LINEAR COMPLEXES AND CONGRUENCES IN MODULAR GEOMETRY.

by

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HISTORICAL SUMMARY.

The subject of line geometry has been extensively studied since the time of Plucker[1] who introduced the idea of the straight line as the element of space. He adopted a coordinate system for lines which had previously been introduced by Cayley and Grassman.

Plucker[2] also is to be credited for the conception of a complex of lines, i.e. the lines which satisfy one given condition, so that one equation exists between the coordinates of each line of a complex.

An important step in the development of the subject was due to Klein[3], who introduced the coordinate system known as the Klein coordinates.

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1. "System der Geometrie des Raumes." Dusseldorf (1846)
2. "Neue Geometrie des Raumes." Leipzig (1868)
3. Vol. II "Mathematische Annalen."

Among others who investigated the subject is Jessop[4] who adopted generally the analytical method of treatment with Klein coordinates.

Veblen and Young[5] present the subject from a purely synthetic standpoint, though they show how the geometric definitions correspond to the ordinary analytic conceptions.

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4. Jessop, C. M. "Treatise on the Line Complex." Cambridge (1903).

5. Veblen, O. and Young, J. W. "Projective Geometry." Vol. I. (1910).

PURPOSE OF THIS PAPER

The purpose of this paper is a study of the classes of complexes, congruences and reguli in a finite projective geometry of modulus K, in a three space. PG (3,K).

This study includes:

- The number of lines in the figures.
- The number of figures in space.
- Equations of general and special complexes.
- Congruences common to two complexes.
- Reguli common to two congruences.
- Tables of lines and points.
- The number of complexes on a congruence.
- The number of complexes on a regulus.
- Examples of complexes and congruences in PG (3,3) and PG (3,2).

BASIS FOR THIS INVESTIGATION.

The following definitions, classifications and theorems established by Veblen and Young[1] will be used as a basis for this paper.

Definitions.

"A regulus is the set of all lines linearly dependent on three linearly independent lines.

"A congruence is the set of all lines linearly dependent on four linearly independent lines.

"A complex is the set of all lines linearly dependent on five linearly independent lines.

"Linear dependence. If two lines are coplanar the lines of the flat pencil containing them both are said to be linearly dependent on them. If two lines are skew the only lines linearly dependent on them are the two lines themselves. On three skew lines are linearly dependent the lines which meet each of the given lines. If l_1, l_2, ----l_n are any number of lines, and m_1, m_2---m_k, are lines such that m_1, is linearly dependent on two or three of l_1, l_2, ----l_n, and m , is linearly dependent on two or three of l_1, l_2, ----l_n, m_1, m_2,----m_k, then m_k is said to be linearly dependent on l_1, l_2,----l_n.

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1. L. C. pp. 311-334.

Classification of Complexes, congruences, and reguli.

Complexes.

General: A flat pencil of lines at every point of space.

Special: A line and all lines of space which meet it.

Congruences.

Elliptic: One and only one line on every point of space.

Hyperbolic: The set of all lines meeting two skew lines.

Parabolic: The set of all tangents and rulers of a given regulus, which meet a fixed directrix of the regulus.

Degenerate: A bundle of lines and a plane of lines, the center of the bundle being on the plane; or the set of all lines meeting two intersecting lines.

Reguli.

Nondegenerate. Mutually skew lines meeting three skew lines.

Degenerate

Bundle of lines,

Plane of lines,

Two pencils with a common line but different planes and centers.

Theorems.

"Thru each point of space there is one and only one line of an elliptic congruence. For hyperbolic and parabolic congruences this statement is true except for points on a directrix.

"A linear complex consists of all lines linearly dependent on the edges of a simple skew pentagon.

"There are two classes of complexes such that all complexes of either class are projectively equivalent, and all complexes belong to these two classes. A complex of one class consists of a line and all lines of space which meet it. This is called a special complex. The lines are linearly dependent on the edges of a simple skew pentagon, four of whose vertices are coplanar. A complex of the other kind is called general. No four vertices of the pentagon which determines it are coplanar. It contains all lines in the flat pencils of lines meeting homologous pairs of lines in the projectivity determined by two projective flat pencils with different centers and planes, but a common self corresponding line, and also the parabolic congruence of lines, whose directrix is the common line of the pencils.

"Two linear complexes have in common a linear congruence.

"The lines of a complex meeting a line not in the complex, form a hyperbolic congruence.

"The lines of a complex, meeting a line in the complex form a parabolic congruence.

"The lines whose coordinates satisfy one linear equation

$$a_{12}p_{12}+ a_{13}p_{13}+ a_{14}p_{14}+ a_{23}p_{23}+ a_{42}p_{42}+ a_{34}p_{34}=0$$

form a linear complex. Those whose coordinates satisfy two independent equations form a linear congruence, and those satisfying three independent linear equations form a regulus. Four independent equations are satisfied by two (distinct or coincident) lines, which may be improper."

On the basis of the above definitions and theorems the following investigation is made for a finite projective geometry of modulus 3, and of modulus 2, in a three space.

THE NUMBER OF LINES IN REGULI, CONGRUENCES AND COMPLEXES.

Reguli.

A. A non degenerate regulus consists of all lines meeting three skew lines, the directrices. On each point of a directrix there is one and only one line meeting the other two directrices. Then there are $\underline{K} + 1$ lines in the regulus.

B. Degenerate Reguli.

1. A bundle of lines consists of all lines on a point, which is $\underline{K}^2+\underline{K}+1$.
2. A plane of lines has $\underline{K}^2+\underline{K}+1$ lines.
3. Two flat pencils with different planes and centers but a common line, have $2(\underline{K}+1)-1=2\underline{K}+1$, lines, since there are $\underline{K}+1$ lines in a flat pencil.

II. Congruences.

A. An elliptic congruence consists of one and only one line on every point of space. Therefore the number of lines is the number of points divided by the number of points on a line, or

$$\frac{\underline{K}^3+\underline{K}^2+\underline{K}+1}{\underline{K}+1} = \underline{K}^2+1.$$

B. A hyperbolic congruence consists of all lines meeting two skew lines, which is $(\underline{K}+1)^2$. The directrices are not lines of the congruence.

C. A parabolic congruence is made up of a regulus together with one directrix and all tangents to the regulus at points of this directrix. At each point of the directrix there is a flat pencil of lines tangent to the regulus, and this pencil includes the directrix and the ruler thru that point. Then since the directrix is in every pencil the number of lines is

$$(\underline{K}+1)\underline{K}+1=\underline{K}^2+\underline{K}+1.$$

D. A degenerate congruence is a plane of lines and a bundle of lines with center on the plane. The bundle and the plane each have $\underline{K}^2+\underline{K}+1$ lines but $\underline{K}+1$ lines of the bundle are in the plane. Then the number of lines in the congruence is

$$2(\underline{K}^2+\underline{K}+1)-(\underline{K}+1)= 2\underline{K}^2+\underline{K}+1.$$

III. <u>Complexes.</u>

A. A general complex consists of a pencil of lines on every one of the $\underline{K}^3+\underline{K}^2+\underline{K}+1$ points of space.

Since each line is on K-1 points, the number of lines in the complex is

$$\frac{(K^3+K^2+K+1)\ (K+1)}{K+1} = K^3+K^2+K+1$$

B. A special complex is a given line, the directrix, and all lines of space which meet it. At each point on the directrix are K^2+K lines, beside the directrix, making a total of

$$(K+1).(K^2+K)+1 = K^3+2K^2+K+1.$$

NUMBER OF LINES IN COMPLEXES, CONGRUENCES, AND REGULI IN PG(3,3) AND PG(3,2).

	PG (3,2)	PG (3,3)
I. Reguli.		
A. Non degenerate regulus	3	4
B. Degenerate reguli		
1. Bundle of lines	7	13
2. Plane of lines	7	13
3. Two flat pencils with a common line	5	7
II. Congruences.		
A. Elliptic	5	10
B. Hyperbolic	9	16
C. Parabolic	7	13
D. Degenerate	11	36
III. Complexes.		
A. General	15	40
B. Special	19	49

THE NUMBER OF COMPLEXES, CONGRUENCES AND REGULI IN PG(3,K).

COMPLEXES.

I. General complexes.

Since a general complex consists of all lines linearly dependent on the edges of a simple skew pentagon, no four vertices of which are coplanar, the total number of general complexes is the number of such pentagons that can be chosen from all the lines of space.

Let the lines of the pentagon be a, b, c, d, e. The number of choices for a is the number of lines in space - $(K^2+1)(K^2+K+1)$.

For b, any line meeting a. Since there are K^2+K+1 lines on a point, there are K^2+K lines meeting a given line at each point. Then the number of choices for b is $K(K+1)^2$;

For c we may choose any line meeting b and not a; b has K points besides the intersection with a. Of the K^2+K lines meeting b at each of the points, K are on the K points of a. Hence for third choice we have K^3.

From each of $\underline{K}$ points of $\underline{c}$, there is one line on each of $\underline{K}$ points of $\underline{b}$ and $\underline{K}$ points of $\underline{a}$, making 2 $\underline{K}$ lines which meet $\underline{a}$ and $\underline{b}$, and leaving $\underline{K}$-$\underline{K}$ on each point of $\underline{c}$ which may be chosen for $\underline{d}$. Thus the number of choices for $\underline{d}$ is

$$\underline{K}^2(\underline{K}-1).$$

The fifth line, $\underline{e}$, must be a line meeting $\underline{a}$ and $\underline{d}$. $\underline{K}$ points of $\underline{a}$, and $\underline{k}$ of $\underline{d}$, are not already vertices of the pentagon but one point of $\underline{d}$ is in the plane of $\underline{a}$ and $\underline{b}$ and no line on that point could be used or we would have four coplanar vertices. Then the number of choices for $\underline{e}$ is

$$\underline{K}(\underline{K}-1)$$

The product of these numbers must be divided by the number of pentagons which can be chosen from the lines of the same complex, since all the lines of the complex are dependent on any five of its linearly independent lines. This number is as follows:

For $\underline{a}$, any line of the complex, $(\underline{K}^2+1)(\underline{K}+1)$

$\underline{b}$, any line of the complex meeting $\underline{a}$ (there is a flat pencil of lines at every point, or $\underline{K}$ lines meeting $\underline{a}$.) $\underline{K}(\underline{K}+1)$.

$\underline{c}$, any one of $\underline{K}$ lines at each of $\underline{K}$ points of $\underline{b}$, $\underline{K}^2$.

$\underline{d}$. At each point of $\underline{c}$ there is a plane of the complex. This plane intersects the plane of $\underline{a}$ and $\underline{b}$ in a line on the point common to $\underline{c}$ and $\underline{b}$ and meeting $\underline{a}$. Then each of the pencils on the $\underline{K}$ points has one line meeting $\underline{a}$ and this line is excluded, leaving as choices for $\underline{d}$,

$$\underline{K}\,(\underline{K}-1).$$

$\underline{e}$. Likewise from each of $\underline{K}$ points of $\underline{d}$ there is one line meeting $\underline{a}$, hence the choices for line $\underline{e}$ are $\underline{K}$.

Accordingly the number of general complexes is

$$\frac{(\underline{K}^2+1)(\underline{K}^2+\underline{K}+1)\cdot\underline{K}(\underline{K}+1)^2\cdot\underline{K}^3\cdot\underline{K}^2(\underline{K}-1)\cdot\underline{K}(\underline{K}-1)}{(\underline{K}^2+1)(\underline{K}+1)\cdot\underline{K}(\underline{K}+1)\cdot\underline{K}^3\cdot\underline{K}(\underline{K}-1)\cdot\underline{K}} =$$

$$\underline{K}^2(\underline{K}-1)(\underline{K}^2+\underline{K}+1) = \underline{K}^2(\underline{K}^3-1)$$

II. Special Complexes. Any line of space may be the directrix of a special complex, so there are as many special complexes as there are lines in space, which is

$$(\underline{K}^2+1)\ (\underline{K}^2+\underline{K}+1).$$

Or the number of special complexes is the number of distinct simple skew pentagons, with four vertices coplanar, that can be formed from the lines of space. Let the sides of the pentagon be $\underline{a},\underline{b},\underline{c},\underline{d},\underline{e}$. The number of choices for each line is as follows:

$\underline{a}$. Any line of space - - -$(\underline{K}^2+1)\ (\underline{K}^2+\underline{K}+1)$

$\underline{b}$. Any line meeting $\underline{a}$ - - $\underline{K}(\underline{K}+1)^2$

$\underline{c}$. Any line in plane of $\underline{a}$ and $\underline{b}$ except on their intersection ----$\underline{K}^2$.

$\underline{d}$. Any line not in plane of $\underline{a}$ and $\underline{b}$, that is on $\underline{c}$, but not on its intersections with $\underline{a}$ or with $\underline{b}$ - - - - - - - - - - - - -$\underline{K}^2(\underline{K}-1)$.

$\underline{e}$. Any line on $\underline{d}$ and $\underline{a}$ except at their intersections with other lines of the pentagon - - - - - - - - - -$\underline{K}(\underline{K}-1)$.

But any pentagon in the two planes of this pentagon, such that the intersections of the two planes will meet all sides of the pentagon, will determine the same complex. And also such a pentagon in any two planes on the same line of intersection will determine the same complex, since the lines dependent on the sides of the pentagon are all lines meeting the intersection of the two planes. Therefore the number of pentagons that can be formed from the lines of space must be divided by the number of pairs of planes on the same intersection, which is $\underline{K}(\underline{K}+1)$ and by the number of such pentagons in each pair of planes.

<u>a</u>. Any line in one plane except intersection of planes - - - - - - - - - - - - - - - $-\underline{K}^2+\underline{K}$

<u>b</u>. Any other line of plane except on intersection of <u>a</u> and the directrix - $\underline{K}^2$

<u>c</u>. Any line of the plane meeting <u>b</u>, except those on interesections of <u>a</u> and the directrix, of <u>a</u> and <u>b</u>, or of <u>b</u> and the directrix - - - - - - - - - - - - - $-(\underline{K}-1)^2$.

d. Any line in second plane on intersection of c and the directrix - - - $\underline{K}$.

e. Any line except directrix, in second plane, meeting e and a - - - - - - - - $\underline{K}$.

The total number of special complexes is

$$\frac{(\underline{K}^2+1)(\underline{K}^2+\underline{K}+1)\ .\ \underline{K}(\underline{K}+1)^2.\underline{K}^2.\ \underline{K}^2(\underline{K}-1).\ \underline{K}(\underline{K}-1)}{(\underline{K}+1)\underline{K}.\ (\underline{K}^2+\underline{K}).\ \underline{K}^2.\ (\underline{K}-1)^2\ .\ \underline{K}^2} =$$

$$(\underline{K}^2+1)(\underline{K}^2+\underline{K}+1).$$

CONGRUENCES.

I. Elliptic. An elliptic congruence is a set of lines linearly dependent on four linearly independent skew lines, such that no one of them meets the regulus of the other three in a proper point. Therefore the number of congruences depends on the number of such sets of lines that can be chosen from the lines of space. Let a, b, c, d, be the four lines.

a. Any line of space may be the first line of such a set - - - $-(K^2+1)(K^2+K+1)$

b. Any line not meeting a - - K^4

c. The number of lines meeting a and b is $2K(K+1)^2-(K+1)^2 = 2K^3+3K^2-1$. Then the number not meeting a and b is - -

$$K(K-1)(K^2-1)$$

d. For fourth line we have any not meeting the regulus of the first three. The number of lines meeting the regulus include the $K+1$ directrices, and also K lines from each point of each of the $K-1$ lines of the regulus to each other of the lines, or $\frac{K^2(K+1)^2}{2}$.

With these and the lines of the regulus itself, there are $\frac{K^2(K+1)}{2} + 2(K+1)$ lines not to be chosen for d, leaving the number -

$$\tfrac{1}{2}(K^2)(K-1)^2.$$

But there are K^2+1 lines in the elliptic congruence and all the lines are linearly dependent on any linearly independent four of its lines. Therefore the total number of sets of lines that can be chosen should be divided by the number of sets found in a congruence, which is as follows,

a.- - Any line of the congruence - - (K^2+1).

b.- - Any other - - - - - - - - - - - - K^2.

c.- - Any not already chosen - - - - K^2-1.

d.- - Any not in regulus of first three - - - - - - - - - - - - - - K^2-K.

The total number of elliptic congruences is

$$\frac{(K^2+1)(K^2+K+1) \cdot K^4 \cdot (K)(K^2-1)(K-1) \cdot \frac{1}{2}K^2(K+1)^2}{(K^2+1) \cdot (K^2) \cdot (K^2-1) \cdot (K^2-K)} =$$

$$\frac{K}{2}(K^3-1)(K-1).$$

II. Hyperbolic. Since a hyperbolic congruence consists of all lines meeting two skew lines, the number of congruences is the number of pairs of skew lines that can be chosen from the lines of space. Any line may be the first and any of the K lines, not meeting the first, may be the second. The number of hyperbolic congruences is therefore

$$\frac{(K^2+1)(K^2+K+1).\ K^4}{2}$$

III. Parabolic. The parabolic congruence consists of a line and a pencil of lines in a different plane at each of its points. Any line may be the first line, or directrix. There are $K+1$ choices for the plane of the first pencil, K for the second, etc. Therefore the number of parabolic congruences is

$$\left[(K^2+K+1)(K^2+1)\right]\ \left[(K-1)!\right]$$

IV. Degenerate. Any plane may be the plane of the degenerate congruence and any point of the plane may be the center of the bundle, therefore the total number of degenerate congruences is the number of planes multiplied by the number of points on a plane.

$$(K^2+K+1)^2(K^2+1)$$

Total number of congruences. Since every two complexes have a congruence in common, the total number of congruences is the number of sets of two that can be chosen from the total number of complexes, divided by the number of sets of two that can be chosen from the K+1 complexes[1] on the same congruence.

This number is

$$\frac{(K^5+K^4+K^3+K^2+K+1)(K^5+K^4+K^3+K^2+K)}{(K+1)K} =$$

$$(K^4+K^2+1)(K^4+K^3+K^2+K+1)$$

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1. See page 51 for proof that there are K+1 complexes on a congruence.

REGULI.

1. Non degenerate. A non degenerate regulus consists of all lines meeting three mutually skew lines. These are dependent on any three lines of the $K+1$ directrices. The number of reguli is therefore

$$\frac{(K^2+1)(K^2+K+1)\ .\ K^4.\ (K^2-1)(K-1)(K)}{(K+1)\ .\ K\ .\ (K-1)} = K^4(K^2+K+1)(K^2+1)(K-1)$$

II. Degenerate.

A. Two pencils with different planes and a common line. Any point may be the center of the first pencil, any other of the second. Any plane on both points may be the plane of the first pencil; any other, on both points, of the second.

$$\frac{(K^2+1)(K+1)\ .\ K(K^2+K+1)\ .\ (K+1)\ K}{2}$$

B. Plane of lines - - -Any plane - $(K^2+1)(K+1)$

C. Bundle of lines - - Any point may be the center of a bundle - - - -$(K^2+1)(K+1)$

NUMBER OF COMPLEXES, CONGRUENCES AND REGULI IN PG(3,3) AND PG (3,2).

	PG(3,2)	PG(3,3)
Complexes.		
I. General	28	234
II. Special	35	130
Total	63	364
Congruences.		
I. Elliptic	56	2,106
II. Hyperbolic	280	5,265
III. Parabolic	210	3,120
IV. Degenerate	105	520
Total	651	11,011
Reguli.		
I. Non degenerate	560	21,060
II. Degenerate		
A. Pencils with common line	630	9,360
B. Planes of lines	15	40
C. Bundles of lines	15	40
Total	1,220	30,500

EQUATIONS OF COMPLEXES IN PG(3,K).

The equation of a complex is of the form

$$a_{12}p_{12} + a_{13}p_{13} + a_{14}p_{14} + a_{23}p_{23} + a_{42}p_{42} + a_{34}p_{34} = 0.$$

The a_{ik} are any coefficients, the p_{ik} are the Plucker line coordinates.

$$p_{ik} = \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}; (x_1, x_2, x_3, x_4) \text{ and } (y_1, y_2, y_3, y_4)$$

being any two points on the line.

The total number of equations, and therefore the total number of complexes, is the number of sets of six numbers that can be chosen for the a's. In a finite geometry of modulus $\underline{K}$, this number is $\frac{\underline{K}^6-1}{\underline{K}-1} = \underline{K}^5+\underline{K}^4+\underline{K}^3+\underline{K}^2+\underline{K}+1$, since any of the $\underline{K}$ marks of may be chosen for any of the six a's, with one exception that they cannot all be zero; division by K-1 is necessary since multiplication or division of an equation by any of the K marks, aside from zero, leaves the equation unchanged.

The analytical condition for special complexes is given by Jessop[1].

1. Jessop, C. M. "A Treatise on the Line Complex." page 27.

If the invariant of the equation is zero, the complex is special. The invariant is

$$2\ (a_{12}a_{34} + a_{14}a_{23} + a_{13}a_{42}).$$

If K=2 the invariant is found to be zero in the following cases:

When all the a's are zero except one, i.e. when the equation has only one term;

When the equation has two terms, unless each subscript of the a's appears once;

When the equation has three terms, if each subscript appears twice;

When the equation has five terms.

Thirty-five equations can be written satisfying these conditions, then there are thirty-five special complexes in PG(3,2) and the remaining twenty-eight are general complexes.

If K = 3, the number of equations of special complexes is found similarly to be one hundred and thirty, the remaining two hundred and thirty-four being those of general complexes.

The following pages contain all the equations of special and general complexes in PG(3,2) and in PG(3,3).

EQUATIONS OF THE SPECIAL COMPLEXES IN PG (3,2).

1. $p_{12} = 0$
2. $p_{13} = 0$
3. $p_{14} = 0$
4. $p_{23} = 0$
5. $p_{42} = 0$
6. $p_{34} = 0$
7. $p_{12} + p_{13} = 0$
8. $p_{12} + p_{14} = 0$
9. $p_{12} + p_{23} = 0$
10. $p_{12} + p_{42} = 0$
11. $p_{13} + p_{14} = 0$
12. $p_{13} + p_{23} = 0$
13. $p_{13} + p_{34} = 0$
14. $p_{14} + p_{42} = 0$
15. $p_{14} + p_{34} = 0$
16. $p_{23} + p_{42} = 0$
17. $p_{23} + p_{34} = 0$
18. $p_{42} + p_{34} = 0$

19. $p_{12} + p_{13} + p_{14} = 0$
20. $p_{12} + p_{13} + p_{23} = 0$
21. $p_{12} + p_{14} + p_{42} = 0$
22. $p_{12} + p_{23} + p_{42} = 0$
23. $p_{13} + p_{14} + p_{34} = 0$
24. $p_{13} + p_{23} + p_{34} = 0$
25. $p_{14} + p_{42} + p_{34} = 0$
26. $p_{23} + p_{42} + p_{34} = 0$
27. $p_{12} + p_{13} + p_{24} + p_{34} = 0$
28. $p_{12} + p_{14} + p_{23} + p_{34} = 0$
29. $p_{13} + p_{14} + p_{23} + p_{42} = 0$
30. $p_{12} + p_{13} + p_{14} + p_{23} + p_{42} = 0$
31. $p_{12} + p_{13} + p_{14} + p_{23} + p_{34} = 0$
32. $p_{12} + p_{13} + p_{14} + p_{42} + p_{34} = 0$
33. $p_{12} + p_{13} + p_{23} + p_{42} + p_{34} = 0$
34. $p_{12} + p_{14} + p_{23} + p_{42} + p_{34} = 0$
35. $p_{13} + p_{14} + p_{23} + p_{42} + p_{34} = 0$

EQUATIONS OF GENERAL COMPLEXES IN PG 3,2.

1. $p_{12} + p_{34} = 0$
2. $p_{13} + p_{42} = 0$
3. $p_{14} + p_{23} = 0$
4. $p_{12} + p + p = 0$
5. $p_{12} + p_{13} + p_{42} = 0$
6. $p_{12} + p_{14} + p_{23} = 0$
7. $p_{12} + p_{14} + p_{34} = 0$
8. $p_{12} + p_{23} + p_{34} = 0$
9. $p_{12} + p_{42} + p_{34} = 0$
10. $p_{13} + p_{14} + p_{42} = 0$
11. $p_{13} + p_{14} + p_{23} = 0$
12. $p_{13} + p_{23} + p_{42} = 0$
13. $p_{13} + p_{42} + p_{34} = 0$
14. $p_{14} + p_{23} + p_{42} = 0$
15. $p_{14} + p_{23} + p_{34} = 0$
16. $p_{12} + p_{13} + p_{14} + p_{23} = 0$
17. $p_{12} + p_{13} + p_{14} + p_{42} = 0$
18. $p_{12} + p_{13} + p_{14} + p_{34} = 0$
19. $p_{12} + p_{13} + p_{23} + p_{42} = 0$
20. $p_{12} + p_{13} + p_{23} + p_{34} = 0$
21. $p_{12} + p_{14} + p_{42} + p_{23} = 0$
22. $p_{12} + p_{14} + p_{42} + p_{34} = 0$
23. $p_{12} + p_{23} + p_{42} + p_{34} = 0$
24. $p_{13} + p_{14} + p_{23} + p_{34} = 0$
25. $p_{13} + p_{14} + p_{42} + p_{34} = 0$
26. $p_{13} + p_{23} + p_{42} + p_{34} = 0$
27. $p_{14} + p_{23} + p_{42} + p_{34} = 0$
28. $p_{12} + p_{13} + p_{14} + p_{23} + p_{42} + p_{34} = 0.$

For convenience the equations of special complexes will be referred to as (S_1), ----(S_{35}); those of general complexes as (G_1), (G_2)----(G_{28}).

EQUATIONS OF THE SPECIAL COMPLEXES IN PG (3,3).

1. $p_{12} = 0$
2. $p_{13} = 0$
3. $p_{14} = 0$
4. $p_{23} = 0$
5. $p_{42} = 0$
6. $p_{34} = 0$

7-8. $p_{12} \pm p_{13} = 0$

9-10. $p_{12} \pm p_{14} = 0$

11-12. $p_{12} \pm p_{23} = 0$

13-14. $p_{12} \pm p_{42} = 0$

15-16. $p_{13} \pm p_{14} = 0$

17-18. $p_{13} \pm p_{23} = 0$

19-20. $p_{13} \pm p_{34} = 0$

21-22. $p_{14} \pm p_{42} = 0$

23-24. $p_{14} \pm p_{34} = 0$

25-26. $p_{23} \pm p_{42} = 0$

27-28. $p_{23} \pm p_{34} = 0$

29-30. $p_{42} + p_{34} = 0$

31-34. $p_{12} \pm p_{13} \pm p_{14} = 0$

35-38. $p_{12} \pm p_{13} \pm p_{23} = 0$

39-42. $p_{12} \pm p_{14} \pm p_{42} = 0$

43-46. $p_{12} \pm p_{23} \pm p_{42} = 0$

47-50. $p_{13} \pm p_{14} \pm p_{34} = 0$

51-54. $p_{13} \pm p_{23} \pm p_{34} = 0$

55-58. $p_{14} \pm p_{42} \pm p_{34} = 0$

59-62. $p_{23} \pm p_{42} \pm p_{34} = 0$

63. $p_{12} + p_{13} + p_{42} - p_{34} = 0$

64. $p_{12} + p_{13} - p_{42} + p_{34} = 0$

65. $p_{12} - p_{13} + p_{42} + p_{34} = 0$

66. $-p_{12} + p_{13} + p_{42} + p_{34} = 0$

67. $p_{12} + p_{14} + p_{23} - p_{34} = 0$

68. $p_{12} + p_{14} - p_{23} + p_{34} = 0$

69. $p_{12} - p_{14} + p_{23} + p_{34} = 0$

70. $-p_{12} + p_{14} + p_{23} + p_{34} = 0$

71. $p_{13} + p_{14} + p_{23} - p_{42} = 0$

72. $p_{13} + p_{14} - p_{23} + p_{42} = 0$

73. $p_{13} - p_{14} + p_{23} + p_{42} = 0$

74. $-p_{13} + p_{14} + p_{23} + p_{42} = 0$

75-76. $\pm p_{12} + p_{13} + p_{14} + p_{23} - p_{42} = 0$

77-78. $\pm p_{12} + p_{13} + p_{14} - p_{23} + p_{42} = 0$

79-80. $\pm p_{12} + p_{13} - p_{14} + p_{23} + p_{42} = 0$

81-82. $\pm p_{12} - p_{13} + p_{14} + p_{23} + p_{42} = 0$

83-84. $p_{12} \pm p_{13} + p_{14} + p_{23} - p_{34} = 0$

85-86. $p_{12} \pm p_{13} + p_{14} - p_{23} + p_{34} = 0$

87-88. $p_{12} \pm p_{13} - p_{14} + p_{23} + p_{34} = 0$

89-90. $-p_{12} \pm p_{13} + p_{14} + p_{23} + p_{34} = 0$

91-92. $p_{12} + p_{13} \pm p_{14} + p_{42} - p_{34} = 0$

93-94. $p_{12} + p_{13} \pm p_{14} - p_{42} + p_{34} = 0$

95-96. $p_{12} - p_{13} \pm p_{14} + p_{42} + p_{34} = 0$

97-98. $-p_{12} + p_{13} \pm p_{14} + p_{42} + p_{34} = 0$

99-100. $p_{12} + p_{13} \pm p_{23} + p_{42} - p_{34} = 0$

101-102. $p_{12} + p_{13} \pm p_{23} - p_{42} + p_{34} = 0$

103-104. $p_{12} - p_{13} \pm p_{23} + p_{42} + p_{34} = 0$

105-106. $-p_{12} + p_{13} \pm p_{23} + p_{42} + p_{34} = 0$

107-108. $p_{12} + p_{14} + p_{23} \pm p_{42} - p_{34} = 0$

109-110. $p_{12} + p_{14} - p_{23} \pm p_{42} + p_{34} = 0$

110-111. $p_{12} - p_{14} + p_{23} \pm p_{42} + p_{34} = 0$

112-113. $-p_{12} + p_{14} + p_{23} \pm p_{42} + p_{34} = 0$

115-116. $p_{13} + p_{14} + p_{23} - p_{42} \pm p_{34} = 0$

117-118. $p_{13} + p_{14} - p_{23} + p_{42} \pm p_{34} = 0$

119-120. $p_{13} - p_{14} + p_{23} + p_{42} \pm p_{34} = 0$

121-122. $-p_{13} + p_{14} + p_{23} + p_{42} \pm p_{34} = 0$

123. $p_{12} + p_{13} + p_{14} + p_{23} + p_{42} + p_{34} = 0$

124. $p_{12} + p_{13} + p_{14} - p_{23} - p_{42} - p_{34} = 0$

125. $p_{12} + p_{13} - p_{14} + p_{23} - p_{42} - p_{34} = 0$

126. $p_{12} - p_{13} + p_{14} - p_{23} + p_{42} - p_{34} = 0$

127. $-p_{12} + p_{13} + p_{14} + p_{23} + p_{42} - p_{34} = 0$

128. $p_{12} - p_{13} + p_{14} + p_{23} - p_{42} + p_{34} = 0$

129. $p_{12} + p_{13} - p_{14} - p_{23} + p_{42} + p_{34} = 0$

130. $-p_{12} + p_{13} + p_{14} - p_{23} - p_{42} + p_{34} = 0$

EQUATIONS OF GENERAL COMPLEXES IN PG (3,3).

1-2. $p_{12} \pm p_{34} = 0$

3-4. $p_{13} \pm p_{42} = 0$

5-6. $p_{14} \pm p_{23} = 0$

7-10. $p_{12} \pm p_{13} \pm p_{42} = 0$

11-14. $p_{12} \pm p_{13} \pm p_{34} = 0$

15-18. $p_{12} \pm p_{14} \pm p_{23} = 0$

19-22. $p_{12} \pm p_{14} \pm p_{34} = 0$

23-26. $p_{12} \pm p_{23} \pm p_{34} = 0$

27-30. $p_{12} \pm p_{42} \pm p_{34} = 0$

31-34. $p_{13} \pm p_{14} \pm p_{23} = 0$

35-38. $p_{13} \pm p_{14} \pm p_{42} = 0$

39-42. $p_{13} \pm p_{23} \pm p_{42} = 0$

43-46. $p_{13} \pm p_{42} \pm p_{34} = 0$

47-50. $p_{14} \pm p_{23} \pm p_{42} = 0$

51-54. $p_{14} \pm p_{23} \pm p_{34} = 0$

55-62. $p_{12} \pm p_{13} \pm p_{14} \pm p_{23} = 0$

62-70. $p_{12} \pm p_{23} \pm p_{14} \pm p_{42} = 0$

71-78. $p_{12} \pm p_{13} \pm p_{14} \pm p_{34} = 0$

79-86. $p_{12} \pm p_{13} \pm p_{23} \pm p_{42} = 0$

87-94. $p_{12} \pm p_{13} \pm p_{23} \pm p_{34} = 0$

95. $p_{12} + p_{13} + p_{42} + p_{34} = 0$

96. $p_{12} + p_{13} - p_{42} - p_{34} = 0$

97. $p_{12} - p_{13} + p_{42} - p_{34} = 0$

98. $p_{12} - p_{13} - p_{42} + p_{34} = 0$

99-106. $p_{12} \pm p_{14} \pm p_{23} \pm p_{42} = 0$

107. $p_{12} + p_{14} + p_{23} + p_{34} = 0$

108. $p_{12} + p_{14} - p_{23} - p_{34} = 0$

109. $p_{12} - p_{14} + p_{23} - p_{34} = 0$

110. $p_{12} - p_{14} - p_{23} + p_{34} = 0$

111-118. $p_{12} \pm p_{14} \pm p_{42} \pm p_{34} = 0$

119-126. $p_{12} \pm p_{23} \pm p_{42} \pm p_{34} = 0$

127-134. $p_{13} \pm p_{14} \pm p_{23} \pm p_{34} = 0$

135. $p_{13} + p_{14} + p_{23} + p_{42} = 0$

136. $p_{13} + p_{14} - p_{23} - p_{42} = 0$

137. $p_{13} - p_{14} + p_{23} - p_{42} = 0$

138. $p_{13} - p_{14} - p_{23} + p_{42} = 0$

139-146. $p_{13} \pm p_{14} \pm p_{42} \pm p_{34} = 0$

147-154. $p_{13} \pm p_{23} \pm p_{42} \pm p_{34} = 0$

155-162. $p_{14} \pm p_{23} \pm p_{42} \pm p_{34} = 0$

163-164. $\pm p_{12} + p_{13} + p_{14} + p_{23} + p_{42} = 0$

165-166. $\pm p_{12} + p_{13} - p_{14} - p_{23} + p_{42} = 0$

167-168. $\pm p_{12} - p_{13} - p_{14} + p_{23} + p_{42} = 0$

169-170. $\pm p_{13} - p_{13} + p_{14} - p_{23} + p_{42} = 0$

171-172. $p_{12} \pm p_{13} + p_{14} + p_{23} + p_{34} = 0$

173-174. $p_{12} \pm p_{13} - p_{14} - p_{23} + p_{34} = 0$

175-176. $-p_{12} \pm p_{13} - p_{14} + p_{23} + p_{34} = 0$

177-178. $-p_{12} \pm p_{13} + p_{14} - p_{23} + p_{34} = 0$

179-180. $p_{12} + p_{13} \pm p_{14} + p_{42} + p_{34} = 0$

181-182. $p_{12} - p_{13} \pm p_{14} - p_{42} + p_{34} = 0$

183-184. $p_{12} + p_{13} \pm p_{14} - p_{42} - p_{34} = 0$

185-186. $p_{12} - p_{13} \pm p_{14} + p_{42} - p_{34} = 0$

187-188. $p_{12} + p_{13} \pm p_{23} + p_{42} + p_{34} = 0$

189-190. $p_{12} - p_{13} \pm p_{23} - p_{42} + p_{34} = 0$

191-192. $p_{12} - p_{13} \pm p_{23} + p_{42} - p_{34} = 0$

193-194. $p_{12} + p_{13} \pm p_{23} - p_{42} - p_{34} = 0$

195-196. $p_{12} + p_{14} + p_{23} \pm p_{42} + p_{34} = 0$

197-198. $p_{12} - p_{14} - p_{23} \pm p_{42} + p_{34} = 0$

199-200. $p_{12} - p_{14} + p_{23} \pm p_{42} - p_{34} = 0$

201-202. $p_{12} + p_{14} - p_{23} \pm p_{42} - p_{34} = 0$

203-204. $p_{13} + p_{14} + p_{23} + p_{42} \pm p_{34} = 0$

205-206. $p_{13} - p_{14} - p_{23} + p_{42} \pm p_{34} = 0$

207-208. $p_{13} - p_{14} + p_{23} - p_{42} \pm p_{34} = 0$

209-210. $p_{13} + p_{14} - p_{23} - p_{42} \pm p_{34} = 0$

211. $p_{12} + p_{13} + p_{14} + p_{23} + p_{42} - p_{34} = 0$

212. $p_{12} + p_{13} + p_{14} + p_{23} - p_{42} + p_{34} = 0$

213. $p_{12} + p_{13} + p_{14} - p_{23} + p_{42} + p_{34} = 0$

214. $p_{12} + p_{13} - p_{14} + p_{23} + p_{42} + p_{34} = 0$

215. $p_{12} - p_{13} + p_{14} + p_{23} + p_{42} + p_{34} = 0$

216. $-p_{12} + p_{13} + p_{14} + p_{23} + p_{42} + p_{34} = 0$

217. $-p_{12} - p_{13} + p_{14} + p_{23} + p_{42} + p_{34} = 0$

218. $-p_{12} + p_{13} - p_{14} + p_{23} + p_{42} + p_{34} = 0$

219. $-p_{12} + p_{13} + p_{14} - p_{23} + p_{42} + p_{34} = 0$

220. $-p_{12} + p_{13} + p_{14} + p_{23} - p_{42} + p_{34} = 0$

221. $p_{12} - p_{13} - p_{14} + p_{23} + p_{42} + p_{34} = 0$

222. $p_{12} - p_{13} + p_{14} - p_{23} + p_{42} + p_{34} = 0$

223. $p_{12} - p_{13} - p_{14} - p_{23} - p_{42} - p_{34} = 0$

224. $p_{12} + p_{13} - p_{14} + p_{23} - p_{42} + p_{34} = 0$

225. $p_{12} + p_{13} - p_{14} + p_{23} + p_{42} - p_{34} = 0$

226. $p_{12} + p_{13} + p_{14} - p_{23} - p_{42} + p_{34} = 0$

227. $p_{12} + p_{13} + p_{14} - p_{23} + p_{42} - p_{34} = 0$

228. $p_{12} + p_{13} + p_{14} + p_{23} - p_{42} - p_{34} = 0$

229. $-p_{12} - p_{13} + p_{14} + p_{23} - p_{42} + p_{34} = 0$

230. $-p_{12} - p_{13} + p_{14} + p_{23} + p_{42} - p_{34} = 0$

231. $-p_{12} + p_{13} - p_{14} - p_{23} + p_{42} + p_{34} = 0$

232. $-p_{12} + p_{13} - p_{14} + p_{23} + p_{42} - p_{34} = 0$

233. $-p_{12} + p_{13} + p_{14} - p_{23} + p_{42} - p_{34} = 0$

234. $-p_{12} + p_{13} + p_{14} + p_{23} - p_{42} - p_{34} = 0$

CONGRUENCES COMMON TO TWO COMPLEXES.

The congruence common to two complexes depends on the kind of complexes and their relative position.

I. Two special complexes. Since each complex consists of all lines meeting a given line, the directrix, the lines common to two complexes are all the lines meeting both directrices. If these directrices are skew, the common congruence is hyperbolic; if they intersect, it is degenerate. Since the directrices either do or do not intersect, the congruence common to two special complexes is either hyperbolic or degenerate.

II. A general and a special complex. If the directrix of the special complex is a line of the general, the lines common to the two complexes form a parabolic congruence[1].

If the directrix of the special complex is not a line of the general complex the common lines form a hyperbolic congruence[1].

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1. See page 9 .

Since the directrix of the special complex either is or is not a line of the general, the congruence common to a special and a general complex is either parabolic or hyperbolic.

III. Two general complexes.

At any point of space the two flat pencils belonging to two general complexes, have a line in common[2]. Then in general the lines common to the two complexes consist of one and only one line on every point of space. This set of lines is an elliptic congruence.

Since some of the lines of a general complex form hyperbolic congruences and some of them parabolic congruences[3], two general complexes may have in common a hyperbolic or a parabolic congruence. A degenerate congruence cannot be common to the two, for neither complex contains the plane of lines or the bundle of lines which make up a degenerate congruence.

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2. L. C. p. 325

3. L. C. p. 321-323.

REGULI COMMON TO TWO CONGRUENCES.

I. Two elliptic congruences. All degenerate reguli contain intersecting lines. Elliptic congruences contain only mutually skew lines and therefore the only kind of a regulus that can be common to two elliptic congruences is the nondegenerate.

II. An elliptic and a parabolic congruence. If the directrix of the parabolic congruence is a line of the elliptic there is no other line common to the two congruences, for all other lines of the parabolic meet the given line and no others of the elliptic do. Otherwise there might be a non degenerate regulus in common.

III. An elliptic and a hyperbolic congruence. If one or both directrices of the hyperbolic congruence are lines of the elliptic, there can be no lines common to the two, for the lines of hyperbolic congruence all meet the directrices.

Otherwise there may be a non degenerate regulus common to the two.

IV. An elliptic and a degenerate congruence. The lines of an elliptic congruence are all skew to each other. One line of the plane of the degenerate congruence and one of the bundle might coincide with lines of the elliptic, but no others could coincide, for every other line of the degenerate congruence would meet one of the first two, and no other line of the elliptic meets either of them. Therefore an elliptic and a degenerate congruence cannot have a regulus in common.

V. Two parabolic congruences. If the directrices of the two congruences coincide, two flat pencils with a common line may be common. If the directrices are skew, there may be one line from each pencil of one congruence meeting the directrix of the other and forming a non-degenerate regulus.

These are the only possibilities since neither of the parabolic congruences contains a bundle of lines, or a plane of lines.

VI. A parabolic and a hyperbolic congruence. The lines of the parabolic congruence form only non-degenerate reguli, or degenerate, consisting of two flat pencils with a common line. If the directrix of the parabolic coincides with one directrix of the hyperbolic the only possible common regulus is the non-degenerate, since the directrix of the hyperbolic is not a line of the congruence.

VII. A parabolic and a degenerate congruence. The degenerate congruence contains no non-degenerate regulus, and the parabolic contains no plane of lines or bundle of lines, so the only possibility in this case is two flat pencils with a common line. This is possible when the directrix of the parabolic is in the plane and on the center of the bundle of the degenerate congruence.

VIII. Two hyperbolic congruences.

Let the two congruences be C_1 with directrices a and b; and C_2, with directrices c and d. If a, b, c, and d are mutually skew, a non-degenerate regulus might be common to C_1 and C_2. This would happen in case d and e coincide with directrices of a regulus contained in C_1.

If b coincides with d, but a and e are skew, then the lines common to the two congruences are the lines meeting three skew lines which form a non-degenerate regulus.

If b coincides with d, but a and e intersect, the pencil with center at the intersection of a and e is common to both congruences, and so also, is the pencil at the point of b=d, that is in the plane of a and e. Then the two congruences have in common two pencils with a common line.

If b and d intersect but a and c are skew, the only line common to the two congruences is the line on the intersection of b and d, meeting both a and e, so there is no common regulus.

If a and b are lines of C_2, there is no regulus in common, the only common lines being the two on the intersections of the directrices.

IX. A hyperbolic and a degenerate congruence.

These can have no non-degenerate regulus in common, since the degenerate congruence contains none; and no bundle of lines or plane of lines since the hyperbolic does not contain one of these. The only possible common regulus therefore is one consisting of two pencils with a common line. This would be possible when a directrix of the hyperbolic is in the plane and on the center of the bundle of the degenerate. A pencil of lines at that point would be common and also one at the point where the other directrix cuts the plane of the degenerate.

X. Two degenerate congruences.

Neither congruence contains a non-degenerate regulus so none can be common to the two. If the planes coincide, and the centers of the bundles are different, the common regulus is a plane of lines.

If the planes of the two are different, but centers of bundles coincide, it is a bundle of lines. If the line of intersection of the two planes is on the centers of both bundles, the common regulus consists of two flat pencils with a common line.

In general, two congruences do not have a regulus in common. A congruence is made up of lines whose coordinates satisfy two equations of the form---$A_{ik} P_{ik} = 0$. If every two congruences had a regulus in common, then, in general, the coordinates of the lines of the regulus would satisfy four equations, which is not the case.[1]

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1. See page 9 .

LINES AND POINTS IN PG (3,K).

In PG (3,K) the points have four coordinates. The number of points on a line is $K-1$; the number of points in space is K^3+K^2+K+1; the number of lines on a point is K^2+K+1 and the number of lines in space is $(K^2+1).(K^2+K+1)$.

If $K = 2$, the marks of the system are 0 and 1. The number of sets of four coordinates that can be formed from the two marks is 15. There are three points on a line, 7 lines on a point and 35 lines in space. The coordinates of the lines are the Plucker coordinates determined by the coordinates of any two points on the line. The points and lines will be numbered according to the value of the coordinates in the binary scale.

If $K = 3$, the marks are 0,1 and 2. There are 4 points on a line; 40 points in space; 13 lines on a point; and 130 lines in space. These points and lines will be numbered according to the value of their coordinates in the ternary scale.

The following tables show the coordinates of the points and lines, the lines on a point, and the points on a line.

POINTS AND LINES IN PG (3,2)

WITH THEIR COORDINATES.

Line co-ordinates.

Points	Pt. co-ordinates.	000001	000010	000011	000100	000101	000110	000111	001000	001001	001010	001011	010000	010001	010100	010101	011000	011001	011110
Lines.-		1	2	3	4	5	6	7	8	9	10	11	16	17	20	21	24	25	30
1	0001	x	x	x					x	x	x	x							
2	0010	x			x	x							x	x	x	x			
3	0011	x					x	x									x	x	x
4	0100		x		x		x												
5	0101		x			x		x											
6	0110			x	x			x											
7	0111			x		x	x												
8	1000								x				x				x		
9	1001								x					x				x	
10	1010									x			x					x	
11	1011									x				x			x		
12	1100										x				x				x
13	1101										x					x			
14	1110											x			x				
15	1111											x				x			x

Line coordinates

0 1 1 1 1 1	1 0 0 0 0 0	1 0 0 0 1 0	1 0 0 1 0 0	1 0 0 1 1 0	1 0 1 0 0 0	1 0 1 0 1 0	1 0 1 1 0 1	1 0 1 1 1 1	1 1 0 0 0 0	1 1 0 0 1 1	1 1 0 1 0 0	1 1 0 1 1 1	1 1 1 0 0 0	1 1 1 0 1 1	1 1 1 1 0 1	1 1 1 1 1 0	Pt. coordinates	Points
31	32	34	36	38	40	42	45	47	48	51	52	55	56	59	61	62	←Lines…	
																	0001	1
																	0010	2
x																	0011	3
	x	x	x	x													0100	4
					x	x	x	x									0101	5
									x	x	x	x					0110	6
													x	x	x	x	0111	7
	x				x				x				x				1000	8
		x				x				x				x			1001	9
			x				x				x				x		1010	10
				x				x				x				x	1011	11
	x					x					x					x	1100	12
x		x			x							x			x		1101	13
x			x					x	x					x			1110	14
				x			x			x			x				1111	15

The 130 Lines in P G (3,3), Their Coordinates, and the Four Points on Each Line.

Line	Coordinates	Points				Line	Coordinates	Points.			
1	000001	1	3	4	5	109	011001	4	29	30	34
3	000010	1	9	10	11	110	011002	4	28	32	33
4	000011	1	15	16	17	123	011120	4	36	40	44
5	000012	1	12	13	14	124	011121	4	38	39	43
9	000100	3	9	12	15	125	011122	4	37	41	42
10	000101	3	11	14	17	129	011210	4	45	49	53
11	000102	3	10	13	16	130	011211	4	47	48	52
12	000110	5	9	14	16	131	011212	4	46	50	51
13	000111	5	11	13	15	135	012000	5	27	32	34
14	000112	5	10	12	17	136	012001	5	29	31	33
15	000120	4	9	13	17	137	012002	5	28	30	35
16	000121	4	11	12	16	147	012110	5	36	41	43
17	000122	4	10	14	15	148	012111	5	38	40	42
27	001000	1	27	28	29	149	012112	5	37	39	44
28	001001	1	30	31	32	159	012220	5	45	50	52
29	001002	1	33	34	35	160	012221	5	47	49	51
30	001010	1	45	46	47	161	012222	5	46	48	53
31	001011	1	48	49	50	243	100000	9	27	36	45
32	001012	1	51	52	53	246	100010	9	28	37	46
33	001020	1	36	37	38	249	100020	9	29	38	47
34	001021	1	39	40	41	252	100100	9	33	42	51
35	001022	1	42	43	44	255	100110	9	34	43	52
81	010000	3	27	30	33	258	100120	9	35	44	53
82	010001	3	29	32	35	261	100200	9	30	39	48
83	010002	3	28	31	34	264	100210	9	31	40	49
90	010100	3	36	39	42	267	100220	9	32	41	50
91	010101	3	38	41	44	270	101000	10	27	37	47
92	010102	3	37	40	43	273	101010	10	28	38	45
99	010200	3	45	48	51	276	101020	10	29	36	46
100	010201	3	47	50	53	281	101102	10	33	43	53
101	010202	3	46	49	52	284	101112	10	34	44	51
108	011000	4	27	31	35	287	101122	10	35	42	52

Lines	Coor-dinates	Points				Lines	Coor-dinates	Points.			
289	101201	10	30	40	50	388	112101	14	33	38	49
292	101211	10	31	41	48	390	112110	14	34	36	50
295	101221	10	32	39	49	395	112122	14	35	37	48
297	102000	11	27	38	46	398	112202	14	30	44	46
300	102010	11	28	36	47	400	112211	14	31	42	47
303	102020	11	29	37	45	402	112220	14	32	43	45
307	102101	11	33	44	52	405	120000	15	27	42	48
310	102111	11	34	42	53	409	120011	15	28	43	49
313	102121	11	35	43	51	413	120022	15	29	44	50
317	102202	11	30	41	49	414	120100	15	33	39	45
320	102212	11	31	39	50	418	120111	15	34	40	46
323	102222	11	32	40	48	422	120122	15	35	41	47
324	110000	12	27	39	51	423	120200	15	30	36	51
329	110012	12	28	40	52	427	120211	15	31	37	52
331	110021	12	29	41	53	431	120222	15	32	38	53
333	110100	12	33	36	48	432	121000	16	27	43	50
338	110112	12	34	37	49	436	121011	16	28	44	48
340	110121	12	35	38	50	440	121022	16	29	42	49
342	110200	12	30	42	45	443	121102	16	33	40	47
347	110212	12	31	43	46	444	121110	16	34	41	45
349	110221	12	32	44	47	448	121121	16	35	39	46
351	111000	13	27	40	53	451	121201	16	30	37	53
356	111012	13	28	41	51	455	121212	16	31	38	51
358	111021	13	29	39	52	456	121220	16	32	36	52
362	111102	13	33	37	50	459	122000	17	27	44	49
364	111111	13	34	38	48	463	122011	17	28	42	50
366	111120	13	35	36	49	467	122022	17	29	43	48
370	111201	13	30	43	47	469	122101	17	33	41	46
372	111210	13	31	44	45	473	122112	17	34	39	47
377	111222	13	32	42	46	474	122120	17	35	40	45
378	112000	14	27	41	52	479	122202	17	30	38	52
383	112012	14	28	39	53	480	122210	17	31	36	53
385	112021	14	29	40	51	484	122221	17	32	37	51

The 40 Points in P G(3,3), Their Coordinates

and the 13 Lines on Each Point.

Pts.	Coor-dinates.	Lines.												
1	0001	1	3	4	5	27	28	29	30	31	32	33	34	35
3	0010	1	9	10	11	81	82	83	90	91	92	99	100	101
4	0011	1	15	16	17	108	109	110	123	124	125	129	130	131
5	0012	1	12	13	14	135	136	137	147	148	149	159	160	161
9	0100	3	9	12	15	243	246	249	252	255	258	261	264	267
10	0101	3	11	14	17	270	273	276	281	284	287	289	292	295
11	0102	3	10	13	16	297	300	303	307	310	313	317	320	323
12	0110	5	9	14	16	324	329	331	333	338	340	342	347	349
13	0111	5	11	13	15	351	356	358	362	364	366	370	372	377
14	0112	5	10	12	17	378	383	385	386	390	395	398	400	402
15	0120	4	9	13	17	405	409	413	414	418	422	423	427	431
16	0121	4	11	12	16	432	436	440	443	444	448	451	455	456
17	0122.	4	10	14	15	459	463	467	469	473	474	479	480	484
27	1000	27	81	108	135	243	270	297	324	351	378	405	432	459
28	1001	27	83	110	137	246	273	300	329	356	383	409	436	463
29	1002	27	82	109	136	249	276	303	331	358	385	413	440	467
30	1010	28	81	109	137	261	289	317	342	370	398	423	451	479
31	1011	28	83	108	136	264	292	320	347	372	400	427	455	480
32	1012	28	82	110	135	267	295	323	349	377	402	431	456	484
33	1020	29	81	110	136	252	281	307	333	362	388	414	443	469
34	1021	29	83	109	135	255	284	310	338	364	390	418	444	473
35	1022	29	82	108	137	258	287	313	340	366	395	422	448	474
36	1100	33	90	123	147	243	276	300	333	336	390	423	456	480
37	1101	33	92	125	149	246	270	303	338	362	395	427	451	484
38	1102	33	91	124	148	249	273	297	340	364	388	431	455	479
39	1110	34	90	124	149	261	295	320	324	358	383	414	448	473
40	1111	34	92	123	148	264	289	323	329	351	385	418	443	474
41	1112	34	91	125	147	267	292	317	331	356	378	422	444	469
42	1120	35	90	125	148	252	287	310	342	377	400	405	440	463
43	1121	35	92	124	147	255	281	313	347	370	402	409	432	467
44	1122	35	91	123	149	258	284	307	349	372	398	413	436	459
45	1200	30	99	129	159	243	273	303	342	372	402	414	444	474
46	1201	30	101	131	161	246	276	297	347	377	398	418	448	469
47	1202	30	100	130	160	249	270	300	349	370	400	422	443	473
48	1210	31	99	130	161	261	292	323	333	364	395	405	436	467
49	1211	31	101	129	160	264	295	317	338	366	388	409	440	459
50	1212	31	100	131	159	267	289	320	340	362	390	413	432	463
51	1220	32	99	131	160	252	284	313	324	356	385	432	455	484
52	1221	32	101	130	159	255	287	307	329	358	378	427	456	479
53	1222	32	100	129	161	258	281	310	331	351	383	431	451	480

NUMBER OF COMPLEXES ON A CONGRUENCE.

Any two equations of complexes determine a congruence. If two such equations are a and b, any equation dependent on the two is of the form $\lambda_1 a + \lambda_2 b$, where $(\lambda_1 \lambda_2)$ are marks of the system. The number of equations dependent on two equations is then the number of sets (λ_1, λ_2) that can be chosen from the marks of the system. Zero is excluded for (0,0) gives no equation, and if either λ_1 or λ_2 is zero, we get one of the original equations. Therefore $(K-1)^2$ is the number of sets of (λ_1, λ_2). But this must be divided by (K-1) since multiplication or division of an equation by any of the marks, except zero, gives no new equation, hence there are (K-1) equations dependent on any two, or (K+1) equations in a set, any two of which give the same congruence. Therefore the number of complexes on a congruence is

$$\underline{K}+1.$$

The $\underline{K}+1$ complexes on a congruence are called a pencil of complexes.

NUMBER OF COMPLEXES ON A REGULUS.

If any three equations (α, β, γ) determine a regulus then the number of complexes on the regulus is the number of equations in the set containing (α, β, γ) such that any one of the set is dependent on two or three of the others.

The number dependent on two of the equations is $(\lambda_1\alpha+\lambda_2\beta) + (\lambda_1\alpha+\lambda_2\gamma) + (\lambda_1\beta+\lambda_2\gamma)$. The number dependent on three is $(\lambda_1\alpha+\lambda_2\beta+\lambda_3\gamma)$. As before the number dependent on any two is $(K-1)$. The number dependent on three is

$$\frac{(\underline{K}-1)^3}{(\underline{K}-1)} = (K-1)^2.$$

Hence the total number of equations in the set is

$$3 + 3(K-1)+(K-1)^2 = \underline{K}^2+\underline{K}+1.$$

Therefore the number of complexes on a regulus is

$$\underline{K}^2+\underline{K}+1.$$

The $\underline{K}^2+\underline{K}+1$ complexes on a regulus are sometimes called a "system of three terms", or a "hyperpencil."

EXAMPLES OF COMPLEXES AND CONGRUENCES IN PG (3,2).

GENERAL COMPLEXES.

The lines of the general complex are dependent on the sides of a simple skew pentagon, no four vertices of which are coplanar. To find the equation of a complex dependent on a given pentagon, we find the equation which is satisfied by the sides of the pentagon.

Since the p_{ik} of the equation,

$a_{12}p_{12}+ a_{13}p_{13} + a_{14}p_{14}+ a_{23}p_{23}+ a_{42}p_{42}+ a_{34}p_{34} = 0$, are the coordinates of the lines, we can find the a_{ik} for each line by substituting the coordinates of each line in the equation.

Let the lines of the given pentagon in PG(3,2) be lines 8,3,62,30,25; with vertices 9,1; 7,12,3. The coordinates of line 8 are 001000. Then the equation of this line is

$$a_{14} = 0.$$

Likewise

for line 3, $a_{34} + a_{42} = 0$.

for line 62, $a_{12} + a_{13} + a_{14} + a_{23} + a_{42} = 0$.

for line 30, $a_{13} + a_{14} + a_{23} + a_{42} = 0$.

for line 25, $a_{13} + a_{14} + a_{34} = 0$.

Solving these equations simultaneously,

$a_{34} = a_{42} = a_{13}$, $a_{12} = a_{14} = a_{23} = 0$.

Since there are five equations in six unknowns, the value of one unknown can be assigned and the rest will be uniquely determined. Then let $a_{34} = 1$ and the equation of the complex is

$$(G_{13}),\ P_{13} + P_{42} + P_{34} = 0.$$

This is the equation of a general complex in which there are fifteen lines, The lines whose coordinates satisfy the equation are 3,4,7,8,11, 17,21,25,30,32,36,40,61,62.

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For another complex let the vertices of the determining pentagon be points 9,1,7,5,10, and the lines 8,3,5,45,25. Then two of the lines will coincide with two of those in the pentagon of (G_{13}).

The equation is determined, as before, to be,

$$(G_{26}), \quad P_{13}+P_{23}+P_{42}+P_{34}=0.$$

The lines which satisfy are lines 3,5,6,8, 11,17,20,25,31,32,38,40,45,52,55.

A pentagon not having two consecutive sides of the pentagon coinciding with that of (G_{13}), is one with sides 9,36,48,7,1, vertices 1,10,14,6,3. The equation is

$$(G_{19}), \quad P_{12}+P_{13}+P_{23}+P_{42}=0.$$

The lines of this complex are, 1,6,7,8,9,20,21, 34,36,42,45,48,55,56,62.

SPECIAL COMPLEXES IN PG(3,2).

For the equation of a special complex, a pentagon is chosen, four vertices of which are coplanar. Let the vertices be points 1, 14, 12, 3, 6. The lines then are lines 11, 20, 30, 7, 3. The equation of the special complex is

$$(S_{25}\text{-}) \qquad P_{14} + P_{42} + P_{43} = 0.$$

Lines of the complex are 3, 4, 7, 9, 10, 16, 20, 25, 32, 30, 36, 42, 45, 48, 51, 52, 55, 61, 62.

Line 52 meets all other lines of the complex and is therefore the directrix.

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Special complex (S_{19}) $\quad P_{12} + P_{13} + P_{14} = 0$ is satisfied by the lines, 1, 2, 3, 4, 5, 6, 7, 24, 25, 30, 31, 40, 42, 45, 47, 48, 51, 52, 55. The directrix is line 7, which has point 6 in common with line 52, the directrix of complex $(S_{25}\text{-})$.

Special complex (S_{20}) $P_{12} + P_{13} + P_{23} = 0$

Lines 1, 2, 3, 8, 9, 10, 11, 20, 21, 31, 36, 38, 45, 47, 48, 51, 56, 59, 30.

The directrix, line 11, has no point on line 52, the directrix of (S_{25}); or on line 7, the directrix of (S_{79}).

CONGRUENCES COMMON TO TWO GENERAL COMPLEXES IN PG (3,2).

The lines common to (G_{13}) and (G_{26}) are the seven lines 3, 8, 11, 17, 25, 32, 40, which form a parabolic congruence. Line 8 meets all the other lines and is therefore the directrix. This gives an example of a parabolic congruence common to two general complexes, which have two consecutive lines of the determining pentagons in common.

Complexes (G_{15}) and (G_{19}), whose determining pentagons do not have two consecutive lines in common, have in common the lines 7, 8, 21, 36, 63, which form an elliptic congruence.

Lines common to (G_{19}) and (G_{26}) are the lines of an elliptic congruence, 6, 8, 20, 45, 55.

EXAMPLES OF COMPLEXES AND CONGRUENCES IN PG (3,3).

GENERAL COMPLEXES.

Equation (G_{43}) $P_{13}+P_{42}+P_{34}=0$.

Lines of determining pentagon, 27, 5, 366, 123, 110.

Vertices of determining pentagon, 28, 1, 13, 36, 4.

Lines 5, 9, 14, 16, 27, 32, 34, 83, 92, 101, 110, 123, 130, 137, 148, 159, 243, 252, 261, 270, 284, 295, 297, 313, 320, 362, 364, 366, 398, 400, 413, 422, 431, 440, 444, 451, 467, 469, 480, 402.

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Equation (G_{147}) $P_{13}+P_{23}+P_{42}+P_{34}=0$.

Lines of pentagon 27, 5, 11, 289, 137.

Vertices of pentagon 28, 1, 13, 10, 30.

Lines 5, 11, 13, 15, 27, 32, 34, 83, 91, 99, 110, 125, 131, 137, 147, 160, 243, 258, 264, 270, 281, 289, 297, 310, 323, 342, 347, 349, 388, 390, 395, 413, 414, 427, 440, 448, 456, 467, 473, 479.

Equation (G_{79}) $P_{12}+P_{13}+P_{23}+P_{42}=0$.

Lines of pentagon 28, 261, 324, 16, 1.

Vertices of pentagon 1, 30, 39, 12, 4.

Lines. 1, 15, 16, 17, 27, 28, 29, 99, 100, 101, 147, 148, 149, 249, 255, 261, 276, 284, 289, 303, 310, 317, 329, 333, 349, 356, 362, 377, 383, 388, 402, 405, 422, 427, 432, 448, 455, 459, 474, 480.

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Equation (G_{153})

Lines of pentagon, 27, 5, 366, 123, 436.

Vertices of pentagon, 29, 1, 13, 36, 44.

Lines. 5, 10, 12, 17, 27, 32, 34, 82, 92, 99, 109, 123, 131, 136, 148, 160, 243, 255, 267, 270, 287, 292, 297, 307, 317, 342, 347, 349, 362, 364, 366, 409, 414, 431, 436, 448, 451, 463, 473, 480.

SPECIAL COMPLEXES IN PG (3,3).

Equation (S_{32}) $P_{12} - P_{13} - P_{14} = 0$.

Lines.

1, 3, 4, 5, 9, 10, 11, 12, 13, 14, 15, 16, 17, 108, 109, 110, 123, 124, 125, 129, 130, 131, 297, 300, 303, 307, 310, 313, 317, 320, 323, 324, 329, 331, 333, 338, 340, 342, 347, 349, 432, 436, 440, 443, 444, 448, 451, 455, 456.

Directrix , line 16.

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Equation (S_{36}) $P_{12} - P_{13} + P_{23} = 0$.

Lines.

1, 3, 4, 5, 27, 28, 29, 30, 31, 32, 33, 34, 35, 90, 91, 92, 123, 124, 125, 147, 148, 149, 261, 264, 267, 289, 292, 295, 317, 320, 323, 324, 329, 331, 351, 356, 358, 378, 383, 385, 414, 418, 422, 443, 444, 448, 469, 473, 474.

Directrix, line 34.

Equation (S_{55}), $P_{14} + P_{42} + P_{34} = 0$.

Lines.

5, 9, 14, 16, 29, 31, 33, 81, 90, 99, 110, 123, 130, 136, 147, 161, 252, 243, 261, 276, 281, 292, 300, 307, 323, 324, 329, 331, 333, 338, 340, 342, 347, 349, 362, 366, 388, 390, 395, 405, 414, 423, 436, 443, 456, 464, 467, 469, 480.

Directrix, line 333.

CONGRUENCE COMMON TO TWO GENERAL COMPLEXES IN PG (3,3).

Complexes (G_{43}) and (G_{147}) are determined by pentagons which have two consecutive lines of one coinciding with two of the other. The common congruence is parabolic.

Lines, 5, 27, 32, 34, 83, 110, 137, 243, 270, 297, 413, 440, 467.

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The pentagons of (G_{43}) and (G_{79}) are distinct. The common congruence is elliptic.

Lines 16, 27, 92, 159, 261, 284, 366, 400, 431, 469.

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The pentagons of (G_{43}) and (G_{153}) have four lines of one coinciding with four lines of the other. The common congruence is hyperbolic.

Lines, 5, 27, 32, 34, 92, 123, 148, 243, 270, 297, 362, 364, 366, 431, 480, 451.

CONGRUENCE COMMON TO TWO SPECIAL COMPLEXES IN PG (3,3).

(S_{32}) and (S_{53}) whose directrices are on a point, have in common the twenty-two lines of a degenerate congruence.

Lines.

5, 9, 14, 16, 110, 130, 300, 307, 323, 324, 329, 331, 333, 338, 340, 342, 347, 349, 436, 443, 446.

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(S_{55}) and (S_{36}) whose directrices are not on a point, have in common the sixteen lines of a hyperbolic congruence.

Lines.

5, 29, 31, 33, 90, 123, 147, 261, 292, 323, 324, 329, 331, 414, 443, 469.

CONGRUENCE COMMON TO A GENERAL AND

A SPECIAL COMPLEX IN PG (3,3).

(S_{32}) and (G_{79}). The directrix of (S) is a line of (G_{79}). The thirteen common lines form a parabolic congruence.

Lines.

1, 15, 16, 17, 303, 310, 317, 329, 333, 349, 432, 448, 455.

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(S_{32}) and(G_{147}). The directrix of (S) is not a line of (G_{147}). The sixteen lines of a hyperbolic congruence are common to the two.

Lines.

5, 11, 13, 15, 110, 125, 131, 297, 310, 323, 342, 347, 349, 440, 448, 456.

PENCIL OF COMPLEXES ON THE SAME CONGRUENCE (3,3).

The four complexes:

$(S_1)\quad P_{12} = 0$

$(S_2)\quad P_{13} = 0$

$(S_7)\quad P_{12} + P_{13} = 0$

$(S_8)\quad P_{12} - P_{13} = 0$

are all on the same congruence, since each of the equations is dependent on two of the others.

The lines whose coordinates satisfy all four equations are the twenty-two lines of a degenerate congruence;

1, 3, 4, 5, 9, 10, 11, 12, 13, 14, 15, 16, 17, 27, 28, 29, 30, 31, 32, 33, 34, 35.

SET OF COMPLEXES ON THE SAME REGULUS IN PG (3,3).

The three complexes

(S_1) $P_{12} = 0$

(S_2) $P_{13} = 0$

(S_3) $P_{14} = 0$

have in common lines

1, 3, 4, 5, 9, 10, 11, 12, 13, 14, 15, 16, 17, which are all on the same plane. This plane of lines is common also to all complexes whose equations are dependent on two or three of these equations. The complete set of $\underline{K}^2+\underline{K}+1 = 13$ equations, is as follows:

(S_1) $P_{12} = 0$

(S_2) $P_{13} = 0$

(S_3) $P_{14} = 0$

(S_7) $P_{12} + P_{13} = 0$

(S_8) $P_{12} - P_{13} = 0$

(S_9) $P_{12} + P_{14} = 0$

$$(S_{10}) \qquad P_{12} - P_{14} = 0$$

$$(S_{15}) \qquad P_{13} + P_{14} = 0$$

$$(S_{16}) \qquad P_{13} - P_{14} = 0$$

$$(S_{91}) \qquad P_{12} + P_{13} + P_{14} = 0$$

$$(S_{92}) \qquad P_{12} - P_{13} + P_{14} = 0$$

$$(S_{93}) \qquad -P_{12} + P_{13} + P_{14} = 0$$

$$(S_{94}) \qquad P_{12} + P_{13} - P_{14} = 0$$