# A SPECIAL RIEMANN SURFACE WITH APPLICATION TO THE HYPER-ELLIPTIC CASE.

by

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# INTRODUCTION.

The purpose of this paper is to consider in detail, for elliptic functions and also briefly for hyper-elliptic functions, a special Riemann surface in three space, obtained as the projection of the intersection of two hyper-surfaces in four space. It will be shown that this surface has all the properties peculiar to a multiple sheeted Riemann surface in two space.

It will be seen that the surface investigated here is of advantage in the fact that it can be easily identified, from the point of view of analysis situs, with a double faced disk having p holes; where $p = \left[\frac{n-1}{2}\right]$*, n being the degree of the function. As is well known, in Riemanns real representation this is obtained only after an artificial and somewhat complicated dissection of the surface, in which the determination of the branch points is a very important factor. In a sense this difficulty may be said in our case to have been merely shifted from such a dissection to the construction of a certain real surface from its equation in three space. This construction in our case can, however, be made very simple. In the ordinary Riemann surface the actual location of the branch points is difficult at best, and is useless so far as the investigations bearing on the surface are concerned. Similarly here, the actual construction

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*$\left[\frac{n-1}{2}\right]$ is understood as the greatest integer in $\frac{n-1}{2}$

of the surface will be avoided, except in the simplest case, namely, the elliptic one, and then, only as much of its outline as is necessary will be obtained. This construction will be found to be comparatively simple.

This paper will be divided into the following main divisions:-

A. General Discussion.

B. A Numerical Example of the Elliptic Case.

C. The General Elliptic Case for which $f(z)$ has real Roots.

D. The General Elliptic Case.

E. A Numerical Example of the Hyper-elliptic Case.

F. Some Considerations of the General Hyper-elliptic Case.

G. Double Curves of the Surface.

## A. GENERAL DISCUSSION.

Let $f(w,z)=0$ be an irreducible polynomial in the two complex variables $w$ and $z$, with either real or imaginary constant coefficients. Substituting $w=u+iv$ and $z=x+iy$ in the above relation and separating reals from imaginaries we obtain the equation,

$$P(x,y,u,v) + iQ(x,y,u,v)=0;$$

Whence,

$$P(x,y,u,v)=0,$$

$$Q(x,y,u,v)=0.$$

The last two equations represent real three dimensional manifolds in the real four space $(x,y,u,v)$. Their intersection in four space will be the surface $\phi$. Assume that $w=w_0$, when $z = z_0$. It is then possible, in the neighborhood $z_0, w_0$, to expand $(w - w_0)$ in powers of $(z - z_0)$, and by analytical continuation to go from the neighborhood of $z_0$ to the neighborhood $z_1$. As $z$ changes from $z_0$ to $z_1$, $w$ will go from $w_0$ into one of the values $w_1$ corresponding to $z_1$. If this process be continued until $z$ by a continuous succession of values returns to $z_0$, $w$ may or may not return to $w_0$. In the first case the representative point on $\phi$ corresponding to a pair of values $(w,z)$ will describe a closed path, while in the second case the path will be open. The obvious one to one correspondence between points of the surface $\phi$ and sets of values $(w,z)$ shows that this surface can play the same role as the ordinary Riemann surface.

If between the surfaces $P=0$ and $Q=0$ $v$ is eliminated there arises the relation,

$$F(x,y,u)=0,$$

which represents in the three space $(x,y,u)$, a surface $F$, viz., the projection of $\phi$ in that space. This surface $F$, as well as $\phi$, can be used as a Riemann image; this being the configuration to be investigated in this paper. We shall limit ourselves, as here stated, to the hyper-elliptic case. It is of course at once evident, that instead of eliminating $v$ any one of the variables $x,y$ or $u$ might have been eliminated and a surface similar to $F$ obtained.

## B. A NUMERICAL EXAMPLE OF THE ELLIPTIC CASE.

Before proceeding with the general cubic a special cubic will be considered in detail, and enough of the resulting surface constructed to show its properties as a Riemann image. (This special cubic is chosen on account of its adaptability to cross-section representation).

Consider the equation:

$$W^2 = Z^3 - 31Z - 30$$

Substituting $w = u+iv$ and $z = x+iy$ in this equation, it becomes

$$U^2 - V^2 - [x^3 - 3XY^2 - 31X - 30 + i(2UV - 3X^2Y + Y^3 + 31Y)] = 0;$$

whence,

$$P = U^2 - V^2 - (X^3 - 3XY^2 - 31X - 30) = 0$$

and

$$Q = 2UV - (3X^2Y - Y^3 - 31Y) = 0$$

The intersection of $P = 0$ and $Q = 0$ in four space is the surface $\phi$. The v projection of $\phi$ in three space has for its equation,

$$F(X,Y,U) = 4U^4 - 4U^2(X^3 - 3XY^2 - 31X - 30) - (3X^2Y - Y^3 - 31Y)^2 = 0$$

This surface is symmetric to both the XU and XY planes. The trace on the XU plane is the XX axis twice and the real curve,

$$U^2 = X^3 - 31X - 30$$

representing all the real pairs (w,z) satisfying the original equation. The latter consists of an infinite branch and an oval. (See Fig.I,a). The XY trace consists of the XX axis twice and the hyperbola,

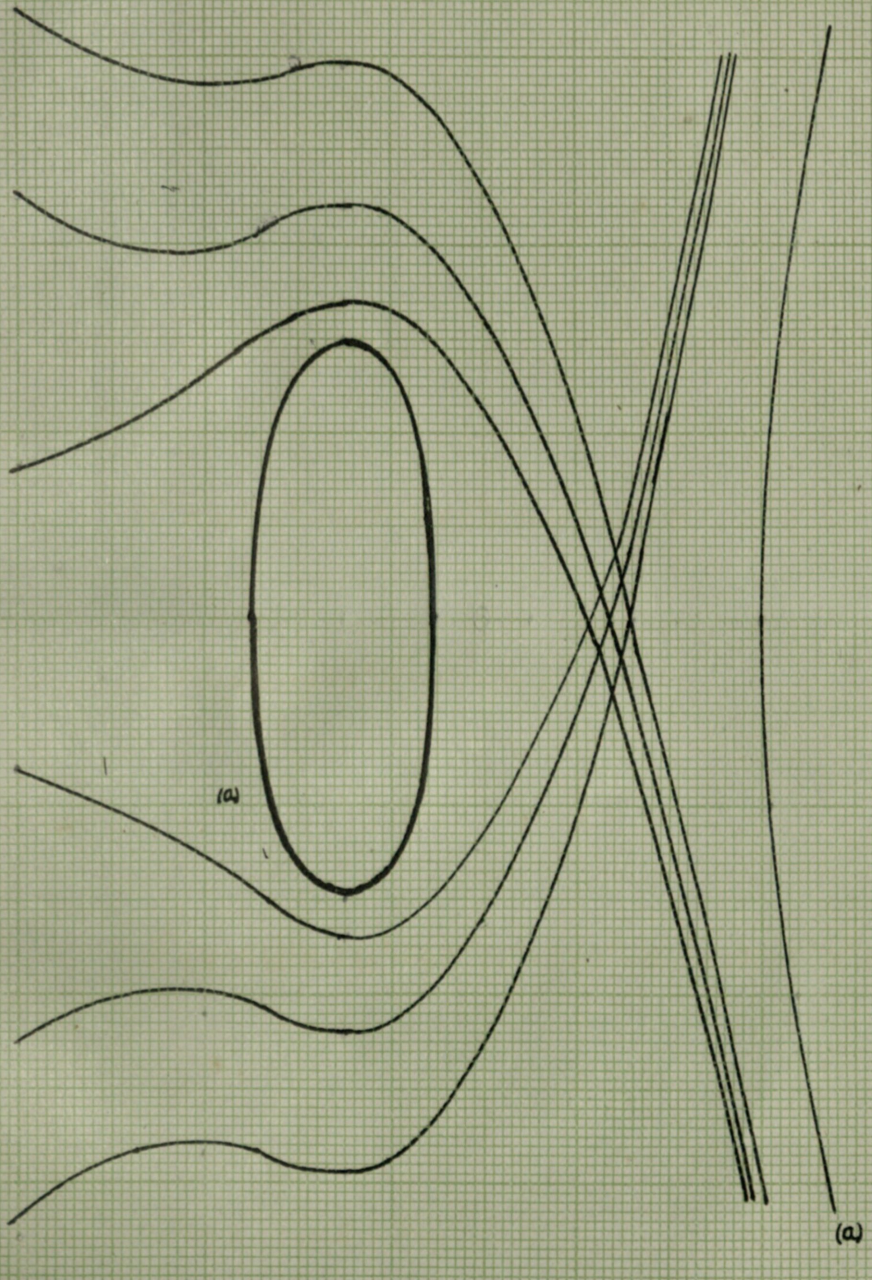


Fig. I

$$3x^2-y^2=31$$

taken twice.(See Fig.II,a). This hyperbola and the XX axis are the only double curves of the surface.

The facts thus far brought out seem to indicate that the oval is a section of a hole in F. That this is actually the case is shown as follows by considering sections of F parallel to the YU plane.Solving F(x,y,u)=0,we obtain

$$u=\pm\frac{\sqrt{2}}{2}\left[S^2+(S^2+T^2)^{1/2}\right]^{1/2}$$

where,

$$S=x^3-3xy^2-31x-30$$

and

$$T=3x^2y-y^3-31y$$

In this expression for u only positive values of the inner radical are considered,as only real points on the surface F are to be considered. For the purpose of examining sections parallel to the YU plane, take,

$$\frac{\partial u}{\partial y}=\pm\frac{\sqrt{2}}{4}y\frac{\left[-6x(S^2+T^2)^{1/2}+3x^4+3y^4+6x^2y^2+124y^2+180x+961\right]}{(S^2+T^2)^{1/2}\left[S+(S^2+T^2)^{1/2}\right]^{1/2}}$$

Then,if y=0, $\frac{\partial u}{\partial y}\equiv 0$ . For all negative values of x and positive values of y, $\frac{\partial u}{\partial y}$ is always positive or always negative according as u is positive or negative. For, $-6x(S^2+T^2)^{1/2}$ is positive and the only negative term in the expression for $\frac{\partial u}{\partial y}$ is 180x;but as is easily seen,

$$3x^4+961>180x$$

for all values of x. By similar reasoning it is seen that

$\frac{\partial u}{\partial y}$ is always negative or always positive, according as u is positive or negative, for all negative values of x and negative values of y. Hence, for all sections parallel to the YU plane, where x is negative, there will be either both a maximum and minimum point, or a double point, for y equal zero and for no other finite value of y. The same reasoning holds for positive values of $x \leq \sqrt{\frac{31}{3}}$ . For $x > \sqrt{\frac{31}{3}}$ , there are other maximum and minimum, or double points. Since the oval is not a double curve all of these sections that cut it will have maximum and minimum points for y equal zero, while all other sections within the region considered will have double points for y equal zero. Since points on the oval are always maximum or minimum points for the sections considered here, according as u is positive or negative, and no other like points exist on these sections, it follows that the projection of F upon the XU plane will be nowhere within the oval, and hence there is a hole in the surface for which the oval is the central section.

From the preceeding discussion and an inspection of the sections, (See Figs. I, III andIV), it is obvious that the surface F is composed of two sheets which pass through each other for values of x from $-\infty$ to -5, from -1 to +6 and along the branch of the double curve T=0 which lies to the right of the YY axis. Along the plane curve,

$$u^2 = x^3 - 31x - 30, \ y = 0$$

we can pass from one sheet to the other.

Taking any section parallel to the XU plane, (See Fig. I)

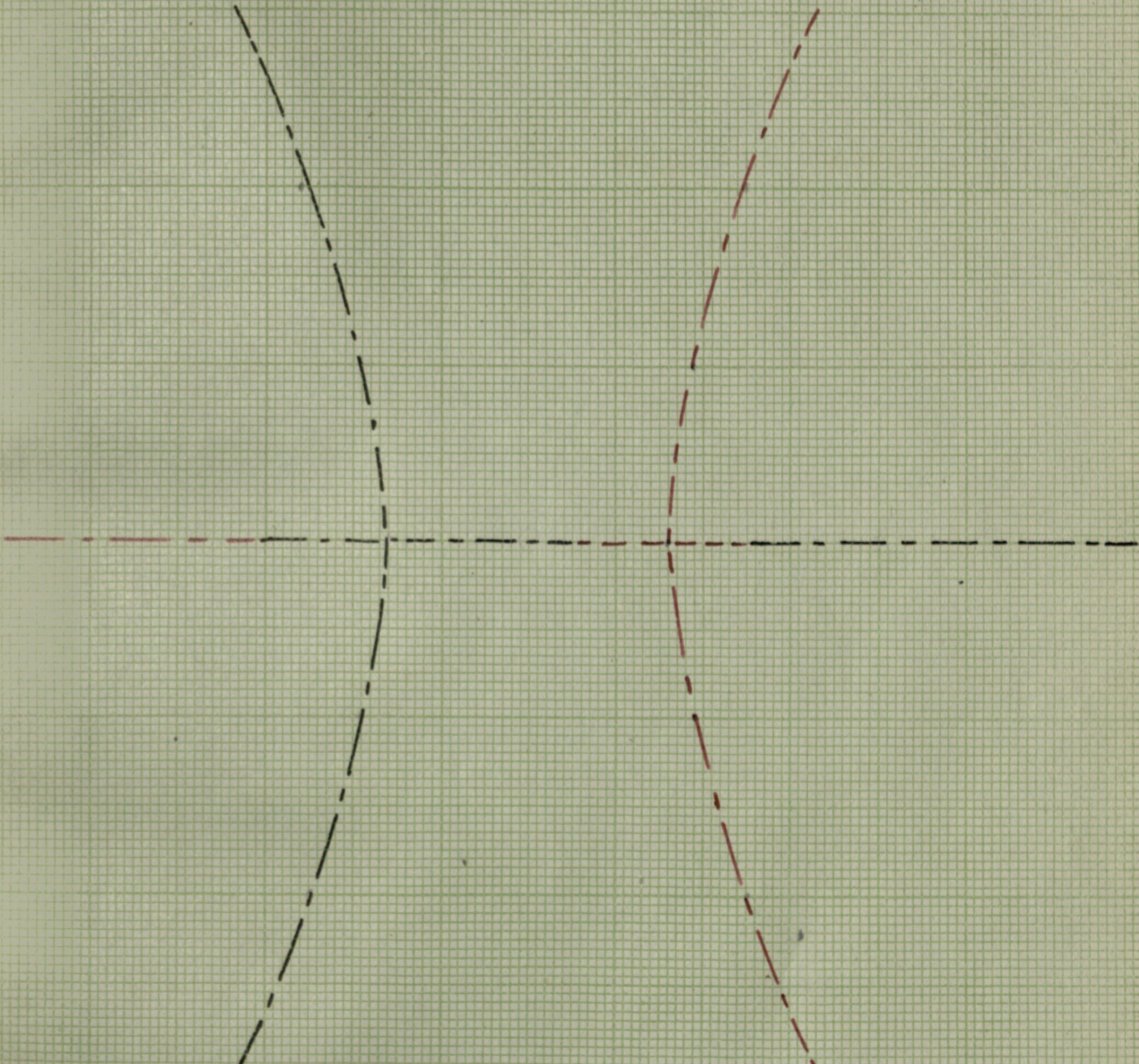

Fig. II

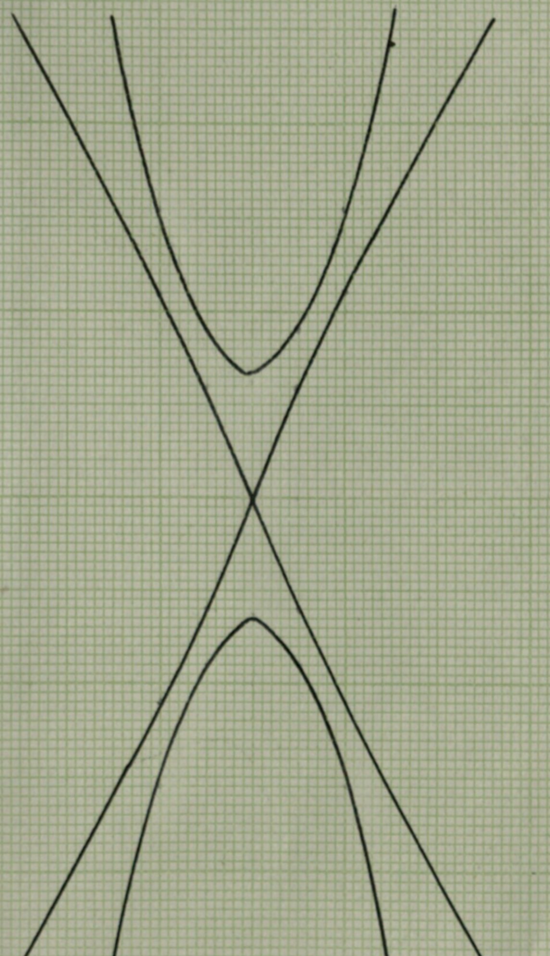


Fig. III

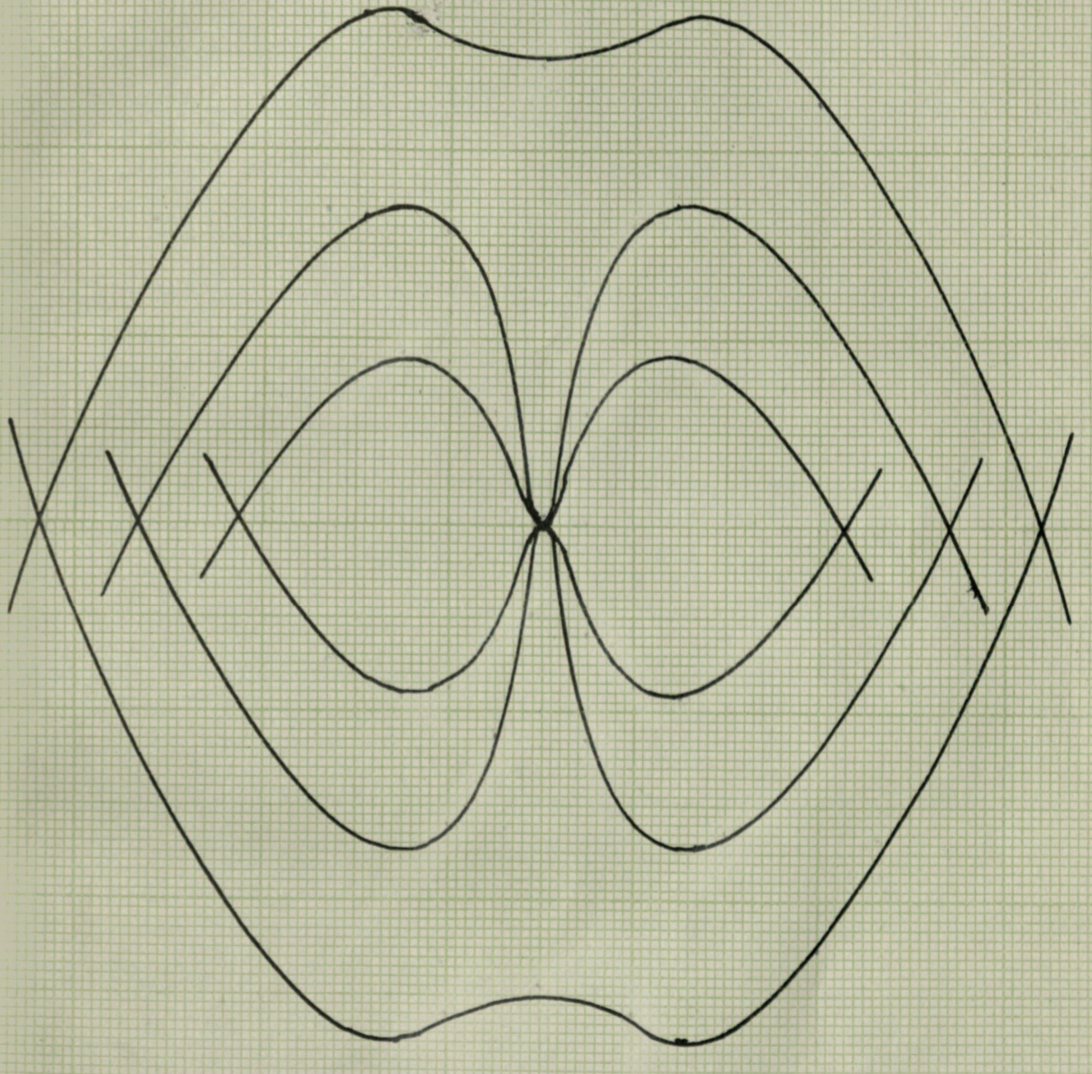

Fig. IV

two branches are obtained which intersect each other in a point on that branch of the double curve $T=0$, which lies to the right of the YU plane. These curves continue to infinity without cutting themselves, or each other, at any other finite point. They are symmetric to the XY plane and unite parabolically at infinity.

The sections parallel to the YU plane are all parabolic in nature, having for their asymtotic direction $y=0$. Likewise, the asymtotic directions of the XU sections are $x=0$ and $u=0$, showing that the two sheets of the surface merge into each other everywhere at infinity.

The surface F may be reduced to a double faced disk with one hole as follows: For all values of x greater than 6, deform the surface in such a way that instead of cutting each other in two distinct points on $T=0$ for each value of x, the two sheets will cut each other in two coincident points. This deformation will be continuous and approach zero in magnitude as x approaches 6 and will nowhere produce a tear in the surface. Having made this deformation project the surface upon the XU plane and the result will be a double faced infinite disk with one hole. It is then evident that if $\infty$ is excluded by means of a large circle the result will be a double faced finite disk with one hole. The two faces of this disk intersect each other; however, by a well known deformation similar to that employed by Neuman, the disk may be transformed into one in which the faces do not intersect each other.

Starting at a point $P_1$ in sheet I and continuing in any direction on the surface F we can always return to $P_1$. This closed path may be all in sheet I or in both sheet I and sheet II. It may or may not pass through or around the oval. In the latter case the circuit can always be reduced to zero, while in the former it cannot be so reduced, unless there be an even number of such passages and they be in opposite directions. Hence any closed circuit on F can be reduced to zero or to sums of multiples of two irreducible circuits. These facts show the elliptic function to be doubly periodic over F.

## C. THE GENERAL ELLIPTIC CASE HAVING REAL ROOTS.

We shall now extend the preceding discussion to any elliptic function of the type,

$$W^2 = z^3 - pz + q$$

where p is positive and q either positive or negative, and where the roots of

$$z^3 - pz + q = 0$$

are all real. It will be shown that the resulting surface $F(x,y,u)=0$ has properties identical with those of the special case already investigated, if judged from the point of view of the investigations of this paper.

We obtain at once, as in the preceding case,

$$F(x,y,u) = 4U^2 - 4U^2S - T^2 = 0$$

where,

$$S = x^3 - 3xy^2 - px + q$$

and

$$T = 3x^2y - y^3 - py$$

The similarity of the XU and XY traces to those in the preceding case is obvious. To investigate this surface by means of sections we proceed as before. The equation

$$F(x, y, u) = 0$$

gives

$$u = \pm \frac{\sqrt{2}}{2}\left[S + (S^2+T^2)^{1/2}\right]^{1/2}$$

and hence,

$$\frac{\partial u}{\partial y} = \pm \frac{\sqrt{2}}{4}\,\frac{y[-6x(S^2+T^2)^{1/2} + 3x^4 + 3y^4 + 6x^2y^2 - 6qx + 4py^2 + p^2]}{(S^2+T^2)^{1/2}[S+(S^2+T^2)^{1/2}]^{1/2}}$$

For $y=0$, $\frac{\partial u}{\partial y} \equiv 0$. For all negative values of x and positive values of y, $\frac{\partial u}{\partial y}$ is always positive or always negative, according as u is positive or negative. It is easily seen that if x is negative $-6x(S^2+T^2)^{1/2}$ is positive and all the other terms are positive with the possible exception of $-6qx$, which, however, is negative only if q is negative. In this case consider the three terms,

$$3x^4 - 6qx + p^2.$$

Itis desired to show that

$$3x^4 + p^2 \geqq 6|qx|. \qquad (1)$$

From the preceding assumption of real and unequal roots of

$$Z^3 - PZ + q = 0$$

it is easily seen that

$$|q| < \tfrac{2}{3}|P\sqrt{\tfrac{P}{3}}|. \qquad (2)$$

Substituting this relation in (1), we obtain

$$|3X^4| - |4P\sqrt{\tfrac{P}{3}}X| + |P^2| \geqq 0 \qquad (3)$$

Since, however, $\sqrt{\frac{P}{3}}$ is a double root of the equation

$$3X^4 - 4P\sqrt{\tfrac{P}{3}}X + P^2 = 0$$

and no other real roots exist, it is evident that the left hand member of (3) is never less than zero, and consequently (1) is always true. The same reasoning shows that for x negative and y negative, $\frac{\partial v}{\partial y}$ is always negative or always positive, according as u is positive or negative. Hence for all sections parallel to the YU plane, where x is negative, there will be a maximum and minimum point, or double point, for y equal zero and for no other finite value of y. The same conditions hold for positive values of $X \gtrless \sqrt{\frac{P}{3}}$. Since the sum of the roots of

$$Z^3 - PZ + q = 0$$

are zero, at least one root must be negative and therefore the oval must at least be partly to the left of the YU plane and $\sqrt{\frac{P}{3}}$ cannot lie to its left. In fact $X = \sqrt{\frac{P}{3}}$ is always to the right of the oval. To show this let $\alpha_1, \alpha_2, \alpha_3$ be the roots of

$$Z^3 - PZ + q = 0$$

where $\alpha_3 > \alpha_2 > \alpha_1$. If $\alpha_1$ and $\alpha_2$ are both negative the proposition is trivial. If $\alpha_2$ is positive take the known relations $\alpha_1+\alpha_2+\alpha_3=0$ and $-\alpha_1\alpha_2\alpha_3=q$. Substitute the value of $\alpha_1$ from the first relation in the second and we have

$$\alpha_2^2\alpha_3+\alpha_2\alpha_3^2=q. \qquad (4)$$

From (2) and (4) it is evident that

$$\tfrac{2}{3}p\sqrt{\tfrac{p}{3}} \geq q > 2\alpha_2^3.$$

Therefore,

$$\tfrac{2}{3}p\sqrt{\tfrac{p}{3}} > 2\alpha_2^3 \text{ or } \sqrt{\tfrac{p}{3}} > \alpha_2 .$$

For $x>\sqrt{\frac{p}{3}}$ there are other maximum and minimum points or double points than for y equal zero. As in the simpler case these sections are parabolic in nature, their asymptotic direction being y=0.

This discussion thus shows that this surface has no important characteristics, from our point of view, not common to the more special case investigated and is therefore always reducible to a double faced disk with one hole.

## D. THE GENERAL ELLIPTIC CASE.

In the previous sections our investigations have been confined to the type

$$W^2=Z^3-pZ+q$$

where p and q were both real, p positive and the roots all real. It will now be shown that no generality is lost by this restriction.

Consider the general elliptic case,

$$W^2 = f(z)$$

where

$$f(z) \equiv a_0(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)$$

and

$$a_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$$

are real or imaginary constants. The elliptic integral resulting from this form may by a well known transformation* of f(z) be made to depend upon an integral of the type,

$$g(z') = b_0(z'^3 - g_2 z' - g_3).$$

No generality is therefore lost by replacing f(z) by $g(z')$. In $g(z')$, $g_2$ and $g_3$ may be positive or negative, real or imaginary constants. If the coefficients $g_2$ and $g_3$ are arbitrarily changed the surface F will undergo a deformation. The only matter of interest in the present paper is whether such a deformation increases or decreases the number of holes in F. It is of course evident if the number of holes is diminished as $g_2$ and $g_3$ assume the values $g_2^\circ$ and $g_3^\circ$ , that as $g_2$ and $g_3$ approach $g_2^\circ$ and $g_3^\circ$ in value, one or more holes in the surface must be continuously decreasing in size in such a way that when $g_2^\circ$ and $g_3^\circ$ are reached the surface has a node at the point $(x_0, y_0, u_0)$ on F, and vice-versa. If $(x_0, y_0, u_0, v_0)$ is the corresponding point on $\Phi$, the latter will also have a node at this point. Therefore, correspond-

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*Boehm, Elliptische Funktionen, Zweiter Teil, page 128-129.

ing to nodes of F are nodes on $\phi$. At such nodes the tangent hyper-planes to

$$P(x,y,u,v)=0$$

and

$$Q(x,y,u,v)=0$$

are coincident. In order to investigate the nature of F at such places write the equations of the tangent hyper-planes to P and Q at the point $(x_0, y_0, u_0, v_0)$, and write the condition for their coincidence. The equations in question are,

$$(x-x_0)P'_{x_0}+(y-y_0)P'_{y_0}+(u-u_0)P'_{u_0}+(v-v_0)P'_{v_0}=0$$

and

$$(x-x_0)Q'_{x_0}+(y-y_0)Q'_{y_0}+(u-u_0)Q'_{u_0}+(v-v_0)Q'_{v_0}=0$$

The condition for these two hyper-planes to be coincident is that

$$\frac{P'_{x_0}}{Q'_{x_0}}=\frac{P'_{y_0}}{Q'_{y_0}}=\frac{P'_{u_0}}{Q'_{u_0}}=\frac{P'_{v_0}}{Q'_{v_0}}$$

It is evident,however,from the relation

$$P(x,y,u,v)+iQ(x,y,u,v)=0$$

that

$$P'_{x_0}=Q'_{y_0},\ P'_{y_0}=-Q'_{x_0},\ P'_{u_0}=Q'_{v_0},\ P'_{v_0}=-Q'_{u_0}.$$

Hence,

$$P'^2_{x_0}+Q'^2_{x_0}=0,\ P'^2_{y_0}+Q'^2_{y_0}=0,\ P'^2_{u_0}+Q'^2_{u_0}=0,\ P'^2_{v_0}+Q'^2_{v_0}=0;$$

and therefore,

$$P'_{x_0}=0,\quad P'_{y_0}=0,\quad P'_{u_0}=0,\quad P'_{v_0}=0$$

$$Q'_{x_0} = 0 \ , \ Q'_{y_0} = 0 \ , \ Q'_{u_0} = 0 \ , \ Q'_{v_0} = 0 .$$

In the above relations

$$P = u^2 - v^2 - S(x,y)$$

and

$$Q = 2uv - T(x,y) ;$$

therefore, if

$$P'_{u_0} = 0 \quad \text{and} \quad P'_{v_0} = 0$$

it follows that $u = 0$ and $v = 0$ and therefore, that $g(z') = 0$. Moreover, since

$$P'_{x_0} + i Q'_{x_0} = 0$$

and

$$P'_{y_0} + i Q'_{y_0} = 0$$

it follows that

$$S'_{x_0} + i T'_{x_0} = 0$$

and

$$S'_{y_0} + i T'_{y_0} = 0 .$$

Therefore, $g'(z') = 0$, showing that $z_0$ is a double root of $g(z') = 0$. It is evident therefore, that the surfaces P and Q, and hence F, may be deformed in any way we please without affecting its analysis situs properties, provided that during this deformation $g(z') = 0$ never acquires any double roots. These conditions allow a deformation that will change complex roots $a+ib, c+id, e+if$ into real and unequal roots $a', c', d',$ without any two roots becoming equal in the process. Hence we may in this manner transform $g(z')$ into $j(z'')$, where the roots of $j(z'')=0$ are real and unequal.

The above conclusions show that no generality is lost in considering the simpler case and thereby avoiding the the difficult task of dealing with imaginary coefficients. The difficulty introduced by imaginary coefficients is that due to the lack of symmetry with respect to the XU plane which was found in our previous discussion.

It is evident now, as was stated in the first section of this paper, that the surface constructed from the simplest possible relation is sufficient for a complete exposition of the Riemann surface properties of the most general elliptic function.

## E. A NUMERICAL EXAMPLE OF THE HYPER-ELLIPTIC CASE.

As an introduction to the general hyper-elliptic case we will consider briefly a simple numerical example. The details of the surface F will be considered sufficiently to show that what has been said about the elliptic case can be carried over in all its essential details to the higher form. For this purpose consider the equation,

$$W^2=(z+3)(z+2)(z+1)(z-1)(z-5).$$

Substituting $w=u+iv$ and $z=x+iy$ and separating the reals from the imaginaries, there arises the equation

$$P(x,y,u,v)+iQ(x,y,u,v)=0.$$

Hence,

$$P(x,y,u,v)=0,$$

and

$$Q(x,y,u,v)=0;$$

where,

$$P=u^2-v^2-(x^5-10x^3y^2+5xy^4-20x^3+60xy^2-30x^2+30y^2+19x+30)=0$$

and

$$Q=2uv-5(x^4y-10x^2y^3+y^5-60x^2y+20y^3-60xy+19y)=0\ .$$

The surface $F(x,y,u)=0$ will be represented by

$$4u^4-4u^2S-T^2=0,$$

where

$$S=(x^5-10x^3y^2+5xy^4-20x^3+60xy^2-30x^2+30y^2+19x+30)$$

and

$$T=5x^4y-10x^2y^3+y^5-60x^2y+20y^3-60xy+19y.$$

The surface F is symmetric to the XU and XY planes. The trace on the XU plane is the XX axis twice and the real curve,

$$v^2=(x+3)(x+2)(x+1)(x-1)(x-5),$$

representing all the real pairs (w,z) satisfying the original equation. The latter consists of the two ovals and an infinite branch. The trace on the XY plane is the XX axis twice and the double curve represented by the equation $T=0$. This curve is composed of four infinite branches which are hyperbolic in form and coaxial. (See Fig.VI).

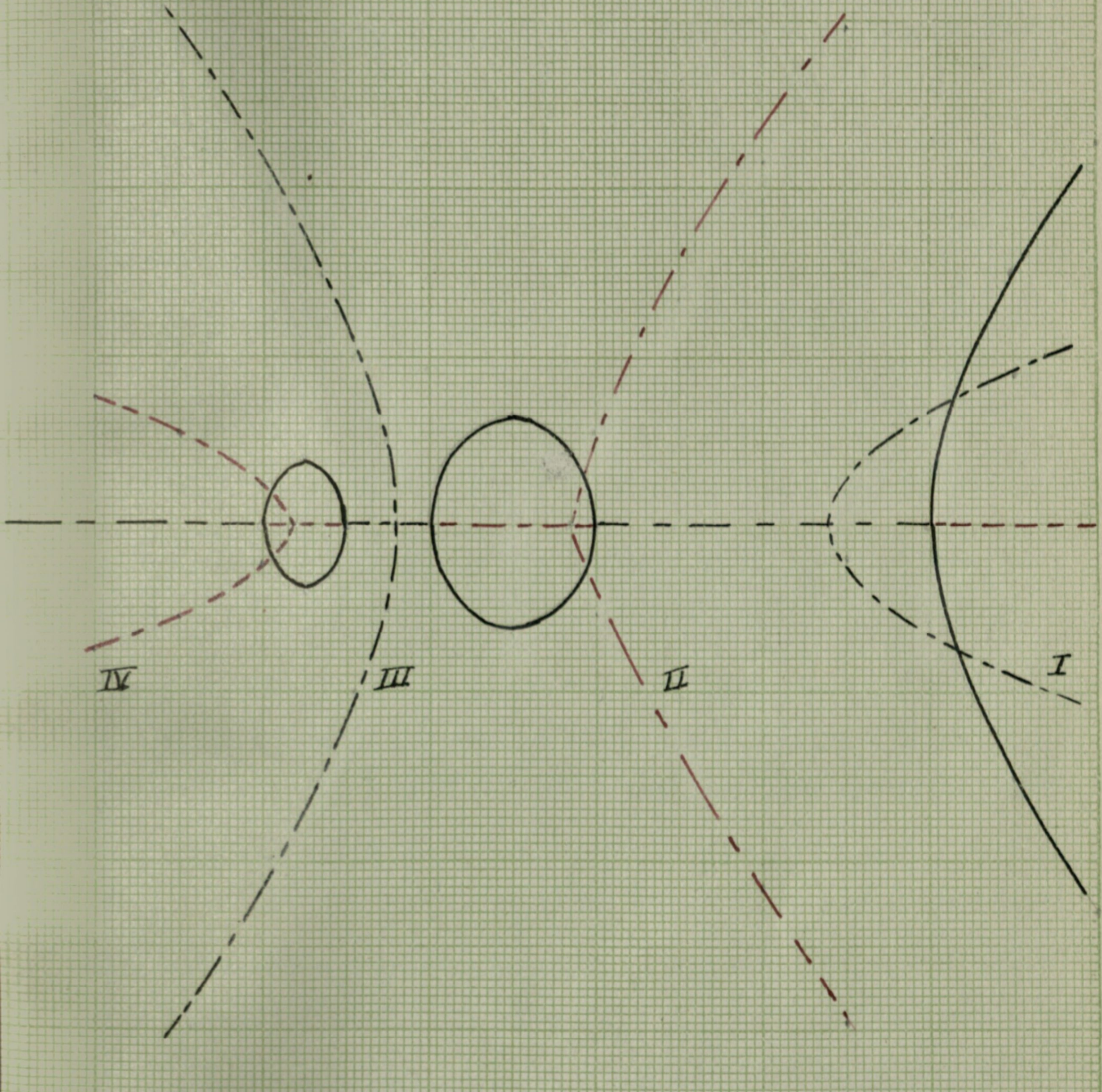


Fig. VI.

Sections parallel to the XU plane give rise to curves which have double points on the bramches I and III of the double curve,as shown in the figure,and nowhere else. This is shown by an investigation of the value of S in the neighborhood of these branches. For the two branches to intersect each other it is necessary that T be equal to zero and S be negative. Every pair of values (x,y) on one of these infinite branches reduces T to zero,but none of these pairs on branch II or IV will cause S to be negative. Therefore the two sheets of the surface F do not cut through each other along either of these branches. The two sheets pass into each other along the curve represented by,

$$V^2 = (x+3)(x+2)(x+1)(x-1)(x-5)\,,\ y=0\,,$$

and cut each other along the XX axis from $-\infty$ to -3,-2 to -1,from+ 1 to +5 and along the two branches I and III of T=0. The symmetry of the surface as shown by the equation

$$V = \pm\frac{\sqrt{2}}{2}\left[S+(S^2+T^2)^{1/2}\right]^{1/2}$$

leads at once to the conclusion that for one sheet to pass into the other T must be zero and at the same time S must be either zero or positive,since for T zero and S negative we always get a cutting of the two sheets. Now S can never be zero when T vanishes, except for the pairs of values (x,y) which are roots of the original expression f(z)=0,but all these pairs have y=0. Suppose S is positive along some branch of the double curve T=0, than this branch of T will be an isolated curve. To prove as in the elliptic case that the two sheets never pass into each other for any finite value of y except zero would be very complicated, and so another method is employed. It is easily seen that any section parallel to the YU plane will give rise to a curve which has a number, say d, double points. But this curve is composed of two branches which intersect in d points in the XY plane. If d is odd the two branches are odd and hence each branch stretches off to infinity in both directions. If d is even each branch is even and hence cuts the line at infinity in an even number of places and is accordingly a closed curve. In the first case (d odd) the XX axis must be composed of intersection points, while in the latter it is not. This leads to the conclusion that all sections which cut the curve $v^2 = f(x)$, $y = 0$ are even branches and all others odd. Hence the former are always reducible to sections of the form, Fig. III, (a), while the latter are always reducible to branches of the form Fig. III, (b). From this will follow as in the elliptic case, that F is two sheeted and contains two holes. By a deformation similar to the one described in the example of the elliptic case it may be brought into the form of a finite double faced disk with two holes. Hence all closed circuits on F may be reduced to zero or to sums of multiples of four irreducible circuits.

## F. SOME CONSIDERATIONS OF THE GENERAL HYPER-ELLIPTIC CASE.

Having considered the elliptic case in detail and investigated briefly a special hyper-elliptic equation, we now proceed to the most general hyper-elliptic function, $w^2=R(z)$, where $R(z)$ is of degree n.

Forming the equation of the surface F in the usual manner there arises the equation $F(x,y,u)=0$, where F is of degree 2n in $(x,y,u)$. The above relation may always be put in the form,

$$4U^4-4U^2S-T^2=0$$

where S and T are polynomials in x and y of degree n. As has been shown in a preceding section, $R(z)$ may be assumed to have only real roots. Hence the surface F is symmetric to the XU and XY planes. The XU trace will consist of the XX axis twice and a real curve consisting of all real pairs $(w,z)$ satisfying the original equation. The latter curve will consist of one or two infinite branches, according as n is odd or even, and p ovals. The XY trace $T=0$ will consist of the XX axis twice and a double curve represented by an equation of degree $(n-1)$. This double curve represents all the real double points of the surface F.

The surface F is composed of two sheets which hang together along the curve,

$$U^2=R(x),\ y=0$$

and cut each other along the XX axis for all real values of x not included in the relation $u=R(x)$, and also along certain other parts of the double curve $T=0$. Corresponding to the p ovals there will be p holes in F. All closed circuits on F may be reduced to sums of multiples of 2p irreducible circuits.

## G. DOUBLE CURVES.

The double curves of the surface F arise as the result of projecting the surface $\phi$ from four space into three space, the center of projection being at infinity. When ever a projecting line cuts $\phi$ in two places a double point occurs on F. If the two points on $\phi$ be real the double point on F will be a real double point connected with the surface F, but ifthe two points on $\phi$ be imaginary the resulting double point on F will be isolated. This gives rise to two classes of double curves, one being on the surface F and the other being related to the surface but isolated from it.

In the elliptic cases studied the double curves consisted of the XX axis and an hyperbola. That part of the XX axis included by the real part of the curve $u = f(x), y=0$ is isolated. Of the hyperbola, that branch lying to the left of the YU plane is isolated.

In the hyper-elliptic example the double curve consists of the XX axis and four infinite branches. What was said of the XX axis for the elliptic case holds here also. Of the four infinite branches two are isolated, (See Fig. VI ), and two are curves of intersection of the two sheets of the surface.

The same conditions will exist in the general hyper-elliptic case, the XX axis always being a double curve with the same law as to isolated points as in the simpler cases. The other double curves will be partly isolated and partly curves of intersection of the two sheets of the surface. The isolated curves separate themselves from the other class in that they always pass through one or more of the ovals, while the curves of intersection of sheets never do.