A GRAPHICAL SOLUTION FOR THE COMPLEX ROOTS OF EQUATIONS

by

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A Graphical Solution for the Complex Roots of Equations

Chapter I

Introduction

1. Object of this thesis.

It is the primary purpose of this thesis to outline briefly the more important graphical methods of obtaining the complex roots of an equation in one unknown and to present a graphical method original with the author.

2. Historical outline.

Burnside and Panton in the 1912 Edition of their Theory of Equations say: "Little attention has been given by writers on the Theory of Equations to the actual determination of the complex roots of numerical equations." As a matter of fact, however, a large number of methods exist for determining to any desired degree of accuracy the complex roots of numerical equations. Horner's method, for example, can be applied as for real roots. Here, however, a first approximation to the complex root is necessary to start. This approximation is obtained usually by graphical means.

It is true, however, that standard treatises on Algebra give little space to methods for determining complex roots. Serret gives[20] an algebraic method based on Newton's. This he admits is very laborious to apply and he then gives another method and by it solves $z^4-z+1=0$. Without indicating how equations of higher degree would be solved he leaves the subject. Weber[24] does not give even Horner's method for real roots. He does give, however, solution for trinomial equations due to Gauss. Neither Serret or Weber give any graphical methods.

Graphical methods of solution for real roots are numerous. Some of these are very ancient. The graphical determination of complex roots was accomplished by P. Hoppe at least as early as 1883. His method used one fixed parabola and determined the complex roots of the cubic and the quartic.

The most extensive discussion available of the imaginary branches of the quadratic, the cubic, and the quartic is given by Phillips and Beebe in their Graphic Algebra published in 1882. While the treatment is not designed to yield graphical solutions giving complex roots their work deserves mention here. No references are given and what part of the work is original and what taken from previous writers can not be determined.

Various machines have been devised some of which may be used to determine the complex roots of cubics and quartics. A paper[6] by Henry Cunynghame describing a mechanical solution of the cubic for both real and imaginary roots was read before the London Physical Society December 12, 1885. This was a modification of a method due[23] to Monge for obtaining the real roots of the cubic $x^3+px+q=0$ by means of $y=x^3$ and $y+px+q=0$.

On the same day C. V. Boys described and exhibited to the same society[2] a machine which would give the roots real or complex of the quadratic. It is not at all clearly described and the diagrams are incomplete and schematic. It may be vaguely described as depending on balances, weights and a movable arm.

To quote the author: "If a quadratic $ax^2+bx+c=0$ is to be solved, weights a, b, and c are placed in the proper pans and beam 2 is slid along until beam 1 shifts its position. The two places where this occurs are the two real roots of the quadratic. When the quadratic has no real roots it can still be employed to find the impossible roots of an equation as will be explained later."

The author indicated how an extension of the principles involved would design a machine for solving cubics and higher equations. He remarked that for higher equations the increased friction would diminish accuracy.

Schultze in his Graphic Algebra (1907) gives a method for finding the complex roots of a cubic which is not widely different from that given by Cunyngname[6] in 1885. Schultze also gives a method of solving the quartic for real roots by means of a parabola and a circle. This method was given earlier by Adler.[1] Schultze, however, modifies the method so as to give the complex roots of the quartic when there are but two.

The graphical determination of the complex roots of a quadratic is simple. For this there are several methods some of which no doubt were devised before those for the cubic and the quartic were worked out. In spite of this, however, little information could be obtained on the earlier work of determining graphically the complex roots of a quadratic.

The method given in detail in Chapter III is original with the author of this thesis. A rather extended search seems to indicate that it is new.

3. Classification of Existing Methods:

1 In $f(z)=0$ let $z=x+iy$. Equate reals and imaginaries to zero. Construct the graphs of the two equations and determine the points of intersection.

2 Same as (1) except $z=r(\cos x + i \sin x)$.

3 Construction of the real and imaginary branches of $f(x) = y$ by plotting with reference to three axes in isometric projection[17]. (Strictly speaking the method was not really applied to solving equations but rather to representing the real and imaginary branches of $y=f(x)$ for real values of y).

4 Special methods:

(a) for quadratics - a fixed parabola and a variable straight line.

(b) for cubics - a fixed cubic parabola $y=x^3$ and a variable straight line.[10]

(c) for quartics - a fixed parabola and a variable circle[11,19] also a method using a fixed parabola and a variable parabola[5].

5 Mechanical methods by specially designed machines[2-8].

Chapter II

4 Method of Intersections of Curves in the Plane of the Complex Variable.

Let $f(z) = 0$ be the given equation and substitute $x+iy$ for z. Then separating real parts and imaginary parts we obtain.

$$u = 0$$
$$v = 0$$

where u and v are real functions of x and y.
If $u = 0$ and $v = 0$ be plotted on rectangular axes the points of intersection will determine values of x and y which makes $z = x + iy$ a root of $f(z) = 0$.

This method soon becomes laborious and it is used mainly to obtain a first approximation of a complex root. This approximate value is of importance in the algebraic methods of Newton, Horner, Schröder and Weddle for computing complex roots of equations[4].

5 Method of Intersection of Curves in the Plane of the Complex Variable (Polar Coördinates).

Let $f(z) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} - - - a_n = 0$
set $z = r(\cos\theta + i\sin\theta)$.
We obtain by separating reals and imaginaries,
$u = u(r,\theta) = a_0r^n\cos n\theta + a_1 r^{n-1}\cos(n-1)\theta - - - a_n = 0.$
$v = v(r,\theta) = a_0r^n\sin n\theta + a_1 r^{n-1}\sin(n-1)\theta - - a_{n-1}\sin\theta = 0.$

The chief advantage of polar over rectangular coordinates is that one can determine asymptotes to these curves and use them as aids in plotting. These curves have been examined carefully[13] so that plotting can be simplified by known results.

6. Pictorial Representation of the Real and the Imaginary Branches of $y=f(x)$ for Real Values of y by Isometric Projection.

(a) Method of Phillips and Beebe

The cubic $x^3+ax^2+bx+c=0$ is put[17] in the form $x^3+px+q=0$. Then since changes in the value of q do not affect the shape of the curve three possibilities arise

$y=x^3-nx$

$y=x^3+nx$

$y=x^3$

The first of these has two distinct "elbows" the second has none and the third is the limiting form of the other two.

Let the roots of $y=x^3-nx$ be $x=a$ and $c \pm bi$

Then since $x^3-nx-y=0$ from the relation between the roots and the coefficients we have

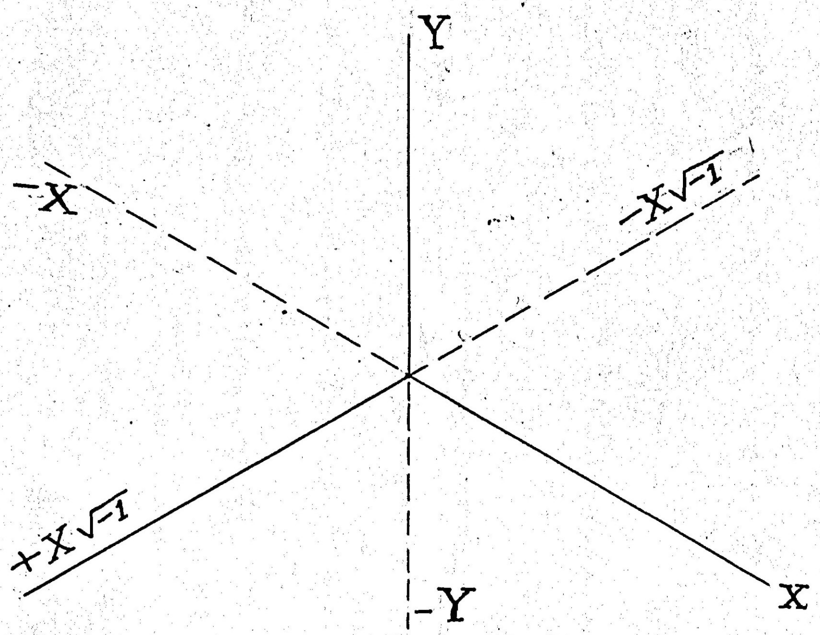
$$y=a(b^2+c^2),$$

$$2b-a=0,$$

$$\text{and} \quad b^2+c^2-2ab=-n.$$

Assuming values for a the corresponding values of b and c can then be found.

To plot the points obtained and thus outline the graph they determine the authors use isometric projection[17]. The resulting view resembles a perspective view but is not one. The axes are drawn thus:



Laying off the real and complex values of x and all values of y on a uniform scale an excellent representation of the real and imaginary branches and a clear notion of their relative positions is secured.

In like manner graphical pictures of $y=x^3+nx$ and $y=x^3$ are obtained.

The quartic $x^4+a_1x^3+a_2x^2+a_3x+a_4=0$.
is put in the form $x^4+ax^2+bx+c=0$
Again c does not affect the <u>shape</u> of the curve.

The authors then consider the special case in which $b=0$. These are

$$y=x^4-ax^2$$
$$y=x^4+ax^2$$
$$y=x^4$$

Let the roots of $x^4-ax^2-y=0$

be $\qquad x=m\pm pi$

and $\qquad x=-m\pm pi$

Then $\qquad y=-(m^2+p^2)^2$

$$\pm a=2p^2-2m^2$$

Values are now assumed for m and the corresponding values of y and p are computed.

These values are plotted by the same method used for the cubic.

For $y=x^4\pm ax^2\pm bx$ the substitution of $x=x\sqrt{a}$ is made and then the resulting equation is divided by a^2.

If $y^1=\frac{y}{a^2}$ and $\Sigma = \frac{b\sqrt{a}}{a^2}$ the following three groups result for $b\neq 0$

$$y^1=x^4-x^2$$
$$y^1=x^4\mp x^2+\Sigma x$$
$$y^1=x^4-x^2-\Sigma x$$

$$y^1=x^4+x^2 \qquad\qquad y^1=x^4$$
$$y^1=x^4+x^2+\Sigma x \qquad\qquad y^1=x^4+\Sigma x$$
$$y^1=x^4+x^2-\Sigma x \qquad\qquad y^1=x^4-\Sigma x$$

The first equation in each of these groups may be plotted as before. The authors then show how to use the line $y=\Sigma x$ to plot the real part of the second equation of each group. Then they show how to use $y=-\Sigma x$ to plot the real part of the third equation of each group.

The imaginary parts of the graphs are obtained thus:

Since x^3 is absent the roots of

$$x^4\pm ax^2\pm bx=0$$

have the form $m\pm pi$ and $-m\pm qi$

Hence the following relations are readily obtained for

$y=x^4 \pm ax^2 \mp bx$
$p^2+q^2-2m^2=\pm 1 \qquad (a=1)$
$2m(p^2-q^2)=\pm\Sigma \qquad (b=\Sigma)$
$-(m^2+p^2)(m^2+q^2)=y$

Assuming various values of m, p and q may first be found and then y can be obtained.

The authors apply this method and give the graphs of:

$y=x^4 \pm x^2$
$y=x^4-x^2-\frac{1}{10}x$
$y=x^4-x^2+\frac{2\sqrt{6}}{9}\ x$
$y=x^4-x^2+.6x$

$y=x^4+x^2 \pm x$

The only equation beyond the fourth degree which is discussed are those[17] coming under the form $x^n=a$. Here it is shown that imaginary branches are obtained by revolving the real curve about the y-axis to the positions it has reached. When it has completed the $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, - - - part of a revolution

The quadratic, the cubic, the quartic and $x^n=a$ are treated. The presentation is mainly for pedagogic purposes. For this use it would gain immensely if a single method were given applicable to the cubic and higher equations. The many special cases into which the general cubic and especially the general quartic is broken up complicates the matter unduly. Then the method of symmetric functions of the roots used in the cubic and the quartic is excessively difficult if not wholly impossible of application for the general quintic. Finally it is intended to display the character of the function $y=f(x)$ rather than offer a means of computing the complex roots of $f(x)=0$.

6. (b) Schultze's Method

This method for finding the complex roots of a cubic is an ingenious modification of an older method due to Monge[23] for finding its real roots.

Given $ax^3+bx+c=0$ (1)

Assume $y=x^3$ (2)

Then $ay+bx+c=0$ (3)

Here the solution of the system (2), (3) produces the real roots of (1). If then the graph of (2) be permanently constructed the solution of any cubic so far as real roots are concerned is effected by drawing the straight line (3).

The complex roots are obtained as follows:

$ax^3+bx+c=0$ can be written

$a(x-m)(x^2+px+q)=0$ if m is a real root of the cubic and $x^2+px+q=0$ has equal roots, Now consider

$$a(x-m)(x^2+px+q+d)=0 \qquad (4)$$

$$a(x-m)(x^2+px+q) =0 \qquad (5)$$

$$a(x-m)(x^2+px+q-d) =0 \qquad (6)$$

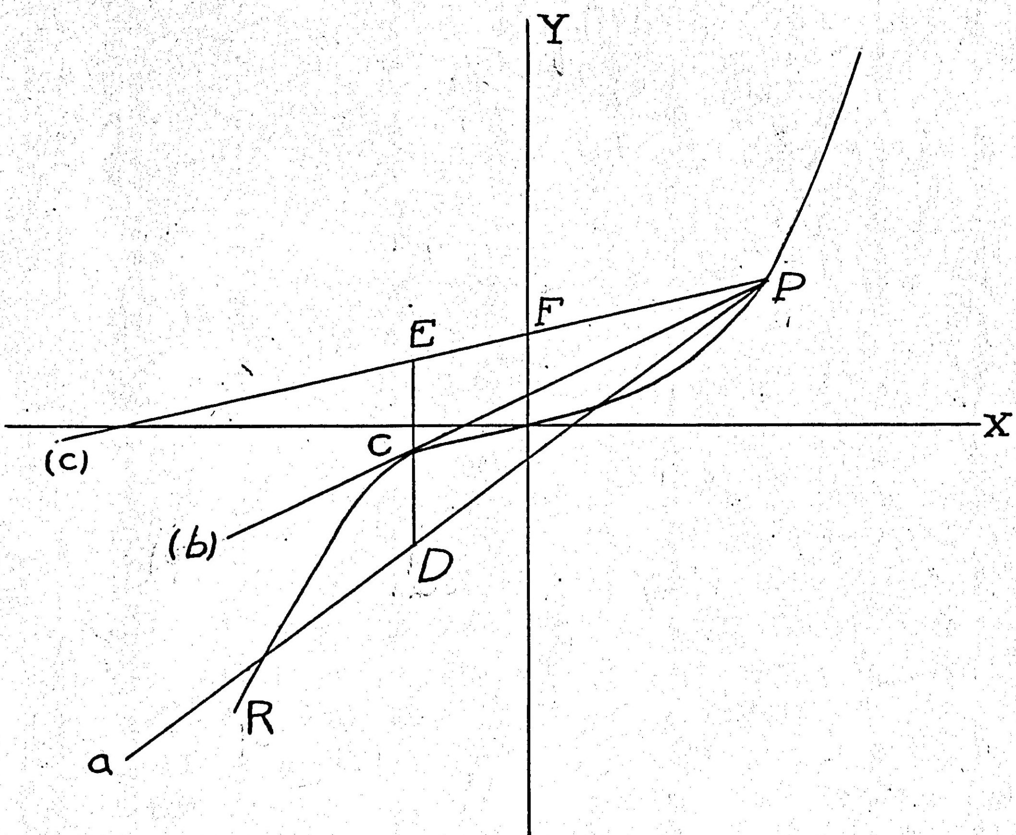
The roots of the last three equations are respectively:

$$m, -\frac{p}{2} \pm \sqrt{d}$$

$$m, -\frac{p}{2} - \frac{p}{2}$$

$$m, -\frac{p}{2} \pm \sqrt{-d}$$

Hence the complex roots of (6) can be obtained by solving (4).



If the three lines which solve (4), (5) and (6) are (c), (b), and (a),

Schultz proves:

(1) The coördinates of P are (m, m^3)

(2) That CD = CE

(3) Line (b) is tangent to the cubic parabola

(4) The abscissas of D, C, and E are $-\frac{m}{2}$. Hence $EF = \frac{1}{2}FP$.

Instead then of finding intersections corresponding to line (c) we find the real intersections of line (a), and the cubic parabola.

This method in sharp contrast with that of Phillips, and Beebe is primarily for purposes of solution. One can find complex roots by this process without having any conception of the complex branches of the cubic, in fact without knowing that they exist.

For the quartic Schultze gives the following method:

Given	$x^4+bx^2+cx+d = 0$	(7)
We may write	$x^4+(b-1)x^2+x^2+cx+d = 0$	(8)
Let	$y = x^2$	(9)
Then	$y^2+(b-1)y+a^2+cx+d = 0$	(10)

The solution of the system (9), (10) is the solution of (8). But (9) is a parabola and (10) is a circle. Therefore the intersections of a parabola and a circle give all the real roots of (8). Formulas for the center and radius of the circle are deduced making the solution easy of application.

Schultze gives an original modification of this method similar to that for the cubic which determines the complex roots provided there are but two. The case of the quartic where all the roots are complex is not discussed. Evidently the author was unable to modify the method to solve this last case.

Schultze's methods as far as real roots are concerned are not new. The method for the cubic is due to Monge. The method for the quartic is the same as that given by Adler.

With regard to complex roots his method for the cubic is akin to that given by Cunyngname in 1885. Schultze's method of obtaining the complex roots of a quartic is very elegant. Both of these he claims as original. They seem to be both original and new.

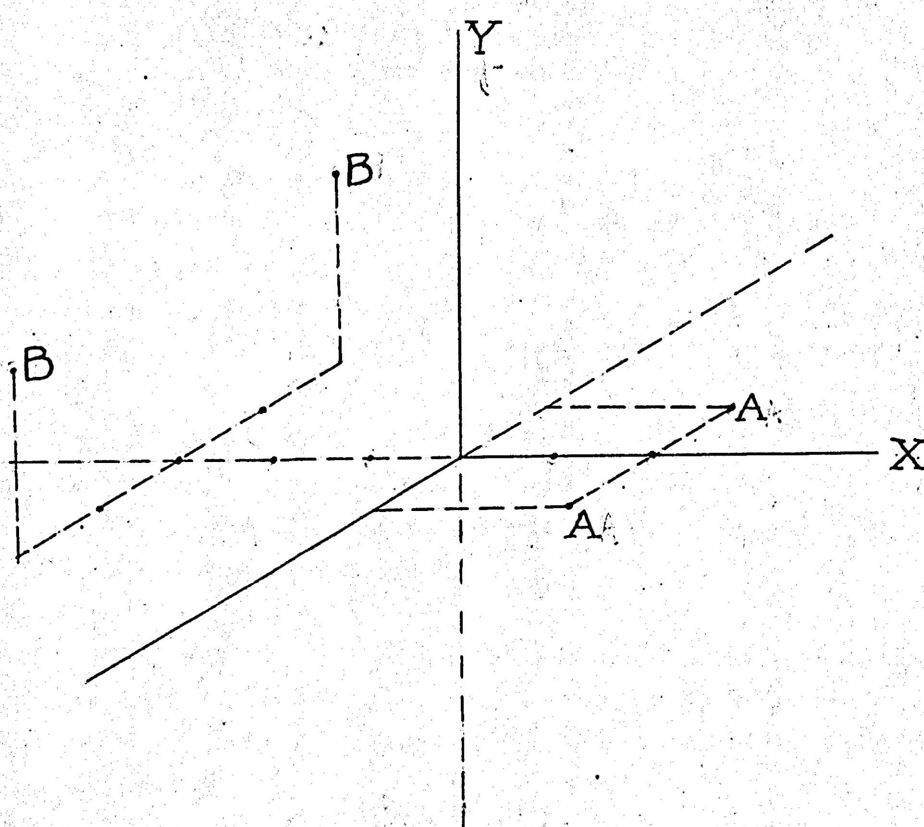
Chapter III

7. Method of Projecting the Imaginary Branches of y=f(x) on the xy plane and on the Iy plane. General description of the method.

Two axes suffice to represent $y=f(x)$ for real values of x and of y. For complex values of x and real values of y these axes are necessary.

Suppose $y=f(x)$ is satisfied by (A) $x=2\pm i$, $y=0$ and (B) $x=3\pm 2i$, $y=2$.

Then the points A are graphed as in the figure and the points B as indicated.



Essentially this means that a point has three co-ordinates as in ordinary analytic geometry.

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The equation

$$f(x)=a_0x^n+a_1x^{n-1} - - - a_n = 0 \qquad (1)$$

may have complex coefficients. In that case

$$f(x) = F(x)+iG(x) = 0 \qquad (2)$$

Multiplying both members of (2) by

$F(x)-iG(x)$ gives

$$(F(x))^2+(G(x))^2 = 0 \qquad (3)$$

Now the roots of (1) are among the roots of (3) and the latter has real coefficients. Therefore in what follows the coefficients will be real numbers.

Let it be assumed that it is possible to find <u>complex</u> values of x for <u>real</u> values of y in

$$y=a_0x^n+a_1x^{n-1} - - - a_n \qquad (4)$$

Later it will be shown how the equivalent of this may be done. Since (4) is satisfied by either real or complex values of x and real values of y, if we locate points as indicated on the preceding page, the resulting graph will be a space curve. Moreover since we have the theorem:

"If a+ib is a root of an equation f(x) = 0 then a-ib is also a root."

it follows that the imaginary branches of (4) will be symmetric with the xy plane.

Hence:

(A) <u>Those values which a may have if a±ib satisfies (4) lie in a real curve $y=f_1(a)$ in the xy plane which is determined by the projection of the imaginary branches upon that plane.</u>

Here $y=f_1(a)$ really is equivalent to

$y=f_1(x)$

Also

Also:

(B) <u>Those values which b may have if $a \pm ib$ satisfies (4) lie in a real curve $f_2(b)$ in the Iy plane which is the projection of the imaginary branch upon that plane.</u>

Here $y = f_2(b)$.

If the two equations $y=f_1(a)$ and $y = f_2(b)$ can be obtained for any equation as (4) then the construction of their graphs gives the roots of $f_1(a) = 0$ and $f_2(b) = 0$. These roots properly associated are the complex roots of (4).

The equations for "a" and for "b" can be found as follows:

$$\text{Given} \qquad y=f(x) \qquad (4)$$

$$\text{Let} \qquad x=a+ib \qquad (5)$$

Substituting from (5) in (4) and separating reals and imaginaries we obtain

$$F_1(a,b)=0 \qquad (6)$$

$$F_2(a,b)=0 \qquad (7)$$

By Sylvester's method of elimination it is always possible to eliminate either a or b between (6) and (7). This gives

$$f_1(a)=0 \qquad (8)$$

$$f_2(b)=0 \qquad (9)$$

The last two equations are the equations of the two cylinders whose intersection determine in space the imaginary branches of $y=f(x)$.

We now construct the real graphs of

$$y=f_1(a) \qquad (10)$$

$$y=f_2(b) \qquad (11)$$

As has been observed $f_1(a)$ is really $f_1(x)$ hence (10) may be plotted on the x and y axes with the original equation $y=f(x)$.

For plotting (11) two courses are open:

(a) We may rotate the Iy plane about the y-axis into coincidence with the xy plane. This means that the three curves

$y=f(x)$, $y=f_1(a)$, and $y=f_1(b)$ would be plotted on the same axes. Some color scheme or other plan of marking the curves to distinguish them would be desirable if not actually necessary.

(b) Since the curves would overlap in the preceding method we may avoid any confusion by having two sets of axes side by side - the y-axes parallel and the "b" axis being a continuation of the x-axis as illustrated in the adjacent figure which is for a cubic.

The points where the graph of (10) cuts the x-axis give the real part of the complex roots, and the points in which the graph of (11) cuts the I or b-axis gives the real coefficient of the imaginary part. Thus we obtain one pair of complex roots $a \pm bi$. The real roots come from the graph of $y=f(x)$ (4)

Hence the graphs of (10), (11), and (4) give all the roots, real and complex of (1).

THE QUADRATIC

Let $ax^2+bx+c=0$ be put in the form

$$x^2+px+q=0 \qquad (1)$$

Set $x^2+px+q=y$ (2)

Assume $x=a+bi$ (3)

Then

$$a^2+2abi-b^2+pa+pbi+q=y \qquad (4)$$

Hence $a^2-b^2+pa+q+y=0$ (5)

and $2ab+pb=0$ (6)

from (6) $a=-\frac{p}{2}$ (7)

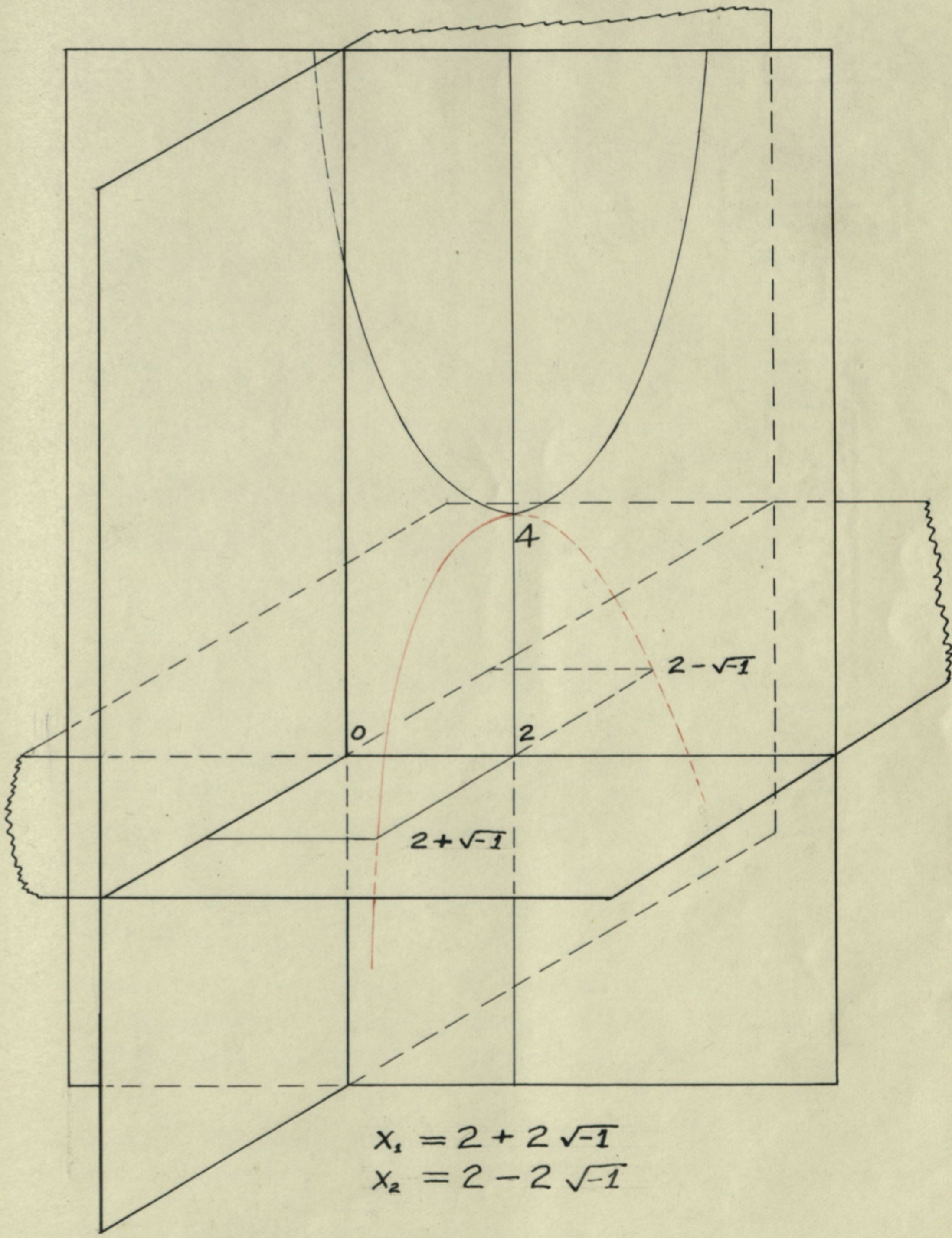
from (5) and (7) $y=\frac{p}{2}+q-b^2$ (8)

Then $b=\pm\sqrt{\frac{p}{2}^2+q-y}$

The real graph of (1) is a parabola. The imaginary part of the graph is obtained by assigning to y values $= <(\frac{p}{2}^2+q)$. The imaginary branch is a plane curve. It is a parabola lying in the plane $x=-\frac{p}{2}$ (or $a=-\frac{p}{2}$) and has the same vertex and axis as the real branch.

A pictorial representation of the complete graph of $y=x^2-4x+8$ is given in the next drawing—the imaginary branch being in red.

Pictorial View of the Complete Graph of $y = x^2 - 4x + 8$



$$x_1 = 2 + 2\sqrt{-1}$$
$$x_2 = 2 - 2\sqrt{-1}$$

The projection of the imaginary branch on the Iy plane is obviously a parabola (8) and is so easily obtained that it will not be given.

THE CUBIC

Let $ax^3+bx^2+cx+d = 0$ be put in the form

$$x^3+px+q = 0 \qquad (1)$$

Set $$x^3+px+q = y \qquad (2)$$

Substituting $$x = a+bi \qquad (3)$$

gives

$$a^3-3ab^2+pa+q-y = 0 \qquad (4)$$

and $$3a^2b-b^3+pb = 0 \qquad (5)$$

from (5) $$a = \pm\sqrt{\frac{b^2-p}{3}} \qquad (6)$$

from (4) and (6), $$y = q-8a^3-2ap \qquad (7)$$

from (6) and (7),

$$y = q \pm \frac{2}{3}\sqrt{\frac{b^2-p}{3}}\,(p-4b^2) \qquad (8)$$

****

Here the graph of (2) is the real branch giving all the real roots which exist.

The graph of (7) gives (x being the same as a) a real cubic curve which is the projection of the imaginary branches on the xy plane where the graph of (7) crosses the x-axis gives the value of a in the complex roots $a\pm bi$ of (1).

This value of a substituted in (6) gives b of the complex roots of (1). Or we may construct the graph of (8) on the I and b axes and obtain the projection on the Ib plane of the imaginary branch of the curve.

In the figure of page 22 a pictorial representation is given for all the curves connected with

$$y=x^3-2x-4 \qquad (9)$$

except that defined by equation (8)

The figure of page 24 gives the xy and the Iy planes side by side, and upon them are shown the real graph of (9) and the projection of its imaginary branches on the two planes.

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Obviously if we put $y=0$ in (7) the resulting equation may be solved for a by Horner's method. The value obtained may then be substituted in (6) and the corresponding values of b for thecomplex roots $a \pm ib$ of (1)obtained. This gives a very easy algebraic solution of (1). There is, to be sure, the easier method which consists of finding the one real root of the cubic and then depressing it to a quadratic and solving the latter.

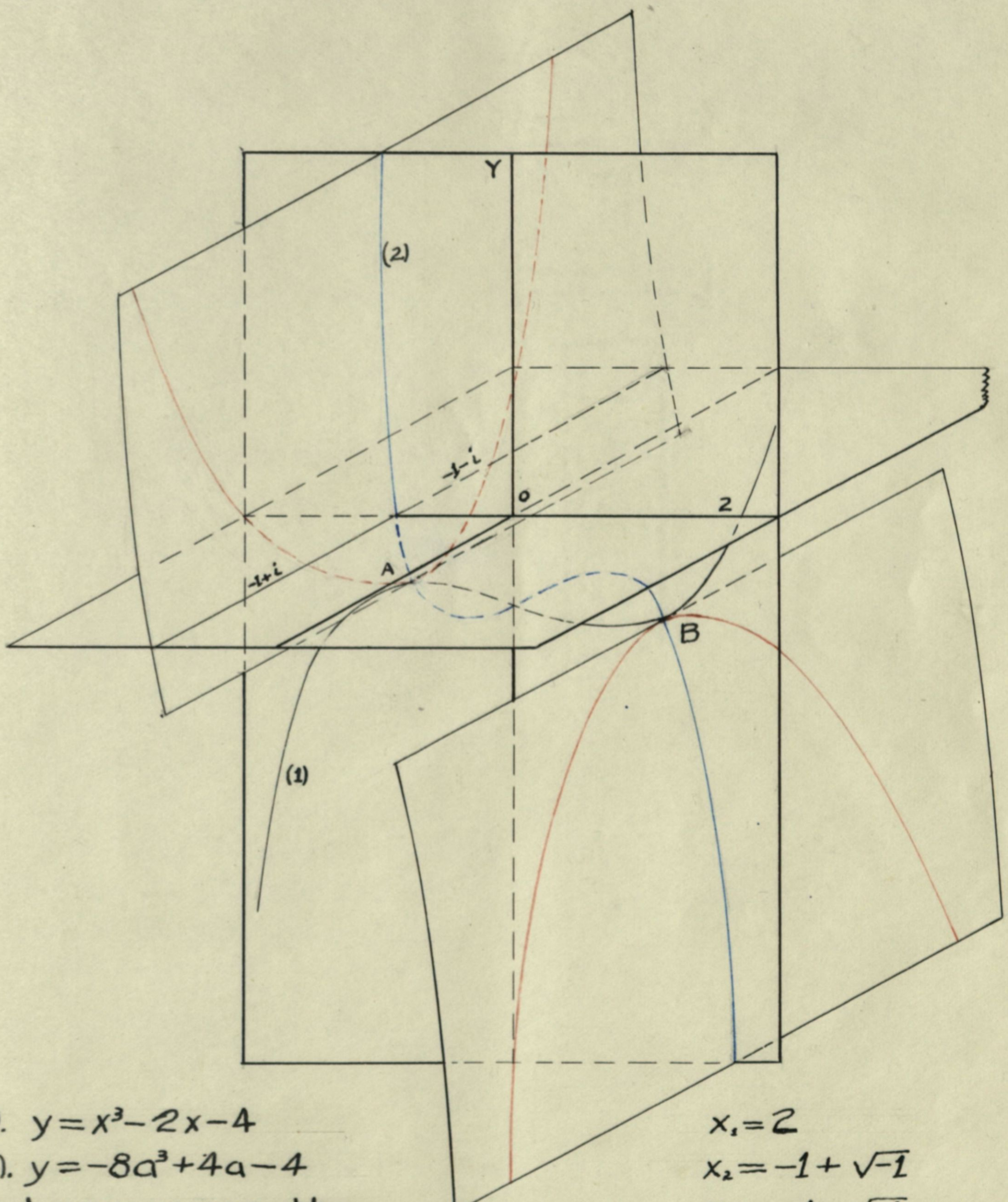
EXAMPLE

As an illustration of how the imaginary branches lie in space they are given below in red for $x^3-2x-4=0$

The projection of the imaginary branches on the xy plane is shown in (2). Its equation is $y=-8a^3+8a-4$.

The projection of the imaginary branches on the Iy plane is not shown.

Pictorial View of the Complete Graph of $y=x^3-2x-4$



(1). $y=x^3-2x-4$

(2). $y=-8a^3+4a-4$

Red curves are the imaginary branches

$x_1=2$

$x_2=-1+\sqrt{-1}$

$x_3=-1-\sqrt{-1}$

(1):. $y = x^3 - 2x - 4$

(2):. $y = -8a^3 + 8a - 4$

(3):. $y = -4 \pm \tfrac{4}{3}\sqrt{\frac{b^2 + 2}{3}}\ (-1-2b^2)$

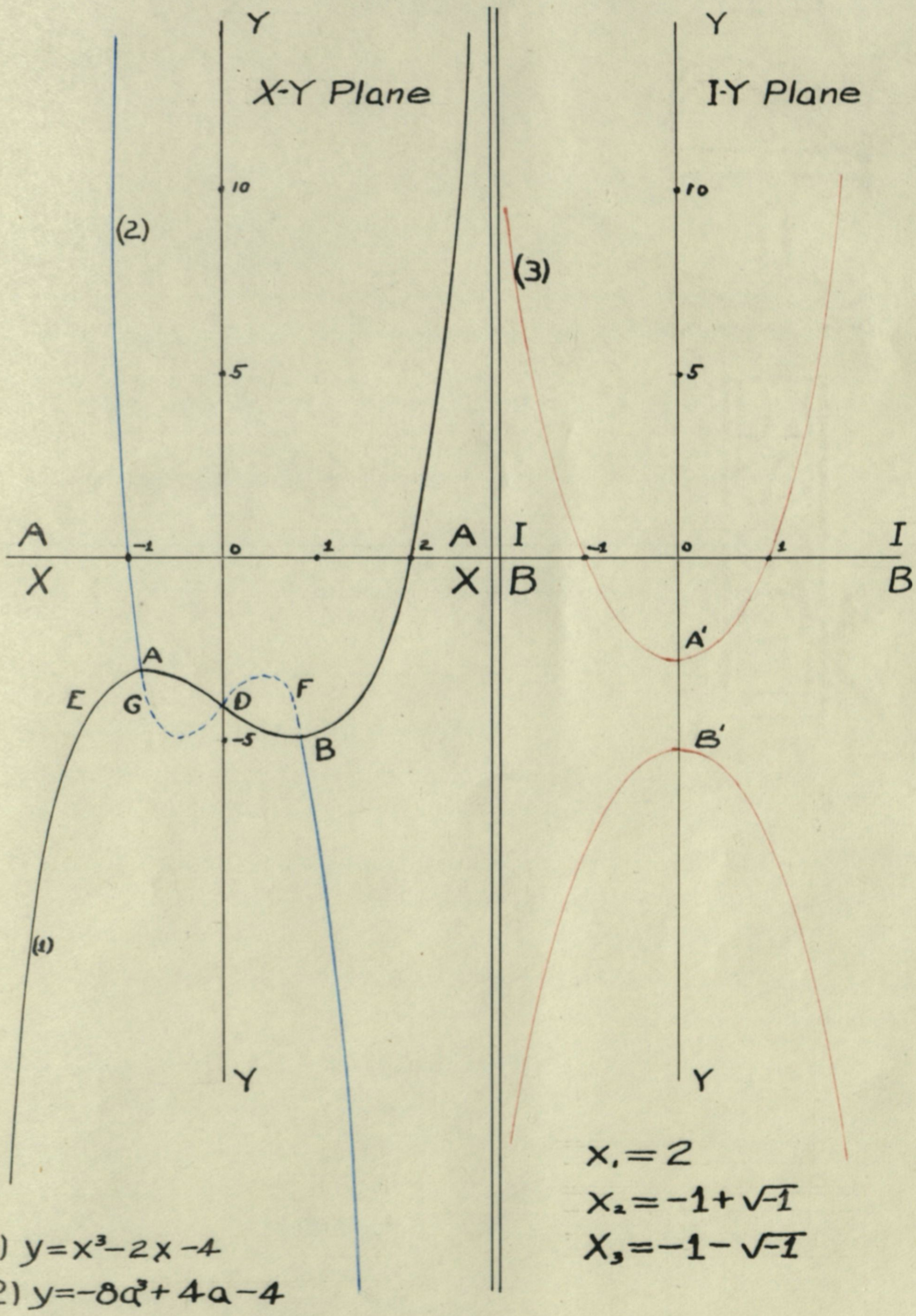
The above special results come from the general results:.

$y = x^3 + px + q$

$y = q - 8a^3 - 2ap$

$y = q \pm \tfrac{2}{3}\sqrt{\frac{b^2 - p}{3}}\ (p - 4b^2)$

Graphical Solution of $y = x^3 - 2x - 4$



$x_1 = 2$

$x_2 = -1 + \sqrt{-1}$

$x_3 = -1 - \sqrt{-1}$

(1) $y = x^3 - 2x - 4$

(2) $y = -8a^3 + 4a - 4$

(3) $y = -4 \pm \frac{4}{3}\sqrt{\frac{b^2+2}{3}}\left(-1-2b^3\right)$

The imaginary branches begin at the extreme points of the "elbows" of the cubic. The equation for "a" is, however, continuous between these points - from A to B in the two preceding graphs. The interpretation to be placed on this is simple and consistent. It is as follows:

Values of a in the interval AB give when substituted in (6) imaginary values of b. Since by hypothesis b is real and measured perpendicular to the x-axis, imaginary values would consistently be measured perpendicular to the I-axis which would be parallel to the x-axis giving points on the real branch of y=f(x) (equation (2)) in the interval AB.

For example if y= -4 in $x^3-2x-4 = y$
$b=\pm\sqrt{-2}$, $\pm\sqrt{-\frac{1}{2}}$. By reference to the graph of page 24 it is seen that these points correspond to the three points on f(x)=y where y= -4. It should be noted that three values of a correspond to y=-4 and the related points on y=f(x) are C, D, and E. Thus:

A at F	$b=\pm .7i$	= C and D
A at D	$b=\pm 1.4i$	= C and E
A at G	$b=\pm .7i$	= E and D

THE QUARTIC

Given $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 = 0$ (1)

Put (1) in the form

$$x^4+px^2+qx+r = 0 \qquad (2)$$

Set

$$x^4+px^2+qx+r = y \qquad (3)$$

Assume

$$x = a+bi \qquad (4)$$

Then

$$a^4+4a^3bi+6a^2(bi)^2+4a(bi)^3+(bi)^4+pa^2+2pabi-pb^2+qa+qbi+r-y=0 \qquad (5)$$

From (5)

$$a^4-6a^2b^2+b^4+pa^2-pb^2+qa+r-y = 0 \qquad (6)$$

$$4a^3b-4ab^3+2abp+bq = 0 \qquad (7)$$

from (7)

$$b^2 = \frac{4a^3+2ap\pm q}{4a} \qquad (8)$$

from (6) and (8) $4a^4+2a^2p+\frac{p^2}{4}-\frac{q^2}{16a^2}+r = -y$ (9)

Obviously the graphical (or algebraic) solution of (9) for a (when y= 0) and then the use of (8) gives all the complex roots of (2).,

It is important to note that it is not necessary to obtain b as a function of y., We can obtain y as an explicit function of "a"., Thensince "b" is an explicit function of a we may proceed as follows:, Fill out the following table by assigning values to a and computing y and b.,

y = (9) above							
a	-3	-2	-1	0	1	2	3
b							

This same method is used in the quintic.,

It may be pointed out here that the transformation which removes the second term of the equation, as in the cubic, the quartic, the quintic, etc., need not be used., It is merely a convenience to do this., For example if the quartic has the form

$$x^4+px^3+qx^2+ix+s = 0$$

we obtain

$$b^2 = \frac{\frac{r}{2} + 3a^2p + 2a^3 + aq}{2a+p}$$

$$\frac{[4a^2(a+p)+aq+\frac{r}{2}][4a(a+p)^2+pq+aq-\frac{r}{2}]}{(2a+p)^2} = s-y$$

These last may be used as were the simpler equations obtained by removing the term x^3.

We may proceed similarly for the quintic and higher equations.

EXAMPLE

Given $x^4-42x^2+64x+105 = y$ (10)

Since $x^4+px^2+qx+r = y$, (2)

$$y = -4a^4-2a^2p-\frac{p^2}{4}+\frac{q^2}{16a^2} + r \qquad (9)$$

becomes $y = -4a^4+84a^2-336+\dfrac{256}{a^2}$ (11)

and $b = \pm\sqrt{\dfrac{p}{2} + a^2 + \dfrac{q}{4a^2}}$ (8)

becomes $b = \pm\sqrt{a^2-21+\dfrac{16}{a}}$ (12)

y	-100	0	-497	-640	-567	-384	-175	0	105	128
x	- 8	-7	-6	-5	-4	-3	-2	-1	0	1

table continued

y	81	0	-55	0	273
x	2	3	4	5	6

Since (11) contains only even powers of a, the values of y below will be the same for corresponding negative values of a.

y	+6067		+671	+167	0	-543	0	124.5
a	.2	..	.5	.75	1	1.5	2	3
b	±7.68	±4.36	±3.35	9.4	Imaginary			

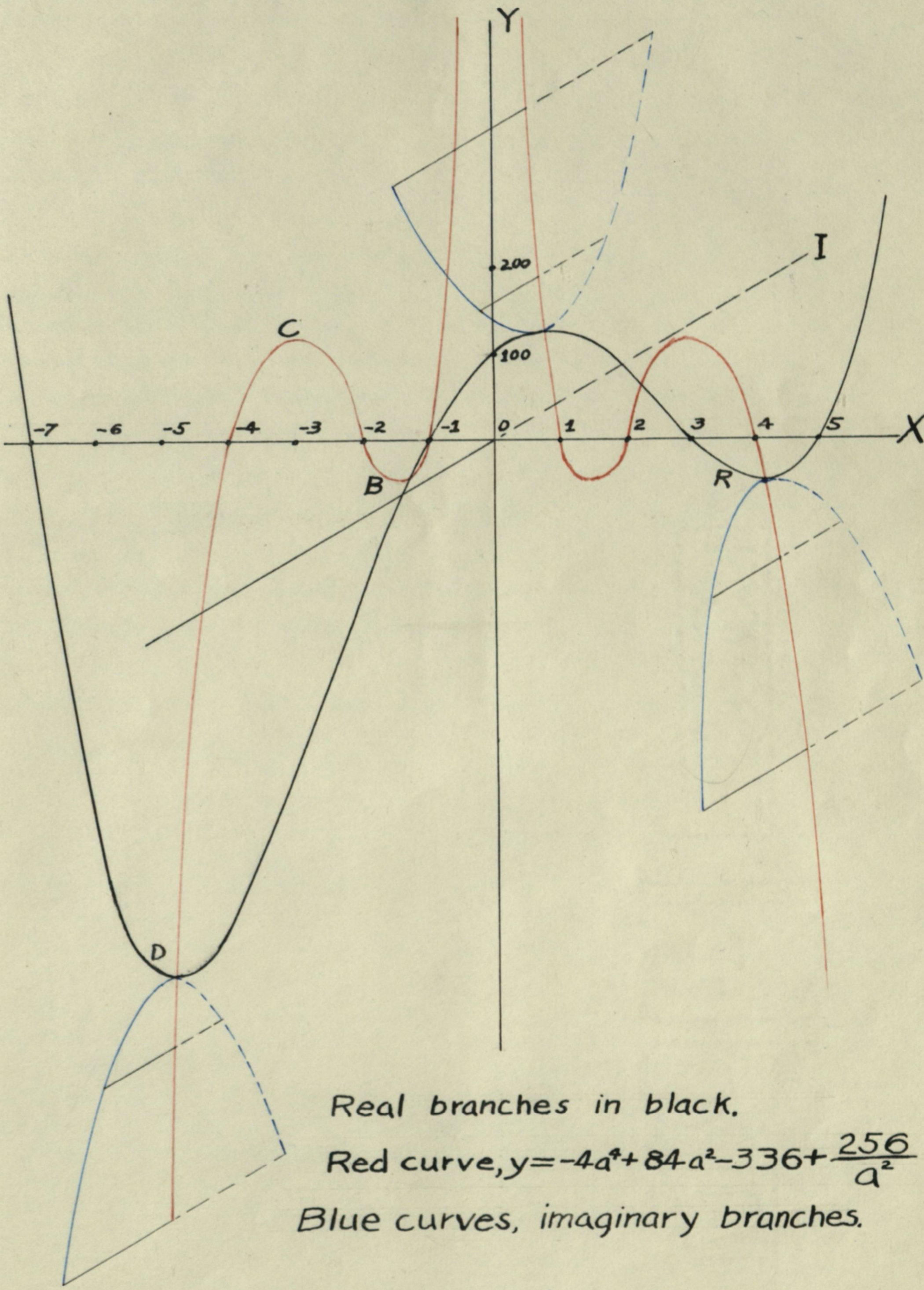
table continued

y	0	-224	-726	-1119	-2489	∞
a	4	4.5	5	5.5	6	0
b	>	±1.67	±2.68	±3.34		

Since b is imaginary for values of a from 1 to 4 inclusive and for negative values of a from 0 to - 4 inclusive, the following are needed.

a	-5	-5.5
b	±.89	±2.44

Pictorial Representation of $y = x^4 - 42x^2 + 64x + 105$



Real branches in black.

Red curve, $y = -4a^4 + 84a^2 - 336 + \dfrac{256}{a^2}$

Blue curves, imaginary branches.

The equation $x^4-42x^2+64x+105=0$ has no imaginary roots., The graph of page gives a good idea of the real graph,, the auxiliary "a" curve and the imaginary branches for the complete quartic curve which has no special irregularities., By increasing or decreasing the constant term the entire graph would be raised or lowered and intersect the Ix plane in four points., Of these either two or four might be complex., Obviously no matter how much the graph is lowered there would always be at least two real roots., For values of a giving the part ABCD the values of b are imaginary., These correspond to points on the real branch $y=f(x)$ on the portion KRLD.,

The imaginary branches projected onthe Ix plane resemble parabolas., This is not always the case., When two "elbows" of the curve approach each other and vanish two imaginary branches approach each other and merge becoming, roughly speaking, of hyperbolic form., The next example illustrates this., It is taken from Murphy's Theory of Equations page 125 where it is used to illustrate this method of determining the complex roots of equations by means of recurring series[16]., For $x^4+px^2+qx+r = 0$ (1)

We have $4a^4+2a^2p+\frac{p^2}{4}+\frac{q^2}{16a^2} = r-y$ (2)

If $x^4+x+10 = 0$ (3)

(2) becomes $y = -4a^4+10+\frac{1}{16a^2}$ (4)

For (1): $b = \pm\sqrt{\frac{p}{2} + a^2 + \frac{q}{4a}}$ (5)

For (3), (5) becomes

$b = \pm\sqrt{a^2 + \frac{1}{4a}}$ (6)

The only extreme point is given by:

$\frac{dy}{dx} = 4x^3+1.$ Whence $x = \frac{1}{2}\sqrt[3]{2} = -.63$

$y=x^4+x+10$

x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	1	$\frac{3}{2}$	2	3
y	88	24	13.5	10	9.5	0	12	16.5	28	94

$y = -4a^4+10+\frac{1}{16a^2}$

y	16.2	11.5	10.6	10.4	10.2	9.65	
a	.1	.2	.3	.4	.5	.6	.63
±b	1.5	1.13	.92	.88	.86	.87	0
a	-.1	-.2	-.3	-.4	-.5	-.6	-.63
±b	Imaginary	-	-	-	-	-	0

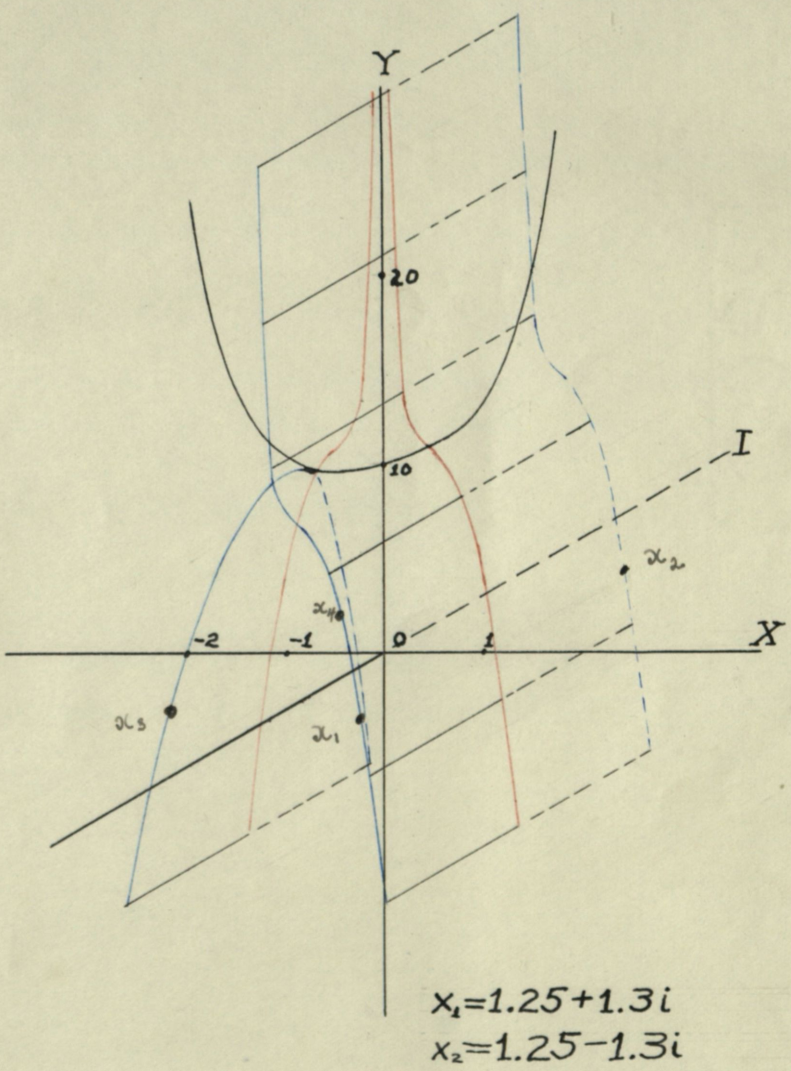
(table Continued)

y	9.16	8.5	7.5	6.06	4.2	1.74	-10.2	∞
a	.7	.8	.9	1.	1.1	1.2	1.5	0
±b	.84	.93	1.08	1.11	1.19	1.28	1.55	∞
a	-.7	-.8	-.9	-10	-1.1	-1.2	-1.5	
±b	.14	.33	.54	.86	.99	1.03	1.44	

Pictorial Representation of $y=x^4+x+10$

As before $y=-4a^4+10+\dfrac{1}{16a^2}$

and $b=\pm\sqrt{a^2+\dfrac{1}{4a}}$



$x_1=1.25+1.3i$

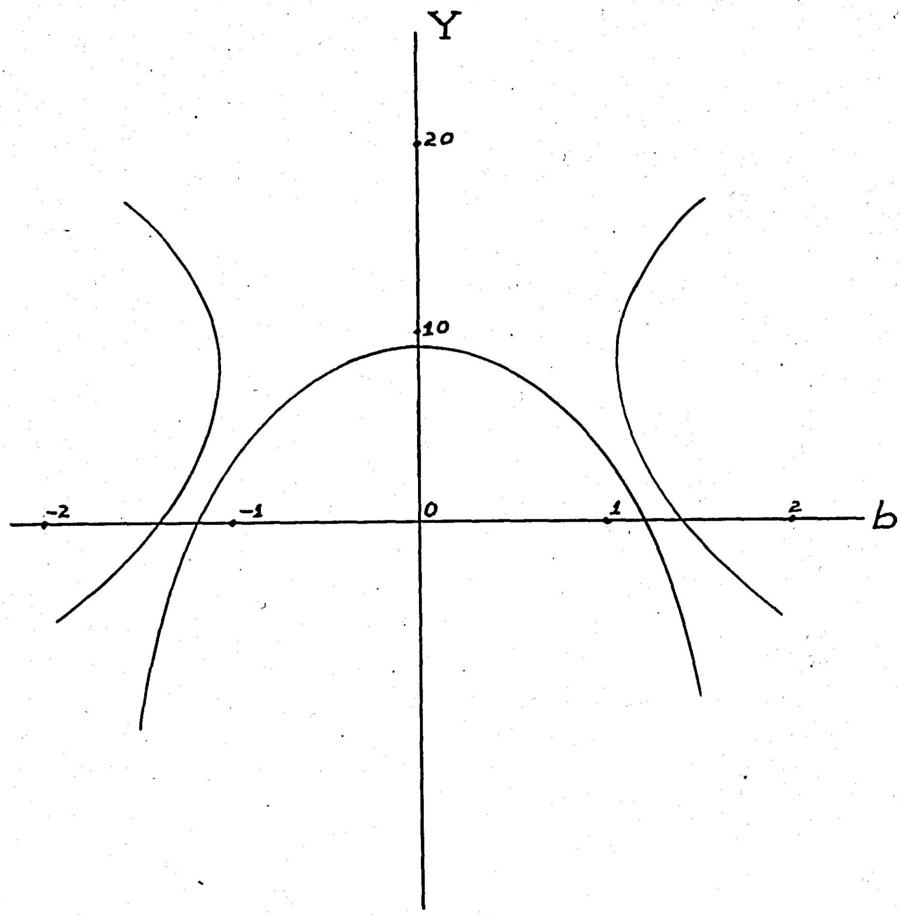
$x_2=1.25-1.3i$

$x_3=-1.25+1.17i$

$x_4=-1.25-1.17i$

"b" Curve for $y=x^4+x+10$

$$b=\pm\sqrt{a^2+\frac{1}{4a}}$$



The next example is taken from Phillips and Beebe's Graphic Algebra page 137.. It illustrates the case where one or more of the complex branches, which are usually doubly curved, becomes a plane curve.. It also illustrates the point that care must be exercised in using the equation

$$b = \pm\sqrt{\frac{p}{2} + a^2 + \frac{q}{4a}}$$

if q and a become zero at the same time.. These curves are shown side by side, a pictorial view being difficult to make clear.. Obviously the roots of $x^4+x^2+a=0$ where a is any constant can be obtained from the graph..

Given $y=x^4+x^2$

As before $y=-4a^4-2a^2-\frac{1}{4}$

$b = \pm \frac{1}{2}\sqrt{\frac{1}{2} + a^2}$ (if $a \neq 0$)

If $a = 0$, we proceed thus:

$$y = -4a^4-2a^2p-\frac{p^2}{4}+\frac{q^2}{16a^2}+r \qquad (9)$$

$$b^2 = \frac{p}{2}+a^2+\frac{q}{4a} \qquad (8)$$

x^4+x^2	80	39	20	2	0	2	20	39	80	$\frac{5}{16}$	$7\frac{15}{16}$
x	3	$-\frac{5}{2}$	-2	-1	0	1	2	$\frac{5}{2}$	3	$\frac{1}{2}$	$\frac{3}{2}$

$-4a^4-2a^2-\frac{1}{4}$	$-72\frac{1}{4}$	$-29\frac{1}{2}$	$-6\frac{1}{4}$	$-\frac{1}{4}$	$-6\frac{1}{4}$
a	-2	$-1\frac{1}{2}$	-1	0	1
$\pm b$ ($b=\pm\sqrt{a^2+\frac{1}{2}}$)	2.12	2.	1.11	.707	1.11

$-4a^4-2a^2-\frac{1}{4}$	$-29\frac{1}{2}$	$-72\frac{1}{4}$	-1
a	$1\frac{1}{2}$	2	$\frac{1}{2}$
$\pm b$ ($b=\pm\sqrt{a^2+\frac{1}{2}}$)	2	2.12	.43

Solving (8) for $\frac{q}{4a}$ and substituting in (9) gives

$y \doteq b^4-b^2$ (when $a = 0$)

Hence when $a = 0$, $y = b^4-b^2$, we have the following:

b^4-b^2					$-\frac{3}{16}$		0		$-\frac{3}{16}$		0	$2\frac{13}{16}$	12
b	2	$-\frac{3}{2}$	-1	$\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2
a	0	0	0	0	0	0	0	0	0	0	0	0	0

Projection of the Imaginary Branches of
$y = x^4 + x^2$
on the XY and IY Planes

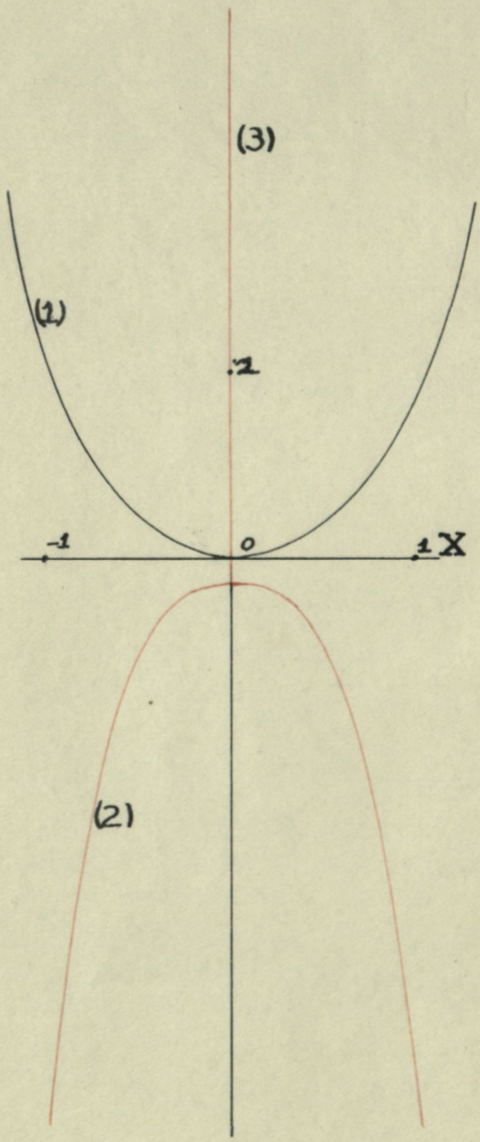
(1) $y = x^4 + x^2$

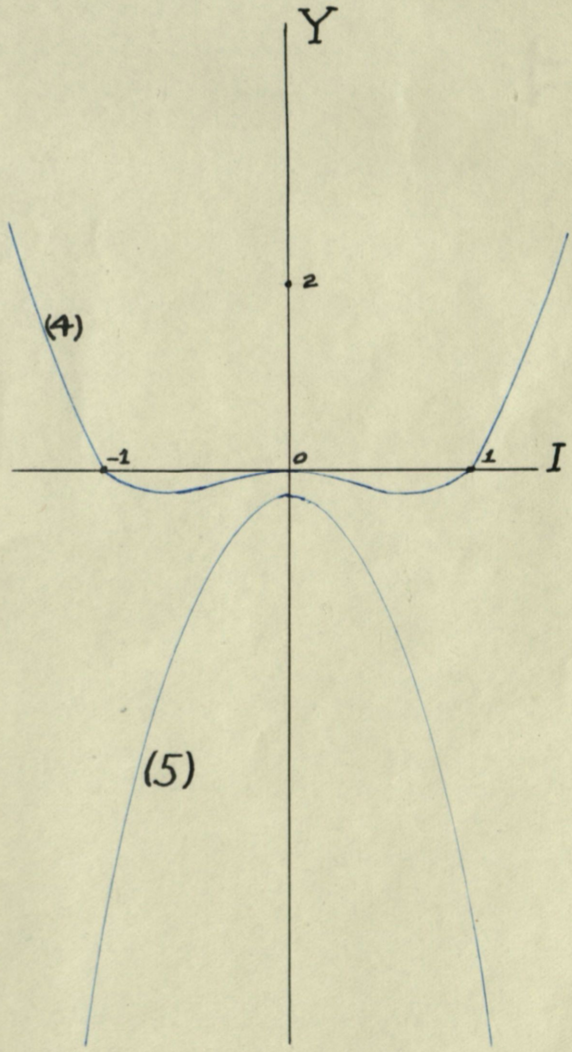
(2) $y = -4a^4 - 2a^2 - 1/4$

(3) $a = 0$

(4) $y = b^4 - b^2$

(5) $b = \pm\sqrt{1/2 + a^2}$





THE QUINTIC

$$x^5+px^3+qx^2+rx+t = 0 \qquad (1)$$

Let $x = a+bi$ then (2)

$$a^5+5a^4(ib)+10a^3(ib)^2+10a^2(ib)^3+5a(ib)^4+(ib)^5$$
$$+p(a^3+3a^2bi+3a(ib)^2+ib)^3$$
$$+q(a^2+2abi+(ib)^2)+ra+rib+t = 0 \qquad (3)$$

$$\therefore a^5-10a^3b^2+5ab^4+pa^3-3apb^2+a^2q-b^2q+ra+t = 0 \qquad (4)$$

and $5a^4b-10a^2b^3+b^5+3a^2bp-pb^3+2abq+bi = 0$ (5)

Then $5a^4-10a^2b^2+b^4+3a^2p-pb^2+2aq+r = 0$ (6)

Eliminating b^4 between (4) and (6):

$$(6)\cdot 5a, \; 25a^5-50a^3b^2+5ab^4+15a^3p-5ab^2p+10a^2q+5ar = 0 \qquad (7)$$

$$(7)-(4), \; 24a^5-40a^3b^2-pa^3+ \quad 15a^3p-2ab^2p+9a^2q+b^2q+4ar-t=0 \qquad (8)$$

$$b^2(40a^3+2ap-q)=24a^5 \quad +14a^3p+9a^2q+4ar-t \qquad (9)$$

$$b^2 = \frac{24a^5 \quad +14a^3p+9a^2q+4ar-t}{40a^3+2ap-q} \qquad (10)$$

We now eliminate b between (6) and (10) obtaining

$$b^4-(10a^2+p)b^2+5a^4+3a^2p+2q+r = 0 \qquad (11)$$

From (4) and (10)

$$5ab^4-(10a^3+3ap+q)b^2+a^5+pa^3+qa^2+ra+t = 0 \qquad (12)$$

For $x^5-25x^3+20x^2+84x-80 = 0$ (13)

we obtain from (12)

$$5ab^4-b^2(10a^3-75a+20)+a^5-25a^3+20a^2+84a-80 = 0 \qquad (14)$$

and from (10)

$$b^2 = \frac{6a^4-3a^3-86a^2+88a+40}{5(2a^2-a-2)} \qquad (15)$$

It may be observed here that a common factor $a+\frac{1}{2}$

appears in the numerator and denominator of the fraction obtained in (10) when p, q, r, and t are substituted from (13). The equation

$$2a^2-a-2 = 0$$

has the roots $\frac{1\pm\sqrt{17}}{4}$. Hence (14) and (15) become infinite at a = 1.28 and a = −.78. These asymtotic lines are shown in the graph page 40

$$y = x^5-25x^3+20x^2+84x-80$$

y	−2240	0	480	417	280	0	−120	min	114
x	−6	−5	−4	−3.5	−3	−2	−1	−.84	−.5

(table continued)

y	−80	36	0	15	0	−80	−92	0	840
x	0	.5	1	1.5	2	3	3.3	4	5

$$b^2 = (6a^4-3a^3-86a^2+88a+40)\div(10a^2-5a-10)$$

$$y = 5ab^4-b^2(10a^3-75a+20)+a^5-25a^3+20x^2+84x-80$$

a	−5	−4	−3	−2	−1.2	−1	−.8	−.78	−.6	−.5
y		559						∞	−1370	−85
b^2	5.9	−.24	−	−	−	−		∞	+12.45	4.95
±b										

(table continued)

a	−.4	−.3	0	.55	.7	1	1.1	1.2	1.3
y			0	−40	−52	0	45.2	1860	
b^2	13.4	−.77	−4	−	−	−9	−	−	+48.7
±b									

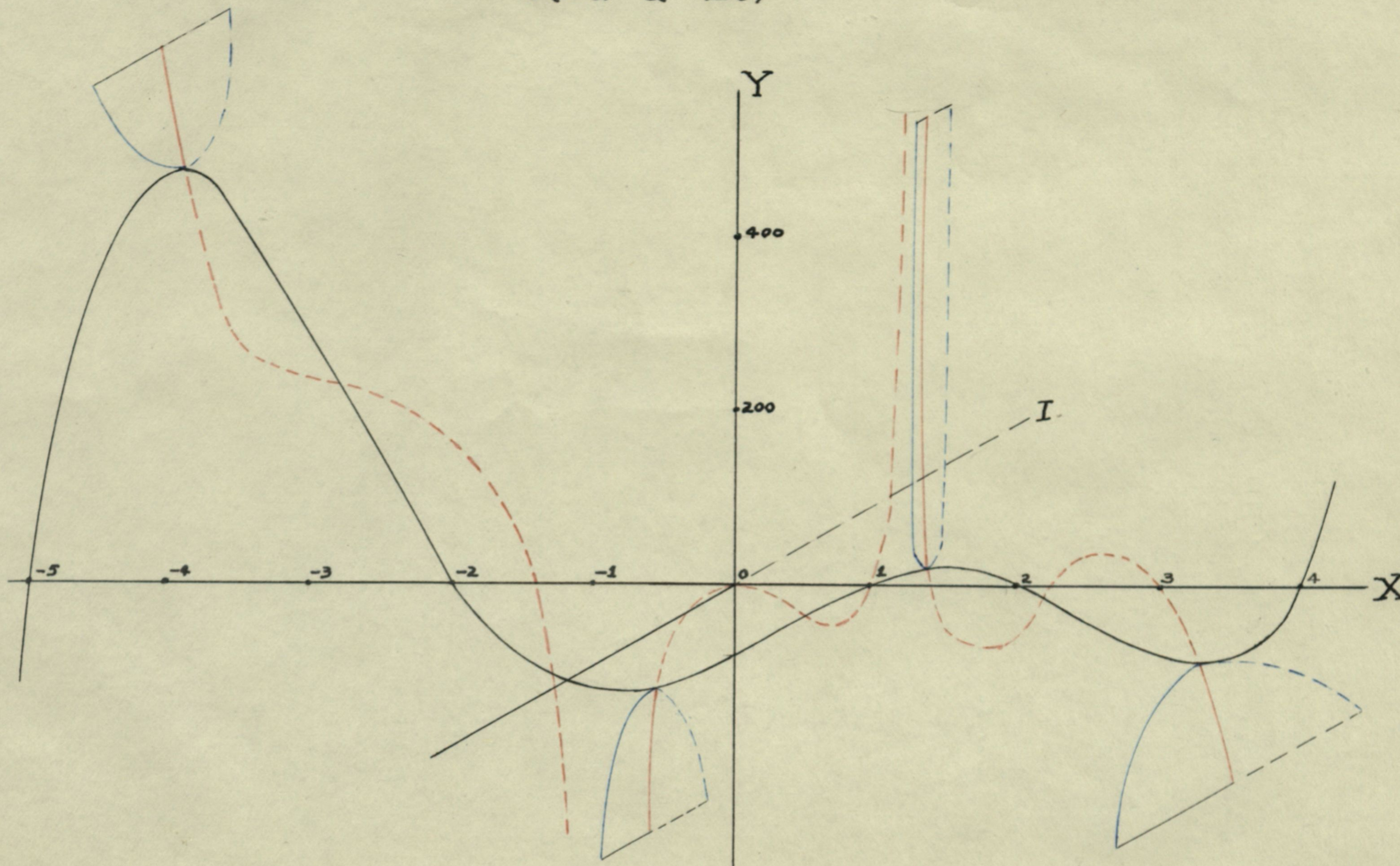
(table continued)

a	1.4	1.5	2	2.5	3	$3\frac{1}{2}$	4
y	-10.3	46.7	-616	38	0	-188	844
b^2	-	-	-	-	-	+1.83	+2.77
$\pm b$							

Pictorial Representation of $y = x^5 - 25x^3 + 20x^2 + 84x - 80$

Red Curve, $y = 5ab^4 - b^2(10a^3 - 75a + 20) + a^5 - 25a^3 + 20a^2 + 84a - 80$

Blue Curves, $b^2 = \dfrac{6a^4 - 3a^3 - 86a^2 + 88a + 40}{5(2a^2 - a - 20)}$



THE SEXTIC

$$x^6+px^4+qx^3+rx^2+tx+s = 0 \qquad (1)$$

Let $x = a+ib$ and we obtain

$$6a^5-20a^3b^2+6ab^4+4a^3p-4ab^2p+3a^2q-b^2q+2a+t = 0 \qquad (2)$$

$$-b^6+(15a^2+p)b^4+(-15a^4-6a^2p-3a^2q-r)b^2$$
$$+a^6+pa^4+qa^3+ra^2+ta+s = 0 \qquad (3)$$

Here we can eliminate b^6 between (2) and (3) giving equation (4). Then between (2) and (4) we can eliminate b^4 giving equation (5). This last can be solved for b^2 giving equation (6) and the result substituted in (3). This gives the "a" equation. The value of b^2 in terms of a given by (6) gives the necessary equations.

We can of course solve (2) for b^2 by quadratics and substitute the result in (3).

We can use Sylvester's method of elimination.

The method first outlined has been found easier in practice than the others.

CONCLUSIONS

The value of the method here presented for obtaining complex roots of equations lies first of all in its generality. The cubic, quartic, quintic and sextic are treated by the same method. The short methods fail for one case of the quartic and entirely for the quintic.

The method is closely analagous to that for real roots. Where the real graph cuts the x-axis giving real roots we have the imaginary branches (the real branch) also graph piercing the Ib plane giving all the roots whether real or complex. This renders it valuable for didactic purposes. The value of real graphs in the exposition of the theory of equations can not be overestimated. The use of graphs of the imaginary branches would be even more illuminating. The method here used displays the totality of values which make y real in $y=f(x)$. This brings the complex roots out of a mysterious, uncharted realm and renders them tangible and visible.

Certain purely mathematical facts or theorems may be pointed out.

(a) The "a" curve always passes through the extreme points of the real curve $y=f(x)$.

(b) If n is the degree of the equation the "a" curve crosses the x-axis $\frac{n}{2}$ times if n is even and $\frac{n-1}{2}$ times if n is odd.

(c) If in $y=f(x)$ we put $x=a+ib$ we obtain

$F_1(A,B)=0$ (1) (Reals = 0)

$F_2(a,b)=$ (2) (Imaginaries = 0)

Now if b^2 be obtained from (1) and (2) its value substituted in (1) will give the required "a" curve (the one passing through the extreme points of $y=f(x)$). If b^2 be substituted in (2) however the resulting "a" curve is not the required one.

Proof: For the first quartic graphed the two equations are:

$$5ab^4-b^2(10a^3-75a+20)+a^5-25a^3+84a-80=0 \qquad (3)$$

$$5a^4-75a^2+40a+84+(25-10a^2)b^2+b^4=0 \qquad (4)$$

Now at the extreme points $b=0$. Put $b=0$ in (3) and (4) and remember that $a=x$ so far as plotting purposes are concerned and we see that (3) passes through the extreme points of $y=f(x)$ and (4) does not.

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