On Kernels, $\beta$-graphs, and $\beta$-graph Sequences of Digraphs

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Submitted to the Department of Mathematics and the
Graduate Faculty of the University of Kansas
in partial fulfillment of the requirements for the degree of
Master of Arts

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Date defended: 5-1-2015
The Master’s Thesis Committee for Kevin D. Adams certifies that this is the approved version of the following master’s thesis:

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Date approved: 5-1-2015
Abstract

We begin by investigating some conditions determining the existence of kernels in various classes of directed graphs, most notably in oriented trees, grid graphs, and oriented cycles. The question of uniqueness of these kernels is also handled. Attention is then shifted to $\gamma$-graphs, structures associated to the minimum dominating sets of undirected graphs. I define the $\beta$-graph of a given digraph analogously, involving the minimum absorbant sets. Finally, attention is given to iterative construction of $\beta$-graphs, with an attempt to characterize for what classes of digraphs these $\beta$-sequences terminate.
Acknowledgements

I would like to thank my advisor Professor Marge Bayer for her patience and guidance, as well as my committee members Professors Jeremy Martin and Yunfeng Jiang for teaching me some very interesting algebra and combinatorics. Thanks to all of the professors from whom I’ve learned so very much during my time at KU.

Thank you to my lovely fiancée Elise for her constant support and encouragement, even across long distances. To my parents, who have been reliable supports and comforts during all the rough spots.

Special thanks to the many graduate students in the department with whom I have shared these past three years, especially Kyle Claassen, Bennet Goeckner, and Josh Fenton for keeping me sane and laughing.

Thanks again to both Kyle Claassen and Jeremy Martin, who thoughtfully scoured my Sage code and gave crucial feedback to help make it better.
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Chapter 1

Introduction

Let a digraph $D = (V, A)$ be any orientation of a simple graph $G = (V, E)$. A subset $S \subseteq V(D)$ is said to be absorbant if for all $v \in V \setminus S$ there exists some $u \in S$ such that $v \to u$. $S \subseteq V(D)$ is said to be a kernel of $D$ if $S$ is both absorbant and independent. It should be noted that we follow both the definition and the (French) spelling of “absorbant” as found in [2]. In Chapter 2, we state some results concerning when a given digraph $D$ does or does not possess a kernel. This question has been handled in the past (see [2], [6]), and many results are known, dependent on Grundy functions or strong connectedness of graphs, though these topics are avoided in this paper.

A subset $S \subseteq V(G)$ is a dominating set of $V$ if every $v \in V \setminus S$ is adjacent to at least one $u \in S$. The domination number of a graph $G$ is $\gamma(G)$, the minimum cardinality of a dominating set. A $\gamma$-set is a dominating set of size $\gamma$. Fricke, et. al. [5] initiated the study of $\gamma$-graphs of graphs, where the $\gamma$-graph of $G$, is the graph $G(\gamma)$ with vertices corresponding one-to-one with $\gamma$-sets, where two $\gamma$-sets $\gamma_1$ and $\gamma_2$ are adjacent if there exist vertices $u \in \gamma_1, v \in \gamma_2$ such that

$$(\gamma_1 \setminus \{u\}) \cup \{v\} = \gamma_2$$

and $u, v$ are adjacent in $G$. It should also be mentioned that a different concept of $\gamma$-graph was developed by [12]. Similarly, the $\beta_0$, $\alpha_0$, and $\omega$-graphs of a graph have been handled (see [8]), where the respective graphs are defined using other graph invariants ($\alpha_0$-graphs deal with minimum
vertex coverings, for example). The beginning of chapter 3 is devoted to discussing the $\gamma$-graph structures defined as in [5]. Chapter 3 also discusses the $k$-dominating graphs $D_k$, with vertices in correspondence with dominating sets of cardinality at most $k$, and the generalized $\gamma$-graphs $X_k$, with vertices corresponding to the dominating sets with cardinality exactly $k$, each defined by Haas and Seyffarth in [7]. Haas and Seyffarth examine connectedness of these graphs and their application to reconfiguration problems, concluding with questions regarding Hamiltonicity of the $D_k$’s, and which graphs $G$ have the property that $G \cong D_k(G)$.

In analogy to the definition of $\gamma$-graphs, in Chapter 3, Section 2, I introduce the $\beta$-graph for a digraph $D$, considering $\beta$-sets of $D$, where $\beta(D)$ is the minimum cardinality of an absorbant set (as defined in [2]). Similar to their undirected counterparts, the edges in $D(\beta)$ are determined by the notion of “adjacent vertex swapping”, i.e., two $\beta$-sets $\beta_1$ and $\beta_2$ of $D$ are adjacent in $D(\beta)$ if there exist vertices $u \in \beta_1$ and $v \in \beta_2$ such that

$$(\beta_1 \setminus \{u\}) \cup \{v\} = \beta_2$$

and either $u \to v$ or $v \to u$ in $D$. Thus, $D(\beta)$ gives a directed graph, with the arc between $\beta_1$ and $\beta_2$ obeying the same direction as the arc between $u$ and $v$. (For example, $u \to v$ implies $\beta_1 \to \beta_2$.) Also defined are directed analogs of those graphs seen in [7], though the main focus is on the true $\beta$-graph (with vertices corresponding to $\beta$-sets).

At the close of [5], the authors give some examples of $\gamma$-graph sequences, noting that many of the sequences terminate in $K_1$. The question is left open, and is not mentioned in the course of [7]. In Chapter 3, Section 3 the study of the sequence of graphs obtained by iterating the $\beta$-graph construction is initiated, with the aim to characterize when these $\beta$-graph sequences terminate in graphs isomorphic to $K_1$. I offer a result on the non-termination of the $\beta$-sequences of cyclicly oriented odd cycles, and a characterization of the definite termination of the $\beta$-sequences of star graphs. The paper closes with a partial description of the structure of the $\beta$-graph of general trees.
Chapter 2

Kernels of Directed Graphs

2.1 Definitions and Preliminaries

Many of the following definitions and more can be found in [6].

**Definition 2.1.1.** Unless otherwise stated, we consider simple graphs \( G = (V, E) \), with \( V \) the vertex set, and \( E \) the collection of edges in \( G \). That is, \( G \) contains no multiple edges or loops.

**Definition 2.1.2.** A digraph \( D = (V, A) \) with underlying simple graph \( G = (V, E) \) is any orientation of the graph \( G \). That is, each arc in \( A \) is an edge of \( E \), together with a prescribed direction. We denote an arc \( (x, y) \in A \) by \( x \to y \).

**Remark 2.1.3.** The restriction to simple graphs \( G \) is by choice, in order that any directed graph with antisymmetric edges is avoided. However, the following discussions may certainly be made without this restriction.

**Definition 2.1.4.** We respectively define the open and closed neighborhoods of a vertex \( v \) by

\[
N(v) = \{ u \in V \mid \text{either } u \to v \text{ or } v \to u \}
\]

and \( N[v] = N(v) \cup \{v\} \).
Definition 2.1.5. Let $D = (V, A)$ be a digraph. Let $v \in V$. Define the outset of $v$ as the set

$$O(v) = \{u \in V \mid v \to u\}.$$ 

Similarly define the inset of $v$ by the set

$$I(v) = \{u \in V \mid u \to v\}.$$ 

We also define the sets $O[v] = O(v) \cup \{v\}$ and $I[v] = I(v) \cup \{v\}$ as the closed outset and closed inset of $v$, respectively.

Definition 2.1.6. Let $D$ be a digraph, and let $v \in V$. $v$ is a sink vertex (or just sink) if $I(v) = N(v)$, i.e., if $v$ is dominated by all of its neighbors. $v$ is a source vertex (or just source) if $O(v) = N(v)$, i.e., if $v$ dominates all of its neighbors.

Remark 2.1.7. Alternatively, we can describe a sink or a source as a vertex $v$ satisfying $O(v) = \emptyset$ or $I(v) = \emptyset$, respectively.

Definition 2.1.8. Let $D = (V, A)$ be a digraph, and $S \subseteq V$. We say that $S$ is absorbant if for each $v \in V \setminus S$, there exists at least one $u \in S$ such that $v \to u$, i.e., each vertex $v$ in the complement of $S$ dominates at least one vertex inside of $S$. We say $S$ is independent if for any pair $x, y \in S$, $x \not\to y$ and $y \not\to x$.

Definition 2.1.9. Let $D = (V, A)$ be a digraph, and $K \subseteq V$ which is both independent and absorbant. We call $K$ a kernel of the digraph $D$.

2.2 Trees and Grids

Lemma 2.2.1. (Theorem 15.8 of [1]) Let $T$ be an oriented tree on $n$ vertices. Then there exists $v \in V(T)$ such that $O(v) = \emptyset$, i.e., every oriented tree on $n$ vertices has a sink.
Proof Let \( u_0 \) be a leaf of \( T \). Either \( I(u_0) = \emptyset \) or \( O(u_0) = \emptyset \). If the latter is true, then take \( v = u_0 \). If \( I(u_0) = \emptyset \), then pick \( u_1 \in O(u_0) \). If \( O(u_1) = \emptyset \), take \( v = u_1 \) and we are done. Otherwise, pick \( u_2 \in O(u_1) \) and continue. Since \( T \) has \( n \) vertices, this process must terminate at some \( u_j \) for \( u_j \in O(u_{j-1}) \), \( 1 \leq j \leq n-1 \). Then \( O(u_j) = \emptyset \), so take \( v = u_j \).

\[ \square \]

It must be noted that the following result is certainly not a new one. Indeed, this was shown by Von Neumann and Morgenstern [13], and has since been improved upon. The following is an alternate proof.

**Proposition 2.2.2.** Any oriented tree on \( n \) vertices has a unique kernel.

**Proof** We induct on \( n \). For \( n = 1 \), the result is trivial. For \( n = 2 \), the tree is a directed path, and we take as the unique kernel the vertex \( v_i \) such that \( O(v_i) = \emptyset \).

Suppose the result holds for all \( k \) up to \( n-1 \). Fix an orientation on \( T_n \), a tree on \( n \) vertices labeled \( \{v_1, \ldots, v_n\} \). By the lemma, there exists \( v_j \in V(T_n) \) such that \( O(v_j) = \emptyset \).

Consider the forest \( F \) induced by the vertices of \( V \setminus N[v_j] \). \( F \) has some number \( m \geq 0 \) of components.

If \( m = 0 \), then \( N[v_j] = V \), in which case \( T_n \) is a star with all pendant edges oriented toward the center node \( v_j \). Thus, we take \( \{v_j\} \) as the unique kernel.

So, suppose \( m > 0 \). Then each component \( F_i \) of \( F \) is an oriented tree on fewer than \( n \) vertices. By our induction hypothesis, each \( F_i \) has a unique kernel \( K_i \) for \( i = 1, \ldots, m \). Let \( K := \bigcup_{i=1}^m K_i \cup \{v_j\} \). Since \( N(v_j) \cap F_i = \emptyset \) for each \( i = 1, \ldots, m \), \( K \) is certainly independent. It is also absorbant, since \( v_j \) is dominated by each element of \( N(v_j) \). Thus \( K \) is a kernel of \( T_n \). To see that this \( K \) is unique, suppose \( K' \) is another kernel of \( T_n \). Since \( O(v_j) = \emptyset \), it must be that \( v_j \in K \) and \( v_j \in K' \).

Define

\[ K'_i := \{v \in V \mid v \in K' \cap F_i\} \]

Then \( K' = \{v_j\} \cup \bigcup_{i=1}^m K'_i \). We claim that \( K'_i \) is a kernel of \( F_i \) for each \( i \). By construction \( K'_i \) is independent. If \( (V \setminus K') \cap F_i = \emptyset \), then \( F_i = K'_i = \{v_k\} \) for some \( k \), and thus \( K'_i \) is trivially absorbant.
Otherwise, since $K'$ is absorbant, it follows that for any vertex $v \in (V \setminus K') \cap F_i$,

$$O(v) \cap K_i' \neq \emptyset$$

That is, $K_i'$ is absorbant and therefore a kernel of $F_i$.

By uniqueness of the $K_i$, it must be that $K_i = K_i'$ for each $i \in \{1, \ldots, m\}$. Therefore,

$$K' = \{v_j\} \cup \bigcup_{i=1}^{m} K_i' = \{v_j\} \cup \bigcup_{i=1}^{m} K_i = K$$

showing that $K$ is the unique kernel of $T_n$.

\[ \blacksquare \]

The following proposition is attributed to Richardson [10]. Berge and Duchet [3] give a simpler proof.

**Proposition 2.2.3.** Any oriented bipartite graph possesses a kernel.

**Proof** Let $D_k$ denote an oriented bipartite graph on $k$ vertices. We proceed by induction on $k$. Clearly, any orientation on $D_2$ possesses a kernel. Suppose the result holds for all $k$ up to $n - 1$. Consider the case when $k = n$.

Fix an orientation on a bipartite graph $D_n$. Let $V_1, V_2$ denote the (nonempty) partite sets of $D_n$, and consider $V_1$. By definition $V_1$ is independent. If it is also absorbant, then $V_1$ is a kernel, and we are done. If this is not the case, then there exists at least one vertex $u \in V_2$ such that $O(u) = \emptyset$, i.e., $u$ is a sink. For the sink $u$, if $I(u) = V_1$, then take $V_2$ as kernel. If $\{u\} = V_2$, then take the union of $V_2$ with all isolated vertices of $V_1$ (if any exist) as the kernel. Otherwise, consider the induced sub-digraph on $D \setminus N[u]$. Here, $D \setminus N[u]$ is a bipartite graph on fewer than $n$ vertices. Then, by the induction hypothesis, there exists a kernel $K'$ in $D \setminus N[u]$.

We claim that in $D$, the union $K = K' \cup \{u\}$ is a kernel. It is clear that $K$ is independent. $K$ is absorbant, since $K'$ is absorbant by definition, and since $u$ is dominated by all vertices in $N(u)$.
Then by definition, $K$ is a kernel of $D$.

**Remark 2.2.4.** Indeed, the previous proposition deals only with existence, and not uniqueness. For example, consider the following:

Then either pair of diametrically opposite vertices is independent and absorbant.

The following are some examples of graphs which necessarily have at least one kernel by the previous proposition.

**Example 2.2.5.** It is well known that the $k$-dimensional cube $Q_n$ is bipartite. Therefore, any orientation of the $k$-dimensional cube $Q_k$ has a kernel. For reference, the unit 3-dimensional cube $Q_3$ is pictured below:

**Definition 2.2.6.** As defined in [5], the step-grid graph $SG(k)$ is the induced subgraph of the $k \times k$ grid graph $P_k \square P_k$ with vertex set

$$V(k) = \{(i, j) : 1 \leq i, j \leq k, i + j \leq k + 2\}$$

and edge set

$$E(k) = \{((i, j), (i', j')) : i = i', j' = j + 1; i' = i + 1, j = j'\}$$
Example 2.2.7. By definition of $E(k)$, for any positive integer $k$, $SG(k)$ has no odd cycles, i.e. is bipartite. Then for all $k$, any orientation of a step-grid graph $SG(k)$ has a kernel.

The graph $SG(3)$ is pictured below on the left, and $SG(4)$ on the right:

2.3 General Results

Proposition 2.3.1. Let $G = (V, E)$ be a finite graph. There exists an orientation $\mathcal{O} = (V(G), A)$ of $G$ which possesses a kernel.

Proof We construct an orientation $\mathcal{O}$ of $G$ as follows: Let $M$ be a maximal independent set. Recall that $N(M)$ denotes the neighborhood of $M$. It is quick to see that $N(M) = V \setminus M$, for if $N(M) \subset V \setminus M$, there must exist some $u \in V \setminus M$ such that $u \notin N(M)$. That is, $u$ is not adjacent to anything in $M$, and thus must be in $M$, a contradiction to the maximality of $M$.

For every $u \in N(M)$, and for every edge $um \in E$ such that $m \in M$, add the arc $u \to m$ to the arc set $A$ of $\mathcal{O}$. Complete the orientation by directing any other edges of $E$ in any direction. Under the orientation $\mathcal{O}$, $M$ is absorbant since we have forced all of $N(M) = V \setminus M$ to dominate at least one element of $M$. Therefore, $M$ is a kernel under the orientation $\mathcal{O}$.

Lemma 2.3.2. Any finite graph with an odd cycle possesses a chordless odd cycle.
Proof. Let $G$ be a finite graph containing some odd cycle $\mathcal{O}$ ordered on the vertices $(u_1, u_2, \ldots, u_{2k+1})$, $k \geq 1$. If $\mathcal{O}$ is itself chordless, we are done. So assume that $\mathcal{O}$ has a chord $e$ between vertices $u_i, u_j$ for (and without loss of generality) $j > i$, $j \neq i + 1$. $e$ divides the cycle into two sub cycles intersecting along $e$, call them $C_1, C_2$. Without loss of generality, if $C_1$ is an even cycle, then $C_2$ is a cycle on the vertices of $(\mathcal{O} \setminus C_1) \cup \{u_i, u_j\}$, of which there are an odd number. That is, $C_2$ is odd. If $C_2$ is chordless, we are done. If not, we repeat the process from above to divide $C_2$ into an even and an odd cycle. Since $G$ is finite, this process must terminate in an odd chordless cycle.

Proposition 2.3.3. Let $G = (V, E)$ be any finite graph possessing an odd cycle. Then there exists an orientation of $G$ for which there is no kernel.

Proof. Let $G$ be any finite graph with an odd cycle $\mathcal{O}$. By the Lemma 2.3.2, $G$ contains some chordless odd cycle $\mathcal{C} = \{u_1, \ldots, u_n\}$ for $n = 2k + 1$, $k \geq 1$. Without loss of generality, let $A$ be an orientation of $G$ such that

$$u_1 \rightarrow u_2 \rightarrow u_3 \rightarrow \ldots \rightarrow u_k \rightarrow u_1$$

and such that for any edge $E$ not contained in $\mathcal{C}$, with $u_i \in E$ for some $i$, $u_i$ is dominated by the other endpoint in $E$.

I claim that $D = (V(G), A)$ has no kernel. Suppose, for the sake of contradiction that $K$ was some kernel of $D$. By design, each $u_j$, $1 \leq j \leq k$ either must be in $K$, or must dominate an element of $K$. Therefore, for any $j$, if $u_j \notin K$, then $u_{j+1(\text{mod} \, n)} \in K$. Form the finite collection of pairs

$$\{\{u_1, u_2\}, \{u_3, u_4\}, \ldots, \{u_{n-2}, u_{n-1}\}\}$$

By the Pigeonhole Principle, vertex $u_n$ is left solitary. But then it follows that

$$u_1 \in K \Rightarrow u_{n-2} \in K \Rightarrow u_{n-1} \notin K \Rightarrow u_n \in K$$

which is a contradiction, since $u_n \rightarrow u_1$. We reach a similar contradiction under assumption that
\[ u_1 \notin K, \text{ since} \]
\[ u_2 \in K \Rightarrow u_{n-1} \in K \Rightarrow u_n \notin K \Rightarrow u_1 \in K \]

Therefore, there exists a vertex \( v \in C \) such that \( v \) can never simultaneously dominate a vertex \( u \in C \) and be independent from \( u \). Therefore, \( K \) cannot be a kernel. Since \( K \) was arbitrary, there can be no independent absorbant set of \( D \).

**Example 2.3.4.** Let \( D \) be the digraph pictured below: It is easy to check that there is no kernel. This is an example of the particular orientation in the previous proposition.

\[ a \quad b \quad c \quad d \quad e \quad f \]

It is a well-known fact in graph theory (attributed to König [9]) that a graph is bipartite if and only if it has no odd cycles. As a consequence, Proposition 2.2.3 and Proposition 2.3.3 together imply the following:

**Proposition 2.3.5.** Every orientation of a graph has a kernel if and only if the graph is bipartite.

**Proof** The backward direction is already proved by Proposition 2.2.3. So, suppose a digraph \( G \) is such that every orientation of \( G \) has a kernel. Suppose for the sake of contradiction that \( G \) is not bipartite. Then \( G \) possesses some odd cycle, and Proposition 2.3.3 gives an immediate contradiction.
Chapter 3

Graphs from Dominating and Absorbant Sets

3.1 $\gamma$-graphs and Generalized $\gamma$-graphs

Fricke, Hedetniemi, Hedetniemi, and Hutson [5] introduce the $\gamma$-graph of a graph. It should be noted that other notions of $\gamma$-graphs have been considered in [12], though the definitions of the graphs are significantly different.

Definition 3.1.1. Let $G = (V, E)$ be a graph (undirected). A set $S \subseteq V$ is called a dominating set of $G$ if for all $v \in V \setminus S$, $N(v) \cap S \neq \emptyset$. The domination number of $G$ is

$$\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}$$

Any dominating set with cardinality equal to $\gamma(G)$ is referred to from here on as a $\gamma$-set.

The authors present the $\gamma$-graph of a graph $G$ as a construct on the set of all $\gamma$-sets, used to find maximum or minimum values of a number of parameters of subgraphs induced by dominating sets, e.g., to maximize the number of isolated vertices in $G[S]$, indicating $S$ as “close” to an independent dominating set.
**Definition 3.1.2.** For a graph $G = (V, E)$, consider the collection of all $\gamma$-sets of $G$. Define the $\gamma$-graph of $G$ as the graph $G(\gamma) = (V(\gamma), E(\gamma))$, with vertex set in one-to-one correspondence with the $\gamma$-sets of $G$, and with $\gamma$-sets $S_1, S_2$ adjacent in $G(\gamma)$ if there is a vertex $v \in S_1$ and a vertex $w \in S_2$ such that:

i) $v, w$ are adjacent in $G$

ii) $S_1 = (S_2 \setminus \{w\}) \cup \{v\}$ and $S_2 = (S_1 \setminus \{v\}) \cup \{w\}$.

In other words, adjacency is defined by the ability to “swap” one and only one pair of $G$-adjacent vertices between dominating sets.

**Example 3.1.3.** Let $G = K_3$. Then $\gamma(G) = 1$, and $G(\gamma) = K_3$

\[ \begin{array}{c}
G & a \\
\triangle & b \\
c & \{a\}
\end{array} \]

In $G(\gamma)$ the vertex corresponding to the $\gamma$-set $\{2, 4\}$ is colored in red (as is $\{2, 4\}$) to highlight the correspondence between $\gamma$-sets of $G$ and vertices in $G(\gamma)$. Then $G(\gamma)$ is as shown below. Note that $G(\gamma) \cong K_{2,4}$.

**Example 3.1.4.** Let $G = C_4$. $\gamma(G) = 2$. The $\gamma$-sets of $G$ are $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. The vertex in $G(\gamma)$ corresponding to the $\gamma$-set $\{2, 4\}$ is colored in red (as is $\{2, 4\}$) to highlight the correspondence between $\gamma$-sets of $G$ and vertices in $G(\gamma)$. Then $G(\gamma)$ is as shown below. Note that $G(\gamma) \cong K_{2,4}$. 

\[ \begin{array}{c}
G & a \\
\triangle & b \\
c & \{a\}
\end{array} \]
One can similarly intuit some definitions of a $\gamma$-graph-like structure in the directed case. We consider two constructions:

**Construction 1:** Let $G = (V, E)$ be a graph, and let $D = (V, A)$ be an orientation of $G$. Define $\overrightarrow{G}(\gamma) = (V(\gamma), A(\gamma))$ to be the graph with vertices in one-to-one correspondence with the $\gamma$ sets of the underlying graph $G$, and a directed edge between $\gamma$ sets $S_1, S_2$ if there exists $v \in S_1, w \in S_2$ such that

1) $v \rightarrow w \in A$ or $w \rightarrow v \in A$

2) $(S_1 \setminus \{v\}) \cup \{w\} = S_2$ and $(S_2 \setminus \{w\}) \cup \{v\} = S_1$

The direction of the edge between $S_1$ and $S_2$ follows the same direction as $v \rightarrow w$ or $w \rightarrow v$, whichever may be in $A$. We note that if both $w \rightarrow v \in A$ and $v \rightarrow w \in A$, then there is a symmetric pair of arcs between $S_1, S_2$.

**Remark 3.1.5.** This construction, however, is not very interesting, as it is the same structure as the undirected $\gamma$-graph. Moreover, some sets which are dominating sets of the underlying graph $G$ may not be dominating sets in the directed graph $D$. See the following example.
Example 3.1.6. Let $G = C_4$. Let $D$ be the cyclic orientation on $C_4$. In the underlying graph $G$, the set of vertices $\{1, 2\}$ is a dominating set of $G$. However, $\{1, 2\}$ is not a dominating set of $D$, since neither 1 nor 2 dominate the vertex 4. Thus, the graph given by construction 1 contains some extraneous information.

\[ G \]

\[ G(\gamma) \]

\[ \triangle \]

For the next construction, we need some extra definitions.

Definition 3.1.7. Let $D = (V, A)$ be a directed graph. As in [4], define an out-dominating set as a set $S \subseteq V$ such that for all $v \in V \setminus S$, there exists some $u \in S$ such that $u \rightarrow v \in A$. The out-domination number $\gamma^+(D)$ of $D$ is the minimum cardinality of an out-dominating set.

Remark 3.1.8. We note that the above definition is equivalent to what is defined as a “dominating set” in [6].

Construction 2: Let $D = (V, A)$ be a directed graph. Define the $\gamma^+$-graph $D(\gamma^+) = (V(\gamma^+), A(\gamma^+))$ to be the graph with vertices in one-to-one correspondence with the $\gamma^+$ sets of $D$, and a directed edge between $\gamma^+$ sets $S_1, S_2$ if there exists $v \in S_1, w \in S_2$ such that

i) $v \rightarrow w \in A$ or $w \rightarrow v \in A$

ii) $(S_1 \setminus \{v\}) \cup \{w\} = S_2$ and $(S_2 \setminus \{v\}) \cup \{v\} = S_1$
The direction of the edge between $S_1$ and $S_2$ follows the same direction as $v \rightarrow w$ or $w \rightarrow v$, whichever may be in $A$. We note that if both $w \rightarrow v \in A$ and $v \rightarrow w \in A$, then there is a symmetric pair of arcs between $S_1, S_2$.

**Remark 3.1.9.** With this construction, it should be noted that the extraneous information present in the first construction is no longer there. The vertices in $D(\gamma^+)$ are precisely the $\gamma^+$ sets of $D$.

**Example 3.1.10.** Let $G = C_4$. Let $D$ be the cyclic orientation on $C_4$. $\gamma^+(D) = 2$, and we have only two $\gamma^+$-sets: $\{\{1, 3\}, \{2, 4\}\}$.

Haas and Seyffarth [7] expand on the study of $\gamma$-graphs, generalizing the definition to $k$-dominating graphs, incorporating dominating sets of any possible size, not just $\gamma$-sets.

**Definition 3.1.11.** The *upper domination number* of a graph $G$ is

$$\Gamma(G) = \max\{|S| : S \text{ is a minimal dominating set of } G\}$$

**Definition 3.1.12.** The $k$-dominating graph of $G$, $D_k(G)$ is the graph with vertices in one-to-one correspondence with the dominating sets of $G$ with cardinality at most $k$. An edge joins two vertices $v, w$ of $D_k(G)$ if and only if the corresponding dominating sets $S_v$ and $S_w$ differ by addition or deletion of a single vertex.
**Remark 3.1.13.** Note that the edges of $D_k(G)$ are only between sets that differ in cardinality by 1, and the notion of “vertex swapping”, seen in the $\gamma$-graph case, is no longer present.

Haas and Seyffarth later describe an analog of the $\gamma$-graph for different sizes of dominating sets.

**Definition 3.1.14.** Define $X_k(G)$ as the graph with vertices in one-to-one correspondence with the dominating sets of $G$ with cardinality $k$, and an edge between two dominating sets $S_1, S_2$ if there exist $v \in S_1$ and $w \in S_2$ such that

i) $v, w$ are adjacent in $G$

ii) $S_1 = (S_2 \setminus \{w\}) \cup \{v\}$ and $S_2 = (S_1 \setminus \{v\}) \cup \{w\}$

**Remark 3.1.15.** It should be noted that $X_\gamma = G(\gamma)$.

**Example 3.1.16.** The graph $G$ is pictured below. Here, $\gamma(G) = 1$, since \{2\} is itself a dominating set. Furthermore, $\Gamma(G) = 2$, since \{1, 3\} is minimal dominating, and any other dominating set of size two or more either contains \{1, 3\}, \{2\}, or \{4\}, and therefore is not minimal.
Similar to the construction given by Haas and Seyffarth, we would like to develop some object involving the absorbant sets of a directed graph. We hope to get some results about kernels by studying this structure. We examine this in the next section.

### 3.2 The $\beta$-graph of a Digraph

**Definition 3.2.1.** Let $G = (V, E)$ be a graph. The *independence number* of $G$ is defined to be

$$\alpha(G) = \max\{|S| : S \text{ is independent}\}$$

We note that any maximally independent set of $G$ has cardinality at most $\alpha(G)$.

**Definition 3.2.2.** Let $D = (V, A)$ be a digraph. The *absorption number* of $D$, defined by Berge [2] is

$$\beta(D) = \min\{|S| : S \text{ is absorbant in } D\}$$

We refer to any absorbant set with cardinality $\beta$ as a $\beta$-set.

**Remark 3.2.3.** We note that Chartrand, Harary, and Yue discuss some results on absorption numbers of digraphs under the name of the “in-domination number”. [4]

**Definition 3.2.4.** Let $D = (V, A)$ be a digraph. Define the $\beta$-graph of $D$ to be the graph $D(\beta)$ with vertices in one-to-one correspondence with the absorbant sets of $D$, and an arc joining absorbant sets $S_1, S_2$ if there exist vertices $v \in S_1$ and $w \in S_2$ such that

i) $v \rightarrow w \in A$ or $w \rightarrow v \in A$ (but not both)

ii) $S_1 = (S_2 \setminus \{w\}) \cup \{v\}$ and $S_2 = (S_1 \setminus \{v\}) \cup \{w\}$

with the direction of the arc between $S_1$ and $S_2$ agreeing with the direction of the arc between $v$ and $w$. 

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Definition 3.2.5. Let $D = (V, A)$ be a digraph. Define the $k$-absorbant graph $A_k(D)$ as the graph with vertices in one-to-one correspondence with the absorbant sets of $D$ with cardinality $k$, and an arc joining absorbant sets $S_1, S_2$ if there exist vertices $v \in S_1$ and $w \in S_2$ such that

i) $v \to w \in A$ or $w \to v \in A$ (but not both)

ii) $S_1 = (S_2 \setminus \{w\}) \cup \{v\}$ and $S_2 = (S_1 \setminus \{v\}) \cup \{w\}$

with the direction of the arc between $S_1$ and $S_2$ agreeing with the direction of the arc between $v$ and $w$.

Remark 3.2.6. For $k = \beta(D)$, $A_k(D) = D(\beta)$.

Definition 3.2.7. Define the absorbant hierarchy of $D$ to be the graph

$$D[\beta] = \bigsqcup_{k=\beta(D)}^{|V(D)|} A_k(D)$$

the disjoint union of all $k$-absorbant graphs of $D$.

Remark 3.2.8. If $D$ possesses kernels, then their corresponding vertices all lie in

$$\bigsqcup_{k=\beta(D)}^{|\alpha(D)|} A_k \subseteq D[\beta]$$

Proposition 3.2.9. Let $D = (V, A)$ be any oriented tree. We know there exists a unique kernel $K$. Let $n = |K|$. Consider $v \in A_n(D)$, the vertex corresponding to the kernel $K$. Then $I(v) = \emptyset$, i.e., $v$ is either isolated, or a source.

Proof For this $v$, suppose $w \in A_n(D)$ such that $v \leftarrow w$. By definition, this means that there exists some $v' \in K$ and $w' \not\in K$ such that

$$(K \setminus v') \cup w'$$

is absorbant, and $v' \leftarrow w'$. Now, $v' \not\in (K \setminus v') \cup w'$, and so must dominate at least one element in $(K \setminus v') \cup w'$. By acyclicity of $D$, $v' \not\to w'$. Thus, if $v'$ dominates anything in $(K \setminus v') \cup w'$, it must
dominate some element in $K \setminus v'$. But then $K$ would not be an independent set, and not a kernel, which is a contradiction. Therefore, $I(v) = \emptyset$. Accordingly, $v$ is either isolated (if $O(v) = \emptyset$), or a source (if $O(v) \neq \emptyset$).

\[\begin{array}{c}
\text{Remark 3.2.10.} \text{ The converse of this is not true. That is, if } \mathcal{A}_k(D) \text{ has a source vertex } v, \text{ the absorbant set corresponding to } v \text{ is not necessarily a kernel. Consider the following example.} \\
\end{array}\]

\[\begin{array}{c}
\text{Example 3.2.11.} \text{ Let } D \text{ be the following directed graph. With the prescribed orientation, it follows from Proposition 2.3.3 above, that } D \text{ possesses no kernel. However, the } \beta \text{-graph of } D \text{ is shown to the right as a } P_3. D(\beta) \text{ has source corresponding to } \{3, 4, 6\}, \text{ which is absorbant, but not independent, and so is not a kernel.} \\
\end{array}\]

\[\begin{array}{c}
\text{D} \\
1 \quad 2 \quad 3 \\
7 \quad 4 \quad 5 \\
\quad 6 \\
\end{array} \quad \begin{array}{c}
\text{D(}\beta\text{)} \\
346 \\
356 \\
\quad 357 \\
\end{array}\]

\[\begin{array}{c}
\triangle \\
\end{array}\]

\[\begin{array}{c}
\text{Proposition 3.2.12.} \text{ Let } D = (V,A) \text{ be a digraph. Suppose } D \text{ possesses multiple kernels } K_1, \ldots, K_n. \text{ Let } v_1, \ldots, v_n \text{ be vertices in } D[\beta] \text{ corresponding to the kernels } K_1, \ldots, K_n, \text{ respectively. Then } \{v_1, \ldots, v_n\} \text{ forms an independent set.} \\
\text{Proof} \text{ To begin, we note that if } K_r, K_s \text{ are kernels of different sizes, then } v_r, v_s \text{ are non-adjacent by} \\
\end{array}\]
the very definition of \( D[\beta] \). So, suppose that there exist some \( i, j, i \neq j \) such that \( v_i \rightarrow v_j \) in \( \mathcal{A}_m(D) \) where \( v_i, v_j \) correspond to some kernels \( K_i, K_j \) of \( D \) each of size \( m \). By definition of \( \mathcal{A}_m(D) \), there exists some vertex \( w_i \in K_i \), and some vertex \( w_j \in K_j \) such that

\[
K_j = (K_i \setminus w_i) \cup \{w_j\}
\]

is absorbant and \( w_i \rightarrow w_j \) in \( D \). But, \( w_j \notin K_i \), and so \( w_j \) dominates some vertex in \( K_i \setminus \{w_i\} \). But this implies that \( K_j = (K_i \setminus \{w_i\}) \cup \{w_j\} \) is not independent, which is a contradiction, since \( K_j \) is a kernel. Therefore, it must be that

\[
\{v_1, \ldots, v_n\}
\]

is independent in \( D[\beta] \).

\[\blacksquare\]

**Proposition 3.2.13.** Let \( u \) be a source in some \( \beta \)-graph. Then \( O(u) \) is independent.

**Proof** The result is clear for the cases when \( |O(u)| = 0, 1 \). So, suppose that \( |O(u)| > 1 \). Consider \( v, w \in O(u) \). We show that there is no arc joining \( v \) and \( w \).

Assume, for the sake of contradiction that \( v \rightarrow w \). Let \( \beta_u, \beta_v, \beta_w \) be the \( \beta \)-sets that correspond to \( u, v, w \). By definition of adjacency in \( \beta \)-graphs, \( u \rightarrow v \) implies that there exist vertices \( t_u \in \beta_u, t_v \in \beta_v \) such that

\[
(\beta_u \setminus \{t_u\}) \cup \{t_v\} = \beta_v
\]

Similarly, since \( u \rightarrow w \), there exist vertices \( t'_u \in \beta_u, t_w \in \beta_w \) such that

\[
(\beta_u \setminus \{t'_u\}) \cup \{t_w\} = \beta_w
\]

Finally, by assumption, \( v \rightarrow w \) implies that there exist vertices \( t'_v \in \beta_v, t'_w \in \beta_w \) such that

\[
(\beta_w \setminus \{t'_w\}) \cup \{t'_v\} = \beta_v
\]
We claim that $t_v = t'_v$, for if not, then

$$(β_u \setminus \{t_u\}) ∩ (β_w \setminus \{t'_w\}) < β - 1$$

which cannot happen since $u → w$ implies that $β_u, β_w$ agree on exactly $β - 1$ vertices.

The following result is analogous to Theorem 21 in [5] about the bipartiteness of $T(γ)$ when $T$ is a tree.

**Proposition 3.2.14.** For any tree $T$, $T(β)$ is $C_n$-free for all odd integers $n ≥ 3$. In other words, $T(β)$ is bipartite.

**Proof** Suppose that the underlying graph $T(β)$ contains some $C_{2k+1}$ on ordered vertices $u_1, u_2, ..., u_{2k+1}$, and let $β_1, β_2, ..., β_{2k+1}$ be the corresponding $β$-sets. Without loss of generality, consider the length $k + 1$ chain of vertex swaps obtained between $u_1, u_2, up to u_{k+2}$. Suppose $1 ≤ q ≤ k + 1$ represents the number of differences in the $β$-sets $β_1$ and $β_{k+2}$. There are two situations to consider.

First, suppose that $q = k + 1$. That is, $β_1$ and $β_{k+2}$ differ in exactly $k + 1$ vertices. Then the chain of vertex swaps from $u_{k+2}$ to $u_{k+3}$ up to $u_1$ can change back at most $k$ of these $k + 1$ differences before reaching $u_1$. That is, $u_1, u_{2k+1}$ are adjacent in $T(β)$, but differ by more than one vertex, which is a contradiction to the definition of $β$-graphs.

So, suppose that $q < k + 1$. Then there exists some $i, j, ℓ$ with $1 ≤ i < j < ℓ ≤ k + 2$ such that the chain of vertex swaps from $u_i$ to $u_ℓ$ is as follows: between $u_i$ and $u_j$, swap $t_i$ for $t_j$, and between $u_j$ and $u_ℓ$, swap the same $t_j$ for $t_ℓ$. Then traversing the underlying cycle $u_i, ..., u_j, ..., u_ℓ, ..., u_i$, there must be some sequence of vertex swaps of the form $t_i, ..., t_j, ..., t_ℓ, ..., t_i$, which is a cycle of vertices in the underlying graph of $T$, yielding a contradiction.

Therefore, if $n$ is odd $T(β)$ must be $C_n$-free, i.e., $T(β)$ is bipartite.

**Remark 3.2.15.** It should be noted that the last paragraph of the proof of Proposition 3.2.14 does not depend on the parity of a cycle. Cycles of length $2k$ may similarly not appear in the $β$-graph if
by moving from some vertex $u_r$ to $u_{r+k+1}$ in the underlying cycle, there exist vertex swaps which share a vertex (in other words, if there are vertices $t_i, t_j, t_\ell$ such that $t_i$ is swapped for $t_j$ and $t_j$ is swapped for $t_\ell$). However, cycles of even length may appear in the $\beta$-graph if for any $1 \leq r \leq 2k$, moving from vertex $u_r$ to $u_{r+k+1}$ yields exactly $k$ differences between the corresponding $\beta$-sets $\beta_r$ and $\beta_{r+k+1}$, evinced in the following example.

**Example 3.2.16.** Let $T$ be the tree given as follows, with $T(\beta)$ displayed on the right.

![Diagram of a tree and a cycle graph](image)

Here, $T(\beta)$ has underlying graph $C_4$. This is fine, even with the previous proposition, since traversing (clockwise) half of the vertices of the cycle in the underlying graph beginning at 1256 to 2346 is given by the following sequence of distinct vertex swaps: 1 to 3, 5 to 4.

\[\triangle\]
3.3 Sequences of $\beta$-graphs

The authors in [5] initiate the study of iteration of $\gamma$-graphs and the termination of these sequences. Several sequences of $\gamma$-graph iterations of various graphs are presented, illustrating the frequency with which these sequences “end” in $K_1$. While [7] does not address sequences for the higher order dominating set graphs, it seems a reasonable question to extend to all $X_k(G)$, and to all $A_k(D)$. Of particular interest are the sequences of $\beta$-graphs of digraphs (i.e., studying sequences of $A_\beta(D)$), though examining the sequences of other strata of $D[\beta]$ is also possible.

**Definition 3.3.1.** Let $D$ be a digraph. We refer to the sequence of digraphs

$$D, D(\beta), D(\beta)(\beta) = D(\beta)^2, D(\beta)(\beta)(\beta) = D(\beta)^3, \ldots$$

as the $\beta$-graph sequence of $D$, or just the $\beta$-sequence. We say that the $\beta$-sequence of $D$ terminates if for some $n$, $D(\beta)^n$ is isomorphic to $K_1$.

**Definition 3.3.2.** We similarly define the absorption number sequence of $D$

$$\beta(D), \beta(D(\beta)), \beta(D(\beta)^2), \ldots$$

We focus attention on the $\beta$-sequence of a digraph $D$, in hopes of characterizing those digraphs with convergent $\beta$-sequences. We begin with some preliminary results.

**Example 3.3.3.** Let $D$ be the cyclicly oriented $K_3$. We compute the first few terms of the $\beta$-
sequence of $D$.

\[
\begin{array}{ccc}
  a & D & ab \\
  c & b & D(\beta) \\
  bc & ac & abbc
\end{array}
\]

As one might guess, this sequence continues infinitely. This is not a coincidence.

$\triangle$

**Lemma 3.3.4.** Let $D$ be a cyclic orientation of a cycle on $2k + 1$ vertices for $k \in \mathbb{N}$. Then $\beta(D) = k + 1$.

**Proof** Suppose $\beta(D) = j < k + 1$. Let $S$ be any set of size $j$. Then among the $2k + 1 - j$ remaining vertices, there is an induced oriented $P_2$, and therefore a vertex which does not dominate any vertex of $S$. Thus no $S$ such that $|S| = j$ is absorbant. So, $\beta(D) \geq k + 1$.

Suppose that $\beta(D) = q > k + 1$. Then for any absorbant set $S$ such that $|S| = q$, there is an induced oriented $P_3$, $v_1 \rightarrow v_2 \rightarrow v_3$, and thus there is a vertex, $v_2$ such that $S \setminus \{v_2\}$ is still absorbant. That is, $\beta(D) \leq k + 1$. By trichotomy, then $\beta(D) = k + 1$.

$\blacksquare$

**Lemma 3.3.5.** Let $D$ be a cyclic orientation of a cycle on $2k + 1$ vertices for $k \in \mathbb{N}$. The subgraph induced by any $\beta$-set of $D$ is $P_3$-free.

**Proof** Suppose that there was some $\beta$-set $S$ of $D$ whose induced subgraph further induced a subgraph of $P_3$ on vertices $v_1, v_2, v_3$ such that $v_1 \rightarrow v_2 \rightarrow v_3$. But $v_2$ dominates $v_3$, so $S \setminus \{v_2\}$ is absorbant, and therefore $S$ cannot be a $\beta$-set. Thus, any $\beta$-set is $P_3$-free.

$\blacksquare$
Lemma 3.3.6. Let $D$ be a cyclic orientation of a cycle on $2k + 1$ vertices. The induced subgraph of any $\beta$-set of $D$ contains exactly one $P_2$.

Proof By Lemma 3.3.4, any $\beta$ set has size $k + 1$. That the $\beta$ set induces at least one $P_2$ follows immediately from the Pigeonhole Principle.

Assume that there is some $\beta$-set of $D$ whose induced subgraph contains two or more $P_2$’s. For any $P_2 = v \rightarrow w$, we refer to the “boundary vertices”, $b_1, b_2$ as those vertices encasing the $P_2$ (i.e., $b_1$ is the predecessor of $v$, $b_2$ the successor of $w$). Denote $A, A'$ as two of the induced $P_2$’s.

There are $j \in \{3, 4\}$ boundary vertices $b_1, \ldots, b_j$ (depending on the value of $\min\{d(v, w) \mid v \in A, w \in A'\}$). By Lemma 6.2, we do not want to select any of the $b_i$, since their inclusion induces a $P_3$, which cannot appear in any $\beta$-set. Denote by $C = V(D) \setminus (V(A) \cup V(A') \cup \{b_1, \ldots, b_j\})$ the remaining $2k - 3 - j$ vertices of $D$. Among these $2k - 3 - j$, we want to choose $k - 3$ vertices to include in the $\beta$-set, making sure to avoid $P_3$’s. Dually, we think of choosing $2k - 3 - (k - 3) = k$ vertices to exclude from the $\beta$-set. (It should be noted that the $j$ boundary vertices are always excluded, so are counted in the $2k - 3$ above.) It follows from the Pigeonhole Principle that there exist two consecutive vertices that are excluded from the $\beta$-set. That is, there exists some vertex that does not dominate anything in the $\beta$-set, which is a contradiction to the sets assumed absorbancy. Thus, any $\beta$-set of $D$ must not have more than one $P_2$ in its induced subgraph. By trichotomy, any $\beta$-set of $D$ has exactly one $P_2$ contained in its induced subgraph.

Proposition 3.3.7. Let $D$ be any cyclicly oriented odd cycle on $2k + 1$ vertices. Then the $\beta$-sequence of $D$ does not terminate. In particular, $D(\beta)^n \cong D$ for all $n \in \mathbb{N}$.

Proof Let the vertices of $D$ be the ordered set $\{u_1, u_2, \ldots, u_{2k+1}\}$. By Lemmas 3.3.4, 3.3.5, and 3.3.6, all $\beta$-sets of $D$ are the size $k + 1$ subsets whose induced subgraphs contain exactly one $P_2$. The number of these $\beta$-sets is given by the number of choices of the two vertices which induce the $P_2$, of which there are exactly $2k + 1$ (we can choose any vertex of $D$ to start the $P_2$.) Thus, $D(\beta)$ has $2k + 1$ vertices. Let $u_{i,i+1} \in V(D(\beta))$ be the vertex corresponding to the $\beta$-set $U_{i,i+1}$ where
$u_i \rightarrow u_{i+1}$ is the induced $P_2$, and let $u_{i-1}, u_{i+2}$ be the boundary vertices. (Here, we note that $u_i$ should be read as $u_i \mod (2k+1)$). By the previous lemmas, $U_{i,i+1}$ contains $u_i, u_{i+1}, u_{i+3}, u_{i+5}, \ldots u_{i-2}$. Consider the $\beta$-set $U_{i+2,i+3}$. Again, by the previous lemmas, we know $U_{i+2,i+3}$ must contain $u_{i+2}, u_{i+3}, u_{i+5}, u_{i+7}, \ldots, u_{i-2}, u_{i}$. That is,

$$U_{i+2,i+3} \setminus U_{i,i+1} = \{u_{i+2}\}$$

and

$$U_{i,i+1} \setminus U_{i+2,i+3} = \{u_{i+1}\}$$

That is, $(U_{i+2,i+3} \setminus \{u_{i+2}\}) \cup \{u_{i+1}\} = U_{i,i+1}$. Furthermore, $u_{i+1} \rightarrow u_{i+2}$ in $D$. Thus, for all $i \in [2k+1]$, $U_{i,i+1} \rightarrow U_{i+2,i+3}$ in the graph $D(\beta)$. It is left to show that these are the only arcs in $D(\beta)$.

Indeed, if $j \neq i+2, j+1 \neq i-1 \mod (2k+1)$, then

$$|U_{j,j+1} \setminus U_{i,i+1}| \geq 2$$

since $U_{j,j+1}$’s $P_2$ contains a vertex not contained in $U_{i,i+1}$ (by the alternation of inclusion and exclusion in $U_{i,i+1}$ of elements in $C = V(D) \setminus \{u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$) and further, since one of either $u_i, u_{i+1}$ is included in $U_{j,j+1}$, while the other is not (again, by the alternation guaranteed by Lemma 3.3.6). Then by definition of $\beta$-graph, no arcs can occur between $U_{i,i+1}, U_{j,j+1}$ for $j \neq i+1, j+1 \neq i-1 \mod (2k+1)$.

Therefore, the only arcs in $D(\beta)$ are those of the form $U_{i,i+1} \rightarrow U_{i+2,i+3}$, so that $D(\beta)$ is again a cycle on $2k+1$ vertices. Thus, $D(\beta) \cong D$. Repeating the above argument for each $\beta$-graph iteration, we see that for all $n \in \mathbb{N}$,

$$D(\beta)^n \cong D$$

Thus, the $\beta$-sequence of $D$ never terminates.

\[\blacksquare\]

**Proposition 3.3.8.** Any orientation of $K_{1,n}$ has a terminating $\beta$-sequence.
Proof We can sort each orientation of $K_{1,n}$ into one of $n + 1$ isomorphism classes. Indeed, for $0 \leq j \leq n$, let $[K_{1,n}]_j$ be the class of orientations of $K_{1,n}$ which have $j$ leaves that are sinks. For example, $[K_{1,n}]_0$ is the orientation of $K_{1,n}$ with all leaves directed toward the center node, while $[K_{1,n}]_n$ is the orientation where all leaves are sinks. By $\beta = \beta ([K_{1,n}]_j)$, we mean the absorption number of any orientation in the class $[K_{1,n}]_j$.

For $j = 0$, $\beta = 1$, and the only $\beta$-set is the center node. Then for $D = [K_{1,n}]_0$, $D(\beta) \cong K_1$.

For $j = n$, $\beta = n$, and the only $\beta$-set is the $n$-fold union of all leaves. Thus, $D \in [K_{1,n}]_n$, so that $D(\beta) \cong K_1$.

Let $v$ denote the central node of $K_{1,n}$. For $1 \leq j \leq n - 2$, $I(v) > 1$, $\beta = j + 1$ and thus must appear in at least one $\beta$-set. In fact, the only $\beta$-set is the union of the $j$ sinks and $v$. (To not include $v$ in a proposed $\beta$-set implies that some $v$-dominating leaf does not point into that set, which implies it is not a $\beta$-set.) So for any $D \in [K_{1,n}]_j$, $D(\beta) \cong K_1$.

Finally, if $j = n - 1$, $I(v) = 1$, $\beta = j + 1 = n$ the sets

$$S_1 = \{v\} \cup \{u \mid u \text{ is a sink-leaf}\}, \quad S_2 = \{w \mid w \text{ is a leaf}\}$$

are both $\beta$-sets. Let $t$ be the single leaf that is not a sink. Then $t \rightarrow v$. Thus, for $D \in [K_{1,n}]_j$, $D(\beta) \cong K_2$ with $v_2 \rightarrow v_1$ where $v_i \in D(\beta)$ corresponds to $S_i$. Then, $\beta (D(\beta)) = 1$, with unique $\beta$-set $v_1$, so that $D(\beta)^2 \cong K_1$.

As the above shows, if $D$ is any orientation of $K_{1,n}$, either $D(\beta)$ or $D(\beta)^2$ is isomorphic to $K_1$. That is, the $\beta$-sequence of $D$ terminates.

3.4 $\beta$-Sequences of Trees

Definition 3.4.1. Let $T$ be any oriented tree. Define the pruning of $T$, denoted as $P(T)$ as the subtree obtained by deleting all of the sink-leaves of $T$.

Remark 3.4.2. It should be noted that the operation of “pruning” as defined above is not necessar-
ily the same as the operation of pruning within data structures.

**Example 3.4.3.**

![Diagram of a tree T and its pruning P(T)]

**Definition 3.4.4.** Let \( T \) be a tree. We call a vertex that is the neighbor of at least one leaf a **meristem**. Every tree has at least one meristem. Further, define the *leaf-neighborhood* (relative to \( T \)) of a meristem \( m \) to be

\[
\mathcal{L}_T(m) = \{ \text{leaves adjacent to } m \}
\]

**Definition 3.4.5.** We often want to talk about the pruning of a tree “around” a meristem. That is, we consider the *restriction of* \( P(T) \) to the meristem \( m \), denoted \( P(T)|_m \). In other words, \( P(T)|_m \) carries out the pruning operation only on the leaf neighborhood of \( m \).

**Definition 3.4.6.** Let \( T \) be a tree, and \( m \) any meristem. We say \( m \) is a **Type A meristem** if \( m \) has exactly one leaf \( \ell \in \mathcal{L}_T(m) \) such that \( \ell \rightarrow m \), \( |\mathcal{L}_T(m)| > 1 \), and \( m \) is a sink in \( P(T) \).

A **Type B meristem** is a meristem \( m \) such that there is exactly one leaf \( \ell \in \mathcal{L}_T(m) \) such that \( \ell \rightarrow m \), \( |\mathcal{L}_T(m)| > 1 \), and there exists some \( u \in N(m) \setminus \mathcal{L}_T(m) \) such that \( m \rightarrow u \).

If \( m \) is such that there exist at least two leaves \( \ell_1, \ell_2 \in \mathcal{L}_T(m) \) such that \( \ell_1 \rightarrow m \) and \( \ell_2 \rightarrow m \), and such that there is at least one vertex \( v \in \mathcal{L}_T(m) \) such that \( m \rightarrow v \), then \( m \) is called a **Type C meristem**.
We say \( m \) is a Type D meristem if for all \( v \in \mathcal{L}_T(m) \) then \( v \rightarrow m \).

Finally, \( m \) is of Type E if for all \( v \in \mathcal{L}_T(m) \) then \( m \rightarrow v \).

**Example 3.4.7.** In any orientation of \( K_{1,n} \), the central node is the only meristem. In the previous example, the tree \( T \) has four meristems; two of Type A: vertices 9 and 15, and two of Type C: vertices 4 and 10. The leaf-neighborhood of the meristem 9 is \( \mathcal{L}_T(9) = \{7, 8\} \).

\( \triangle \)

**Example 3.4.8.** Let \( T \) be the graph shown below. Then \( T \) has two meristems; vertex 3 is of Type E, and vertex 5 is of Type D.

We want to characterize the \( \beta \)-graph of trees. The following theorem attempts to do just that. To make the proof a bit easier, the following lemmas are warranted.

**Lemma 3.4.9.** Let \( T \) be an oriented tree, and suppose that \( m \) is a Type A meristem. Then \( P(T)|_m \) has fewer \( \beta \)-sets than \( T \).

**Proof** Let \( m \) be a Type A meristem. That is, there exists exactly one \( \ell \in \mathcal{L}_T(m) \) such that \( \ell \rightarrow m \), and \( \mathcal{L}_T(m) \setminus \{\ell\} \neq \emptyset \). In any \( \beta \)-set \( B \) of \( T \), it is necessary that each sink-leaf in \( \mathcal{L}_T(m) \setminus \{\ell\} \) is present. Since \( \ell \) must either belong to \( B \) or dominate something in \( B \), we have a choice that either \( \ell \in B \) or \( m \in B \) (but not both, since this would contradict minimality).
Now, suppose that $B_\ell$ is a $\beta$-set which contains the vertex $\ell$. Consider the set $B_m = (B_\ell \setminus \{\ell\}) \cup \{m\}$. Since we swapped a single vertex for one other vertex, $B_m$ is still size $\beta$. I claim that $B_m$ is absorbant. Note that because $\ell \rightarrow m$, and all sink-leaves of $m$ are in $B_m$, if $B_m$ is not absorbant, then there exists some $u \in N(m) \setminus \mathcal{L}_T(m)$ such that $u$ dominates nothing in $B_m$. But if this were true, then $u$ would also not dominate anything in $B_\ell$, contradicting absorption of $B_\ell$. Thus, $B_m$ is a $\beta$-set. In particular, for $B$ any $\beta$-set containing $\ell$, there is a corresponding $\beta$-set $B'$ containing $m$ such that $(B \setminus \{\ell\}) \cup \{m\} = B'$.

By pruning $T$, we remove all of $\mathcal{L}_T(m) \setminus \ell$, which makes $m$ a sink of $P(T)$ (since it is Type A). Thus, in any $\beta$-set of $P(T)$, $m$ must always be present. Furthermore, $\ell$ can not be a part of any $\beta$-set of $P(T)$, since its presence would necessarily contradict minimality (since $m$ is also present). As a result, $P(T)$ has fewer $\beta$-sets than $T$.

---

**Lemma 3.4.10.** Let $T$ be an oriented tree with $m$ a Type D meristem. Then $P(T)|_m$ yields no changes to any $\beta$-sets.

**Proof** By definition, $P(T)|_m$ removes all sink-leaves of $m$, but since $m$ is Type D, there are no sink-leaves to be removed, so that $P(T)|_m = T$. Thus, any $\beta$-set of $T$ is a $\beta$-set of $P(T)|_m$.

---

**Theorem 3.4.11.** Let $T$ be an oriented tree on $[n]$, and $P(T)$ its pruning. Suppose that $T$ has Type A meristems $\{m_1, \ldots, m_j\}$, and suppose $T$ has no Type B, Type C, or Type E meristems. Then $T(\beta)$ is a subgraph of the digraph:

$$P(T)(\beta) \square \left( \boxplus_{k=1}^{j} P_2 \right)$$

where $P_2$ is isomorphic to the graph on two vertices $v_1, v_2$ with $v_1 \rightarrow v_2$.

**Proof** First, let us note that for all $i \in [j]$, $m_i$ is a sink-meristem in $P(T)$ (by the nature of Type A meristems), and so is necessary for all $\beta$-sets of $P(T)$.

Let us think of reversing the pruning process. Start with $P(T)(\beta)$. Necessarily, each $m_j$ is
present in every $\beta$-set. By Lemma 3.4.10, pruning Type D meristems does nothing, so any $\beta$-set of $T$ which contains (or does not contain) a Type D meristem, when pruned, yields a $\beta$-set of $P(T)$ which contains (or does not) the meristem as well. However, reintroducing sink-leaves to Type A meristems causes some non-trivial changes. We describe the algorithm to construct $T(\beta)$ from $P(T)(\beta)$ as follows:

**Step 1:** Consider $m_1$. By “un-pruning” $P(T)$ around $m_1$, we re-introduce all of $m_1$’s sink-leaves. This has the effect of relaxing the constraint that $m_1$ appear in every $\beta$-set of $P(T)$. Suppose $P(T)(\beta)$ has $N$ vertices $v_1,m_1,v_2,m_1,\ldots,v_N,m_1$ corresponding one-to-one to the $\beta$-sets: $B_{1,m_1},B_{2,m_1},\ldots,B_{N,m_1}$. Relaxing the constraint that $m_1$ must be in every $\beta$-set of $P(T)$ and instead allowing $\ell_1$ (where $\ell_1$ is the guaranteed source-leaf of $m_1$) to appear creates a copy of $P(T)(\beta)$, with $N$ vertices $v_1,\ell_1,v_2,\ell_1,\ldots,v_N,\ell_1$ corresponding to $\beta$-sets: $B_{1,\ell_1},B_{2,\ell_1},\ldots,B_{N,\ell_1}$. Since $\ell_1 \rightarrow m_1$ by assumption, and $|B_{i,\ell_1} \cap B_{i,m_1}| = \beta - 1$, then for each $i \in \{1,\ldots,N\}$,

$$v_{i,\ell_1} \rightarrow v_{i,m_1}$$

Thus, “un-pruning” around $m_1$ corresponded to taking the Cartesian product $P(T)(\beta) \square (\ell_1 \rightarrow m_1)$. Define $T_1(\beta) = P(T)(\beta) \square (\ell_1 \rightarrow m_1)$.

**Step 2:** Consider $m_2$. We now “un-prune” $T_1(\beta)$ around $m_2$. This reintroduces all of $m_2$’s sink-leaves, which again relaxes the condition that $m_2$ must appear in every set of $T_1(\beta)$, and allows $\ell_2$ to appear instead, where $\ell_2$ is the unique source-leaf of $m_2$. $T_1(\beta)$ has $2N$ vertices $v_1,m_2,\ldots,v_{2N},m_2$ corresponding to $\beta$-sets: $B_{1,m_2},\ldots,B_{2N,m_2}$. By “un-pruning”, we make a copy of $T_1(\beta)$ with $2N$ vertices $v_1,\ell_2,\ldots,v_{2N},\ell_2$ corresponding to size $\beta$ sets: $B_{1,\ell_2},\ldots,B_{2N,\ell_2}$. Since $\ell_2 \rightarrow m_2$, and $|B_{i,\ell_2} \cap B_{i,m_2}| = 1$, it follows that for each $i \in \{1,\ldots,2N\}$,

$$v_{i,\ell_2} \rightarrow v_{i,m_2}$$

So, we have formed the new digraph $T_2(\beta) = T_1(\beta) \square (\ell_2 \rightarrow m_2)$.
Step j Consider \( m_j \). By “un-pruning” \( T_{j-1}(\beta) \) around \( m_j \), we reintroduce all of \( m_j \)’s sink-leaves, which relaxes the condition that \( m_j \) must appear in all 2\( j-1 \) vertices \( v_{1,m_j} \) to \( v_{2j-1,N,m_j} \) of \( T_{j-1}(\beta) \), and allows \( \ell_j \) to appear instead, where \( \ell_j \) is the unique source-leaf of \( m_j \). The vertices of \( T_{j-1}(\beta) \) correspond to: \( B_{1,m_j}, \ldots, B_{2j-1,N,m_j} \). “Un-pruning” makes a copy of \( T_{j-1}(\beta) \), with 2\( j-1 \) vertices \( v_{1,\ell_j} \) to \( v_{2j-1,N,\ell_j} \) corresponding to size \( \beta \) sets: \( B_{1,\ell_j}, \ldots, B_{2j-1,\ell_j} \). Since \( \ell_j \to m_j \), and \( |B_{i,\ell_j} \cap B_{i,m_j}| = 1 \), it follows that for each \( i \in \{1, \ldots, 2^{j-1}N\} \),

\[ v_{i,\ell_j} \to v_{i,m_j} \]

So, we have formed a new digraph: \( T_j(\beta) = T_{j-1}(\beta) \cup (\ell_j \to m_j) \).

End Algorithm

We claim that \( T(\beta) \) is a subgraph of \( T_j(\beta) \). Suppose, for the sake of contradiction that this was not the case. Then there must exist some \( \beta \)-set \( Q \) of \( T \) such that \( Q \) does not correspond to any vertex in \( T_j(\beta) \). Since \( Q \) is a \( \beta \)-set of \( T \), it must contain all the sink-leaves of \( T \). Consider the set \( Q' = Q \setminus \{ \text{all sink-leaves of } T \} \). This is certainly a subset of the vertices of \( P(T) \) of size \( \beta(P(T)) \). If \( Q' \) is absorbant, then we are done, since then \( Q' \) must correspond to a vertex in \( P(T)(\beta) \), of which there is an isomorphic copy in \( T_j(\beta) \), yielding a contradiction. Therefore, we consider the case that \( Q' \) is not absorbant in \( P(T) \). Then it must be the case that \( Q' \) does not contain some subset of the Type A meristems of \( T \). That is, \( Q' \) does not contain \( \{m_{k_1}, m_{k_2}, \ldots, m_{k_q}\} \) for \( q \geq 1 \). Instead, \( \{\ell_{k_1}, \ell_{k_2}, \ldots, \ell_{k_q}\} \subset Q' \). Choose the \( \beta \)-set \( R' \) of \( P(T) \) such that \( R' \) contains all Type A meristems of \( T \), and \( Q' \setminus \{\ell_{k_1}, \ell_{k_2}, \ldots, \ell_{k_q}\} = R' \setminus \{m_{k_1}, m_{k_2}, \ldots, m_{k_q}\} \). Define \( R = R' \cup \{ \text{all sink-leaves of } T \} \), and let \( v_R \) be the vertex in \( T_j(\beta) \) corresponding to \( R \). Via the algorithm above, there exists a path in the underlying graph of \( T_j(\beta) \) such that moving along each vertex of the path corresponds to swapping \( m_{k_i} \) for \( \ell_{k_i} \) for each \( i \in [q] \). Following this path until each of the \( m_{k_i} \)'s are swapped terminates at a vertex \( v \in T_j(\beta) \) which corresponds to the set

\[ (R \setminus \{m_{k_1}, m_{k_2}, \ldots, m_{k_q}\}) \cup \{\ell_{k_1}, \ell_{k_2}, \ldots, \ell_{k_q}\} = Q \]
But this is a contradiction, since $Q$ was assumed not to correspond to any vertex in $T_j(\beta)$. Thus, it must be that our assumption of the existence of such a $Q$ was incorrect. Therefore, $T(\beta)$ is a subgraph of $T_j(\beta)$.

\[\square\]

**Remark 3.4.12.** Care should be taken with the $\beta$ symbol in the proof of the previous theorem. In particular, note that at each step of the algorithm, $\beta$ is increasing, due to the re-inclusion of the sink-leaves of the meristems.

The following example shows that $T(\beta)$ may in fact only be a proper subgraph of $P(T(\beta)) \bigotimes \left(\square_{i=1}^{j} P_2\right)$.

**Example 3.4.13.** Let $T$ be the tree as shown below. The pruning $P(T)$ is shown to the right of $T$:

By examination, $T$ has two Type A meristems, namely: vertices 7 and 12, and two Type D meristems: vertices 3 and 9. Thus, $T$ satisfies the assumptions in the theorem above. By computing the $\beta$-graphs of both $T$ and $P(T)$, we see that

\[T(\beta) \nRightarrow P(T(\beta)) \bigotimes P_2 \bigotimes P_2\]
So, we indeed see that for some $T$ which satisfy the necessary assumptions, $T(\beta)$ is a proper sub-
graph of $P(T)(\beta) \square P_2 \square P_2$.

The example was computed using the code from the Appendix, and using the following com-
mands:

```python
T = DiGraph({1:[3], 2:[3], 3:[4], 4:[13], 7:[6], 13:[12,7], 5:[7], 8:[9], 9:[13], 10:[12], 12:[11]});
T.show(); Beta_graph(T).show();  # display the trees T and beta graph
Pruner(T);                        # returns the pruned tree T
Pruner(T).show(); Beta_graph(Pruner(T)).show();  # display P(T), and its beta graph
```

It certainly is possible that a tree $T$ is such that $T(\beta) \cong P(T)(\beta) \square \left( \bigoplus_{k=1}^{i} P_2 \right)$, as shown by the following.

**Example 3.4.14.** Recall the tree $T$ from the first example of the section. Running $T$ and its pruning $P(T)$ through the Sage code provided in the Appendix (this takes a bit of time to complete), we
see the $\beta$-graphs of each below.

\[ T(\beta) \]

\[ P(T)(\beta) \]

It is easy to see that $T(\beta) \cong P(T)(\beta) \Box P(T)(\beta) \Box P(T)(\beta)$. As this shows, $T(\beta)$ may be isomorphic to the full Cartesian product of the graphs as mentioned in the statement of Theorem 3.4.11. As the next example shows, this is not always the case.

\[ \triangle \]
Chapter 4

Open Questions

4.1 Further Study

Theorem 3.4.11 is a small first step in a full characterization of $\beta$-sequences of trees. It would be worthwhile to handle $\beta$-sequences of trees that have Type B, Type C, or Type E meristems, which also cause changes to $\beta$-sets under pruning. For such trees $T$, one would hope that a similar relationship would exist between $T(\beta)$ and $P(T)(\beta)$.

This paper arose from the study of kernels of digraphs, which undoubtedly play a critical role in studying $\beta$-graphs, $\mathcal{A}_k(D)$-graphs for any $\beta \leq k \leq |D|$, and in $\beta$-sequences. By Proposition 3.3.7, we know there exists a class of digraphs (without kernels, in particular) whose $\beta$-sequences will never terminate. Interestingly enough, the lack of a kernel is not enough to guarantee an infinite $\beta$-sequence, as evinced in the following example (which was shown to have no kernel):

Example 4.1.1. Let $D$ be the digraph pictured below. By Proposition 2.3.3, $D$ has no kernel. Interestingly, though, $D(\beta)$ terminates after only two iterations.
Of course, since $def$ is a sink in $D(\beta)$, $D(\beta)^2 \cong K_1$.

This suggests the following conjecture:

**Conjecture 4.1.2.** Let $D = (V, A)$ be any oriented graph. If there exists an $n \in \mathbb{N}$, such that $D(\beta)^n$ has a kernel, then $D$ has a terminating $\beta$-sequence.

Were this conjecture true, it seems that most digraphs would have terminating $\beta$-sequences.

The following are some open questions related to $\beta$-sequences:

1. Are there bounds on the length of the $\beta$-sequence of a digraph $D$? Some digraphs have $\beta$-sequences that converge almost instantly. Are the number of iterations required for termination closely tied to some parameter of $D$?

2. The graph underlying $D(\beta)$ is always a subgraph of the $G(\gamma)$, where $G$ is the underlying graph of $D$. The termination of $\gamma$-graph sequences is still open-ended. Does this relationship between $D(\beta)$ and $G(\gamma)$ give any indication of the convergence of $\gamma$-sequences? That is, if $D$ has a terminating $\beta$-sequence, must $G$ also have a terminating $\gamma$-sequence?

3. The Cartesian product digraph constructed in the proof of Theorem 3.4.11 possibly contains extraneous vertices (corresponding to size $\beta$ sets which are not absorbant). Is there a way to count the number of these extraneous vertices?
References


Appendix A

Sage Code

To examine more complicated $\beta$ sequences, I have developed the following Sage [11] program, which can compute $\beta$-graphs, the $n$th term along the $\beta$-sequence, prunings, $\beta$-sets, and the absorption number $\beta$ of a directed graph. There is still much room for optimization to diminish computing time for larger examples.

```python
def vert_abs(S1,v,DiGraph):
    return [ vert for vert in S1 if vert in DiGraph.neighbors_in(v)]

def set_absorb(S1,S2,DiGraph):
    M = []
    for v in S1:
        M = M+vert_abs(S2,v,DiGraph)
    return Set(M)

def max_in_deg(DiGraph):
    M = max(DiGraph.in_degree_sequence())
    return M

def absorb_chk(Set1,Set2,DiGraph):
    if set_absorb(Set1,Set2.difference(Set1),DiGraph).intersection(Set2.difference(Set1)) == Set2.difference(Set1):
        return True
    else:
        return False

def min_abs_coll(Set1,Set2,DiGraph):
    t = len(DiGraph.vertices())-max_in_deg(DiGraph)+1
```
for i in range(t):
    M = [W for W in Combinations(Set1,i) if absorb_chk(Set(W),Set2,DiGraph)==True]
    if M != []:
        break
return M

def beta(DiGraph):
    Y = Set(DiGraph.vertices())
    return len(min_abs_coll(Y,Y,DiGraph)[0])

def b_set(DiGraph):
    Y = Set(DiGraph.vertices())
    betaD = [Set(y) for y in min_abs_coll(Y,Y,DiGraph)]
    return betaD

def card_chk(listy,num1,num2):
    if len(listy[num1] & listy[num2]) == len(listy[num1])-1:
        return True
    else:
        return False

def edge_chk(graphy,listy,num1,num2):
    W = (listy[num1]-(listy[num1] & listy[num2]))[0]
    V = (listy[num2]-(listy[num2] & listy[num1]))[0]
    return graphy.has_edge(W,V)

def Pruner(tree):
    foo = deepcopy(tree)
    foo.delete_vertices([y for y in foo.vertices() if foo.in_degree(y)==1 and foo.out_degree(y)==0])
    return foo

def Beta_graph(Gr):
    g = DiGraph([[1..len(b_set(Gr))], lambda i,j: i!=j and card_chk(b_set(Gr),i-1,j-1)==True and edge_chk(Gr,b_set(Gr),i-1,j-1)==True])
    return g

def Beta_seq(B,N):
    C = deepcopy(B)
    for j in range(N):
        C = Beta_graph(C)
    return C