For any $p \in \operatorname{spec} R$ and any $X \in \mathcal{X}$ and $Y \in \operatorname{mod}(R_p)$, X_p and Y satisfy the depth formula, that is if $\operatorname{Tor}^{R_p}_{>0}(X_p, Y) = 0$, then*

$$\operatorname{depth} X_{\mathfrak{p}} \otimes Y + \operatorname{depth} R_p = \operatorname{depth} X_p + \operatorname{depth} Y.$$

2. *If $\operatorname{depth} R_p = 0$, then $\mathcal{X}_p$ has is UTAC and has UTAC index 0.*

3. *The category $\mathcal{X}$ is closed under syzygies.*

Then $\mathcal{X}$ is UTAC with UTAC index less than $\operatorname{depth} R$.

Moreover, if R is CM, then $\mathcal{X}$ is also UAC. Also if $Y \in \operatorname{mod}(R)$ is maximal Cohen-Macaulay and $X \in \mathcal{X}$, then $\operatorname{Tor}_{\gg 0}(X, Y) = 0$ implies that $\operatorname{Tor}_{>0}(X, Y) = 0$.

Proof. We proceed by induction on $\dim R$. If $\dim R = 0$, then $\operatorname{depth} R = 0$ and we are done. So assume that $\dim R > 0$. Thus, we know that for all $p \in \operatorname{spec} R$ with $p \neq \mathfrak{m}$, $\mathcal{X}_p$ is UTAC. Furthermore, we may assume that $\operatorname{depth} R > 0$. Set $d = \operatorname{depth} R$. Let $\mathcal{Y}$ be the class of modules of the form $\Omega^d M$ with $M \in \operatorname{mod}(R)$. Take any $Y \in \mathcal{Y}$ and $X \in \mathcal{X}$, such that $\operatorname{Tor}_{\gg 0}(X, Y) = 0$. The result follows if we show that $\operatorname{Tor}_{>0}(X, Y) = 0$. So by way of contradiction, suppose not, i.e. suppose that $\operatorname{Tor}_i(X, Y) \neq 0$ for some $i \neq 0$. By replacing Y with some syzygy, we may assume that $\operatorname{Tor}_1(X, Y) \neq 0$ and $\operatorname{Tor}_{>1}(X, Y) = 0$.

Now $Y_p = \Omega^d_{R_p} M_p$ for some $M \in \operatorname{mod}(R)$. We thus have $\operatorname{Tor}^{R_p}_{\gg 0}(X_p, M_p) = 0$. Since the the UTAC index of $\mathcal{X}_p$ is less than $\operatorname{depth} R_p < \operatorname{depth} R$, it follows that

$$0 = \operatorname{Tor}^{R_p}_{>d}(X_p, M_p) = \operatorname{Tor}^{R_p}_{>0}(X_p, Y_p) = \operatorname{Tor}_{>0}(X, Y)_p$$

and in particular, $\operatorname{Tor}_1(X, Y)_p = 0$. It follows that $\operatorname{Tor}_1(X, Y)$ has finite length.

Since $\operatorname{Tor}_{>0}(\Omega X, Y) = 0$ and $\Omega X \in X$, we have

$$\operatorname{depth} \Omega X \otimes Y + \operatorname{depth} R = \operatorname{depth} X + \operatorname{depth} Y$$

and since $\operatorname{depth} Y = \operatorname{depth} R$ and $\operatorname{depth} \Omega X = \min\{\operatorname{depth} X + 1, \operatorname{depth} R\} > 0$, we also have $\operatorname{depth} \Omega X \otimes Y > 0$. However, we also have the exact sequence

$$0 \to \operatorname{Tor}_1(X, Y) \to \Omega X \otimes Y \to Y^n \to X \otimes Y \to 0$$

where $0 \to \Omega X \to R^n \to X \to 0$ is exact. However, since $\operatorname{Tor}_1(X, Y)$ has finite length, we also have $\operatorname{depth} \Omega X \otimes Y = 0$. This is a contradiction, proving the claim

Now suppose that R is Cohen-Macaulay. By Lemma 6.3, we know that $\mathcal{X}$ satisfies UAC. We can prove the last statement using the above argument by replacing the category $\mathcal{Y}$ with MCM.

□

Remark 5.6.5. The following problem was on the final exam for the homological algebra course taught by by Hailong Dao in the fall of 2012: Assuming that R i Cohen-Macaulay, if $\operatorname{pd} X < \infty$ and Y is maximal Cohen-Macaulay, then $\operatorname{Tor}_{>0}(M, Y) = 0$. Proposition 6.4 is a generalization of this problem.

We now give some immediate corollaries.

Corollary 5.6.6. *Suppose the depth formula holds for all modules at every localization. Suppose further that R_p is UTAC with UTAC index 0 for all primes with $\operatorname{depth} R_p = 0$. Then R is UTAC at every localization with UTAC index $\operatorname{depth} R$.*

Corollary 5.6.7. *Suppose R is Cohen-Macaulay and at every localization the depth formula holds for all pairs of modules. Suppose further that R satisfies R_0 i.e. R_p is regular when $\operatorname{ht} p = 0$. Then R is UAC at every localization.*

The following gives a converse to a special case of (Christensen & Jorgensen, 2015, Theorem 4.2).

Theorem 5.6.8. *Let R be a Gorenstein isolated singularity with $\dim R > 0$. The following are equivalent.*

1. *Every pair of R-modules satisfies the derived depth formula (see definition 4.2)*
2. *Every pair of R-modules satisfies the depth formula*
3. *Every pair of R-modules X and Y satisfy the depth formula if X is maximal Cohen-Macaulay*
4. *R is AB*

Proof. As remarked upon in the introduction, if $\operatorname{Tor}_{>0}(X, Y) = 0$, then $X \otimes^{\mathbf{L}} Y$ is bounced, showing that (1) implies (2). Clearly (2) implies (3). By Lemma 6.4, (3) implies that the MCM is UTAC. But since R is Cohen-Macaulay, this shows that every Maximal Cohen-Macaulay module has finite AB-dimension. It follows that every module has finite AB-dimension, and so the ring is AB. Lastly, (4) implies (1) by (Christensen & Jorgensen, 2015, Theorem 4.2).

□

There are no known examples of a pair of modules which do not satisfy the depth formula. There is an example in Jorgensen & Şega (2004) of an Artinian Gorenstein ring which is not AB. For this reason we pose the following question.

Question 4. *Does there exists a one-dimensional Gorenstein domain that is not AB?*

Such a ring is the simplest nontrivial ring meeting the hypothesis of Theorem 6.8. Thus a positive answer would give an example of a ring where the depth formula fails. This question is also interesting when compared with the Huneke Weigand conjecture in Huneke & Wiegand (1994).

Conjecture 5.6.9 (Huneke Weigand)**.** *Let R be a one-dimensional Gorenstein domain. If M is finitely generated and not free, then $M \otimes M^*$ has a nontrivial torsion submodule.*

If Question 4 is false and the Huneke Weigand conjecture is true, then for any torsion-free module M, $\operatorname{Tor}_i(M, M^*) \neq 0$ for arbitrarily large i. Indeed, M and M^* would be maximal

Cohen-Macaulay in this case. So if Question 4 is false, then R is AB, which means that $\operatorname{Tor}_{\gg 0}(M, M^*) = 0$ would imply that $\operatorname{Tor}_{>0}(M, M^*) = 0$. Therefore, since the depth formula also holds when Question 4 is false, $M \otimes M^*$ is Cohen-Macaulay, and hence also torsion-free.

One would like to give a version of Theorem 6.8 for UAC rings. However, Theorem 5.1 does not show that UAC rings satisfy the depth formula. We address this difficulty with the following question.

Question 5. *Suppose that* $\operatorname{Tor}_{\gg 0}(X, Y) = 0$*. Then does there exist a semidualizing module* C *such that* $X \in \overline{\mathcal{G}_C}$ *and* $Y \in \mathcal{A}_C$*?*

A positive answer would imply that Cohen-Macaulay UAC rings and UTAC rings have a similar relationship with the depth formula that AB rings enjoy.

Theorem 5.6.10. *Suppose that the answer to Question 5 for a ring* R *is yes. Furthermore, suppose that* R *is an isolated singularity with* $\operatorname{depth} R > 0$*. Then the following are equivalent.*

1. *every pair of* R *modules satisfies the derived depth formula*

2. *Every pair of* R*-modules satisfies the depth formula*

3. R *is UTAC with UTAC index* $\operatorname{depth} R$*.*

4. R *is UTAC*

Proof. The earlier arguments show (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). Thus we need only to show that (4) implies (1). The remainder of the proof is similar to that of (Christensen & Jorgensen, 2015, Lemma 4.1). Suppose $\operatorname{Tor}_{\gg 0}(X, Y) = 0$. Since we assume that the answer to Question 5 is yes, there is a semidualizing module C such that $X \in \overline{G_C}$ and $Y \in \mathcal{A}_C$. By Theorem 5.1, we need only to show that $\widehat{\operatorname{Tor}}_i^C(X, Y) = 0$ for all $i \in \mathbb{Z}$.

Using (Auslander & Buchweitz, 1989, Theorem 1.1), there exists an exact sequence

$$0 \to W \to M \to X \to 0$$

with $M \in \mathcal{G}_C$, and W has a resolution of the form

$$0 \to L_n \to \cdots \to L_0 \to W \to 0$$

with each $L_j \in \operatorname{add} C$. Since $Y \in \mathcal{A}_C$, $\widehat{\operatorname{Tor}}_i^C(L_j, Y) = 0$ for all i and j. It using Lemma 4.2 one can easily show that $\widehat{\operatorname{Tor}}_i^C(W, Y) = 0$ by inducting on n. Hence, by Lemma 4.2 it suffices to show that $\widehat{\operatorname{Tor}}_i^C(M, Y) = 0$. Furthermore, since $Y \in \mathcal{A}_C$, we may use the resolution $L_\bullet$ to compute $\operatorname{Tor}_i(W, Y)$. Doing so shows that $\operatorname{Tor}_{\gg 0}(W, Y) = 0$. We thus also have $\operatorname{Tor}_{\gg 0}(M, Y) = 0$.

Using Corollary 4.3, it suffices to show that $\widehat{\operatorname{Tor}}_i^C(\Omega M, Y) = 0$ for all $i \in \mathbb{Z}$. Therefore, by replacing M with a high syzygy, we may assume that $\operatorname{Tor}_{>0}(M, Y) = 0$. Let $P_\bullet$ be a projective resolution of M. Let $Q_\bullet$ be a projective resolution of $M^\dagger$. Since M is in $\mathcal{G}_C$, $Q_\bullet^\dagger$ is an $\operatorname{add} C$ coresolution of M. As in Example 3.6, we may splice these resolutions together

$$\cdots \to P_1 \to P_0 \to Q_0^\dagger \to Q_1^\dagger \to \cdots$$

to construct $T_C(M)$. Set $M^i = \ker \partial_i^{Q^\dagger}$. Note that $M^0 = M$. Since $Y \in \mathcal{A}_C$, we may use the resolution

$$\cdots \to P_1 \to P_0 \to Q_0^\dagger \to Q_1^\dagger \to \cdots \to Q_{n-1}^\dagger \to M^n \to 0$$

to compute $\operatorname{Tor}_i(M^n, Y)$. Doing so shows that $\operatorname{Tor}_{i+n}(M^n, Y) \cong \operatorname{Tor}_i(M, Y)$. Therefore $\operatorname{Tor}_{\gg 0}(M^n, Y) = 0$ for all $n \in \mathbb{Z}$. Letting η the the UTAC index for R, we have $\operatorname{Tor}_{>\eta}(M^n, Y) = 0$. In particular, for each $n \in \mathbb{N}$ and $i > 0$, we have $\operatorname{Tor}_i(M^n, Y) \cong \operatorname{Tor}_{i+\eta}(M^{n+\eta}, Y) = 0$. Taking n arbitrarily large shows that $T_C(M) \otimes Y$ is exact. It follows that $\widehat{\operatorname{Tor}}_i^C(M, Y) = 0$ for all $i \in \mathbb{Z}$ as desired.

□

By Lemma 6.3, the following corollary is immediate.

Corollary 5.6.11. *Let R be a Cohen-Macaulay isolated singularity with $\dim R > 0$, and* *suppose the answer to Question 5 is yes. The following are equivalent.*

1. *Every pair of R-modules satisfy the derived depth formula*
2. *Every pair of R-modules satisfy the depth formula*
3. *Every pair of R-modules X and Y satisfy the depth formula if X is maximal Cohen-Macaulay*
4. *R is UAC*

We now give two consequences of this result.

Corollary 5.6.12. *Let R be a Cohen-Macaulay isolated singularity with $\dim R > 0$ such that the answer to Question 5 is yes. Then if R has finite Cohen-Macaulay type, then the depth formula holds for all pairs of modules.*

Proof. Since Cohen-Macaulay rings with finite Cohen-Macaulay type rings are UAC by (Christensen & Holm, 2012, Theorem 1.2), the corollary follows from the previous result.

□

Corollary 5.6.13. *Let R be a Cohen-Macaulay isolated singularity with $\dim R > 0$ and the answer to Question 5 is yes. If every pair of modules satisfies the depth formula, then the Auslander Reiten Conjecture holds.*

Proof. By (Christensen & Holm, 2010, Theorem A), the UAC condition implies that the Artin Reiten conjecture holds. Thus the result follows from Corollary 6.11.

□

Specializing to the case where R is Gorenstein, we have the following.

Corollary 5.6.14. *Suppose R is a Gorenstein isolated singularity with $\dim R > 0$. If every pair of modules satisfies the depth formula, then the Auslander Reiten Conjecture holds.*

Chapter 6

A derived equivalence

Note that this chapter is joint work with Sarang Sane.

6.1 Introduction

Let R be a commutative noetherian ring R. Let $\mathcal{P}(R)$ be the category of finitely generated projective modules, $\overline{\mathcal{P}(R)}$ be the category of finite projective dimension and $\mathcal{M}_{\text{fl}}$ be the category of finite length modules. Let $D^b_{\text{fl}}(\mathcal{P}(R))$ be the full subcategory of $D^b(\mathcal{P}(R))$ consisting of bounded complexes with finite length homologies. In this chapter, we prove the following statement in Theorem 5.7.

Theorem 6.1.1. *Let $\mathcal{A}$ be a resolving subcategory of* $\text{mod}(R)$*, $\overline{\mathcal{A}}$ the category of modules of finite $\mathcal{A}$-dimension, and $\mathcal{L}$ a Serre subcategory satisfying condition (*), (see Definition 2.2). Then there is an equivalence of derived categories*

$$D^b(\overline{\mathcal{A}} \cap \mathcal{L}) \simeq D^b_{\mathcal{L}}(\mathcal{A}).$$

As a corollary, we obtain the following interesting result.

Corollary 6.1.2. *Let R be a local ring, $\mathcal{A} = \mathcal{P}(R)$ and $\mathcal{L} = \mathcal{M}_{\text{fl}}$. Then R is Cohen-Macaulay* *if and only if there is an equivalence of derived categories*

$$D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}}) \simeq D^b_{\mathrm{fl}}(\mathcal{P}(R)).$$

If R is Cohen-Macaulay, Theorem 1.1 implies the equivalence of categories holds. Conversely, if R is not Cohen-Macaulay, the new intersection theorem Roberts (1987) asserts that such a ring R never admits a finite length, finite projective dimension module. Thus $D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}}) = 0$. However, the Hopkins-Neeman classification in Hopkins (1987),Neeman92 of thick subcategories of $D^b(\overline{\mathcal{P}(R)})$ states that thick subcategories are in bijective correspondence with specialization closed subsets of spec(R) and hence $D^b_{\mathrm{fl}}(\overline{\mathcal{P}(R)})$ is never zero.

Using the above equivalence, we can conclude in Theorem 6.1 that the non-connective $\mathbb{K}$-theory spectra are homotopy equivalent. Special cases of this result for the connective K-theory spectrum can be obtained from (Thomason & Trobaugh, 1990, Ex. 5.7), and similar results comparing the K_0 groups are in ((Roberts & Srinivas, 2003, Proposition 2), Foxby & Halvorsen (2009)). The homotopy equivalence of the $\mathbb{K}$-theory spectra is obtained by putting together two equivalences in Theorem 5.5 and Lemma 5.6 , both of which are induced through natural functors of the chain complex categories (a zigzag of equivalences induced from chain complex functors).

In the special case where R is Cohen-Macaulay and local, and $\mathcal{L} = \mathcal{M}_{\mathrm{fl}}$, both categories have matching dualities, which induces isomorphisms of triangular Witt and Grothendieck-Witt groups in Theorems 6.5 and 6.7.

Let X be a noetherian scheme, c a non-negative integer, coh(X) the category of coherent $\mathcal{O}_X$-modules, coh$(X)^c$ the subcategory of modules with codimension at least c, and $D^c(\mathrm{coh}(X))$ the subcategory of $D^b(\mathrm{coh}(X))$ consisting of complexes with homologies in coh$(X)^c$. Using the natural coniveau filtration by codimension, one obtains the classical Brown-Gersten-Quillen spectral sequences of $\mathbb{K}$-groups which abut to the $\mathbb{K}$-theory of coh(X). Applying Quillen's localization and dévissage theorems, the terms occurring in these sequences can be identified with $\mathbb{K}$-groups of the residue fields of the points. Classically, these spectral sequences were applied in the case when X was Noetherian, regular and separated, in which case they converged to the $\mathbb{K}$-groups of X, Quillen (1973). This is because for such X there is a well-known equivalence $D^c(VB_X) \overset{\xi}{\rightsquigarrow} D^c(\text{coh}(X))$ (since there is an ample family of line bundles) which yields $D^b(Coh(X)^c) \simeq D^c(VB_X)$, where VB_X is the category of locally free sheaves over X. The philosophy thus is that if one can understand the $\mathbb{K}$-groups for fields and transfer maps between them, one can compute the global $\mathbb{K}$-groups.

However, without the regularity assumption, ξ is not an equivalence and hence the classical spectral sequences can be used to compute only the coherent $\mathbb{K}$-groups (better known as G-groups) but not the usual $\mathbb{K}$-groups. This entire discussion applies for several other generalized cohomology theories, for example triangular Witt groups in Balmer & Walter (2002) and Grothendieck-Witt groups in Walter (2003), Schlichting (2012). When X is Gorenstein, the corresponding result for coherent Witt groups is in Gille (2002).

In Balmer (2009), niveau and coniveau spectral sequences are established for the usual $\mathbb{K}$-theory over a (topologically) Noetherian scheme with a bounded dimension function. Similarly weak Gersten complexes are defined. However, the terms occurring in these spectral sequences involve abstract derived categories with support $\mathbb{K}_i(D^b(\mathcal{O}_{X,x} \text{ on } \{x\}))$. Now using Theorem 1.1, these terms can be identified with the $\mathbb{K}$-groups of an actual category of modules, that is, we can rewrite these as $\mathbb{K}_i(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{\text{fl}}(\mathcal{O}_{X,x}))$, thus obtaining refined spectral sequences and Gersten complexes in Theorem 6.4. Thus, the philosophy can now be changed to understanding the $\mathbb{K}$-groups of $\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\text{fl}}(R)$ when R is Cohen-Macaulay local ring, and maps between them.

Theorem 1.1 also allows us to compare a recent definition Mandal & Sane (2014) of triangular Witt groups for certain subcategories of triangulated categories with duality (for example $W^i(D^b_{\overline{\mathcal{P}(R)}}(\mathcal{P}(R)))$ in Theorem 6.8.

Corollary 5.11 of our main theorem 1.1 generalizes a consequence of the Hopkins-Neeman theorem Hopkins (1987); Neeman (1992).

A brief word on the organization of the Chapter. In Section 2, we give discuss some categories relevant to this chapter. Section 3 gives technical results which will be used in later sections to manipulate objects in the derived category. The proof of Theorem 1.1 uses a trick, Lemma 4.1, to reduce the lengths of complexes via a suitable Koszul complex. The rest of Section 4 is then devoted to proving the technical theorems 4.2, 4.3 which make the trick somewhat functorial and amenable to use in the derived category. These theorems are crucially used in Section 5, where we prove the main theorems 5.5 and 5.7. In Section 6, we use the main theorems to obtain the natural consequences for generalized cohomology theories (specifically $\mathbb{K}$-theory and Witt groups). In Section 7, we list several questions and examples of interest.

6.2 Serre Subcategories

Hitherto in this work, we have focused on thick and resolving subcategories. However, in this chapter, another type of category comes into play, namely Serre subcategories. These categories naturally arise while studying the support of modules and complexes.

Definition 6.2.1. A *Serre subcategory* $\mathcal{L}$ of an abelian category is a full subcategory such that if $0 \to M'' \to M \to M' \to 0$ is a short exact sequence in the ambient category, then $M \in \mathcal{L}$ if and only if $M', M'' \in \mathcal{L}$.

The category $\mathcal{M}_{\text{fl}}$ is an easy example of a Serre subcategory. More generally, for any specialization closed subset $V \subseteq \operatorname{spec} R$ (i.e. if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in V$, then $\mathfrak{q} \in V$), the category $\mathcal{L}_v = \{M \in \operatorname{mod}(R) \mid \operatorname{Supp} M \subseteq V\}$ is Serre. In fact every Serre subcategory of $\operatorname{mod}(R)$ is obtained in this manner from from a specialization-closed subset of $\operatorname{spec}(R)$, Gabriel (1962). For example, $\mathcal{M}_{\text{fl}}$ is obtained from the specialization closes set $V = \operatorname{MaxSpec}(R)$. Let V be a closed subset of $\operatorname{spec}(R)$ and c be any integer. The main example we will consider in this chapter is the full subcategory of $\mathrm{mod}(R)$ with modules

$$\mathcal{L} = \{M|\operatorname{codim}(\operatorname{Supp}(M)) \geq c, \operatorname{Supp}(M) \subseteq V\}.$$

Essential to our results is the following condition on Serre subcategories.

Definition 6.2.2. A Serre subcategory satisfies condition (*) if, given an ideal $I \subseteq R$ such that $\frac{R}{I} \in \mathcal{L}$, there exists a regular sequence $(a_1, a_2, \ldots, a_c) \in \cap I$ such that $\frac{R}{(a_1, a_2, \ldots, a_c)} \in \mathcal{L}$.

When R is Cohen-Macaulay, there are an abundance of Serre subcategories satisfying condition (*) because of the following result.

Theorem 6.2.3. *(Bruns & Herzog, 1993, Corollary 2.1.4) Every ideal I of height r in a Cohen-Macaulay ring R contains a regular sequence of length r.*

Important example 6.2.4. The following Serre subcategories $\mathcal{L} \subseteq \mathrm{mod}(R)$ satisfy condition (*).

1. If R is equicodimensional and Cohen-Macaulay, $\mathcal{L} = \mathcal{M}_{\mathrm{fl}}$.

2. If R is Cohen-Macaulay, V is a set theoretic complete intersection, i.e. there exists a complete intersection ideal $J = (b_1, b_2, \ldots, b_r)$ such that $V = V(J)$, c is any integer, $\mathcal{L} = \{M|\operatorname{codim}(\operatorname{Supp}(M)) \geq c, \operatorname{Supp}(M) \subseteq V\}$.

3. An important special case of the previous example is when $V = \operatorname{spec}(R) = V(\emptyset)$. Then $\mathcal{L}$ is the category of modules supported in codimension at least c.

4. V is a set theoretic complete intersection, $\mathcal{L} = \{M|\operatorname{Supp}(M) \subseteq V\}$.

Remark 6.2.5. We briefly mention what these definitions will help us achieve. When $\mathcal{T}$ is thick in $\mathrm{mod}(R)$ and $\mathcal{L}$ is a Serre subcategory of $\mathrm{mod}(R)$, there is a natural functor $\mathrm{Ch}^b(\mathcal{T} \cap \mathcal{L}) \rightsquigarrow \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{T})$ which induces the natural functor $D^b(\mathcal{T} \cap \mathcal{L}) \rightsquigarrow D^b_{\mathcal{L}}(\mathcal{T})$. In Theorem 5.5, we will prove that this is an equivalence when $\mathcal{L}$ satisfies condition (*).

When $\mathcal{T} = \overline{\mathcal{A}}$ where $\mathcal{A}$ is resolving, there is a natural functor $\mathrm{Ch}^b_{\mathcal{L}}(\mathcal{A}) \rightsquigarrow \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{T})$ which induces an equivalence $D^b_{\mathcal{L}}(\mathcal{A}) \overset{\sim}{\rightsquigarrow} D^b_{\mathcal{L}}(\mathcal{T})$.

To a reader somewhat lost in the notations, we highlight the two important special cases which might shed light on what this achieves. Let R be equicodimensional and Cohen-Macaulay.

1. $\mathcal{A} = \mathcal{P}(R)$, $\mathcal{T} = \overline{\mathcal{A}} =$ finite projective dimension modules, $\mathcal{L} = \mathcal{M}_{\mathrm{fl}}$. Then $\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}} = \mathcal{T} \cap \mathcal{L}$ and hence the equivalences in 2.5 yield $D^b_{fl}(\mathcal{P}(R)) \simeq D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}})$.
2. $\mathcal{A} = \mathrm{mod}(R)$, $\mathcal{T} = \overline{\mathcal{A}} = \mathrm{mod}(R), \mathcal{L} = \mathcal{M}_{\mathrm{fl}}$. Then the comparison in 2.5 yields the well-known equivalence $D^b_{\mathrm{fl}}(\mathrm{mod}(R)) \simeq D^b(\mathcal{M}_{\mathrm{fl}})$.

Remark 6.2.6. We emphasize an immediate consequence of the definitions 2.7 and 2.1. For a thick subcategory $\mathcal{T}$ of $\mathrm{mod}(R)$ and a Serre subcategory $\mathcal{L}$ of $\mathrm{mod}(R)$ the intersection $\mathcal{T} \cap \mathcal{L}$ has all the properties of a thick subcategory except possibly that it contains R. In particular it has the 2-out-of-3 property.

For the rest of the chapter, we fix a resolving subcategory $\mathcal{A} \subseteq \mathrm{mod}(R)$, a thick subcategory $\mathcal{T}$ of $\mathrm{mod}(R)$ and a Serre subcategory $\mathcal{L}$ of $\mathrm{mod}(R)$ satisfying condition (*).

6.3 Chain complexes

In this section, we state and prove some general lemmas for chain complexes of resolving (and thick) subcategories and their derived categories. We begin by introducing some notations for chain complexes.

Definition 6.3.1. Let $P_\bullet \in \mathrm{Ch}^b(\mathrm{mod}(R))$, i.e. chain complexes with elements in $\mathrm{mod}(R)$.

1. $\min_c(P_\bullet)$ is defined as $\sup\{n \mid P_i = 0 \quad \forall\, i < n\}$.
2. $\max_c(P_\bullet)$ is defined as $\inf\{n \mid P_i = 0 \quad \forall\, i > n\}$.

3. $\min(P_\bullet) = \sup\{n \mid H_i(P_\bullet) = 0 \quad \forall i < n\}$

4. $\max(P_\bullet) = \inf\{n \mid H_i(P_\bullet) = 0 \quad \forall i > n\}$

5. $\mathrm{Width}(P_\bullet) = \max(P_\bullet) - \min(P_\bullet)$. $\mathrm{Width}(0_\bullet)$ is defined to be 0.

6. $Z_n = \ker(\partial_n), B_n = \partial_{n+1}(P_{n+1})$.

7. $\mathrm{Supph}(P_\bullet) = \{n \mid H_n(P_\bullet) \neq 0\}$.

We prove a lemma which will be useful for changing complexes in $D^b(\mathcal{T}\cap\mathcal{L})$ to complexes beginning with projective modules.

Lemma 6.3.2. *1. Let $P_\bullet \in \mathrm{Ch}^b(\mathcal{T}\cap\mathcal{L})$. Then there exists a quasi-isomorphism $P'_\bullet \xrightarrow{\phi} P_\bullet$ with $P'_\bullet \in \mathrm{Ch}^b(\mathcal{T}\cap\mathcal{L})$, $\min_c(P'_\bullet) = m$ and $P'_m \twoheadrightarrow Z^P_m$.*

2. Let $P_\bullet \in \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{T})$. Let t be any integer and m be an integer such that $\min(P_\bullet) \geq m$. Then there exists a quasi-isomorphism $T_\bullet \xrightarrow{\phi} P_\bullet$ with $T_\bullet \in \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{T})$ such that $T_i \in \mathcal{P}(R)$ for all $i < t$ and $\min_c(T_\bullet) = m$ and $T_m \twoheadrightarrow Z_m$.

Proof. Consider the complex $P'_\bullet : \ldots \to P_{m+2} \to P_{m+1} \to Z_m \to 0\ldots$ which is in the required category in both cases since $\mathcal{T}$ and $\mathcal{T}\cap\mathcal{L}$ satisfy the 2-out-of-3 property by Remark 2.6. Then there is a quasi-isomorphism $P'_\bullet \to P_\bullet$, and so we can assume that $\min(P_\bullet) = m$ and $P_m = Z_m$. This proves the first part.

Let $n = \max\{t-m, d+1\}$. Choose a resolution $Q_{i\bullet}$ of P_i of length n such that $Q_{m0} \neq 0$, $Q_{ij} \in \mathcal{P}(R)$ for $j \leq n-1$ and $Q_{in} = \Omega^n P_i$ and hence lies in $\mathcal{T}$. Hence, the differentials of $P_\bullet$ lift to give a double complex $Q_{\bullet\bullet}$. Then the total complex $T_\bullet$ satisfies all the required properties. □

We end this section by proving a result for maps in $D^b_{\mathcal{L}}(\mathcal{T})$.

Lemma 6.3.3. *Let $P_\bullet, Q_\bullet \in D^b_{\mathcal{L}}(\mathcal{T})$ such that $\min(P_\bullet) > \max(Q_\bullet)$. Then we have*

$$\mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(P_\bullet, Q_\bullet) = 0.$$

Proof. To see this, let $f \in \mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(P_\bullet, Q_\bullet)$. Then f is represented by a roof diagram $P_\bullet \overset{q}{\leftarrow} P'_\bullet \overset{g}{\rightarrow} Q_\bullet$ where q is a quasi-isomorphism. By Lemma 3.2, we can assume that $\min_c(P'_\bullet) = \min(P_\bullet)$ and $P'_i \in \mathcal{P}(R)$ for all $i < \max_c(Q_\bullet)$. Since $\min_c(P'_\bullet) = \min(P_\bullet) > \max(Q_\bullet)$, the map g is null-homotopic. Hence $f = 0$. $\square$

6.4 The Koszul construction

We begin this section with a basic construction on Koszul complexes. This construction first appears in an unpublished preprint Foxby (1982), and is used in (Roberts & Srinivas, 2003, Lemma 1) and in more generality in Foxby & Halvorsen (2009). Since we have not ourselves seen a complete proof in the literature, we have taken the liberty of proving it below.

Lemma 6.4.1. *Let $P_\bullet \in \mathrm{Ch}(\mathrm{mod}(R))$. Let $f_1, f_2, \ldots, f_k$ be a regular sequence in $\bigcap_{i \in \mathrm{Supph}(P_\bullet)} \mathrm{ann}(H_i(P_\bullet))$. Suppose there exist $\alpha_i, 0 \le i \le k-1$ such that*

$$\begin{array}{ccccccccccccc}
 & & 0 & \longrightarrow & R & \longrightarrow & R^k & \longrightarrow \ldots & R^k & \longrightarrow & R & \longrightarrow & 0 \longrightarrow \ldots \\
 & & & & & & \downarrow{\scriptstyle \alpha_{m+k-1}} & & \downarrow{\scriptstyle \alpha_{m+1}} & & \downarrow{\scriptstyle \alpha_m} & & \downarrow \\
P_{m+k+1} & \xrightarrow{\partial_{m+k+1}} & P_{m+k} & \xrightarrow{\partial_{m+k}} & P_{m+k-1} & \xrightarrow{\partial_{m+k-1}} & \ldots & & P_{m+1} & \xrightarrow{\partial_{m+1}} & P_m & \xrightarrow{\partial_m} & P_{m-1} \longrightarrow \ldots
\end{array}$$

where the top complex is the Koszul complex $\mathrm{Kos}(f_1^n, f_2^n, \ldots, f_k^n)$ with $n \ge k$ and all the squares commute. Then there exists α_k such that there is a morphism of complexes

$$\mathrm{Kos}(f_1^n, f_2^n, \ldots, f_k^n) \to P_\bullet.$$

In particular, $im(R) \subseteq im(\partial_{m+k})$.

Proof. The above diagram induces a map $\beta : R \to Z_{m+k-1}$ so that the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & R & \longrightarrow & R^k & \longrightarrow \ldots & R^k & \longrightarrow & R & \longrightarrow & 0 \longrightarrow \ldots \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha_{m+k-1}} & & \downarrow{\scriptstyle \alpha_{m+1}} & & \downarrow{\scriptstyle \alpha_m} & & \downarrow \\
0 & \longrightarrow & Z_{m+k-1} & \longrightarrow & P_{m+k-1} & \xrightarrow{\partial_{m+k}} \ldots & P_{m+1} & \xrightarrow{\partial_{m+1}} & P_m & \xrightarrow{\partial_m} & P_{m-1} \longrightarrow \ldots
\end{array}$$

commutes. To define α_k , it is enough to check that $\beta\left(e_1 \wedge e_2 \wedge \ldots \wedge e_k\right) \in B_{m+k-1}$. We proceed to do so. Fix $t \in \{1, 2, \ldots, k-1\}$ and $1 \leq i_1 < i_2 < \ldots < i_t \leq k$. By induction on t, we will define elements $u_{i_1,i_2,\ldots,i_t}, v_{i_1,i_2,\ldots,i_t}, w_{i_1,i_2,\ldots,i_t}$ so that

1. $v_{i_1,i_2,\ldots,i_t} = \left(\prod_{s\neq i_1,i_2,\ldots i_t} f_s\right)^{n-k+t} \alpha_{m+t}\left(\bigwedge_{j=1}^{t} e_{i_j}\right) + \sum_{j=1}^{t}(-1)^{j+1} f_{i_j}^{k-t} u_{\hat{i_j}} \in Z_{m+t} \subseteq P_{m+t}$
2. $w_{i_1,i_2,\ldots,i_t} = \left(\prod_{s\neq i_1,i_2,\ldots i_t} f_s\right) v_{i_1,i_2,\ldots,i_t} \in B_{m+t} \subseteq Z_{m+t} \subseteq P_{m+t}$
3. $u_{i_1,i_2,\ldots,i_t} \in \partial_{m+t+1}^{-1}\left(w_{i_1,i_2,\ldots,i_t}\right) \subseteq P_{m+t+1}$ is an arbitrary lift of $w_{i_1,i_2,\ldots,i_t}$

where $u_{\hat{i_j}}$ denotes $u_{i_1,i_2,\ldots,i_{j-1},i_{j+1},\ldots i_t}$ and $u_{\hat{i_l},\hat{i_j}}$ denotes $u_{i_1,i_2,\ldots,i_{l-1},i_{l+1},i_{l+2},\ldots,i_{j-1},i_{j+1},\ldots i_t}$. The same conventions apply to the v's and w's. The induction above begins at

$$v = \alpha_m(1) \in Z_m \qquad w = \left(\prod_{j=1}^{k} f_j\right)^{n-k} \alpha_m(1) \qquad u \in \partial_{m+1}^{-1}\left(\left(\prod_{j=1}^{k} f_j\right)^{n-k} \alpha_m(1)\right).$$

Since $t \leq k-1$, $\prod_{s\neq i_1,i_2,\ldots i_t} f_s$ is a non-trivial product and hence $w_{i_1,i_2,\ldots,i_t} \in B_{m+t}$ follows if we establish $v_{i_1,i_2,\ldots,i_t} \in Z_{m+t}$. So we prove $v_{i_1,i_2,\ldots,i_t} \in Z_{m+t}$.

$$\begin{aligned}
\partial_{m+t}\left(v_{i_1,i_2,\ldots,i_t}\right) &= \left(\prod_{s\neq i_1,i_2,\ldots i_t} f_s\right)^{n-k+t} \partial_{m+t}\left(\alpha_{m+t}\left(\bigwedge_{j=1}^{t} e_{i_j}\right)\right) + \sum_{j=1}^{t}(-1)^{j+1} f_{i_j}^{k-t} \partial_{m+t}\left(u_{\hat{i_j}}\right) \\
&= \left(\prod_{s\neq i_1,i_2,\ldots i_t} f_s\right)^{n-k+t} \sum_{j=1}^{t}(-1)^{j} f_{i_j}^{n} \alpha_{m+t-1}\left(\bigwedge_{\substack{l=1\\ l\neq j}}^{t} e_{i_l}\right) + \sum_{j=1}^{t}(-1)^{j+1} f_{i_j}^{k-t} w_{\hat{i_j}} \\
&= \left(\prod_{s\neq i_1,i_2,\ldots i_t} f_s\right)^{n-k+t} \sum_{j=1}^{t}(-1)^{j} f_{i_j}^{n} \alpha_{m+t-1}\left(\bigwedge_{\substack{l=1\\ l\neq j}}^{t} e_{i_l}\right) \\
&\quad + \sum_{j=1}^{t}(-1)^{j+1} f_{i_j}^{k-t}\left(\prod_{\substack{s\neq i_1,i_2,\ldots,i_{j-1},\\ i_{j+1},\ldots,i_t}} f_s\right) v_{\hat{i_j}}
\end{aligned}$$

$$
= \left(\prod_{s \neq i_1, i_2, \ldots i_t} f_s \right)^{n-k+t} \sum_{j=1}^{t} (-1)^j f_{i_j}^n \alpha_{m+t-1} \left(\bigwedge_{\substack{l=1 \\ l \neq j}}^{t} e_{i_l} \right)
$$

$$
+ \sum_{j=1}^{t} (-1)^{j+1} f_{i_j}^{k-t} \left(\prod_{\substack{s \neq i_1, i_2, \ldots, i_{j-1}, \\ i_{j+1}, \ldots, i_t}} f_s \right) \left(\prod_{\substack{s \neq i_1, i_2, \ldots, i_{j-1}, \\ i_{j+1}, \ldots, i_t}} f_s \right)^{n-k+t-1} \alpha_{m+t-1} \left(\bigwedge_{\substack{l=1 \\ l \neq j}}^{t} e_{i_l} \right)
$$

$$
+ \sum_{j=1}^{t} (-1)^{j+1} f_{i_j}^{k-t} \left(\prod_{\substack{s \neq i_1, i_2, \ldots, i_{j-1}, \\ i_{j+1}, \ldots, i_t}} f_s \right) \sum_{l=1}^{j-1} (-1)^{l+1} f_{i_l}^{k-t+1} u_{\hat{i_l}, \hat{i_j}}
$$

$$
+ \sum_{j=1}^{t} (-1)^{j+1} f_{i_j}^{k-t} \left(\prod_{\substack{s \neq i_1, i_2, \ldots, i_{j-1}, \\ i_{j+1}, \ldots, i_t}} f_s \right) \sum_{l=j+1}^{t} (-1)^{l} f_{i_l}^{k-t+1} u_{\hat{i_j}, \hat{i_l}}
$$

$$
= \left[\left(\prod_{s \neq i_1, i_2, \ldots i_t} f_s \right)^{n-k+t} (-1)^j f_{i_j}^n + (-1)^{j+1} f_{i_j}^{k-t} \left(\prod_{\substack{s \neq i_1, i_2, \ldots, i_{j-1}, \\ i_{j+1}, \ldots, i_t}} f_s \right)^{n-k+t} \right] \alpha_{m+t-1} \left(\bigwedge_{\substack{l=1 \\ l \neq j}}^{t} e_{i_l} \right)
$$

$$
+ \sum_{\substack{a=1 \\ a<b}}^{t} \sum_{b=1}^{t} \left[(-1)^{b+1} f_{i_b}^{k-t} \left(\prod_{\substack{s \neq i_1, i_2, \ldots, i_{b-1}, \\ i_{b+1}, \ldots, i_t}} f_s \right) (-1)^{a+1} f_{i_a}^{k-t+1} \right] u_{\hat{i_a}, \hat{i_b}}
$$

$$
+ \quad \sum_{\substack{a=1 \\ a<b}}^{t} \sum_{b=1}^{t} \left[(-1)^{a+1} f_{i_a}^{k-t} \left(\prod_{s \neq i_1, i_2, \ldots, i_{a-1}, i_{a+1}, \ldots, i_t} f_s \right) (-1)^{b} f_{i_b}^{k-t+1} \right] u_{\hat{i_a}, \hat{i_b}} = 0.
$$

Hence, $\partial_{m+t} \left(v_{i_1, i_2, \ldots, i_t} \right) = 0$. Thus, we have elements $v_{\hat{i}}, w_{\hat{i}} \in P_{m+t-1}$ and elements $u_{\hat{i}} \in P_{m+t}$.

But

$$
\begin{aligned}
\partial_{m+k}\left(\sum_{i=1}^{k}(-1)^i u_{\hat{i}}\right) &= \sum_{i=1}^{k}(-1)^i w_{\hat{i}} = \sum_{i=1}^{k}(-1)^i f_i v_{\hat{i}} \\
&= \sum_{i=1}^{k}(-1)^i f_i f_i^{n-1}\alpha_{m+k-1}\left(\bigwedge_{j\neq i} e_j\right) \\
&\quad + \sum_{i=1}^{k}(-1)^i f_i\left[\sum_{j=1}^{i-1}(-1)^{j+1} f_j^{k-k+1} u_{\hat{j},\hat{i}} + \sum_{j=i+1}^{k}(-1)^j f_j^{k-k+1} u_{\hat{i},\hat{j}}\right] \\
&= \sum_{i=1}^{k}(-1)^i f_i^n \alpha_{m+k-1}\left(\bigwedge_{j\neq i} e_j\right) = \beta\left(\bigwedge_{i=1}^{k} e_i\right).
\end{aligned}
$$

Thus $\beta\left(\bigwedge_{i=1}^{k} e_i\right) \in B_{m+k}$ and hence we can lift it to to obtain α_k. □

The next two results are the technical tools that are used in proving the main results. Recall that $\mathcal{T} \subseteq \mathrm{mod}(R)$ is a thick subcategory, and $\mathcal{L} \subseteq \mathrm{mod}(R)$ is a Serre subcategory satisfying condition (*).

Theorem 6.4.2. *Let $P_\bullet \in \mathrm{Ch}_{\mathcal{L}}(\mathcal{T})$ and let $\min(P_\bullet) \geq m$. Let $M \in \mathcal{T} \cap \mathcal{L}$ and ψ be a morphism $Z_m \xrightarrow{\psi} M$. Then there exists a complex $K_\bullet \in \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{P}(R))$ and a chain complex morphism $\alpha : K_\bullet \to P_\bullet$ with the following properties*

- $\min_c(K_\bullet) = m$, $\mathrm{Supph}(K_\bullet) = \{m\}$
- $K_m \overset{\alpha_m}{\twoheadrightarrow} Z_m$,
- $H_m(\alpha)$ *is surjective but not injective*
- α_m *induces a map $H_m(K_\bullet) \to M$.*

Proof. By using m and $t = m+1$ as in Lemma 3.2, we can assume that $P_\bullet$ has the property that $\min_c(P_\bullet) = m$ and $P_m = Z_m$ is projective. Since

$$\frac{R}{\mathrm{ann}(M)}, \frac{R}{\mathrm{ann}(H_i(P_\bullet))} \in \mathcal{L}$$

for all $i \in \operatorname{Supph}(P_\bullet)$ we have

$$\frac{R}{\bigcap\limits_{i \in \operatorname{Supph}(P_\bullet)} \operatorname{ann}(H_i(P_\bullet)) \cap \operatorname{ann}(M)}$$

by Gabriel's theorem, Gabriel (1962). Since $\mathcal{L}$ satisfies (*), we can choose a maximal regular sequence

$$f_1, f_2, \ldots, f_c \in \bigcap_{i \in \operatorname{Supph}(P_\bullet)} \operatorname{ann}(H_i(P_\bullet)) \cap \operatorname{ann}(M)$$

such that $\dfrac{R}{(f_1, f_2, \ldots, f_c)}$ is in $\mathcal{L}$. Let

$$K_\bullet = \operatorname{Kos}(f_1^n, f_2^n, \ldots, f_c^n)[m] \otimes_R P_m$$

where $n \geq c$. Then $K_i = 0$ for all $i < m$,

$$H_m(K_\bullet) = \frac{P_m}{(f_1^n, f_2^n, \ldots, f_k^n)P_m}$$

and $\operatorname{Supph}(K_\bullet) = \{m\}$. Also, $H_m(K_\bullet) \in \mathcal{L}$. We will inductively define maps $K_i \to P_i$. Define $\alpha_m = id_{P_m}$ and $\alpha_i = 0$ for all $i < m$. Assume now that we have defined maps α_i for $i < m + k$. Then we have a commutative diagram

$$\begin{array}{ccccccccccccc}
K_{m+k+1} & \longrightarrow & K_{m+k} & \longrightarrow & K_{m+k-1} & \longrightarrow & \ldots K_{m+1} & \longrightarrow & K_m & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha_{m+k-1}} & & \downarrow{\scriptstyle \alpha_{m+1}} & & \downarrow{\scriptstyle \alpha_m} & & \downarrow & & \\
0 & \longrightarrow & Z_{m+k-1} & \longrightarrow & P_{m+k-1} & \xrightarrow{\partial_{m+k}} & \ldots P_{m+1} & \xrightarrow{\partial_{m+1}} & P_m & \xrightarrow{\partial_m} & 0 & \longrightarrow & \ldots
\end{array}$$

with β being the map induced by the projectivity of K_{m+k}. Now let $p_j \in P_m$ be the free generators of P_m. Then $p_j \otimes \left(\bigwedge\limits_{t=1}^{k} e_{i_t} \right)$ are the free generators of K_{m+k}. Consider the Koszul subcomplex of $K_\bullet$ given by $\operatorname{Kos}(f_{i_1}^n, f_{i_2}^n, \ldots, f_{i_k}^n)[m] \otimes_R Rp_j$. Composing with the maps in the above diagram, we get a diagram as in Lemma 4.1. But then, by Lemma 4.1, the last term which is generated by $p_j \otimes \left(\bigwedge_{t=1}^{k} e_{i_t} \right)$ maps into the image B_{m+k-1}. Hence, $\beta(K_{m+k}) \subseteq B_{m+k-1}$. But since K_{m+k} is free, we can lift β to a map $K_{m+k} \xrightarrow{\alpha_{m+k}} P_{m+k}$. Hence, we construct α inductively. It is now clear that α_m is a surjection and hence $H_m(\alpha)$ is surjective. By choosing a large enough n we can ensure that $H_m(\alpha)$ is not injective.

Now notice that α_m gives a map $K_m = P_m \to M$ and that by our choice of the regular sequence, $(f_1^n, f_2^n, \dots, f_k^n)P_m$ maps to zero in M. Hence, there is an induced map from $H_m(K_\bullet)$ to M. $\square$

Theorem 6.4.3. *Let $X_\bullet \xrightarrow{g} Y_\bullet$ be a morphism in $D^b_{\mathcal{L}}(\mathcal{T})$ such that $X_\bullet, Y_\bullet$ are complexes in $\mathrm{Ch}^b(\mathcal{T} \cap \mathcal{L})$ and $\min(X_\bullet \oplus Y_\bullet) = m$. Then there exist complexes $M_\bullet^X$ and $M_\bullet^Y$ in $\mathrm{Ch}^b(\mathcal{T} \cap \mathcal{L})$ and chain maps $M_\bullet^X \xrightarrow{\beta^X} X_\bullet$, $M_\bullet^Y \xrightarrow{\beta^Y} Y_\bullet$ and $M_\bullet^X \xrightarrow{g'} M_\bullet^Y$ such that*

- *$M_\bullet^X, M_\bullet^Y \in \mathrm{Ch}^b(\mathcal{T} \cap \mathcal{L})$ are concentrated in degree m*
- *there is a commutative square in $D^b_{\mathcal{L}}(\mathcal{T})$*

$$\begin{array}{ccc} M_\bullet^X & \xrightarrow{\beta^X} & X_\bullet \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet \end{array}$$

- *$H_m(\beta^X)$ and $H_m(\beta^Y)$ are surjective and not injective.*

Proof. By Lemma 3.2, there exists $T_\bullet^X \in \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{T})$ such that $T_m^X \in \mathcal{P}(R)$ and $\min_c(T_\bullet) = m$ and a quasi-isomorphism $T_\bullet^X \xrightarrow{\phi^X} X_\bullet$ such that $T_m^X \twoheadrightarrow Z_m^X$. Similarly there exists $T_\bullet^Y$ and ϕ^Y. So we have an induced map $T_\bullet^X \xrightarrow{g_1} T_\bullet^Y$ in $D^b_{\mathcal{L}}(\mathcal{T})$.

Since Z_m^Y is in $\mathcal{T} \cap \mathcal{L}$, there is a complex $K_\bullet^Y \in \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{P}(R))$ and a chain complex morphism $\alpha^Y : K_\bullet^Y \to T_\bullet^Y$ by Lemma 4.2, with the following properties

- $\min_c(K_\bullet^Y) = m$, $\mathrm{Supph}(K_\bullet^Y) = \{m\}$
- $K_m^Y \overset{\alpha_m}{\twoheadrightarrow} T_m^Y$,

- $H_m(\alpha)$ is surjective but not injective.

- α_m induces a map $H_m(K^Y_\bullet) \xrightarrow{\theta^Y} Z^Y_m$.

Let g_1 be given by a roof diagram $T^X_\bullet \xleftarrow{q} T_\bullet \xrightarrow{f} T^Y_\bullet$ where q is a quasi-isomorphism. By Lemma 3.2, we may assume $\min_c(T_\bullet) = m$. Now, let $T'_\bullet$ be the pull-back

$$\begin{array}{ccc} T'_\bullet & \dashrightarrow & T_\bullet \\ \downarrow & & \downarrow g \\ K^Y_\bullet & \xrightarrow{\beta^Y} & T^Y_\bullet \end{array}$$

Then we have $\min_c(T'_\bullet) = m$ and $T'_m \twoheadrightarrow T_m$. Composing further, we get a morphism

$$T'_m \twoheadrightarrow T_m \to T^X_m \twoheadrightarrow Z^X_m.$$

Using Lemma 4.2 again, we can choose a complex $K^X_\bullet \in \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{P}(R))$ and a chain complex morphism $K^X_\bullet \xrightarrow{\lambda} T'_\bullet$ with the properties

- $\min_c(K^X_\bullet) = m$, $\mathrm{Supph}(K^X_\bullet) = \{m\}$

- $K^X_m \overset{\lambda_m}{\twoheadrightarrow} T'_m$

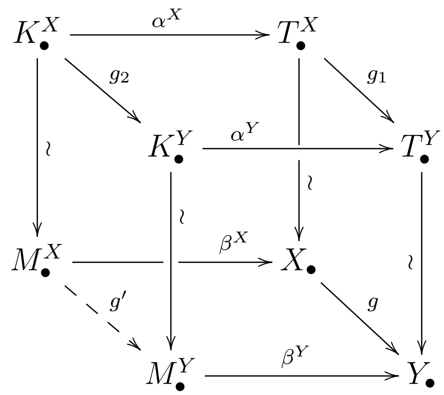
- the induced map $H_m(K^X_\bullet) \to H_m(T'_\bullet)$ is surjective but not injective

- there is an induced map $H_m(K^X_\bullet) \xrightarrow{\theta^X} Z^X_m$

Let α^X be the induced homomorphism $K^X_\bullet \to T^X_\bullet$. Since $K^X_m \twoheadrightarrow T_m$, it follows that $H_m(\alpha^X)$ is surjective and not injective. Thus, we obtain the following commutative diagram

$$\begin{array}{ccccc} K^X_\bullet & \xrightarrow{\alpha^X} & T^X_\bullet & \longrightarrow & X_\bullet \\ g_2 \downarrow & & g_1 \downarrow & & g \downarrow \\ K^Y_\bullet & \xrightarrow[\alpha^Y]{} & T^Y_\bullet & \longrightarrow & Y_\bullet \end{array}$$

where g_2 is a chain map. Now let $M^X_\bullet$ and $M^Y_\bullet$ be the chain complexes concentrated in degree m with modules $H_m(K^X_\bullet)$ and $H_m(K^Y_\bullet)$ respectively. Clearly these complexes are in $\mathrm{Ch}^b(\mathcal{T} \cap \mathcal{L})$. Then θ^X and θ^Y induce chain maps β^X and β^Y so that we get the following commutative diagram



where g' exists because of the vertical isomorphisms. Note that $H_m(\beta^X)$ and $H_m(\beta^Y)$ are surjective and not injective and g' is a chain map. This proves the theorem. $\square$

We collect together a few inequalities about the sizes of the various complexes in Theorem 4.3 and their cones.

Lemma 6.4.4. *Assume the hypothesis of Theorem 4.3 and consider the commutative diagram constructed*

$$\begin{array}{ccc} M^X_\bullet & \xrightarrow{\beta^X} & X_\bullet \\ \downarrow & & \downarrow g \\ M^Y_\bullet & \xrightarrow{\beta^Y} & Y_\bullet \end{array}$$

Assume that neither $X_\bullet$ nor $Y_\bullet$ is $0_\bullet$ and suppose $\mathrm{Width}(X_\bullet \oplus Y_\bullet) = k > 0$. Complete β^X and β^Y to exact triangles

$$T^{-1}C^X_\bullet \to M^X_\bullet \xrightarrow{\beta^X} X_\bullet \xrightarrow{\gamma^X} C^X_\bullet \qquad T^{-1}C^Y_\bullet \to M^Y_\bullet \xrightarrow{\beta^Y} Y_\bullet \xrightarrow{\gamma^Y} C^Y_\bullet .$$

Then we have the following :

1. $\min(C^X_\bullet) = \min(C^Y_\bullet) = m + 1$

2. $\min(T^{-1}C^X_\bullet) = \min(T^{-1}C^Y_\bullet) = m$

3. $\max(T^{-1}C^X_\bullet) = \max\{m, \max(X_\bullet) - 1\}$

4. $\max(T^{-1}C^Y_\bullet) = \max\{m, \max(Y_\bullet) - 1\}$

5. $\mathrm{Width}(C^X_\bullet \oplus C^Y_\bullet) < k$.

Proof. Note that $H_i(M^X_\bullet) = 0$ for all $i \neq m$, thus $H_i(X_\bullet) \cong H_i(C^X_\bullet)$ for all $i \neq m, m+1$. Furthermore, $H_m(\beta^X)$ is surjective but not injective and so $H_m(C^X_\bullet) = 0$ and $H_{m+1}(C^X_\bullet) \neq 0$. The same considerations hold for $H_i(Y_\bullet)$. The above statements now follow from these observations. □

6.5 The equivalence of derived categories

In this section, we prove the promised equivalence of the derived categories $D^b(\mathcal{T} \cap \mathcal{L})$ and $D^b_{\mathcal{L}}(\mathcal{T})$ and then consider the case where $\mathcal{T} = \overline{\mathcal{A}}$, yielding our main theorem. We begin by defining the natural functors.

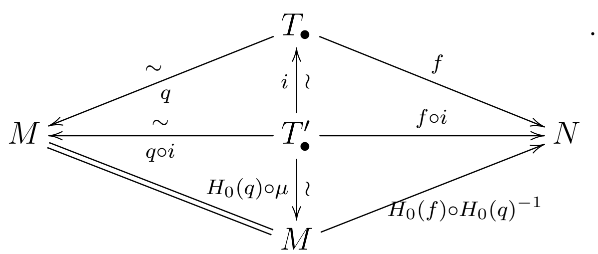
Definition 6.5.1. Let $\iota : D^b(\mathcal{T} \cap \mathcal{L}) \rightsquigarrow D^b_{\mathcal{L}}(\mathcal{T})$ be the natural functor induced by inclusion $\mathrm{Ch}^b(\mathcal{T} \cap \mathcal{L}) \hookrightarrow \mathrm{Ch}^b_{\mathcal{L}}(\mathcal{T})$. Abusing notation, we will sometimes write $X_\bullet \in D^b_{\mathcal{L}}(\mathcal{T})$ when we mean $\iota(X_\bullet)$. Similarly, for a morphism $f \in D^b_{\mathcal{L}}(\mathcal{T})$, we will write $f \in D^b(\mathcal{T} \cap \mathcal{L})$ when we mean $f = \iota(g)$ for some $g \in D^b(\mathcal{T} \cap \mathcal{L})$.

Lemma 6.5.2. *Viewing the modules $M, N \in \mathcal{T} \cap \mathcal{L}$ as complexes, we have*

$$\mathrm{Hom}_{\mathcal{T}\cap\mathcal{L}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{D^b(\mathcal{T}\cap\mathcal{L})}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(\iota(M), \iota(N)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{L}}(M, N).$$

Proof. The maps above are natural, the last one being induced by taking homologies. The composition of the maps is the identity as both sets are $\mathrm{Hom}_R(M, N)$. Thus the first map is injective. Let $\beta \in \mathrm{Hom}_{D^b(\mathcal{T}\cap\mathcal{L})}(M, N)$. Let β be given by the roof diagram $M \overset{q}{\leftarrow} T_\bullet \overset{f}{\rightarrow} N$

where q is a quasi-isomorphism. By Lemma 3.2, we can get $T'_\bullet \xrightarrow{i} T_\bullet$ so that $\min_c(T'_\bullet) = 0$ and i is a quasi-isomorphism. Then $H_0(T_\bullet) = H_0(T'_\bullet)$ and let $T'_\bullet \xrightarrow{\mu} H_0(T_\bullet)$ be the obvious map. Then the following diagram commutes



This tells us that β is equivalent to the map induced by $H_0(f) \circ H_0(q)^{-1}$. So the first map is an isomorphism and the second map an injection. The same argument applies to the second map, proving that all the above maps are isomorphisms. $\square$

Lemma 6.5.3. *Let* $X_\bullet, Y_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ *such that* $\mathrm{Supph}(Y_\bullet) = \{m\}$ *and* $\min(X_\bullet) = m$. *Then* $\mathrm{Hom}_{D^b(\mathcal{T}\cap\mathcal{L})}(X_\bullet, Y_\bullet) \to \mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(X_\bullet, Y_\bullet)$ *is injective.*

Proof. By Lemma 3.2, there exists $Y'_\bullet \in D^b(\mathcal{T}\cap\mathcal{L})$ such that $Y'_\bullet \hookrightarrow Y_\bullet$ is a quasi-isomorphism and $\min_c(Y'_\bullet) = m$. Hence, there is a quasi-isomorphism $Y'_\bullet \to H_m(Y_\bullet)$ where the latter complex is the module $H_m(Y_\bullet) \in \mathcal{T} \cap \mathcal{L}$, concentrated in degree m. Hence, we can assume that $Y_\bullet$ is a module B concentrated in degree m. Let $g \in \mathrm{Hom}_{D^b(\mathcal{T}\cap\mathcal{L})}(X_\bullet, B)$. Then g is given by a roof diagram $X_\bullet \xleftarrow{q} A_\bullet \xrightarrow{f} B$ where q is a quasi-isomorphism. It is enough to show that $\iota(f) = 0$ implies $f = 0$. We can assume, by Lemma 3.2, that $\min_c(A_\bullet) = \min(X_\bullet) = m$. Note that $\iota(f) = 0$ implies $H_m(f) = 0$. Since $\min_c(A_\bullet) = m$, there is a natural surjection $A_m \overset{h}{\twoheadrightarrow} H_m(A_\bullet)$. Further, since B is concentrated in degree m, $f_m = H_m(f) \circ h = 0$. This finishes the proof. $\square$

We now have all the ingredients to prove the equivalence of the derived categories $D^b_{\mathcal{L}}(\mathcal{T})$ and $D^b(\mathcal{T} \cap \mathcal{L})$.

Proposition 6.5.4. *The functor* ι *is essentially surjective and full.*

Proof. We prove the following set of statements by induction on k.

1. For every $P_\bullet \in D^b_{\mathcal{L}}(\mathcal{T})$, with $\mathrm{Width}(P_\bullet) = k$, there exists $\tilde{P}_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ such that $\iota(\tilde{P}_\bullet) \cong P_\bullet$.

2. For every $X_\bullet, Y_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ with $\mathrm{Width}(X_\bullet \oplus Y_\bullet) = k$, the map induced by $\mathrm{Hom}_{D^b(\mathcal{T}\cap\mathcal{L})}(X_\bullet, Y_\bullet) \to \mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(X_\bullet, Y_\bullet)$ is surjective.

When $k = 0$, we get that $\mathrm{Supph}(P_\bullet) \subseteq \{m\}$ for some m. By Lemma 3.2, we can assume that $P_\bullet$ is a resolution of $H_m(P_\bullet)$. Note that $H_m(P_\bullet) \in \mathcal{T} \cap \mathcal{L}$ and so $\iota(\tilde{P}_\bullet) \cong P_\bullet$ where $\tilde{P}_\bullet$ is the complex concentrated in degree m with term $H_m(P_\bullet)$, proving the first part. The second part follows from Lemma 5.2 after replacing the complexes by their homology.

Now suppose $k > 0$ and the statement is true for all $k' < k$. Let $\min(P_\bullet) = m$. By Lemma 4.2, there exists a morphism $K_\bullet \xrightarrow{\beta} P_\bullet$ in $\mathrm{Ch}_{\mathcal{L}}(\mathcal{T})$ such that $\mathrm{Supph}(K_\bullet) = \{m\}$ and further $H_m(\beta)$ is surjective but not injective. Note that this means $K_\bullet \xrightarrow{\sim} H_m(K_\bullet)$ is a quasi-isomorphism in $D^b_{\mathcal{L}}(\mathcal{T})$. Extend this to an exact triangle

$$T^{-1}C_\bullet \to K_\bullet \xrightarrow{\beta} P_\bullet \xrightarrow{\gamma} C_\bullet \qquad .$$

Now by Lemma 4.4, we have $\mathrm{Width}(T^{-1}C_\bullet \oplus K_\bullet) < k$ and $\mathrm{Width}(C_\bullet) < k$. Therefore, by induction, there exists $\tilde{C}_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ such that $\iota(\tilde{C}_\bullet) \cong C_\bullet$, and the following map is a surjection

$$\mathrm{Hom}_{D^b(\mathcal{T}\cap\mathcal{L})}(T^{-1}\tilde{C}_\bullet, H_m(K_\bullet)) \twoheadrightarrow \mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(\iota(T^{-1}\tilde{C}_\bullet), \iota(H_m(K_\bullet))) \cong \mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(T^{-1}\tilde{C}_\bullet, K_\bullet)$$

Because of this surjection, there exists $\tilde{\alpha} : T^{-1}\tilde{C}_\bullet \to H_m(K_\bullet)$ in $D^b(\mathcal{T} \cap \mathcal{L})$ with cone $\tilde{P}_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ and maps $\tilde{\beta}$ and $\tilde{\gamma}$ so that the following morphism of exact triangles commutes

$$\begin{array}{ccccccc} T^{-1}C_\bullet & \xrightarrow{\alpha} & K_\bullet & \xrightarrow{\beta} & P_\bullet & \xrightarrow{\gamma} & C_\bullet \\ \downarrow\wr & & \downarrow\wr & & & & \downarrow\wr \\ \iota(T^{-1}\tilde{C}_\bullet) & \xrightarrow{\iota(\tilde{\alpha})} & \iota(H_m(K_\bullet)) & \xrightarrow{\iota(\tilde{\beta})} & \iota(\tilde{P}_\bullet) & \xrightarrow{\iota(\tilde{\gamma})} & \iota(\tilde{C}_\bullet) \end{array} \tag{6.1}$$

It follows that there is an isomorphism $\iota(\tilde{P}_\bullet) \cong P_\bullet$. This shows the first part of the induction statement.

We now prove the second part of the statement. Let $X_\bullet, Y_\bullet \in D^b(\mathcal{T} \cap \mathcal{L})$ and $f \in \mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(X_\bullet, Y_\bullet)$. Let $\min(X_\bullet \oplus Y_\bullet) = m$. From Theorem 4.3, we have chain complex maps $M^X_\bullet \xrightarrow{\beta^X} X_\bullet$, $M^Y_\bullet \xrightarrow{\beta^Y} Y_\bullet$ and $M^X_\bullet \xrightarrow{\kappa} M^Y_\bullet$ such that

- $M^X_\bullet, M^Y_\bullet \in \mathrm{Ch}^b(\mathcal{T} \cap \mathcal{L})$ are concentrated in degree m
- $H_m(\beta^X)$ and $H_m(\beta^Y)$ are surjective and not injective.
- there is a commutative diagram in $D^b_{\mathcal{L}}(\mathcal{T})$

$$\begin{array}{ccc} M^X_\bullet & \xrightarrow{\beta^X} & X_\bullet \\ \downarrow\kappa & & \downarrow f \\ M^Y_\bullet & \xrightarrow{\beta^Y} & Y_\bullet \end{array}.$$

Taking cones $C^X_\bullet, C^Y_\bullet$ of β^X, β^Y respectively in $D^b(\mathcal{T}\cap\mathcal{L})$, we get a morphism of triangles in $D^b_{\mathcal{L}}(\mathcal{T})$ (as mentioned earlier, we drop the ι)

$$\begin{array}{ccccccc} T^{-1}C^X_\bullet & \xrightarrow{\alpha^X} & M^X_\bullet & \xrightarrow{\beta^X} & X_\bullet & \xrightarrow{\gamma^X} & C^X_\bullet \\ \downarrow T^{-1}\lambda & & \downarrow\kappa & & \downarrow f & & \downarrow\lambda. \\ T^{-1}C^Y_\bullet & \xrightarrow{\alpha^Y} & M^Y_\bullet & \xrightarrow{\beta^Y} & Y_\bullet & \xrightarrow{\gamma^Y} & C^Y_\bullet \end{array} \tag{6.2}$$

We emphasize again that all the objects are in $D^b(\mathcal{T}\cap\mathcal{L})$ and the horizontal maps and κ are morphisms from $D^b(\mathcal{T} \cap \mathcal{L})$. By Lemma 4.4, $\mathrm{Width}(C^X_\bullet \oplus C^Y_\bullet) < k$, and so it follows from the induction hypothesis that there exists $\widetilde{\lambda} : C^X_\bullet \to C^Y_\bullet$ such that $\iota(\widetilde{\lambda}) \simeq \lambda$. We now have a diagram whose rows are exact triangles in $D^b(\mathcal{T} \cap \mathcal{L})$

$$\begin{array}{ccccccc} T^{-1}C_\bullet^X & \xrightarrow{\alpha^X} & M_\bullet^X & \xrightarrow{\beta^X} & X_\bullet & \xrightarrow{\gamma^X} & C_\bullet^X \\ \downarrow{\scriptstyle T^{-1}\widetilde{\lambda}} & & \downarrow{\scriptstyle \kappa} & & & & \downarrow{\scriptstyle \tilde{\lambda}}. \\ T^{-1}C_\bullet^Y & \xrightarrow{\alpha^Y} & M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet & \xrightarrow{\gamma^Y} & C_\bullet^Y \end{array} \tag{6.3}$$

A priori the left square in diagram (8) may not be commutative. Note however that

$$\iota(\alpha^Y \circ T^{-1}\widetilde{\lambda} - \kappa \circ \alpha^X) = \iota(\alpha^Y \circ T^{-1}\widetilde{\lambda}) - \iota(\kappa \circ \alpha^X) = \alpha^Y \circ T^{-1}\lambda - \kappa \circ \alpha^X = 0.$$

Note that Lemma 5.3 applies and hence the left square in diagram (8) commutes. Thus diagram (8) induces $X_\bullet \xrightarrow{g} Y_\bullet$ which gives a morphism of triangles in $D^b(\mathcal{T} \cap \mathcal{L})$

$$\begin{array}{ccccccc} T^{-1}C_\bullet^X & \xrightarrow{\alpha^X} & M_\bullet^X & \xrightarrow{\beta^X} & X_\bullet & \xrightarrow{\gamma^X} & C_\bullet^X \,. \\ \downarrow{\scriptstyle T^{-1}\widetilde{\lambda}} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \tilde{\lambda}} \\ T^{-1}C_\bullet^Y & \xrightarrow{\alpha^Y} & M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet & \xrightarrow{\gamma^Y} & C_\bullet^Y \end{array}$$

This means if we replace f with $\iota(g)$ in diagram (7), the diagram remains commutative. Unfortunately, it is not true a priori that $f = \iota(g)$, see (Kashiwara & Schapira, 2006, Proposition 10.1.17).

Set $\delta = f - \iota(g)$. We now have a morphism of triangles in $D^b_{\mathcal{L}}(\mathcal{T})$.

$$\begin{array}{ccccccc} T^{-1}C_\bullet^X & \xrightarrow{\alpha^X} & M_\bullet^X & \xrightarrow{\beta^X} & X_\bullet & \xrightarrow{\gamma^X} & C_\bullet^X \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle 0} \\ T^{-1}C_\bullet^Y & \xrightarrow{\alpha^Y} & M_\bullet^Y & \xrightarrow{\beta^Y} & Y_\bullet & \xrightarrow{\gamma^Y} & C_\bullet^Y \end{array}$$

Hence, there exists $X_\bullet \xrightarrow{u} K_\bullet^Y$ such that $\delta = \beta^Y \circ u$. Similarly there exists a $C_\bullet^X \xrightarrow{v} Y_\bullet$ such that $\delta = v \circ \gamma^X$. Hence, we have

$$\begin{array}{ccccccc} T^{-1}C^X_\bullet & \xrightarrow{\alpha^X} & M^X_\bullet & \xrightarrow{\beta^X} & X_\bullet & \xrightarrow{\gamma^X} & C^X_\bullet \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & \swarrow{\scriptstyle u} & \downarrow{\scriptstyle \delta} & \swarrow{\scriptstyle v} & \downarrow{\scriptstyle 0} \\ T^{-1}C^Y_\bullet & \xrightarrow[\alpha^Y]{} & M^Y_\bullet & \xrightarrow[\beta^Y]{} & Y_\bullet & \xrightarrow[\gamma^Y]{} & C^Y_\bullet \end{array}$$

Note that since $k > 0$, $\max(C^X_\bullet \oplus Y_\bullet) = \max(X_\bullet \oplus Y_\bullet)$. Since $\min(C^X_\bullet) = m + 1$, we obtain

$$\mathrm{Width}(C^X_\bullet \oplus Y_\bullet) = \begin{cases} k & \min(Y_\bullet) = m \\ k - 1 & \text{otherwise.} \end{cases}$$

We break the proof into 3 easy cases.

Case (i) If $\min(X_\bullet) > m$, then Lemma 3.3 tells us that $u = 0$. Hence, $\delta = \beta^Y \circ u = 0$. So $f = \iota(g) \in D^b(\mathcal{T} \cap \mathcal{L})$. Thus, we have shown that whenever there are complexes with $\min(X_\bullet) > \min(Y_\bullet) = m$ and $\mathrm{Width}(X_\bullet \oplus Y_\bullet) = k$, $\mathrm{Hom}_{D^b(\mathcal{T}\cap\mathcal{L})}(X_\bullet, Y_\bullet) \twoheadrightarrow \mathrm{Hom}_{D^b_{\mathcal{L}}(\mathcal{T})}(X_\bullet, Y_\bullet)$.

Case (ii) If $\min(X_\bullet) = \min(Y_\bullet) = m$, then $\mathrm{Width}(C^X_\bullet \oplus Y_\bullet) = k$, and $\min(C^X_\bullet) = m + 1 > m = \min(Y_\bullet)$. Then $C^X_\bullet, Y_\bullet$ satisfy the hypothesis of the already proved case(i) and so we get that $v \in D^b(\mathcal{T}\cap\mathcal{L})$. Hence, $\delta = v \circ \gamma^X \in D^b(\mathcal{T}\cap\mathcal{L})$. Therefore, $f = \iota(g) + \delta \in D^b(\mathcal{T}\cap\mathcal{L})$.

Case (iii) If $\min(X_\bullet) = m < \min(Y_\bullet)$, then $\mathrm{Width}(C^X_\bullet \oplus Y_\bullet) = k - 1$, then $v \in D^b(\mathcal{T} \cap \mathcal{L})$ by the induction hypothesis, and, as in case(ii), $f \in D^b(\mathcal{T} \cap \mathcal{L})$.

This finishes the proof. □

Theorem 6.5.5. *The functor* $\iota : D^b(\mathcal{T} \cap \mathcal{L}) \to D^b_{\mathcal{L}}(\mathcal{T})$ *is an equivalence.*

Proof. Let $\mathcal{K} = \{X \in D^b(\mathcal{T} \cap \mathcal{L}) \mid \iota(X) = 0\}$. Since ι is full and essentially surjective, it induces an equivalence between the Verdier quotient $D^b(\mathcal{T} \cap \mathcal{L})/\mathcal{K}$ and $D^b_{\mathcal{L}}(\mathcal{T})$. However, ι is injective on objects, so $\mathcal{K} = 0$. □

We now come to the case when $\mathcal{T} = \overline{\mathcal{A}}$ where $\mathcal{A}$ is a resolving subcategory of $\mathrm{mod}(R)$.

Lemma 6.5.6. *There is an equivalence of categories induced by chain complex functors* $\iota' : D^b_{\mathcal{L}}(\mathcal{A}) \overset{\sim}{\rightsquigarrow} D^b_{\mathcal{L}}(\overline{\mathcal{A}})$.

Proof. We only sketch the proof since this is an often used idea. The main point is that given any complex $P_\bullet \in D^b_{\mathcal{L}}(\overline{\mathcal{A}})$, we take a resolution of length n of each term by projective modules and affix the n^{th} syzygy at the end. For large enough n, this gives us a double complex whose total complex $T_\bullet$ is in $D^b_{\mathcal{L}}(\mathcal{A})$ and there is a quasi-isomorphism $T_\bullet \to P_\bullet$. This already proves ι' is essentially surjective. The proof that ι' fully faithful is easy to obtain and standard. □

As a straightforward consequence of Theorem 5.6 and Theorem 5.5 when $\mathcal{T} = \overline{\mathcal{A}}$, we can now obtain the main result.

Theorem 6.5.7. *There is an equivalence of categories* $D^b(\overline{\mathcal{A}} \cap \mathcal{L}) \simeq D^b_{\mathcal{L}}(\mathcal{A})$.

We get the following important corollary.

Corollary 6.5.8. *When R is Cohen-Macaulay and equicodimensional, by setting $\mathcal{L} = \mathcal{M}_{\mathrm{fl}}$ and $\mathcal{A} = \mathcal{P}(R)$, we obtain*

$$D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}}) \simeq D^b_{\mathrm{fl}}(\mathcal{P}(R)).$$

Furthermore, our result holds for any resolving subcategory in Example 2.2 and Serre subcategory in Example 2.4. We explicitly state some important cases. In the special case when R is Cohen-Macaulay and equicodimensional, $\mathcal{A} = \mathrm{mod}(R)$ and $\mathcal{L} = \mathcal{M}_{\mathrm{fl}}$, we obtain the well-known equivalence used in most dévissage statements (refer to (Keller, 1999, 1.15(Lemma,Ex. (b)))).

Corollary 6.5.9.

$$D^b(\mathcal{M}_{\mathrm{fl}}) \simeq D^b_{\mathrm{fl}}(\mathrm{mod}(R)).$$

Note that this equivalence is known even without the assumption that R is Cohen-Macaulay.

Corollary 6.5.10. *Let R be Cohen-Macaulay. Let V be any set theoretic complete intersection in* $\mathrm{spec}(R)$ *and c be any integer. Let $\overline{\mathcal{P}(R)}_V^c$ denote the category of modules with finite projective dimension supported on V and in codimension at least c. Let $D_V^c(\mathcal{P}(R))$ denote the derived category with chain complexes of projective modules with homologies supported on V and in codimension at least c. Then*

$$D^b(\overline{\mathcal{P}(R)}_V^c) \simeq D_V^c(\mathcal{P}(R)).$$

Note that without c, the above result holds even without R being Cohen-Macaulay.

The main theorem, Theorem 5.7 is also related to an interesting corollary of the oft-quoted Hopkins-Neeman theorem Hopkins (1987); Neeman (1992) for perfect complexes. Let $\mathcal{L}$ be any Serre subcategory of $\mathrm{mod}(R)$. A consequence of the Hopkins-Neeman theorem is that $thick_{D^b(\mathrm{mod}(R))}(\overline{\mathcal{P}(R)} \cap \mathcal{L}) \simeq D^b_{\mathcal{L}}(\mathcal{P}(R))$ where *thick* is the thick closure (note that here we use thick in the triangulated sense). We generalize this as follows.

Corollary 6.5.11. *Let $\mathcal{L}$ be a Serre subcategory satisfying condition (*). Let $\mathcal{T}$ be any thick subcategory of* $\mathrm{mod}(R)$. *Then the thick closure (in the triangulated sense) of $\mathcal{T} \cap \mathcal{L}$ in* $D^b(\mathrm{mod}(R))$ *is* $D^b_{\mathcal{L}}(\mathcal{T})$ *(after completion with respect to isomorphisms).*

Proof. Note that there is a commutative square

$$\begin{array}{ccc} K^+_{\mathcal{T}}(\mathcal{P}(R)) & \longrightarrow & K^+(\mathcal{P}(R)) \\ \downarrow \wr & & \downarrow \wr \\ D^+(\mathcal{T}) & \longrightarrow & D^+(\mathrm{mod}(R)). \end{array}$$

The top horizontal arrow is a full embedding, hence so is the bottom. Hence, restricting to the bounded category, $D^b(\mathcal{T})$ is a thick subcategory of $D^b(\mathrm{mod}(R))$ and hence so is $D^b_{\mathcal{L}}(\mathcal{T})$ (after completing them with respect to isomorphisms). However, up to completion with respect to isomorphisms,

$$\mathrm{image}(D^b(\mathcal{T} \cap \mathcal{L})) \subseteq \mathrm{Thick}_{D^b(\mathrm{mod}(R))}(\mathcal{T} \cap \mathcal{L}) \subseteq D^b_{\mathcal{L}}(\mathcal{T}).$$

But by Theorem 5.5, we then get the required result. □

6.6 Results on homological functors

Now that we have proved the equivalence of the two categories, we can compare various generalized cohomology theories which are invariants for derived equivalences. In this work, we restrict our attention to $\mathbb{K}$-theory and triangular Witt groups, but essentially the same statements will work for any generalized cohomology theory with some (natural) conditions.

$\mathbb{K}$-theory comparisons and results

Since K-theoretic invariants need not always be preserved by equivalences of derived categories (Schlichting (2002)), we will need to view the categories above with some more structure. While the original and several other articles Waldhausen (1985); Thomason & Trobaugh (1990); Neeman (2005); Balmer (2001a,b); Walter (2003); Schlichting (2006, 2012) serve as good references for this part, we will refer to the articles Schlichting (2011); Toën (2011) for the terminology and results.

The categories $\mathrm{Ch}^b_{\mathcal{L}}(\mathcal{A}), \mathrm{Ch}^b(\mathcal{T} \cap \mathcal{L})$ and $\mathrm{Ch}^b_{\mathcal{L}}(\mathcal{T})$ are strongly pretriangulated dg-categories and the natural functors are functors of such categories. In particular, with the usual choices of weak equivalences as quasi-isomorphisms, these are all complicial exact categories with weak equivalences, and the natural functors preserve weak equivalences. Assume, as usual, that $\mathcal{L}$ also satisfies condition (*).

Let $\mathbb{K}$ be the nonconnective K-theory spectrum. Applying Theorem 5.5 and (Schlichting, 2011, Theorem 3.2.29), we get that ι induces homotopy equivalences of $\mathbb{K}$-theory spectra.

Similarly, applying Theorem 5.6 and (Schlichting, 2011, Theorem 3.2.29), we get that ι' induces homotopy equivalences of $\mathbb{K}$-theory spectra.

Putting these together and further using (Schlichting, 2011, 3.2.30), we obtain

Theorem 6.6.1. *The spectra $\mathbb{K}(\overline{\mathcal{A}} \cap \mathcal{L})$ and $\mathbb{K}(D^b_{\mathcal{L}}(\mathcal{A}))$ are homotopy equivalent. Hence*

$$\mathbb{K}_i(\overline{\mathcal{A}} \cap \mathcal{L}) \simeq \mathbb{K}_i(D^b_{\mathcal{L}}(\mathcal{A})) \quad \forall i \in \mathcal{Z}.$$

Once again, this result holds for any resolving subcategory in Example 2.2 and Serre subcategory in Example 2.4. We list the most important corollary.

Corollary 6.6.2. *Let R be Cohen-Macaulay and equicodimensional. Then there is a homotopy equivalence between $\mathbb{K}(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\text{fl}})$ and $\mathbb{K}(D^b_{\text{fl}}(\mathcal{P}(R)))$. Hence*

$$\mathbb{K}_i(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\text{fl}}) \simeq \mathbb{K}_i(D^b_{\text{fl}}(\mathcal{P}(R)))$$

As mentioned in Section 1, special cases of Corollary 6.2 were known earlier.

When R is Cohen-Macaulay and equicodimensional of dimension d, the special case of $\mathcal{A} = \text{mod}(R)$ and $c = d$ in Theorem 6.1 gives us the well-known equivalence for coherent $\mathbb{K}$-theory.

Remark 6.6.3. Let X be a (topologically) noetherian scheme with a bounded generalized dimension function as in Balmer (2009). Then coniveau and niveau spectral sequences are defined in (Balmer, 2009, Theorem 1, Theorem 2) converging to the $\mathbb{K}$-groups of X. The q^{th} row on the E_1 page consists of unaugmented Gersten-like complexes with terms

$$\bigoplus_{x \in X^{(p)}} \mathbb{K}_{-p-q}(\mathcal{O}_{X,x} \text{ on } x) \qquad \text{and} \qquad \bigoplus_{x \in X_{(-p)}} \mathbb{K}_{-p-q}(\mathcal{O}_{X,x} \text{ on } x)$$

respectively. Further, there is an augmented weak Gersten complex for the usual codimension and dimension functions, as defined in Balmer (2009). In the regular situation, Quillen's dévissage theorem can be applied to rewrite these terms as the K-theories of the fields and thus they remained as abstract K-groups of derived categories over the local rings at the points.

Now under the further assumption that all the local rings $\mathcal{O}_{X,x}$ are Cohen-Macaulay (i.e. X is Cohen-Macaulay), we can apply Theorem 6.2 and rewrite these spectral sequences in terms of the $\mathbb{K}$-groups of the category of finite length, finite projective dimension modules over the local rings at the points. Thus the computation of global $\mathbb{K}$-groups can now be reduced to computing $\mathbb{K}$-groups of the category $\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}}(R)$ for a Cohen-Macaulay local ring R.

To summarize, we obtain the following.

Theorem 6.6.4. *For a Cohen-Macaulay scheme X of dimension d, we have spectral sequences*

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathbb{K}_{-p-q}(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{\mathrm{fl}}(\mathcal{O}_{X,x})) \overset{p+q=n}{\Longrightarrow} \mathbb{K}_{-n}(X) \qquad \textit{and}$$

$$E_1^{p,q} = \bigoplus_{x \in X_{(p)}} \mathbb{K}_{-p-q}(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{\mathrm{fl}}(\mathcal{O}_{X,x})) \overset{p+q=n}{\Longrightarrow} \mathbb{K}_{-n}(X)$$

and augmented weak Gersten complexes for each $q \in \mathcal{Z}$

$$\mathbb{K}_q(X) \to \bigoplus_{x \in X^{(0)}} \mathbb{K}_q(\overline{P(\mathcal{O}_{X,x})} \cap \mathcal{M}_{\mathrm{fl}}(\mathcal{O}_{X,x})) \to$$

$$\bigoplus_{x \in X^{(1)}} \mathbb{K}_{q-1}(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{\mathrm{fl}}(\mathcal{O}_{X,x})) \to \ldots \to \bigoplus_{x \in X^{(d)}} \mathbb{K}_{q-d}(\overline{\mathcal{P}(\mathcal{O}_{X,x})} \cap \mathcal{M}_{\mathrm{fl}}(\mathcal{O}_{X,x})).$$

Witt and Grothendieck-Witt group comparisons and results

Let R be an equicodimensional Cohen-Macaulay ring of dimension d. We now consider the situation where the category $\mathcal{A}$ is a duality-closed thick subcategory of $\mathcal{G}_C$, the category arising from a semidualizing module C as defined in Section 3 with duality given by $\mathrm{Hom}(\_, C)$. Let $\mathcal{L} = \mathcal{M}_{\mathrm{fl}}$. We assume further that 2 is invertible in the ring R.

There is a duality on the category $\overline{\mathcal{A}} \cap \mathcal{L}$ given by $\mathrm{Ext}^d_R(\_, C)$ which induces a duality on $D^b(\overline{\mathcal{A}} \cap \mathcal{L})$. Similarly, there is a duality on $D^b_{\mathcal{L}}(\mathcal{A})$ given by $\mathrm{Hom}_R(\_, C)$ (or $\dagger$) respectively.

The arguments in either of (Balmer & Walter, 2002, Lemma 6.4), Gille (2002) or Mandal & Sane (2014) go through with minimal modifications showing that these are indeed dualities and that they are preserved by the "resolution functor", the composite functor $\iota \circ \iota'^{-1}$ from $D^b(\overline{\mathcal{A}} \cap \mathcal{L})$ to $D^b_{\mathcal{L}}(\mathcal{A})$.

A direct application of (Balmer & Walter, 2002, Lemma 4.1(c)) now yields

Theorem 6.6.5. *There is an isomorphism of triangular Witt groups*

$$W(\overline{\mathcal{A}} \cap \mathcal{L}) \xrightarrow{\sim} W^0(D^b(\overline{\mathcal{A}} \cap \mathcal{L})) \xrightarrow{\sim} W^d(D^b_{\mathcal{L}}(\mathcal{A})).$$

The case $\mathcal{A} = \mathcal{P}(R)$ results in the following new corollary.

Corollary 6.6.6. *When R is equicodimensional, there is an isomorphism of triangular Witt groups*

$$W(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}}) \xrightarrow{\sim} W^d(D^b_{\mathrm{fl}}(\mathcal{P}(R)))$$

The special case of R being equicodimensional and $C = D$ a dualizing module (or complex) gives us the result in the well-known coherent case, Gille (2002).

Remark 6.6.7. Similarly, a direct application of (Walter, 2003, Theorem 2.1) yields the same results for Grothendieck-Witt groups as for Witt groups above.

In Mandal & Sane (2014), the authors define a new Witt group for exact subcategories of triangulated categories closed under duality. They further prove another form of dévissage for triangular Witt groups, namely

$$\begin{array}{ccccc} W(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}}) & \xrightarrow[\alpha]{\sim} & W(D^b_{\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}}}(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}})) & \xrightarrow[\alpha']{\sim} & W(D^b(\overline{\mathcal{P}(R)} \cap \mathcal{M}_{\mathrm{fl}})) \\ & & \beta \big\downarrow \wr & & \\ & & W^d(D^b_{\mathrm{fl}}(\mathcal{P}(R))) & & \end{array}$$

With our notations as above, we can now generalize and improve upon this picture to obtain the following theorem.

Theorem 6.6.8. *There are natural isomorphisms of Witt groups*

$$\begin{array}{ccccc} W(\overline{\mathcal{A}} \cap \mathcal{L}) & \xrightarrow[\alpha]{\sim} & W(D^b_{\overline{\mathcal{A}} \cap \mathcal{L}}(\overline{\mathcal{A}} \cap \mathcal{L})) & \xrightarrow[\alpha']{\sim} & W(D^b_{\mathcal{L}}(\overline{\mathcal{A}} \cap \mathcal{L})) \,. \\ & & \beta \big\downarrow \wr & & \gamma \big\downarrow \wr \\ & & W^d(D^b_{\overline{\mathcal{A}} \cap \mathcal{L}}(\mathcal{A})) & \xrightarrow[\delta]{\sim} & W^d(D^b_{\mathcal{L}}(\mathcal{A})) \end{array}$$

Proof. We only give a sketch of the proof. The isomorphism γ already occurs in Theorem 6.5 and β is also a similar direct consequence of Theorem 5.7 by restricting the equivalence of categories to the categories with support $\overline{\mathcal{A}} \cap \mathcal{L}$ (or by following the arguments in Mandal & Sane (2014)). The arguments in Mandal & Sane (2014) generalize directly to give the isomorphisms α and α'. The commutativity of the diagram shows that δ is also an isomorphism. □

Remark 6.6.9. The remark 6.3 and succeeding theorem 6.4 works as in (Balmer, 2009, Remark 3) for any cohomology theory which induces long exact sequences on short exact sequences of triangulated categories. In particular, for Witt theory, we would get spectral sequences and augmented weak Gersten-Witt complexes for triangular Witt groups tensored with $\mathcal{Z}[\frac{1}{2}]$ as in Theorem 6.4.

6.7 Examples and questions

An advantage of working with arbitrary Serre subcategories is the ability to deal with supports. We are able to deal with supports only when the Serre subcategory satisfies the condition (*). In contrast, in the coherent picture (i.e. G theory or coherent Witt groups), theorems similar to the ones in section 5 exist with arbitrary supports, i.e. supports in any specialization closed subset, in particular over any closed subset V of spec(R). This is one reason why smoothness has played a crucial role in results for $\mathbb{K}$-theory or more generally, for generalized cohomology theories, since both coherent and usual theories coincide. Thus the following question is natural :

Question 6. *Is $D^b(\mathcal{T} \cap \mathcal{L}) \rightsquigarrow D^b_{\mathcal{L}}(\mathcal{T})$ an equivalence for any thick subcategory $\mathcal{T}$ of* $\mathrm{mod}(R)$ *and any Serre subcategory $\mathcal{L}$ of* $\mathrm{mod}(R)$*?*

As we noted in Section 1, when R is local and not Cohen-Macaulay, this is always false with $\mathcal{L} = \mathcal{M}_{\mathrm{fl}}$ and $\mathcal{T} = \overline{\mathcal{P}(R)}$. However, it is still plausible that the result holds for a more general class of Serre subcategories.

Next, we consider a question about quotients. Supports allow one to write localization exact sequences. Comparing supports with quotients is a powerful tool, for example it is known that $\mathbb{K}_i\left(\mathcal{M}\left(\dfrac{R}{(a)}\right)\right) \simeq \mathbb{K}_i(\{\text{modules supported on V(a)}\})$ where a is a nonzero divisor. This is because one can either apply dévissage directly, or, for other generalized cohomology theories, the spectral sequence and dévissage reduces one to the case of residue fields of points and both sides have the same residue fields. This leads to the following question :

Question 7. *Let $\mathcal{L}$ be the Serre subcategory of modules supported on $V(I)$ where I is a set theoretic complete intersection ideal. Let $\mathcal{T}$ be a thick subcategory in* $\mathrm{mod}(R)$*. Is* $\mathbb{K}_i\left(D^b\left(\mathcal{T} \cap \mathcal{M}\left(\frac{R}{I}\right)\right)\right) \to \mathbb{K}_i\left(D^b_{\mathcal{L}}(\mathcal{T})\right)$ *an isomorphism?*

We present a rather simple example which answers this question negatively.

Example 6.7.1. Let $R = \dfrac{k[X]}{(X^2)}$. Let $I = (X)$, so $V(I) = \text{spec}(R)$. Note that $V(I) = V(\emptyset)$ and so $V(I)$ is a set theoretic complete intersection. Let $\mathcal{T} = \overline{\mathcal{P}(R)}$. Then $\mathcal{T} \cap \mathcal{M}(R/I) = \{0\}$ while $D^b_{\mathcal{L}}(\mathcal{T}) = D^b(\overline{\mathcal{P}(R)}) \neq \{0\}$. Then $K_0(\mathcal{T} \cap \mathcal{M}(R/I)) = 0$ and $K_0(D^b_{\mathcal{L}}(\mathcal{T})) = \mathcal{Z}$.

Clearly going modulo any ideal I does not work. We specialize to the case when $I = (a)$ where a is a nonzero divisor.

Question 8. *Let a be a nonzero divisor. Let $\mathcal{L}$ be the Serre subcategory of modules supported on $V(a)$. Is $\mathbb{K}_i\left(\overline{\mathcal{A}} \cap \mathcal{M}\left(\frac{R}{(a)}\right)\right) \to \mathbb{K}_i(\overline{\mathcal{A}} \cap \mathcal{L})$ an isomorphism?*

One natural way to answer this question would involve the following two steps

1. Reprove Quillen's dévissage theorem for full subcategories of $\text{mod}(R)$ satisfying the 2-out-of-3 property (it is known for abelian subcategories).

2. Find a natural filtration for a module $M \in \overline{\mathcal{A}} \cap \mathcal{L}$ so that the quotients belong to $\overline{\mathcal{A}} \cap \mathcal{M}\left(\frac{R}{(a)}\right)$.

For every module M on the right there exists an n such that $a^n M = 0$ and thus a natural filtration $M \supseteq aM \ldots \supset a^{n-1}M \supset 0$ and $\dfrac{a^i M}{a^{i+1}M}$ is in $\mathcal{M}(\frac{R}{(a)})$. At first sight, this might seem like an answer to the second part. However, it turns out that even though $M \in \mathcal{A}$, its quotient $\dfrac{M}{aM}$ need not be. Hailong Dao provided the author with the following example in the best possible case of a polynomial variable.

Example 6.7.2. Let $R = \dfrac{k[[X,Y]]}{(XY)}[Z]$. Let $M = \dfrac{R}{(X-Z, Y-Z)}$. Then M has finite projective dimension over R, $Z^2 M = 0$ but $\dfrac{M}{(Z)M} = k$ does not have finite projective dimension over $\dfrac{k[[X,Y]]}{(XY)}$.

Since the module M has length 2, there is no other option of a filtration. Thus, the most natural naive arguments provide no answer. If Question 8 has a positive answer, it would yield nice long exact sequences and be useful in computations.

Finally, let R be equicodimensional and Cohen-Macaulay of dimension d, $\mathcal{A} \subseteq \mathcal{G}_C$ a thick subcategory closed under the duality, and $\mathcal{L}$ the Serre subcategory of finite length modules supported on a set theoretic complete intersection V. Since we have an equivalence of triangular Witt groups and triangular Grothendieck-Witt groups 6.5,6.7 and both are obtained from a Grothendieck-Witt spectrum, it begs the natural question :

Question 9. *Are the Grothendieck-Witt spectra of $D^b(\overline{\mathcal{A}} \cap \mathcal{L})$ and $D^b_{\mathcal{L}}(\mathcal{A})$ homotopy equivalent?*

If a suitable intermediate category can be found which has duality, then we can answer this in the affirmative. However, there is in general no duality on $\overline{\mathcal{A}}$. Also, we do not know if the category of double complexes constructed in the proof of (Balmer & Walter, 2002, Lemma 6.4) arises as the homotopy category of some suitable pretriangulated category.

Chapter 7

The geometry of cohomological supports

Note that this chapter is joint work with Hailong Dao.

7.1 Introduction

Cohomological supports encode into geometric objects the homological behavior of modules over complete intersection rings. However, serious geometric tools have not been used to study cohomological supports. Such an approach could potentially be beneficial to both geometers and algebraists. In this chapter, we attempt such an undertaking by studying the cohomological support of the tensor product of two modules using the geometric join. The geometric join of two varieties is currently the subject of much research. Of particular interest is computing the generating set of the defining ideal and also the dimension of the join of two intersecting varieties.

In this chapter, we prove the following result, which appears as Theorem 3.4.

Theorem 7.1.1. *If* $\operatorname{Tor}_{\geq 0}(M,N) = 0$*, then* $\mathrm{V}^*(M \otimes N) = \operatorname{Join}(V^*(M), V^*(N))$.

In section 2, we discuss some background information regarding the geometric join. In Section 3 we prove the main result, Theorem 3.4. In Section 4, we discuss several corollaries of this result. In particular, we discuss the implications on the relation of thick subcategories and tensor products. We also give a version of Theorem 3.4 for Ext. In Section 5, we show that for Cohen-Macaulay modules M and N, $\operatorname{Tor}_{\gg 0}(M,N)=0$ implies that $\operatorname{Tor}_{>0}(M,N)=0$ when R is an AB ring, allowing us to give a corollary of our main theorem. We discuss in Section 6 questions and examples regarding the effects of removing the hypothesis that $\operatorname{Tor}_{>0}(M,N)=0$ from Theorem 3.4.

Unless otherwise stated, for the entirety of this chapter, $(R,\mathfrak{m},k)$ will be a local complete intersection of codimension c such that $\hat{R}=Q/(f_1,\dots,f_c)$ for a regular local ring Q and a regular sequence $\underline{f}=f_1,\dots,f_c$ not contained in the square of the maximal ideal of Q. Let $\tilde{k}$ be the algebraic closure of k.

7.2 Geometric join

In this subsection we give attention to another construction central to this paper.

Definition 7.2.1. Let $U,V\subseteq\mathbb{P}^n_{\tilde{k}}$ be Zariski closed sets. We define the *join* of two sets to be

$$\operatorname{Join}(U,V)=\overline{\bigcup_{u\in U\ v\in V\ u\neq v}\operatorname{line}(u,v)}$$

where $\operatorname{line}(u,v)$ is the projective line containing u and v. Furthermore, in the case when $U=V$, we set $\sec V=\operatorname{Join}(V,V)$ which, when V is a variety, we refer to it as the secant variety of V.

Remark 7.2.2. When U and V are disjoint Zariski closed sets, we may simplify this definition to

$$\operatorname{Join}(U,V)=\bigcup_{u\in U\ v\in V}\operatorname{line}(u,v).$$

and we still obtain a closed set, (Harris, 1995, Proposition 6.3,Example 6.14). In most contexts in this paper, we will take the join of disjoint sets.

Remark 7.2.3. By convention, we set $\operatorname{Join}(\emptyset,V)=V$. To justify this, in affine space

Join$(0, V) = V$ if V is a cone. Since projectivization of cones should commute with joins, this convention makes sense.

Remark 7.2.4. In order to make the our main result be true for all modules, we set the convention that $\operatorname{Join}(V^*(0), V^*(M)) = 0$ for any module M over a complete intersection ring.

To visualize the join, consider the following easy examples. The join of two distinct points is a projective line, and the join of two skew lines is a three dimensional projective linear space. In fact, the join of any two linear spaces is the smallest linear space containing them. In particular, the secant variety of any linear space is itself. The join is not always linear: the join of a point and circle (not containing the point) is a double cone.

Theorem 7.2.5 ((Ådlandsvik, 1987, 1.1))**.** *For two closed sets $U, V \subseteq \mathbb{P}^n_k$, we have*

$$\dim \operatorname{Join}(U, V) \leq \dim U + \dim V + 1$$

and if $U \cap V = \emptyset$, then

$$\dim \operatorname{Join}(U, V) = \dim U + \dim V + 1.$$

The converse is not true, and, in fact, when it is not known in general when $\dim \operatorname{Join}(U, V) = \dim U + \dim V + 1$ when $U \cap V \neq \emptyset$.

7.3 Joins of cohomological supports

The goal of this section is to prove the main result of this paper, namely Theorem 3.4.

Lemma 7.3.1. *If R has codimension two and $\operatorname{Tor}_{>0}(M, N) = 0$, then*

$$V^*(M \otimes N) = \operatorname{Join}(V^*(M), V^*(N)).$$

Proof. Let k be the algebraic closure of the residue field of R. We may also assume that R is complete. If $\operatorname{pd} M, \operatorname{pd} N < \infty$, then $\operatorname{pd} M \otimes N < \infty$ and the conclusion is clear.

Now assume that $\operatorname{pd} M = \operatorname{pd} N = \infty$. Since $V^*(M)$ and $V^*(N)$ are disjoint, nonempty, and lie in $\mathbb{P}^1_k$, we know that $\dim V^*(M) = \dim V^*(N) = 0$. Therefore, the complexity of both M and N is one. Since $\operatorname{Tor}_{>0}(M, N) = 0$, the complexity of $M \otimes N$ is two, and thus $\dim V^*(M \otimes N) = 1$. This means that $V^*(M \otimes N) = \mathbb{P}^1_k = \operatorname{Join}(V^*(M), V^*(N))$.

We now assume that $\operatorname{pd} M < \infty$ and $\operatorname{pd} N = \infty$. Letting

$$0 \to R^{n_t} \to \cdots \to R^{n_0} \to M \to 0$$

be a free resolution, the sequence

$$0 \to N^{n_t} \to \cdots \to N^{n_0} \to M \otimes N \to 0$$

is exact. Thus $V^*(M \otimes N) \subseteq V^*(N)$. Now since $\operatorname{Tor}_{>0}(M, N) = 0$, we have

$$\operatorname{cx}(M \otimes N) = \operatorname{cx} M + \operatorname{cx} N = \operatorname{cx} N.$$

Therefore $\dim V^*(M \otimes N)$ is the same as $\dim V^*(N)$. So if $V^*(N)$ is irreducible, we are done. If in particular $\dim V^*(N) = 1$, then this is the case and we are done. So suppose $\dim V^*(N) = 0$, that is $V^*(N) = \{p_1, \ldots, p_n\}$ where p_i are points. The short exact sequence $0 \to \Omega N \to R^m \to N \to 0$ yields the short exact sequence

$$0 \to M \otimes \Omega N \to M^m \to M \otimes N \to 0.$$

But since $V^*(M)$ is empty, we have $V^*(M \otimes \Omega N) = V^*(M \otimes N)$. Thus, since $V^*(N) = V^*(\Omega M)$ we may assume that N is maximal Cohen-Macaulay. Therefore, by Theorem 3.1 of Bergh (2007), we may write $N = N_1 \oplus \cdots \oplus N_n$ with $V^*(N_i) = \{p_i\}$, which is irreducible.

Hence we have

$$V^*(M \otimes N) = V^*(M \otimes N_1 \oplus \cdots \oplus M \otimes N_n)$$
$$= V^*(M \otimes N_1) \cup \cdots \cup V^*(M \otimes N_n) = \{p_1\} \cup \cdots \cup \{p_n\} = V^*(N)$$

which completes the proof.

□

Proposition 7.3.2. *If* $\operatorname{Tor}_{>0}(M,N) = 0$, *then* $\operatorname{Join}(V^*(M), V^*(N)) \subseteq V^*(M \otimes N)$.

Proof. By Lemma 5.18 and Lemma 5.19, we may assume that R is complete and k is algebraically closed. We proceed by induction on the c, the codimension of R. When $c = 1$, the cohomological supports lie in $\mathbb{P}^0_k$, a point. Thus a cohomological support is either that point or empty, depending on whether or not the module has finite projective dimension. Since $\operatorname{Tor}_{>0}(M,N) = 0$, the result follows from the equality $\operatorname{cx} M + \operatorname{cx} N = \operatorname{cx} M \otimes N$. If $c = 2$, the statement is true by Lemma 3.1.

Now suppose that $c \geq 2$. It suffices to show that for any hyperplane $H \subseteq \mathbb{P}^{c-1}_k$ we have

$$\operatorname{Join}(V^*(M), V^*(N)) \cap H \subseteq V^*(M \otimes N) \cap H$$

Since $c \geq 2$, any hyperplane is a linear space with dimension at least one. Therefore, for any $x \in V^*(M) \cap H$ and $y \in V^*(N)$, the projective line between x and y is also in H. Because of this, we have

$$\operatorname{Join}(V^*(M), V^*(N)) \cap H = \operatorname{Join}(V^*(M) \cap H, V^*(N) \cap H).$$

Thus we need to show that

$$\operatorname{Join}(V^*(M) \cap H, V^*(N) \cap H) \subseteq V^*(M \otimes N) \cap H.$$

To that end, we fix a hyperplane H. Now we write $R = Q/(f_1, \dots, f_c)$, and after a change of coordinates, by Proposition 5.12 we may assume that $H = V(x_1)$ where $k[x_1, \dots, x_c]$ is the ring of cohomological operators. Now we set $T = Q/(f_2, \dots, f_c)$ and $f = f_1$, thus T is a complete intersection with $\operatorname{codim} T = c - 1$, f is regular on T, and $R = T/(f)$. Note that for any module $X \in \operatorname{mod}(R)$, $V_R^*(X) \cap H = V_T^*(X)$ by Corollary 5.7. Therefore we need to show that

$$\operatorname{Join}(V_T^*(M), V_T^*(N)) \subseteq V_T^*(M \otimes N).$$

Since $\operatorname{Tor}_{>0}^R(M, N) = 0$, by (Avramov, 2010, Lemma 9.3.8), we have $\operatorname{Tor}_1^T(M, N) = M \otimes N$ and $\operatorname{Tor}_{>1}^T(M, N) = 0$. It follows that $\operatorname{Tor}_{>0}^T(M, \Omega_T N) = 0$, and, after tensoring with $0 \to \Omega_T N \to T^t \to N \to 0$, we get the exact sequence

$$0 \to M \otimes N \to M \otimes \Omega_T N \to M^t \to M \otimes N \to 0.$$

Thus, by induction, we have

$$\operatorname{Join}(V_T^*(M), V_T^*(N)) = \operatorname{Join}(V_T^*(M), V_T^*(\Omega_T N)) \subseteq V_T^*(M \otimes \Omega_T N) \subseteq V_T^*(M \otimes N) \cup V_T^*(M).$$

Note, we may assume that N is not a free T-module, and so $\Omega_T N$ is not zero. A similar argument using $\Omega_T M$ gives us $\operatorname{Join}(V_T^*(M), V_T^*(N)) \subseteq V_T^*(M \otimes N) \cup V_T^*(N)$. This implies that

$$\operatorname{Join}(V_T^*(M), V_T^*(N)) \subseteq V_T^*(M \otimes N) \cup (V_T^*(M) \cap V_T^*(N)) = V_T^*(M \otimes N)$$

proving the claim.

□

Before we show that the reverse inclusion holds, we need a lemma.

Lemma 7.3.3. *Suppose that $c = \operatorname{codim} R \geq 2$ and that R is complete and algebraically closed. Fix an $M \in \operatorname{mod}(R)$ such that $V_R^*(M) = q \in \mathbb{P}_k^{c-1}$. For any $p \in \mathbb{P}_k^{c-1}$ distinct from*

q, there exists an $L \in MCM$ such that $V_R^*(L) = p$ and $V_R^*(M \otimes L) = \operatorname{Join}(p, q)$.

Proof. As R is complete, we write $R = Q/(f_1, \ldots, f_c)$ where Q is a regular local ring and $\underline{f}$ is a regular sequence. After a change of coordinates, we may assume that $p = (1, 0, 0, \ldots, 0)$. Set $T = Q/(f_1)$. Set $X = \Omega_T^{d-1} k$ where $d = \dim Q$. Then $V_T^*(X) = p$ when viewed as a subspace of $\mathbb{P}_k^{c-1}$. Now we set $L = X/(f_2, \ldots, f_c)X$. We prove that L is our desired module.

For a regular sequence $\underline{g}$ on T, we set $X_{\underline{g}} := X/(\underline{g})X$. We claim that $V_{T/(\underline{g})}^*(X_{\underline{g}}) = \{p\}$. This and Proposition 5.5 would show that $\operatorname{Tor}_{>0}^{T/(\underline{g})}(X_{\underline{g}}, M) = 0$, because $X_{\underline{g}}$ is maximal Cohen-Macaulay over $T/(\underline{g})$. Take any point $p \in V_{T/(\underline{g})}^*(k)$ such that $p' \neq p$. By (Bergh, 2007, Theorem 2.2), there exists a $Y \in \operatorname{mod}(T/(\underline{g}))$ such that $p' = V_{T/(\underline{g})}^*(Y)$. Then we have $\operatorname{Tor}_i^{T/(\underline{g})}(Y, X_{\underline{g}}) \cong \operatorname{Tor}_i^T(Y, X)$ for all $i > 0$ and thus $\operatorname{Tor}_{\gg 0}^{T/(\underline{g})}(Y, X_{\underline{g}}) = 0$. Therefore $V_{T/(\underline{g})}^*(Y) \cap V_{T/(\underline{g})}^*(X_{\underline{g}}) = \emptyset$ and so $p' \notin V_{T/(\underline{g})}^*(X_{\underline{g}})$. Therefore $V_{T/(\underline{g})}^*(X_{\underline{g}}) = \{p\}$ as claimed.

Let $l = \operatorname{Join}(p, q)$, and let $\underline{g}$ be a regular sequence on T that can be extended to a regular sequence that generates $(f_2, \ldots, f_c)$. By Proposition 5.12, $\underline{g}$ corresponds to a linear subspace of $\mathbb{P}_k^{c-1}$ containing p. We show the following by induction on $|\underline{g}|$.

$$V_{T/(\underline{g})}^*(X_{\underline{g}} \otimes M) = \operatorname{Join}(V_{T/(\underline{g})}^*(X_{\underline{g}}), V_{T/(\underline{g})}^*(M)) = \begin{cases} l & q \in V_{T/(\underline{g})}^*(k) \\ p & q \notin V_{T/(\underline{g})}^*(k) \end{cases}$$

When $|\underline{g}| = c - 1$, then $T/(\underline{g}) = R$, proving the desired result.

Assume that $|\underline{g}| = 1$, i.e. $\underline{g} = g$. Since X_g is maximal Cohen-Macaulay, by Proposition 5.5 we have $\operatorname{Tor}_{>0}^{T/(g)}(X_g, M) = 0$. Since $T/(g)$ is a codimension 2 complete intersection, the claim follows from Lemma 3.1.

Now suppose that $|\underline{g}| > 1$. It suffices to show that for any hyperplane $H \subseteq V_{T/(\underline{g})}^*(k)$ containing p, that $V_{T/(\underline{g})}^*(X_{\underline{g}} \otimes M) \cap H$ is l if $q \in H$ or p otherwise. After a change of coordinates, Proposition 5.12 we may assume that $H = V(\chi_{|\underline{g}|})$ where $k[\chi_1, \ldots, \chi_{|\underline{g}|}]$ is the ring of cohomological operators and then set $\underline{g}' = g_2, \ldots, g_{|\underline{g}|-1}$. Now $X_{\underline{g}} \otimes M \cong X_{\underline{g}'} \otimes M$ as

$T/(\underline{g'})$-modules. By induction, since $H = V^*_{T/(\underline{g'})}(k)$, we have the following.

$$V^*_{T/(\underline{g})}(X_{\underline{g}} \otimes M) \cap H = V^*_{T/(\underline{g'})}(X_{\underline{g}} \otimes M) = V^*_{T/(\underline{g'})}(X_{\underline{g'}} \otimes M) = \begin{cases} l & q \in V^*_{T/(\underline{g'})}(k) \\ p & q \notin V^*_{T/(\underline{g'})}(k) \end{cases} = \begin{cases} l & q \in H \\ p & q \notin H \end{cases}$$

This completes the proof.

□

Theorem 7.3.4. *If* $\operatorname{Tor}_{>0}(M, N) = 0$*, then* $V^*(M \otimes N) = \operatorname{Join}(V^*(M), V^*(N))$*.*

Proof. By Lemma 5.18 and Lemma 5.19, we may assume that R is complete and k is algebraically closed. First, we note that we may assume that neither M nor N is zero since otherwise the statement is trivial. Proposition 3.2 gives us one containment. Now we show the reverse containment. We induct on $\alpha(M, N) = 2 \operatorname{depth} R - \operatorname{depth} M - \operatorname{depth} N$. Assume for the moment that we have given equality when $\alpha(M, N) = 0$, which is precisely the case when both M and N are maximal Cohen-Macaulay. Suppose that $\alpha(M, N) > 0$. Then one of the modules, say M, is not maximal Cohen-Macaulay, and so $\alpha(\Omega M, N) = \alpha(M, N) - 1$. Tensoring the short exact sequence $0 \to \Omega M \to R^s \to M \to 0$ with N yields

$$0 \to \Omega M \otimes N \to N^s \to M \otimes N \to 0.$$

By induction, we have the following.

$$V^*(M \otimes N) \subseteq V^*(N) \cup V^*(\Omega M \otimes N) = V^*(N) \cup \operatorname{Join}(V^*(\Omega M), V^*(N)) = \operatorname{Join}(V^*(M), V^*(N))$$

Therefore we may assume that $\alpha(M, N) = 0$ or equivalently that M and N are maximal Cohen-Macaulay modules. First we show the theorem when $V^*(M)$ is simply a single point, say q. Suppose by way of contradiction that the containment is strict. So take

$$p \in V^*(M \otimes N) \setminus \operatorname{Join}(V^*(M), V^*(N)).$$

By Lemma 3.3, there exists maximal Cohen-Macaulay module L such that $V^*(L) = p$ and $V^*(M \otimes L) = \text{Join}(p, q) = \text{Join}(V^*(M), V^*(L))$. However, since $p \notin \text{Join}(V^*(M), V^*(N))$, there are no lines containing p, q and a point in $V^*(N)$, and therefore $V^*(M \otimes L)$ and $V^*(N)$ is disjoint. Since M, N, L are all maximal Cohen Macaulay, this shows that $\text{Tor}_{>0}(M, L) = 0$ and also $\text{Tor}_{>0}(M \otimes L, N) = 0$. Now let $A_\bullet, B_\bullet, C_\bullet$ be free resolutions of L, M, N respectively. But then, $(M \otimes L) \otimes C_\bullet$ is quasi-isomorphic to $\text{Tot}_\bullet(A_\bullet \otimes B_\bullet \otimes C_\bullet)$ which is quasi-isomorphic to $A_\bullet \otimes (M \otimes N)$. Therefore $\text{Tor}_i(M \otimes L, N) \cong \text{Tor}_i(L, M \otimes N)$, which means that the latter eventually vanishes. This implies that $V^*(M \otimes N)$ does not contain $p = V^*(L)$, a contradiction.

Now show the general case. We again proceed by contradiction and assume that there exists a point $p \in V^*(M \otimes N) \setminus \text{Join}(V^*(M), V^*(N))$. By (Bergh, 2007, Corollary 2.3), there exists an $L \in \text{mod}\, R$ with $V^*(L) = p$. However, by the previous paragraph shows that $V^*(M \otimes L) = \text{Join}(V^*(M), V^*(L))$. Therefore, the previous argument still holds, completing the proof.

□

7.4 Corollaries

We now state some interesting corollaries of Theorem 3.4. The following is immediate.

Corollary 7.4.1. *If N is not zero and $\text{Tor}_{>0}(M, N) = 0$, then $V^*(M) \subseteq V^*(M \otimes N)$.*

From this we are able to prove a plethora of other corollaries.

Corollary 7.4.2. *If $N \neq 0$ and $\text{Tor}_{>0}(M, N) = 0$, then the following hold.*

1. $\text{Ext}^{\gg 0}(M \otimes N, L) = 0 \Rightarrow \text{Ext}^{\gg 0}(M, L) = 0$
2. $\text{Ext}^{\gg 0}(L, M \otimes N) = 0 \Rightarrow \text{Ext}^{\gg 0}(L, M) = 0$
3. $\text{Tor}_{\gg 0}(M \otimes N, L) = 0 \Rightarrow \text{Tor}_{\gg 0}(M, L) = 0$

Proof. The previous corollary shows that $V^*(M \otimes N, L) = \emptyset$ implies that $V^*(M, L) = \emptyset$.

□

Corollary 7.4.3. *Suppose* $\operatorname{Tor}_{>0}(M, N) = 0$*. If* $\operatorname{Sing} R \subseteq \operatorname{Supp} N \cup (\operatorname{spec} R \setminus \operatorname{Supp} M)$*, then* M *is in* $M \otimes N$*. In particular, if* R *is an isolated singularity, then* $\operatorname{Tor}_{>0}(M, N) = 0$ *implies that* $M, N \subseteq \operatorname{Thick} M \otimes N$ *when* $M, N \neq 0$.

Proof. First note that $\operatorname{Tor}_{>0}(M_p, N_p) = 0$ for every $p \in \operatorname{spec} R$. Let $p \in \operatorname{Sing} R$. Then either $p \in \operatorname{Supp} N$ or $p \notin \operatorname{Supp} M$. Then Corollary 4.1 implies that $V^*(M_p) \subseteq V^*((M \otimes N)_p)$ for all $p \in \operatorname{spec} R$. The result then follows by Remark 5.16.

□

We may also use Theorem 3.4 to construct modules with linear cohomological supports. This proves a very special case of (Bergh, 2007, Corollary 2.2).

Corollary 7.4.4. *Assume that* k *is algebraically closed and* R *is complete. Set* $p_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{P}_k^{c-1}$ *be the point that is one in the* i*th position and zeros elsewhere. Let* L *be the affine span of* $p_1, \ldots, p_n$*. Set*

$$X_i = \frac{\Omega^{d-1}_{Q/(f_i)} k}{(\Omega^{d-1}_{Q/(f_i)} k)(f_1, \ldots, \hat{f}_i, \ldots, f_c)}$$

where $d = \dim Q$*. Then* $X_1 \otimes \cdots \otimes X_n$ *is maximal Cohen-Macaulay and* $L = V^*(X_1 \otimes \cdots \otimes X_n)$.

Note that by Proposition 5.12, any linear space of $\mathbb{P}_k^{c-1}$ is of the form of L.

Proof. By the proof of Lemma 3.3, $V^*(X_i) = \{p_i\}$. We work by induction on n. When $n = 1$, we are done. So assume the statement is true for $n - 1$. Let L' be the affine span of $p_1, \ldots, p_{n-1}$. Then, by Proposition 5.5, $\operatorname{Tor}_{>0}(X_1 \otimes \cdots \otimes X_{n-1}, X_n) = 0$. Then $X_1 \otimes \cdots \otimes X_n$ is maximal Cohen-Macaulay and by Theorem 3.4, we have

$$V^*(X_1 \otimes \cdots \otimes X_n) = \operatorname{Join}(V^*(X_1 \otimes \cdots \otimes X_{n-1}), V^*(X_n)) = \operatorname{Join}(L', p_n) = L$$

proving the claim.

□

The main result of this paper also prevents certain tor modules from vanishing.

Corollary 7.4.5. *Suppose $M_1, \ldots, M_{c+1}$ are nonfree maximal Cohen Macaulay modules. Then for some $i \in \{1, \ldots, c\}$,*

$$\operatorname{Tor}_n(M_1 \otimes \cdots \otimes M_i, M_{i+1}) \neq 0$$

for infinitely many n.

Proof. Proceeding by contradiction, suppose that

$$\operatorname{Tor}_{\gg 0}(M_1 \otimes \cdots \otimes M_i, M_{i+1}) = 0$$

for each $1 \leq i \leq c$. Inducting on i, we will show that

$$V^*(M_1 \otimes \cdots \otimes M_i) = \operatorname{Join}(V^*(M_1), \ldots, V^*(M_i))$$

for each i in $\{1, \ldots, c\}$ and that $V^*(M_1 \otimes \ldots \otimes M_i)$ contains a linear space of dimension $i - 1$. When $i = 1$, the statement is trivial. So suppose the statement is true for i. Since R is a complete intersection and each M_{i+1} is maximal Cohen-Macaulay, by Proposition 5.5 $\operatorname{Tor}_{>0}(M_1 \otimes \cdots \otimes M_i, M_{i+1}) = 0$. By Theorem 3.4, it follows that

$$\begin{aligned} V^*(M_1 \otimes \cdots \otimes M_i \otimes M_{i+1}) &= \operatorname{Join}((V^*(M_1), \ldots, V^*(M_i)), V^*(M_{i+1})) \\ &= \operatorname{Join}(V^*(M_1), \ldots, V^*(M_{i+1})) \end{aligned}$$

Furthermore, let L be the dimension i linear space in $V^*(M_1 \otimes \cdots \otimes M_i)$ guaranteed by the induction hypothesis. Take $x \in V^*(M_{i+1})$ which exists since M_{i+1} is not free. Now x is not in L and so

$$\text{Join}(L, x) \subseteq V^*(M_1 \otimes \cdots \otimes M_{i+1}).$$

But $\text{Join}(L, x)$ is a linear space of dimension $i + 1$, proving the claim.

Now the contradiction is clear, for there is a c-dimensional linear space contained in $V^*(M_1 \otimes \cdots \otimes M_{c+1})$ which is a closed subset of $\mathbb{P}^{c-1}$.

□

We also show that an analogue of Theorem 3.4 holds for Ext.

Corollary 7.4.6. *If M and N are maximal Cohen-Macaulay and* $\text{Ext}^{>0}(M, N) = 0$*, then* $V^*(\text{Hom}(M, N)) = \text{Join}(V^*(M), V^*(N))$.

Proof. Let $X^* = \text{Hom}(X, R)$. By (Huneke & Jorgensen, 2003, Theorem 2.1), $\text{Tor}_{\gg 0}(M, N^*) = 0$, and thus Proposition 5.5 implies that $\text{Tor}_{>0}(M, N^*) = 0$ since N^* is maximal Cohen-Macaulay. Therefore, $M \otimes N^*$ is also maximal Cohen-Macaulay. First, we have

$$(M \otimes N^*)^* = \text{Hom}(\text{Hom}(M \otimes N^*), R) \cong \text{Hom}(M, \text{Hom}(N^*, R)) \cong \text{Hom}(M, N)$$

since N is reflexive. By (Avramov & Buchweitz, 2000b, Theorem 5.6), for any maximal Cohen-Macaulay module X, we have $V^*(X) = V^*(X^*)$. Thus, Theorem 3.4 yields

$$\begin{aligned} V^*(\text{Hom}(M, N)) &= V^*((M \otimes N^*)^*) = V^*(M \otimes N^*) \\ &= \text{Join}(V^*(M), V^*(N^*)) = \text{Join}(V^*(M), V^*(N)). \end{aligned}$$

□

7.5 Vanishing of Tor

In the section, we prove Theorem 5.7 which gives conditions for when eventual vanishing of tor implies all the higher tor vanish. We then apply this result to Theorem 3.4. Although we are interested in complete intersection rings, for the sake of generality, we prove the result in terms of AB-dimension which was introduced by Araya in (Araya, 2012b, Definition 2.2).

Definition 7.5.1. For modules $M, N \in \operatorname{mod} R$, set $r(M,N) = \inf\{n \in \mathbb{N} \mid \operatorname{Ext}^n(M,N) \neq 0\}$. We set

$$\text{AB-dim}\, M = \sup\{\text{G-dim}\, M\} \cup \{r(M,N) \mid r(M,N) < \infty\}.$$

We have the following basic facts on AB-dimension.

Theorem 7.5.2 ((Araya, 2012b, Theorem 1.2))**.** *We have the following, with $M \in \operatorname{mod}(R)$.*

1. *A ring is AB if every finitely generated module has finite AB-dimension.*

2. *If $\text{AB-dim}\, M < \infty$, then we have $\text{AB-dim}\, M + \operatorname{depth} M = \operatorname{depth} R$.*

3. *We have $\text{G-dim}\, M \leq \text{AB-dim}\, M \leq \mathrm{CI} - \dim M$ with equality between finite terms.*

We prove a simple lemma regarding AB-dimension.

Lemma 7.5.3. *If $0 \to L \to M \to N \to 0$ is exact, and $\operatorname{pd} M < \infty$, then $\text{AB-dim}\, L < \infty$ if and only if $\text{AB-dim}\, N < \infty$. In particular, if $\text{AB-dim}\, N < \infty$, then $\text{AB-dim}\, \Omega N < \infty$.*

Proof. Take a module $X \in \operatorname{mod} R$. Using the induced long exact sequence of Ext modules, since $\operatorname{Ext}^{>\operatorname{pd} M}(M,X) = 0$, it is clear that $r(L,X) < \infty$ if and only if $r(N,X) < \infty$. Furthermore, if $r(L,X), r(N,X) < \infty$ it is easy to see that

$$r(L,X) \leq \max\{\operatorname{pd} M, r(N,X)\} \qquad r(N,X) \leq \max\{\operatorname{pd} M, r(M,X)\} + 1$$

so $r(L,X)$ is bounded if and only if $r(N,X)$ is too. Since $\text{G-dim}\, L < \infty$ if and only if $\text{G-dim}\, N < \infty$, the lemma is proven.

□

Although AB-dimension is defined in terms of ext, it detects invariants regarding tor.

Lemma 7.5.4. *Suppose R is Cohen-Macaulay. There exists an $\eta \geq 0$, depending only on the ring R, such that if* $\text{AB-dim}\, M < \infty$ *and* $\text{Tor}_{\gg 0}(M, N) = 0$*, then we have* $\text{Tor}_{>\eta}(M, N) = 0$.

This lemma follows from Lemma 6.3.

Proposition 7.5.5. *If R is Cohen-Macaulay,* $\text{AB-dim}\, N < \infty$*, and* $\text{Tor}_{\gg 0}(M, N) = 0$*, then* $\text{Tor}_{>\text{AB-dim}\, N}(M, N) = 0$*. In particular, if N is maximal Cohen-Macaulay, these assumptions imply that* $\text{Tor}_{>0}(M, N) = 0$.

Proof. Because $\text{AB-dim}\, N = \text{depth}\, R - \text{depth}\, N$ and any syzygy of N has finite AB-dimension, it suffices to prove the second statement. It follows from Theorem 5.2 that N is totally reflexive, and thus has inverse syzygies. Letting η be the bound guaranteed by Lemma 5.4, we have $\text{Tor}_i(M, N) \cong \text{Tor}_{\eta+1}(M, \Omega^{i-\eta-1} N) = 0$ for all $i > 0$.

□

When M or N is Cohen-Macaulay and R is AB, Proposition 5.5 essentially specializes to (Huneke & Jorgensen, 2003, Proposition 3.1) by Huneke and Jorgensen.

Corollary 7.5.6. *Assume R is Cohen-Macaulay. Then finite AB-dimension implies the depth formula. In other words, if* $\text{AB-dim}\, N < \infty$ *and* $\text{Tor}_{>0}(M, N) = 0$*, then the depth formula holds, i.e.* $\text{depth}\, M \otimes N + \text{depth}\, R = \text{depth}\, M + \text{depth}\, N$.

Proof. Since N has finite AB-dimension, it has finite Gorenstein dimension. By Christensen and Jorgensen's result on the depth formula, Christensen & Jorgensen (2015), we need only show that the Tate homology $\widehat{\text{Tor}}_i(M, N)$ vanishes for all i. By Lemma 5.3, every syzygy of N has finite AB-dimension, and so we may assume that N is maximal Cohen-Macaulay and $\text{Tor}_{\gg 0}(M, N) = 0$. Thus N is totally reflexive, and thus has inverse syzygies. Proposition 5.5 implies that $\widehat{\text{Tor}}_i(M, N) \cong \text{Tor}_1(M, \Omega^{-i-1} N) = 0$, completing the proof.

□

We now proceed to the main result of this section.

Theorem 7.5.7. *Suppose* AB-dim $N < \infty$ *and* M*,* N*, and* R *are Cohen-Macaulay. Also suppose that* $(R, \mathfrak{m}, k)$ *is local. Then if* $\operatorname{Tor}_{\gg 0}(M, N) = 0$ *and*

$$\dim M \otimes N + \dim R \leq \dim M + \dim N$$

then $\operatorname{Tor}_{>0}(M, N) = 0$.

Key to the proof, is the following observation.

Lemma 7.5.8. *Suppose* $(R, \mathfrak{m}, k)$ *is local and* M*,* N*, and* R *are Cohen-Macaulay. If*

$$\dim M \otimes N + \dim R \leq \dim M + \dim N$$

then there exists a sequence $x_1, \ldots, x_n \in \operatorname{ann} N$ *regular on* M *and* R *such that* N *is maximal Cohen-Macaulay over* $R/(x_1, \ldots, x_n)$.

Proof. We work by induction on $r = \dim R - \dim N$. When $r = 0$, the statement is trivial. So suppose $r > 0$. We divide the proof into two cases. First, suppose there is an $x \in \operatorname{ann} N$ which is regular on R and M. Since $\dim R/xR = \dim R - 1$ and $\dim M/xM = \dim M - 1$, we have $\dim(M/xM) \otimes N + \dim R/xR = \dim M/xM + \dim N$. Thus by induction, there exists a regular sequence $x_2, \ldots, x_n$ on M/xM and R/xR such that N is maximal Cohen-Macaulay over $R/(x, x_2, \ldots, x_n)$. Thus $x, x_2, \ldots, x_n$ is our desired regular sequence.

For the second case, suppose there is no $x \in \operatorname{ann} N$ which is regular on R and M. In other words, suppose that

$$\operatorname{ann} N \subseteq \left(\bigcup_{p \in \operatorname{ass} M} p \right) \cup \left(\bigcup_{q \in \operatorname{ass} N} q \right).$$

Then $\operatorname{ann} N \subseteq p$ with either $p \in \operatorname{ass} R$ or $p \in \operatorname{ass} M$. If $p \in \operatorname{ass} R = \min R$, then $\dim N = \dim R$, and the empty regular sequence suffices. So suppose $p \in \operatorname{ass} M$. Then p is in $\operatorname{Supp} M \cap \operatorname{Supp} N = \operatorname{Supp} M \otimes N$, thus $\dim M \otimes N \geq \dim R - \operatorname{ht} p$. But since M is Cohen-Macaulay, $\dim M = \dim R - \operatorname{ht} p \leq \dim M \otimes N$. Since $\dim M \otimes N + \dim R \leq \dim M + \dim N$, this means that $\dim R \leq \dim N$, which implies that N is maximal Cohen-Macaulay, proving the claim.

$\square$

Proof of Theorem 5.7. By the previous lemma, we may let $x_1, \ldots, x_n \in \operatorname{ann} N$ be a regular sequence on M and R such that N is maximal Cohen-Macaulay over $R/(x_1, \ldots, x_n)$. Set $I = (x_1, \ldots, x_n)$ and $\bar{R} = R/I$. Now since AB-$\dim N < \infty$, N has finite Gorenstein dimension over R. Thus, N has finite Gorenstein dimension over $\bar{R}$ and hence is a totally reflexive $\bar{R}$-module, Auslander & Bridger (1969)[Lemma 4.32]. Therefore, there exists a long exact sequence of the form

$$0 \to N \to \bar{R}^{m_0} \xrightarrow{\partial^0} \bar{R}^{m_1} \xrightarrow{\partial^1} \bar{R}^{m_2} \xrightarrow{\partial^2} \cdots$$

Set $N^i = \Omega_{\bar{R}}^{-i} N = \ker \partial^i$. Note that $N^0 = N$. Now $\operatorname{Tor}^R_{>0}(M, \bar{R}) = 0$ since $x_1, \ldots, x_n$ is regular on M. Thus, considering the short exact sequence $0 \to N^i \to \bar{R}^m \to N^{i+1} \to 0$, we see that $\operatorname{Tor}_j(M, N^i) \cong \operatorname{Tor}_{j+1}(M, N^{i+1})$. In particular, $\operatorname{Tor}_{\gg 0}(M, N^i) = 0$ for all i.

By Lemma 5.3, each N^i has finite AB dimension. Furthermore, Lemma 5.4 guarantees an $\eta \geq 0$ such that that $\operatorname{Tor}^R_{>\eta}(M, N^i) = 0$ for all i. Thus for all $j > 0$, we have

$$\operatorname{Tor}^R_j(M, N) \cong \operatorname{Tor}^R_{j+\eta+1}(M, N^{\eta+1}) = 0$$

completing the proof.

$\square$

The previous result and Theorem 3.4 give this immediate corollary.

Corollary 7.5.9. *Suppose R is a local complete intersection and M and N are Cohen-Macaulay. Then if $V^*(M) \cap V^*(N) = 0$ and $\dim M \otimes N + \dim R \leq \dim M + \dim N$, then $V^*(M \otimes N) = \operatorname{Join}(V^*(M), V^*(N))$.*

This corollary gives a relation between the actual support of a module and also the cohomological support.

7.6 Questions and examples

The first immediate question is what happens to Theorem 3.4 if we remove the assumption that all the Tor modules vanish. The following two examples show that in general neither containment holds.

Example 7.6.1. Let k be a field and set $R = k[x,y]/(xy)$. Now the modules $M = R/(x+y)$ and $N = R/(x-y)$ have finite projective dimension. However, we have

$$\operatorname{Join}(V^*(M), V^*(N)) = \emptyset \not\supseteq V^*(M \otimes N) = \mathbb{P}^0_{\tilde{k}}$$

showing that one containment does not always hold.

Example 7.6.2. Set $R = \mathbb{Q}[a,b,c]/(a^2 - b^2, b^3 - c^3)$ and

$$M = \operatorname{coker} \begin{bmatrix} 8ab^2c^2 + 4abc^3 + 6b^2c^3 + 8ac^4 + 6bc^4 + c^5 & 3ab + 4b^2 + 7ac + 7bc + 4c^2 \\ 4ab^2c^2 + 6abc^3 + 9b^2c^3 + ac^4 + 9bc^4 + 4c^5 & 4ab + 5b^2 + 3ac + 5bc + 5c^2 \end{bmatrix}$$

$$N = \frac{R}{(8ab^2c + 8b^2c^2 + 6ac^3 + 5bc^3 + c^4, 3ab + 2b^2 + 3ac + 2bc + 9c^2)}.$$

An easy computation in Macaulay2 shows that

$$\operatorname{cx} M = 0 \qquad \operatorname{cx} N = 2 \qquad \operatorname{cx} M \otimes N = 1.$$

This shows that $\operatorname{Join}(V^*(M), V^*(N)) = V^*(N) \not\subseteq V^*(M \otimes N)$.

In both of these examples $\operatorname{Tor}_{\gg 0}(M,N) = 0$ and also $V^*(M)$ is empty. Examples 6.3 and 6.4 show that both containment fails even when neither $V^*(M)$ and $V^*(N)$ are empty and the tor modules do not eventually vanish.

Example 7.6.3. Set $R = \mathbb{Q}[a,b,c,d]/(a^2 - b^2, b^2 - c^2, d^2)$ and define the ideal

$$I = \left(\frac{3}{5}a + \frac{8}{7}b + \frac{5}{2}c, 2a + \frac{1}{2}b + 3c, d\right).$$

A simple computation on Macaulay2 shows that

$$V^*(I) = V(3740x_1 + 477x_2)$$

$$V^*(I \otimes I) = V(0) = \mathbb{P}^2_{\tilde{\mathbb{Q}}}$$

Were $\tilde{\mathbb{Q}}[x_1, x_2, x_3]$ is the ring of cohomological operators over the algebraic closure of $\mathbb{Q}$. Since $V^*(I)$ is linear, we have

$$\mathrm{Join}(V^*(I), V^*(I)) = V^*(I) \not\supseteq V^*(I \otimes I).$$

Example 7.6.4. Let $R = \mathbb{Q}[a,b,c]/(a^2, b^2, c^2)$ and $I = (b)$ and $J = (ab)$. An easy computation yields $V^*(R/I) = V(x_1, x_3)$ and $V^*(R/J) = V(x_1)$ where $\tilde{\mathbb{Q}}[x_1, x_2, x_3]$ is the ring of cohomological operators over the algebraic closure of $\mathbb{Q}$. Now because $V^*(R/J)$ is a linear space containing $V^*(R/I)$, we have

$$\mathrm{Join}(V^*(R/I), V^*(R/J)) = V^*(R/J) \not\subseteq V^*(R/J \otimes R/I) = V^*(R/(I+J)) = V^*(I).$$

The authors wondered if there was a relation between the stable behavior of $V^*(\mathrm{Tor}_i(M,N))$ and $\mathrm{Join}(V^*(M), V^*(N))$. Investigations using Macaulay2 compelled them to ask the following questions.

Question 10. *Does*

$$\bigcup_{i=0}^{n} V^*(\mathrm{Tor}_i(M,N))$$

stabilize as n tends to infinity?

Question 11. *For any modules M and N, do we have the following?*

$$\mathrm{Join}(V^*(M), V^*(N)) \subseteq \bigcup_{i=0}^{\infty} V^*(\mathrm{Tor}_i(M, N))$$

Proposition 7.6.5. *Questions 10 and 11 are true when* $\mathrm{Tor}_{\gg 0}(M, N) = 0$.

Proof. The first question is trivially true in this case. We prove that the second question is true in this case using induction on the minimal n such that $\mathrm{Tor}_{>n}(M, N) = 0$. When $n = 0$, the the statement follows from Theorem 3.4. So suppose $n > 0$. Then we have $\mathrm{Tor}_{>n-1}(\Omega M, N) = 0$ and so by induction, we have

$$\begin{aligned}\mathrm{Join}(V^*(M), V^*(N)) &= \mathrm{Join}(V^*(\Omega M), V^*(N)) \\ &\subseteq \bigcup_{i=0}^{\infty} V^*(\mathrm{Tor}_i(\Omega M, N)) = \bigcup_{i=2}^{\infty} V^*(\mathrm{Tor}_i(M, N)) \cup V^*(\Omega M \otimes N).\end{aligned}$$

Note that M is not free and so ΩM is not zero. The short exact sequence

$$0 \to \Omega M \to R^m \to M \to 0$$

yields

$$0 \to \mathrm{Tor}_1(M, N) \to \Omega M \otimes N \to N^m \to M \otimes N \to 0.$$

It follows that $V^*(\Omega M \otimes N) \subseteq V^*(N) \cup V^*(M \otimes N) \cup V^*(\mathrm{Tor}_1(M, N))$ and hence

$$\mathrm{Join}(V^*(M), V^*(N)) \subseteq \bigcup_{i=0}^{\infty} V^*(\mathrm{Tor}_i(M, N)) \cup V^*(N).$$

Similarly, we have

$$\mathrm{Join}(V^*(M), V^*(N)) \subseteq \bigcup_{i=0}^{\infty} V^*(\mathrm{Tor}_i(M, N)) \cup V^*(M)$$

but since $V^*(M) \cap V^*(N) = \emptyset$, this shows the desired result.

□

Question 10 is also true when R is a hypersurface, because over such rings $\operatorname{Tor}_i(M,N)$ is eventually periodic. The following example shows that even over a hypersurface we cannot weaken Question 10 to asking if $V^*(\operatorname{Tor}_i(M,N))$ stabilizes.

Example 7.6.6. Let k be a field and set $R = k[x,y]/(xy)$. It is easy to show that $\operatorname{Tor}_{\text{odd}}(R/(x), R/(y)) = 0$ and $\operatorname{Tor}_{\text{even}}(R/(x), R/(y)) \cong k$. The projective dimension of the former is obviously finite, and the projective dimension of the latter is infinite. Thus $V^*(\operatorname{Tor}_i(R/(x), R/(y)))$ cannot stabilize.

Note that Example 6.3 shows that we cannot hope to replace the containment in Question 11 with equality. We now show some potential applications.

Proposition 7.6.7. *Suppose Question 10 is true for a pair of modules M and N. If we have*

$$\operatorname{Join}(V^*(M), V^*(N)) = \bigcup_{i=0}^{\infty} V^*(\operatorname{Tor}_i(M,N))$$

then we have the inequality

$$\max_i\{\operatorname{cx} Tor_i(M,N)\} \leq \operatorname{cx} M + \operatorname{cx} N \leq \operatorname{cx} \sum_{i+j=n} \beta_j(\operatorname{Tor}_i(M,N)).$$

Proof. If we assume that Question 10 holds, then the assumptions imply

$$\dim \operatorname{Join}(V^*(M), V^*(N)) = \max\{\dim V^*(\operatorname{Tor}_i(M,N))\}.$$

This yields the inequality

$$\max\{\operatorname{cx} \operatorname{Tor}_i(M,N)\} - 1 = \max\{\dim V^*(\operatorname{Tor}_i(M,N))\} \leq \dim V^*(M) + V^*(N) + 1 = \operatorname{cx} M + \operatorname{cx} N - 1$$

It suffices to show the second inequality. By Miller (1998), we have

$$\operatorname{cx} \sum_{i+j=n} \beta_i(M)\beta_j(N) = \operatorname{cx} M + \operatorname{cx} N,$$

hence the second inequality follows from the next lemma, Lemma 6.8.

□

Lemma 7.6.8. *We have the inequality*

$$\sum_{i+j=n} \beta_i(M)\beta_j(N) \leq \sum_{i+j=n} \beta_j(\operatorname{Tor}_i(M,N)).$$

Proof. Let $F_\bullet$ and $G_\bullet$ be minimal free resolutions of M and N and $P_\bullet$ a free resolution of k. Let $T_\bullet$ be the total complex $F_\bullet \otimes G_\bullet$. Let E^0 be the double complex $T \otimes P_\bullet$. Computing the spectral sequence using the vertical filtration gives $E^1_{i,0} = T_i \otimes k$ and $E^1_{i,j} = 0$ for $j \neq 0$. Since $F_\bullet$ and $G_\bullet$ are minimal, the differential of T are given by matrices whose entries lie in $\mathfrak{m}$. Therefore the differentials of E^1 are zero, and so the spectral sequence collapses. Computing the spectral sequence using the horizontal filtration gives $E^1_{i,j} = \operatorname{Tor}_i(M,N) \otimes P_j$ and $E^2_{i,j} = \operatorname{Tor}_j(\operatorname{Tor}_i(M,N),k)$. We thus have

$$k^{\beta_j(\operatorname{Tor}_i(M,N))} \implies \bigoplus_{i'+j'=i+j} k^{\beta_{i'}(M)\beta_{j'}(N)}.$$

However, since E^∞_n is a graded vector space whose associated graded space is a quotient of a subspace of $\bigoplus_{i+j=n} k^{\beta_j(\operatorname{Tor}_i(M,N))}$, the desired inequality follows.

□

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Appendix A

Appendix of notation

We list some notation which is consistently used throughout this document.

1. R is a commutative Noetherian ring
2. $\mathrm{mod}(R)$ is the category of finitely generated R-modules
3. $\mathrm{Mod}(R)$ is the category of all R-modules
4. $\mathrm{Ch}^b(R)$ the category of bounded complexes of finitely generated R-modules
5. $\mathrm{Ch}(R)$ is the category of complexes of all (not necessarily finitely generated) R-modules
6. $K^b(R)$ the homotopy category of bounded complexes of finitely generated R-modules
7. $K(R)$ is the homotopy category of all complexes of all (not necessarily finitely generated) R-modules
8. $D^b(R)$ the derived category of bounded complexes of finitely generated R-modules
9. $D(R)$ is the derived category of all complexes of all (not necessarily finitely generated) R-modules
10. $\mathcal{M}_{\mathrm{fl}}$ will denote the category of finite length modules

11. $\mathcal{P}(R)$ and $\mathcal{P}$ denote the projective R-modules

12. $\overline{\mathcal{P}(R)}$ or $\bar{\mathcal{P}}$ will denote the category of modules of finite projective dimension

13. Γ is the set of grade consistent functions

14. $\mathcal{C}\text{-dim}\, M$ is the dimension of M with respect to the category $\mathcal{C} \subseteq \operatorname{mod}(R)$

15. $\bar{\mathcal{C}}$ will denote the category of (finitely generated) modules M such that $\mathcal{C}\text{-dim}\, M < \infty$ where $\mathcal{C} \subseteq \operatorname{mod}(R)$ is a a subcategory

16. $\operatorname{add} \mathcal{C}$ is the smallest category closed under direct sums and summands containing $\mathcal{C} \subseteq \operatorname{mod}(R)$

17. $\Lambda(\mathcal{C})(f) = \{M \in \operatorname{mod}(R) \mid \operatorname{add} \mathcal{C}_p\text{-dim}\, M_p \leq f(p) \quad \forall p \in \operatorname{spec} R\}$ with $f \in \Gamma$

18. $\Phi_{\mathcal{C}}(\mathcal{X})(p) = \sup\{\operatorname{add} \mathcal{C}_p\text{-dim}\, X_p \mid X \in \mathcal{X}\}$ with $\mathcal{C}, \mathcal{X} \subseteq \operatorname{mod}(R)$ categories

19. $\mathfrak{R}(\mathcal{C}) = \{\mathcal{X} \subseteq \operatorname{mod}(R) \mid \mathcal{C} \subseteq \mathcal{X} \subseteq \bar{\mathcal{C}} \quad \mathcal{X} \text{ is resolving}\}$

20. $\mathfrak{R}$ is the collection of all resolving subcategories

21. $\operatorname{res} \mathcal{X}$ is the smallest resolving subcategory of $\operatorname{mod}(R)$ containing $\mathcal{X} \subseteq \operatorname{mod}(R)$

22. $\operatorname{Thick}_{\mathcal{C}}(\mathcal{X})$ is the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{X}$ with $\mathcal{C}, \mathcal{X} \subseteq \operatorname{mod}(R)$

23. $\operatorname{Thick}(\mathcal{X})$ is the smallest thick subcategory of $\operatorname{mod}(R)$ containing $\mathcal{X}$

24. $\overline{\mathcal{X}}$ is the category of modules with finite $\mathcal{X}$-dimension

25. $\mathcal{G}_C$ is the collection of totally C-reflexive modules

26. $M^\dagger = \operatorname{Hom}(M, C)$ where C is a semidualizing module

27. ρ denotes projective resolutions

28. $\simeq$ denotes quasi-isomorphism

29. $-^{\#} = \mathbf{R}\mathrm{Hom}(-, C)$ for a fixed semidualizing module C

30. $\mathcal{G}_C = \mathcal{G}_C(R)$ is the category of totally C-reflexive R-modules

31. $\mathcal{A}_C = \mathcal{A}_C(R)$ is the Auslander class

32. $\mathcal{B}_C = \mathcal{B}_C(R)$ is the Bass class

33. $D\mathcal{G}_C$ is the derived subcategory of totally C-reflexive complexes.

34. $D\mathcal{A}_C = D\mathcal{A}_C(R)$ is the derived analogue of the Auslander class

35. $D\mathcal{B}_C = D\mathcal{B}_C(R)$ is the derived analogue of the Bass class

36. $\min_c(P_\bullet) = \sup\{n \mid P_i = 0 \quad \forall\ i < n\}$ for a complex $P_\bullet$

37. $\max_c(P_\bullet)$ is defined as $\inf\{n \mid P_i = 0 \quad \forall\ i > n\}$ for a complex $P_\bullet$

38. $\min(P_\bullet) = \sup\{n \mid H_i(P_\bullet) = 0 \quad \forall i < n\}$ for a complex $P_\bullet$

39. $\max(P_\bullet) = \inf\{n \mid H_i(P_\bullet) = 0 \quad \forall i > n\}$ for a complex $P_\bullet$

40. $\mathrm{Width}(P_\bullet) = \max(P_\bullet) - \min(P_\bullet)$ for a complex $P_\bullet$

41. $\mathrm{Width}(0)$ is defined to be 0

42. $\mathrm{Supph}(P_\bullet) = \{n \mid H_n(P_\bullet) \neq 0\}$

43. $V^*(M)$ is the cohomological support of a module over a complete intersection ring

44. $\mathrm{Join}(V, U)$ is the closure of the union of lines of the form $\mathrm{line}(u, v)$ with $u \in U$, $v \in V$ and $u \neq v$

Also, unless otherwise stated, all rings will be assumed to be commutative and Noetherian, and all modules will be finitely generated.