Graphs of Polytopes

By

William Espenschied

Margaret Bayer, Chairperson

Jeremy Martin

Jack Porter

Saul Stahl

Nancy Kinnersley

Date defended: 09-02-2014

The Dissertation Committee for William Espenschied certifies
that this is the approved version of the following dissertation :

Graphs of Polytopes

_____
Margaret Bayer, Chairperson

Date approved: 09-02-2014

Abstract

The graph of a polytope is the graph whose vertex set is the set of vertices of the polytope, and whose edge set is the set of edges of the polytope. Several problems concerning graphs of polytopes are discussed. The primary result is a set of bounds (Theorem 39) on the maximal size of an anticlique (sometimes called a coclique, stable set, or independent set) of the graph of a polytope based on its dimension and number of vertices.

Two results concerning properties preserved by certain operations on polytopes are presented. The first is that the Gale diagram of a join of polytopes is the direct sum of the Gale diagrams of the polytopes and dually, that the Gale diagram of a direct sum of polytopes is the join of their Gale diagrams (Theorem 23). The second is that if two polytopes satisfy a weakened form of Gale's evenness condition, then so does their product (Theorem 32).

It is shown, by other means, that, with only two exceptions, the complete bipartite graphs are never graphs of polytopes (Theorem 47). The techniques developed throughout are then used to show that the complete 3-partite graph $K_{1,n,m}$ is the graph of a polytope if and only if $K_{n,m}$ is the graph of a polytope (Theorem 49). It is then shown that $K_{2,2,3}$ and $K_{2,2,4}$ are never graphs of polytopes. A conjecture is then stated as to precisely when a complete multipartite graph is the graph of a polytope. Finally, a section is devoted to results concerning the dimensions for which the graph of a crosspolytope is the graph of a polytope.

Acknowledgements

Contents

List of Figures

List of Tables

Chapter 1

Polytopes

This chapter will provide relevant background material in the theory of convex polytopes. Proofs of any theorems which appear can be found in McMullen & Shephard (1971) (unless otherwise noted).

The set $\{1, 2, \ldots, n\}$ will be denoted $[n]$. Throughout, $\mathbb{R}^d$ will denote the real inner product space of column vectors of length d with real entries and the Euclidean inner product. The set of vectors $\{e_i \mid i \in [d]\} \subseteq \mathbb{R}^d$ is called the *standard basis* and e_i has jth entry $(e_i)_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$.

The vectors $\mathbf{0}_d, \mathbf{1}_d \in \mathbb{R}^d$ have jth entry $(\mathbf{0})_j = 0$, and $(\mathbf{1})_j = 1$ respectively. When no confusion may arise, these vectors shall be denoted simply $\mathbf{0}$ and $\mathbf{1}$. If $x, y \in \mathbb{R}^n$, then $\langle x, y \rangle$ will denote the inner product of x and y, $\|x\| = \langle x, x \rangle^{1/2}$ is the Euclidean norm, and $B_\varepsilon(x) = \{y \mid \|x - y\| \leq \varepsilon\}$ is the closed ball of radius $\varepsilon > 0$ centered at x. No distinction shall be made between the vector space $\mathbb{R}^d$ and its dual $(\mathbb{R}^d)^*$. Therefore all vectors will be considered to be column vectors. If $A \subseteq \mathbb{R}$ and $d \in \mathbb{N}$, then $A^d = \{x \in \mathbb{R}^d \mid \langle x, e_i \rangle \in A \text{ for all } i \in [d]\}$.

1.1   Hulls; or, The Parts That Touch the Water

If $\{x_1, x_2, \ldots, x_k\} \subseteq \mathbb{R}^n$, then a vector v is an *affine combination* of $x_1, x_2, \ldots, x_k$ if there are elements $\lambda_i \in \mathbb{R}$ such that $v = \sum_{i \in [k]} \lambda_i x_i$ and $\sum_{i \in [k]} \lambda_i = 1$. If $A \subseteq \mathbb{R}^n$, then the *affine hull* of A is the

set[1]

$$\text{aff}(A) = \left\{ \sum_{i \in [k]} \lambda_i \boldsymbol{x}_i \ \middle| \ \boldsymbol{x}_i \in A, \ \lambda_i \in \mathbb{R} \text{ and } \sum_{i \in [k]} \lambda_i = 1 \right\}.$$

A set which is closed under taking affine combinations of its elements (that is, $\text{aff}(A) = A$) is called an *affine set*. The intersection of a family of affine sets is again an affine set. Every affine subset of $\mathbb{R}^n$ is a translation of a linear subspace of $\mathbb{R}^n$. If A is an affine set, and L is a linear subspace of $\mathbb{R}^n$ such that $A = L + \boldsymbol{b}$, then the *dimension* of A is $\dim A = \dim L$. For example: $\dim \varnothing = -1$; $\dim \{\boldsymbol{x}\} = 0$; and $\dim \text{aff} \{\boldsymbol{x}, \boldsymbol{y}\} = 1$ for $\boldsymbol{x} \neq \boldsymbol{y}$.

A set $K \subseteq \mathbb{R}^n$ is *convex* if for each $\boldsymbol{x}, \boldsymbol{y} \in K$ the set $\{\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \mid 0 \leq \lambda \leq 1\} \subseteq K$. If $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\} \subseteq \mathbb{R}^n$, then a vector $\boldsymbol{v}$ is a *convex combination* of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ if there are elements $\lambda_i \in \mathbb{R}$ such that $\boldsymbol{v} = \sum_{i \in [k]} \lambda_i \boldsymbol{x}_i$, with $\lambda_i \geq 0$ and $\sum_{i \in [k]} \lambda_i = 1$. If $A \subseteq \mathbb{R}^n$, then the *convex hull* of A is the set

$$\text{conv}(A) = \left\{ \sum_{i \in [k]} \lambda_i \boldsymbol{x}_i \ \middle| \ \boldsymbol{x}_i \in A, \ \lambda_i \geq 0 \text{ and } \sum_{i \in [k]} \lambda_i = 1 \right\}.$$

If K is convex, then $\text{conv} K = K$. For each set A, $\text{conv} A \subseteq \text{aff} A$. The intersection of a family of convex sets is again a convex set. The *dimension* of a convex set K is the dimension of its affine hull, $\dim K = \dim \text{aff} K$.

Both aff and conv are closure operators, that is:

- $A \subseteq \text{aff} A$ and $A \subseteq \text{conv} A$;

- if $A \subseteq B$, then $\text{aff} A \subseteq \text{aff} B$ and $\text{conv} A \subseteq \text{conv} B$; and

- $\text{aff}(\text{aff} A) = \text{aff} A$ and $\text{conv}(\text{conv} A) = \text{conv} A$.

Both $\text{aff} A$ and $\text{conv} A$ are minimal in the sense that $\text{aff} A$ is the intersection of all affine sets which contain A, and $\text{conv} A$ is the intersection of all convex sets which contain A.

If $H \subseteq \mathbb{R}^n$ is an affine set with $\dim H = n - 1$, then H is called a *hyperplane*, and there is a pair $(\boldsymbol{\xi}, t)$ such that $\boldsymbol{\xi} \in \mathbb{R}^n$ with $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = 1$ and $t \in \mathbb{R}$ with $H = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = t\}$ (here, $\boldsymbol{\xi}$ is not

---

[1]Note, in this definition (as well as that of the convex hull), A need not be a finite set.

unique unless $n = 0$ (if $n \neq 0$, then there are exactly two such vectors, and the other one is $-\boldsymbol{\xi}$)). The pair $(H, \boldsymbol{\xi})$ is called an *oriented hyperplane*, and the *positive* and *negative closed halfspaces* of H are defined by, respectively, $H^+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle \geq t\}$ and $H^- = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle \leq t\}$. Thus $H = H^+ \cap H^-$. The *positive* and *negative open half-spaces* $H^{(+)}$ and $H^{(-)}$ are defined by making the defining inequalities strict. Alternatively, $H^{(+)} = H^+ \setminus H$ and $H^{(-)} = H^- \setminus H$. A set of the form H^+ or H^- is called a *closed half-space*.

1.2   Polytopes (not 'Galeorhinus Galeus')

In this section convex polytopes[2] will be defined in two different ways. The proof of equivalence of these two definitions can be found in Brøndsted (1983), Grünbaum (2003), McMullen & Shephard (1971) or Ziegler (1995).

An *$\mathcal{H}$-polytope* is a bounded set of the form $P = \bigcap \mathcal{H}$ where

$$\mathcal{H} = \left\{ H_1^+, H_2^+, \ldots, H_k^+, H_{k+1}^-, H_{k+2}^-, \ldots, H_\ell^- \right\}$$

is a finite collection of closed halfspaces. A *$\mathcal{V}$-polytope* is the convex hull of a finite set of points in some Euclidean space $\mathbb{R}^k$. A set P is an $\mathcal{H}$-polytope if and only if it is a $\mathcal{V}$-polytope. Henceforth the prefix will be omitted, and P shall be called a polytope. If P is a polytope, then the set $\operatorname{vert} P$ is the (unique) minimal (under inclusion) set such that $P = \operatorname{conv}(\operatorname{vert} P)$.

The dimension of a polytope is its dimension as a convex set. A polytope of dimension d is called a *d-polytope*. If P is a d-polytope, then a hyperplane H is called a *supporting hyperplane* of P if $P \cap H \neq \varnothing$ and either $P \subseteq H^+$ or $P \subseteq H^-$. A *face* of a polytope P is a subset F of P such that one of the following holds:

1. $F = \varnothing$; or

2. there is some supporting hyperplane H of P with $F = H \cap P$; or

---

[2]Throughout, all polytopes will be convex so the adjective "convex" shall be omitted.

3. $F = P$.

If F is a face of a polytope P, then F is itself a polytope since $F = \text{conv}(F \cap \text{vert} P)$, further $\text{vert} F = F \cap \text{vert} P$. If $\dim F = r$, then F is called an *r-face* of P. A face of a face of a polytope is again a face of the polytope. A face of a polytope is called *proper* if it is neither empty, nor the whole polytope. If P is a d-polytope, then a face of dimension

- 0 is called a *vertex*,

- 1 is called an *edge*,

- $d - 2$ is called a *ridge*,

- $d - 1$ is called a *facet*.

If P is a polytope, and $\{x\}$ is a vertex of P, then no distinction is made between the vertex $\{x\}$ and the point x. If P is a polytope, then $\text{vert} P = \{x \in P \mid x \text{ is a vertex of } P\}$. Each face is the intersection of all facets which contain it.

When ordered by inclusion, the faces of a d-polytope P form a lattice of rank $d + 1$, called the *face lattice* of P and denoted $\mathscr{F}(P)$. The join of two faces is the minimal face containing both of them, and the meet of two faces is their intersection. Two polytopes are said to be *combinatorially equivalent* if their face lattices are isomorphic. Combinatorial equivalence is an equivalence relation on the class of all polytopes. The *ith face number* $f_i(P)$ of P is the number of faces F of P with $\dim F = i$. The *f-vector* of P is the vector $\boldsymbol{f}(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$. The nonproper faces correspond to the numbers $f_{-1}(P) = 1 = f_d(P)$.

If $X \subseteq \mathbb{R}^d$, then the *relative interior* of X is the interior of X relative to the topological space $\text{aff} X$, that is,

$$\text{relint} X = \{x \in X \mid \text{there is some } \varepsilon > 0 \text{ such that } B_\varepsilon(x) \cap \text{aff}(X) \subseteq X\}.$$

If P is a k-polytope in $\mathbb{R}^d$ such that $\mathbf{0} \in \operatorname{relint} P$, then the *polar dual* of P is the set

$$P^\Delta = \left\{ \mathbf{x} \in \operatorname{aff} P \mid \langle \mathbf{x}, \mathbf{y} \rangle \le 1 \text{ for all } \mathbf{y} \in P \right\}.$$

The polar dual of a polytope is again a polytope, and $\dim P^\Delta = \dim P$. Thus any polytope which is combinatorially equivalent to P^Δ is called a *dual* of P. Furthermore, $(P^\Delta)^\Delta = P$, and $\mathscr{F}(P^\Delta)$ is the dual (as a lattice) of $\mathscr{F}(P)$. Thus $f_i(P) = f_{d-i-1}(P^\Delta)$.

If $P \subseteq \mathbb{R}^d$, and $\mathbf{x} \in \mathbb{R}^d$, then

$$P + \mathbf{x} = \{ \mathbf{p} + \mathbf{x} \mid \mathbf{p} \in P \}$$

is called the *translation* of P by $\mathbf{x}$. Similarly, define $P - \mathbf{x} = P + (-\mathbf{x})$. Any translation of any polytope is again a polytope, and is combinatorially equivalent to P. Furthermore, the equality $\operatorname{vert}(P + \mathbf{x}) = \operatorname{vert}(P) + \mathbf{x}$ holds. If $P \in \mathbb{R}^d$ is a k-polytope and $\mathbf{0} \notin \operatorname{relint} P$ with $\mathbf{x} \in \operatorname{relint} P$, then a *dual* of P is any polytope which is combinatorially equivalent to $(P - \mathbf{x})^\Delta$. This definition is independent of the choice of $\mathbf{x} \in \operatorname{relint} P$.

1.3 Examples of Polytopes

1.3.1 Simplices

The *standard d-simplex* is

$$\Delta_d = \operatorname{conv}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d+1}\} = \left\{ \mathbf{x} \in \mathbb{R}^{d+1} \;\middle|\; \langle \mathbf{x}, \mathbf{1} \rangle = 1, \text{ and } \langle \mathbf{x}, \mathbf{e}_i \rangle \ge 0 \right\}.$$

A d-polytope which is combinatorially equivalent to the standard d-simplex is called a *d-simplex*.

All d-simplices have $d + 1$ vertices, and moreover any d-polytope with $d + 1$ vertices is a d-simplex. The face lattice of a d-simplex is the Boolean lattice on $d + 1$ elements. Equivalently, if P is a simplex, then for any $A \subseteq \operatorname{vert} P$, the set $\operatorname{conv} A$ is a face of P. A 0-dimensional simplex is a point; a 1-simplex is a line segment; a 2-simplex is a triangle; and a 3-simplex is a tetrahedron.

The *f*-vector of a *d*-simplex is given by

$$f_k(\Delta_d) = \binom{d+1}{k+1}.$$

Consequently, the dual of a *d*-simplex is also a *d*-simplex (since $f_0(\Delta_d^\Delta) = f_{d-1}(\Delta_d) = d+1$).
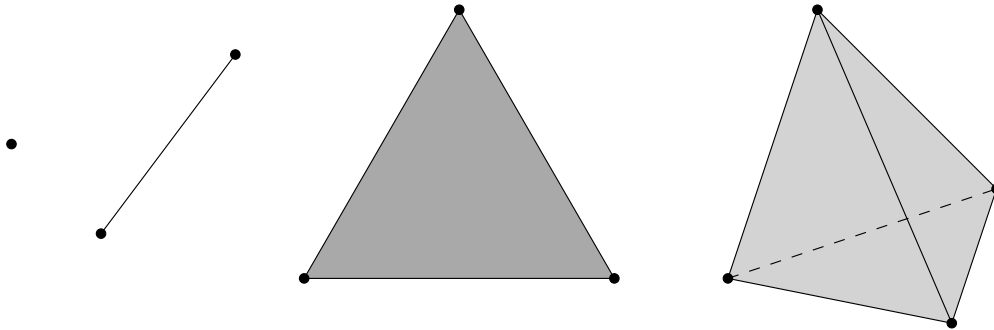
Figure 1.1: Simplices of dimensions 0, 1, 2, and 3.

1.3.2   Cubes

The *standard d-cube* is

$$Q_d = \operatorname{conv}\left\{\{-1,1\}^d\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^d \;\middle|\; |\langle \boldsymbol{x}, \boldsymbol{e}_i\rangle| \le 1\right\}.$$

A *d*-polytope which is combinatorially equivalent to the standard *d*-cube is called a *d-cube*.

Any face of a cube is also a cube. A 0-cube is a point; a 1-cube is a line segment; and a 2-cube is a convex quadrilateral. Every vertex of a *d*-cube lies in exactly *d* facets. The *f*-vector of a *d*-cube is given by

$$f_k(Q_d) = 2^{d-k}\binom{d}{k}$$

for $k \in [d] \cup \{0\}$.
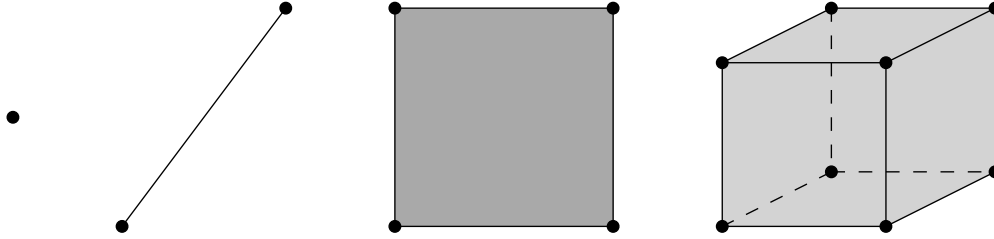
Figure 1.2: Cubes of dimensions 0, 1, 2, and 3.

1.3.3 Crosspolytopes

The *standard d-crosspolytope* is

$$X_d = \text{conv}\left(\{e_i \mid i \in [d]\} \cup \{-e_i \mid i \in [d]\}\right) = \left\{x \in \mathbb{R}^d \;\middle|\; \sum_{i \in [d]} |\langle x, e_i \rangle| \le 1\right\} = Q_d^\Delta.$$

A *d*-polytope which is combinatorially equivalent to the standard *d*-crosspolytope is called a *d-crosspolytope*.

Any proper face of a crosspolytope is a simplex. A 0-crosspolytope is a point; a 1-crosspolytope is a line segment; a 2-crosspolytope is a convex quadrilateral, and a 3-crosspolytope is a polytope which is combinatorially equivalent to a regular octahedron. The *f*-vector of a *d*-crosspolytope is given by

$$f_k(X_d) = 2^{k+1} \binom{d}{k+1}$$

for $k \in [d-1] \cup \{-1, 0\}$.
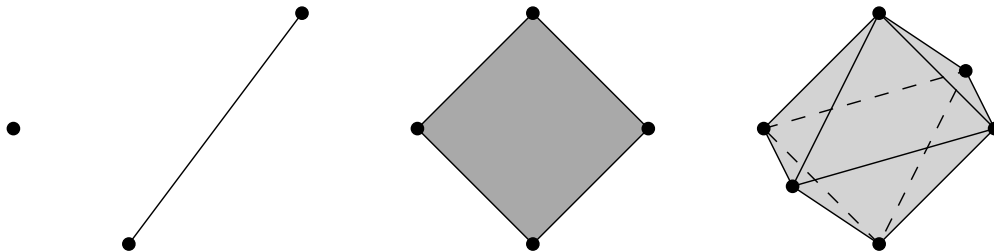
Figure 1.3: Crosspolytopes of dimensions 0, 1, 2, and 3.

1.3.4 Simplicial and Simple Polytopes

A polytope is called *simplicial* if each of its proper faces is a simplex (equivalently, each facet is a simplex). A polytope P is called *simple* if its dual P^Δ is simplicial.

In any simplicial polytope, a proper k-face has $k+1$ vertices; dually, in any simple polytope a proper j-face is contained in exactly $d-j$ facets. If a polytope is both simplicial and simple, then it is either a 2-polytope or a simplex. Examples of simplicial polytopes are: all 2-polytopes; the regular icosahedron; all simplices; and all crosspolytopes. Examples of simple polytopes are: all 2-polytopes; the regular dodecahedron; all simplices; and all cubes.

1.3.5 Cyclic and Gale Polytopes

Let M_d be the curve in $\mathbb{R}^d$ which is defined parametrically by the equation

$$
\boldsymbol{r}(t) = \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^d \end{bmatrix},
$$

for $t \in \mathbb{R}$. If V is any set of $n > d$ distinct points on M_d, say $V = \{\boldsymbol{r}(t_i) \mid t_1 < t_2 < \cdots < t_n\}$, then $\mathrm{conv}\,V$ is a d-dimensional polytope with n vertices (that is, $V = \mathrm{vert}(\mathrm{conv}\,V)$). All such polytopes with n vertices in dimension d are combinatorially equivalent and any polytope which is combinatorially equivalent to such a polytope will be denoted by $C_d(n)$ and called a *cyclic polytope*.

All cyclic polytopes are simplicial, and, for $d \geq 3$, are simple if and only if $n = d+1$ (in which case they are simplices). Since the combinatorial type of $C_d(n)$ is independent of the choices for the t_i, the vertex $\boldsymbol{r}(t_i)$ will be identified with the integer i (assuming that $t_1 < t_2 < \cdots < t_n$), and the face $\mathrm{conv}\,\{i_1, i_2, \ldots, i_k\}$ with the set $\{i_1, i_2, \ldots, i_k\}$ (with $i_1 i_2 \ldots i_k$ if there is no chance for any confusion).

Theorem 1 (Gale's Evenness Condition). *A subset $S \subseteq [n]$ with $|S| = d$ forms a facet of $C_d(n)$ if and only if*

$$|\{k \mid k \in S \text{ and } i < k < j\}| \text{ is even for all } i < j, \text{ with } \{i, j\} \cap S = \varnothing.$$

For a proof, see Grünbaum (2003), McMullen & Shephard (1971) or Ziegler (1995).

Example 2. Table 1.1 shows a method of visualizing Gale's evenness condition for the polytope $C_3(6)$. The facets are given by the rows of the table, and in each row, an asterisk in column i denotes that i is in the facet, and a dash denotes an element of $[6]$ that is not in the facet. Gale's evenness condition can then be interpreted as: between any two dashes there are an even number of asterisks.
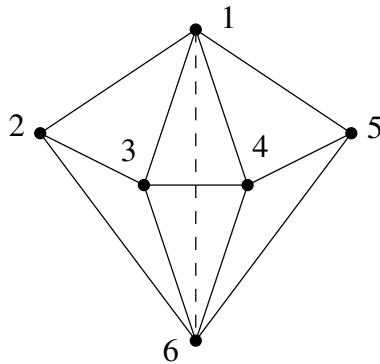
Figure 1.4: The polytope $C_3(6)$

facet	1	2	3	4	5	6
123	*	*	*	-	-	-
126	*	*	-	-	-	*
134	*	-	*	*	-	-
145	*	-	-	*	*	-
156	*	-	-	-	*	*
236	-	*	*	-	-	*
346	-	-	*	*	-	*
456	-	-	-	*	*	*

Table 1.1: A chart to determine the facets of $C_3(6)$

If P is a d-polytope and there is an ordering of $\text{vert}\, P$ such that for each facet F of P, $\text{vert}\, F$ satisfies Gale's evenness condition (other than the requirement $|\text{vert}\, F| = d$), then P is said to be a *Gale polytope*. Gale polytopes will be developed more in Chapter 5.

For $d > 3$, a subset $S \subseteq [n]$ with $|S| \leq \frac{d}{2}$ forms a face of $C_d(n)$. In particular, for $d > 3$, the convex hull of any two vertices of $C_d(n)$ forms an edge of $C_d(n)$. The f-vector of $C_d(n)$ is given

by

$$f_k(C_{2d}(n)) = \sum_{j=1}^{d} \frac{n}{n-j} \binom{n-j}{j} \binom{j}{k+1-j} \qquad \text{if } k \in [2d-1] \cup \{0\}$$

$$f_k(C_{2d+1}(n)) = \sum_{j=0}^{d} \frac{k+2}{n-j} \binom{n-j}{j+1} \binom{j+1}{k+1-j} \qquad \text{if } k \in [2d] \cup \{0\}$$

The following theorem shall not be used; it is included as an example of the usefulness of cyclic polytopes. For a proof see either McMullen & Shephard (1971), or Ziegler (1995).

Upper Bound Theorem (McMullen 1971)**.** *If P is a d-polytope with n vertices, then*

$$f_k(P) \le f_k(C_d(n))$$

for each $k \in [d] \cup \{-1, 0\}$.

1.4 Neoteric Polytopes From Erstwhile Polytopes

1.4.1 Pyramids

If $P \subseteq \mathbb{R}^n$ is a d-polytope (assuming $n > d$), and $\boldsymbol{v} \in \mathbb{R}^n \setminus \operatorname{aff} P$, then $\operatorname{conv}(P \cup \{\boldsymbol{v}\})$ is a $(d+1)$-polytope. This polytope is called a *pyramid* over P with *apex* $\boldsymbol{v}$. If $P \subseteq \mathbb{R}^d$, then P can be embedded in $\mathbb{R}^{d+1}$ by appending a -1 to the end of each point in P. Thus, one way of realizing $\operatorname{pyr}(P)$ is as:

$$\operatorname{conv}\left(\left\{ \begin{bmatrix} \boldsymbol{x} \\ -1 \end{bmatrix} \,\middle|\, \boldsymbol{x} \in P \right\} \cup \left\{ \begin{bmatrix} \boldsymbol{0}_d \\ 1 \end{bmatrix} \right\} \right).$$

The faces of $\operatorname{pyr}(P) = \operatorname{conv}(P \cup \{\boldsymbol{v}\})$ are the copies of the faces of P (that are not P themselves), as well as the pyramids over these faces with apex $\boldsymbol{v}$ (this includes $\boldsymbol{v}$ since $\boldsymbol{v} = \operatorname{pyr}(\varnothing)$). Thus

$$f_k(\operatorname{pyr} P) = f_k(P) + f_{k-1}(P)$$

10

where $f_j(P) = 0$ if either $j < -1$ or $j > d$. As an example, a d-simplex is a pyramid over a $(d-1)$-simplex.

It is possible, and often useful, to discuss the polytope that arises as a result of repeatedly pyramiding over some polytope. Thus, define $\mathrm{pyr}_1(P) = \mathrm{pyr}(P)$, and for $k \in \mathbb{N}$ with $k > 1$, set $\mathrm{pyr}_k(P) = \mathrm{pyr}(\mathrm{pyr}_{k-1}(P))$. Then any polytope that is combinatorially equivalent to $\mathrm{pyr}_k(P)$ is called a *k-fold pyramid over P*.

1.4.2 Prisms

If $P \subseteq \mathbb{R}^n$ is a d-polytope, and $P' \subseteq \mathbb{R}^n$ is a translation of P such that $\mathrm{aff}(P) \cap \mathrm{aff}(P') = \varnothing$ (this requires that $n > d$), then a *prism* with base P is a polytope $\mathrm{prism}(P)$ combinatorially equivalent to the polytope $\mathrm{conv}(P \cup P')$. The proper faces of a prism are:

1. proper faces of either P or its translate; as well as

2. prisms over proper faces of P; as well as

3. the polytope P or the translate of P.

Hence

$$f_k(\mathrm{prism}\,P) = \begin{cases} 1 & \text{if } k \in \{-1, d+1\} \\ 2f_k(P) & \text{if } k = 0 \\ 2f_k(P) + f_{k-1}(P) & \text{if } k \in [d] \end{cases}$$

are the face numbers of a prism over a d-polytope P. As an example, a d-cube is a prism over a $(d-1)$-cube.

1.4.3 Bipyramids

If $P \subseteq \mathbb{R}^n$ is a d-polytope, and $Q \subseteq \mathbb{R}^n$ is a 1-polytope such that $|\mathrm{relint}(P) \cap \mathrm{relint}(Q)| = 1$, then any $(d+1)$-polytope that is combinatorially equivalent to the $(d+1)$-polytope $\mathrm{conv}(P \cup Q)$ is called

a *bipyramid* over P. If $P \subseteq \mathbb{R}^d$ is a d-polytope with $\mathbf{0} \in \text{relint}\, P$, then a bipyramid over P can be obtained in $\mathbb{R}^{d+1}$ as

$$\text{conv}\left(\left\{\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix} \,\middle|\, \mathbf{x} \in P\right\} \cup \left\{\begin{bmatrix} \mathbf{0}_d \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{0}_d \\ -1 \end{bmatrix}\right\}\right).$$

If P is a d-polytope, and $Q = \text{conv}\,\{\mathbf{a}, \mathbf{b}\}$ is a 1-polytope such that $B = \text{conv}(P \cup Q)$ is a bipyramid over P, then the proper faces of B are:

1. the vertices $\mathbf{a}$ or $\mathbf{b}$; as well as

2. proper faces of P; as well as

3. pyramids over a proper face of P with apex either $\mathbf{a}$ or $\mathbf{b}$.

Hence

$$f_k(B) = \begin{cases} 1 & \text{if } k \in \{-1, d+1\} \\ 2f_{k-1}(P) + f_k(P) & \text{if } k \in [d-1] \cup \{0\} \\ 2f_{k-1}(P) & \text{if } k = d \end{cases}$$

are the face numbers of a bipyramid B over a d-polytope P. As an example, a d-crosspolytope is a bipyramid over a $(d-1)$-crosspolytope. An alternative definition could be that a bipyramid over a polytope P is the dual of a prism over a dual of P, that is, $\left(\text{prism}(P^\Delta)\right)^\Delta$.

1.4.4 Product

If P is a d_1-polytope with $\text{vert}\, P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\} \subseteq \mathbb{R}^{d_1}$, and Q is a d_2-polytope with $\text{vert}\, Q = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\} \subseteq \mathbb{R}^{d_2}$, then the *(Cartesian) product* of P and Q, denoted $P \times Q$, is a $(d_1 + d_2)$-polytope that is combinatorially equivalent to the polytope with vertex set

$$\text{vert}(P \times Q) = \left\{\begin{bmatrix} \mathbf{p}_i \\ \mathbf{q}_j \end{bmatrix} \,\middle|\, i \in [n],\, j \in [m]\right\}.$$

The nonempty faces of a product $P \times Q$ are products of nonempty faces of P with nonempty faces of Q. Hence

$$
f_i(P \times Q) = \begin{cases} 1 & \text{if } i \in \{-1, d_1 + d_2\} \\ \displaystyle\sum_{\substack{j+k=i \\ j \in [d_1] \cup \{0\} \\ k \in [d_2] \cup \{0\}}} f_j(P) f_k(Q) & \text{if } i \in [d_1 + d_2 - 1] \cup \{0\} \end{cases}
$$

are the face numbers of a product of the d_1-polytope P and the d_2-polytope Q. In particular, if the facets of P are $F_1, F_2, \ldots, F_s$, and the facets of Q are $G_1, G_2, \ldots, G_t$, then the facets of $P \times Q$ are

$$
P \times G_1, P \times G_2, \ldots, P \times G_t, F_1 \times Q, F_2 \times Q, \ldots, F_s \times Q.
$$

As an example, prisms are products of a polytope with a 1-polytope.

1.4.5 Join

In this and the following section, the fundamental definitions of the operations will be in terms of point sets rather than polytopes.

If $X = \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^{d_1}$ and $Y = \{q_1, q_2, \ldots, q_m\} \subseteq \mathbb{R}^{d_2}$, then the *join* of X and Y is the set

$$
X \vee Y = \left\{ \begin{bmatrix} p_i \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix} \middle| \; i \in [n] \right\} \cup \left\{ \begin{bmatrix} \mathbf{0}_{d_1} \\ q_j \\ 1 \end{bmatrix} \middle| \; j \in [m] \right\} \subseteq \mathbb{R}^{d_1 + d_2 + 1}
$$

If P and Q are polytopes of dimensions d_1 and d_2 respectively such that $\operatorname{vert} P = X$ and $\operatorname{vert} Q = Y$, then define the join of P and Q to be a polytope combinatorially equivalent to $P \vee Q = \operatorname{conv}(X \vee Y)$. In this case, $\operatorname{vert}(P \vee Q) = X \vee Y$ and $\dim(P \vee Q) = d_1 + d_2 + 1$. Intuitively, the join of two polytopes is obtained by placing both polytopes in a high enough dimensional space so that they can be arranged with their affine hulls nonintersecting, and then taking a convex hull.

The *i*-faces of $P \vee Q$ are joins of faces of P and faces of Q such that the sums of the dimensions

of these faces is $i - 1$. Hence

$$f_i(P \vee Q) = \sum_{\substack{j+k+1=i \\ j \in [d_1] \cup \{0,-1\} \\ k \in [d_2] \cup \{0,-1\}}} f_j(P) f_k(P)$$

are the face numbers of a join of the d_1-polytope P and the d_2-polytope Q. As an example, pyramids are joins of a polytope with a 0-polytope. Moreover, the join of two simplices is again a simplex.

1.4.6 Direct Sum

If $X = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_n\} \subseteq \mathbb{R}^{d_1}$ and $Y = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \dots, \boldsymbol{q}_m\} \subseteq \mathbb{R}^{d_2}$, then the *direct sum* of X and Y is the set

$$X \oplus Y = \left\{ \begin{bmatrix} \boldsymbol{p}_i \\ \boldsymbol{0}_{d_2} \end{bmatrix} \,\middle|\, i \in [n] \right\} \cup \left\{ \begin{bmatrix} \boldsymbol{0}_{d_1} \\ \boldsymbol{q}_j \end{bmatrix} \,\middle|\, j \in [m] \right\} \subseteq \mathbb{R}^{d_1+d_2}$$

If P and Q are polytopes of dimensions d_1 and d_2 respectively such that $\operatorname{vert} P = X$ and $\operatorname{vert} Q = Y$, then define the direct sum of P and Q to be a polytope combinatorially equivalent to $P \oplus Q = \operatorname{conv}(X \oplus Y)$. In this case, $\operatorname{vert}(P \oplus Q) = X \oplus Y$ and $\dim(P \oplus Q) = d_1 + d_2$. Geometrically, the direct sum of two polytopes is obtained by placing both polytopes in a high enough dimensional space so that they can be arranged with their relative interiors intersecting in a single point, and then taking the convex hull of their union.

The i-faces of $P \oplus Q$ are joins of faces of P and faces of Q (but not P or Q themselves) such that the sums of the dimensions of these faces is $i - 1$. Hence

$$f_i(P \oplus Q) = \sum_{\substack{j+k+1=i \\ j \in [d_1-1] \cup \{0,-1\} \\ k \in [d_2-1] \cup \{0,-1\}}} f_j(P) f_k(P)$$

are the face numbers of a direct sum of the d_1-polytope P and the d_2-polytope Q. As an example, the bipyramid over a polytope P is the direct sum of P with a 1-polytope. Moreover, the direct sum of two crosspolytopes is again a crosspolytope.

1.4.7 Vertex Figures

Suppose P is a polytope, $\boldsymbol{v}$ is a vertex of P, and $H = \{\boldsymbol{x} \mid \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = t\}$ is a supporting hyperplane of $\boldsymbol{v}$ with $P \subseteq H^+$. Let $m = \min\{\langle \boldsymbol{\xi}, \boldsymbol{w} \rangle \mid \boldsymbol{w} \in \mathrm{vert}(P) \setminus \{\boldsymbol{v}\}\}$. This is a positive number since H is a supporting hyperplane of the face $\boldsymbol{v}$.
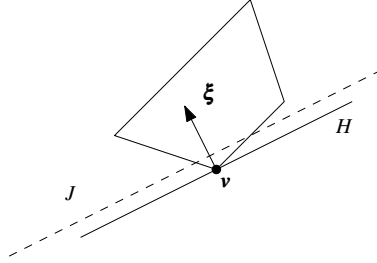
Figure 1.5: The hyperplane J.

Now set $J = \left\{\boldsymbol{x} \mid \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = t + \frac{m}{2}\right\} = H + \frac{m}{2}\boldsymbol{\xi}$ (see Figure 1.5). Then $\boldsymbol{v} \in J^{(-)}$, and $\boldsymbol{w} \in J^{(+)}$ for every other vertex $\boldsymbol{w}$ of P. Furthermore, $J \cap P$ is a polytope since the intersection of a polytope and a plane is again a polytope (this can be easily proved by using the $\mathcal{H}$-polytope formulation of the definition of a polytope). A *vertex figure of P at $\boldsymbol{v}$* is any polytope combinatorially equivalent to $P_{\boldsymbol{v}} = J \cap P$.

The face lattice of a vertex figure is isomorphic to the interval $[\boldsymbol{v}, P]$ in the face lattice of P. Thus

$$f_i(P_{\boldsymbol{v}}) = |\{F \in \mathcal{F}(P) \mid \dim F = i+1 \text{ and } \boldsymbol{v} \in F\}|.$$

Some examples of vertex figures are: vertex figures of simple polytopes are simplices of one dimension lower; vertex figures of crosspolytopes are crosspolytopes of one dimension lower; a vertex figure of a regular icosahedron is a pentagon; and a vertex figure of a pyramid at its apex is the base of the pyramid.
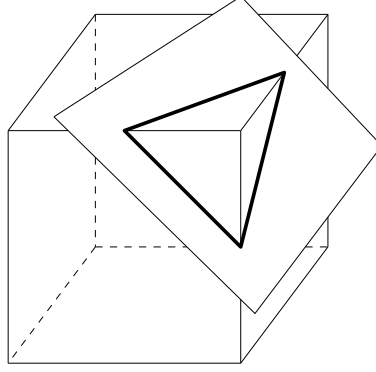
Figure 1.6: A vertex figure of a 3-cube is a triangle.

1.4.8 Kleetopes

Let P be a d-polytope in $\mathbb{R}^d$, and for each facet F of P, let $\boldsymbol{x}_F$ be the unit vector that is normal to aff(F) such that $\langle \boldsymbol{x}_F, \boldsymbol{v} \rangle \leq 0$ for each $\boldsymbol{v} \in P$. Let F_0 be a fixed facet of P. Now choose a point $\boldsymbol{x}_0$ in the region of $\mathbb{R}^d$ where $\langle \boldsymbol{x}_{F_0}, \boldsymbol{v} \rangle > 0$, and $\langle \boldsymbol{x}_F, \boldsymbol{v} \rangle < 0$ for every other facet F. Then the *stellar subdivision* $K(P; F_0)$ is any polytope that is combinatorially equivalent to conv$(P \cup \{\boldsymbol{x}_0\})$. The combinatorial type of $K(P; F_0)$ is independent of the choice of $\boldsymbol{x}_0$. The vertex set of $K(P; F_0)$ is vert$(P) \cup \{\boldsymbol{x}_0\}$. Geometrically, $K(P; F_0)$ is P with a shallow pyramid over F_0 "glued" to F_0. Thus

$$f_i(K(P; F_0)) = \begin{cases} f_i(P) + f_{i-1}(F_0) & \text{if } i \in [d-2] \cup \{0\} \\ f_{d-1}(P) + f_{d-2}(F_0) - 1 & \text{if } i = d-1 \\ 1 & \text{if } i \in \{-1, d\} \end{cases}.$$

If $F_0, F_1, \ldots, F_n$ is a sequence of distinct facets of P, and Φ_i is the subsequence of the first $i+1$ terms, then for $i \in [n] \setminus \{1\}$, define $K(P; \Phi_i) = K(K(P; \Phi_{i-1}); F_i)$. The combinatorial type of the polytope $K(P; F_0, F_1, \ldots, F_n)$ is independent of the order of the sequence. Thus the sequence can be considered to just be a set of facets Φ of P. If Φ is the set of all facets of P, then $K(P; \Phi)$ is denoted P^K, and called the *Kleetope* of P (named after Victor Klee). See Figure 1.7 for an example (the original polytope is bolded in the Kleetope for clarity). Note that the Kleetope of a simplicial polytope is a simplicial polytope, as is the Kleetope of any 3-polytope.

If P is a d-simplex, then the combinatorial type of $K(P; \Phi)$ is completely determined by $k = |\Phi|$,

and thus is denoted $K(\Delta_d; k)$ for $k \leq d+1$.

prism(Δ_2)   (prism(Δ_2))^K
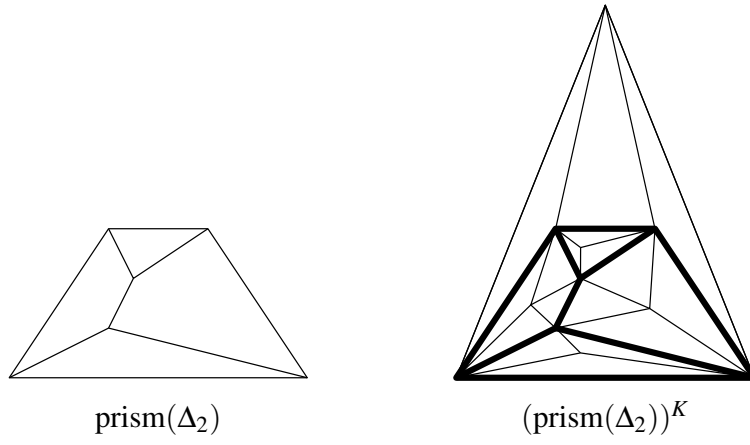
Figure 1.7: A prism over a triangle, and its Kleetope.

Chapter 2

Graphs

This chapter will provide relevant background material in the theory of graphs.

If X is a set, and $k \in \mathbb{N}$ is a natural number (including zero), then $\binom{X}{k} = \{A \subseteq X \mid |A| = k\}$ is the set of unordered k-tuples of elements of X.

2.1 Definition and Examples

A graph is an ordered pair $G = (V, E)$ where V is a finite set, and $E \subseteq \binom{V}{2}$ is a set of unordered pairs of elements of V. The elements of $V = V(G)$ are called *vertices*, and the elements of $E = E(G)$ are called *edges*. Note that either V or E is allowed to be empty. If $V = \varnothing$, then $E = \varnothing$, and G is called the *empty graph*. If $e = \{x, y\} \in E$ and no confusion may arise, the edge e may be denoted $e = xy = yx = \{x, y\}$. Two vertices $v, w \in V$ are called *neighbors*, or *adjacent* if $vw \in E$. The *neighborhood* of a vertex v is the set $N(v) = \{w \in V \mid vw \in E\}$ of all neighbors of v. The *degree* of a vertex v is the number of edges on which v lies, that is, $\deg v = |\{e \in E \mid v \in e\}| = |N(v)|$.

Two graphs $G = (V, E)$ and $H = (W, F)$ are said to be *isomorphic* if a bijection $\phi : V \to W$ exists with the additional property: $xy \in E$ if and only if $\phi(x)\phi(y) \in F$. If G and H are isomorphic, no distinction will be made between them. It is often useful to draw a graph by depicting its vertices as points in the plane, and its edges by continuous curves between vertices.

Examples 3.

1. A *discrete* graph on n vertices is a graph isomorphic to $D_n = ([n], \varnothing)$.

2. A *cycle* of *length* $n \geq 3$ is a graph isomorphic to C_n where $V(C_n) = [n]$ and

$$E(C_n) = \{ij \mid i - j \equiv 1 \pmod{n}\}.$$
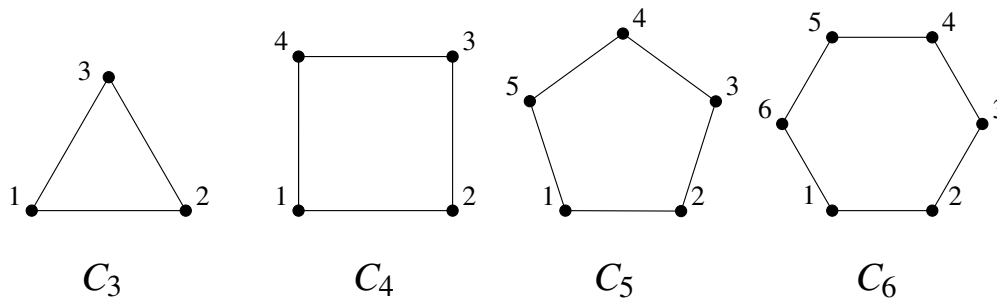
Figure 2.1: Several cycle graphs.

3. A graph is *complete* if $E = \binom{V}{2}$. If $|V| = n$, then G is denoted K_n.
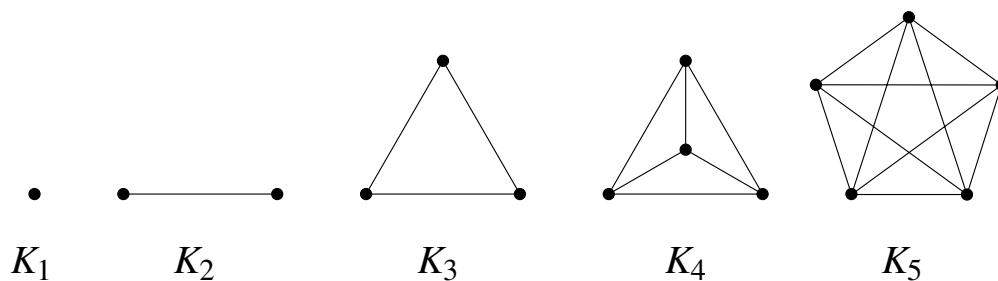
Figure 2.2: Several complete graphs.

4. A graph G is said to be *bipartite* if $V(G) = V_1 \cup V_2$ with $V_1 \neq \varnothing \neq V_2$ and $V_1 \cap V_2 = \varnothing$ such that $E(G) \cap \binom{V_1}{2} = E(G) \cap \binom{V_2}{2} = \varnothing$. If G is a bipartite graph satisfying

$$E(G) = \left\{ xy \in \binom{V}{2} \;\middle|\; x \in V_1 \text{ and } y \in V_2 \right\},$$

19

then G is *complete bipartite*, and is denoted $K_{n,m}$ where $|V_1| = n$ and $|V_2| = m$. Note that a complete bipartite graph must have at least one edge.

$$K_{2,2} \qquad K_{2,3} \qquad K_{3,3}$$
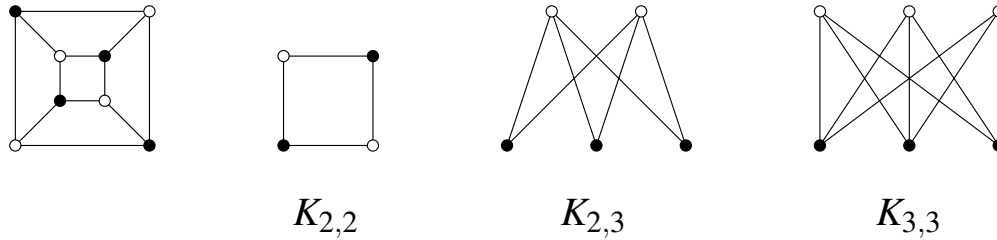
Figure 2.3: Several bipartite graphs. The first vertex set is colored white, and the second black.

5. More generally, a graph is *n-partite* if its vertex set can be written $V(G) = \bigcup_{i \in [n]} V_i$ such that:

   (a) $V_i \neq \varnothing$ for all $i \in [n]$;

   (b) $V_i \cap V_j = \varnothing$ for $i \neq j$; and

   (c) $E(G) \cap \binom{V_i}{2} = \varnothing$ for each $i \in [n]$.

   In this case, $V(G)$ will be written as $\bigsqcup_{i \in [n]} V_i$ and the sets V_i are called *color classes*.

   If, additionally, $E(G) = \left\{ xy \in \binom{V}{2} \mid \{x,y\} \nsubseteq V_i \text{ for each } i \in [n] \right\}$, then G is called *complete n-partite*, and denoted $K_{m_1,m_2,\ldots,m_n}$ where $|V_i| = m_i$ for each $i \in [n]$. Note that for any permutation σ of $[n]$ the graphs $K_{m_1,m_2,\ldots,m_n}$ and $K_{m_{\sigma 1},m_{\sigma 2},\ldots,m_{\sigma n}}$ are isomorphic. Hence complete n-partite graphs will be assumed to satisfy $m_1 \leq m_2 \leq \cdots \leq m_n$.

$$K_{2,2,2} \qquad K_{1,2,2} \qquad K_{1,1,2} \qquad K_{1,2,3}$$
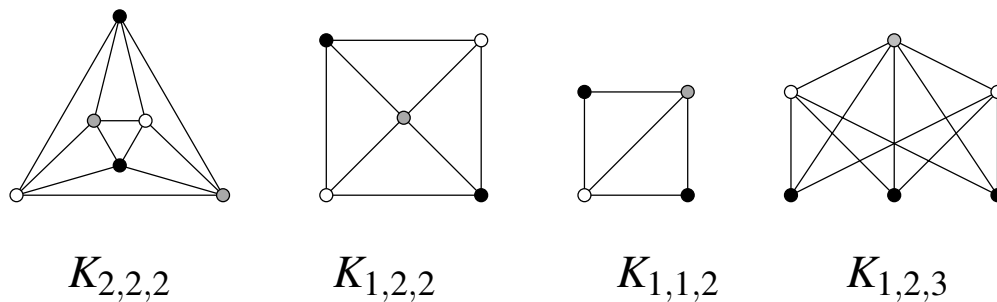
Figure 2.4: Several complete 3-partite graphs.

If G and H are graphs, then their *join* is the graph $G \vee H$ that is defined by

$$V(G \vee H) = V(G) \cup V(H)$$

$$E(G \vee H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G) \text{ and } h \in V(H)\}.$$

If r, s are positive integers, then $K_{r,s} = D_r \vee D_s$, and more generally,

$$K_{r_1, r_2, \dots, r_t} = D_{r_1} \vee D_{r_2} \vee \cdots \vee D_{r_t}.$$

2.2 Subgraphs and Minors

If $G = (V, E)$ is a graph, then a *subgraph H of G* is a graph of the form $H = (W, F)$ with $W \subseteq V$, and $F \subseteq E \cap \binom{W}{2}$. If, additionally, for all $xy \in E(G)$ the containment $\{x, y\} \subseteq V(H)$ implies the containment $xy \in E(H)$, that is, $F = E \cap \binom{W}{2}$, then H is said to be an *induced subgraph* of G. When working with graphs, subgraphs are not generally sufficient for dealing with statements of theorems; the idea of a minor of a graph is also needed:

Definitions. Let $G = (V, E)$ be a graph, and $W \subseteq V$.

1. The *restriction* of G to W, denoted by $G|_W$, is the graph defined by $V(G|_W) = W$, and $E(G|_W) = E \cap \binom{W}{2}$.

2. The *deletion* of W, denoted $G \setminus W$, is the restriction of G to $V \setminus W$, that is $G|_{V \setminus W}$.

3. If $xy = e \in E$, then the *contraction* of e, denoted by G/e, is the graph defined by $V(G/e) = (V \setminus \{x, y\}) \cup \{\tilde{e}\}$ and

$$E(G/e) = E(G|_{V \setminus \{x,y\}}) \cup \{\tilde{e}z \mid xz \in E \text{ or } yz \in E \text{ and } z \notin \{x, y\}\}$$

4. If $F \subseteq E$, then the *deletion* of F, denoted $G \setminus F$, is the graph with $V(G \setminus F) = V$ and $E(G \setminus F) = E \setminus F$.

21

5. A graph H is a *minor* of a graph G if H is isomorphic to a graph that is obtainable from G through a sequence of deletions and contractions.
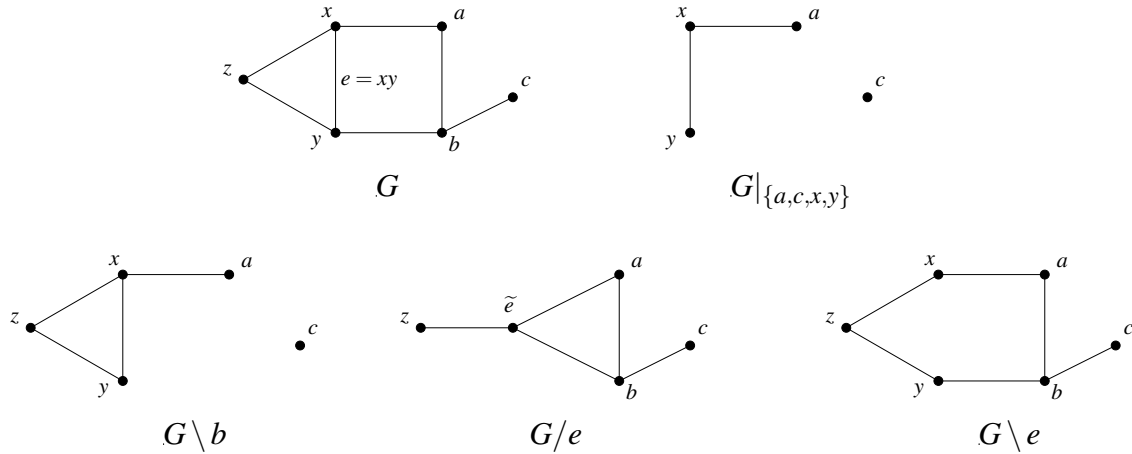
Figure 2.5: Examples of some minors of a graph G.

The word deletion is defined in two different ways above; however no confusion should arise since one definition was in regards to a set of vertices, whereas the other refers to a set of edges.

Example 4. The graph $K_{3,3}$ has a K_4 minor. Let $V(K_{3,3}) = \{a_1, a_2, a_3\} \sqcup \{b_1, b_2, b_3\}$. First contract the edge $a_2 b_2$, then contract the edge $a_3 b_3$. See Figure 2.6.
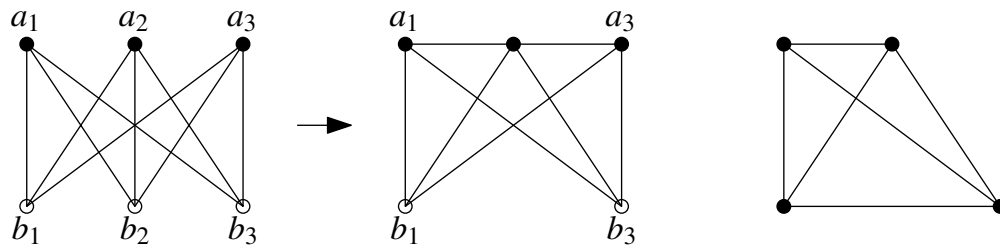
Figure 2.6: $K_{3,3}$ has a K_4 minor.

If $F = \{e_1, e_2, \ldots, e_n\} \subseteq E(G)$, then it can be shown that $(\cdots((G/e_1)/e_2)\cdots)/e_n$ is isomorphic to $(\cdots((G/e_{\sigma 1})/e_{\sigma 2})\cdots)/e_{\sigma n}$ for any permutation σ of $[n]$. For details see Diestel (2010). This graph is denoted G/F.

2.3 Planarity and Connectivity

When drawing a graph G in the plane, a natural question to ask is, "Is it possible to draw G such that no two edges intersect unless they share a vertex, and in that case only at the shared vertex?" If the answer to this question is yes, then the graph is called *planar*. The following gives a characterization of planarity that avoids topology. For a proof, see Wagner (1937) or Diestel (2010).

Theorem 5 (Wagner 1937). *A graph G is planar if and only if G has no minor isomorphic to either K_5 or $K_{3,3}$.*

As mentioned above, the statement of Theorem 5 must be in terms of minors; the Peterson graph P (Figure 2.7) has neither a K_5 nor $K_{3,3}$ subgraph; however, it does contain each as a minor. The graph K_5 can be realized by contracting each of the edges $v_i w_i$ for $i \in [5]$, while $K_{3,3}$ can be realized as $(P \setminus \{w_2 w_3, v_5 w_5\}) / \{w_4 w_5, v_2 v_5, w_1 w_2, v_3 w_3\}$.
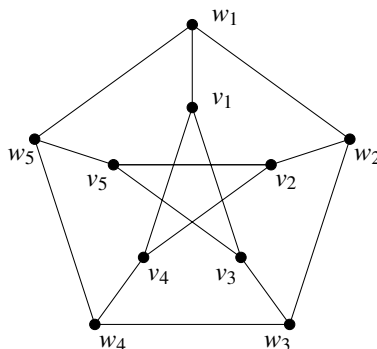
Figure 2.7: The Peterson graph.

Whenever planarity is needed, the above conditions will be checked. Thus the above definition will be treated as the definition of planarity, and the previous discussion will be reserved for geometric intuition.

A graph G is said to be *connected* if either $|V| = 1$, or for any pair $x, y \in V$ there exists a sequence of vertices $x = x_1, x_2, \ldots, x_{r+1} = y$ such that $x_i x_{i+1} \in E$ for $i \in [r]$. If, additionally, $x_i \neq x_j$ for $i \neq j$, then $x_1, x_2, \ldots, x_{r+1}$ is called a *path* from x to y of *length* r with *endpoints* x and y. If $d \in \mathbb{N}$, then a graph G is said to be *d-connected* if $|V| > d$, and for $W \subseteq V$ with $|W| < d$ the graph

$G \backslash W$ is connected. Every non-empty graph is 0-connected, and the 1-connected graphs are exactly the connected graphs with at least two vertices.

Two paths $x_1, x_2, \ldots, x_i$ and $y_1, y_2, \ldots, y_j$ are said to be *disjoint* if the only intersections of the paths are at endpoints of each. The following theorem (equivalent to Menger's Theorem, but first stated in this form by Whitney) gives a characterization of d-connectedness in terms of disjoint paths. For a proof, see either Whitney (1932) or Diestel (2010).

Theorem 6 (Whitney 1932). *A graph G is d-connected if and only if for all $a, b \in V$ with $a \neq b$ there exist d pairwise disjoint paths from a to b.*

Example 7. The graph $K_{2,2,2}$ is 4-connected. Let $V(K_{2,2,2}) = \{a_{1,1}, a_{2,1}, a_{3,1}, a_{1,2}, a_{2,2}, a_{3,2}\}$, and $E(K_{2,2,2}) = \{a_{i,j} a_{k,l} \mid i \neq k\}$. Then a pair of vertices $a_{i,j} a_{k,l}$ is either adjacent, or not. In the latter case assume, without loss of generality, that $i = j = k = 1$, and $l = 2$. The following are four pairwise disjoint paths from $a_{1,1}$ to $a_{1,2}$:

$$a_{1,1}, a_{2,1}, a_{1,2} \qquad\qquad a_{1,1}, a_{2,2}, a_{1,2}$$

$$a_{1,1}, a_{3,1}, a_{1,2} \qquad\qquad a_{1,1}, a_{3,2}, a_{1,2}$$

If, on the other hand, $i \neq k$, then assume again that $i = j = 1$, and without loss of generality, assume $k = 3$ and $l = 2$. Then the following are four pairwise disjoint paths from $a_{1,1}$ to $a_{3,2}$:

$$a_{1,1}, a_{3,2} \qquad\qquad a_{1,1}, a_{2,1}, a_{3,2}$$

$$a_{1,1}, a_{2,2}, a_{3,2} \qquad\qquad a_{1,1}, a_{3,1}, a_{1,2}, a_{3,2}.$$

If G is a graph, then the *connectivity*[1] of G is $\kappa(G) = \max\{k \mid G \text{ is } k\text{-connected}\}$, and the *Hadwiger number* of G is $h(G) = \max\{n \mid K_n \text{ is a minor of } G\}$.

---

[1]This is usually called the *edge-connectivity* of the graph. However no other types of connectivity will be considered, so no confusion should arise.
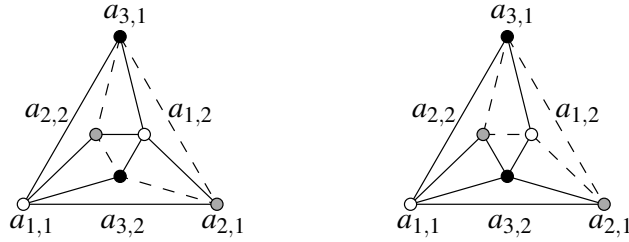
Figure 2.8: $K_{2,2,2}$ is 4-connected.

2.4   Graphs of Polytopes

If P is a polytope, then the *graph* of P, denoted $\mathscr{G}(P)$, is the graph with $V(\mathscr{G}(P)) = \operatorname{vert} P$, and $E(\mathscr{G}(P)) = \left\{ xy \in \binom{\operatorname{vert} P}{2} \mid \operatorname{conv}\{x, y\} \text{ is an edge of } P \right\}$. The following theorem is a special case of a more general theorem proven in Grünbaum (2003).

Theorem 8 (Grünbaum). *If P is a d-polytope, then $\mathscr{G}(P)$ has a K_{d+1} minor.*

Proof. If $\dim P = -1$, then $\mathscr{G}(P)$ is empty, i.e., K_0. If $\dim P = 0$, then $\mathscr{G}(P)$ has one vertex and no edges , i.e., K_1. If $\dim P = 1$, then $\mathscr{G}(P)$ has two vertices and one edge , i.e., K_2. The proof now proceeds by induction on d.

Suppose for some $d \geq 2$ that K_d is a minor of the graph of every $(d-1)$-polytope. Let P be a d-polytope and F be a facet of P. By the induction hypothesis, F has a K_d minor. Fix one such minor. Each vertex in this minor comes from a vertex of F. Let $\{v_1, v_2, \ldots, v_n\}$ be a set of vertices of F such that each v_i corresponds to a different vertex in the complete minor. Since $\dim P \geq 2$, each v_i is on an edge with another vertex $w_i \in \operatorname{vert} P \setminus \operatorname{vert} F$ (it is possible that $\{w_1, w_2, \ldots, w_n\}$ has cardinality less than n). Now, in $\mathscr{G}(P)$ first perform a sequence of deletions and contractions to obtain the chosen K_d minor of $\mathscr{G}(F)$, and then contract all edges in this new graph that are not incident to any vertex in the copy of K_d. This leaves the K_d minor and one other vertex. This vertex is adjacent to each vertex in the K_d minor. Thus a K_{d+1} minor of $\mathscr{G}(P)$ has been constructed. $\square$

Example 9. Let P be the triangular prism whose graph is shown in Figure 2.9 and F be the facet $\operatorname{conv}\{v_1, v_2, v_3, v_4\}$. Note that F has a K_3 minor that is obtained by contracting $v_3 v_4$. The first arrow indicates forming that K_3 minor by contracting the edge $v_3 v_4$ in $\mathscr{G}(P)$. Now the only edge

with no vertices corresponding to any of v_1, v_2, v_3 is e. The second arrow indicates the contraction of e. This results in a K_4 minor.
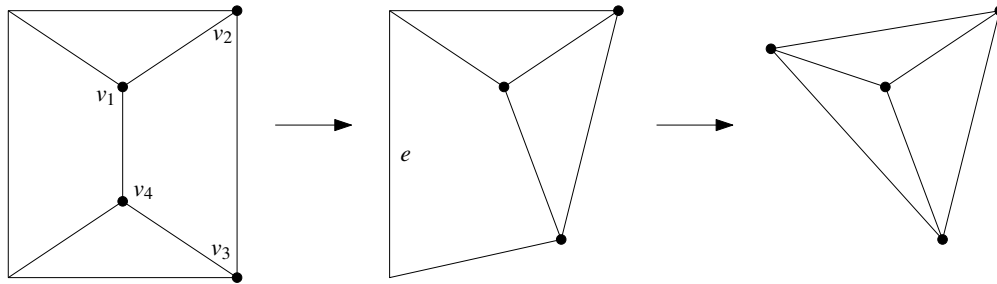
Figure 2.9: Example of Theorem 8.

The following theorem about the connectivity of graphs of d-polytopes was proved in Balinski (1961) using linear programming techniques.

Theorem 10 (Balinski 1961)**.** *If P is a d-polytope, then $\mathscr{G}(P)$ is d-connected.*

The converse is not true. The 3-dimensional crosspolytope has graph $K_{2,2,2}$, and is 4-connected (see Example 7). It is also planar, and therefore does not contain a K_5 minor. Thus it is not the graph of any 4-dimensional polytope.

Theorems 8 and 10 imply:

Corollary 1. *If G is the graph of a d-polytope, then*

$$d \leq \min \left\{ h(G) - 1, \kappa(G) \right\}.$$

This bound will be important in Chapter 7. Notice that these two numbers are, in general, incomparable. Let F be any facet of $C_4(6)$, and $P = K(C_4(6); F)$. Then $h(P) = 6$, and $\kappa(P) = 4$. On the other hand, if R is the regular icosahedron, then $h(R) = 4$, and $\kappa(R) = 5$.

The following theorem can be found in Ziegler (1995), albeit without a proof.

Theorem 11. *If F is a face of a polytope P, then $\mathscr{G}(F)$ is an induced subgraph of $\mathscr{G}(P)$.*

Proof. If $F = \varnothing$, then $\mathscr{G}(F)$ is the empty graph which is an induced subgraph of every graph. If $F = P$, then $\mathscr{G}(F) = \mathscr{G}(P)$ and any graph is an induced subgraph of itself.

26

Thus, suppose F is a proper face of P, and write

$$\varnothing \neq \operatorname{vert} F = \{v_1, v_2, \ldots, v_n\} \subsetneqq \operatorname{vert} P$$

If $v_i v_j \in E(\mathscr{G}(P))$, then there is a hyperplane H such that $P \cap H = \operatorname{conv}\{v_i, v_j\}$ and $P \subseteq H^+$. Hence,

$$
\begin{aligned}
\operatorname{conv}\{v_i, v_j\} &= \operatorname{conv}\{v_i, v_j\} \cap H \\
&\subseteq \operatorname{conv}\{v_1, v_2, \ldots, v_n\} \cap H \\
&= F \cap H \\
&\subseteq P \cap H \\
&= \operatorname{conv}\{v_i, v_j\}
\end{aligned}
$$

where the first equality follows since $\operatorname{conv}\{v_i, v_j\} \subseteq H$. Therefore equality holds throughout. Hence $F \cap H = \operatorname{conv}\{v_i, v_j\}$, whence H is a supporting hyperplane of the face $\operatorname{conv}\{v_i, v_j\}$ of F. Thence $v_i v_j \in E(\mathscr{G}(F))$. $\qquad \square$

If G is a graph, and P a d-polytope with $\mathscr{G}(P) = G$, then G is said to be *d-realizable*. A major open problem in the theory of polytopes is to give a complete characterization of the graphs that are d-realizable for a fixed d. One can similarly ask, for a fixed graph G, for which d is G d-realizable? The cyclic polytopes provide examples of graphs that are d-realizable for multiple values of d. Grünbaum asks in Grünbaum (2003) a question, that in the case of graphs becomes: if G is both d-realizable and d'-realizable, then is it d''-realizable for every d'' between d and d'?

The following theorem gives a complete characterization of the graphs that are 3-realizable. Proofs can be found in Steinitz (1922), Steinitz & Rademacher (1976), Grünbaum (2003), or Ziegler (1995).

Theorem 12 (Steinitz 1922). *A graph G is the graph of a 3-polytope if and only if G is both planar and 3-connected.*

Corollary 1. *If G is a 3-connected graph, then G has a K_4 minor.*

Proof. If G is planar, then it is 3-realizable, and thus by Theorem 8 it has a K_4 minor.

On the other hand, if G is not planar, then it has either a $K_{3,3}$ or a K_5 minor. In the former case, Example 4 shows that G has a K_4 minor. In the latter, K_m is a minor of K_n if and only if $m \leq n$, so that K_4 is a minor of K_5. □

A complete characterization of the graphs of polytopes with dimension less than or equal to three can be found in Table 2.1.

Dimension	Characterization of Graph
-1	The only (-1)-dimensional polytope is the empty polytope, and it has the empty graph.
0	The only combinatorial type of 0-dimensional polytope is a single vertex. This has graph K_1.
1	The only combinatorial type of 1-dimensional polytope is a closed line segment. This has graph K_2.
2	The 2-polytopes are exactly the convex n-gons lying in a plane, and therefore have graphs that are cycle graphs. Further, every cycle graph is the graph of a 2-polytope.
3	Theorem 12 gives a complete characterization of the graphs of 3-polytopes as the planar 3-connected graphs.

Table 2.1: Characterizations of graphs of d-polytopes for $d \leq 3$.

Chapter 3

Gale Transformations and Diagrams

A Gale transformation is a method of encoding the set of all of the affine dependencies of a d-dimensional set of points with cardinality $n \geq d+1$ into a space of dimension $n-d-1$. This is exceptionally useful if n is not much larger than d. Usually this process is only applied to the vertex set of a polytope.

3.1 Definition of Gale Transformation

Let P be a d-polytope in $\mathbb{R}^d$ with $\operatorname{vert} P = V = \{v_1, v_2, \ldots, v_n\}$ and consider the set of affine dependencies of V, that is,

$$\operatorname{dep} V = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \,\middle|\, \sum_{i \in [n]} \lambda_i v_i = 0 \text{ and } \sum_{i \in [n]} \lambda_i = 0 \right\}.$$

Note that $\operatorname{dep} V$ is an $(n-d-1)$-dimensional vector space. Let $\{a_1, a_2, \ldots, a_{n-d-1}\}$ be a basis of $\operatorname{dep} V$, and write $a_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n})$ for $i \in [n-d-1]$. Now, let A be the $(n-d-1) \times n$

matrix whose ith row is $\boldsymbol{a}_i$ for $i \in [n-d-1]$, and let $\overline{\boldsymbol{v}}_j$ be the jth column of A for $j \in [n]$, that is,

$$
A = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{n-d-1} \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{v}}_1 & \overline{\boldsymbol{v}}_2 & \cdots & \overline{\boldsymbol{v}}_n \end{bmatrix}.
$$

Then the (multi)set $\overline{V} = \{\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \ldots, \overline{\boldsymbol{v}}_n\} \subseteq \mathbb{R}^{n-d-1}$ is a *Gale transformation* of V with $\overline{\boldsymbol{v}}_i$ corresponding to $\boldsymbol{v}_i$. Define as well, for a subset $X \subseteq V$ the (multi)set $\overline{X} = \{\overline{\boldsymbol{v}} \in \overline{V} \mid \boldsymbol{v} \in X\}$.

Note that there is not a unique Gale transformation for a given vertex set since a choice of basis was necessary. This does nothing to detract from the usefulness of the construction. Also, note that the definition did not require that the points $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be the vertex set of a polytope. It was only necessary that $\dim \operatorname{aff} V = d$. Thus Gale diagrams can be defined for point sets satisfying this condition.

3.1.1 Computing a Gale Transformation

One method for actually computing a Gale transformation is as follows:

Let P be a d-polytope in $\mathbb{R}^d$ with $\operatorname{vert} P = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ ordered such that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}$ are affinely independent. Then

$$
\operatorname{rref} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} I_{d+1} & \big| & N \end{bmatrix}
$$

where N is some $(d+1) \times (n-d-1)$ matrix. Setting

$$
\begin{bmatrix} -N^{\mathsf{T}} & \big| & I_{n-d-1} \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{v}}_1 & \overline{\boldsymbol{v}}_2 & \cdots & \overline{\boldsymbol{v}}_n \end{bmatrix},
$$

yields a Gale transformation of $\operatorname{vert} P$, that is, the ordered (multi)set $\{\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \ldots, \overline{\boldsymbol{v}}_n\}$.

3.2 A Preliminary Result

The following theorem will be crucial in the proof of Theorem 15, which provides the answer to the question, "Why should I care about Gale transformations?". The proof closely follows that in Thomas (2006).

Theorem 13. *If P is a d-polytope with $V = \text{vert}(P) = \{v_1, v_2, \ldots, v_n\} \subseteq \mathbb{R}^d$ and $F \subseteq V$, then $\text{conv}\,F$ is a face of P if and only if $\text{conv}(V \setminus F) \cap \text{aff}(F) = \varnothing$.*

Proof. Suppose, without loss of generality, $F = \{v_1, v_2, \ldots, v_k\}$ and $\text{conv}(F)$ is a face of P. Further, let H be a supporting hyperplane of F (so that $\text{aff}(F) \subseteq H$), say, $H = \{w \mid \langle \xi, w \rangle = t\}$ with $P \subseteq H^+$. Then the inclusions $v_j \in P \cap (\mathbb{R}^d \setminus H)$ for all $j > k$ imply that $\langle \xi, v_j \rangle > t$ for $j > k$. Let $x \in \text{conv}(V \setminus F)$ with

$$x = \sum_{j \in [n] \setminus [k]} \alpha_j v_j, \qquad \sum_{j \in [n] \setminus [k]} \alpha_j = 1, \qquad \alpha_j \geq 0 \text{ for all } j \in [n] \setminus [k].$$

Then

$$\langle \xi, x \rangle = \sum_{j \in [n] \setminus [k]} \alpha_j \langle \xi, v_j \rangle > \sum_{j \in [n] \setminus [k]} \alpha_j t = t.$$

Thus $x \in H^{(+)} = H^+ \setminus H$. Therefore, the inclusion $\text{aff}(F) \subseteq H$ implies $\text{conv}(V \setminus F) \cap \text{aff}(F) = \varnothing$.

On the other hand, suppose $\text{conv}(V \setminus F) \cap \text{aff}(F) = \varnothing$, and let $y_0 \in \text{conv}(V \setminus F)$. Then

$$\inf_{x \in \text{aff}(F)} \|x - y_0\|$$

is attained at x_0 when $\text{aff}\{x_0, y_0\}$ is perpendicular to $\text{aff}(F)$. Let H be the hyperplane through x_0 normal to $x_0 - y_0$. Then

1. $\text{conv}(V \setminus F) \cap H = \varnothing$ and

2. $\text{aff}(F) \subseteq H$.

Thus $H \cap P = \operatorname{conv} F$ is a face of P. □

Example 14. In Figure 3.1, the set $\operatorname{conv}\{x_1, x_4\}$ is not a face of $\operatorname{conv}\{x_1, x_2, x_3, x_4, x_5\}$ since $\operatorname{conv}\{x_2, x_3, x_5\} \cap \operatorname{aff}\{x_1, x_4\} \neq \varnothing$.
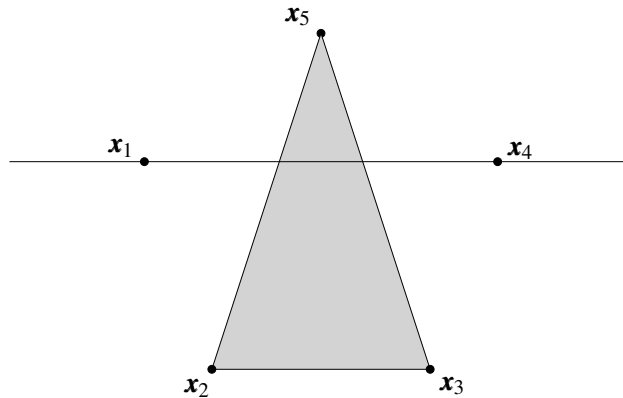
Figure 3.1: Example of Theorem 13.

3.3 What is it good for?

It turns out that it is more convenient to phrase the main theorem about Gale transformations in terms of the complement of a face of a polytope than the face itself. This section follows McMullen & Shephard (1971).

Definition. Let P be a polytope with vertex set V. A subset X of the vertices is called a *coface* if $\operatorname{conv}(V \setminus X)$ is a face of P. If $\operatorname{conv}(V \setminus X)$ is a facet of P, then X is called a *cofacet*.

Definition. Let X be a set of points in $\mathbb{R}^d$, and x be any point of $\mathbb{R}^d$. Then X is said to *capture* x if $x \in \operatorname{relint} \operatorname{conv} X$, that is, x is in the relative interior of the convex hull of X.

The following theorem gives a characterization, in terms of a Gale transformation, of the cofaces of a polytope. A proof of only one implication of the equivalence is given here. The proof of the other implication can be found in Grünbaum (2003), McMullen & Shephard (1971), or Thomas (2006).[1]

---

[1] The proof of the implication given here is not explicitly given in any of these sources.

Theorem 15. *Let P be a polytope, $X \subseteq V = \text{vert}\,P$, and $\overline{V}$ be a Gale transformation of V. Then X is a coface of P if and only if either $\overline{X}$ captures the origin or $X = \varnothing$.*

Proof. Let P be a d-polytope with $V = \text{vert}\,P = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \subseteq \mathbb{R}^d$.

If $X = \varnothing$, then $\text{vert}(P) \setminus X = \text{vert}\,P$, and $P = \text{conv}\,\text{vert}\,P$ is a face of P. Thus, suppose $X \neq \varnothing$.

Let $I = \{i_1, i_2, \ldots, i_r\} \subseteq [n]$, $J = [n] \setminus I$, and $X = \{\boldsymbol{v}_i \mid i \in I\}$. Also, let

$$
\begin{bmatrix} \overline{\boldsymbol{v}}_1 \\ \overline{\boldsymbol{v}}_2 \\ \vdots \\ \overline{\boldsymbol{v}}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_{n-d-1} \end{bmatrix}
$$

where the $\overline{\boldsymbol{v}}_i$'s are regarded as row vectors, and the $\boldsymbol{w}_i$'s are regarded as column vectors.

Suppose $\boldsymbol{0} \notin \text{relint}\,\text{conv}\,\overline{X}$. Then there is some hyperplane $H = \{\boldsymbol{y} \mid \langle \boldsymbol{\xi}, \boldsymbol{y} \rangle = 0\} \subseteq \mathbb{R}^{n-d-1}$ with $\overline{\boldsymbol{v}}_i \in H^+$ for each $i \in I$ and there is some $i_0 \in I$ with $\overline{\boldsymbol{v}}_{i_0} \in H^{(+)}$. Write $\boldsymbol{\xi} = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_{n-d-1} \end{bmatrix}^{\text{T}}$.

For each $i \in [n]$, let $\alpha_i = \langle \boldsymbol{\xi}, \overline{\boldsymbol{v}}_i \rangle$ and $S = \sum_{i \in I} \alpha_i$. The inequalities $\langle \boldsymbol{\xi}, \overline{\boldsymbol{v}}_i \rangle \geq 0$ for $i \in I$ and $\langle \boldsymbol{\xi}, \overline{\boldsymbol{v}}_{i_0} \rangle > 0$ imply that $S > 0$. Therefore, let $\lambda_i = \alpha_i / S$, and $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}^{\text{T}} \in \mathbb{R}^n$. Note that $\boldsymbol{\lambda} \in \text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-d-1}\} = \text{dep}(V)$; indeed:

$$
\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \frac{1}{S}\begin{bmatrix} \langle \boldsymbol{\xi}, \overline{\boldsymbol{v}}_1 \rangle \\ \langle \boldsymbol{\xi}, \overline{\boldsymbol{v}}_2 \rangle \\ \vdots \\ \langle \boldsymbol{\xi}, \overline{\boldsymbol{v}}_n \rangle \end{bmatrix} = \frac{1}{S}\begin{bmatrix} \overline{\boldsymbol{v}}_1 \\ \overline{\boldsymbol{v}}_2 \\ \vdots \\ \overline{\boldsymbol{v}}_n \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-d-1} \end{bmatrix} = \frac{1}{S}\begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_{n-d-1} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-d-1} \end{bmatrix}
$$

$$
= \frac{1}{S}\sum_{i \in [n-d-1]} \xi_i \boldsymbol{w}_i.
$$

Ergo, $\sum_{i \in [n]} \lambda_i \boldsymbol{v}_i = \boldsymbol{0}$ and $\sum_{i \in [n]} \lambda_i = 0$. Set

$$
\boldsymbol{z} = \sum_{i \in I} \lambda_i \boldsymbol{v}_i = \sum_{i \in J}(-\lambda_i)\boldsymbol{v}_i
$$

and note that

$$\sum_{i \in I} \lambda_i = \sum_{i \in I} \frac{\alpha_i}{S} = 1$$

and $\lambda_i \geq 0$ for $i \in I$. Hence, $z \in \text{conv} X$. Further, $\sum_{i \in J} (-\lambda_i) = 1$, whence $z \in \text{aff}(V \setminus X)$. Thence, $\text{aff}(V \setminus X) \cap \text{conv}(X) \neq \varnothing$. Therefore, by Theorem 13, X is not a coface of P. $\qquad \square$

If P is a d-simplex in $\mathbb{R}^d$, then $|\text{vert} P| = d + 1$, and thus a Gale transformation of $\text{vert} P$ is a subset of $\mathbb{R}^0 = \{0\}$. Thus the only Gale transformation of a d-simplex is the multiset $\{0, 0, \ldots, 0\}$ of $d + 1$ equal points.

Corollary 1. *Let $P \subseteq \mathbb{R}^d$ be a d-polytope with vertex set $V = \text{vert} P$ of cardinality $n > d + 1$. Also, let $\overline{V}$ be a Gale transformation of P, and H be a hyperplane in $\mathbb{R}^{n-d-1}$ with $\mathbf{0} \in H$. Then*

$$\left| \overline{V} \cap H^{(+)} \right| \geq 2.$$

Proof. Let P be a d-polytope in $\mathbb{R}^d$ that is not a simplex, and $V = \text{vert} P = \{v_1, v_2, \ldots, v_n\}$ be its vertex set. Then each v_i is a face of P, so that $V_i = V \setminus \{v_i\}$ is a coface of P. Hence, $\overline{V}_i$ captures the origin. If there were some hyperplane H containing $\mathbf{0}$ such that $\overline{V} \cap H^{(+)} = \{\overline{v}_i\}$, then this could not happen. $\qquad \square$

If P is a polytope with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, then a pair of vertices v_i, v_j is a *nonedge* of P if $\text{conv} \{v_i, v_j\}$ is not a face of P.

Corollary 2. *If $\overline{V}$ is a Gale transformation of the vertices $V = \text{vert} P$ of a d-polytope $P \subseteq \mathbb{R}^d$, then a pair of vertices v_i, v_j forms a nonedge of P if and only if there is some hyperplane H such that $\overline{V} \cap H^{(+)} = \{\overline{v}_i, \overline{v}_j\}$.*

Proof. Let $Q = \text{conv}(\overline{V} \setminus \{\overline{v}_i, \overline{v}_j\})$.

If there is some hyperplane H containing the origin such that $\overline{V} \cap H^{(+)} = \{\overline{v}_i, \overline{v}_j\}$, then the set $\overline{V} \setminus \{\overline{v}_i, \overline{v}_j\}$ must have at least two points in $H^{(-)}$, and none in $H^{(+)}$ and therefore cannot capture the origin.

The set $\operatorname{conv}\{\boldsymbol{v}_i,\boldsymbol{v}_j\}$ is a nonedge of P if and only if the set $\overline{V}\setminus\{\overline{\boldsymbol{v}}_i,\overline{\boldsymbol{v}}_j\}$ does not capture the origin. Since Q is a convex set, it does not capture the origin if and only if there is some hyperplane H containing the origin with $Q\subseteq H^-$. Notice that $H^{(+)}$ must contain at least two points of $\overline{V}$ since $\overline{V}$ is a Gale transformation. However, $H^{(+)}\cap Q=\varnothing$, and therefore these two points must be $\{\overline{\boldsymbol{v}}_i,\overline{\boldsymbol{v}}_j\}$. $\qquad\square$

The next theorem gives two conditions that, together, guarantee that a set of n points in $\mathbb{R}^{n-d-1}$ is a Gale transformation of the vertex set of some polytope. The proof given closely follows that in McMullen & Shephard (1971).

Theorem 16. *Let $X=\{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n\}$ be a set of points in $\mathbb{R}^k$ such that*

1. $\sum_{i\in[n]}\boldsymbol{x}_i=\boldsymbol{0}$ and

2. for all hyperplanes H containing $\boldsymbol{0}$ each open half-space contains at least two points of X.

Then X is a Gale transformation of the vertex set of some d-polytope.

Proof. Let $X=\{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n\}$ be a set of points in $\mathbb{R}^k$ satisfying the two conditions. Also, let

$$A=\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n\end{bmatrix}$$

be the matrix whose ith column is $\boldsymbol{x}_i$.

The second condition guarantees that X cannot be contained in any hyperplane containing the origin. Thus $\dim\operatorname{span}X=k$, and therefore by the Rank-Nullity Theorem of Linear Algebra the dimension of the kernel of A is $\dim\ker A=n-k$.

Note that by the first condition $\boldsymbol{1}_n\in\ker A$, and hence the set $\{\boldsymbol{1}_n\}$ can be extended to a basis of

$\ker A$, say $\{\mathbf{1}_n, \widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_2, \ldots, \widetilde{\mathbf{y}}_{n-k-1}\}$. Let

$$B = \begin{bmatrix} \mathbf{1}_n & \widetilde{\mathbf{y}}_1 & \widetilde{\mathbf{y}}_2 & \cdots & \widetilde{\mathbf{y}}_{n-k-1} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{x}}_1 \\ \widetilde{\mathbf{x}}_2 \\ \vdots \\ \widetilde{\mathbf{x}}_n \end{bmatrix}.$$

That is, the vectors $\widetilde{\mathbf{x}}_i$ are the rows of the matrix B whose columns are $\mathbf{1}_n, \widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_2, \ldots, \widetilde{\mathbf{y}}_{n-k-1}$. Then, by definition, $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a Gale transformation of the set $\widetilde{X} = \{\widetilde{\mathbf{x}}_1, \widetilde{\mathbf{x}}_2, \ldots, \widetilde{\mathbf{x}}_n\}$. Now, by the second condition, and the previous theorem each $\widetilde{\mathbf{x}}_i$ is a vertex of $\operatorname{conv}\widetilde{X}$, and therefore $\widetilde{X}$ is the vertex set of some polytope. $\qquad \square$

Example 17. Let $X = \{1, t, -t, -1\}$ where $t > 0$, and note that X satisfies the hypotheses of Theorem 16. Following the proof, set

$$B = \begin{bmatrix} 1 & -t & t \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then X is a Gale transformation of the quadrilateral with vertices $\begin{bmatrix} -t \\ t \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3.4   Gale Diagrams

If $\overline{V}$ is a Gale transformation of the vertex set V of some polytope, and only the face lattice of the polytope is being considered, then the condition $\sum_{\overline{\mathbf{v}} \in \overline{V}} \overline{\mathbf{v}} = \mathbf{0}$ is superfluous. The following definitions are made in light of this.

Definition. Suppose the multisets $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \subseteq \mathbb{R}^k$ and $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\} \subseteq \mathbb{R}^k$ both capture $\mathbf{0}$. Then X and Y are called *consubstantial* if for each $J \subseteq [n]$ the sets $\{\mathbf{x}_j \mid j \in J\}$ and $\{\mathbf{y}_j \mid j \in J\}$ either both capture, or both do not capture $\mathbf{0}$.

36

Consubstantiality is an equivalence relation, and for a fixed polytope P the Gale transformations of $\operatorname{vert} P$ all lie in the same equivalence class. However, if P is not a simplex, then there are multisets consubstantial to a Gale transformation of $\operatorname{vert} P$ that are not themselves Gale transformations. For example, in $\mathbb{R}$, the multiset $\{1, 1, -1, -2\}$ is consubstantial to the multiset $\{1, 1, -1, -1\}$. The former is not a Gale transformation of a polytope since $1 + 1 - 1 - 2 \neq 0$. However, the latter is (see Example 17).

Definition. Suppose $P \subseteq \mathbb{R}^d$ is a d-polytope with n vertices, and $\overline{V}$ is a Gale transformation of $V = \operatorname{vert} P$. If $\Gamma \subseteq \mathbb{R}^{n-d-1}$ and $\overline{V}$ are consubstantial, then Γ is called a *Gale diagram* of P, and the equivalence class of all Gale diagrams of P is denoted $\operatorname{gale}(P)$.

The following theorem shows the usefulness of Gale diagrams.

Theorem 18. *Two polytopes P, Q are combinatorially equivalent if and only if $\operatorname{gale}(P) = \operatorname{gale}(Q)$.*

Proof. Suppose P and Q are combinatorially equivalent, that is, there is an isomorphism φ of face lattices $\varphi \colon \mathscr{F}(P) \to \mathscr{F}(Q)$ (and therefore $|\operatorname{vert} P| = |\operatorname{vert} Q|$). Write $\operatorname{vert} P = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$ and $\operatorname{vert} Q = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n\}$ ordered such that $\varphi(\boldsymbol{p}_i) = \boldsymbol{q}_i$ for each $i \in [n]$.

The set $\Gamma = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\} \in \operatorname{gale}(P)$ is a Gale diagram of P if and only if for each $I \subseteq [n]$ such that $\{\boldsymbol{g}_i \mid i \in I\}$ captures the $\boldsymbol{0}$ the set $\{\boldsymbol{p}_i \mid i \in I\}$ is a coface of P. This happens if and only if $\{\boldsymbol{q}_i \mid i \in I\}$ is a coface of Q (via the isomorphism φ). This is equivalent to $G \in \operatorname{gale}(Q)$. Thus $\operatorname{gale}(Q) = \operatorname{gale}(P)$.

On the other hand, suppose that $\operatorname{gale}(P) = \operatorname{gale}(Q)$ (and therefore that $|\operatorname{vert} P| = |\operatorname{vert} Q|$). Also, let $G = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\} \in \operatorname{gale}(P) = \operatorname{gale}(Q)$, and order the sets $\operatorname{vert} P = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$ and $\operatorname{vert} Q = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n\}$ such that $\boldsymbol{g}_i$ corresponds to both $\boldsymbol{p}_i$ and $\boldsymbol{q}_i$ for each $i \in [n]$.

Define $\vartheta \colon 2^{\operatorname{vert} P} \to 2^{\operatorname{vert} Q}$ (where 2^X denotes the power set of the set X) by, for $i \subseteq [n]$, $\vartheta(\{\boldsymbol{p}_i \mid i \in I\}) = \{\boldsymbol{q}_i \mid i \in I\}$. Then $\{\boldsymbol{p}_i \mid i \in I\}$ is a face of P if and only if $\{\boldsymbol{g}_i \mid i \in [n] \setminus I\}$ captures $\boldsymbol{0}$. This happens if and only if $\vartheta(\{\boldsymbol{p}_i \mid i \in I\}) = \{\boldsymbol{q}_i \mid i \in I\}$ is a face of Q. Hence ϑ is an invertible map that sends faces of P to faces of Q. Furthermore, $\{\boldsymbol{p}_j \mid j \in J\} \subseteq \{\boldsymbol{p}_i \mid i \in I\}$ if and only if

$J \subseteq I$ if and only if

$$\vartheta(\{\boldsymbol{p}_j \mid j \in J\}) = \{\boldsymbol{q}_j \mid j \in J\} \subseteq \{\boldsymbol{q}_i \mid i \in I\} = \vartheta(\{\boldsymbol{p}_i \mid i \in I\}),$$

whence ϑ is order preserving. Thence, ϑ induces an isomorphism $\eta \colon \mathscr{F}(P) \to \mathscr{F}(Q)$. $\qquad \square$

The following theorem follows immediately from Theorem 16 and can be found in McMullen & Shephard (1971). It is the condition that the program in Appendix A uses to check whether or not a set of points is a Gale diagram of some polytope.

Theorem 19. *Suppose $n \geq 0$ and $d \geq -1$ are integers such that $n \geq d+1$. Then a set of points $\Gamma = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\} \subseteq \mathbb{R}^{n-d-1}$ is a Gale diagram of some d-polytope P with $|\operatorname{vert} P| = n$ if and only if for each hyperplane $H \subseteq \mathbb{R}^{n-d-1}$ with $\boldsymbol{0} \in H$ the cardinality $\left| G \cap H^{(+)} \right| \geq 2$.*

In practice, one "only" needs to check the hyperplanes through the origin that are the span of $n - d - 2$ points in G, that is, at most $\binom{n}{n-d-2} = \binom{n}{d+2}$ distinct hyperplanes. Further, since both orientations of a hyperplane need to be checked, after computing $\left| G \cap H^{(+)} \right|$, one can immediately compute $\left| G \cap H^{(-)} \right|$.

The following theorem gives a characterization of apices of pyramids and can be found in McMullen & Shephard (1971).

Theorem 20. *If $\Gamma \in \operatorname{gale}(P)$, then P is a pyramid with apex $\boldsymbol{x}$ if and only if $\bar{\boldsymbol{x}} = \boldsymbol{0}$.*

Proof. The point $\bar{\boldsymbol{x}} = \boldsymbol{0}$ if and only if $\operatorname{relint} \operatorname{conv} \{\bar{\boldsymbol{x}}\} = \{\boldsymbol{0}\}$ if and only if $\operatorname{conv}(\operatorname{vert}(P) \setminus \{\boldsymbol{x}\})$ is a face of P. Hence P is a pyramid with apex $\boldsymbol{x}$. $\qquad \square$

Suppose $\{\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2, \ldots, \bar{\boldsymbol{x}}_n\}$ is a Gale transformation of a d-polytope $P \subseteq \mathbb{R}^d$ with vertex set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathbb{R}$ is a set of positive real numbers. Then the (multi)set $\{\alpha_1 \bar{\boldsymbol{x}}_1, \alpha_2 \bar{\boldsymbol{x}}_2, \ldots, \alpha_n \bar{\boldsymbol{x}}_n\}$ is a Gale diagram of P. Similarly, if $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ is a Gale diagram of P, then there are positive real numbers $\beta_1, \beta_2, \ldots, \beta_n$ such that $\sum_{i \in [n]} \beta_i \boldsymbol{g}_i = \boldsymbol{0}$. Therefore, $\{\beta_1 \boldsymbol{g}_1, \beta_2 \boldsymbol{g}_2, \ldots, \beta_n \boldsymbol{g}_n\} \in \operatorname{gale}(P)$ is a Gale transformation of P. Hence one can easily move between Gale transformations and Gale diagrams if necessary.

3.5   Standard Gale Diagrams

Gale diagrams (like most things) are easier to work with when they lie in a nice subset of the ambient space[2]. Points corresponding to apices of pyramids cannot be moved in a Gale diagram (Theorem 20), however each other point in a Gale diagram can be moved along the ray from the origin passing through that point (and possibly through a larger set), thus a natural construction is to performing the rescaling

$$
\boldsymbol{g}_i \longmapsto
\begin{cases}
\boldsymbol{0} & \text{if } \boldsymbol{g}_i = \boldsymbol{0} \\[2mm]
\boldsymbol{g}_i / \|\boldsymbol{g}_i\| & \text{if } \boldsymbol{g}_i \neq \boldsymbol{0}
\end{cases}
$$

which places all points corresponding to non-apices on the unit sphere $\mathbb{S}^{n-d-2} \subseteq \mathbb{R}^{n-d-1}$. A Gale diagram that is a subset of $\mathbb{S}^{n-d-2} \cup \{\boldsymbol{0}\}$ is called a *standard* Gale diagram.

3.6   Examples

3.6.1   Crosspolytopes

Let $\boldsymbol{e}_i$ be the ith unit basis vector in $\mathbb{R}^d$ and consider $X_d = \mathrm{conv}\{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \dots, \pm\boldsymbol{e}_d\}$, the standard d-crosspolytope. Computing a Gale transformation of X_d requires finding a basis for the kernel of the matrix

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\
\boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_{d-1} & \boldsymbol{e}_d & -\boldsymbol{e}_d & -\boldsymbol{e}_{d-1} & \cdots & -\boldsymbol{e}_1
\end{bmatrix}.
$$

First, note that this is a $(d+1) \times (2d)$ matrix with rank $d+1$. Thus, the kernel has dimension $d-1$. The set $\{\boldsymbol{e}_i + \boldsymbol{e}_{2d+1-i} - \boldsymbol{e}_1 - \boldsymbol{e}_{2d} \mid i \in [d] \setminus \{1\}\}$ of $d-1$ vectors in $\mathbb{R}^{2d}$ is linearly independent, and is a subset of the kernel. Thus this set forms a basis. Forming the matrix that has these vectors for its columns, and extracting the row vectors yields the Gale transformation $\{-\boldsymbol{1}, \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{d-2}, \boldsymbol{e}_{d-1}, \boldsymbol{e}_{d-1}, \boldsymbol{e}_{d-2}, \dots, \boldsymbol{e}_2, \boldsymbol{e}_1, -\boldsymbol{1}\} \subseteq \mathbb{R}^{d-1}$. See Figure 3.2 for an illustration of the 3-dimensional case.

---

[2]Here, 'nice' has the completely circular meaning of a subset that makes a Gale diagram easy to work with.
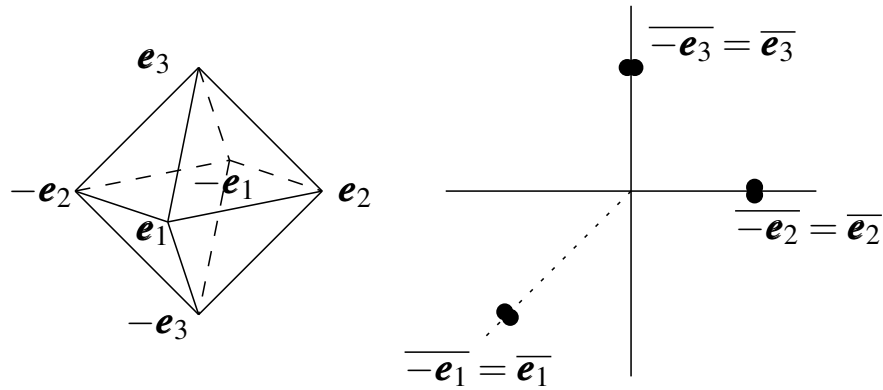
Figure 3.2: The polytope X_3 and a Gale transformation of its vertices.
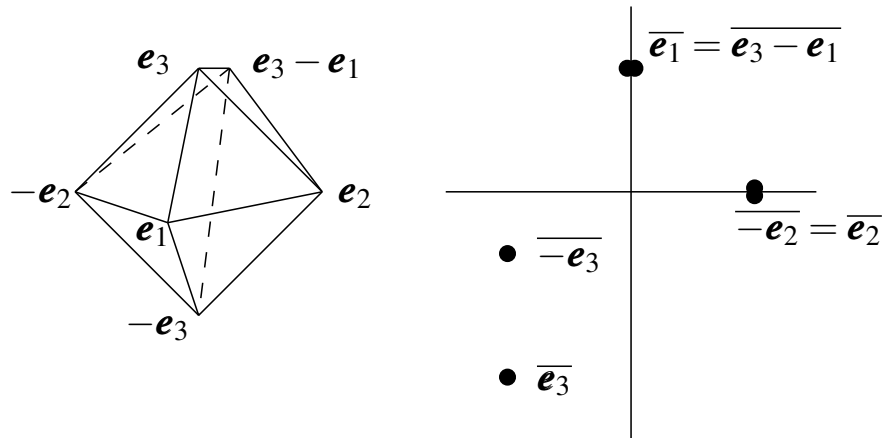
Figure 3.3: A 3-dimensional crosspolytope and a Gale transformation of its vertices.

Thus a Gale diagram of the d-dimensional crosspolytope can be given by the vertices of a $(d-1)$-simplex with the origin in its relative interior where each point occurs twice and corresponds to a pair of vertices that are not joined by an edge. These are not all of the Gale diagrams of crosspolytopes; any doubled point can be split into two points as long as the two points are not moved "too far" apart [3] (compare Figures 3.2 and 3.3).

3.6.2 Prisms over Simplices

A prism over a d-simplex is a $(d+1)$-dimensional polytope with $2d+2$ vertices. It thus has a d-dimensional Gale diagram.

---

[3]Here, "too far" has nothing to do with distance. It is entirely possible that one point can be moved freely within an unbounded region if all other points are held fixed, e.g., in Figure 3.3 the point $\overline{e_3}$ could be moved anywhere within the open third quadrant without changing the combinatorial type of the polytope.

Let P be the prism over a d-simplex with vertices

$$\begin{bmatrix} e_1 \\ 1 \end{bmatrix}, \begin{bmatrix} e_2 \\ 1 \end{bmatrix}, \ldots \begin{bmatrix} e_d \\ 1 \end{bmatrix}, \begin{bmatrix} -\mathbf{1} \\ 1 \end{bmatrix}, \begin{bmatrix} -\mathbf{1} \\ -1 \end{bmatrix}, \begin{bmatrix} e_d \\ -1 \end{bmatrix}, \begin{bmatrix} e_{d-1} \\ -1 \end{bmatrix}, \ldots \begin{bmatrix} e_1 \\ -1 \end{bmatrix},$$

where e_i is the ith standard basis vector in $\mathbb{R}^d$.

Then the vectors

$$\begin{bmatrix} -e_i \\ 1 \\ -1 \\ e_i \end{bmatrix}$$

form a basis for the kernel of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ e_1 & e_2 & \cdots & e_d & -\mathbf{1} & -\mathbf{1} & e_d & e_{d-1} & \cdots & e_1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

so that the set

$$\{-e_1, -e_2, \ldots, \mathbf{1}, -\mathbf{1}, e_d, e_{d-1}, \ldots, e_1\} \subseteq \mathbb{R}^d$$

is a Gale transformation of P.

In general a Gale diagram of P is given by the vertices of a simplex with the origin in its relative interior, along with the negatives of these points. The pairs of points that are antipodal are points of the polytope that correspond to the same point in the original simplex.
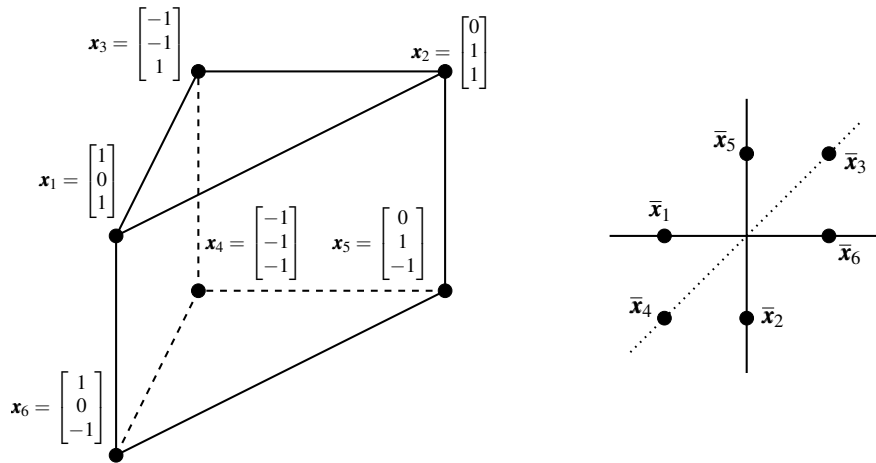
Figure 3.4: The polytope prism Δ_2 and a Gale transformation of its vertices.

3.7 Oriented Matroids

This section will focus mainly on oriented matroids that arise from point configurations in a Euclidean space. Such oriented matroids are called *realizable*. The definition of a general oriented matroid will be given in the penultimate subsection, though it is not necessary for the rest of the narrative.

This section will closely follow Ziegler (1995) and Björner et al. (1999).

3.7.1 Sign Vectors and Orthogonality

If $n \in \mathbb{N}$, then a *sign vector of length n* is an element of the set $\{-,0,+\}^n$. If $\boldsymbol{v} \in \mathbb{R}^n$, then the *sign vector of $\boldsymbol{v}$*, denoted $\text{SIGN}\,\boldsymbol{v}$, is the sign vector of length n with coordinates

$$
(\text{SIGN}\,\boldsymbol{v})_i = \begin{cases} - & \text{if } v_i < 0 \\ 0 & \text{if } v_i = 0 \\ + & \text{if } v_i > 0. \end{cases}
$$

Define, also, $-\,\textsc{sign}\,\boldsymbol{v}$ by

$$
(-\,\textsc{sign}\,\boldsymbol{v})_i = \begin{cases} + & \text{if } v_i < 0 \\[2mm] 0 & \text{if } v_i = 0 \\[2mm] - & \text{if } v_i > 0. \end{cases}
$$

The sign vector with all coordinates 0 is denoted $\boldsymbol{0}$, that is, $\boldsymbol{0} = \textsc{sign}\,\boldsymbol{0}$. No confusion should arise from this slight abuse of notation. If $V \subseteq \mathbb{R}^n$, then set $\textsc{sign}\,V = \{\textsc{sign}\,\boldsymbol{v} \mid \boldsymbol{v} \in V\}$. The support of an n-dimensional sign vector $\boldsymbol{v}$ is the set $\operatorname{supp}\boldsymbol{v} = \{i \in [n] \mid v_i \neq 0\}$. If S is a set of sign vectors, then $\operatorname{supp}S = \cup_{\boldsymbol{s} \in S}\operatorname{supp}\boldsymbol{s}$. A sign vector $\boldsymbol{s} \in S$ is said to be of *minimal support* if for each $\boldsymbol{t} \in S$ the inclusion $\operatorname{supp}\boldsymbol{t} \subseteq \operatorname{supp}\boldsymbol{s}$ implies $\operatorname{supp}\boldsymbol{t} = \operatorname{supp}\boldsymbol{s}$.

Definition. If $\boldsymbol{s}$ and $\boldsymbol{t}$ are two sign vectors of length n, then they are said to be *orthogonal* if either:

1. for all $i \in [n]$ either $s_i = 0$, or $t_i = 0$; or

2. there is a pair $i, j \in [n]$ with $s_i = t_i \neq 0$ and $s_j = -t_j \neq 0$.

   This definition of orthogonality is reasonable, since two sign vectors $\boldsymbol{s}, \boldsymbol{t}$ of length n are orthogonal if and only if there are two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ with $\boldsymbol{s} = \textsc{sign}\,\boldsymbol{v}$ and $\boldsymbol{t} = \textsc{sign}\,\boldsymbol{w}$ such that $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$.

   The notation $\boldsymbol{s} \perp \boldsymbol{t}$ signifies that $\boldsymbol{s}$ is orthogonal to $\boldsymbol{t}$. Notice that $\boldsymbol{s} \perp \boldsymbol{s}$ if and only if $\boldsymbol{s}$ is the vector that is 0 in each coordinate. Orthogonality is also a symmetric relationship, i.e., $\boldsymbol{s} \perp \boldsymbol{t}$ if and only if $\boldsymbol{t} \perp \boldsymbol{s}$. Orthogonality of sign vectors is not transitive, just as orthogonality of vectors in $\mathbb{R}^n$ is not transitive.

   If S is a set of sign vectors of length n, then the *dual* of S is the set

$$
S^\perp = \{\boldsymbol{t} \in \{-, 0, +\}^n \mid \boldsymbol{t} \perp \boldsymbol{s} \text{ for all } \boldsymbol{s} \in S\}.
$$

Notice that $\boldsymbol{0} \in S^\perp$ so that $S^\perp \neq \varnothing$.

3.7.2 Realizable Oriented Matroids

Let $X = \{x_1, x_2, \ldots, x_n\} \subseteq \mathbb{R}^d$, and suppose that $\operatorname{aff} X = \mathbb{R}^d$. Denote by $[X]$ the matrix whose ith column is x_i. Consider the set of affine dependencies of X, that is:

$$\operatorname{dep} X = \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \;\middle|\; [X]\boldsymbol{\lambda} = \mathbf{0} \text{ and } \sum_{i \in [n]} \lambda_i = 0 \right\}.$$

The set of *vectors of X* is the set $\mathscr{V}(X) = \operatorname{SIGN}(\operatorname{dep} X)$ of sign vectors of the points in $\operatorname{dep} X$.[4]

A sign vector $\boldsymbol{s} \in \mathscr{V}(X)$ of minimal support is called a *circuit* of X. The set of all circuits of X is denoted

$$\mathscr{C}(X) = \{\boldsymbol{s} \in \mathscr{V}(X) \mid \boldsymbol{s} \text{ is of minimal support}\}.$$

The set of *covectors of X* is the set $\mathscr{V}(X)^\perp$, and the set of *cocircuits* is the set of covectors of minimal support.

$$\mathscr{C}(X)^\perp = \left\{ \boldsymbol{s} \in \mathscr{V}(X)^\perp \;\middle|\; \boldsymbol{s} \text{ is of minimal support} \right\}.$$

Any one of the sets $\mathscr{V}(X), \mathscr{V}(X)^\perp, \mathscr{C}(X), \mathscr{C}(X)^\perp$ can be used to define a realizable oriented matroid, in the same way that a topology on a set can be defined by either open or closed sets. That is, any one of these sets can be obtained from each of the others. For details, see (Ziegler, 1995, section 6.3).

3.7.2.1 Geometric Interpretation of $\mathscr{V}(X)$

Every $\boldsymbol{\lambda} \in \operatorname{dep} X \setminus \{\mathbf{0}\}$ corresponds to a point $\widetilde{\boldsymbol{\lambda}}$ that lies in $\operatorname{conv} X$ in the following way:

Let $\boldsymbol{\lambda} \in \operatorname{dep}(X) \setminus \{\mathbf{0}\}$. Define the positive and negative parts of $\boldsymbol{\lambda}$ to be the sets of coordinates such that $\boldsymbol{\lambda}$ is positive or negative respectively:

$$\mathrm{P}(\boldsymbol{\lambda}) = \{i \mid \lambda_i > 0\} \qquad\qquad \mathrm{N}(\boldsymbol{\lambda}) = \{i \mid \lambda_i < 0\}.$$

---

[4]The elements of X are themselves vectors in that they are elements of a vector space. They are not, regrettably, *the vectors of X*. This terminology, however unfortunate it may be, is standard in the study of oriented matroids.

Now define $\lambda_+ = \sum_{i \in P(\boldsymbol{\lambda})} \lambda_i$ and $\lambda_- = \sum_{i \in N(\boldsymbol{\lambda})} \lambda_i$. The equality $\sum \lambda_i = 0$ implies $\lambda_+ = -\lambda_-$; further, $\boldsymbol{\lambda} \neq \boldsymbol{0}$ implies $\lambda_+ \neq 0$. Finally, the equality $[X]\boldsymbol{\lambda} = \boldsymbol{0}$ yields that

$$\sum_{i \in P(\boldsymbol{\lambda})} \frac{\lambda_i}{\lambda_+} \boldsymbol{x}_i = - \sum_{i \in N(\boldsymbol{\lambda})} \frac{\lambda_i}{\lambda_+} \boldsymbol{x}_i.$$

Call this point $\widetilde{\boldsymbol{\lambda}}$. The previous paragraph shows that

$$\widetilde{\boldsymbol{\lambda}} \in \operatorname{conv}\left\{\boldsymbol{x}_i \mid i \in P(\boldsymbol{\lambda})\right\} \cap \operatorname{conv}\left\{\boldsymbol{x}_i \mid i \in N(\boldsymbol{\lambda})\right\}.$$

Conversely each of the points in the above intersections corresponds to at least one affine dependence of X. Furthermore, every point in a particular intersection of the type above has the same sign vector.

3.7.2.2 Geometric Interpretation of $\mathscr{V}^\perp(X)$

Intuitively, a sign vector in $\mathscr{V}^\perp(X)$ keeps track of which side of some hyperplane each point of X lies on (or if it lies on the hyperplane itself).

To this end, let $H = \{\boldsymbol{x} \mid \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = t\}$ be some hyperplane in $\mathbb{R}^d$, and define the sign vector $\boldsymbol{s}(H)$ as follows:

$$(\boldsymbol{s}(H))_i = \begin{cases} + & \text{if } \langle \boldsymbol{\xi}, \boldsymbol{x}_i \rangle > t \\ 0 & \text{if } \langle \boldsymbol{\xi}, \boldsymbol{x}_i \rangle = t \\ - & \text{if } \langle \boldsymbol{\xi}, \boldsymbol{x}_i \rangle < t. \end{cases}$$

Then this sign vector encodes which side of H each point of X lies on. Notice that

$$\boldsymbol{s}(H) = \operatorname{SIGN}\left([X]^{\mathrm{T}}\boldsymbol{\xi} - t\boldsymbol{1}\right).$$

Therefore

$$\mathscr{V}^{\perp}(X) = \left\{ \text{SIGN}\left([X]^{\mathsf{T}}\boldsymbol{\xi} - t\mathbf{1}\right) \;\middle|\; \boldsymbol{\xi} \in \mathbb{R}^d \text{ and } t \in \mathbb{R} \right\}.$$

Example 21. Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5\} \subseteq \mathbb{R}^2$ where

$$\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{x}_5 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

and consider the affine dependence $\boldsymbol{\lambda} = (4, -1, -1, 4, -6)^{\mathsf{T}}$. In this case, $P(\boldsymbol{\lambda}) = \{1,4\}$, $N(\boldsymbol{\lambda}) = \{2,3,5\}$ and $\lambda_+ = 8$. Hence

$$\widetilde{\boldsymbol{\lambda}} = \frac{\lambda_1 x_1 + \lambda_4 x_4}{\lambda_+} = \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$$

which is a point along the line segment from $\boldsymbol{x}_1$ to $\boldsymbol{x}_4$ as well as a point inside the triangle with vertices $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_5$.
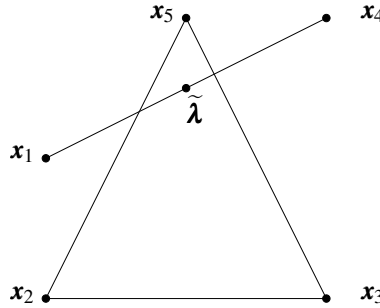
Further: $\boldsymbol{s} = \text{SIGN}\,\boldsymbol{\lambda} = \begin{bmatrix} + & - & - & + & - \end{bmatrix}^{\mathsf{T}} \in \mathscr{V}(X)$; $|\mathscr{V}(X)| = 21$; and

$$\mathscr{C}(X) = \left\{ \pm\begin{bmatrix} 0 & + & - & + & - \end{bmatrix}^{\mathsf{T}}, \pm\begin{bmatrix} + & - & 0 & + & - \end{bmatrix}^{\mathsf{T}}, \pm\begin{bmatrix} + & - & + & - & 0 \end{bmatrix}^{\mathsf{T}}, \right.$$
$$\left. \pm\begin{bmatrix} + & - & + & 0 & - \end{bmatrix}^{\mathsf{T}}, \pm\begin{bmatrix} + & 0 & - & + & - \end{bmatrix}^{\mathsf{T}} \right\}.$$

Note that $\boldsymbol{s} \notin \mathscr{C}(X)$ since $\boldsymbol{t} = \begin{bmatrix} + & - & 0 & + & - \end{bmatrix}^{\mathsf{T}} \in \mathscr{V}(X)$ and $\text{supp}\,\boldsymbol{t} \subsetneq \text{supp}\,\boldsymbol{s}$. As for the dual

sets, $\left|\mathscr{V}^{\perp}(X)\right| = 83$, and

$$\mathscr{C}^{\perp}(X) = \left\{ \pm \begin{bmatrix} 0 & 0 & + & + & + \end{bmatrix}^{\mathsf{T}}, \pm \begin{bmatrix} 0 & + & 0 & - & - \end{bmatrix}^{\mathsf{T}}, \pm \begin{bmatrix} 0 & + & + & 0 & - \end{bmatrix}^{\mathsf{T}}, \right.$$

$$\pm \begin{bmatrix} 0 & + & + & + & 0 \end{bmatrix}^{\mathsf{T}}, \pm \begin{bmatrix} + & 0 & - & - & 0 \end{bmatrix}^{\mathsf{T}}, \pm \begin{bmatrix} + & 0 & - & 0 & + \end{bmatrix}^{\mathsf{T}},$$

$$\pm \begin{bmatrix} + & 0 & 0 & + & + \end{bmatrix}^{\mathsf{T}}, \pm \begin{bmatrix} + & + & 0 & - & 0 \end{bmatrix}^{\mathsf{T}}, \pm \begin{bmatrix} + & + & 0 & 0 & + \end{bmatrix}^{\mathsf{T}},$$

$$\left. \pm \begin{bmatrix} + & + & + & 0 & 0 \end{bmatrix}^{\mathsf{T}} \right\}.$$

3.7.3 General Oriented Matroids

This section closely follows (Ziegler, 1995, section 7.4). Before giving the definition of a general oriented matroid, three more definitions are needed for collections of sign vectors.

Definition. Let s, t, u be sign vectors of length n.

1. The *composition* of s and t is the sign vector $s \circ t$ with coordinates

$$(s \circ t)_i = \begin{cases} s_i & , \ s_i \neq 0 \\ t_i & , \ s_i = 0. \end{cases}$$

2. The separation set of s and t is the set

$$\mathrm{sep}(s, t) = \{ i \in [n] \mid s_i = -t_i \neq 0 \}.$$

3. If $j \in \mathrm{sep}(s, t)$, then u *eliminates j between s and t* if $u_j = 0$ and for every $i \notin \mathrm{sep}(s, t)$, the equality $u_i = (s \circ t)_i$ holds.

Notice that $\mathrm{sep}(s, t) = \mathrm{sep}(t, s)$ and composition is not a commutative operation (Example 22).

47

It is an associative operation since

$$(s \circ (t \circ u))_i = ((s \circ t) \circ u)_i = \begin{cases} s_i & , \; s_i \neq 0 \\ t_i & , \; s_i = 0 \neq t_i \\ u_i & , \; s_i = 0 = t_i \end{cases}.$$

Example 22. Let $s = \begin{bmatrix} + & + & + & 0 & - \end{bmatrix}^{\mathrm{T}}$, and $t = \begin{bmatrix} + & - & 0 & + & + \end{bmatrix}^{\mathrm{T}}$. Then

- $s \circ t = \begin{bmatrix} + & + & + & + & - \end{bmatrix}^{\mathrm{T}}$,

- $\mathrm{sep}(s,t) = \{2,5\}$, and

- u eliminates 2 between s and t if and only if u is of the form $\begin{bmatrix} + & 0 & + & + & \alpha \end{bmatrix}^{\mathrm{T}}$ for $\alpha \in \{-,0,+\}$.

On the other hand

- $t \circ s = \begin{bmatrix} + & - & + & + & + \end{bmatrix}^{\mathrm{T}}$ and

- u eliminates 2 between t and s if and only if u is of the form $= \begin{bmatrix} + & 0 & + & + & \beta \end{bmatrix}^{\mathrm{T}}$ for $\beta \in \{-,0,+\}$.

Definition. A set $\mathscr{V}^{\perp} \subseteq \{-,0,+\}^n$ is the set of *covectors of an oriented matroid* if it satisfies the following:

1. $\mathbf{0} \in \mathscr{V}^{\perp}$;

2. if $s \in \mathscr{V}^{\perp}$, then $-s \in \mathscr{V}^{\perp}$;

3. if $s,t \in \mathscr{V}^{\perp}$, then $s \circ t \in \mathscr{V}^{\perp}$

4. if $s,t \in \mathscr{V}^{\perp}$ and $j \in \mathrm{sep}(s,t)$, then there is some $u \in \mathscr{V}^{\perp}$ such that u eliminates j between s and t.

48

Covectors as described in section 3.7.2 satisfy the properties above, and therefore do form the set of covectors of an oriented matroid.

3.7.4   The Dual of an Oriented Matroid

For proofs of the results in this subsection, see either Ziegler (1995) or Björner et al. (1999).

Recall that the definition of orthogonality and dual were in terms of sign vectors. Thus it makes sense to apply these operations to the covectors in a general oriented matroid. In general, $(S^{\perp})^{\perp} = S$. Thus taking duals of a set leads to at most two distinct sets. For $\mathscr{V}^{\perp}$ the set of covectors of a general oriented matroid, define the *set of vectors* to be $\mathscr{V} = (\mathscr{V}^{\perp})^{\perp}$. One can define circuits and cocircuits analogously. Similar to the case of realizable oriented matroids, any of these four sets can be used to determine the others.

A nontrivial result of oriented matroid theory is that the set of vectors of an oriented matroid is also the set of covectors of a different[5] oriented matroid. This oriented matroid is called the *dual* of the original oriented matroid.

In the case of a realizable oriented matroid (realized by the set of points V), the dual is also realizable, and is realized by a Gale diagram $\Gamma \in \mathrm{gale}\, V$.

---

[5] As long as $\mathscr{V} \neq \{\mathbf{0}\}$. This follows since $\mathscr{V} \cap \mathscr{V}^{\perp} = \{\mathbf{0}\}$.

Chapter 4

Modifying Gale Diagrams

This chapter explores certain operations on Gale diagrams and how these operations affect the polytopes.

4.1   Joins and Direct Sums

Recall, from Sections 1.4.5 and 1.4.6, the definitions of join and direct sum for point sets. If $X = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\} \subseteq \mathbb{R}^{d_1}$ and $Y = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_m\} \subseteq \mathbb{R}^{d_2}$, then

$$
X \vee Y = \left\{ \begin{bmatrix} \boldsymbol{p}_i \\ \boldsymbol{0}_{d_2} \\ -1 \end{bmatrix} \; \middle| \; i \in [n] \right\} \cup \left\{ \begin{bmatrix} \boldsymbol{0}_{d_1} \\ \boldsymbol{q}_j \\ 1 \end{bmatrix} \; \middle| \; j \in [m] \right\} \subseteq \mathbb{R}^{d_1 + d_2 + 1}
$$

and

$$
X \oplus Y = \left\{ \begin{bmatrix} \boldsymbol{p}_i \\ \boldsymbol{0}_{d_2} \end{bmatrix} \; \middle| \; i \in [n] \right\} \cup \left\{ \begin{bmatrix} \boldsymbol{0}_{d_1} \\ \boldsymbol{q}_j \end{bmatrix} \; \middle| \; j \in [m] \right\} \subseteq \mathbb{R}^{d_1 + d_2}.
$$

Since these operations are defined for general point sets, it makes sense to ask what happens if you perform them on two Gale diagrams.

Theorem 23. *If P and Q are polytopes, $\Gamma \in \text{gale}(P)$, and $\Lambda \in \text{gale}(Q)$, then*

$$\Gamma \oplus \Lambda \in \text{gale}(P \vee Q)$$

and

$$\Gamma \vee \Lambda \in \text{gale}(P \oplus Q).$$

Proof. Suppose the following:

$$\dim P = d_1; \qquad\qquad \dim Q = d_2;$$

$$P \subseteq \mathbb{R}^{d_1}; \qquad\qquad Q \subseteq \mathbb{R}^{d_2};$$

$$\text{vert}\, P = \{v_1, v_2, \ldots, v_n\}; \qquad\qquad \text{vert}\, Q = \{w_1, w_2, \ldots, w_m\};$$

$$\mathbf{0}_{d_1} \in \text{relint}\, P; \qquad\qquad \mathbf{0}_{d_2} \in \text{relint}\, Q;$$

$$\dim \text{aff}\, \{v_1, v_2, \ldots, v_{d_1+1}\} = d_1; \qquad\qquad \dim \text{aff}\, \{w_1, w_2, \ldots, w_{d_2+1}\} = d_2;$$

and

$$\text{rref}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} I_{d_1+1} & A \end{bmatrix}; \qquad \text{rref}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ w_1 & w_2 & \cdots & w_m \end{bmatrix} = \begin{bmatrix} I_{d_2+1} & B \end{bmatrix}.$$

Then order the vertices of $P \vee Q$ as follows:

$$\left\{ \begin{bmatrix} v_1 \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix}, \ldots, \begin{bmatrix} v_{d_1+1} \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{d_1} \\ w_1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{d_1} \\ w_{d_2+1} \\ 1 \end{bmatrix}, \begin{bmatrix} v_{d_1+2} \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix}, \ldots, \begin{bmatrix} v_n \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{d_1} \\ w_{d_2+2} \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{d_1} \\ w_m \\ 1 \end{bmatrix} \right\}.$$

Here, the first $d_1 + d_2 + 2 = \dim(P \vee Q) + 1$ vertices are affinely independent. Now, use the tech-

niques of Section 3.1.1 to compute a Gale transformation of $P \vee Q$ as follows: Form the matrix

$$
Z = \left[
\begin{array}{ccc|ccc|ccc|ccc}
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\mathbf{v}_1 & \cdots & \mathbf{v}_{d_1+1} & \mathbf{0}_{d_1} & \cdots & \mathbf{0}_{d_1} & \mathbf{v}_{d_1+2} & \cdots & \mathbf{v}_n & \mathbf{0}_{d_1} & \cdots & \mathbf{0}_{d_1} \\
\hline
\mathbf{0}_{d_2} & \cdots & \mathbf{0}_{d_2} & \mathbf{w}_1 & \cdots & \mathbf{w}_{d_2+1} & \mathbf{0}_{d_2} & \cdots & \mathbf{0}_{d_2} & \mathbf{w}_{d_2+1} & \cdots & \mathbf{w}_m \\
-1 & \cdots & -1 & 1 & \cdots & 1 & -1 & \cdots & -1 & 1 & \cdots & 1
\end{array}
\right].
$$

First, add the first row to the last row, and then perform, in the first $d_1 + 1$ rows, the operations that transformation the matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$ into $\begin{bmatrix} I_{d_1+1} & A \end{bmatrix}$. Note that in the $1,2$ and $1,4$ blocks each column will be the same. Call this common (d_1+1)-dimensional vector $\mathbf{v}$. Further, denote an $r \times s$ matrix with all entries 0 by $[0]_s^r$. That is,

$$
M = \mathrm{rref}\, Z = \mathrm{rref} \left[
\begin{array}{c|ccc|c|ccc}
I_{d_1+1} & \mathbf{v} & \cdots & \mathbf{v} & A & \mathbf{v} & \cdots & \mathbf{v} \\
\hline
[0]_{d_1+1}^{d_2+1} & \begin{matrix}\mathbf{w}_1 \\ 2\end{matrix} & \cdots & \begin{matrix}\mathbf{w}_{d_2+1} \\ 2\end{matrix} & [0]_{n-d_1+1}^{d_2+1} & \begin{matrix}\mathbf{w}_{d_2+2} \\ 2\end{matrix} & \cdots & \begin{matrix}\mathbf{w}_m \\ 2\end{matrix}
\end{array}
\right].
$$

Next, use the bottom row of the matrix to turn the $1,2$ and $1,4$ blocks into all zeros. Then move the bottom row to the (d_1+2)th row, divide it by 2 and shift each of the remaining rows down, that is:

$$
M = \mathrm{rref} \left[
\begin{array}{c|c|c|c}
I_{d_1+1} & [0]_{d_2+1}^{d_1+1} & A & [0]_{m-d_2-1}^{d_1+1} \\
\hline
[0]_{d_1+1}^{d_2+1} & \begin{matrix} 1 & \cdots & 1 \\ \mathbf{w}_1 & \cdots & \mathbf{w}_{d_2+1} \end{matrix} & [0]_{n-d_1+1}^{d_2+1} & \begin{matrix} 1 & \cdots & 1 \\ \mathbf{w}_{d_2+2} & \cdots & \mathbf{w}_m \end{matrix}
\end{array}
\right].
$$

Now, in the bottom $d_2 + 1$ rows, perform the row operations that transformation the matrix

52

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ w_1 & w_2 & \cdots & w_m \end{bmatrix} \text{ into } \begin{bmatrix} I_{d_2+1} & B \end{bmatrix}. \text{ Thus}$$

$$M = \begin{bmatrix} I_{d_1+d_2+2} & \begin{array}{c|c} A & [0] \\ \hline [0] & B \end{array} \end{bmatrix}$$

and so a Gale transformation of $P \vee Q$ is given by the multiset of the columns of the matrix

$$\begin{bmatrix} -A^{\mathrm{T}} & [0] & I & [0] \\ \hline [0] & -B^{\mathrm{T}} & [0] & I \end{bmatrix}.$$

These are exactly the elements of $\Gamma \oplus \Lambda$.

The second result holds by oriented matroid duality. $\qquad\square$

4.1.1 d-Polytopes with $d+2$ Vertices

As an application of Theorem 23, all d-polytopes with $d+2$ vertices are classified. This section follows the discussion found in McMullen & Shephard (1971).

Let P be a d-polytope with $d+2$ vertices, and let Γ be a standard Gale diagram of P. Since $\mathbb{S}^0 = \{-1, 1\} \subseteq \mathbb{R}$, as a set $\{-1, 1\} \subseteq \Gamma \subseteq \{-1, 0, 1\}$. Because Γ is a Gale diagram,

$$p = |\{x \in \Gamma \mid \bar{x} = -1\}| \geq 2$$

$$q = |\{x \in \Gamma \mid \bar{x} = 1\}| \geq 2.$$

This leaves $a = d + 2 - p - q$ points in Γ, and they must all be at the origin. A point at the origin in a Gale diagram corresponds to the apex of a pyramid in the polytope (Theorem 20). Thus P is an a-fold pyramid over a polytope Q whose Gale diagram Λ has $p \geq 2$ points on one side of the origin, and $q \geq 2$ points on the other.

The Gale diagram Λ is the join of the Gale diagrams of a $(p-1)$-simplex and a $(q-1)$-simplex.

Hence, by Theorem 23, P is combinatorially equivalent to the polytope

$$\mathrm{pyr}_a \left(\Delta_{p-1} \oplus \Delta_{q-1} \right).$$

where $a + p + q = d + 2$ and $p, q \geq 2$.

Note that the combinatorial type of each d-polytope with $d + 2$ vertices can be specified by the unordered pair (p, q). So to count the number of combinatorial types of these polytopes, one only needs to count the number of such pairs. Assume, without loss of generality, that $p \leq q$. The pairs are then of the following form:

$$(2, q), \qquad 2 \leq q \leq d$$
$$(3, q), \qquad 3 \leq q \leq d - 1$$
$$\vdots$$
$$\left(\left\lfloor \frac{d}{2} \right\rfloor, q \right), \qquad \left\lfloor \frac{d}{2} \right\rfloor \leq q \leq \left\lceil \frac{d}{2} \right\rceil.$$

Thus, if d is odd, then there are

$$2 + 4 + \cdots + (d - 1) = 2 \left(1 + 2 + \cdots + \frac{d-1}{2} \right) = \frac{d^2}{4} - \frac{1}{4} = \left\lfloor \frac{d^2}{4} \right\rfloor$$

combinatorial types of these polytopes. If d is even, then there are

$$1 + 3 + \cdots + (d - 1) = 2 + 4 + \cdots + d - \frac{d}{2} = \frac{d^2}{4} = \left\lfloor \frac{d^2}{4} \right\rfloor$$

types of these polytopes.

In McMullen & Shephard (1971), the following lemma is proved.

Lemma 24. *Let Γ be a Gale diagram of a d-polytope P with n vertices. Then P is a simplicial*

polytope if and only if for every hyperplane $H \in \mathbb{R}^{n-d-1}$ *containing* $\mathbf{0}$ *the following holds:*

$$\mathbf{0} \notin \operatorname{relint} \operatorname{conv}(\Gamma \cap H).$$

Combining this with the above discussion yields:

Theorem 25. *There are* $\lfloor d^2/4 \rfloor$ *combinatorial types of d-polytopes with* $d + 2$ *vertices. Further-more, each of these polytopes is of the form* $\operatorname{pyr}_a \left(\Delta_{p-1} \oplus \Delta_{q-1} \right)$ *where* $a + p + q = d + 2$ *and* $p, q \geq 2$. *Moreover,* $\lfloor d/2 \rfloor$ *of these polytopes are simplicial.*

4.2   Adding Points to Gale Diagrams

The first thing one should do in a section entitled "Adding Points to Gale Diagrams" is discuss under what conditions adding a point to a Gale diagram yields a Gale diagram. Thus, suppose $\Gamma \in \operatorname{gale}(P)$ is a Gale diagram of some d-polytope P with n vertices, and let $\mathbf{x} \in \mathbb{R}^{n-d-1}$ be any point. Then $\Gamma \cup \{\mathbf{x}\}$ is a Gale diagram of some $(d + 1)$-polytope with $n + 1$ vertices. This is true since every hyperplane still has at least two points in both of its open half-spaces. It should be clear that the location of the new point does matter in determining the new polytope. Adding a point to a Gale diagram increases the number of points by 1, but not the dimension of the space in which the Gale diagram lies. Thus the dimension of the new polytope is 1 greater than the dimension of the old polytope.

Points can be added to Gale diagrams in many ways. Unfortunately, it is possible given two Gale diagrams $\Gamma, \Lambda \in \operatorname{gale}(P)$ to add a point to both Γ and Λ in the same way (such as adding a point antipodal to the point corresponding to a particular vertex), yet not have the two new polytopes be combinatorially equivalent.
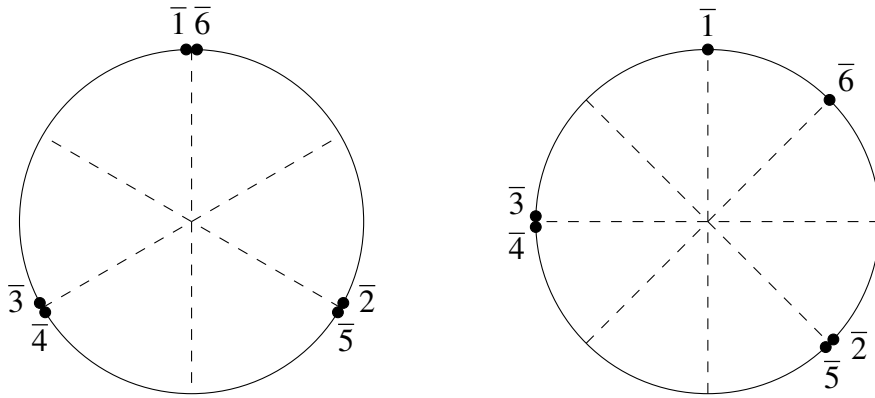
Figure 4.1: Two Gale diagrams of X_3.
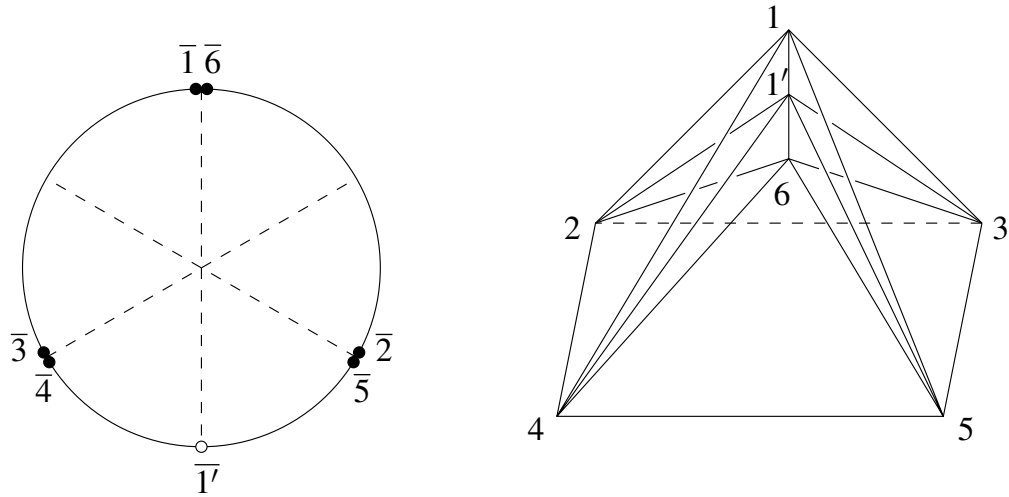
Figure 4.2: Adding an antipode $\overline{1}'$ to the point $\overline{1}$ in a Gale diagram of X_3. (cf. Figure 4.3)
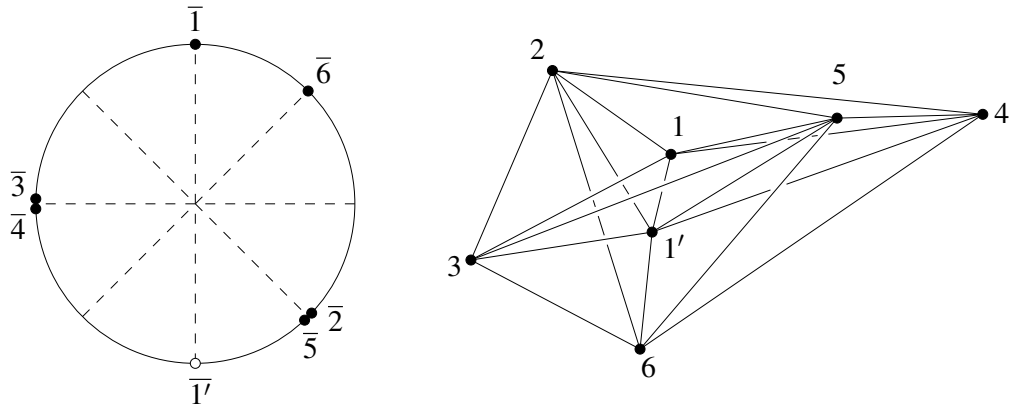
Figure 4.3: Adding an antipode $\overline{1}'$ to the point $\overline{1}$ in a Gale diagram of X_3. (cf. Figure 4.2)

4.2.1 Adding an Antipode to a Gale Diagram

As an example; consider the crosspolytope X_3. Figure 4.1 shows two consubstantial standard Gale diagrams for X_3. If a point $\overline{1'}$ is added that is antipodal to $\overline{1}$ in each of these Gale diagrams, then Figures 4.2 and 4.3 show the graphs of the polytopes with each of these Gale diagrams. Let P be a polytope with the Gale diagram in Figure 4.2, and Q be a polytope with the Gale diagram in Figure 4.3. Note that conv $\{2,5\}$ is not an edge in P, whereas in Q it is. Hence, P and Q cannot be combinatorially equivalent.

The facets of P are the 3-dimensional pyramids 23456 and 12345 each with base 2345, as well as the following 3-simplices.

$11'45$	$11'35$	$11'24$	$11'23$
$1'456$	$1'356$	$1'246$	$1'236$

The facets of Q are the triangular bipyramid 23456 with base 256, as well as the following 3-simplices.

$11'45$	$11'35$	$11'24$	$11'23$
$1'456$	$1'356$	$1'246$	$1'236$
1245	1235		

4.2.2 Duplicating a Point in a Gale Diagram

On the other hand, an operation that is well defined is that of adding a duplicate of an existing point to a Gale diagram. Let P be a d-polytope with vertex set vert $P = \{v_1, v_2, \ldots, v_n\}$. Let Γ be a Gale diagram of P, and let Q be a $(d+1)$-polytope with vertex set vert $Q = \{w_{1'}, w_1, w_2, \ldots, w_n\}$ whose Gale diagram Λ satisfies $\overline{w}_{1'} = \overline{v}_1$ and $\overline{w}_i = \overline{v}_i$ otherwise.

Let Y be a cofacet of Q, that is, a minimal coface and consider the following four cases:

1. ($\{\boldsymbol{w}_1, \boldsymbol{w}_{1'}\} \subseteq Y$) This case cannot happen since the set $\overline{Y} \setminus \{\overline{\boldsymbol{w}}_1\}$ captures the origin, and $Y \setminus \{\boldsymbol{w}_1\} \subsetneq Y$.

2. ($\boldsymbol{w}_1 \in Y$ and $\boldsymbol{w}_{1'} \notin Y$) For this case, let $X = \{\boldsymbol{v}_i \mid \boldsymbol{w}_i \in Y\}$. Then since $\overline{Y} = \overline{X}$, the set Y is a cofacet if and only if X is a cofacet.

3. ($\boldsymbol{w}_1 \notin Y$ and $\boldsymbol{w}_{1'} \in Y$) Here, set $X = \{\boldsymbol{v}_i \mid \boldsymbol{w}_i \in Y \text{ or } i = 1\}$. In this case, $\overline{Y} = \overline{X}$ again, and so Y is a cofacet if and only if X is a cofacet.

4. ($\{\boldsymbol{w}_1, \boldsymbol{w}_{1'}\} \cap Y = \varnothing$) For the final case, again set $X = \{\boldsymbol{v}_i \mid \boldsymbol{w}_i \in Y\}$. Once more, $\overline{Y} = \overline{X}$, so that Y is a cofacet if and only if X is a cofacet.

Thus the cofacets (and hence the facets) of Q are determined by the cofacets (and hence the cofacets) of P and vice versa. Geometrically, Q is combinatorially equivalent to a polytope with the following vertices:

$$\boldsymbol{w}_1 = \begin{bmatrix} \boldsymbol{v}_1 \\ 1 \end{bmatrix} \qquad \boldsymbol{w}_{1'} = \begin{bmatrix} \boldsymbol{v}_1 \\ -1 \end{bmatrix} \qquad \boldsymbol{w}_j = \begin{bmatrix} \boldsymbol{v}_j \\ 0 \end{bmatrix}, j \in [n] \setminus \{1\}$$

Conceptually, first place P in $\mathbb{R}^{d+1}$. Then the vertex $\boldsymbol{v}_1$ is replaced by a line segment whose relative interior passes through $\boldsymbol{v}_1$, and whose affine hull intersects that of the original polytope at precisely the point $\boldsymbol{v}_1$.

If P is a 2-polytope with n vertices, then performing this operation at any vertex yields a polytope that is combinatorially equivalent to performing the operation at any other vertex. The resulting polytope is combinatorially equivalent to the cyclic polytope $C_3(n+1)$. See Figure 4.4 for an example. This process is called a *simplicial wedge* in Ayzenberg (2013).
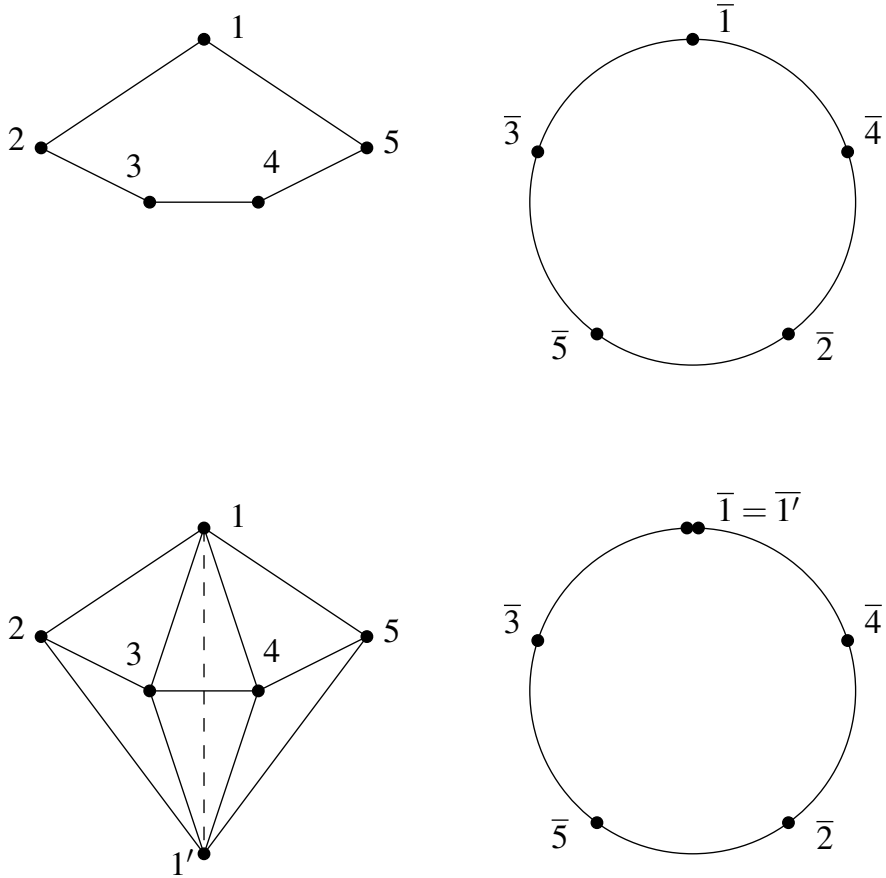
Figure 4.4: Duplicating the point $\overline{1}$ in a Gale diagram of $C_5(2)$.

Chapter 5

Gale Polytopes

Recall Gale's evenness condition, which exactly describes the facets of the cyclic polytope $C_d(n)$ of dimension d with n vertices.

Gale's Evenness Condition. *A subset $S \subseteq [n]$ with $|S| = d$ forms a facet of $C_d(n)$ if and only if*

$$|\{k \mid k \in S \text{ and } i < k < j\}| \text{ is even for all } i < j, \{i, j\} \cap S = \varnothing$$

It is reasonable to ask if a noncyclic polytope satisfies a weakened form of Gale's evenness condition, namely:

Definition. A d-polytope P with vertex set $\text{vert}(P) = \{v_1, v_2, \ldots, v_n\}$ is called a *Gale* polytope if there is an ordering of $\text{vert}(P)$ such that each facet satisfies the following property:

If $S \subseteq [n]$ and F is a facet of P with $\text{vert}(F) = \{v_s \mid s \in S\}$, then

$$|\{k \mid k \in S \text{ and } i < k < j\}| \text{ is even for all } i < j, \{i, j\} \cap S = \varnothing.$$

An equivalent characterization is as follows:

Let P be a polytope, $v_1, v_2, \ldots, v_n$ be an ordering of $\text{vert}(P)$ and $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq \text{vert}(P)$ be such that $i_1 < i_2 < \cdots < i_k$. Then a *block* of S is a subset C of S such that if $t = \min\{i_j \mid v_{i_j} \in C\}$,

then there is some $r \in \mathbb{N}$ such that $C = \{v_t, v_{t+1}, \ldots, v_{t+r-1}\}$ and $v_{t+r} \notin S$. In this case, the *length* of a block is $|C| = r$.

Definition. A polytope P is *Gale* if there is an ordering of $\mathrm{vert}(P)$ such that if F is a facet of P, then for each block G of $\mathrm{vert}\,F$ one of the following holds:

1. the length of G is even; or

2. $v_1 \in G$; or

3. $v_n \in G$.

Such an ordering of the vertices is called a *Gale ordering*. A block which contains v_1 is called an *initial block*. A block which contains v_n is called a *terminal block*. A block which is neither initial, nor terminal is called *internal*.

In terms of blocks, a polytope is Gale if and only if there is an ordering of the vertices such that in each facet the only blocks of odd length are either initial, or terminal. Note that a facet need not have either an initial block, or a terminal block.

An immediate consequence of either definition is: If P is a Gale polytope with n vertices, and $v_1, v_2, \ldots, v_n$ is a Gale ordering of $\mathrm{vert}(P)$, then so is $v_n, v_{n-1}, \ldots, v_1$. Note that all cyclic polytopes are Gale, and therefore all polytopes of dimension less than or equal to 2 are Gale.

Theorem 26. *If P is a Gale polytope, then so is* $\mathrm{pyr}(P)$, *a pyramid over P.*

Proof. Let $v_1, v_2, \ldots, v_n$ be a Gale ordering of $\mathrm{vert}(P)$ and $w_1, w_2, \ldots, w_{n+1}$ be the ordering of $\mathrm{vert}(\mathrm{pyr}(P))$ with w_i corresponding to v_i for $i \in [n]$, and with w_{n+1} as the apex of the pyramid.

Let F be a face of $\mathrm{pyr}(P)$. Then either $\mathrm{vert}(F) = \{w_1, w_2, \ldots, w_n\}$, or $w_{n+1} \in \mathrm{vert}(F)$. In the first case, F contains only an initial block.

In the second case, write $\mathrm{vert}(F) = \{w_{i_1}, w_{i_2}, \ldots, w_{i_k}, w_{n+1}\}$ with $i_1 < i_2 < \cdots < i_k < n+1$. Then either $i_k = n$, or $i_k < n$. If $i_k = n$, then each block C of $\{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$ satisfies at least one of the following: C is internal of even length; C is initial; or $w_n \in C$. Thus, after adding in

61

$\boldsymbol{w}_{n+1}$, each block C' satisfies one of the following respectively: C' is internal of even length; C' is initial; or C' is terminal. In the case that $i_k < n$, each block of $\{\boldsymbol{w}_{i_1}, \boldsymbol{w}_{i_2}, \ldots, \boldsymbol{w}_{i_k}\}$ is either initial, or is of even length. Ergo, when $\boldsymbol{w}_{n+1}$ is added the only change is that a terminal block of length 1 is added. $\qquad\square$

Theorem 27. *Suppose P is a Gale polytope, and let X be a collection of pairwise disjoint facets that each have an odd number of vertices. Then $|X| \leq 2$.*

Proof. Let P be a Gale polytope, and F be a facet of P which has an odd number of vertices.

Consider the blocks of F. since F has an odd number of vertices, it must have either an initial or terminal block. If this were not the case, then there would be an internal block with an odd number of vertices. $\qquad\square$

Example 28. The above theorem shows that both the regular dodecahedron and regular icosahedron are not Gale polytopes. The shaded facets in Figure 5.1 are all disjoint with an odd number of vertices, and there are more than two in each polytope.
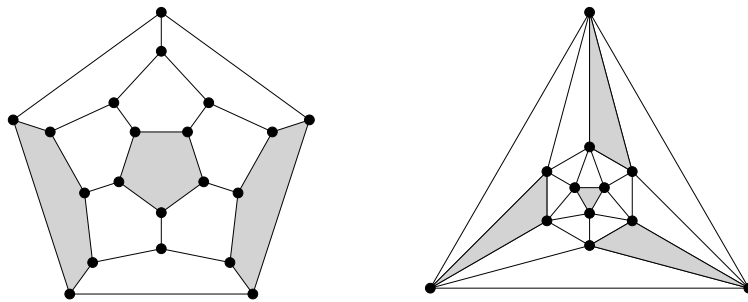
Figure 5.1: The regular dodecahedron and icosahedron are not Gale polytopes.

If P is a 3-polytope with five or fewer vertices then it is combinatorially equivalent to one of Δ_3, $\mathrm{pyr}(C_4(2))$, or $C_5(3)$ each of which is Gale. However, there is a 3-polytope with six vertices which is not Gale. In order to demonstrate this, we will use Theorem 30 which is a consequence of Lemma 29. For a proof, see Klee & Minty (1972).

Lemma 29 (Klee, Minty 1972)**.** *Let X and Y be polytopes having the same number m of vertices, the vertices of X being $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ and those of Y being $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m$. Suppose that for each*

index set $I \subseteq [m]$,

$$\text{conv}\{x_i \mid i \in I\} \text{ is a facet of } X \text{ implies } \text{conv}\{y_i \mid i \in I\} \text{ is a facet of } Y.$$

Then the reverse implications hold, whence X and Y are combinatorially equivalent.

The following theorem is found in Bayer & Bisztriczky (2007).

Theorem 30. *If P is a simplicial Gale polytope, then P is a cyclic polytope.*

Proof. First, recall that if $\text{vert}(C_d(n)) = \{v_1, v_2, \ldots, v_n\}$, then the set facets of $C_d(n)$ is

$$X = \left\{ \text{conv}\left\{v_{i_1}, v_{i_2}, \ldots, v_{i_{d+1}}\right\} \mid \left\{v_{i_1}, v_{i_2}, \ldots, v_{i_{d+1}}\right\} \text{ satisfies Gale's evenness condition}\right\}.$$

Let P be a simplicial Gale polytope of dimension d with $|\text{vert}\,P| = n$. Then there is some ordering $w_1, w_2, \ldots, w_n$ of $\text{vert}\,P$ such that if $\text{conv}\{w_1, w_2, \ldots, w_{i_d}\}$ is a facet of P, then the set $\text{conv}\{v_{i_1}, v_{i_2}, \ldots, v_{i_d}\}$ is a facet of $C_d(n)$ since the set of facets of $C_d(n)$ is X.

Thus P and $C_d(n)$ satisfy the hypothesis of Lemma 29. Hence P is combinatorially equivalent to $C_d(n)$. $\qquad\square$

Thus, for example, X_3 is not a Gale polytope since each vertex of X_3 lies on four edges, whereas $C_6(3)$ has two vertices which each lie on only three edges.

Since simplices are cyclic polytopes, and cyclic polytopes are simplicial, each facet of a cyclic polytope satisfies Gale's evenness condition. However, this need not happen for a general Gale polytope. The following is an example of a Gale polytope with a facet which is not Gale.

Example 31. Consider the 5-dimensional cyclic polytope with vertices $x_1, x_2, x_3, x_4, x_5, x_6, x_7$. Let P be a vertex figure of $C_5(7)$ at x_4. Then P is a 4-dimensional simplicial polytope with 6 vertices $z_1, z_2, z_3, z_5, z_6, z_7$ (where z_i corresponds to the edge $x_4 x_i$).

The facets of P are the convex hulls of the following sets:

$$\{z_1, z_2, z_3, z_5\} \qquad \{z_1, z_2, z_3, z_7\} \qquad \{z_1, z_2, z_5, z_7\} \qquad \{z_1, z_3, z_5, z_6\}$$

$$\{z_1, z_3, z_6, z_7\} \qquad \{z_1, z_5, z_6, z_7\} \qquad \{z_2, z_3, z_5, z_7\} \qquad \{z_3, z_5, z_6, z_7\}.$$

Let Q be a pyramid over P with apex $\mathbf{a}$. Then Q is a Gale polytope with Gale ordering

$$z_1, z_2, z_3, \mathbf{a}, z_5, z_6, z_7.$$

However, P (which is the base of the pyramid Q, and therefore also a facet of Q) is not Gale since it is a simplicial 4-polytope whose graph is not complete (it is missing the edge conv $\{z_2, z_6\}$) and therefore P is not cyclic.

Notice that this construction generalizes to give infinitely many polytopes with this property. Consider the cyclic polytope $C_d(2n+1)$ (of dimension at least 4) with an odd number of vertices. Let P be a vertex figure of $C_d(2n+1)$ at the vertex $n+1$, and Q be a pyramid over Q.

Theorem 32. *If P is a Gale d-polytope, and Q is a Gale d'-polytope, then $P \times Q$ is a Gale $(d+d')$-polytope.*

Proof. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a Gale ordering of $\text{vert}(P)$, and $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m$ is a Gale ordering of $\text{vert} Q$. Then define an ordering of $\text{vert}(P \times Q)$ as follows:

Let $(V_1, \mathbf{w}_i)$ and $(V_{-1}, \mathbf{w}_i)$ respectively be the sequences:

$$(\mathbf{v}_1, \mathbf{w}_i), (\mathbf{v}_2, \mathbf{w}_i), \ldots, (\mathbf{v}_n, \mathbf{w}_i)$$

$$(\mathbf{v}_n, \mathbf{w}_i), (\mathbf{v}_{n-1}, \mathbf{w}_i), \ldots, (\mathbf{v}_1, \mathbf{w}_i).$$

Define similarly, $(B_1, \mathbf{w}_i)$ and $(B_{-1}, \mathbf{w}_i)$ for B a set of vertices of P.

Claim: $(V_1, \mathbf{w}_0), (V_{-1}, \mathbf{w}_1), \ldots, (V_{(-1)^m}, \mathbf{w}_m)$ is a Gale ordering of $P \times Q$.

First, consider the facets of $P \times Q$ of the form $P \times G$ where G is a facet of Q. The vertex $\mathbf{w}_j$ is a

vertex of G if and only if $(\boldsymbol{v}_i, \boldsymbol{w}_j)$ is a vertex of the facet $P \times G$ for each $\boldsymbol{v}_i \in \mathrm{vert}(P)$. In particular, the sequence $(V_{(-1)^j}, \boldsymbol{w}_j)$ is part of a block of $P \times G$.

If n is even, then each of these parts is of even length, and hence every block of $P \times G$ is even.

If n is odd, then consider the blocks of G. If G has an initial block $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_t$, then $P \times G$ has initial block $(V_1, \boldsymbol{w}_0), (V_{-1}, \boldsymbol{w}_1), \ldots, (V_{(-1)^t}, \boldsymbol{w}_t)$ (since $(\boldsymbol{v}_1, \boldsymbol{w}_{t+1})$ is not a vertex of $P \times G$). A similar statement applies for terminal blocks.

Suppose that G has an internal block $\boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_{k+s-1}$. Since this is an internal block, s must be even. Thus the sequence $(V_{(-1)^k}, \boldsymbol{w}_k), (V_{(-1)^{k+1}}, \boldsymbol{w}_{k+1}), \ldots, (V_{(-1)^{k+s-1}}, \boldsymbol{w}_{k+s-1})$ is an internal block of $P \times G$ of length $|\mathrm{vert}(P)| \cdot s$ (this is an even number). This follows since each element of the above sequence is a vertex of $P \times G$, and none of the vertices

$$(\boldsymbol{v}_1, \boldsymbol{w}_{k-1}), (\boldsymbol{v}_n, \boldsymbol{w}_{k-1}), (\boldsymbol{v}_n, \boldsymbol{w}_{k+s}), (\boldsymbol{v}_1, \boldsymbol{w}_{k+s})$$

is a vertex of $P \times G$.

Now, consider the facets of $P \times Q$ of the form $F \times Q$ with F a facet of P. In this case, similar to above, $\boldsymbol{v}_i \in F$ if and only if $(\boldsymbol{v}_i, \boldsymbol{w}_j) \in F \times Q$ for each $j \in [m] \cup \{0\}$. The sequence $B = \boldsymbol{v}_i, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_{i+k-1}$ is part of a block of F if and only if

$$\begin{cases} (B_1, \boldsymbol{w}_j) = (\boldsymbol{v}_i, \boldsymbol{w}_j), (\boldsymbol{v}_{i+1}, \boldsymbol{w}_j), \ldots, (\boldsymbol{v}_{i+k-1}, \boldsymbol{w}_j) & , j \text{ even} \\ (B_{-1}, \boldsymbol{w}_j) = (\boldsymbol{v}_{i+k-1}, \boldsymbol{w}_j), (\boldsymbol{v}_{i+k-2}, w_j), \ldots, (\boldsymbol{v}_i, \boldsymbol{w}_j) & , j \text{ odd} \end{cases}$$

is part of a block of $F \times Q$. If B is an internal block of F, then the above are internal blocks of $F \times Q$.

Suppose that B is an initial block of F (in particular, $i = 1$, and B has length k). Then:

- $(B_1, \boldsymbol{w}_1)$ is an initial block of $F \times Q$;

- if m is odd, then $(B_{-1}, \boldsymbol{w}_m)$ is a terminal block of $F \times Q$;

- if m is even, then $(B_{-1}, \boldsymbol{w}_{m-1}), (B_1, \boldsymbol{w}_m)$ is an internal block of length $2k$.

Further, $(B_{-1}, \boldsymbol{w}_{2\ell-1}), (B_1, \boldsymbol{w}_{2\ell})$ is an internal block of length $2k$ for $\ell \in [\lfloor m/2 \rfloor]$.

The case that B is a terminal block is handled similarly. $\qquad\square$

Chapter 6

Anticliques in Graphs of Polytopes

Let $G = (V, E)$ be a graph. An *anticlique* is a subset A of V (of cardinality at least 2) such that $E \cap \binom{A}{2} = \varnothing$. If $|A| = k$, then A is said to be a k-anticlique. A 2-anticlique is simply a nonedge.

Recall that if G is the graph of a d-polytope P with n vertices, and $\Gamma \in \mathrm{gale}(P)$ is a Gale diagram of P, then $\{v_i, v_j\}$ is a nonedge of P if and only if there is some hyperplane $H_{i,j} \subseteq \mathbb{R}^{n-d-1}$ containing $\mathbf{0}$ such that $H_{i,j}^{(+)} \cap \Gamma = \{\bar{v}_i, \bar{v}_j\}$. Such a hyperplane is called a *separating* hyperplane. The normal vector to $H_{i,j}$ which has positive inner product with $\bar{v}_i, \bar{v}_j$ and norm 1 is denoted $x_{i,j}$.

If P is a d-polytope with $d+k$ vertices, then Theorem 10 in Section 2.4 implies that each vertex of $\mathscr{G}(P)$ must have degree at least d. If a graph H with $d+k$ vertices has a $(k+1)$-anticlique, then each vertex in the anticlique has degree at most $d-1$, and hence H is not d-realizable. However, the graph $\left([d+k], \binom{[d+k]}{2} \setminus \binom{[k]}{2}\right)$ is d-connected, and furthermore has a K_{d+1} minor (the induced subgraph on the vertex set $\{k, k+1, \ldots, k+d\}$ is complete) and thus *could* be the graph of a d-polytope. It will be shown that this cannot happen. More precisely for $k \geq 2$, let

$$f(d,k) = \max \{n \mid \text{there is some } d\text{-polytope } P \text{ with } d+k \text{ vertices and an } n\text{-anticlique}\}.$$

Then the goal is to show that $f(d,k) < k$ for $k > 2$.

6.1 An Upper Bound on f

First note that f is weakly increasing in both arguments.

Theorem 33.

1. $f(d,k) \leq f(d+1,k)$

2. $f(d,k) \leq f(d,k+1)$

Proof. Let P be a d-polytope with $d+k$ vertices, and an $f(d,k)$-anticlique.

1. The $(d+1)$-polytope $\mathrm{pyr}(P)$ has $(d+k)+1 = (d+1)+k$ vertices and an $f(d,k)$-anticlique.

2. Let F be any facet of P. The d-polytope $K(P;F)$ (defined in section 1.4.8) has $d+(k+1)$ vertices and an $f(d,k)$-anticlique.

$\square$

Recall that a standard Gale diagram is one in which each point is either on the unit sphere, or at the origin. Further if a point in the Gale diagram is at the origin, then the polytope is a pyramid with apex the corresponding point. Since the main question is concerned with sizes of anticliques, and the anticliques of a pyramid are exactly those of its base, nothing is lost or gained in the assumption that a polytope is not a pyramid.

Theorem 34. *Let P be a d-polytope with vertex set $\{v_1, v_2, \ldots, v_n\}$; $A = \{i_1, i_2, \ldots, i_k\} \subseteq [n]$; $\{v_i \mid i \in A\}$ be an anticlique; Γ be a standard Gale diagram of P; and $\{i, j\} \subseteq A$. Then*

1. $\bar{v}_i \neq \mathbf{0}$.

2. $\bar{v}_i \neq -\bar{v}_j$.

Furthermore if $k \geq 3$, then

3. *if $\bar{v}_i = \bar{v}_j$, then $i = j$.*

Proof.

1. If $\bar{v}_i = \mathbf{0}$, then conv $\{v_i, v_t\}$ is a face of P for each $t \in [n] \setminus \{i\}$. In particular, conv $\{v_i, v_j\}$ is an edge of $G(P)$.

2. If $\bar{v}_i = -\bar{v}_j$, and $H_{i,j}$ is a separating hyperplane for $\bar{v}_i, \bar{v}_j$ with normal vector $x_{i,j}$, then

$$0 < \langle x_{i,j}, \bar{v}_i \rangle = -\langle x_{i,j}, \bar{v}_j \rangle < 0.$$

3. Let $H_{a,b}$ be a separating hyperplane for $\bar{v}_a, \bar{v}_b$ with normal $x_{a,b}$ for each pair $a, b \in A$. Suppose $\bar{v}_i = \bar{v}_j$ with $i \neq j$. Then for $r \in A \setminus \{i, j\}$,

$$0 < \langle x_{j,r}, \bar{v}_j \rangle = \langle x_{j,r}, \bar{v}_i \rangle \leq 0.$$

Hence $i = j$. $\qquad\square$

Theorem 35. $f(d, 2) = 2$

Proof. If P is a d-polytope with $d + 2$ vertices, then P has a 1-dimensional Gale diagram. Since a nonedge requires that there be a separating hyperplane, and in this case there is only one possible hyperplane (namely the origin), each vertex can have at most one nonneighbor. This implies that $f(d, 2) \leq 2$. The polytope $\mathrm{pyr}_{d-2}(X_2)$ is d-dimensional with $d + 2$ vertices and a 2-anticlique. Thus $f(d, 2) = 2$. $\qquad\square$

Theorem 36. $f(d, 3) < 3$

Proof. Let P be a d-polytope with $d + 3$ vertices, and let Γ be a standard Gale diagram of P. Suppose that P has a 3-anticlique $\{v_1, v_2, v_3\}$, and let v_4 be any other vertex of P. Note that since $\Gamma \subseteq \mathbb{S}^1$ (assuming again that P is not a pyramid), by writing each $\bar{v}_i \in \Gamma$ in polar coordinates only the angle $\varphi_i \in [0, 2\pi)$ is necessary to specify $\bar{v}_i$. By Theorem 34 part 3, none of $\varphi_1, \varphi_2, \varphi_3$ are equal, so assume $\varphi_1 < \varphi_2 < \varphi_3$. Further, if $\varphi_1 > 0$, then rotating each point in Γ clockwise by an angle of φ_1 produces a consubstantial standard Gale diagram, so assume

$$0 = \varphi_1 < \varphi_2 < \varphi_3 < 2\pi.$$

69
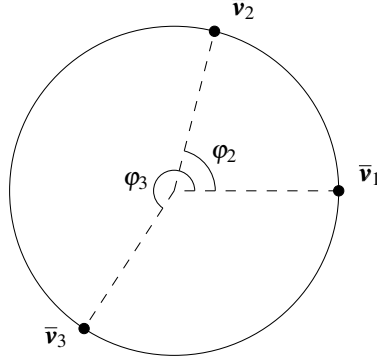
Since $\boldsymbol{v}_1\boldsymbol{v}_2$ is not an edge of P, there is some hyperplane H_3 through the origin such that $H_3^{(+)} \cap \Gamma = \{\bar{\boldsymbol{v}}_1, \bar{\boldsymbol{v}}_2\}$. Therefore, $\varphi_4 \notin [0, \varphi_2]$. Similarly, $\varphi_4 \notin [\varphi_2, \varphi_3]$, and $\varphi_4 \notin [\varphi_3, 2\pi)$. Since Γ is a standard Gale diagram, this means that $\bar{\boldsymbol{v}}_4 = \boldsymbol{0}$. But this contradicts the assumption that P is not a pyramid. $\qquad \square$

Lemma 37. *If*

- *P is a d-polytope with vertex set $V \cup Y$ of cardinality $d + k$,*

- *$|V| = d$ and $|Y| = k$,*

- *Y is an anticlique of $\mathcal{G}(P)$, and*

- *$\boldsymbol{v} \in V$, then*

$\boldsymbol{v}\boldsymbol{y}$ is an edge of $\mathcal{G}(P)$ for each $y \in Y$.

Proof. Let $\boldsymbol{v} \in V$, $\boldsymbol{y} \in Y$. Since P is a d-polytope, it is d-connected, therefore each vertex of $G(P)$ is adjacent to at least d other vertices. But since Y is an anticlique, $\boldsymbol{y}$ is adjacent to at most d other vertices. Hence $\boldsymbol{y}$ is adjacent to exactly d other vertices, and since none of these other vertices is in Y, $\boldsymbol{v}\boldsymbol{y}$ is an edge of $G(P)$. $\qquad \square$

The following is a general fact about polytopes. A proof is included for completeness, but

Lemma 38. *Let P be a d-polytope, $y \in \text{vert}\,P$ with $\deg y = d$, and $V = N(y) = \{v_1, v_2, \ldots, v_d\}$. Then for each subset X of V with cardinality $d - 1$ there is a unique facet F of P with $X \subseteq \text{vert}\,F$.*

Proof. Each vertex of P lies in at least d facets since P is a d-polytope. Let F be a facet containing y. Since F is a $(d-1)$-polytope, each vertex of $G(F)$ has degree at least $d - 1$. Thus y must have either $d - 1$ or d neighbors in F. If y had each of its d neighbors in F, then y would only lie in one facet. Thus each facet in which y lies contains exactly $d - 1$ neighbors of y. Further, each element of $\binom{V}{d-1}$ determines a unique facet of P since y lies in at least d facets. $\qquad\square$

Notice that this argument also shows that the set V above is affinely independent, and that V is not contained in any proper face of P.

Theorem 39. *If $k > 2$, then $f(d, k) < k$.*

Proof. Proceed by induction on d. For the base case: the graph of a 2-polytope with n vertices is a cycle of length n, so a maximal anticlique can be obtained by taking every other vertex. Therefore $f(2, k) = 1 + \lfloor k/2 \rfloor < k$. Thus, suppose for some d that $f(d - 1, k) < k$ for all $k > 2$. The proof of the contrapositive will be shown, that is, if $f(d, k) = k$, then $k = 2$.

Let P be a d-polytope with $d + k$ vertices and, without loss of generality, suppose that P is not a pyramid. Suppose further, that P has a k-anticlique $Y = \{y_1, y_2, \ldots, y_k\}$. Write $\text{vert}(P) = \{v_1, v_2, \ldots, v_d\} \cup Y$, let $V = \{v_1, v_2, \ldots, v_d\}$, and $V_i = V \setminus \{v_i\}$ for $i \in [d]$. Fix some $i \in [d]$, and for each $j \in [k]$, let $F_{i,j}$ be the facet of P which contains $V_i \cup \{y_j\}$ (Lemmata 37 and 38), and G_i be the smallest face of P which contains V_i.

Notice that $G_i \subseteq F_{i,j}$ for each $j \in [k]$, and that G_i is either a facet or a ridge since V is an affinely independent set.

1. (G_i is a facet.) In this case, for each $j \in [k]$, $G_i = F_{i,j} = \text{conv}(Y \cup V_i)$. Thus P is a pyramid over G_i with apex v_i. This is a contradiction.

2. (G_i is a ridge.) Let F_1, F_2 be the facets containing G_i. In this case, Y can be partitioned into

two nonempty sets

$$Y_1 = \left\{ \mathbf{y}_{a_1}, \mathbf{y}_{a_2}, \ldots, \mathbf{y}_{a_r} \right\} \qquad\qquad Y_2 = \left\{ \mathbf{y}_{b_1}, \mathbf{y}_{b_2}, \ldots, \mathbf{y}_{b_s} \right\}$$

such that

$$Y_1 \subseteq F_{i,a_1} = F_1 \qquad\qquad Y_2 \subseteq F_{i,b_1} = F_2.$$

If $|Y_1| = 2$, then F_1 is a $(d-1)$-polytope with $(d-1)+2$ vertices and G_i is a facet of F_1. Thus a Gale diagram of F_1 is one dimensional with $d+1$ vertices, and both $\bar{\mathbf{y}}_{a_1}$ and $\bar{\mathbf{y}}_{a_2}$ are on the same side of the origin. However, $\mathbf{0} \in \operatorname{relint} \operatorname{conv} \left(\overline{\operatorname{vert}(F_1) \setminus Y_1} \right)$. Similarly, $|Y_2| \neq 2$.

If $|Y_1| \geq 3$, then F_{i,a_1} is a $(d-1)$-polytope with $(d-1)+|Y_1|$ vertices, and a $|Y_1|$-anticlique, that is, $f(d-1, |Y_1|) \geq |Y_1|$. This contradicts the inductive hypothesis.

Hence $|Y_1| = |Y_2| = 1$, whence P is a d-polytope with $d+2$ vertices and a 2-anticlique. Thence $k = 2$.

$\square$

6.2   A Lower Bound on f

Lemma 40. *If*

- *P is a d-polytope with $d+k$ vertices,*

- *$\mathbf{v}$ is a vertex of P such that each facet containing $\mathbf{v}$ is a simplex, and*

- *$\mathbf{v}$ is contained in a q-anticlique of P, then*

$f(d,k+i) \geq q+i-1 \text{ for } i \in [d].$

For example, such a vertex exists when P is a simplex (in which case each vertex has this property) or if P is of the form $P = K(P';F)$ where F is a facet of P' which is a simplex, so the Lemma is not vacuous.

Proof. Let $F_1, F_2, \ldots, F_d$ be facets of P containing $\boldsymbol{v}$, and set

$$P_0(\boldsymbol{v}) = P$$

$$P_i(\boldsymbol{v}) = K(P; F_1, F_2, \ldots, F_i)$$

$$\{\boldsymbol{v}_i\} = \text{vert}(P_i(\boldsymbol{v})) \setminus \text{vert}(P_{i-1}(\boldsymbol{v})).$$

Further, let A' be a q-anticlique of P which contains $\boldsymbol{v}$, and set $A = A' \setminus \{\boldsymbol{v}\}$.

For a fixed $i \in [d]$, the inclusion $N(\boldsymbol{v}_i) \subseteq N(\boldsymbol{v}) \cup \{\boldsymbol{v}\}$ in $\mathscr{G}(P_i(\boldsymbol{v}))$ implies that $A \cup \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_i\}$ is a $(q-i-1)$-anticlique in $\mathscr{G}(P_i(\boldsymbol{v}))$. $\square$

Theorem 41. *If $k, d > 2$, then $k - 1 - \left\lfloor \dfrac{k-3}{d} \right\rfloor \leq f(d, k) \leq k - 1$.*

Proof. Fix $d > 2$ and set $t = \lfloor (k-3)/d \rfloor$ and let $Q_0 = K(\Delta_d; F)$ for some facet F of Δ_d. Let $\boldsymbol{v}_0$ be a vertex of Q_0 and $F_{0,1}, F_{0,2}, \ldots, F_{0,d}$ be the facets of Q_0 which contain $\boldsymbol{v}_0$. Then for $k \in [d]$, the polytope $K(Q_0; F_{0,1}, F_{0,2}, \ldots, F_{0,k})$ is a d-polytope with $d + 2 + k$ vertices and a $(k+1)$-anticlique. Now, inductively define, for $n \in \mathbb{N} \setminus \{0\}$, the polytope

$$Q_n = K(Q_{n-1}; F_{n-1,1}, F_{n-1,2}, \ldots, F_{n-1,d})$$

where $\{F_{n-1,1}, F_{n-1,2}, \ldots, F_{n-1,d}\}$ is the set of facets that contain some fixed vertex in a maximal anticlique of Q_{n-1}.

Finally, for $k \in [d]$ the polytope $K(Q_n; F_{n,1}, F_{n,2}, \ldots, F_{n,k})$ is a d-polytope with $d + (nd + 2 + k)$ vertices and an anticlique of cardinality $nd + 2 + k - n - 1$. $\square$

Notice that if $\lfloor (k-3)/d \rfloor = 0$ (that is, $k \leq d+2$), then the upper and lower bounds on f agree.

6.3 The Value of f in Dimension 3

If $d = 3$, then Theorem 41 says that $k - 1 - \lfloor (k-3)/3 \rfloor \leq f(3,k) \leq k - 1$. In this case, the lower bound can be rewritten as $k - 1 - \lfloor (k-3)/3 \rfloor = \lceil 2k/3 \rceil$.

6.3.1 Euler's Theorem

Euler's Theorem states that the f-vector of a d-polytope lies on a certain hyperplane in $\mathbb{R}^d$. For a proof, see Grünbaum (2003), McMullen & Shephard (1971), or Ziegler (1995). All that will be needed is the $d = 3$ case:

Theorem 42 (Euler). *If P is a 3-polytope, then $f_0(P) - f_1(P) + f_2(P) = 2$.*

Euler's Theorem can be proven for planar graphs in general; in this case f_2 counts the number of regions into which the graph divides the plane.

Since each region of a planar graph is bounded by at least 3 edges, and each edge lies on at most 2 facets, $3R \leq 2|E(G)|$ where R is the number of regions into which the graph divides the plane. If G is the graph of a polytope P, then $R = f_2(P)$. If the number of edges bounding a region is at least 4, then this inequality becomes $4R \leq 2|E(G)|$, i.e., $2R \leq |E(G)|$. Combining this with Euler's Theorem yields $|E(G)| \leq 3|V(G)| - 6$ for a general planar graph, and $|E(G)| \leq 2|V(G)| - 4$ for a planar graph with no 3-cycles. In particular, if G is bipartite, then it has no 3-cycles.

6.3.2 Bipartite Graphs Induced by Anticliques

If P is a 3-polytope and A is an anticlique of $G = \mathscr{G}(P)$, then A induces a bipartite graph G_A whose vertex set is that of G, and whose edge set is the set of edges in G which contain a vertex of the anticlique A. That is:

$$V(G_A) = V(G)$$

$$E(G_A) = \{ax \mid a \in A\} \cap E(G).$$

Note that G_A is planar since G is planar, and G_A is a subgraph of G.

Theorem 43. $f(3,k) = \lceil 2k/3 \rceil$

Proof. Suppose that P is a 3-polytope with $3+k$ vertices and a $(\lceil 2k/3 \rceil + 1)$-anticlique A. Let $G = \mathscr{G}(P)$ and $a \in A$. Since each edge of G on which a lies is also an edge of G_A, and a lies on at least 3 edges of G, it follows that

$$|E(G_A)| \geq 3(\lceil 2k/3 \rceil + 1) \geq 3(2k/3 + 1) = 2k + 3$$

On the other hand, G_A is a bipartite planar graph with $3+k$ vertices. It therefore has at most $2(3+k) - 4 = 2k + 2$ edges. □

d \ k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
3	2	2	3	4	4	5	6	6	7	8	8	9	10	10	11
4	2	2	3	4	5	**5**	**6**	**7**	**8**	8	**9**	**10**	**11**	**11**	**12**
5	2	2	3	4	5	6	**6**	**7**	8	**9**	**10**	10	**11**	**12**	**13**
6	2	2	3	4	5	6	7	**7**	8	**9**	10	**11**	**12**	12	13
7	2	2	3	4	5	6	7	8	**8**	9	10	**11**	**12**	**13**	**14**
8	2	2	3	4	5	6	7	8	9	**9**	**10**	**11**	**12**	**13**	**14**
9	2	2	3	4	5	6	7	8	9	10	**10**	**11**	**12**	**13**	**14**
10	2	2	3	4	5	6	7	8	9	10	11	**11**	**12**	**13**	**14**
11	2	2	3	4	5	6	7	8	9	10	11	12	**12**	**13**	**14**

Table 6.1: Values of $f(d,k)$. Numbers in bold are lower bounds.

Chapter 7

Realizability of Complete Multipartite Graphs

The aim of this chapter is to answer the question, "When is a complete multipartite graph realizable?". A complete answer will not be given, but progress will be made toward an answer. In particular, a complete multipartite graph is the graph of a polytope if and only if if is either $K_{1,1}$ or $K_{2,2}$. A characterization of all of the possible 2- and 3-faces will also be given.

7.1 Hereditary Classes of Graphs

A class of graphs $\mathfrak{G}$ called a *hereditary* if $G \in \mathfrak{G}$, and H is an induced subgraph of G implies $H \in \mathfrak{G}$.

Example 44. Throughout these examples, $0 \notin \mathbb{N}$.

1. Let $q \in \mathbb{N}$. The class of discrete graphs with at most q vertices is hereditary, and is denoted $\mathfrak{D}_q = \{D_n \mid n \leq q\}$.

2. The class of all discrete graphs is hereditary, and is denoted $\mathfrak{D} = \{D_n \mid n \in \mathbb{N}\}$.

3. The class of all complete graphs is hereditary, and is denoted $\mathfrak{K} = \{K_n \mid n \in \mathbb{N}\}$.

4. The class of all complete bipartite graphs (along with the discrete graphs) is hereditary, and is denoted $\mathfrak{K}_2 = \{K_{n,m} \mid n,m \in \mathbb{N}\} \cup \mathfrak{D}$.

5. Let $3 \leq q \in \mathbb{N}$. The class of all complete multipartite graphs with fewer than q parts (along with the discrete graphs) is hereditary, and is denoted

$$\mathfrak{K}_q = \left\{ K_{n_1,\ldots,n_q} \mid n_i \in \mathbb{N} \text{ for each } i \in [q] \right\} \cup \mathfrak{K}_{q-1}.$$

6. Let $a_1 \leq a_2 \leq \cdots \leq a_m$ be a sequence of natural numbers. Then the class

$$\mathfrak{K}[a_1, a_2, \ldots, a_m] = \{K_{n_1,n_2,\ldots,n_m} \mid n_i \leq a_i\} \cup \mathfrak{K}_{m-1}$$

is hereditary.

7. Let $a_1 \leq a_2 \leq \cdots \leq a_m$ be a sequence of natural numbers. Then the class

$$\mathfrak{K}(a_1, a_2, \ldots, a_m) = \left\{ K_{n_1,n_2,\ldots,n_q} \mid q \leq m \text{ and } \exists i_1 < i_2 < \cdots < i_q \leq m \text{ with } n_{i_j} \leq a_j \right\} \cup \mathfrak{D}_{a_m}$$

is hereditary. More over it is exactly the set of induced subgraphs of the complete multipartite graph $K_{a_1,a_2,\ldots,a_m}$.

Theorem 45. *Suppose $\mathfrak{G}$ is a hereditary class of graphs and there is some $d \in \mathbb{N}$ such that no $G \in \mathfrak{G}$ is d-realizable. Then for every $d' > d$ there is no $G \in \mathfrak{G}$ which is d'-realizable.*

Proof. Suppose $\mathfrak{G}$ is a hereditary class of graphs and there is some $d \in \mathbb{N}$ such that no $G \in \mathfrak{G}$ is d-realizable. Suppose further that there is some $d' > d$ and a $G \in \mathfrak{G}$ such that G is d'-realizable.

Let P be any d'-polytope whose graph is G, and let F be any d-face of P. By Theorem 11, the graph of F is an induced subgraph of G. Since $G \in \mathfrak{G}$ and $\mathfrak{G}$ is hereditary, the graph of F must also be an element of $\mathfrak{G}$. However this contradicts the assumption that no graph in $\mathfrak{G}$ is d-realizable. $\qquad\square$

7.2 Complete Bipartite Graphs

This section will answer the question "For which values of n, m, d is the graph $K_{n,m}$ d-realizable?".

First, recall the convention that a complete multipartite graph $K_{n_1, n_2, \ldots, n_t}$ is written such that $n_1 \leq n_2 \leq \cdots \leq n_t$. Then it will be shown that the answer to the above question is, "only if $n = m = d \in \{1, 2\}$". These are the graphs of Δ_1 and X_2. Recall that the connectivity of a graph $\kappa(G)$ is the largest q such that G is q-connected.

Lemma 46. $\kappa(K_{n,m}) = n$.

Proof. Write $V(K_{n,m}) = P \cup Q$ with $P = \{p_1, p_2, \ldots, p_n\}$, $Q = \{q_1, q_2, \ldots, q_m\}$ and $E(K_{n,m}) = \{p_i q_j \mid p_i \in P \text{ and } q_j \in Q\}$. Then $\deg(q_j) = n$ for every $j \in [m]$ and $\deg(p_i) = m$ for every $i \in [n]$. Thus the minimum degree of a vertex of $K_{n,m}$ is $n = \min\{n, m\}$. Hence $\kappa(K_{n,m}) \leq n$.

To show that $\kappa(K_{n,m}) \geq n$, a set of n disjoint paths will be constructed for each pair of vertices of $K_{n,m}$. There are three cases: a pair of vertices from P; a pair of vertices from Q; and a pair with one vertex in P and the other in Q.

In the case that both vertices of the pair are in P, say p_i, p_j, the paths

$$p_i, q_t, p_j \qquad t \in [n]$$

are disjoint, and there are n of them.

In the case that both vertices of the pair are in Q, say q_i, q_j, the paths

$$q_i, p_t, q_j \qquad t \in [n]$$

are disjoint, and there are n of them.

For the case that the pair of vertices is of the form p_i, q_j with $p_i \in P$ and $q_j \in Q$, assume, without loss of generality, that $i = j = n$. Then the paths

$$p_n, q_t, p_t, q_n \qquad t \in [n-1]$$

are disjoint, and there are $n-1$ of them. These paths, together with the path p_n, q_n form n disjoint paths. $\qquad \square$

Theorem 47. $K_{n,m}$ is d-realizable if and only if $n = m = d \in \{1,2\}$.

Proof. Since $|V(K_{n,m})| \geq 2$, $K_{n,m}$ is not (-1)-, or 0-realizable.

Suppose $n = 1$. Then $\kappa(K_{1,m}) = 1$, and thus $K_{1,m}$ is realizable only if $d = 1$. There is only one combinatorial type of 1-polytope, and it has graph $K_{1,1}$.

Suppose $n = 2$. Then $\kappa(K_{2,m}) = 2$, and thus $K_{2,m}$ is realizable only if $d \leq 2$. However, $K_{2,n}$ is never 1-realizable, so if $K_{2,m}$ is to be d-realizable, then $d = 2$. Each 2-polytope has a graph which is a cycle, and thus each vertex has degree 2. The only complete bipartite graph for which each vertex has degree 2 is $K_{2,2}$, and it is the graph of X_2, the 2-crosspolytope.

Suppose $n \geq 3$. Then $K_{n,m}$ is at least 3-connected. Suppose that $K_{n,m}$ is 3-realizable. Steinitz's Theorem (Theorem 12) thus implies that $K_{n,m}$ must be planar. However this is not possible since $K_{n,m}$ has an induced $K_{3,3}$ for $n \geq 3$ (and therefore a $K_{3,3}$ minor). Thus $K_{n,m}$ is never 3-realizable.

Ergo, by Theorem 45, $K_{n,m}$ is not d-realizable for $d \geq 3$. $\qquad \square$

7.3 Complete 3-partite Graphs

The proof of the following lemma is similar to that of lemma 46.

Lemma 48. $\kappa(K_{1,n,m}) = n + 1$

Theorem 49. $K_{1,n,m}$ is d-realizable if and only if $K_{n,m}$ is $(d-1)$-realizable.

Proof. Suppose that $n = 1$. Then $\kappa(K_{1,1,m}) = 2$, and $K_{1,1,m}$ is only a cycle if $m = 1$. In which case it is the graph of Δ_2, the 2-simplex.

Suppose that $n = 2$. The graph $K_{1,2,2}$ is the graph of $\mathrm{pyr}\, X_2$. If $m \geq 3$, then $K_{1,2,m}$ cannot be the graph of a 3-polytope since it would then be a 3-polytope with $3 + m$ vertices and an m-anticlique (see Theorem 39). The graph $K_{1,2,m}$ also cannot be d-realizable for $d > 3$ since $\kappa(K_{1,2,m}) = 3$.

Suppose that $n = 3$. The graph $K_{1,3,m}$ then has an induced $K_{3,3}$, and therefore cannot be planar (hence is not 3-realizable). Furthermore, $K_{1,3,m}$ cannot be the graph of a 4-polytope since it has $1 + 3 + m = 4 + m$ vertices and an m-anticlique.

Suppose that $n \geq 4$, and write

$$V(K_{1,n,m}) = \{a\} \cup \{b_i \mid i \in [n]\} \cup \{c_i \mid i \in [m]\}.$$

where each set in the union above is one of the sets of vertices in the definition of a complete bipartite graph. Here again, $K_{1,n,m}$ has an induced $K_{3,3}$, and is therefore not 3-realizable. Suppose that $K_{1,n,m}$ is the graph of a 4-polytope P. Then the set of induced subgraphs of $K_{1,n,m}$ is $\mathfrak{K}(1,n,m)$ which is hereditary. The only graph in $\mathfrak{K}(1,n,m)$ that is 3-realizable is $K_{1,2,2}$, and therefore each facet of P must be a pyramid over a quadrilateral. In order to obtain an induced $K_{1,2,2}$ from $K_{1,n,m}$, the set of vertices must be of the form $\left\{a, b_{i_1}, b_{i_2}, c_{j_1}, c_{j_2}\right\}$. However this forces the vertex a to be in each facet, and this is impossible.

Thus no graph of the form $K_{1,n,m}$ is 4-realizable, and therefore no graph $K_{1,n,m}$ is d-realizable for $d \geq 4$. $\qquad\square$

The graphs $K_{2,2,m}$ are not 4-realizable since they would then have $2 + 2 + m = 4 + m$ vertices and an m-anticlique. More generally, $K_{2,n,m}$ is not $(2+n)$-realizable.

The $K_{2,n,m}$ case is more difficult. First, note that a graph of this form is only planar if $n = m = 2$, and in this case is the graph of X_3, the 3-crosspolytope.

Suppose that $K_{2,n,m}$ is the graph of some polytope P. Working with the hereditary class of graphs $\mathfrak{K}(2,n,m)$ (this is the set of induced subgraphs of $K_{2,n,m}$) shows that the 2-faces of P must be either 2-simplices, or 2-crosspolytopes. Similarly, the 3-faces of P can only be pyramids over quadrilaterals, or 3-crosspolytopes. Further, P has at most one facet that is combinatorially equivalent to X_3 since each X_3 facet must contain both vertices in the color class of cardinality 2. This restriction of the allowable facets does not show that there is no such polytope. A proof of the following theorem can be found in Perles & Shephard (1967).

Theorem 50 (Perles, Shephard). *If P is a d-polytope with $d+2$ or fewer vertices, then there is a $(d+1)$-polytope all of whose facets are combinatorially equivalent to P.*

The authors also prove, in the same paper, that for $d \geq 6$ there is no $(d+1)$-polytope all of whose facets are combinatorially equivalent to X_d.

Suppose P is a 4-polytope with graph $K_{2,3,3}$ and that P has a facet combinatorially equivalent to X_3. Then by a careful consideration of the possible ridges, it can be shown that there must be a ridge that is contained in only one facet. Therefore there is no 4-polytope with graph $K_{2,3,3}$, and hence no polytope with graph $K_{2,3,3}$. Alternatively, the two papers Altshuler & Steinberg (1984) and Altshuler & Steinberg (1985) provide a complete list of all 4-polytopes with 8 vertices. These papers show that any such polytope is either quasisimplicial (all of its facets are simplicial), or has a facet that is a simplex. Since a realization of $K_{2,3,3}$ could satisfy neither of these properties, there can be no such polytope.

7.3.1 Complete *k*-partite Graphs $(k > 3)$

In general, if P is a realization of a complete multipartite graph, then every one of its 2-faces must be combinatorially equivalent to either Δ_2 or X_2 since these are the only complete multipartite graphs that are cycles. Similarly, every one of its 3-faces must be combinatorially equivalent to one of the following:

- a 3-simplex, or

- a pyramid over X_2, or

- a bipyramid over a 2-simplex, or

- a 3-crosspolytope

since these are the only planar, 3-connected complete multipartite graphs.

A direct consequence of Theorem 39 is that if $K_{n_1,n_2,...,n_t}$ is d-realizable, then $d < \sum_{i \in [t-1]} n_i$. If $K_{n_1,n_2,...,n_t}$ is d-realizable as a polytope P, then $K_{1,n_1,n_2,...,n_t}$ is $d+1$-realizable as a pyramid over P.

The converse is, unfortunately, not true since $K_{1,1,1,2}$ is the graph of $C_3(5)$ (i.e., a bipyramid over a 2-simplex) but $K_{1,1,2}$ is not the graph of any polytope. Similarly, $K_{n_1,n_2,...,n_t,2}$ is realizable as a bipyramid over P.

Thus, if (as a set) $\{n_1, n_2, \ldots, n_t\} \subseteq \{1,2\}$, then $K_{n_1,n_2,...,n_t}$ is the graph of a polytope if and only if (as a multiset) $\{n_1, n_2, \ldots, n_t\}$ is not equal to either $\{1,2\}$ or $\{1,1,2\}$.

Conjecture. *If $K_{n_1,n_2,...,n_t}$ is the graph of a polytope, then $\{n_1, n_2, \ldots, n_t\} \subseteq \{1,2\}$ as sets,.*

Note that the conjecture does not make mention of the dimension of the polytope. This is because the graphs of higher-dimensional crosspolytopes can be realized in multiple dimensions.

7.4 Graphs of Crosspolytopes

Throughout this section, let G_n denote the graph of the n-dimensional crosspolytope for $n \geq 2$, that is

$$G_n = \mathscr{G}(X_n) = K_{\underbrace{2,2,\ldots,2}_{n}}.$$

The goal of this section will be to establish for which d the graph G_n is d-realizable.

The graphs G_2 and G_3 are realizable only in dimensions 2 and 3 respectively. Thus assume throughout that $n \geq 4$. The first thing to do is establish an upper bound on the values d such that G_n is d-realizable. Recall the Hadwiger number of a graph G is $h(G) = \max\{n \mid K_n \text{ is a minor of } G\}$. Theorem 8 then implies:

Corollary. *If a graph G is d-realizable, then $d \leq h(G) - 1$.*

Thus a natural question to ask is "What is $h(G_n)$?". This question was answered in Halin (1966).

Theorem 51 (Halin 1966).

$$h(G_n) = \left\lfloor \frac{3n}{2} \right\rfloor.$$

The following gives, for each n, an explicit construction of a maximal complete minor of G_n.

Write $V(G_n) = V_1 \cup V_2$ where $V_1 = \{v_1^1, v_2^1, \ldots, v_n^1\}$ and $V_2 = \{v_1^2, v_2^2, \ldots, v_n^2\}$ are disjoint sets such that

$$E(G_n) = \left\{ v_j^i v_l^k \;\middle|\; i,k \in [2], \, j,l \in [n], \text{ and if } i \neq k, \text{ then } j \neq l \right\}$$
$$= \binom{V(G_n)}{2} \setminus \left\{ v_j^1 v_j^2 \;\middle|\; j \in [n] \right\}.$$

Set

$$D = \left\{ v_j^i v_k^2 \;\middle|\; i \in [2], \, j,k > \lfloor n/2 \rfloor \right\} \cup \left\{ v_j^i v_k^2 \;\middle|\; i \in [2], \, j \leq \lfloor n/2 \rfloor \text{ and } k > \lfloor n/2 \rfloor \text{ with } j+k \neq n+1 \right\}$$

and

$$C = \left\{ v_i^2 v_j^2 \;\middle|\; i+j = n+1 \right\}.$$

Then $(G_n \setminus D)/C$ is a complete graph with $\lfloor 3n/2 \rfloor$ vertices. The paths

$v_i^1 v_j^1$	$i,j \in [n]$
$v_i^2 v_j^2$	$i,j \leq \lfloor n/2 \rfloor$
$v_i^1 v_j^2$	$i \in [n], \, j \leq \lfloor n/2 \rfloor$ with $i \neq j$
$v_i^1 v_{n+1-i}^2 v_i^2$	$i \leq \lfloor n/2 \rfloor$

contract to form the edges in the complete graph. Thus $h(G_n) \geq \lfloor 3n/2 \rfloor$. This construction is included here because Halin (1966) does not give an explicit construction of a maximal complete minor. See Figures 7.1 and 7.2 for examples of these constructions.
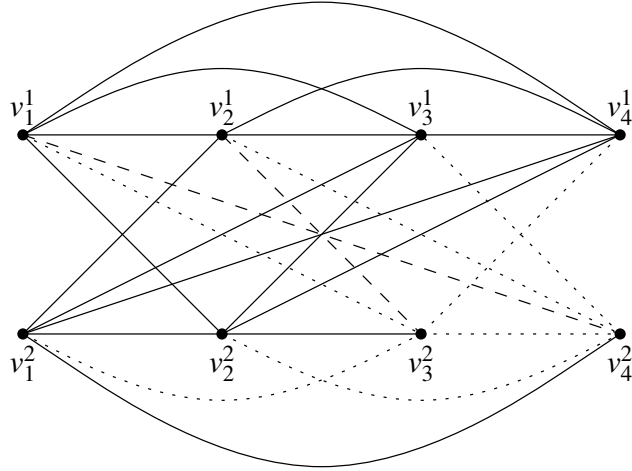
Figure 7.1: G_4, edges in the deletion set are dotted, and those in the contraction set are dashed.
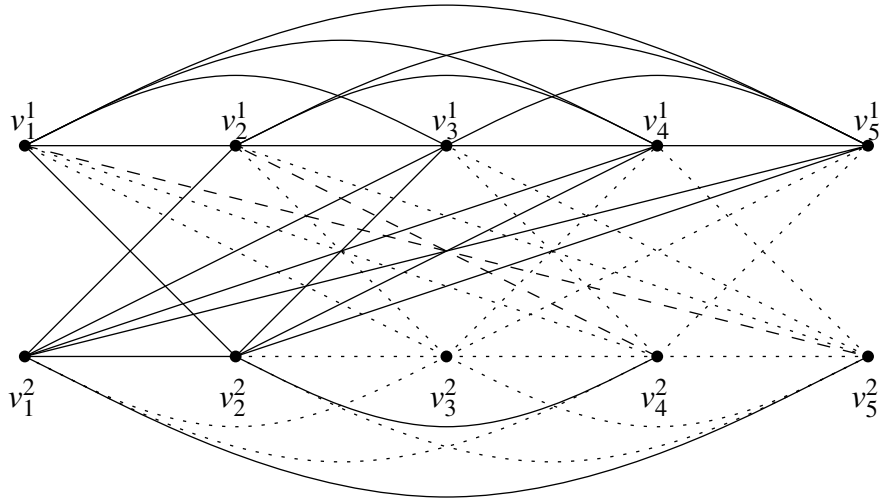
Figure 7.2: G_5, edges in the deletion set are dotted, and those in the contraction set are dashed.

7.4.1 High Dimensional Realizations

The graph G_n *could* thus be realizable in each dimension from 4 to $\lfloor 3n/2 \rfloor - 1$. Before giving explicit constructions of d-polytopes with graph G_n for $n \leq d < \lfloor 3n/2 \rfloor$, notice that if $p_i \in \mathbb{N}$ with $p_i \geq 2$ and $\sum_{i \in [m]} p_i = q$, then

$$\mathscr{G}(X_{p_1} \vee X_{p_2} \vee \cdots \vee X_{p_m}) = G_q$$

and

$$\dim(X_{p_1} \vee X_{p_2} \vee \cdots \vee X_{p_m}) = \sum_{i \in [m]} p_i + m - 1.$$

In particular, let X_2^k be the join of k copies of X_2, so that $\mathcal{G}(X_2^k) = G_{2k}$ and $\dim X_2^k = 3k - 1$. Thus the polytope $X_{3n-2d} \vee X_2^{d-n}$ is a d-polytope with graph G_n. These polytopes are by no means unique with these properties. Others can be constructed by similar means by taking joins of crosspolytopes (or even polytopes with these graphs) and forcing the joins to have the required number of vertices and be of the required dimension. That is:

Theorem 52. *The graph G_n is d-realizable for $n \le d < \left\lfloor \dfrac{3n}{2} \right\rfloor$.*

This settles the question of d-realizability of G_n for $d \ge n$. However, the number of combinatorial types of d-polytopes with graph G_n and $d \ge n$ is not known. These constructions merely provide a lower bound for that number.

The following is a Corollary to Theorems 39 and 51.

Corollary. *Let P be a simple d-polytope that is not a simplex. If $\mathcal{G}(P) = K_{n_1, n_2, \ldots, n_s}$, then $P = X_2$.*

Proof. If P is simple, then $\deg \boldsymbol{v} = d$ for every $\boldsymbol{v} \in \mathrm{vert}\, P$. Therefore $\sum_{j \in [s] \setminus \{i\}} n_j = d$ for every $i \in [s]$. Hence $n_1 = n_2 = \cdots = n_s = k$. Since P is not a simplex, $k \neq 1$. Notice that P is a d-polytope with $d + k$ vertices whose graph has a k-anticlique. Thus $k = 2$. Now, P is a $2(s-1)$-polytope with graph G_s. Therefore

$$2(s-1) \le \left\lfloor \frac{3s}{2} \right\rfloor - 1 \le \frac{3s}{2} - 1.$$

Solving for s yields $s \le 2$. Hence $s = 2$, whence $\mathcal{G}(P) = G_2$. Thence $P = X_2$. $\qquad\square$

7.4.2 Low Dimensional Realizations

In Grünbaum (2003), a construction of a 4-polytope Z with graph G_5 is given. Repeated bipyramids over this polytope yield $(n-1)$-realizations of G_n for $n \ge 5$. Note that $Z \vee X_2$ is a 7-polytope with graph G_7 and is not X_7. Thus, in general, the graph G_n is not the graph of a unique n-polytope.

The polytope Z has an additional property that will be useful in talking about low dimensional realizations of G_n.

Definition. A polytope P is called *centrally symmetric* if there is a point $\boldsymbol{p} \in P$ such that $\boldsymbol{x} \in P - \boldsymbol{p}$ implies $-\boldsymbol{x} \in P - \boldsymbol{p}$.

It will be assumed throughout that $\boldsymbol{p} = \boldsymbol{0}$. The 4-realization Z of G_5 is centrally symmetric and much of the work on low-dimensional realizations of G_n concerns centrally symmetric realizations. In Grünbaum (2003) it is shown that if G_6 is 4-realizable by a polytope P, then P cannot be centrally symmetric.

Let P be a centrally symmetric d-polytope with graph G_n with $n < d$. The polytope $P \vee X_2$ is a $(d+3)$-polytope with graph G_{n+2}. However, $P \vee X_2$ is not centrally symmetric.

It is shown in Barvinok & Novik (2008) that if P is a centrally symmetric d-polytope with N vertices, then

$$f_1(P) \leq \frac{N^2}{2}(1 - 2^{-d}).$$

In the case that $\mathscr{G}(P) = G_n$, this implies that $\operatorname{lb} n \leq d$ where $\operatorname{lb} n$ is the binary logarithm $\log_2 n$.

Let $m \in \mathbb{N}$. The paper Barvinok et al. (2013a) gives an explicit construction of a d-dimensional centrally symmetric polytope with graph G_n where

$$d = 4m + 6$$

$$n = 2 \cdot 3^{m+1}.$$

This construction, in the case $m = 0$, yields that the polytope X_6 has graph G_6. Setting $m = 1$ gives a centrally symmetric 10-polytope with graph G_{18}. By considering bipyramids over these polytopes, one obtains centrally symmetric polytopes of dimension $4m + 7$ with graph G_n where $n = 2 \cdot 3^{m+1} + 1$. Furthermore, taking a join with X_2 yields a polytope of dimension $4m + 11$ and with graph G_n where $n = 2 \cdot 3^{m+1} + 2$.

Chapter 8

Conclusion

The first chapter with original work in it was Chapter 4. It was shown that the join of polytopes has Gale diagram the direct sum of the Gale diagrams of the polytopes. This was then used to give a quick classification of the combinatorial types of d-polytopes with $d+2$ vertices (this was already in the literature in McMullen & Shephard (1971)). Next, the question, "What happens if you duplicate a vertex in a Gale diagram?" was answered.

In Chapter 5 the idea of a Gale polytope was discussed (these were first introduced in Bayer & Bisztriczky (2007)), and several properties were examined: a reversal of a Gale ordering is still a Gale ordering; no Gale polytope has more than two disjoint facets with an odd number of vertices; and the product of Gale polytopes is again a Gale polytope, and a pyramid over a Gale polytope is again Gale. This last property leads to the following question.

Question. Is the join of two Gale polytopes a Gale polytope?

Chapter 6 attempts to answer the question, "How big can an anticlique in the graph of a polytope be?". To answer this, first there was an exploration of what properties points in an anticlique have in a Gale diagram. These properties were then used to show that the graph of a d-polytope with $d+k$ vertices can have an anticlique with at most $k-1$ vertices. A sequence of d-polytopes with anticliques that are "large" is then described, and it is conjectured that these polytopes do in fact have anticliques with largest possible cardinality. Recall the function $f(d,k)$, defined as the

largest size of an anticlique for a d-polytope with $d+k$ vertices then the conjecture states that:

Conjecture. $f(d,k) = k - 1 - \lfloor (k-3)/d \rfloor$.

An alternative approach that was abandoned (but could still yield useful results) is as follows. Assume that P is a d-polytope with $d+k$ vertices, and a standard Gale diagram $\Gamma \subseteq \mathbb{S}^{k-1} \cup \{\mathbf{0}\}$. If two points are in an anticlique, then there must be a hyperplane through the origin that separates them from all other points in Γ. This introduces a "forbidden" zone in $\mathbb{S}^{k-2}$. How many of these forbidden zones are necessary to cover all of $\mathbb{S}^{k-2}$? Whatever this number is, it gives an upper bound on the size of an anticlique. Part of this approach was an attempt to prove the following:

Conjecture. *Let P be a d-polytope with $d+k$ vertices an anticlique A, and a Gale diagram Γ. Then for each $B \subseteq A$ with $2 \leq |B| \leq |A| - 1$, there is a hyperplane H through the origin in $\mathbb{R}^{k-1}$ such that $H^{(+)} \cap \Gamma = \overline{B}$.*

Chapter 7 is the chapter that motivated everything before it. The original motivating question was the following, found in Grünbaum (2003).

Conjecture (Grünbaum)**.** *If the k-complex $\mathscr{C}$ is the k-skeleton of both a d-polytope and a d'' polytope, where $d \leq d''$, then $\mathscr{C}$ is the k-skeleton of a d'-polytope for every d' satisfying $d \leq d' \leq d''$.*

Rather than attempt to attack the problem for all k, the simplest case $k = 1$ was considered. The field of search was then narrowed even further to simply asking for the dimensions in which the graph of a d-crosspolytope can be realized. It is shown in Halin (1966) that the graph G_d of a d-crosspolytope cannot be the graph of a polytope of dimension $\lfloor 3d/2 \rfloor$ or greater. In Chapter 7 polytopes of dimension k are exhibited with graph G_d for every $d \leq k \leq \lfloor 3d/2 \rfloor - 1$. This leaves the question of k-realizability for $4 \leq k \leq d - 1$. A survey of the state of current research into this question is then given.

This search into realizability of graphs of crosspolytopes naturally turned into a question of the realizability of the graphs of all complete multipartite graphs. The case of complete bipartite graphs is easily handled, and it is shown that only $K_{1,1}$ and $K_{2,2}$ are graphs of polytopes. Next, complete

3-partite graphs are considered, and several steps are taken toward completing the classification of realizable complete 3-partite graphs. The following conjecture is made regarding realizability of complete multipartite graphs:

Conjecture. *$K_{n_1,n_2,\ldots,n_s}$ is the graph of a polytope if and only if, as sets, $\{n_1,n_2,\ldots,n_s\} \subseteq \{1,2\}$ and, as multisets, $\{n_1,n_2,\ldots,n_s\}$ is not equal to either $\{1,2\}$ or $\{n_1,n_2,\ldots,n_s\} \neq \{1,1,2\}$.*

References

Altshuler, A. & Steinberg, L. (1984). Enumeration of the quasisimplicial 3-spheres and 4-polytopes with eight vertices. *Pacific J. Math.*, 113(2), 269–288.

Altshuler, A. & Steinberg, L. (1985). The complete enumeration of the 4-polytopes and 3-spheres with eight vertices. *Pacific J. Math.*, 117(1), 1–16.

Ayzenberg, A. (2013). Simplicial complexes Alexander dual to boundaries of polytopes. *ArXiv e-prints*.

Balinski, M. L. (1961). On the graph structure of convex polyhedra in n-space. *Pacific J. Math.*, 11, 431–434.

Barvinok, A., Lee, S. J., & Novik, I. (2013a). Centrally symmetric polytopes with many faces. *Israel J. Math.*, 195(1), 457–472.

Barvinok, A., Lee, S. J., & Novik, I. (2013b). Explicit constructions of centrally symmetric k-neighborly polytopes and large strictly antipodal sets. *Discrete Comput. Geom.*, 49(3), 429–443.

Barvinok, A., Lee, S. J., & Novik, I. (2013c). Neighborliness of the symmetric moment curve. *Mathematika*, 59(1), 223–249.

Barvinok, A. & Novik, I. (2008). A centrally symmetric version of the cyclic polytope. *Discrete Comput. Geom.*, 39(1-3), 76–99.

Bayer, M. M. & Bisztriczky, T. (2007). On Gale and braxial polytopes. *Arch. Math. (Basel)*, 89(4), 373–384.

Bayer, M. M. & Lee, C. W. (1993). Combinatorial aspects of convex polytopes. In *Handbook of convex geometry, Vol. A, B* (pp. 485–534). North-Holland, Amsterdam.

Björner, A., Las Vergnas, M., Sturmfels, B., White, N., & Ziegler, G. M. (1999). *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge: Cambridge University Press, second edition.

Brøndsted, A. (1983). *An introduction to convex polytopes*, volume 90 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.

Diestel, R. (2010). *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Heidelberg: Springer, fourth edition.

Ewald, G. (1996). *Combinatorial convexity and algebraic geometry*, volume 168 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

Grünbaum, B. (2003). *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, second edition. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

Halin, R. (1966). Zu einem Problem von B. Grünbaum. *Arch. Math. (Basel)*, 17, 566–568.

Henk, M., Richter-Gebert, J., & Ziegler, G. M. (1997). Basic properties of convex polytopes. In *Handbook of discrete and computational geometry*, CRC Press Ser. Discrete Math. Appl. (pp. 243–270). Boca Raton, FL: CRC.

Hibi, T. (1992). *Algebraic Combinatorics on Convex Polytopes*. Glebe, N.S.W., Australia: Carslaw Publications.

Kalai, G. (1988). A simple way to tell a simple polytope from its graph. *J. Combin. Theory Ser. A*, 49(2), 381–383.

Klee, V. & Minty, G. J. (1972). How good is the simplex algorithm? In *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin)* (pp. 159–175). Academic Press, New York.

McMullen, P. & Shephard, G. C. (1971). *Convex polytopes and the upper bound conjecture*. London: Cambridge University Press. Prepared in collaboration with J. E. Reeve and A. A. Ball, London Mathematical Society Lecture Note Series, 3.

Perles, M. A. & Shephard, G. C. (1967). Facets and nonfacets of convex polytopes. *Acta Math.*, 119, 113–145.

Stanley, R. P. (1999). *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

Stanley, R. P. (2012). *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press, second edition.

Steinitz, E. (1922). Polyeder und raumeinteilungen. *Enzykl. math. Wiss.*, 3, 1–139.

Steinitz, E. & Rademacher, H. (1976). *Vorlesungen über die Theorie der Polyeder unter Einschluss der Elemente der Topologie*. Berlin: Springer-Verlag. Reprint der 1934 Auflage, Grundlehren der Mathematischen Wissenschaften, No. 41.

Thomas, R. R. (2006). *Lectures in geometric combinatorics*, volume 33 of *Student Mathematical Library*. Providence, RI: American Mathematical Society. IAS/Park City Mathematical Subseries.

Wagner, K. (1937). Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*, 114(1), 570–590.

West, D. B. (1996). *Introduction to graph theory*. Upper Saddle River, NJ: Prentice Hall Inc.

Whitney, H. (1932). Congruent Graphs and the Connectivity of Graphs. *Amer. J. Math.*, 54(1), 150–168.

Ziegler, G. M. (1995). *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.

Appendix A

Python Code to Determine Whether or not a Point Configuration is a Gale Diagram

This program was written in Python version 2.7.2.

```
#################################################################
##                                                             ##
##   This program checks whether or not a point configuration  ##
##   is the Gale diagram of some polytope.                     ##
##                                                             ##
##   That is, whether for each hyperplane passsing through the ##
##   origin there are at least two points on either side of it.##
##                                                             ##
#################################################################
from sympy import *
from fractions import Fraction
from itertools import *
from sys import exit
```

```python
#####
#    This function checks some trivial cases
#    to see if it is possible for the point
#    configuration to be a Gale diagram.
#####
def dim_check(v, g):
    if min([v,g])<0:              # If either v, or g is negative,
        return 0                  # then return a command to kill
                                  # the program.
    else:
        if v in set([1,2,3]):     # All polytopes with 1, 2, or 3
            if g==0:              # vertices are simplices, and
                return 1          # therefore have a Gale diagram
            else:                 # of dimension 0.
                return 0
        else:
            if v-g-1>=2:          # In a general polytope P,
                return 1          # dim(P)=v-g-1.  Already checked
            else:                 # if it is a 0- or 1-polytope,
                return 0          # so this must be >=2.


#####
#    This function gets the vectors in the
#    configuration and puts them in a
#    (g)x(v) matrix.
#####
def get_matrix(height, width):
```

```python
    M=Matrix(height, width, lambda i,j: 0)
    for j in range(width):
        print "Please input the coordinates of point %s: " % (j
            +1)
        for i in range(height):
            M[i,j]=Fraction(raw_input("    Entry %s: " % (i+1)))
    return M


#####
#    This function checks whether or not a
#    matrix M has rank rnk.
#####
def rank_ok(rnk, N):
    return len(N.rref()[1])==rnk




###########################
###########################
### Start the program. ###
###########################
###########################


# Get the dimension of the point confuguration
g = int(raw_input("Dimension of point configuration: "))
# Get the number of points in the configuration
v = int(raw_input("Number of points in the configuration: "))
```

96

```python
if dim_check(v,g)==0:
    if v==1:
        print("There is no Gale Diagram with 1 point in a %s
            dimensional space." % (g))
        quit
    else:
        print("There is no Gale Diagram with %s points in a %s
            dimensional space." % (v,g))
        quit
else:
    M=get_matrix(g,v)
    print M
    #print M[1,2]
    if raw_input("Is this matrix correct?(y/n) ")!="y":
        quit
    else:
        if len(M.rref()[1])!=g:
            print "All of your points lie in a hyperplane."
        else:
            if g==1:                        # In the g=1 case,
                pos=neg=0                   # just need 2 points
                for i in range(v):          # on either side of
                    if M[i]>0:              # the origin.
                        pos+=1
                    elif M[i]<0:
                        neg+=1
                if pos>=2 and neg>=2:
```

97

```python
            print('This␣is␣the␣Gale␣diagram␣of␣a␣polytope
                ␣of␣'
                    'dimension␣%s.' % (v−g−1))
            quit
        else:
            print('This␣is␣not␣the␣Gale␣diagram␣of␣a␣
                polytope.')
            quit
else:
    N=Matrix(g, g−1, lambda i,j:0)
    for comb in combinations(range(v),g−1):
        for j in comb:
            for i in range(g):
                N[i,comb.index(j)]=M[i,j]
        if len(N.rref()[1])!=g−1:
            continue
        else:
            nml=Matrix(g,1, lambda i,j:0)
            for d in range(g):
                P=N.T
                P.col_del(d)
                nml[d]=(−1)**d*P.det()
            pos=neg=0
            for k in range(v):
                ip=0
                for b in range(g):
                    ip+=nml[b]*M[b,k]
```

98

```python
            if ip >0:
                pos+=1
            elif ip <0:
                neg+=1
        if pos <2:
            print "The hyperplane perpendicular
                to"
            print nml,
            print "has an open half space with",
                pos,
            print "point(s) contained in it."
            exit
        elif neg <2:
            print "The hyperplane perpendicular
                to"
            print nml,
            print "has an open half space with",
                neg,
            print "point(s) contained in it."
            exit
    print "This is the Gale diagram of a polytope of
        dimension",
    print v-g-1
```