Commutators of Multilinear Singular Integral Operators with Pointwise Multiplication

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Abstract

In this dissertation we further develop the theory of commutators of multilinear singular integral operators with pointwise multiplication. Generally speaking, the commutator of two operators is itself an operator that measures the changes which occur when switching the order in which the commuted operators are being applied. They have proven to be significant historically, and can be useful in the study of PDE.

Our first main contribution is the completion of the characterization of the space functions with bounded mean oscillation (BMO) in terms of the boundedness of the corresponding commutator in an appropriate set of Lebesgue spaces. It is already known in a variety of settings that a function being in BMO is sufficient to conclude the boundedness of the commutator, we were able to show that this condition is in fact necessary, a long standing open question.

Our characterization opened the door for us to obtain our second main result, namely the necessary and sufficient conditions which guarantee the compactness of the commutator of the bilinear singular integral with pointwise multiplication in appropriate weighted Lebesugue spaces.

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Contents

Introduction

Calderón-Zygmund theory has proven to be an ideal tool for many problems involving linear operators in many areas of analysis and partial differential equations, and the theory has been expanded in recent years to treat multilinear operators. The problems treated are natural in the context of harmonic analysis, but have also found applications in other areas, such as the use of multilinear operators which arose in the study of Navier-Stokes systems of Germain in [14]. In this work we deal extensively with one object which can arise in Calderón-Zygmund theory: commutators.

In the linear setting, a commutator of an operator T with pointwise multiplication by a function b acting on a function f is defined as

$$[b,T](f)(x) := b(x)T(f)(x) - T(bf)(x).$$

In [12], Coifman, Rochberg, and Weis showed that if T is the Hilbert Transform, then $[b,T]$ is bounded if and only if $b \in \mathrm{BMO}$, the John-Nirenberg space. Note that for $f \in L^p$ and $g \in L^{p'}$ we have

$$\langle [b,T](f), g\rangle = \langle b, T(f)g - fT^*(g)\rangle,$$

where T^* denotes the transpose of T. In this light, we see that the characterization of the boundedness of the commutator with BMO functions means $T(f)g - fT^*(g)$, which is clearly in L^1 since T maps L^p to itself, is in fact in the Hardy space H^1, the pre-dual of BMO. This allowed Coifman et al. to achieve a factorization of H^1 in a higher dimensional setting than had previously been done. Janson and Uchiyama each extended this characterization of BMO, in [18] and [27] respectively, to commutators of Calderón-Zygmund operators of convolution type with smooth homogeneous kernels, and Chanillo, [7], did the same for commutators of the fractional integral operator with the restriction that $n-\alpha$ be an even integer. In [19], Lacey partially extended Chanillo's result to the multiparameter setting. He looked at commutators with iterations of the 1-dimensional I_α in each variable of a function in $\mathbb{R}^m$. In particular, if one reduces his arguments to the one parameter situation, and only in one dimension, one obtains the characterization of boundedness of the commutator in $\mathbb{R}$ for I_α, $0 < \alpha < 1$, which was not covered in Chanillo's work. The boundedness of commutators in the multilinear setting has been extensively studied already, as in Pérez and Torres' [24], Tang's [26], Lerner, Ombrosi, Pérez, Torres, and Trujillo-González's [20], Chen and Xue's [9], and Pérez, Pradolini, Torres, and Trujillo-González's [23] to name a few. However it has been an open question until now whether they can be used to characterize BMO($\mathbb{R}^n$). We answer this question in the affirmative for a large class of multilinear singular integral operators which includes both fractional integrals and homogeneous Calderón-Zygmund operators of convolution type. Our result for fractional integral operators in the linear setting expands on the works of both Chanillo and the single parameter results of Lacey by characterizing the boundedness of the commutators for any $0 < \alpha < n$, with no requirement that $n-\alpha$ be an even integer or that $n = 1$. This result will be immediately put to use to prove our second main result, one related to compactness.

Historically, the first result on compactness of commutators of singular integrals with point-wise multiplication is due to Uchiyama [27]. He refined the boundedness results of

Coifman et al. on the commutator with symbols in BMO to compactness. This is achieved by requiring the symbol not to be just in BMO, but rather in CMO, which is the closure in BMO of the space of C^∞ functions with compact support. For linear fractional integrals, the compactness is credited in Chen, Ding and Wang [10] to Wang [28]. As in the case of singular integrals of Calderón-Zygmund type, the conditions are again that the symbol is respectively in BMO or CMO.

In the multilinear setting, the compactness of commutators of Calderón-Zygmund operators and fractional integrals started to receive attention only a few years ago. In particular, Bényi and Torres [3] and Bényi et al. [2] showed that symbols in CMO again produce compact commutators. As before, none of these results indicated that the symbol being in CMO was actually necessary for compactness, and our second result does exactly this in the fractional integral setting. We obtain a result characterizing compactness of commutators with bilinear fractional integrals on certain weighted Lebesgue spaces which includes the unweighted case. This last result complements the results of Bényi et al. [1] for bilinear Calderón-Zygmund operators and it is of interest in its own. Formally, the characterization results for $\alpha = 0$ would correspond to the case of bilinear Calderón-Zygmund operators. Some of the techniques employed do not apply to Calderón-Zygmund operators, mainly because they lack positive kernels. We are actively studying the case for Calderón-Zygmund operators as a topic for future work.

At this time the author would like to note that while he has contributed to other research projects during his stay at the University of Kansas, their subject matter differs greatly from commutators and each other. With this in mind, he chose to omit them for the sake of the cohesiveness of the work.

Chapter 1

Preliminaries

In this chapter we will set notation, define certain spaces and objects, and recall certain results which are used throughout this work.

1.1 Geometric Notation

First, while we will be working primarily in $\mathbb{R}^n$, for this chapter alone, when our definition or result holds in either $\mathbb{R}$ or $\mathbb{C}$ we will use $\mathbb{F}$ to highlight that it could be either field.

In this work, the use of $|\cdot|$ appears in many different situations, and so we will now make clear what we mean in each situation. For a number, $x \in \mathbb{F}$, we denote the modulus $|x| = \sqrt{x\bar{x}}$, in the case that x is real, this is of course the absolute value. For $n \in \mathbb{N}$, and x in $\mathbb{R}^n$, we denote the Euclidean distance as

$$|x| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

For a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$, we use $|\alpha|$ to denote the size of α in terms of the ℓ^1

norm, namely

$$|\alpha| = \sum_{i=1}^{m} \alpha_i,$$

keeping in mind that multi-indices have non-negative entries.

Finally, for a Lebesgue measurable set, E, we use $|E|$ to denote the Lebesgue measure of the set.

With regards to geometry, we denote by $B(x,r) \subset \mathbb{F}^n$ the ball or radius r, centered at x. That is to say,

$$B(x,r = \{y \in \mathbb{F}^n : |x-y| < r\}).$$

For cubes, we will use the convention that sides will always be parallel to the axes, and use $Q(x,r)$ to be the cube centered at x with side length r, explicitly, this set is

$$Q(x,r) = \{y \in \mathbb{F}^n | \max\{|x_1 - y_1|, |x_1 - y_2|, ..., |x_n - y_n|\} < \frac{r}{2}\}.$$

1.2 Functions and Function Spaces

First, all our integrals will be with respect to the Lebesgue measure, and so when we are integrating with respect to the x variable, we will use dx as our differential for clarity. When the variable with respect to which we are integrating is clear, we will often omit the differential all together. For a function f, we will denote by f_E the average of f on the set E. When the expression we have for our function is more complicated to express, we may instead use $⨍_E f$ to denote this. So to summarize,

$$f_E = ⨍_E f = \frac{1}{|E|} \int_E f.$$

We say a function, f, is **homogeneous of degree** k if $f(\lambda x) = \lambda^k f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda \neq 0$. This definition of homogeneity is frequently stronger than necessary, and often unattained by otherwise excellent functions, and so we define the weaker condition of positive homogeneity. A function f is **positively homogeneous of degree** k if we instead have $f(\lambda x) = \lambda^k f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$. Positive homogeneity is what we will ultimately find ourselves using as a requirement for the kernels of our operators, which we will discuss in the next section, and it is the strongest form of homogeneity one can hope for if, for instance, f is radial or positive.

For a measure space X, recall that the Lebesgue spaces, denoted $L^p(X)$, or just L^p when the measure space is clear, are the spaces of functions, identified with each other if they agree outside of a set of measure zero, for which

$$\|f\|_{L^p} = \left(\int_X |f|^p \right)^{\frac{1}{p}} < \infty.$$

If $p \geq 1$ then L^p is a Banach space, and $\|\cdot\|_{L^p}$ is indeed its norm, as the notation suggests. If $f \in L^1(K)$ for any $K \subset X$ compact, we say $f \in L^1_{loc}(X)$.

One classical result which plays a key role in some of our proofs is Hölder's Inequality, which in its simplest form is simply that

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}},$$

where p and p' are greater than or equal to 1 and such that $\frac{1}{p} + \frac{1}{p'} = 1$. In the remainder of this work, whenever a we write p' or q', it is to be understood that it is the number which satisfies this Hölder relation with p or q, respectively. It can quickly be seen that for

$r, p_1, \ldots, p_m$ positive, such that $\frac{1}{r} = \sum_{i=1}^{m} \frac{1}{p_i}$, Hölder's Inequality can be generalized to

$$\|f_1 \cdot \ldots \cdot f_m\|_{L^r} = \prod_{i=1}^{m} \|f_i\|_{L^{p_i}}.$$

In this work we will deal extensively with the John-Nirenberg space, BMO. To define it, we begin by observing that $⨍_Q |b(x) - b_Q| dx$ measures the mean of the oscillation of a function over Q. BMO($\mathbb{R}^n$) is simply the space of all $b \in L^1_{loc}$, identified up to vertical shifts, such that this mean oscillation over all cubes is bounded, or more simply,

$$\|b\|_* = \sup_Q ⨍_Q |b(x) - b_q| dx < \infty$$

Note that since vertical shifts do not affect the oscillation, it makes sense to identify two functions with one another if the differ by a constant, and because we have quotiented out by this equivalence, we get that BMO is in fact a Banach space with norm $\|\cdot\|_*$. One thing worth noting is that we can replace the b_Q in the integrand with any constant, and still control the BMO norm as defined, specifically, for any constant C we have

$$⨍_Q |b(x) - b_Q| dx \leq 2 ⨍_Q |b(x) - C| dx.$$

As has already been mentioned, the closure of C^∞ functions with compact support is denoted CMO, and in [27], Uchiyama was able to characterize CMO as follows:

Theorem 1.2.1 *(Uchiyama) A function $b \in$BMO is in CMO if an only if,*

$$\lim_{a\to 0} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = 0, \tag{1.1}$$

$$\lim_{a\to \infty} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = 0, \tag{1.2}$$

$$\lim_{|y|\to \infty} \frac{1}{|Q|} \int_Q |b(x+y) - b_Q| dx = 0, \textit{ for each } Q. \tag{1.3}$$

This will be an invaluable tool to us later, when we use it to characterize the compactness of the bilinear commutator.

1.3 Operators, Linear and Multilinear

We begin by introducing an operator that has great importance and which serves in some sense as a template for similar operators which we will define as they are needed. For a given function f, the **Hardy-Littlewood maximal function**, denoted Mf, is given by

$$Mf(x) := \sup_{Q: x \in Q} \fint_Q |f(y)| dy.$$

This operator is bounded on L^p for all $1 < p \leq \infty$, and has certain extremely useful boundedness properties which we will discuss as they become relevant in the next section.

The operators with which we are primarily concerned are the multilinear Calderón-Zygmund operators and the mutilinear fractional integral operators. We will begin by defining their linear counterparts.

We say a function K is a **standard kernel** if it satisfies

1. $|K(x,y)| \lesssim \frac{1}{|x-y|^n}$ for $x \neq y$.

2. For some $\delta > 0$, we have $|K(x,y) - K(x',y)| \lesssim \frac{|x-x'|^\delta}{|x-y|^{n+\delta}}$ whenever $|x-x'| \leq \frac{1}{2}|x-y|$.

3. For the same δ, the above regularity condition also holds in the y variable.

If K is a standard kernel, then an operator $T : \mathscr{S} \to \mathscr{S}'$ is associated with K if for all $x \notin \text{supp}(f)$, we have

$$T(f)(x) := \int_{\mathbb{R}^n} K(x,y) f(y)\, dy.$$

If T is bounded on L^2, we call it a **Calderón-Zygmund Operator**, and we denote the class of such operators as CZO. One of the many useful properties of operators in CZO which is worth mentioning here, is that it can be shown that they are in fact bounded on L^p for any $1 < p < \infty$. Because of this, when we are discussing operators in CZO it will be in the more general setting of L^p. It is clear from the definition of standard kernels that operators in CZO my have non-integrable singularities, and so often rely on subtle cancellation properties to achieve boundedness. There is a subtype of kernel with which we will deal extensively, called convolution kernels. A **convolution kernel** is one for which $K(x,y) = \tilde{K}(x-y)$ for some $\tilde{K}$. When K is a convolution kernel, we will often write $K(x-y)$ in favor of $K(x,y)$.

For $0 < \alpha < n$, the fractional integral operator, $I_\alpha : L^p \to L^q$, is defined by

$$I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

While I_α is similar in some ways to a Calderón-Zygmund operator with a positively homogeneous convolution kernel, there are a few key differences worth noting. First, these operators are not bounded on L^p to itself, but are instead bounded from L^p to L^q whenever

p and q satisfy the Sobolev relation

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},$$

formally, we can view the case of $\alpha = 0$ as recovering the L^p boundedness of CZO. Second, the kernel is positive, meaning on the one hand that it can be easier to work with directly, but on the other hand, there are no cancellation properties which can be taken advantage of. However, this itself somewhat compensated for by the fact that the singularity is locally integrable.

Given an operator T, and a function b, we define the **commutator of T and b** by

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

If T is defined by integration against a kernel for certain x, such as when $T \in$ CZO, we have that this becomes

$$[b,t](f)(x) = \int_{\mathbb{R}^n} (b(x) - b(y))K(x,y)f(y)dy$$

for all x for which the integral representation of T holds.

These commutators encapsulate certain cancellation properties which have proven to be both intrinsically interesting, historically useful as in [12], and applicable to the study of certain PDE's (see [17] and the references therein). It is worth noting that for a constant

C, if T is linear we have,

$$\begin{aligned}[b+C,T](f) &= (b+C)T(f) - T((b+C)f) \\ &= bT(f) + CT(f) - T(bf) - CT(f) \\ &= [b,T](f).\end{aligned}$$

This leads one to intuitively look to spaces for which we identify functions which differ by constants, and so it is no surprise that $b \in$ BMO or CMO has had the most historical significance.

With these operators and definitions in tow, we are ready to expand our theory and define their multilinear counterparts. For readability we will, for the most part, restrict ourselves to studying the bilinear versions of these objects, but it will be clear from context how to obtain the fully generalized multilinear results.

In the bilinear setting, $K(x,y,z)$ is a **standard kernel** if it satisfies the similar size and regularity estimates as in the linear setting, namely,

- $|K(x,y,z)| \lesssim \frac{1}{(|x-y|+|x-z|)^{2n}}$ away from $x=y=z$.
- For some $\delta > 0$, we have $|K(x,y,z) - K(x',y,z)| \lesssim \frac{|x-x'|^{\delta}}{(|x-y|+|x-z|)^{2n+\delta}}$ whenever $|x - x'| \leq \frac{1}{2}\max\{|x-y|, |x-z|\}$.
- For the same δ, the above regularity condition also holds in the y and z variables.

And as before, if an operator T can be represented as

$$T(f,g)(x) = \int\int K(x,y,z)f(y)g(z)\, dydz$$

for all $x \notin \text{supp}(f) \cap \text{supp}(g)$, with K a standard kernel, and if

$$T : L^{p_1} \times L^{p_2} \to L^p$$

for

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},$$

we say T is a **bilinear Calderón Zygmund operator**. As in the linear case, boundedness for one p guarantees the boundedness for all p satisfying the appropriate Hölder relations. In this setting, the notion of a "convolution kernel" is adjusted to a kernel K such that $K(x,y,z) = \tilde{K}(x-y, x-z)$ for some function $\tilde{K}$. This corresponds to simultaneous translations in each variable, as opposed to allowing a separate translation in each position, and as before, we will simply write $K(x-y, x-z)$.

It is worth noting that these convolution type operators are invariant under simultaneous translations, and can be realized (at least formally) as a pointwise multiplication operator by a function σ on the Fourier transform side, in other words,

$$T(f,g)(x) = \int\int \sigma(\xi_1, \xi_2) \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2.$$

More generally, one can look at certain classes of multilinear pseudodifferential operators, which are defined by,

$$T(f,g)(x) = \int\int \sigma(x, \xi_1, \xi_2) \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2.$$

For example, in [15] it was shown that these are Calderón-Zygmund operators, provided that

$$|\partial_x^{\alpha}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_1}\sigma(z,\xi_1,\xi_2)| \le C_{\alpha,\beta}(1+|\xi_1|+|\xi_2|)^{|\alpha|-|\beta_1|-|\beta_2|},$$

and $T^{*j}\left(e^{2\pi i\xi_1\cdot(\cdot)}, e^{2\pi i\xi_2\cdot(\cdot)}\right)$ are in BMO uniformly for $\xi_i \in \mathbb{R}^n$. Here, T^{*j} is j^{th} transpose of T, defined by

$$\langle T^{*1}(f,g),h\rangle = \langle T(h,g),f\rangle \quad \text{and} \quad \langle T^{*2}(f,g),h\rangle = \langle T(f,h),g\rangle.$$

For $0<\alpha<2n$, the bilinear fractional integral operator can be defined in one of two equivalent ways,

$$\mathscr{I}_{\alpha}(f,g)(x) = \int\int \frac{f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}},$$

or

$$I_{\alpha}(f,g)(x) := \int\int \frac{f(y)g(z)}{(|x-y|^2+|x-z|^2)^{n-\alpha/2}}dydz.$$

As our choice of notation hints, when considering the multilinear fractional integrals we will focus on the I_α version of the operator, as the differentiability of the denominator will at times be beneficial.

As for commutators, there are now two different ones to be considered, namely

$$[b,T]_1(f,g)(x) := b(x)T(f,g)(x) - T(bf,g)(x)$$

and

$$[b,T]_2(f,g)(x) := b(x)T(f,g)(x) - T(f,bg)(x)$$

For many operators, symmetry of the kernel allows one to only consider $[b,T]_1$. Indeed, for the operators in this work this will always be the case.

Finally, the notion of compactness must be discussed. Let X, Y, and Z be Banach spaces, and let B_X and B_Y be the unit balls in their respective spaces. We say that a bilinear operator $T : X \times Y \to Z$ is compact if the image of those balls, $T(B_X, B_Y)$, is a precompact set in Z. The work of Bényi and Torres in [3] contains many natural properties of compact bilinear operator, but there are three in particular worth noting here, which we will state as a lemma without proof.

Lemma 1.3.1 *Suppose T is a compact bilinear operator from $L^{p_1} \times L^{P_2}$ to L^p,*

- $T : L^{p_1} \times L^{P_2} \to L^p$.
- *For $f \in L^{p_1}$, $g \in L^{p_2}$, we have that $T(\cdot, g)$ and $T(f, \cdot)$ are compact linear operators.*
- *The subspace of compact operators is closed in the operator norm.*

It should be pointed out that there exist operators which are compact in each position which are not compact, similar to how a function can be continuous in each variable, but not continuous as a whole. The last property of this lemma will be the most critical to this work, as it will allow us to use a convergent sequence of compact operators when the time comes. In the next section we will describe a criterion for showing that a multilinear operator is compact in an appropriately weighted Lebesgue setting.

1.4 Weights

For $1 < p < \infty$, recall that the class of **Muckenhoupt A_p weights** consists of all non-negative, locally integrable, functions w such that

$$[w]_{A_p} := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{1-p'} \right)^{\frac{p}{p'}} < \infty;$$

for $1 < p < \infty$, and

$$[w]_{A_1} := \inf\{C : Mw(x) \leq Cw(x)\ \forall x\}.$$

A quick application of Hölder's inequality gives us that for any $p \geq 1$, if $w \in A_p$ then $w \in A_q$ for all $q > p$. With this in mind we define the following class of weights,

$$A_\infty = \cup_{1<p<\infty} A_p.$$

Since the constant functions satisfy the A_1 condition, we have that the unweighted setting is in fact a special case of the weighted one for any Lebesgue space.

These weights are of particular use for a variety of reasons, but there are two which stand out, namely the boundedness of M on $L^p(w)$, and extrapolation. To be more precise, A_p weights are precisely the weights for which the Hardy Littlewood maximal function is bounded, so if one can obtain pointwise control of their operator by the maximal function, then they can immediately conclude boundedness for all weights, for all $1 < p < \infty$. Should this prove difficult we have the following theorem, first proved by J. L. Rubio de Francia in [25].

Theorem 1.4.1 *Suppose that for a fixed $1 < p < \infty$ an operator T is bounded from $L^p(w) \to L^p(w)$ for all $w \in A_p$, then T is bounded from $L^q(w)$ to $L^q(w)$ for any $w \in A_q$, for any $1 < q < \infty$.*

The benefit of these two facts are immediately clear, especially with previous observation that the constant function 1 is an A_p weight for all p, so we gain the standard unweighted bounds on L^p.

We now move on to defining variations of these Muckenhoupt weights first introduced by Muckenhoupt and Wheeden in [22]. For $1 < p \leq q < \infty$, the weight w is in $A_{p,q}$ if

$$[w]_{A_{p,q}} := \sup_Q \left(\fint_Q w^q \right) \left(\fint_Q w^{-p'} \right)^{q/p'} < \infty,$$

and it can quickly be shown that

$$[w]_{A_{p,q}} = [w^q]_{A_{1+q/p'}}.$$

These are exactly the weights for which the fractional integral operators are bounded from L^p to L^q, and are also the weights for which the so-called fractional maximal function,

$$M_\alpha f(x) := \sup_{Q: x \in Q} \frac{1}{|Q|^{1-\alpha}} \int_Q |f(y)| dy,$$

is bounded on those same spaces. In both of these instances, the Sobolev relation

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

must be satisfied.

We now extend these to the vector weights used in the bilinear setting, and as usual, it is clear from context how one would generalize them to fully suit the m-linear setting. For $1 < p_1, p_2 < \infty$, $\mathbf{P} = (p_1, p_2)$, and p such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, a vector weight $\mathbf{w} = (w_1, w_2)$ belongs to $\mathbf{A_P}$ if

$$[\mathbf{w}]_{\mathbf{A_P}} := \sup_Q \left(\fint_Q w_1^{p/p_1} w_2^{p/p_2} \right) \left(\fint_Q w_1^{1-p_1'} \right)^{p/p_1'} \left(\fint_Q w_2^{1-p_2'} \right)^{p/p_2'} < \infty.$$

For brevity, we will often use the notation $\nu_{\mathbf{w}} = w_1^{p/p_1} w_2^{p/p_2}$ in the first integral. In [20] it was shown by Lerner et al. that for $\mathbf{w} \in \mathbf{A_P}$, it holds that

$$\nu_{\mathbf{w}} \in A_{2p},$$

$$w_j^{1-p_j'} \in A_{2p_j'},\ j = 1,2,$$

and that

$$A_{p_1} \times A_{p_2} \subsetneq \mathbf{A_P} \subsetneq \mathbf{A}_{c\mathbf{P}},$$

for $c > 1$.

As with our other weights, there is a corresponding maximal function which is bounded precisely on these weights. In this case it is given by

$$\mathscr{M}(f,g)(x) := \sup_{Q:x\in Q} \left(\fint_Q |f| \right) \left(\fint_Q |g| \right).$$

A strong indication that this class of weights is in some sense the correct class of weights to consider for multilinear Calderón-Zygmund operators, is that they are also precisely the weights for which multilinear Calderón-Zygmund operators are bounded. In this light, it makes sense that these multilinear weights have been used as a template to obtain a multilinear analog to the $A_{p,q}$.

For $1 < p_1, p_2 < \infty$, $\mathbf{P} = (p_1, p_2)$, $0 < \alpha < 2n$, $\frac{\alpha}{n} < \frac{1}{p_1} + \frac{1}{p_2}$, and q such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$, a vector weight $\mathbf{w} = (w_1, w_2)$ belongs to $\mathbf{A}_{\mathbf{P},q}$ if

$$[\mathbf{w}]_{\mathbf{A}_{\mathbf{P},q}} := \sup_Q \left(\fint_Q w_1^q w_2^q \right) \left(\fint_Q w_1^{-p_1'} \right)^{q/p_1'} \left(\fint_Q w_2^{-p_2'} \right)^{q/p_2'} < \infty.$$

As with the $A_{\mathbf{P}}$ weights, for brevity we will use $\mu_{\mathbf{w}} = w_1^q w_2^q$. We reiterate for clarity that will use $\nu_{\mathbf{w}}$ when dealing with $\mathbf{A_P}$ classes and $\mu_{\mathbf{w}}$ with $\mathbf{A}_{\mathbf{P},q}$ ones. It was shown by Moen in [21] that if $\mathbf{w} \in \mathbf{A}_{\mathbf{P},q}$ then $w_i^{-p_i'} \in A_{2p_i'}$ and $\mu_{\mathbf{w}} \in A_{2q}$. In addition, as in the linear case, we have that the weights in $\mathbf{A}_{\mathbf{P},q}$ are precisely those for which

$$I_\alpha : L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^q(\mu_{\mathbf{w}})$$

is bounded.

Also mirroring the linear situation, we have that the following maximal function

$$\mathscr{M}_\alpha(f,g)(x) = \sup_{Q:x\in Q} |Q|^{\alpha/n} \left(⨍_Q |f(y)|\,dy \right) \left(⨍_Q |g(z)|\,dz \right),$$

also satisfies the weighted bounds

$$\mathscr{M}_\alpha : L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^q(\mu_{\mathbf{w}})$$

for the same parameter as I_α, for any weights in $\mathbf{A}_{\mathbf{P},q}$. This was again shown by Moen in [21].

The classes $\mathbf{A}_{\mathbf{P},q}$ are also the natural ones to establish the boundedness of commutators of bilinear fractional integral operators. It was first shown in [9] that given $0 < \alpha < 2n$, $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/p - \alpha/n$, if $(w_1^r, w_2^r) \in \mathbf{A}_{\mathbf{P}/r,q/r}$ for some $r > 1$ with $0 < r\alpha < 2n$, and $\mu_{\mathbf{w}} \in A_\infty$, then

$$[b, I_\alpha]_j : L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^q(\mu_{\mathbf{w}}).$$

Moreover, the operator norm satisfies

$$\|[b, I_\alpha]_j\| \lesssim \|b\|_{BMO}. \tag{1.4}$$

This result was later improved in [8], and they were able to remove the above bump condition involving $r > 1$. This requires a simple argument based on reverse Hölder inequality, as used in the work [20] when dealing with similar situation for the classes $\mathbf{A_P}$. In fact, such condition is always satisfied: for $(w_1, w_2) \in \mathbf{A}_{\mathbf{P},q}$ there exist an appropriate $r > 1$, depending on (w_1, w_2), such that $(w_1^r, w_2^r) \in \mathbf{A}_{\mathbf{P}/r,q/r}$; while it is also true that $(w_1^r, w_2^r) \in \mathbf{A}_{\mathbf{P}/r,q/r}$ always implies $(w_1, w_2) \in \mathbf{A}_{\mathbf{P},q}$ for all $r > 1$.

We now show two important properties of the weights which we will be using in our final chapter.

Lemma 1.4.2 *Let $1 < p_1, p_2 < \infty$, $\mathbf{P} = (p_1, p_2)$, $0 < \alpha < 2n$, $\frac{\alpha}{n} < \frac{1}{p_1} + \frac{1}{p_2}$, and q such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$. Suppose that $w_1^{\frac{p_1 q}{p}}, w_2^{\frac{p_2 q}{p}} \in A_p$. Then,*

(i) $\mathbf{w} = (w_1, w_2) \in \mathbf{A}_{\mathbf{P},q}$,

(ii) $\mu_{\mathbf{w}} = w_1^q w_2^q \in A_p \subset A_q$.

Proof: Note that $p < \min\{p_1, p_2\}$, so $(w_1^{\frac{p_1 q}{p}}, w_2^{\frac{p_2 q}{p}}) \in \mathbf{A_P}$, and we have

$$\begin{aligned}
&\left(\fint_Q (w_1 w_2)^q\right) \prod_{i=1}^{2} \left(\fint_Q w_i^{-p_i'}\right)^{q/p_i'} \\
&\qquad = \left(\fint_Q \left(w_1^{p_1 q/p}\right)^{p/p_1} \left(w_2^{p_2 q/p}\right)^{p/p_2}\right) \prod_{i=1}^{2} \left(\fint_Q w_i^{-p_i'}\right)^{q/p_i'} \\
&\qquad \leq \left(\fint_Q \left(w_1^{p_1 q/p}\right)^{p/p_1} \left(w_2^{p_2 q/p}\right)^{p/p_2}\right) \prod_{i=1}^{2} \left(\fint_Q \left(w_i^{p_i q/p}\right)^{1-p_i'}\right)^{p/p_i'} \\
&\qquad = \left[\left(w_1^{p_1 q/p}, w_2^{p_2 q/p}\right)\right]_{\mathbf{A_P}} < \infty.
\end{aligned}$$

A quick application of Hölder's inequality to the A_p condition shows that

$$[\mu_{\mathbf{w}}]_{A_p} = \sup_Q \left(\fint_Q w_1^q w_2^q \right) \left(\fint_Q (w_1 w_2)^{q(1-p')} \right)^{p/p'}$$

$$\leq \left[w_1^{\frac{p_1 q}{p}} \right]_{A_p}^{\frac{p}{p_1}} \left[w_2^{\frac{p_2 q}{p}} \right]_{A_p}^{\frac{p}{p_2}} < \infty,$$

and since $q > p$, we also have $w_1^q w_2^q \in A_q$.

□

Next, we introduce an operator that is of great use for our purposes in this weighted setting. Namely, as is often the case when dealing with compactness of singular integrals (see [2] and the references therein), we find it convenient to use smooth truncations of I_α. Following the construction in [2] it is possible to approximate I_α by operators I_α^δ defined by a smooth kernel $K^\delta(x,y,z)$ in $\mathbb{R}^{3n}$ such that

$$K^\delta(x,y,z) = \frac{1}{(|x-y|^2 + |x-z|^2)^{n-\alpha/2}}$$

for $\max(|x-y|, |x-z|) > 2\delta$;

$$K^\delta(x,y,z) = 0$$

for $\max(|x-y|, |x-z|) < \delta$; and

$$|\partial^\gamma K^\delta(x,y,z)| \lesssim \frac{1}{(|x-y| + |x-z|)^{2n-\alpha+|\gamma|}}$$

for all (x,y,z) and all multi-indexes with $|\gamma| \leq 1$. It is important to note that all of the constants associated with the above estimates are completely independent of δ.

The operators I_α^δ approximate I_α in the following sense.

Lemma 1.4.3 *If $b \in C_c^\infty$ and $\mathbf{w} \in \mathbf{A}_{\mathbf{P},q}$, then*

$$\lim_{\delta \to 0} \|[b, I_\alpha^\delta] - [b, I_\alpha]\|_{L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^p(\mu_{\mathbf{w}})} = 0.$$

The proof of this result is very similar to that of Lemma 2.1 in [2] and so we omit it here.

Finally, we conclude our preliminaries by stating the criteria for compactness which was promised in the previous section.We use the following definition of compactness in the bilinear setting. In weighted L^q spaces we can conclude the compactness of an operator by the following weighted version of the Frechét-Kolmogorov-Riesz theorem. We refer to the works by Hanche-Olsen and Holden[16] and Clop and Cruz [11], and present it as it appeared in the latter work.

Theorem 1.4.4 *(Clop-Cruz) Let $1 < q < \infty$ and $w \in A_q$ and let $\mathscr{K} \subset L^q(w)$. If*

$$\mathscr{K} \text{ is bounded in } L^q(w); \tag{1.5}$$

$$\lim_{A \to \infty} \int_{|x|>A} |f(x)|^q w(x)\, dx = 0 \text{ uniformly for } f \in \mathscr{K}; \tag{1.6}$$

$$\lim_{t \to 0} \|f(\cdot + t) - f\|_{L^q(w)} = 0 \text{ uniformly for } f \in \mathscr{K}; \tag{1.7}$$

then $\mathscr{K}$ is pre-compact in $L^q(w)$.

Chapter 2

Necessity of BMO for the boundedness of commutators

In this chapter we characterize BMO in terms of the boundedness of commutators of various bilinear singular integral operators with pointwise multiplication. In particular, we study commutators of a wide class of bilinear operators of convolution type, including bilinear Calderón-Zygmund operators and the bilinear fractional integral operators. The work in this chapter will appear in [4].

2.1 Theorem and Initial Remarks

We begin by stating our main theorem for this chapter.

Theorem 2.1.1 *For $b \in L^1_{loc}(\mathbb{R}^n)$, and T a bilinear operator defined on $L^{p_1} \times L^{p_2}$ which can be represented as*

$$T(f,g)(x) = \int K(x-y, x-z) f(y) g(z) dy dz$$

for all $x \notin \text{supp}(f) \cap \text{supp}(g)$, where K is a positively homogeneous kernel of degree $-2n+\alpha$, and such that on some ball, $B \subset \mathbb{R}^{2n}$ we have that the Fourier series of $\frac{1}{K}$ is absolutely convergent. We then have that for $1 > \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$, and for $j = 1$ or 2,

$$[b,T]_j : L^{p_1} \times L^{p_2} \to L^q \implies b \in BMO(\mathbb{R}^n).$$

It is worth noting that the condition on the Fourier coefficients of the kernel will, for example, be satisfied if K is smooth, and this is the assumption that similar arguments have used in the past. For $\alpha = 0$, this theorem includes the case where the operator is a bilinear Calderón-Zygmund operator, whereas if $0 < \alpha < 2n$, it includes the case where it is either of the bilinear fractional integral operators.

2.2 Proof of the theorem

The proof of theorem 2.1.1 uses techniques applied by Janson in [18], modified to suit the multilinear setting and extended for kernels with different homogeneity. We note that by symmetry, it is enough to prove this for $[b,T]_1$.

Proof: [Proof of Theorem 2.1.1] Let $B = B((y_0,z_0), \delta\sqrt{2n}) \subset \mathbb{R}^{2n}$, be the ball for which we can express $\frac{1}{K(y,z)}$ as an absolutely convergent Fourier series of the form

$$\frac{1}{K(y,z)} = \sum_j a_j e^{\nu_j \cdot (y,z)}.$$

The specific vectors, ν_j, will not play a role in this proof. Note that due to the homogeneity of K, we can take (y_0,z_0) such that $|(y_0,z_0)| > 2\sqrt{n}$ and take $\delta < 1$ small such that $\overline{B} \cap \{0\} = \emptyset$. As was said above, we do not care about the specific vectors $\nu_j \in \mathbb{R}^{2n}$, but we will at times express them as $\nu_j = (\nu_j^1, \nu_j^2) \in \mathbb{R}^n \times \mathbb{R}^n$.

Set $y_1 = \delta^{-1} y_0$ and $z_1 = \delta^{-1} z_0$, and note that

$$\left(|y-y_1|^2 + |z-z_1|^2\right)^{1/2} < \sqrt{2n} \implies \left(|\delta y - y_0|^2 + |\delta z - z_0|^2\right)^{1/2} < \delta\sqrt{2n},$$

and so for all (y,z) satisfying the inequality on the left, we have

$$\frac{1}{K(y,z)} = \frac{\delta^{-2n+\alpha}}{K(\delta y, \delta z)} = \delta^{-2n+\alpha} \sum_j a_j e^{i\delta \nu_j \cdot (y,z)}.$$

Let $Q = Q(x_0, r)$ be an arbitrary cube in $\mathbb{R}^n$. Set $\tilde{y} = x_0 + ry_1$, $\tilde{z} = x_0 + rz_1$, and take $Q' = Q(\tilde{y}, r) \subset \mathbb{R}^n$ and $Q'' = Q(\tilde{z}, r) \subset \mathbb{R}^n$. Then for any $x \in Q$ and $y \in Q'$, we have

$$\left|\frac{x-y}{r} - y_1\right| \leq \left|\frac{x-x_0}{r}\right| + \left|\frac{y-\tilde{y}}{r}\right| \leq \sqrt{n}.$$

The same estimate holds for $x \in Q$ and $z \in Q''$, and so we have

$$\left(\left|\frac{x-y}{r} - y_1\right|^2 + \left|\frac{x-z}{r} - z_1\right|^2\right)^{1/2} \leq \sqrt{2n}.$$

Let $\sigma(x) = \text{sgn}(b(x) - b_{Q'})$.

To restate what we did above, if we are given a ball on which our Fourier series is absolutely convergent, by homogeneity we can send that ball far from the origin. From this we can choose a ball, B, contained inside it for which the Fourier series of $\frac{1}{K}$ is still absolutely convergent, and which is both small enough and far enough from the origin to have beneficial properties which we will be able to take advantage of. Then, given an arbitrary cube $Q = Q(x_0, r)$, we developed an explicit way of generating other cubes, Q' and Q'' of the same size as Q such that $x \in Q$, $y \in Q'$, and $z \in Q''$ guarantees that $\left(\frac{x-y}{r}, \frac{x-z}{r}\right)$ lies in a ball for which the Fourier series of $\frac{1}{K}$ is absolutely convergent. We will see that the initial distance condition on B is enough to guarantee that $Q \cap Q' \cap Q'' = \emptyset$, and so we will be able to use the integral representation of our operator for functions supported on these cubes when the time comes.

We now have the following,

$$\begin{aligned}
&\int_Q |b(x) - b_{Q'}|\, dx \\
&\qquad = \int_Q (b(x) - b_{Q'})\sigma(x)\, dx \\
&\qquad = \frac{1}{|Q''|}\frac{1}{|Q'|}\int_Q \int_{Q'} \int_{Q''} (b(x) - b(y))\sigma(x)\, dzdydx \\
&\qquad = r^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{r^{2n-\alpha} K(x-y, x-z)}{K\left(\frac{x-y}{r}, \frac{x-z}{r}\right)} \\
&\qquad\qquad \cdot \sigma(x)\chi_Q(x)\chi_{Q'}(y)\chi_{Q''}(z)\, dzdydx \\
&\qquad = \delta^{-2n+\alpha} r^{-\alpha} \int\int\int (b(x) - b(y)) K(x-y, x-z) \sum_j a_j e^{i\frac{\delta}{r}\nu_j \cdot (x-y, x-z)} \\
&\qquad\qquad \cdot \sigma(x)\chi_Q(x)\chi_{Q'}(y)\chi_{Q''}(z)\, dzdydx
\end{aligned}$$

Let

$$\begin{aligned}
f_j(y) &= e^{-i\frac{\delta}{r}\nu_j^1 \cdot y}\chi_{Q'}(y) \\
g_j(z) &= e^{-i\frac{\delta}{r}\nu_j^2 \cdot z}\chi_{Q''}(z) \\
h_j(x) &= e^{i\frac{\delta}{r}\nu_j \cdot (x,x)}\sigma(x)\chi_Q(x)
\end{aligned}$$

Note that Each of the above functions has an L^q norm of $|Q|^{1/q}$ for any $q \geq 1$. Recall that Q, Q', and Q'' all have side length r, we will now show that $Q \cap Q' \cap Q'' = \emptyset$ by showing that either $|x_0-\tilde{y}|>r\sqrt{n}$ or $|x_0-\tilde{z}|>r\sqrt{n}$. Note that the size condition on (y_0,z_0) means that either $|y_0|>\sqrt{n}$ or $|z_0|>\sqrt{n}$. If $|y_0|>\sqrt{n}$, then

$$|x_0-\tilde{y}|=\left|x_0-x_0+r\frac{y_0}{\delta}\right|\geq r|y_0|>r\sqrt{n},$$

with an identical calculation if $z_0>\sqrt{n}$. Therefore we have that $Q\cap Q'\cap Q''=\emptyset$ since at least one of Q' and Q'' must be disjoint from Q, and so for all x, y, and z in the supports of their respective characteristic functions, $(x-y,x-z)$ avoids the singularity of K. In particular, this means that the use of the kernel representation of $[b,T](f_j,g_j)$ is valid for all $x\in Q$. Continuing with our above calculations, we have,

$$\begin{aligned}
\int_Q &|b(x)-b_{Q'}|\,dx \\
&=\delta^{-2n+\alpha}r^{-\alpha}\sum_j a_j\int h_j(x)\int\int(b(x)-b(y)) \\
&\qquad\cdot K(x-y,x-z)f_j(y)g_j(z)\,dzdydx \\
&=\delta^{-2n+\alpha}|Q|^{-\frac{\alpha}{n}}\sum_j a_j\int h_j(x)[b,T](f_j,g_j)(x)\,dx \\
&\leq\delta^{-2n+\alpha}|Q|^{-\frac{\alpha}{n}}\sum_j |a_j|\int |h_j(x)||[b,T](f_j,g_j)(x)|\,dx \\
&\leq\delta^{-2n+\alpha}|Q|^{-\frac{\alpha}{n}}\sum_j |a_j|\left(\int|h_j(x)|^{q'}\,dx\right)^{\frac{1}{q'}}\left(\int|[b,T](f_j,g_j)(x)|^{q}\,dx\right)^{\frac{1}{q}} \\
&\leq\delta^{-2n+\alpha}|Q|^{-\frac{\alpha}{n}}\sum_j |a_j|\|h_j\|_{L^{q'}}\|[b,T]\|_{L^{p_1}\times L^{p_2}\to L^p}\|f_j\|_{L^{p_1}}\|g_j\|_{L^{p_2}} \\
&=\delta^{-2n+\alpha}\|[b,T]\|_{L^{p_1}\times L^{p_2}\to L^p}\sum_j |a_j||Q|^{\frac{1}{q'}}|Q|^{\frac{1}{p_1}}|Q|^{\frac{1}{p_2}}|Q|^{-\frac{\alpha}{n}} \\
&=\delta^{-2n+\alpha}|Q|\|[b,T]\|_{L^{p_1}\times L^{p_2}\to L^p}\sum_j |a_j|
\end{aligned}$$

Recall that $\frac{1}{|Q|}\int_Q |b(x)-b_Q|\,dx \leq \frac{2}{|Q|}\int_Q |f(x)-C|$ for any C, and so this gives us that for any arbitrary $Q \subset \mathbb{R}^n$ we have

$$\frac{1}{|Q|}\int_Q |b(x)-b_Q| \leq \frac{2}{|Q|}\int_Q |b(x)-b_{Q'}|\,dx \leq 2\|[b,T]\|_{L^{p_1}\times L^{p_2}\to L^p}\sum_j |a_j|.$$

Therefore $b \in \mathrm{BMO}(\mathbb{R}^n)$ □

2.3 Corollaries and remarks

In Proposition 3.1 of [24], C. Pérez and R. H. Torres showed that $b \in BMO$ was sufficient to show the boundedness of commutators with m-linear Calderón-Zygmund operators, which we state in a simpler bilinear format without proof.

Proposition 2.3.1 *If T is a bilinear Calderón-Zygmund operator and $b \in BMO$, then $[b,T]_j : L^{p_1}\times L^{p_2} \to L^p$, for $j = 1$ or 2 and $\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}$ with $1 < p, p_1, p_2 < \infty$.*

This, combined with Theorem 2.1.1, immediately gives us the following.

Corollary 2.3.2 *Let $b \in L^1_{loc}(\mathbb{R}^n)$, and T a bilinear Calderón-Zygmund operator of convolution type with kernel, K, a homogeneous function of degree $-2n$. Suppose that on some ball, B, in $\mathbb{R}^{2n}$ we have that the Fourier series of $\frac{1}{K}$ is absolutely convergent. Then for $1 > \frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}$, and $j = 1$ or 2,*

$$[b,T]_j : L^{p_1}\times L^{p_2} \to L^p \iff b \in BMO(\mathbb{R}^n).$$

Note that the kernel conditions in the statements of this corollary is not present in the statement of Proposition 2.3.1, whereas the requirement that T be a Calderón-Zygmund operator is not needed for Theorem 2.1.1.

For $T = I_\alpha$, the sufficiency of $b \in$BMO to conclude the boundedness of $[b, I_\alpha]_i$ was shown by X. Chen and Q. Xue in [9], Theorem 2.7. As before, we state without proof a particular case of this theorem which suits our needs,

Proposition 2.3.3 *Let $0 < \alpha < 2n$, and $1 \leq p_1,\ p_2$, and q be such that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q}$. Then*

$$\|[b, I_\alpha]_j(f, g)\|_{L^q} \lesssim \|b\|_* \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for $j = 1$ or 2.

The kernel of I_α has precisely the homogeneity required by Theorem 2.1.1, and the reciprocal of the convolution kernel of I_α, $(|y|^2 + |z|^2)^{n-\alpha/2}$, is smooth away from the origin and so its Fourier series will indeed have regions on which it is absolutely convergent. These facts give us the following result.

Corollary 2.3.4 *For $b \in L^1_{loc}$, $0 < \alpha < 2n$ and $1 < p_1, p_2$, and q satisfying*

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q} < 1,$$

then

$$\|[b, I_\alpha]_j\|_{p_1 \times p_2 \to q} \approx \|b\|_* \quad \text{for } j = 1 \text{ or } 2$$

In particular, for $j = 1$ or 2,

$$[b, I_\alpha]_j : L^{p_1} \times L^{p_2} \to L^q \iff b \in BMO.$$

With regards to our main theorem, a few key things should be noted. First, the proof easily generalizes to commutators with the m-linear operators and homogeneous kernels of degree $-mn + \alpha$. The original statements of Proposition 2.3.1 in [24] and Proposition

2.3.3 in [9] ares for m-linear commutators, so Corollaries 2.3.2 and 2.3.4 hold in the m-linear setting as well. As a quick aside, in the linear case we recall that in [7], the necessity that $b \in BMO$ for the boundedness of the commutator was only shown when $n - \alpha$ was an even integer. Chanillo then went on to remark that this restriction seemed artificial and it should be true for any α. Indeed, others have taken the truth of this more general version for granted, as in [19]. In that work Lacey provided a new proof that did achieve a full characterization of boundedness in terms of the BMO norm of b, but this was only in the dimension one setting. As far as the author is aware, it has never been shown explicitly in the past for functions defined on $\mathbb{R}^n$. Our proof also applies to the linear case $m = 1$, hence giving the result of Chanillo for all α and all dimensions.

Second we observe that since our proof required the use of Hölder's inequality with q and q', the exponent in our target space *must* be larger than 1. As a result, it remains unclear whether or not it is possible to characterize BMO in terms of the boundedness of commutators for $L^{p_1} \times L^{p_2} \to L^q$ for $\frac{1}{2} < q < 1$. This is of interest because bounds of this form have indeed been shown. In particular, in [20], Lerner et al. showed that commutators with m-linear Calderón-Zygmund operators are bounded from $\prod_{j=1}^{m} L^{p_j}$ to L^p, for any $1 < p_1, \ldots, p_m$ such that $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$, provided that $b \in$ BMO. In [26], Tang obtained this result for commutators of vector valued multilinear Calderón-Zygmund operators, again without the restriction that p be greater than 1.

Finally, we note –perhaps as nothing more than a novelty– that we only assume the commutator is bounded, not the underlying operator, and the non-negativity of α never played a role. On the surface this is immaterial, as one does not often look at commutators of unbounded operators, and indeed, the operators corresponding to negative α are hyper-singular, and not bounded in any Lebesgue space. However, this result means that if an operator can be formally expressed as

$$T(f)(x) = \int (b(x) - b(y)) K(x-y) f(y) dy,$$

for some function b, and K with appropriate homogeneity corresponding to a negative α, the boundedness of T would require that in addition to any other conditions exhibited by b it must also be in BMO. As was stated at the beginning of this paragraph, this is perhaps nothing more than a novelty; the p values would have to be large and $|\alpha|$ would have to be correspondingly small to guarantee that $q > 1$, and it does not seem likely that such an operator could be bounded or useful, but it is perhaps an interesting notion to consider.

Chapter 3

Characterization of CMO in terms of compactness of $[b, I_\alpha]_i$

In this chapter the compactness of our commutators with bilinear fractional integral operators is characterized in terms of appropriate mean oscillation properties of their symbols, and the compactness of the commutators when acting on product of weighted Lebesgue spaces is also studied. The work appearing in this chapter is from [5], a joint work with R. H. Torres.

3.1 Theorem and Proof

As before, we begin by stating our main theorem for the chapter.

Theorem 3.1.1 *Let $1 < p_1, p_2 < \infty$, $\mathbf{P} = (p_1, p_2)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $0 < \alpha < 2n$, $\frac{\alpha}{n} < \frac{1}{p_1} + \frac{1}{p_2}$, and q such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$ and $1 < p, q < \infty$. Then the following are equivalent,*

(i) *$b \in CMO$.*

(ii) $[b,I_\alpha]_1 : L^{p_1}(w^{p_1}) \times L^{p_2}(w^{p_2}) \to L^q(w_1^q w_2^q)$ *is a compact operator for all* $\mathbf{w} = (w_1, w_2)$ *such that* $w_1^{\frac{p_1 q}{p}}, w_2^{\frac{p_2 q}{p}} \in A_p$.

(iii) $[b,I_\alpha]_1 : L^{p_1} \times L^{p_2} \to L^q$ *is a compact operator.*

Proof: Recall that as was shown in [3] and like in the linear setting, the limit of compact operators in the operator norm is compact. Since by definition CMO is the closure of C^∞ with compact support, the estimate in Proposition 2.3.3 implies that it is enough to prove results for b smooth. Furthermore, because of Lemma 1.4.3, we can consider $[b,I_\alpha^\delta]$ with b smooth wherever it is convenient to do so. We will show that the image of $B_1(L^{p_1}(w_1^{p_1})) \times B_1(L^{p_2}(w_2^{p_2}))$ under $[b,I_\alpha^\delta]_1$ verifies the Frechét-Kolmogorov-Riesz conditions in $L^q(\mu_{\mathbf{w}})$. The approach for this part is similar to the one taken by Bényi et al. in [2], but we need to carefully use the properties of the weights established in Lemma 1.4.2.

For the convenience of the reader, we repeat these conditions as they appeared in our first chapter, though with the notation of our current setting.

Let $1 < q < \infty$ *and* $\mu_{\mathbf{w}} \in A_q$ *and let*

$$\mathscr{K} = [b,I_\alpha^\delta]_1 \left(B_1(L^{p_1}(w_1^{p_1})) \times B_1(L^{p_2}(w_2^{p_2}))\right) \subset L^q(\mu_{\mathbf{w}}).$$

If

$$\mathscr{K} \text{ is bounded in } L^q(\mu_{\mathbf{w}}); \tag{1.5}$$

$$\lim_{A\to\infty} \int_{|x|>A} |[b, I_\alpha^\delta]_1(f,g)(x)|^q \, \mu_{\mathbf{w}}(x)\, dx = 0$$

$$\text{uniformly for } f \in B_1(L^{p_1}(w_1^{p_1}),\ g \in B_1(L^{p_2}(w_2^{p_2})); \tag{1.6}$$

$$\lim_{t\to 0} \|[b, I_\alpha^\delta]_1(f,g)(\cdot + t) - [b, I_\alpha^\delta]_1(f,g)\|_{L^q(w)} = 0$$

$$\text{uniformly for } f \in B_1(L^{p_1}(w_1^{p_1}),\ g \in B_1(L^{p_2}(w_2^{p_2})); \tag{1.7}$$

then $\mathscr{K}$ *is pre-compact in* $L^q(\mu_{\mathbf{w}})$.

Note that (1.5) is immediate since for $b \in C_c^\infty$, $[b, I_\alpha^\delta]_1$ is bounded from $L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2})$ to $L^q(\mu_{\mathbf{w}})$, because $\mathbf{w} \in \mathbf{A}_{\mathbf{P},q}$ by Lemma 1.4.2.

To show that (1.6) holds, choose r large so that $\operatorname{supp} b \subset B_r(0)$, then for $|x| > R \geq \max\{2r, 1\}$, we have

$$
\begin{aligned}
|[b,I_\alpha^\delta](f,g)(x)| &\lesssim \int_{\operatorname{supp} b} \int_{\mathbb{R}^n} \frac{|b(y)||f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-\alpha}} dzdy \\
&\lesssim \|b\|_\infty \int_{\operatorname{supp} b} |f(y)| \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x|+|x-z|)^{2n-\alpha}} dzdy \\
&\lesssim \|b\|_\infty \|f\|_{L^{p_1}(w_1^{p_1})} \left(\int_{B_r(0)} w_1^{-p_1'} dy \right)^{1/p_1'} \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x|+|x-z|)^{2n-\alpha}} dz \\
&\lesssim \frac{\|b\|_\infty}{|x|^{n-\alpha}} \|f\|_{L^{p_1}(w_1^{p_1})} \left(\int_{B_r(0)} w_1^{-p_1'} dy \right)^{1/p_1'} \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x|+|x-z|)^{n}} dz \\
&\lesssim \frac{\|b\|_\infty}{|x|^{n-\alpha}} \|f\|_{L^{p_1}(w_1^{p_1})} \left(\int_{B_r(0)} w_1^{-p_1'} dy \right)^{1/p_1'} \int_{\mathbb{R}^n} \frac{|g(z)|}{(1+|z|)^{n}} dz \\
&\lesssim \frac{\|b\|_\infty}{|x|^{n-\alpha}} \|f\|_{L^{p_1}(w_1^{p_1})} \|g\|_{L^{p_2}(w_2^{p_2})} \left(\int_{B_r(0)} w_1^{-p_1'} dy \right)^{\frac{1}{p_1'}} \left\| \frac{1}{(1+|z|)^n} \right\|_{L^{p_2'}(w_2^{-p_2'})}.
\end{aligned}
$$

Note now, that since $w_2^{p_2 q/p} \in A_p \subset A_{p_2}$, we have that $w_2^{-\frac{q}{p} p_2'} = w_2^{(p_2 q/p)(1-p_2')}$ is in $A_{p_2'}$, and since $q/p > 1$, we have that $w_2^{-p_2'} \in A_{p_2'}$ as well. This gives us that

$$
\int_{\mathbb{R}^n} \frac{w_2^{-p_2'}}{(1+|z|)^{np_2'}} dz < \infty,
$$

due to a well known result which can, for instance, be found in page 412 of [13]. And so we have

$$
|[b,I_\alpha^\delta](f,g)(x)| \lesssim \frac{1}{|x|^{n-\alpha}}.
$$

Raising both sides of the last inequality to the power q and integrating over $|x| > R$ we have

$$\int_{|x|>R} |[b, I_\alpha^\delta](f,g)(x)|^q \mu_{\mathbf{w}}\, dx \lesssim \int_{|x|>R} \frac{\mu_{\mathbf{w}}}{|x|^{(n-\alpha)q}} dx = \int_{|x|>R} \frac{\mu_{\mathbf{w}}}{|x|^{\frac{n-\alpha}{n-p\alpha}np}} dx.$$

Note now that $\frac{n-\alpha}{n-p\alpha} > 1$, and that $\mu_{\mathbf{w}}$ is an A_p weight by Lemma 1.4.2, so this quantity tends to zero as $R \to \infty$.

To show (1.7), notice that by adding and subtracting

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} b(x+t) K^\delta(x,y,z) f(y) g(z)\, dydz,$$

we can compute

$$\begin{aligned}
&[b, I_\alpha^\delta](f,g)(x+t) - [b, I_\alpha^\delta](f,g)(x) \\
&\quad = (b(x) - b(x+t)) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K^\delta(x,y,z) f(y) g(z)\, dydz \\
&\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(y) - b(x+t)) f(y) g(z) (K^\delta(x+t,y,z) - K^\delta(x,y,z))\, dydz \\
&\quad = I(x,t) + II(x,t).
\end{aligned}$$

For I, we simply have

$$|I(x,t)| \leq |t| \|\nabla b\|_\infty I_\alpha^\delta(f,g)(x),$$

and since I_α^δ is bounded from $L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2})$ to $L^q(\mu_{\mathbf{w}})$, we have

$$\|I(\cdot,t)\|_{L^q(\mu_{\mathbf{w}})} \lesssim |t|.$$

We now move on to the control of II. We can assume $t < \delta/4$. Hence, because of the properties of K^δ, if $\max(|x-y|,|x-z|) \le \delta/2$ we have

$$K^\delta(x+t,y,z) - K^\delta(x,y,z) = 0,$$

while for $\max(|x-y|,|x-z|) > \delta/2$ we have $\max(|x-y|,|x-z|) > 2t$. We can then estimate II by

$$\begin{aligned}
&\left|\iint (b(y)-b(x+t))(K^\delta(x+t,y,z)-K^\delta(x,y,z))f(y)g(z)\,dydz\right| \\
&\lesssim \|b\|_\infty |t| \iint_{\max\{|x-y|,|x-z|\}>\delta/2} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-\alpha+1}}\,dydz \\
&\lesssim \|b\|_\infty |t| \sum_{j\ge 0} \iint_{2^{j-1}\delta<\max\{|x-y|,|x-z|\}\le 2^j\delta} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-\alpha+1}}\,dydz \\
&\lesssim \|b\|_\infty |t| \sum_{j\ge 0} \left(\int_{2^{j-1}\delta\le|x-y|\le 2^j\delta} \frac{|f(y)|}{|x-y|^{2n-\alpha+1}}\,dy \int_{|x-z|\le 2^j\delta} |g(z)|\,dz \right. \\
&\left. + \int_{|x-y|\le 2^j\delta} |f(y)|\,dy \int_{2^{j-1}\delta\le|x-z|\le 2^j\delta} \frac{|g(z)|}{|x-z|^{2n-\alpha+1}}\,dz \right) \\
&\lesssim \|b\|_{L^\infty} |t| \sum_{j\ge 0} (2^j\delta)^{-2n+\alpha-1} \left(\int_{|x-y|\lesssim 2^j\delta} |f(y)|\,dy \int_{|z-y|\lesssim 2^j\delta} |g(z)|\,dz \right) \\
&\lesssim \|b\|_{L^\infty} \frac{|t|}{\delta} \sum_{j\ge 0} 2^{-j}(2^j\delta)^\alpha \left(⨍_{|x-y|\lesssim 2^j\delta} |f(y)|\,dy ⨍_{|z-y|\lesssim 2^j\delta} |g(z)|\,dz \right) \\
&\lesssim \|b\|_{L^\infty} \frac{|t|}{\delta} \mathscr{M}_\alpha(f,g)(x).
\end{aligned}$$

It follows that

$$\|II(\cdot,t)\|_{L^q(\mu_{\mathbf{w}})} \lesssim |t|.$$

Obviously (ii) implies (iii). So it remains to show that (iii) implies (i). To do so we will adapt some arguments from [10], which in turn are based on the original work in [27]. Recall Theorem 1.2.1 due Uchiyama, which states that for $b \in \mathrm{BMO}$, conditions (1.1)-(1.3) characterize CMO. As we did before, we will repeat the statement of the theorem here for the convenience of the reader.

A function b ∈BMO is in CMO if an only if,

$$\lim_{a\to 0}\sup_{|Q|=a}\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx=0, \tag{1.1}$$

$$\lim_{a\to \infty}\sup_{|Q|=a}\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx=0, \tag{1.2}$$

$$\lim_{|y|\to \infty}\frac{1}{|Q|}\int_Q |b(x+y)-b_Q|dx=0, \textit{ for each } Q. \tag{1.3}$$

The approach is as follows: we will show that if we assume that $[b,I_\alpha]_1$ is compact, and b, which we know must be in BMO by theorem 2.1.1, fails to satisfy one of the conditions (1.1)-(1.3), then one can construct sequences of functions, $\{f_j\}_j$ uniformly bounded on L^{p_1}, and $\{g_j\}_j$ uniformly bounded on L^{p_2}, such that $\{[b,I_\alpha]_1(f_j,g_j)\}_j$ has no convergent subsequence. This contradicts the compactness assumption, and so it then follows that if $[b,I_\alpha]_1$ is compact, b must satisfy all three conditions (1.1)-(1.3) and hence be an element of CMO.

Before we construct the sequences, we observe that by linearity in b, it is enough to prove that (iii) implies (i) for b real valued and with $\|b\|_*=1$. So we will assume such conditions.

Given a cube Q_j such that

$$\frac{1}{|Q_j|}\int_{Q_j}|b(x)-b_{Q_j}|dx\geq \varepsilon, \tag{3.1}$$

for some $\varepsilon > 0$, we define

$$f_j(y) = |Q_j|^{-1/p_1} \left(\operatorname{sgn}(b(y) - b_{Q_j}) - c_0\right) \chi_{Q_j}(y),$$

where $c_0 = |Q_j|^{-1} \int_{Q_j} \operatorname{sgn}(b(y) - b_{Q_j}) dy$. Note that $-1 < c_0 < 1$, and from this we see that f_j has the following properties,

$$\operatorname{supp} f_j \subset Q_j,$$

$$f_j(y)(b(y) - b_{Q_j}) \geq 0,$$

$$\int f_j(y) dy = 0,$$

$$\int (b(y) - b_{Q_j}) f_j(y) dy = |Q_j|^{-1/p_1} \int_{Q_j} |b(y) - b_{Q_j}| dy$$

$$|f_j(y)| \leq 2|Q_j|^{-1/p_1}$$

This last property gives us that $\|f_j\|_{L^{p_1}} \leq 2$. For the other functions, we will simply define

$$g_j = \frac{\chi_{Q_j}}{|Q_j|^{1/p_2}},$$

which satisfies $\|g_j\|_{L^{p_2}} = 1$.

Next we establish several technical estimates. For a cube Q_j with center y_j and satisfying (3.1) for some $\varepsilon > 0$, f_j and g_j as above, and all $x \in \left(2\sqrt{n} Q_j\right)^c$, the following point-wise estimates hold:

$$|I_\alpha((b-b_{Q_j})f_j,g_j)(x)| \lesssim |Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}}|x-y_j|^{-2n+\alpha}, \tag{3.2}$$

$$|I_\alpha((b-b_{Q_j})f_j,g_j)(x)| \gtrsim \varepsilon|Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}}|x-y_j|^{-2n+\alpha}, \tag{3.3}$$

$$|I_\alpha(f_j,g_j)(x)| \lesssim |Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}+\frac{1}{n}}|x-y_j|^{-2n+\alpha-1}, \tag{3.4}$$

where the constants involved are independent of b, f_j, g_j and ε.

To prove (3.2), we use that $|x-y_j| \approx |x-y|$ for all $y \in Q_j$ and that $\|b\|_* = 1$ to obtain

$$\begin{aligned}
|I_\alpha((b-b_{Q_j})f_j,g_j)(x)| &= \left|\int\int \frac{(b(y)-b_{Q_j})f_j(y)g_j(z)}{(|x-y|^2+|x-z|^2)^{n-\alpha/2}}dydz\right| \\
&\lesssim \frac{1}{|Q_j|^{\frac{1}{p_1}+\frac{1}{p_2}}}|x-y_j|^{-2n+\alpha}\int_{Q_j}\int_{Q_j}|b(y)-b_{Q_j}|dydz \\
&\lesssim |Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}}|x-y_j|^{-2n+\alpha}.
\end{aligned}$$

Using that $(b(y)-b_{Q_j})f_j(y) \geq 0$, we can also estimate

$$\begin{aligned}
|I_\alpha((b-b_{Q_j})f_j,g_j)(x)| &= \left|\int\int \frac{(b(y)-b_{Q_j})f_j(y)g_j(z)}{(|x-y|^2+|x-z|^2)^{n-\alpha/2}}dydz\right| \\
&\gtrsim |Q_j|^{1-\frac{1}{p_2}}|x-y_j|^{-2n+\alpha}\left|\int_{Q_j}(b(y)-b_{Q_j})f_j(y)dy\right| \\
&= |Q_j|^{1-\frac{1}{p_2}}|x-y_j|^{-2n+\alpha}\int_{Q_j}(b(y)-b_{Q_j})f_j(y)dy \\
&= |Q_j|^{1-\frac{1}{p_2}}|x-y_j|^{-2n+\alpha}|Q_j|^{1-\frac{1}{p_1}}\frac{1}{|Q_j|}\int_{Q_j}|(b(y)-b_{Q_j})|dy \\
&\geq |Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}}|x-y_j|^{-2n+\alpha}\varepsilon,
\end{aligned}$$

which gives (3.3). Finally using that f_j has mean zero we obtain (3.4) in the following way,

$$
\begin{aligned}
&|I_\alpha(f_j,g_j)(x)| = \left| \int\int \frac{f_j(y)g_j(z)}{(|x-y|^2+|x-z|^2)^{n-\alpha/2}} dydz \right| \\
&= \left| \int \left(\int \frac{f_j(y)g_j(z)}{(|x-y|^2+|x-z|^2)^{n-\alpha/2}} - \frac{f_j(y)g_j(z)}{\left(|x-y_j|^2+|x-z|^2\right)^{n-\alpha/2}} dy \right) dz \right| \\
&\lesssim \int\int \frac{|y-y_j||f_j(y)|g_j(z)}{\left(|x-y_j|+|x-z|\right)^{2n-\alpha+1}} dydz \\
&\lesssim \frac{|Q_j|^{\frac{1}{n}}}{|x-y_j|^{2n-\alpha+1}} \int\int |f_j(y)|g_j(z)dydz \\
&\lesssim |Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}+\frac{1}{n}} |x-y_j|^{-2n+\alpha-1}.
\end{aligned}
$$

Following [27] and [10], we now use the above point-wise estimates (3.2)-(3.4) to prove some L^q-norm inequalities for $[b,I_\alpha]_1(f_j,g_j)$.

For a cube Q_j with center y_j, side length d_j, and satisfying (3.1) for some $\varepsilon > 0$; and f_j and g_j defined as above; there exist constants $\gamma_2 > \gamma_1 > 2$, and $\gamma_3 > 0$, depending only on $p_1,\ p_2,\ n$, and ε, such that

$$
\left(\int_{\gamma_1 d_j < |x-y_j| < \gamma_2 d_j} |[b,I_\alpha]_1(f_j,g_j)(y)|^q dy \right)^{1/q} \geq \gamma_3 \tag{3.5}
$$

$$
\left(\int_{|x-y_j| > \gamma_2 d_j} |[b,I_\alpha]_1(f_j,g_j)(y)|^q dy \right)^{1/q} \leq \frac{\gamma_3}{4} \tag{3.6}
$$

Starting with some $\tilde{\gamma}_1 > 16$, using (3.4) and the fact that $2n - \alpha - n/q > 0$ (since $\frac{1}{p_1} + \frac{1}{p_2} < 2$), we have,

$$
\begin{aligned}
&\left(\int_{|x-y_j|>\tilde{\gamma}_1 d_j} |(b(x)-b_{Q_j})I_\alpha(f_j,g_j)(x)|^q dx\right)^{\frac{1}{q}} \\
&\le C|Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}+\frac{1}{n}} \sum_{s=\lfloor \log_2(\tilde{\gamma}_1)\rfloor}^{\infty} \left(\int_{2^s d_j<|x-y_j|<2^{s+1}d_j} \frac{|b(x)-b_{Q_j}|^q}{|x-y_j|^{q(2n-\alpha+1)}} dx\right)^{\frac{1}{q}} \\
&\le C|Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}+\frac{1}{n}} \times \\
&\quad \sum_{s=\lfloor \log_2(\tilde{\gamma}_1)\rfloor}^{\infty} 2^{-s(2n-\alpha+1)}|Q_j|^{-2+\frac{\alpha}{n}-\frac{1}{n}} \left(\int_{2^s d_j<|x-y_j|<2^{s+1}d_j} |b(x)-b_{Q_j}|^q dx\right)^{\frac{1}{q}} \\
&\le C \sum_{s=\lfloor \log_2(\tilde{\gamma}_1)\rfloor}^{\infty} s 2^{-s(2n-\alpha-\frac{n}{q}+1)} \\
&\le C \sum_{s=\lfloor \log_2(\tilde{\gamma}_1)\rfloor}^{\infty} 2^{-s(2n-\alpha-\frac{n}{q}+\frac{1}{2})},
\end{aligned}
$$

where we have used that for $b \in \mathrm{BMO}$,

$$
\left(\int_{2^s d_j<|x-y_j|<2^{s+1}d_j} |b(x)-b_{Q_j}|^q dx\right)^{\frac{1}{q}} \lesssim s2^{sn/q}|Q_j|^{1/q},
$$

and that $s \le 2^{s/2}$ for $4 \le \lfloor \log_2(\tilde{\gamma}) \rfloor \le s$. We thus obtain

$$
\left(\int_{|x-y_j|>\tilde{\gamma}_1 d_j} |(b(x)-b_{Q_j})I_\alpha(f_j,g_j)(x)|^q dx\right)^{\frac{1}{q}} \le C\tilde{\gamma}_1^{-(2n-\alpha-\frac{n}{q}+\frac{1}{2})}. \tag{3.7}
$$

Next, for $\tilde{\gamma}_2 > \tilde{\gamma}_1$, using (3.3) and (3.7), we obtain the following,

$$\begin{aligned}
&\left(\int_{\tilde{\gamma}_1 d_j<|x-y_j|<\tilde{\gamma}_2 d_j} |[b,I_\alpha]_1(f_j,g_j)(x)|^q dx\right)^{\frac{1}{q}} \\
&\geq C\left(\int_{\tilde{\gamma}_1 d_j<|x-y_j|<\tilde{\gamma}_2 d_j} |I_\alpha\left((b-b_Q)f_j,g_j\right)(x)|^q dx\right)^{\frac{1}{q}} \\
&\quad -C\left(\int_{\tilde{\gamma}_1 d_j<|x-y_j|} |(b(x)-b_Q)I_\alpha(f_j,g_j)(x)|^q dx\right)^{\frac{1}{q}} \\
&\geq C\varepsilon|Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}}\left(\int_{\tilde{\gamma}_1 d_j<|x-y_j|<\tilde{\gamma}_2 d_j} |x-y_j|^{q(-2n+\alpha)} dx\right)^{\frac{1}{q}} \\
&\quad -C\tilde{\gamma}_1^{(-2n+\alpha+n/q-1/2)} \\
&\geq C\varepsilon\left(\tilde{\gamma}_1^{-2nq+n+\alpha q}-\tilde{\gamma}_2^{-2nq+n+\alpha q}\right)^{\frac{1}{q}} - C\tilde{\gamma}_1^{(-2n+\alpha+n/q-1/2)}. \qquad (3.8)
\end{aligned}$$

Using (3.7) and (3.8) we see that we can select γ_1, γ_2 in place of $\tilde{\gamma}_1, \tilde{\gamma}_2$, with $\gamma_2 >> \gamma_1$, so that (3.5) and (3.6) are verified for some $\gamma_3 > 0$.

The final technical estimate we need is the following. Given γ_1, γ_2 in (3.5) and (3.6), there exists a $0 < \beta << \gamma_2$ depending only on $p_1,\ p_2,\ n$, and ε such that for any E measurable such that

$$E \subset \{x : \gamma_1 d_j < |x-y_j| < \gamma_2 d_j\}$$

and $|E|/|Q_j| < \beta^n$, we have

$$\left(\int_E |[b,I_\alpha]_1(f_j,g_j)(y)|^q dy\right)^{1/q} \leq \frac{\gamma_3}{4}. \qquad (3.9)$$

To prove this last inequality we note that if $E \subset \{x : \gamma_1 d_j < |x - y_j| < \gamma_2 d_j\}$ is measurable, we can use (3.2) and (3.4) to get,

$$\begin{aligned}\left(\int_E |[b,I_\alpha]_1(f_j,g_j)(x)|^q dx\right)^{\frac{1}{q}} &\lesssim |Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}}\left(\int_E |x-y_j|^{-q(2n-\alpha)}dx\right)^{\frac{1}{q}} \\ &\quad + |Q_j|^{\frac{1}{p_1'}+\frac{1}{p_2'}+\frac{1}{n}}\left(\int_E \frac{|b(x)-b_{Q_j}|}{|x-y_j|^{q(2n-\alpha+1)}}dx\right)^{\frac{1}{q}} \\ &\lesssim \left(\frac{|E|^{1/q}}{|Q_j|^{1/q}} + \left(\frac{1}{|Q_j|}\int_E |b(x)-b_{Q_j}|^q dx\right)^{\frac{1}{q}}\right). \qquad (3.10)\end{aligned}$$

From here the arguments in [10] can be followed identically, and it shown there that there exists some positive constant $\tilde{C}$ depending on $\gamma_1,\ \gamma_2$, and b such that

$$(3.10) \lesssim \frac{|E|^{1/q}}{|Q_j|^{1/q}}\left(1+\log\left(\frac{\tilde{C}|Q_j|}{|E|}\right)\right)^{\frac{\lfloor q\rfloor+1}{q}}$$

(see [10, p.309]). Clearly we can now select $0 < \beta < \min(\tilde{C}^{1/n}, \gamma_2)$ and sufficiently small so that (3.9) holds.

Before we continue, we would like to take a minute to discuss the technical estimates. The first simply shows that on an annulus determined by Q_j, $[b,I_\alpha](f_j,g_j)$ is large, and that it is small outside of this annulus. Given that our goal is to show that a sequence of these functions has no convergent subsequences, the benefit of this is immediately clear. We can use these to obtain a uniform lower bound on the distance between two such functions, provided that their respective annuli don't overlap. The second estimate says that even if they do overlap, as long as the intersection is small, the function will be quantitatively small on the intersection. As we will see, it is possible to construct a sequence of functions for which the overlap of these annuli is in fact small, and this will allow us to reach our conclusion.

We will have to proceed by cases, depending upon which of the conditions (1.1)-(1.3) b is supposed to fail to satisfy. The arguments are again borrowed from [10] but adapted to our bilinear situation.

If b does not satisfy (1.1), then there exists some $\varepsilon > 0$ and a sequence $\{Q_j\}$ with $|Q_j| \to 0$ as $j \to \infty$ such that for every j,

$$\varepsilon \leq \frac{1}{|Q_j|} \int_{Q_j} |b(y) - b_{Q_j}| dy. \tag{3.11}$$

We then can pick a subsequence, which we will denote $\{Q_j^{(i)}\}$, so that

$$\frac{d_{j+1}^{(i)}}{d_j^{(i)}} < \frac{\beta}{2\gamma_2}.$$

We also let $f_j^{(i)}$ and $g_j^{(i)}$ be the sequences associated to the selected cubes $Q_j^{(i)}$ as defined earlier on.

For fixed k and m, we define the following sets,

$$\begin{aligned} G &= \{x : \gamma_1 d_k^{(i)} < |x - y_k^{(i)}| < \gamma_2 d_k^{(i)}\}, \\ G_1 &= G \setminus \{x : |x - y_{k+m}^{(i)}| \leq \gamma_2 d_{k+m}^{(i)}\}, \\ G_2 &= \{x : |x - y_{k+m}^{(i)}| > \gamma_2 d_{k+m}^{(i)}\}. \end{aligned}$$

Note that since $G_1 = G \cap G_2$, we have,

$$G_1 \subset G_2 \tag{3.12}$$

$$G_1 = G \setminus (G_2^c \cap G). \tag{3.13}$$

Also, by construction and our choice of $Q_j^{(i)}$'s, one can easily see that

$$\frac{|G_2^c \cap G|}{|Q_k^{(i)}|} \leq \beta^n, \tag{3.14}$$

see [10, p.307]. It follows that

$$\begin{aligned}
&\|[b,I_\alpha]_1(f_k^{(i)},g_k^{(i)}) - [b,I_\alpha]_1(f_{k+m}^{(i)},g_{k+m}^{(i)})\|_{L^q} \\
&\quad \geq \left(\int_{G_1} |[b,I_\alpha]_1(f_k^{(i)},g_k^{(i)}) - [b,I_\alpha]_1(f_{k+m}^{(i)},g_{k+m}^{(i)})|^q\right)^{\frac{1}{q}} \\
&\quad \geq \left(\int_{G_1} |[b,I_\alpha]_1(f_k^{(i)},g_k^{(i)})|^q\right)^{\frac{1}{q}} - \left(\int_{G_1} |[b,I_\alpha]_1(f_{k+m}^{(i)},g_{k+m}^{(i)})|^q\right)^{\frac{1}{q}} \\
&\quad \geq \left(\int_{G_1} |[b,I_\alpha]_1(f_k^{(i)},g_k^{(i)})|^q\right)^{\frac{1}{q}} - \left(\int_{G_2} |[b,I_\alpha]_1(f_{k+m}^{(i)},g_{k+m}^{(i)})|^q\right)^{\frac{1}{q}} \\
&\quad = \left(\int_{G} |[b,I_\alpha]_1(f_k^{(i)},g_k^{(i)})|^q - \int_{G_2^c \cap G} |[b,I_\alpha]_1(f_k^{(i)},g_k^{(i)})|^q\right)^{\frac{1}{q}} \\
&\qquad - \left(\int_{G_2} |[b,I_\alpha]_1(f_{k+m}^{(i)},g_{k+m}^{(i)})|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Using (3.5), (3.9), and (3.6) in each of the three terms above we finally arrive at

$$\begin{aligned}
\|[b,I_\alpha]_1(f_k^{(i)},g_k^{(i)}) - [b,I_\alpha]_1(f_{k+m}^{(i)},g_{k+m}^{(i)})\|_{L^q} &\geq \left(\gamma_3^q - \frac{\gamma_3^q}{4^q}\right)^{\frac{1}{q}} - \frac{\gamma_3}{4} \\
&\gtrsim \frac{\gamma_3}{2}.
\end{aligned}$$

Since every pair of terms in the sequence $\{[b,I_\alpha]_1(f_j^{(i)},g_j^{(i)})\}$ are at least $C\gamma_3$ apart from each other, there can be no convergent subsequence, and therefore $[b,I_\alpha]_1$ would not be compact. So b must satisfy (1.1).

If b violates (1.2), we again have that there exists ε and sequence of cubes $\{Q_j\}$, this time with $|Q_j| \to \infty$ as $j \to \infty$, such that (3.11) is satisfied. This time we take the subsequence $\{Q_j^{(ii)}\}$ so that

$$\frac{d_j^{(ii)}}{d_{j+1}^{(ii)}} < \frac{\beta}{2\gamma_2}.$$

We can use a similar method as in the previous case, but since our diameters are increasing instead of decreasing, we simply define our sets in a 'reversed' order, so for fixed k and m, we have

$$G = \{x : \gamma_1 d_{k+m}^{(ii)} < |x - y_{k+m}^{(ii)}| < \gamma_2 d_{k+m}^{(ii)}\},$$
$$G_1 = G \setminus \{x : |x - y_k^{(ii)}| \leq \gamma_2 d_k^{(ii)}\},$$
$$G_2 = \{x : |x - y_k^{(ii)}| > \gamma_2 d_k^{(ii)}\}.$$

As before we have that (3.12)-(3.14) hold, and so from here, the calculations are identical to those in the first case.

Finally, if (1.3) is not satisfied, there exists some cube Q with diameter d, and some sequence $\{y_j\}$, with $|y_j| \to \infty$, such that (3.11) holds for $\{Q_j := Q + y_j\}$. We then let $B_j = \{x \in \mathbb{R}^n : |x - y_j| < \gamma_2 d\}$, and choose $\{Q_j^{(iii)}\}$ so that $B_j \cap B_k = \emptyset$ for $j \neq k$.

Note that by the construction of the balls B_j, if we define G, G_1, and G_2 as in (i), we in fact have that $G = G_1 = G \cap G_2$, and so $G_2^c \cap G = \emptyset$. This means that while the calculations for this case could certainly be simplified, it is sufficient to once again repeat the steps preformed in the first case to obtain the desired result. $\square$

3.2 Remarks

It is worth noting that our proof of estimate (3.3) relies on the fact that I_α has a positive kernel, and estimate (3.4) is due primarily to the fact that the gradient of I_α is easy to work with. With this in mind, our above proof that (iii) implies (1) becomes impossible in the general case that T is in CZO, unless we are able to in some way regain these two inequalities for x in some large set. This means that while a result analogous to the above (i) implies (ii) is known, it is too soon to for us complete the characterization.

Future Work and Closing Remarks

My current and future work is extending in several different directions at the current moment.

In the multilinear setting the broader classes of vector weights in general require stronger regularity conditions on the kernel of an operator in order to guarantee its boundedness. One of my current projects, [6], a joint work with R. H. Torres and X. Wu, expands our understanding of the differences of the regularity required for operators to be bounded, tuning up the amount of regularity imposed with the classes of weights used.

I am also currently working on the issue raised in the closing remark of the previous chapter. More specifically, for the multilinear Riesz transforms, R_j, defined by

$$R_j(f,g)(x) = \int\int \frac{x_j - y_j}{(|x-y|^2 + |x-z|^2)^{n+1/2}} f(y)g(z)dydz,$$

I am working to show that in order for $[b,R_j]_i$ to be compact, it is necessary for b to be in CMO. It is my hope that doing so will provide a template for working with other Calderón-Zygmund operators future.

As has already been mentioned, the proof technique of 2.1.1 requires the target space to have index greater than 1, and in turn this requirement was inherited in 3.1.1. However many results in m-linear Calderón-Zygmund theory show boundedness or compactness for the full range of the target space, for all $p > \frac{1}{m}$. These quasi-Banach spaces require very different techniques to work with, and many tools which can be taken for granted in Banach spaces are no longer available. Moving forward, I intend to further develop my results and extend them to the full range of $p > \frac{1}{m}$.

Another aspect of the results which we would like to improve upon is that the classes of weights used for the compactness result of the commutator with I_α are much more restricted than the ones for which boundedness results hold. This restriction is in part because of Theorem 1.4.4, due to Clop and Cruz, since to use it we need the weight in the target space to be in A_q. For more general classes, the weight $\mu_{\mathbf{w}}$ is still in A_∞ but not in A_q. It would be interesting to see if Freshet-Kolmogorov result could be extended to a larger class of weights in A_∞, which would in turn allow us to extend Theorem 3.1.1.

Lastly, there has been much work done in recent years on variable Lebesgue spaces, ones for which the exponent is itself a function, and I am currently involved in joint projects to continue my work with commutators to those settings.

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