On Kernels, β-graphs, and β-graph Sequences of Digraphs

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Abstract

We begin by investigating some conditions determining the existence of kernels in various classes of directed graphs, most notably in oriented trees, grid graphs, and oriented cycles. The question of uniqueness of these kernels is also handled. Attention is then shifted to γ-graphs, structures associated to the minimum dominating sets of undirected graphs. I define the β-graph of a given digraph analogously, involving the minimum absorbant sets. Finally, attention is given to iterative construction of β-graphs, with an attempt to characterize for what classes of digraphs these β-sequences terminate.

Acknowledgements

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Chapter 1

Introduction

Let a digraph $D = (V, A)$ be any orientation of a simple graph $G = (V, E)$. A subset $S \subseteq V(D)$ is said to be *absorbant* if for all $v \in V \setminus S$ there exists some $u \in S$ such that $v \to u$. $S \subseteq V(D)$ is said to be a *kernel* of D if S is both absorbant and independent. It should be noted that we follow both the definition and the (French) spelling of "absorbant" as found in [2]. In Chapter 2, we state some results concerning when a given digraph D does or does not possess a kernel. This question has been handled in the past (see [2], [6]), and many results are known, dependent on Grundy functions or strong connectedness of graphs, though these topics are avoided in this paper.

A subset $S \subseteq V(G)$ is a *dominating set* of V if every $v \in V \setminus S$ is adjacent to at least one $u \in S$. The *domination number* of a graph G is $\gamma(G)$, the minimum cardinality of a dominating set. A γ-set is a dominating set of size γ. Fricke, et. al. [5] initiated the study of γ-graphs of graphs, where the γ-graph of G, is the graph $G(\gamma)$ with vertices corresponding one-to-one with γ-sets, where two γ-sets γ_1 and γ_2 are adjacent if there exist vertices $u \in \gamma_1$, $v \in \gamma_2$ such that

$$(\gamma_1 \setminus \{u\}) \cup \{v\} = \gamma_2$$

and u, v are adjacent in G. It should also be mentioned that a different concept of γ-graph was developed by [12]. Similarly, the β_0, α_0, and ω-graphs of a graph have been handled (see [8]), where the respective graphs are defined using other graph invariants (α_0-graphs deal with minimum vertex coverings, for example). The beginning of chapter 3 is devoted to discussing the γ-graph structures defined as in [5]. Chapter 3 also discusses the k-dominating graphs D_k, with vertices in correspondence with dominating sets of cardinality at most k, and the generalized γ-graphs X_k, with vertices corresponding to the dominating sets with cardinality exactly k, each defined by Haas and Seyffarth in [7]. Haas and Seyffarth examine connectedness of these graphs and their application to reconfiguration problems, concluding with questions regarding Hamiltonicity of the D_k's, and which graphs G have the property that $G \cong D_k(G)$.

In analogy to the definition of γ-graphs, in Chapter 3, Section 2, I introduce the β-graph for a digraph D, considering β-sets of D, where $\beta(D)$ is the minimum cardinality of an absorbant set (as defined in [2]). Similar to their undirected counterparts, the edges in $D(\beta)$ are determined by the notion of "adjacent vertex swapping", i.e., two β-sets β_1 and β_2 of D are adjacent in $D(\beta)$ if there exist vertices $u \in \beta_1$ and $v \in \beta_2$ such that

$$(\beta_1 \setminus \{u\}) \cup \{v\} = \beta_2$$

and either $u \to v$ or $v \to u$ in D. Thus, $D(\beta)$ gives a directed graph, with the arc between β_1 and β_2 obeying the same direction as the arc between u and v. (For example, $u \to v$ implies $\beta_1 \to \beta_2$.) Also defined are directed analogs of those graphs seen in [7], though the main focus is on the true β-graph (with vertices corresponding to β-sets).

At the close of [5], the authors give some examples of γ-graph sequences, noting that many of the sequences terminate in K_1. The question is left open, and is not mentioned in the course of [7]. In Chapter 3, Section 3 the study of the sequence of graphs obtained by iterating the β-graph construction is initiated, with the aim to characterize when these β-graph sequences terminate in graphs isomorphic to K_1. I offer a result on the non-termination of the β-sequences of cyclicly oriented odd cycles, and a characterization of the definite termination of the β-sequences of star graphs. The paper closes with a partial description of the structure of the β-graph of general trees.

Chapter 2

Kernels of Directed Graphs

2.1 Definitions and Preliminaries

Many of the following definitions and more can be found in [6].

Definition 2.1.1. Unless otherwise stated, we consider simple graphs $G = (V, E)$, with V the vertex set, and E the collection of edges in G. That is, G contains no multiple edges or loops.

Definition 2.1.2. A digraph $D = (V, A)$ with underlying simple graph $G = (V, E)$ is any orientation of the graph G. That is, each arc in A is an edge of E, together with a prescribed direction. We denote an arc $(x, y) \in A$ by $x \to y$.

Remark 2.1.3. The restriction to simple graphs G is by choice, in order that any directed graph with antisymmetric edges is avoided. However, the following discussions may certainly be made without this restriction.

Definition 2.1.4. We respectively define the open and closed neighborhoods of a vertex v by

$$N(v) = \{u \in V \mid \text{either } u \to v \text{ or } v \to u\}$$

and $N[v] = N(v) \cup \{v\}$.

Definition 2.1.5. Let $D = (V,A)$ be a digraph. Let $v \in V$. Define the *outset* of v as the set

$$O(v) = \{u \in V \,|\, v \to u\}.$$

Similarly define the *inset* of v by the set

$$I(v) = \{u \in V \,|\, u \to v\}.$$

We also define the sets $O[v] = O(v) \cup \{v\}$ and $I[v] = I(v) \cup \{v\}$ as the closed outset and closed inset of v, respectively.

Definition 2.1.6. Let D be a digraph, and let $v \in V$. v is a *sink* vertex (or just sink) if $I(v) = N(v)$, i.e., if v is dominated by all of its neighbors. v is a *source* vertex (or just source) if $O(v) = N(v)$, i.e., if v dominates all of its neighbors.

Remark 2.1.7. Alternatively, we can describe a sink or a source as a vertex v satisfying $O(v) = \emptyset$ or $I(v) = \emptyset$, respectively.

Definition 2.1.8. Let $D = (V,A)$ be a digraph, and $S \subseteq V$. We say that S is *absorbant* if for each $v \in V \setminus S$, there exists at least one $u \in S$ such that $v \to u$, i.e., each vertex v in the complement of S dominates at least one vertex inside of S. We say S is *independent* if for any pair $x, y \in S$, $x \not\to y$ and $y \not\to x$.

Definition 2.1.9. Let $D = (V,A)$ be a digraph, and $K \subseteq V$ which is both independent and absorbant. We call K a *kernel* of the digraph D.

2.2 Trees and Grids

Lemma 2.2.1. ***(Theorem 15.8 of [1])*** *Let T be an oriented tree on n vertices. Then there exists $v \in V(T)$ such that $O(v) = \emptyset$, i.e., every oriented tree on n vertices has a sink.*

Proof Let u_0 be a leaf of T. Either $I(u_0) = \emptyset$ or $O(u_0) = \emptyset$. If the latter is true, then take $v = u_0$. If $I(u_0) = \emptyset$, then pick $u_1 \in O(u_0)$. If $O(u_1) = \emptyset$, take $v = u_1$ and we are done. Otherwise, pick $u_2 \in O(u_1)$ and continue. Since T has n vertices, this process must terminate at some u_j for $u_j \in O(u_{j-1})$, $1 \leq j \leq n-1$. Then $O(u_j) = \emptyset$, so take $v = u_j$.

■

It must be noted that the following result is certainly not a new one. Indeed, this was shown by Von Neumann and Morgenstern [13], and has since been improved upon. The following is an alternate proof.

Proposition 2.2.2. *Any oriented tree on n vertices has a unique kernel.*

Proof We induct on n. For $n = 1$, the result is trivial. For $n = 2$, the tree is a directed path, and we take as the unique kernel the vertex v_i such that $O(v_i) = \emptyset$.

Suppose the result holds for all k up to $n-1$. Fix an orientation on T_n, a tree on n vertices labeled $\{v_1, \ldots, v_n\}$. By the lemma, there exists $v_j \in V(T_n)$ such that $O(v_j) = \emptyset$.

Consider the forest F induced by the vertices of $V \setminus N[v_j]$. F has some number $m \geq 0$ of components.

If $m = 0$, then $N[v_j] = V$, in which case T_n is a star with all pendant edges oriented toward the center node v_j. Thus, we take $\{v_j\}$ as the unique kernel.

So, suppose $m > 0$. Then each component F_i of F is an oriented tree on fewer than n vertices. By our induction hypothesis, each F_i has a unique kernel K_i for $i = 1, \ldots, m$. Let $K := \bigcup_{i=1}^{m} K_i \cup \{v_j\}$. Since $N(v_j) \cap F_i = \emptyset$ for each $i = 1, \ldots, m$, K is certainly independent. It is also absorbant, since v_j is dominated by each element of $N(v_j)$. Thus K is a kernel of T_n. To see that this K is unique, suppose K' is another kernel of T_n. Since $O(v_j) = \emptyset$, it must be that $v_j \in K$ and $v_j \in K'$. Define

$$K'_i := \{v \in V \mid v \in K' \cap F_i\}.$$

Then $K' = \{v_j\} \cup \bigcup_{i=1}^{m} K'_i$. We claim that K'_i is a kernel of F_i for each i. By construction K'_i is independent. If $(V \setminus K') \cap F_i = \emptyset$, then $F_i = K'_i = \{v_k\}$ for some k, and thus K'_i is trivially absorbant.

Otherwise, since K' is absorbant, it follows that for any vertex $v \in (V \setminus K') \cap F_i$,

$$O(v) \cap K'_i \neq \emptyset$$

That is, K'_i is absorbant and therefore a kernel of F_i.

By uniqueness of the K_i, it must be that $K_i = K'_i$ for each $i \in \{1, \ldots, m\}$. Therefore,

$$K' = \{v_j\} \cup \bigcup_{i=1}^{m} K'_i = \{v_j\} \cup \bigcup_{i=1}^{m} K_i = K$$

showing that K is the unique kernel of T_n.

■

The following proposition is attributed to Richardson [10]. Berge and Duchet [3] give a simpler proof.

Proposition 2.2.3. *Any oriented bipartite graph possesses a kernel.*

Proof Let D_k denote an oriented bipartite graph on k vertices. We proceed by induction on k. Clearly, any orientation on D_2 possesses a kernel. Suppose the result holds for all k up to $n-1$. Consider the case when $k = n$.

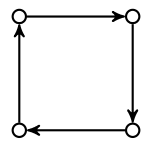
Fix an orientation on a bipartite graph D_n. Let V_1, V_2 denote the (nonempty) partite sets of D_n, and consider V_1. By definition V_1 is independent. If it is also absorbant, then V_1 is a kernel, and we are done. If this is not the case, then there exists at least one vertex $u \in V_2$ such that $O(u) = \emptyset$, i.e., u is a sink. For the sink u, if $I(u) = V_1$, then take V_2 as kernel. If $\{u\} = V_2$, then take the union of V_2 with all isolated vertices of V_1 (if any exist) as the kernel. Otherwise, consider the induced sub-digraph on $D \setminus N[u]$. Here, $D \setminus N[u]$ is a bipartite graph on fewer than n vertices. Then, by the induction hypothesis, there exists a kernel K' in $D \setminus N[u]$.

We claim that in D, the union $K = K' \cup \{u\}$ is a kernel. It is clear that K is independent. K is absorbant, since K' is absorbant by definition, and since u is dominated by all vertices in $N(u)$.

Then by definition, K is a kernel of D.

■

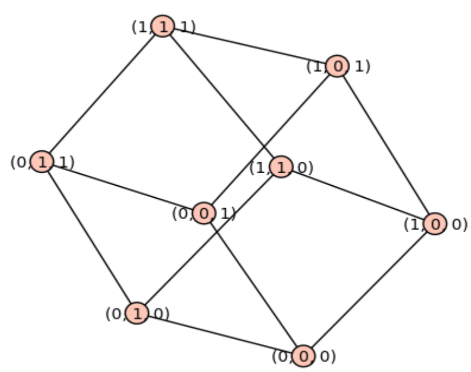
Remark 2.2.4. Indeed, the previous proposition deals only with existence, and not uniqueness. For example, consider the following:



Then either pair of diametrically opposite vertices is independent and absorbant.

The following are some examples of graphs which necessarily have at least one kernel by the previous proposition.

Example 2.2.5. It is well known that the k-dimensional cube Q_n is bipartite. Therefore, any orientation of the k-dimensional cube Q_k has a kernel. For reference, the unit 3-dimensional cube Q_3 is pictured below:



Definition 2.2.6. As defined in [5], the *step-grid* graph $SG(k)$ is the induced subgraph of the $k \times k$ grid graph $P_k \square P_k$ with vertex set

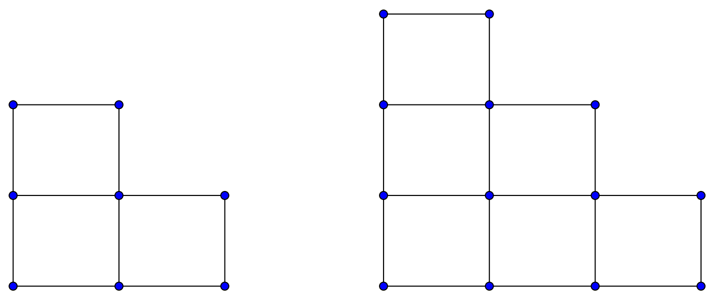
$$V(k) = \{(i, j) : 1 \leq i, j \leq k, i + j \leq k + 2\}$$

and edge set

$$E(k) = \{((i, j), (i', j')) : i = i', j' = j + 1; i' = i + 1, j = j'\}$$

Example 2.2.7. By definition of $E(k)$, for any positive integer k, $SG(k)$ has no odd cycles, i.e. is bipartite. Then for all k, any orientation of a step-grid graph $SG(k)$ has a kernel.

The graph $SG(3)$ is pictured below on the left, and $SG(4)$ on the right:



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2.3 General Results

Proposition 2.3.1. *Let $G = (V,E)$ be a finite graph. There exists an orientation $\mathscr{O} = (V(G),A)$ of G which possesses a kernel.*

Proof We construct an orientation $\mathscr{O}$ of G as follows: Let M be a maximal independent set. Recall that $N(M)$ denotes the neighborhood of M. It is quick to see that $N(M) = V \setminus M$, for if $N(M) \subset V \setminus M$, there must exist some $u \in V \setminus M$ such that $u \notin N(M)$. That is, u is not adjacent to anything in M, and thus must be in M, a contradiction to the maximality of M.

For every $u \in N(M)$, and for every edge $um \in E$ such that $m \in M$, add the arc $u \to m$ to the arc set A of $\mathscr{O}$. Complete the orientation by directing any other edges of E in any direction. Under the orientation $\mathscr{O}$, M is absorbant since we have forced all of $N(M) = V \setminus M$ to dominate at least one element of M. Therefore, M is a kernel under the orientation $\mathscr{O}$.

$\blacksquare$

Lemma 2.3.2. *Any finite graph with an odd cycle possesses a chordless odd cycle.*

Proof Let G be a finite graph containing some odd cycle $\mathfrak{O}$ ordered on the vertices $(u_1, u_2, \ldots, u_{2k+1})$, $k \geq 1$. If $\mathfrak{O}$ is itself chordless, we are done. So assume that $\mathfrak{O}$ has a chord e between vertices u_i, u_j for (and without loss of generality) $j > i$, $j \neq i+1$. e divides the cycle into two sub cycles intersecting along e, call them C_1, C_2. Without loss of generality, if C_1 is an even cycle, then C_2 is a cycle on the vertices of $(\mathfrak{O} \setminus C_1) \cup \{u_i, u_j\}$, of which there are an odd number. That is, C_2 is odd. If C_2 is chordless, we are done. If not, we repeat the process from above to divide C_2 into an even and an odd cycle. Since G is finite, this process must terminate in an odd chordless cycle.

■

Proposition 2.3.3. *Let $G = (V, E)$ be any finite graph possessing an odd cycle. Then there exists an orientation of G for which there is no kernel.*

Proof Let G be any finite graph with an odd cycle $\mathfrak{O}$. By the Lemma 2.3.2, G contains some chordless odd cycle $\mathscr{C} = \{u_1, \ldots, u_n\}$ for $n = 2k+1$, $k \geq 1$. Without loss of generality, let A be an orientation of G such that

$$u_1 \to u_2 \to u_3 \to \ldots \to u_k \to u_1$$

and such that for any edge E not contained in $\mathscr{C}$, with $u_i \in E$ for some i, u_i is dominated by the other endpoint in E.

I claim that $D = (V(G), A)$ has no kernel. Suppose, for the sake of contradiction that K was some kernel of D. By design, each u_j, $1 \leq j \leq k$ either must be in K, or must dominate an element of K. Therefore, for any j, if $u_j \notin K$, then $u_{j+1 \pmod{n}} \in K$. Form the finite collection of pairs

$$\{\{u_1, u_2\}, \{u_3, u_4\}, \ldots, \{u_{n-2}, u_{n-1}\}\}$$

By the Pigeonhole Principle, vertex u_n is left solitary. But then it follows that

$$u_1 \in K \Rightarrow u_{n-2} \in K \Rightarrow u_{n-1} \notin K \Rightarrow u_n \in K$$

which is a contradiction, since $u_n \to u_1$. We reach a similar contradiction under assumption that

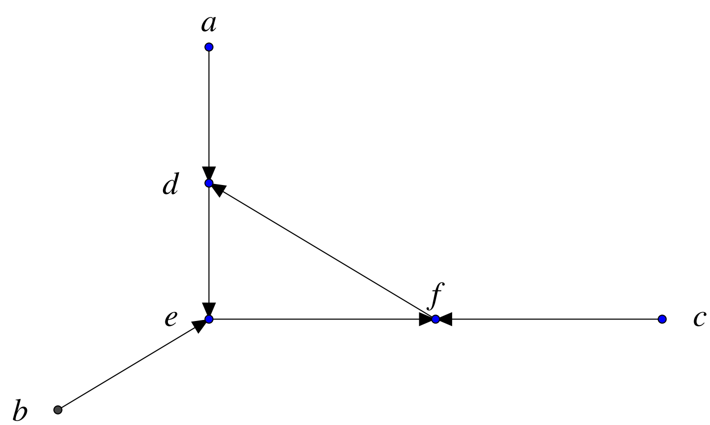
$u_1 \notin K$, since

$$u_2 \in K \Rightarrow u_{n-1} \in K \Rightarrow u_n \notin K \Rightarrow u_1 \in K$$

Therefore, there exists a vertex $v \in \mathscr{C}$ such that v can never simultaneously dominate a vertex $u \in \mathscr{C}$ and be independent from u. Therefore, K cannot be a kernel. Since K was arbitrary, there can be no independent absorbant set of D.

■

Example 2.3.4. Let D be the digraph pictured below: It is easy to check that there is no kernel. This is an example of the particular orientation in the previous proposition.



△

It is a well-known fact in graph theory (attributed to König [9]) that a graph is bipartite if and only if it has no odd cycles. As a consequence, Proposition 2.2.3 and Proposition 2.3.3 together imply the following:

Proposition 2.3.5. *Every orientation of a graph has a kernel if and only if the graph is bipartite.*

Proof The backward direction is already proved by Proposition 2.2.3. So, suppose a digraph G is such that every orientation of G has a kernel. Suppose for the sake of contradiction that G is not bipartite. Then G possesses some odd cycle, and Proposition 2.3.3 gives an immediate contradiction.

■

Chapter 3

Graphs from Dominating and Absorbant Sets

3.1 γ-graphs and Generalized γ-graphs

Fricke, Hedetniemi, Hedetniemi, and Hutson [5] introduce the γ-graph of a graph. It should be noted that other notions of γ-graphs have been considered in [12], though the definitions of the graphs are significantly different.

Definition 3.1.1. Let $G = (V, E)$ be a graph (undirected). A set $S \subseteq V$ is called a *dominating set* of G if for all $v \in V \setminus S$, $N(v) \cap S \neq \emptyset$. The *domination number* of G is

$$\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}$$

Any dominating set with cardinality equal to $\gamma(G)$ is referred to from here on as a γ-set.

The authors present the γ-graph of a graph G as a construct on the set of all γ-sets, used to find maximum or minimum values of a number of parameters of subgraphs induced by dominating sets, e.g., to maximize the number of isolated vertices in $G[S]$, indicating S as "close" to an independent dominating set.

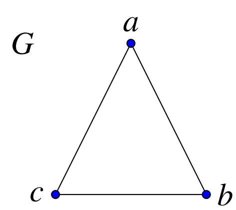
Definition 3.1.2. For a graph $G = (V,E)$, consider the collection of all γ-sets of G. Define the γ-graph of G as the graph $G(\gamma) = (V(\gamma), E(\gamma))$, with vertex set in one-to-one correspondence with the γ-sets of G, and with γ-sets S_1, S_2 adjacent in $G(\gamma)$ if there is a vertex $v \in S_1$ and a vertex $w \in S_2$ such that:

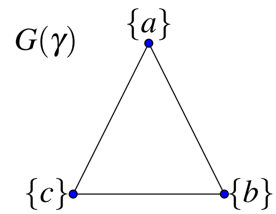
i) v,w are adjacent in G

ii) $S_1 = (S_2 \setminus \{w\}) \cup \{v\}$ and $S_2 = (S_1 \setminus \{v\}) \cup \{w\}$.

In other words, adjacency is defined by the ability to "swap" one and only one pair of G-adjacent vertices between dominating sets.

Example 3.1.3. Let $G = K_3$. Then $\gamma(G) = 1$, and $G(\gamma) = K_3$





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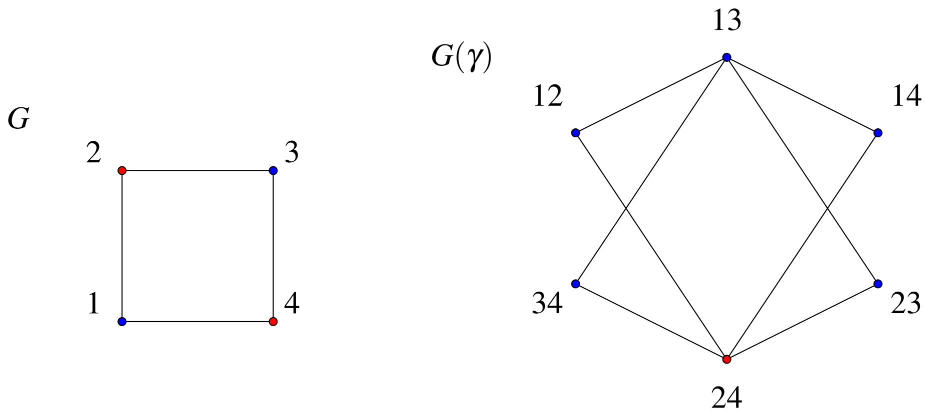
Example 3.1.4. Let $G = C_4$. $\gamma(G) = 2$. The γ-sets of G are $\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$. The vertex in $G(\gamma)$ corresponding to the γ-set $\{2,4\}$ is colored in red (as is $\{2,4\}$) to highlight the correspondence between γ-sets of G and vertices in $G(\gamma)$. Then $G(\gamma)$ is as shown below. Note that $G(\gamma) \cong K_{2,4}$.



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One can similarly intuit some definitions of a γ-graph-like structure in the directed case. We consider two constructions:

Construction 1: Let $G = (V,E)$ be a graph, and let $D = (V,A)$ be an orientation of G. Define $\overrightarrow{G}(\gamma) = (V(\gamma), A(\gamma))$ to be the graph with vertices in one-to-one correspondence with the γ sets of the underlying graph G, and a directed edge between γ sets S_1, S_2 if there exists $v \in S_1$, $w \in S_2$ such that

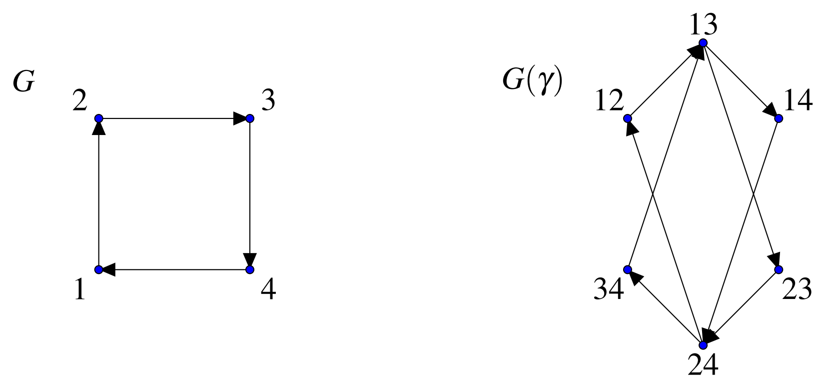
i) $v \to w \in A$ or $w \to v \in A$

ii) $(S_1 \setminus \{v\}) \cup \{w\} = S_2$ and $(S_2 \setminus \{w\}) \cup \{v\} = S_1$

The direction of the edge between S_1 and S_2 follows the same direction as $v \to w$ or $w \to v$, whichever may be in A. We note that if both $w \to v \in A$ and $v \to w \in A$, then there is a symmetric pair of arcs between S_1, S_2.

Remark 3.1.5. This construction, however, is not very interesting, as it is the same structure as the undirected γ-graph. Moreover, some sets which are dominating sets of the underlying graph G may not be dominating sets in the directed graph D. See the following example.

Example 3.1.6. Let $G = C_4$. Let D be the cyclic orientation on C_4. In the underlying graph G, the set of vertices $\{1,2\}$ is a dominating set of G. However, $\{1,2\}$ is not a dominating set of D, since neither 1 nor 2 dominate the vertex 4. Thus, the graph given by construction 1 contains some extraneous information.



△

For the next construction, we need some extra definitions.

Definition 3.1.7. Let $D = (V,A)$ be a directed graph. As in [4], define an *out-dominating set* as a set $S \subseteq V$ such that for all $v \in V \setminus S$, there exists some $u \in S$ such that $u \to v \in A$. The *out-domination number* $\gamma^+(D)$ of D is the minimum cardinality of an out-dominating set.

Remark 3.1.8. We note that the above definition is equivalent to what is defined as a "dominating set" in [6].

Construction 2: Let $D = (V,A)$ be a directed graph. Define the γ^+-graph $D(\gamma^+) = (V(\gamma^+), A(\gamma^+))$ to be the graph with vertices in one-to-one correspondence with the γ^+ sets of D, and a directed edge between γ^+ sets S_1, S_2 if there exists $v \in S_1$, $w \in S_2$ such that

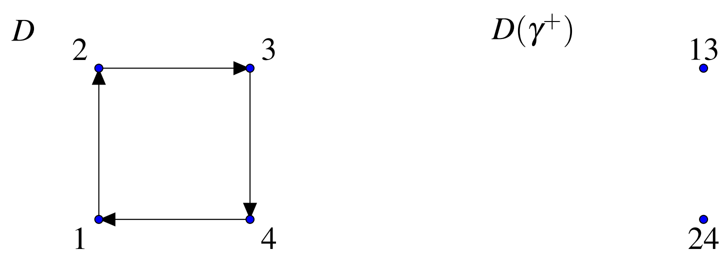
i) $v \to w \in A$ or $w \to v \in A$

ii) $(S_1 \setminus \{v\}) \cup \{w\} = S_2$ and $(S_2 \setminus \{w\}) \cup \{v\} = S_1$

The direction of the edge between S_1 and S_2 follows the same direction as $v \to w$ or $w \to v$, whichever may be in A. We note that if both $w \to v \in A$ and $v \to w \in A$, then there is a symmetric pair of arcs between S_1, S_2.

Remark 3.1.9. With this construction, it should be noted that the extraneous information present in the first construction is no longer there. The vertices in $D(\gamma^+)$ are precisely the γ^+ sets of D.

Example 3.1.10. Let $G = C_4$. Let D be the cyclic orientation on C_4. $\gamma^+(D) = 2$, and we have only two γ^+-sets: $\{\{1,3\},\{2,4\}\}$.



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Haas and Seyffarth [7] expand on the study of γ-graphs, generalizing the definition to k-dominating graphs, incorporating dominating sets of any possible size, not just γ-sets.

Definition 3.1.11. The *upper domination number* of a graph G is

$$\Gamma(G) = \max\{|S| : \text{S is a } \textit{minimal} \text{ dominating set of } G\}$$

Definition 3.1.12. The *k-dominating graph* of G, $D_k(G)$ is the graph with vertices in one-to-one correspondence with the dominating sets of G with cardinality at most k. An edge joins two vertices v, w of $D_k(G)$ if and only if the corresponding dominating sets S_v and S_w differ by addition or deletion of a single vertex.

Remark 3.1.13. Note that the edges of $D_k(G)$ are only between sets that differ in cardinality by 1, and the notion of "vertex swapping", seen in the γ-graph case, is no longer present.

Haas and Seyffarth later describe an analog of the γ-graph for different sizes of dominating sets.

Definition 3.1.14. Define $X_k(G)$ as the graph with vertices in one-to-one correspondence with the dominating sets of G with cardinality k, and an edge between two dominating sets S_1, S_2 if there exist $v \in S_1$ and $w \in S_2$ such that

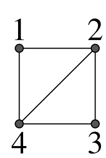
i) v, w are adjacent in G

ii) $S_1 = (S_2 \setminus \{w\}) \cup \{v\}$ and $S_2 = (S_1 \setminus \{v\}) \cup \{w\}$

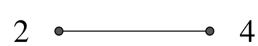
Remark 3.1.15. It should be noted that $X_\gamma = G(\gamma)$.

Example 3.1.16. The graph G is pictured below. Here, $\gamma(G) = 1$, since $\{2\}$ is itself a dominating set. Furthermore, $\Gamma(G) = 2$, since $\{1,3\}$ is minimal dominating, and any other dominating set of size two or more either contains $\{1,3\}$, $\{2\}$, or $\{4\}$, and therefore is not minimal.

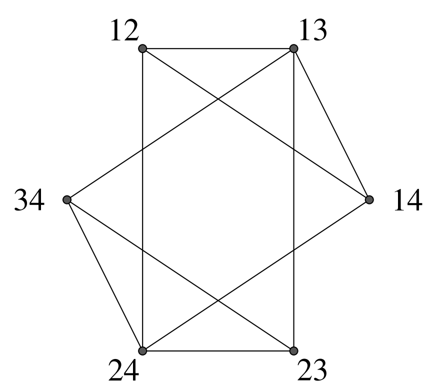
G



$X_1(G) = G(\gamma)$



$X_2(G)$



△

Similar to the construction given by Haas and Seyffarth, we would like to develop some object involving the absorbant sets of a directed graph. We hope to get some results about kernels by studying this structure. We examine this in the next section.

3.2 The β-graph of a Digraph

Definition 3.2.1. Let $G = (V,E)$ be a graph. The *independence number* of G is defined to be

$$\alpha(G) = \max\{|S| : S \text{ is independent}\}$$

We note that any maximally independent set of G has cardinality at most $\alpha(G)$.

Definition 3.2.2. Let $D = (V,A)$ be a digraph. The *absorption number* of D, defined by Berge [2] is

$$\beta(D) = \min\{|S| : S \text{ is absorbant in } D\}$$

We refer to any absorbant set with cardinality β as a β-set.

Remark 3.2.3. We note that Chartrand, Harary, and Yue discuss some results on absorption numbers of digraphs under the name of the "in-domination number". [4]

Definition 3.2.4. Let $D = (V,A)$ be a digraph. Define the β-graph of D to be the graph $D(\beta)$ with vertices in one-to-one correspondence with the absorbant sets of D, and an arc joining absorbant sets S_1, S_2 if there exist vertices $v \in S_1$ and $w \in S_2$ such that

i) $v \to w \in A$ or $w \to v \in A$ (but not both)

ii) $S_1 = (S_2 \setminus \{w\}) \cup \{v\}$ and $S_2 = (S_1 \setminus \{v\}) \cup \{w\}$

with the direction of the arc between S_1 and S_2 agreeing with the direction of the arc between v and w.

Definition 3.2.5. Let $D = (V,A)$ be a digraph. Define the k-absorbant graph $\mathscr{A}_k(D)$ as the graph with vertices in one-to-one correspondence with the absorbant sets of D with cardinality k, and an arc joining absorbant sets S_1, S_2 if there exist vertices $v \in S_1$ and $w \in S_2$ such that

i) $v \to w \in A$ or $w \to v \in A$ (but not both)

ii) $S_1 = (S_2 \setminus \{w\}) \cup \{v\}$ and $S_2 = (S_1 \setminus \{v\}) \cup \{w\}$

with the direction of the arc between S_1 and S_2 agreeing with the direction of the arc between v and w.

Remark 3.2.6. For $k = \beta(D)$, $\mathscr{A}_k(D) = D(\beta)$.

Definition 3.2.7. Define the *absorbant hierarchy* of D to be the graph

$$D[\beta] = \bigsqcup_{k=\beta(D)}^{|V(D)|} \mathscr{A}_k(D)$$

the disjoint union of all k-absorbant graphs of D.

Remark 3.2.8. If D possesses kernels, then their corresponding vertices all lie in

$$\bigsqcup_{k=\beta(D)}^{\alpha(D)} \mathscr{A} \subset D[\beta]$$

Proposition 3.2.9. *Let $D = (V,A)$ be any oriented tree. We know there exists a unique kernel K. Let $n = |K|$. Consider $v \in \mathscr{A}_n(D)$, the vertex corresponding to the kernel K. Then $I(v) = \emptyset$, i.e., v is either isolated, or a source.*

Proof For this v, suppose $w \in \mathscr{A}_n(D)$ such that $v \leftarrow w$. By definition, this means that there exists some $v' \in K$ and $w' \notin K$ such that

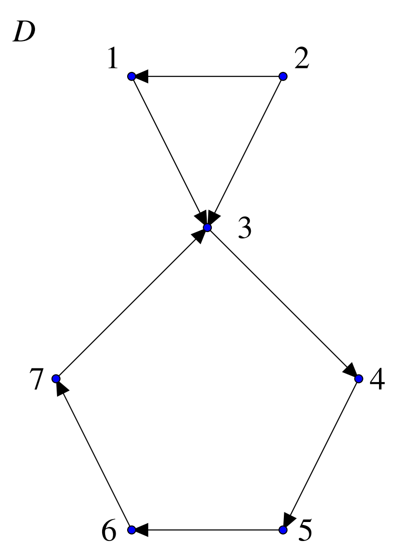
$$(K \setminus v') \cup w'$$

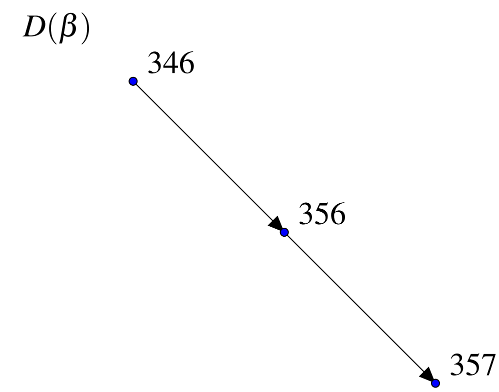
is absorbant, and $v' \leftarrow w'$. Now, $v' \notin (K \setminus v') \cup w'$, and so must dominate at least one element in $(K \setminus v') \cup w'$. By acyclicity of D, $v' \not\to w'$. Thus, if v' dominates anything in $(K \setminus v') \cup w'$, it must dominate some element in $K \setminus v'$. But then K would not be an independent set, and not a kernel, which is a contradiction. Therefore, $I(v) = \emptyset$. Accordingly, v is either isolated (if $O(v) = \emptyset$), or a source (if $O(v) \neq \emptyset$).

■

Remark 3.2.10. The converse of this is not true. That is, if $\mathscr{A}_k(D)$ has a source vertex v, the absorbant set corresponding to v is not necessarily a kernel. Consider the following example.

Example 3.2.11. Let D be the following directed graph. With the prescribed orientation, it follows from Proposition 2.3.3 above, that D possesses no kernel. However, the β-graph of D is shown to the right as a P_3. $D(\beta)$ has source corresponding to $\{3,4,6\}$, which is absorbant, but not independent, and so is not a kernel.





△

Proposition 3.2.12. *Let $D = (V,A)$ be a digraph. Suppose D possesses multiple kernels $K_1,\ldots,K_n$. Let $v_1,\ldots,v_n$ be vertices in $D[\beta]$ corresponding to the kernels $K_1,\ldots,K_n$, respectively. Then $\{v_1,\ldots,v_n\}$ forms an independent set.*

Proof To begin, we note that if K_r, K_s are kernels of different sizes, then v_r, v_s are non-adjacent by the very definition of $D[\beta]$. So, suppose that there exist some i, j, $i \neq j$ such that $v_i \to v_j$ in $\mathscr{A}_m(D)$ where v_i, v_j correspond to some kernels K_i, K_j of D each of size m. By definition of $\mathscr{A}_m(D)$, there exists some vertex $w_i \in K_i$, and some vertex $w_j \in K_j$ such that

$$K_j = (K_i \setminus w_i) \cup \{w_j\}$$

is absorbant and $w_i \to w_j$ in D. But, $w_j \notin K_i$, and so w_j dominates some vertex in $K_i \setminus \{w_i\}$. But this implies that $K_j = (K_i \setminus \{w_i\}) \cup \{w_j\}$ is not independent, which is a contradiction, since K_j is a kernel. Therefore, it must be that

$$\{v_1, \ldots, v_n\}$$

is independent in $D[\beta]$.

■

Proposition 3.2.13. *Let u be a source in some β-graph. Then $O(u)$ is independent.*

Proof The result is clear for the cases when $|O(u)| = 0, 1$. So, suppose that $|O(u)| > 1$. Consider $v, w \in O(u)$. We show that there is no arc joining v and w.

Assume, for the sake of contradiction that $v \to w$. Let $\beta_u, \beta_v, \beta_w$ be the β-sets that correspond to u, v, w. By definition of adjacency in β-graphs, $u \to v$ implies that there exist vertices $t_u \in \beta_u, t_v \in \beta_v$ such that

$$(\beta_u \setminus \{t_u\}) \cup \{t_v\} = \beta_v$$

Similarly, since $u \to w$, there exist vertices $t'_u \in \beta_u, t_w \in \beta_w$ such that

$$\left(\beta_u \setminus \{t'_u\}\right) \cup \{t_w\} = \beta_w$$

Finally, by assumption, $v \to w$ implies that there exist vertices $t'_v \in \beta_v, t'_w \in \beta_w$ such that

$$\left(\beta_w \setminus \{t'_w\}\right) \cup \{t'_v\} = \beta_v$$

We claim that $t_v = t'_v$, for if not, then

$$(\beta_u \setminus \{t_u\}) \cap \left(\beta_w \setminus \{t'_w\}\right) < \beta - 1$$

which cannot happen since $u \to w$ implies that β_u, β_w agree on exactly $\beta - 1$ vertices.

■

The following result is analogous to Theorem 21 in [5] about the bipartiteness of $T(\gamma)$ when T is a tree.

Proposition 3.2.14. *For any tree T, $T(\beta)$ is C_n-free for all odd integers $n \geq 3$. In other words, $T(\beta)$ is bipartite.*

Proof Suppose that the underlying graph $T(\beta)$ contains some C_{2k+1} on ordered vertices $u_1, u_2, \ldots, u_{2k+1}$, and let $\beta_1, \beta_2, \ldots, \beta_{2k+1}$ be the corresponding β-sets. Without loss of generality, consider the length $k+1$ chain of vertex swaps obtained between u_1, u_2, up to u_{k+2}. Suppose $1 \leq q \leq k+1$ represents the number of differences in the β-sets β_1 and β_{k+2}. There are two situations to consider.

First, suppose that $q = k+1$. That is, β_1 and β_{k+2} differ in exactly $k+1$ vertices. Then the chain of vertex swaps from u_{k+2} to u_{k+3} up to u_1 can change back at most k of these $k+1$ differences before reaching u_1. That is, u_1, u_{2k+1} are adjacent in $T(\beta)$, but differ by more than one vertex, which is a contradiction to the definition of β-graphs.

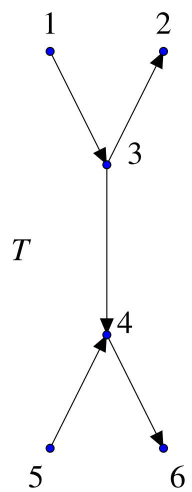
So, suppose that $q < k+1$. Then there exists some i, j, ℓ with $1 \leq i < j < \ell \leq k+2$ such that the chain of vertex swaps from u_i to u_ℓ is as follows: between u_i and u_j, swap t_i for t_j, and between u_j and u_ℓ, swap the same t_j for t_ℓ. Then traversing the underlying cycle $u_i, \ldots, u_j, \ldots, u_\ell, \ldots, u_{i-1}, u_i$, there must be some sequence of vertex swaps of the form $t_i, \ldots, t_j, \ldots, t_\ell, \ldots, t_i$, which is a cycle of vertices in the underlying graph of T, yielding a contradiction.

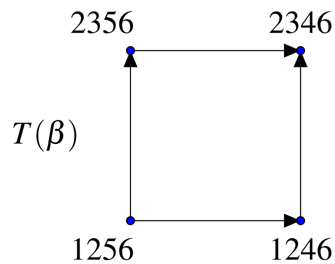
Therefore, if n is odd $T(\beta)$ must be C_n-free, i.e., $T(\beta)$ is bipartite.

■

Remark 3.2.15. It should be noted that the last paragraph of the proof of Proposition 3.2.14 does not depend on the parity of a cycle. Cycles of length $2k$ may similarly not appear in the β-graph if by moving from some vertex u_r to u_{r+k+1} in the underlying cycle, there exist vertex swaps which share a vertex (in other words, if there are vertices t_i, t_j, t_ℓ such that t_i is swapped for t_j and t_j is swapped for t_k). However, cycles of even length may appear in the β-graph if for any $1 \leq r \leq 2k$, moving from vertex u_r to u_{r+k+1} yields exactly k differences between the corresponding β-sets β_r and β_{r+k+1}, evinced in the following example.

Example 3.2.16. Let T be the tree given as follows, with $T(\beta)$ displayed on the right.





Here, $T(\beta)$ has underlying graph C_4. This is fine, even with the previous proposition, since traversing (clockwise) half of the vertices of the cycle in the underlying graph beginning at 1256 to 2346 is given by the following sequence of distinct vertex swaps: 1 to 3, 5 to 4.

$\triangle$

3.3 Sequences of β-graphs

The authors in [5] initiate the study of iteration of γ-graphs and the termination of these sequences. Several sequences of γ-graph iterations of various graphs are presented, illustrating the frequency with which these sequences “end" in K_1. While [7] does not address sequences for the higher order dominating set graphs, it seems a reasonable question to extend to all $X_k(G)$, and to all $\mathscr{A}_k(D)$. Of particular interest are the sequences of β-graphs of digraphs (i.e., studying sequences of $\mathscr{A}_\beta(D)$), though examining the sequences of other strata of $D[\beta]$ is also possible.

Definition 3.3.1. Let D be a digraph. We refer to the sequence of digraphs

$$D, D(\beta), D(\beta)(\beta) = D(\beta)^2, D(\beta)(\beta)(\beta) = D(\beta)^3, \ldots$$

as the *β-graph sequence of D*, or just the β-sequence. We say that the β-sequence of D *terminates* if for some n, $D(\beta)^n$ is isomorphic to K_1.

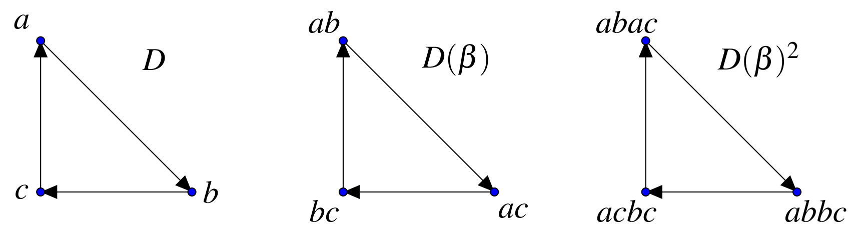
Definition 3.3.2. We similarly define the *absorption number sequence* of D

$$\beta(D), \beta(D(\beta)), \beta(D(\beta)^2), \ldots$$

We focus attention on the β-sequence of a digraph D, in hopes of characterizing those digraphs with convergent β-sequences. We begin with some preliminary results.

Example 3.3.3. Let D be the cyclicly oriented K_3. We compute the first few terms of the β-

sequence of D.



As one might guess, this sequence continues infinitely. This is not a coincidence.

$\triangle$

Lemma 3.3.4. *Let D be a cyclic orientation of a cycle on $2k+1$ vertices for $k \in \mathbb{N}$. Then $\beta(D) = k+1$.*

Proof Suppose $\beta(D) = j < k+1$. Let S be any set of size j. Then among the $2k+1-j$ remaining vertices, there is an induced oriented P_2, and therefore a vertex which does not dominate any vertex of S. Thus no S such that $|S| = j$ is absorbant. So, $\beta(D) \geq k+1$.

Suppose that $\beta(D) = q > k+1$. Then for any absorbant set S such that $|S| = q$, there is an induced oriented P_3, $v_1 \to v_2 \to v_3$, and thus there is a vertex, v_2 such that $S \setminus \{v_2\}$ is still absorbant. That is, $\beta(D) \leq k+1$. By trichotomy, then $\beta(D) = k+1$.

$\blacksquare$

Lemma 3.3.5. *Let D be a cyclic orientation of a cycle on $2k+1$ vertices for $k \in \mathbb{N}$. The subgraph induced by any β-set of D is P_3-free.*

Proof Suppose that there was some β-set S of D whose induced subgraph further induced a subgraph of P_3 on vertices v_1, v_2, v_3 such that $v_1 \to v_2 \to v_3$. But v_2 dominates v_3, so $S \setminus \{v_2\}$ is absorbant, and therefore S cannot be a β-set. Thus, any β-set is P_3-free.

$\blacksquare$

Lemma 3.3.6. *Let D be a cyclic orientation of a cycle on $2k+1$ vertices. The induced subgraph of any β-set of D contains exactly one P_2.*

Proof By Lemma 3.3.4, any β set has size $k+1$. That the β set induces at least one P_2 follows immediately from the Pigeonhole Principle.

Assume that there is some β-set of D whose induced subgraph contains two or more P_2's. For any $P_2 = v \to w$, we refer to the "boundary vertices", b_1, b_2 as those vertices encasing the P_2 (i.e., b_1 is the predecessor of v, b_2 the successor of w). Denote A, A' as two of the induced P_2's.

There are $j \in \{3,4\}$ boundary vertices $b_1, \ldots, b_j$ (depending on the value of $\min\{d(v,w) \mid v \in A, w \in A'\}$). By Lemma 6.2, we do not want to select any of the b_i, since their inclusion induces a P_3, which cannot appear in any β-set. Denote by $C = V(D) \setminus \big(V(A) \cup V(A') \cup \{b_1, \ldots, b_j\}\big)$ the remaining $2k-3-j$ vertices of D. Among these $2k-3-j$, we want to choose $k-3$ vertices to include in the β-set, making sure to avoid P_3's. Dually, we think of choosing $2k-3-(k-3) = k$ vertices to exclude from the β-set. (It should be noted that the j boundary vertices are always excluded, so are counted in the $2k-3$ above.) It follows from the Pigeonhole Principle that there exist two consecutive vertices that are excluded from the β-set. That is, there exists some vertex that does not dominate anything in the β-set, which is a contradiction to the sets assumed absorbancy. Thus, any β-set of D must not have more than one P_2 in its induced subgraph. By trichotomy, any β-set of D has exactly one P_2 contained in its induced subgraph.

■

Proposition 3.3.7. *Let D be any cyclicly oriented odd cycle on $2k+1$ vertices. Then the β-sequence of D does not terminate. In particular, $D(\beta)^n \cong D$ for all $n \in \mathbb{N}$.*

Proof Let the vertices of D be the ordered set $\{u_1, u_2, \ldots, u_{2k+1}\}$. By Lemmas 3.3.4,3.3.5, and 3.3.6, all β-sets of D are the size $k+1$ subsets whose induced subgraphs contain exactly one P_2. The number of these β-sets is given by the number of choices of the two vertices which induce the P_2, of which there are exactly $2k+1$ (we can choose any vertex of D to start the P_2.) Thus, $D(\beta)$ has $2k+1$ vertices. Let $u_{i,i+1} \in V(D(\beta))$ be the vertex corresponding to the β-set $U_{i,i+1}$ where

$u_i \to u_{i+1}$ is the induced P_2, and let u_{i-1}, u_{i+2} be the boundary vertices. (Here, we note that u_i should be read as $u_{i \,(\mathrm{mod}\, 2k+1)}$). By the previous lemmas, $U_{i,i+1}$ contains $u_i, u_{i+1}, u_{i+3}, u_{i+5}, \ldots u_{i-2}$. Consider the β-set $U_{i+2,i+3}$. Again, by the previous lemmas, we know $U_{i+2,i+3}$ must contain $u_{i+2}, u_{i+3}, u_{i+5}, u_{i+7}, \ldots, u_{i-2}, u_i$. That is,

$$U_{i+2,i+3} \setminus U_{i,i+1} = \{u_{i+2}\}$$

and

$$U_{i,i+1} \setminus U_{i+2,i+3} = \{u_{i+1}\}$$

That is, $(U_{i+2,i+3} \setminus \{u_{i+2}\}) \cup \{u_{i+1}\} = U_{i,i+1}$. Furthermore, $u_{i+1} \to u_{i+2}$ in D. Thus, for all $i \in [2k+1]$, $U_{i,i+1} \to U_{i+2,i+3}$ in the graph $D(\beta)$. It is left to show that these are the only arcs in $D(\beta)$.

Indeed, if $j \neq i+2$, $j+1 \neq i-1 \,(\mathrm{mod}\, 2k+1)$, then

$$\left|U_{j,j+1} \setminus U_{i,i+1}\right| \geq 2$$

since $U_{j,j+1}$'s P_2 contains a vertex not contained in $U_{i,i+1}$ (by the alternation of inclusion and exclusion in $U_{i,i+1}$ of elements in $C = V(D) \setminus \{u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$) and further, since one of either u_i, u_{i+1} is included in $U_{j,j+1}$, while the other is not (again, by the alternation guaranteed by Lemma 3.3.6). Then by definition of β-graph, no arcs can occur between $U_{i,i+1}, U_{j,j+1}$ for $j \neq i+1$, $j+1 \neq i-1 \,(\mathrm{mod}\, 2k+1)$.

Therefore, the only arcs in $D(\beta)$ are those of the form $U_{i,i+1} \to U_{i+2,i+3}$, so that $D(\beta)$ is again a cycle on $2k+1$ vertices. Thus, $D(\beta) \cong D$. Repeating the above argument for each β-graph iteration, we see that for all $n \in \mathbb{N}$,

$$D(\beta)^n \cong D$$

Thus, the β-sequence of D never terminates.

$\blacksquare$

Proposition 3.3.8. *Any orientation of $K_{1,n}$ has a terminating β-sequence.*

Proof We can sort each orientation of $K_{1,n}$ into one of $n+1$ isomorphism classes. Indeed, for $0 \le j \le n$, let $[K_{1,n}]_j$ be the class of orientations of $K_{1,n}$ which have j leaves that are sinks. For example, $[K_{1,n}]_0$ is the orientation of $K_{1,n}$ with all leaves directed toward the center node, while $[K_{1,n}]_n$ is the orientation where all leaves are sinks. By $\beta = \beta\left([K_{1,n}]_j\right)$, we mean the absorption number of any orientation in the class $[K_{1,n}]_j$.

For $j = 0$, $\beta = 1$, and the only β-set is the center node. Then for $D = [K_{1,n}]_0$, $D(\beta) \cong K_1$.

For $j = n$, $\beta = n$, and the only β-set is the n-fold union of all leaves. Thus, $D \in [K_{1,n}]_n$, so that $D(\beta) \cong K_1$.

Let v denote the central node of $K_{1,n}$. For $1 \le j \le n-2$, $I(v) > 1$, $\beta = j+1$ and thus must appear in at least one β-set. In fact, the only β-set is the union of the j sinks and v. (To not include v in a proposed β-set implies that some v-dominating leaf does not point into that set, which implies it is not a β-set.) So for any $D \in [K_{1,n}]_j$, $D(\beta) \cong K_1$.

Finally, if $j = n-1$, $I(v) = 1$, $\beta = j+1 = n$ the sets

$$S_1 = \{v\} \cup \{u \,|\, u \text{ is a sink-leaf}\}, \quad S_2 = \{w \,|\; w \text{ is a leaf}\}$$

are both β-sets. Let t be the single leaf that is not a sink. Then $t \to v$. Thus, for $D \in [K_{1,n}]_j$, $D(\beta) \cong K_2$ with $v_2 \to v_1$ where $v_i \in D(\beta)$ corresponds to S_i. Then, $\beta(D(\beta)) = 1$, with unique β-set v_1, so that $D(\beta)^2 \cong K_1$.

As the above shows, if D is any orientation of $K_{1,n}$, either $D(\beta)$ or $D(\beta)^2$ is isomorphic to K_1. That is, the β-sequence of D terminates.

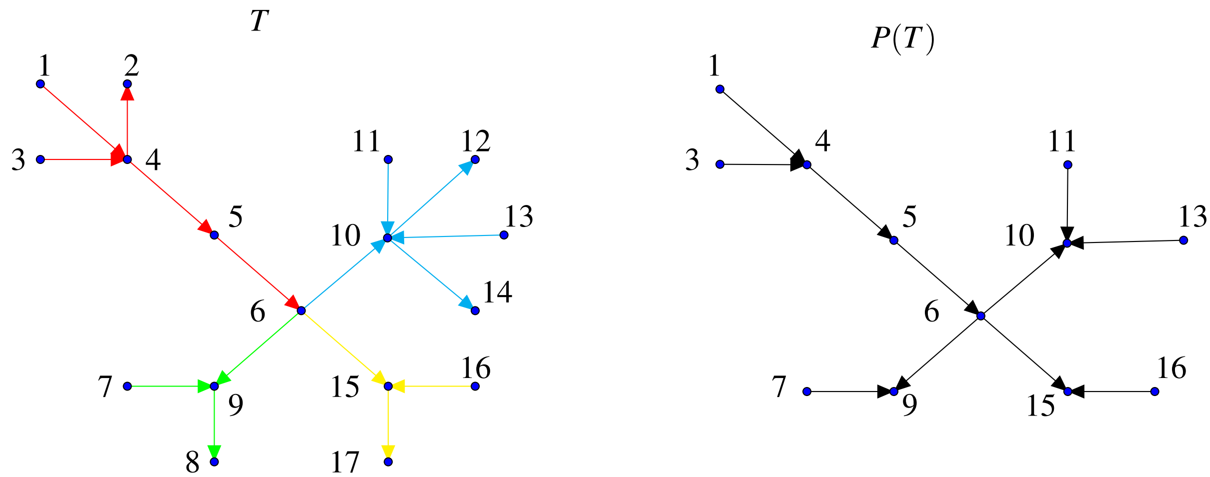
■

3.4 β-Sequences of Trees

Definition 3.4.1. Let T be any oriented tree. Define the *pruning* of T, denoted as $P(T)$ as the subtree obtained by deleting all of the sink-leaves of T.

Remark 3.4.2. It should be noted that the operation of "pruning" as defined above is not necessarily the same as the operation of pruning within data structures.

Example 3.4.3.



$\triangle$

Definition 3.4.4. Let T be a tree. We call a vertex that is the neighbor of at least one leaf a *meristem*. Every tree has at least one meristem. Further, define the *leaf-neighborhood* (relative to T) of a meristem m to be

$$\mathscr{L}_T(m) = \{\text{ leaves adjacent to } m\}$$

Definition 3.4.5. We often want to talk about the pruning of a tree "around" a meristem. That is, we consider the *restriction of* $P(T)$ to the meristem m, denoted $P(T)|_m$. In other words, $P(T)|_m$ carries out the pruning operation only on the leaf neighborhood of m.

Definition 3.4.6. Let T be a tree, and m any meristem. We say m is a *Type A meristem* if m has exactly one leaf $\ell \in \mathscr{L}_T(m)$ such that $\ell \to m$, $|\mathscr{L}_T(m)| > 1$, and m is a sink in $P(T)$.

A *Type B meristem* is a meristem m such that there is exactly one leaf $\ell \in \mathscr{L}_T(m)$ such that $\ell \to m$, $|\mathscr{L}_T(m)| > 1$, and there exists some $u \in N(m) \setminus \mathscr{L}_T(m)$ such that $m \to u$.

If m is such that there exist at least two leaves $\ell_1, \ell_2 \in \mathscr{L}_T(m)$ such that $\ell_1 \to m$ and $\ell_2 \to m$, and such that there is at least one vertex $v \in \mathscr{L}_T(m)$ such that $m \to v$, then m is called a *Type C meristem*.

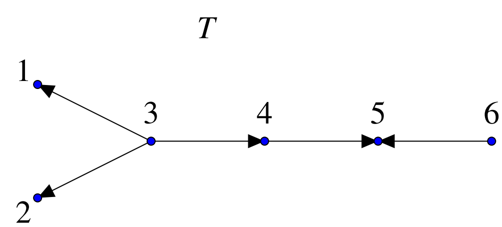
We say m is a *Type D meristem* if for all $v \in \mathscr{L}_T(m)$ then $v \to m$.

Finally, m is of *Type E* if for all $v \in \mathscr{L}_T(m)$ then $m \to v$.

Example 3.4.7. In any orientation of $K_{1,n}$, the central node is the only meristem. In the previous example, the tree T has four meristems; two of Type A: vertices 9 and 15, and two of Type C: vertices 4 and 10. The leaf-neighborhood of the meristem 9 is $\mathscr{L}_T(9) = \{7,8\}$.

△

Example 3.4.8. Let T be the graph shown below. Then T has two meristems; vertex 3 is of Type E, and vertex 5 is of Type D.



△

We want to characterize the β-graph of trees. The following theorem attempts to do just that. To make the proof a bit easier, the following lemmas are warranted.

Lemma 3.4.9. *Let T be an oriented tree, and suppose that m is a Type A meristem. Then $P(T)|_m$ has fewer β-sets than T.*

Proof Let m be a Type A meristem. That is, there exists exactly one $\ell \in \mathscr{L}_T(m)$ such that $\ell \to m$, and $\mathscr{L}_T(m) \setminus \{\ell\} \neq \emptyset$. In any β-set B of T, it is necessary that each sink-leaf in $\mathscr{L}_T(m) \setminus \{\ell\}$ is present. Since ℓ must either belong to B or dominate something in B, we have a choice that either $\ell \in B$ or $m \in B$ (but not both, since this would contradict minimality).

Now, suppose that B_ℓ is a β-set which contains the vertex ℓ. Consider the set $B_m = (B_\ell \setminus \{\ell\}) \cup \{m\}$. Since we swapped a single vertex for one other vertex, B_m is still size β. I claim that B_m is absorbant. Note that because $\ell \to m$, and all sink-leaves of m are in B_m, if B_m is not absorbant, then there exists some $u \in N(m) \setminus \mathscr{L}_T(m)$ such that u dominates nothing in B_m. But if this were true, then u would also not dominate anything in B_ℓ, contradicting absorption of B_ℓ. Thus, B_m is a β-set. In particular, for B any β-set containing ℓ, there is a corresponding β-set B' containing m such that $(B \setminus \{\ell\}) \cup \{m\} = B'$.

By pruning T, we remove all of $\mathscr{L}_T(m) \setminus \ell$, which makes m a sink of $P(T)$ (since it is Type A). Thus, in any β-set of $P(T)$, m must always be present. Furthermore, ℓ can not be a part of any β-set of $P(T)$, since its presence would necessarily contradict minimality (since m is also present). As a result, $P(T)$ has fewer β-sets than T.

■

Lemma 3.4.10. *Let T be an oriented tree with m a Type D meristem. Then $P(T)|_m$ yields no changes to any β-sets.*

Proof By definition, $P(T)|_m$ removes all sink-leaves of m, but since m is Type D, there are no sink-leaves to be removed, so that $P(T)|_m = T$. Thus, any β-set of T is a β-set of $P(T)|_m$.

■

Theorem 3.4.11. *Let T be an oriented tree on $[n]$, and $P(T)$ its pruning. Suppose that T has Type A meristems $\{m_1, \ldots, m_j\}$, and suppose T has no Type B, Type C, or Type E meristems. Then $T(\beta)$ is a subgraph of the digraph:*

$$P(T)(\beta) \square \left(\square_{k=1}^{j} P_2 \right)$$

where P_2 is isomorphic to the graph on two vertices v_1, v_2 with $v_1 \to v_2$.

Proof First, let us note that for all $i \in [j]$, m_i is a sink-meristem in $P(T)$ (by the nature of Type A meristems), and so is necessary for all β-sets of $P(T)$.

Let us think of reversing the pruning process. Start with $P(T)(\beta)$. Necessarily, each m_j is present in every β-set. By Lemma 3.4.10, pruning Type D meristems does nothing, so any β-set of T which contains (or does not contain) a Type D meristem, when pruned, yields a β-set of $P(T)$ which contains (or does not) the meristem as well. However, reintroducing sink-leaves to Type A meristems causes some non-trivial changes. We describe the algorithm to construct $T(\beta)$ from $P(T)(\beta)$ as follows:

Step 1: Consider m_1. By "un-pruning" $P(T)$ around m_1, we re-introduce all of m_1's sink-leaves. This has the effect of relaxing the constraint that m_1 appear in every β-set of $P(T)$. Suppose $P(T)(\beta)$ has N vertices $v_{1,m_1}, v_{2,m_1}, \dots v_{N,m_1}$ corresponding one-to-one to the β-sets: $\mathscr{B}_{1,m_1}, \mathscr{B}_{2,m_1}, \dots, \mathscr{B}_{N,m_1}$. Relaxing the constraint that m_1 must be in every β-set of $P(T)$ and instead allowing ℓ_1 (where ℓ_1 is the guaranteed source-leaf of m_1) to appear creates a copy of $P(T)(\beta)$, with N vertices $v_{1,\ell_1}, v_{2,\ell_1}, \dots, v_{N,\ell_1}$ corresponding to β-sets: $\mathscr{B}_{1,\ell_1}, \mathscr{B}_{2,\ell_1}, \dots, \mathscr{B}_{N,\ell_1}$. Since $\ell_1 \to m_1$ by assumption, and $\left|\mathscr{B}_{i,\ell_1} \cap \mathscr{B}_{i,m_1}\right| = \beta - 1$, then for each $i \in \{1, \dots, N\}$,

$$v_{i,\ell_1} \to v_{i,m_1}$$

Thus, "un-pruning" around m_1 corresponded to taking the Cartesian product $P(T)(\beta)\square(\ell_1 \to m_1)$. Define $T_1(\beta) = P(T)(\beta)\square(\ell_1 \to m_1)$.

Step 2: Consider m_2. We now "un-prune" $T_1(\beta)$ around m_2. This reintroduces all of m_2's sink-leaves, which again relaxes the condition that m_2 must appear in every set of $T_1(\beta)$, and allows ℓ_2 to appear instead, where ℓ_2 is the unique source-leaf of m_2. $T_1(\beta)$ has $2N$ vertices $v_{1,m_2}, \dots, v_{2N,m_2}$ corresponding to β-sets: $\mathscr{B}_{1,m_2}, \dots, \mathscr{B}_{2N,m_2}$. By "un-pruning", we make a copy of $T_1(\beta)$ with $2N$ vertices $v_{1,\ell_2}, \dots, v_{2N,\ell_2}$ corresponding to size β sets: $\mathscr{B}_{1,\ell_2}, \dots, \mathscr{B}_{2N,\ell_2}$. Since $\ell_2 \to m_2$, and $\left|\mathscr{B}_{i,\ell_2} \cap \mathscr{B}_{i,m_2}\right| = 1$, it follows that for each $i \in \{1 \dots, 2N\}$,

$$v_{i,\ell_2} \to v_{i,m_2}$$

So, we have formed the new digraph $T_2(\beta) = T_1(\beta)\square(\ell_2 \to m_2)$.

$\vdots$

Step j Consider m_j. By "un-pruning" $T_{j-1}(\beta)$ around m_j, we reintroduce all of m_j's sink-leaves, which relaxes the condition that m_j must appear in all $2^{j-1}N$ vertices $v_{1,m_j}, \dots v_{2^{j-1}N,m_j}$ of $T_{j-1}(\beta)$, and allows ℓ_j to appear instead, where ℓ_j is the unique source-leaf of m_j. The vertices of $T_{j-1}(\beta)$ correspond to: $\mathscr{B}_{1,m_j}, \dots, \mathscr{B}_{2^{j-1}N,m_j}$. "Un-pruning" makes a copy of $T_{j-1}(\beta)$, with $2^{j-1}N$ vertices $v_{1,\ell_j}, \dots, v_{2^{j-1}N,\ell_j}$ corresponding to size β sets: $\mathscr{B}_{1,\ell_j}, \dots, \mathscr{B}_{2^{j-1},\ell_j}$. Since $\ell_j \to m_j$, and $\left|\mathscr{B}_{i,\ell_j} \cap \mathscr{B}_{i,m_j}\right| = 1$, it follows that for each $i \in \{1, \dots, 2^{j-1}N\}$,

$$v_{i,\ell_j} \to v_{i,m_j}$$

So, we have formed a new digraph: $T_j(\beta) = T_{j-1}(\beta)\square(\ell_j \to m_j)$.

End Algorithm

We claim that $T(\beta)$ is a subgraph of $T_j(\beta)$. Suppose, for the sake of contradiction that this was not the case. Then there must exist some β-set Q of T such that Q does not correspond to any vertex in $T_j(\beta)$. Since Q is a β-set of T, it must contain all the sink-leaves of T. Consider the set $Q' = Q \setminus \{\text{all sink-leaves of } T\}$. This is certainly a subset of the vertices of $P(T)$ of size $\beta(P(T))$. If Q' is absorbant, then we are done, since then Q' must correspond to a vertex in $P(T)(\beta)$, of which there is an isomorphic copy in $T_j(\beta)$, yielding a contradiction. Therefore, we consider the case that Q' is not absorbant in $P(T)$. Then it must be the case that Q' does not contain some subset of the Type A meristems of T. That is, Q' does not contain $\{m_{k_1}, m_{k_2}, \dots, m_{k_q}\} \subseteq \{m_1, \dots, m_j\}$ for $q \geq 1$. Instead, $\{\ell_{k_1}, \ell_{k_2}, \dots, \ell_{k_q}\} \subset Q'$. Choose the β-set R' of $P(T)$ such that R' contains all Type A meristems of T, and $Q' \setminus \{\ell_{k_1}, \ell_{k_2}, \dots, \ell_{k_q}\} = R' \setminus \{m_{k_1}, m_{k_2}, \dots, m_{k_q}\}$. Define $R = R' \cup \{\text{all sink-leaves of } T\}$, and let v_R be the vertex in $T_j(\beta)$ corresponding to R. Via the algorithm above, there exists a path in the underlying graph of $T_j(\beta)$ such that moving along each vertex of the path corresponds to swapping m_{k_i} for ℓ_{k_i} for each $i \in [q]$. Following this path until each of the m_{k_i}'s are swapped terminates at a vertex $v \in T_j(\beta)$ which corresponds to the set

$$(R \setminus \{m_{k_1}, m_{k_2}, \dots, m_{k_q}\}) \cup \{\ell_{k_1}, \ell_{k_2}, \dots, \ell_{k_q}\} = Q$$

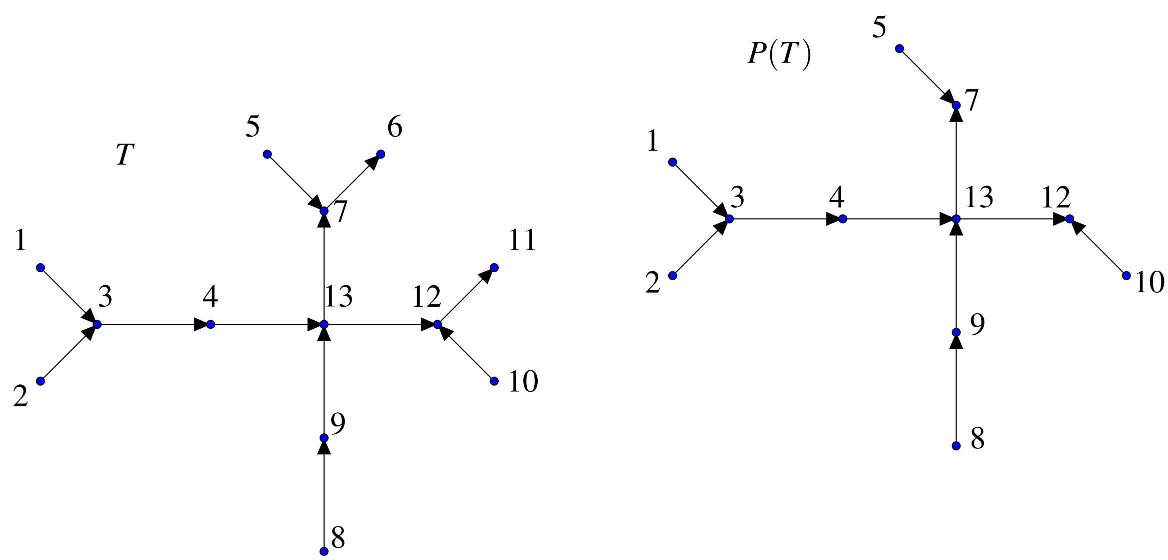
But this is a contradiction, since Q was assumed not to correspond to any vertex in $T_j(\beta)$. Thus, it must be that our assumption of the existence of such a Q was incorrect. Therefore, $T(\beta)$ is a subgraph of $T_j(\beta)$.

$\blacksquare$

Remark 3.4.12. Care should be taken with the β symbol in the proof of the previous theorem. In particular, note that at each step of the algorithm, β is increasing, due to the re-inclusion of the sink-leaves of the meristems.

The following example shows that $T(\beta)$ may in fact only be a proper subgraph of $P(T)(\beta)\Box\left(\Box_{k=1}^{j} P_2\right)$.

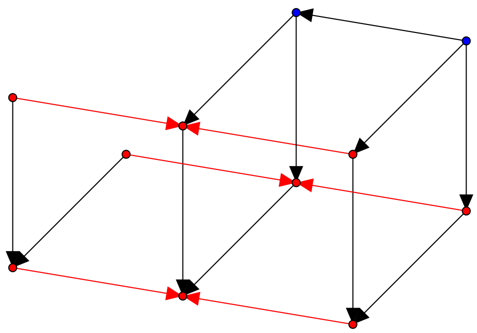
Example 3.4.13. Let T be the tree as shown below. The pruning $P(T)$ is shown to the right of T:



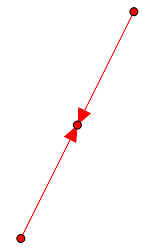
By examination, T has two Type A meristems, namely: vertices 7 and 12, and two Type D meristems: vertices 3 and 9. Thus, T satisfies the assumptions in the theorem above. By computing the β-graphs of both T and $P(T)$, we see that

$$T(\beta) \ncong P(T)(\beta)\Box P_2\Box P_2$$

$T(\beta)$



$P(T)(\beta)$



So, we indeed see that for some T which satisfy the necessary assumptions, $T(\beta)$ is a proper subgraph of $P(T)(\beta)\square P_2\square P_2$.

The example was computed using the code from the Appendix, and using the following commands:

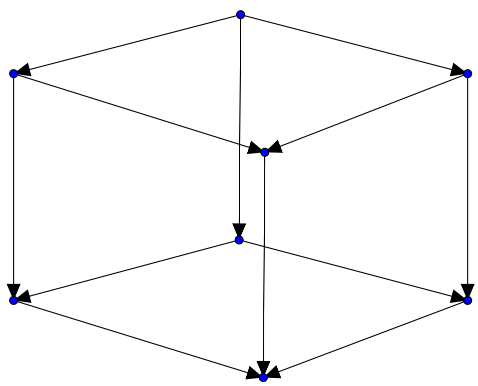
```
T = DiGraph({1:[3],2:[3],3:[4],4:[13],7:[6],13:[12,7],5:[7],8:[9],9:[13],10:[12],12:[11]});
T.show(); Beta_graph(T).show();                        ##display the trees T and beta graph
Pruner(T);                                               ## returns the pruned tree T
Pruner(T).show();Beta_graph(Pruner(T)).show();         ##display P(T), and its beta graph
```

$\triangle$

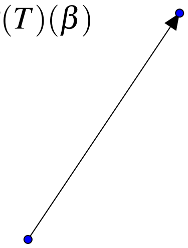
It certainly is possible that a tree T is such that $T(\beta) \cong P(T)(\beta)\square\left(\square_{k=1}^{j} P_2\right)$, as shown by the following.

Example 3.4.14. Recall the tree T from the first example of the section. Running T and its pruning $P(T)$ through the Sage code provided in the Appendix (this takes a bit of time to complete), we see the β-graphs of each below.

$T(\beta)$



$P(T)(\beta)$



It is easy to see that $T(\beta) \cong P(T)(\beta) \square P(T)(\beta) \square P(T)(\beta)$. As this shows, $T(\beta)$ may be isomorphic to the full Cartesian product of the graphs as mentioned in the statement of Theorem 3.4.11. As the next example shows, this is not always the case.

$\triangle$

Chapter 4

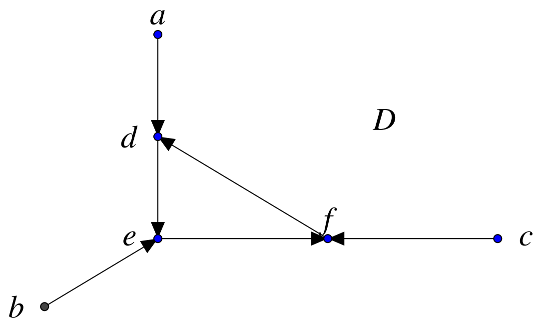
Open Questions

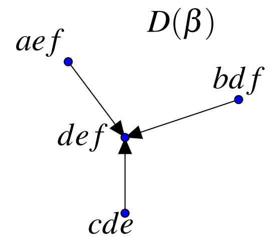
4.1 Further Study

Theorem 3.4.11 is a small first step in a full characterization of β-sequences of trees. It would be worthwhile to handle β-sequences of trees that have Type B, Type C, or Type E meristems, which also cause changes to β-sets under pruning. For such trees T, one would hope that a similar relationship would exist between $T(\beta)$ and $P(T)(\beta)$.

This paper arose from the study of kernels of digraphs, which undoubtedly play a critical role in studying β-graphs, $\mathscr{A}_k(D)$-graphs for any $\beta \leq k \leq |D|$, and in β-sequences. By Proposition 3.3.7, we know there exists a class of digraphs (without kernels, in particular) whose β-sequences will never terminate. Interestingly enough, the lack of a kernel is not enough to guarantee an infinite β-sequence, as evinced in the following example (which was shown to have no kernel):

Example 4.1.1. Let D be the digraph pictured below. By Proposition 2.3.3, D has no kernel. Interestingly, though, $D(\beta)$ terminates after only two iterations.





Of course, since def is a sink in $D(\beta)$, $D(\beta)^2 \cong K_1$.

$\triangle$

This suggests the following conjecture:

Conjecture 4.1.2. *Let $D = (V,A)$ be any oriented graph. If there exists an $n \in \mathbb{N}$, such that $D(\beta)^n$ has a kernel, then D has a terminating β-sequence.*

Were this conjecture true, it seems that most digraphs would have terminating β-sequences.

The following are some open questions related to β-sequences:

1. Are there bounds on the *length* of the β-sequence of a digraph D? Some digraphs have β-sequences that converge almost instantly. Are the number of iterations required for termination closely tied to some parameter of D?

2. The graph underlying $D(\beta)$ is always a subgraph of the $G(\gamma)$, where G is the underlying graph of D. The termination of γ-graph sequences is still open-ended. Does this relationship between $D(\beta)$ and $G(\gamma)$ give any indication of the convergence of γ-sequences? That is, if D has a terminating β-sequence, must G also have a terminating γ-sequence?

3. The Cartesian product digraph constructed in the proof of Theorem 3.4.11 possibly contains extraneous vertices (corresponding to size β sets which are not absorbant). Is there a way to count the number of these extraneous vertices?

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Appendix A

Sage Code

To examine more complicated β sequences, I have developed the following Sage [11] program, which can compute β-graphs, the nth term along the β-sequence, prunings, β-sets, and the absorption number β of a directed graph. There is still much room for optimization to diminish computing time for larger examples.

```
def vert_abs(S1,v,DiGraph):
    return [ vert for vert in S1 if vert in DiGraph.neighbors_in(v)]

def set_absorb(S1,S2,DiGraph):
    M = []
    for v in S1:
        M = M+vert_abs(S2,v,DiGraph)
    return Set(M)

def max_in_deg(DiGraph):
    M = max(DiGraph.in_degree_sequence())
    return M

def absorb_chk(Set1,Set2,DiGraph):
    if set_absorb(Set1,Set2.difference(Set1),DiGraph).intersection(Set2.difference(Set1)) ==
Set2.difference(Set1):
        return True
    else:
        return False

def min_abs_coll(Set1,Set2,DiGraph):
    t = len(DiGraph.vertices())-max_in_deg(DiGraph)+1
```

```
    for i in range(t):
        M = [W for W in Combinations(Set1,i) if absorb_chk(Set(W),Set2,DiGraph)==True]
        if M != []:
            break
    return M

def beta(DiGraph):
    Y = Set(DiGraph.vertices())
    return len(min_abs_coll(Y,Y,DiGraph)[0])

def b_set(DiGraph):
    Y = Set(DiGraph.vertices())
    betaD = [Set(y) for y in min_abs_coll(Y,Y,DiGraph)]
    return betaD

def card_chk(listy,num1,num2):
    if len(listy[num1] & listy[num2]) == len(listy[num1])-1:
        return True
    else:
        return False

def edge_chk(graphy,listy,num1,num2):
    W = (listy[num1]-(listy[num1] & listy[num2]))[0]
    V = (listy[num2]-(listy[num2] & listy[num1]))[0]
    return graphy.has_edge(W,V)

def Pruner(tree):
    foo = deepcopy(tree)
    foo.delete_vertices([y for y in foo.vertices() if foo.in_degree(y)==1 and foo.out_degree(y)==0])
    return foo

def Beta_graph(Gr):
    g = DiGraph([[1..len(b_set(Gr))], lambda i,j: i!=j and card_chk(b_set(Gr),i-1,j-1)==True and
edge_chk(Gr,b_set(Gr),i-1,j-1)==True])
    return g

def Beta_seq(B,N):
    C = deepcopy(B)
    for j in range(N):
        C = Beta_graph(C)
    return C
```