THE PRINCIPLE OF DUALITY IN A PROJECTIVE SPACE OF N DIMENSIONS

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SECTION 1, INTRODUCTION

Purpose

1. The purpose of this paper is to state and prive the principle of duality in a projective space of any number of dimensions.

2. History of the Development of the Principle of Duality.

The principle of duality has had its entire development as a principle during the last century, altho various theorems in which the idea of duality was noticed were discovered many centuries ago. One might easily conclude that Euclid must have noticed some kind of duality in such statements as "Two points determine a line." and "Two intersecting lines determine a point." Until recently, however, the principle was used chiefly as a means for discovering new propositions, without even a glimpse of the wonderful law itself.

Its development as a formal principle had its beginning in Poncelet's ideas of central projection (a) and reciprocal polars (b) together with a study of invariant properties of figures, introduced by Desargues.

Fink (c) summarizes the development of the principle of duality as follows: "In 1811 Servoir had used the expression 'pole of a straight line' and in 1813 Gergonne the terms 'polar of a point' and 'duality', but in 1818

(a) Traité des Propriétés des Figures, second edition, Paris, 1865, Section I, ChapitreI, page 3.
(b) Traité des Propriétés des Figures, Section II Chapitre II, page 116.
(c) Geschichte der Elementar -Mathematik, by Dr. Karl Fink, translated by Beman and Smith, Chicago, 1900, page 249.

Poncelet developed some observations made by Lahire in 1685, upon the mutual correspondence of pole and polar in the case of conics, into a method of transforming figures into their reciprocal polars. Gergonne recognized in this theory of reciprocal polars a principle whose beginnings were known by Vieta, Lansberg and Snellius, from spherical geometry. He called it the 'principle of duality', 1826. In 1827 Gergonne associated dualistically with the notion of order of a plane curve, that of its class (a). While in France, Chasles alone interested himself thoroughly in its advancement, this new theory found its richest development in the third decade of the nineteenth century upon German soil, where almost at the same time the three great investigators, Möbius, Plücker and Steiner entered the field. From this time on the synthetic and more constructive tendency followed by Steiner, von Staudt and Möbius (b) diverges from the analytic side of Modern Geometry which, Plücker, Hesse, Aronhold and Clebsche had especially developed."

Emch (c) and E. H. Smart (d) tell practically the same story of the development of the principle of duality.

(a) Baltzer.
(b) Brill, A., Antrittsrede, Tübingen, 1884
(c) Introduction to Projective Geometry and its Applications, by Arnold Emch, New York, 1905, page 68, "Historic Note."
(d) A First Course in Projective Geometry by E. H. Smart, Macmillan Co., 1913, page 20.

Florian Cojori (a) says, in part, "Gergonne and Poncelet carried on an intense controversy on the priority of discovering the principle of duality. No doubt, Poncelet entered this field earlier, while Gergonne had a deeper grasp of the principle. To Jean Victor Poncelet in Traite des Propriétés projectives des figures, 1822, we owe the Principle of Duality as a consequence of reciprocal polars. As an independent principle it is due to Gergonne." "Julius Plüker in the Analytisch-geometrische Entwicklungen (1828 & 1831) second volume, the principle of duality is formulated analytically." "Steiner's Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander (1832) is the first book in which the principle of duality is introduced at the outset."

Cremona (b) says "Professor Reye remarks, with justice in the preface of his book, that Geometry affords nothing so stirring to the beginner, nothing so likely to stimulate him to original work, as the principle of duality, and for this reason it is very important to make him acquainted with it as soon as possible, and to accustom him to enploy it with confidence." But Cremona makes no attempt to prove the principle, tell where a proof may be found, or to give any instructions about its use.

(a) A History of Mathematics, Second Edition, 1919, pages 288, 290, and 310.
(b) Elements of Projective Geometry, edited by Luigi Cremona, Oxford, 1885, translated by Charles Leudesdorf, page VII .

Veblen and Young (a) state that "The Principle of duality was first stated explicitly by Gergonne (1826), but was led up to by the writings of Poncelet and others during the first quarter of the nineteenth century. It should be noted that this principle was for several years after its publication the subject of discussion and often acromonious dispute, and the treatment of this principle in many standard texts is far from convincing."

Most writers of Modern Geometry, even of recent texts, use the principle of duality rather extensively; but very few of them make any mention of whether it is to be assumed or might be proved, or even to what extent it is valid. A thorough treatment of the subject may be found tho, in Veblen and Young's Projective Geometry, volume I, pages 15 to 34. Here we find enough of the proof of the theorem in a space of two dimensions to make a complete proof of the theorem in a space of three dimensions. This is followed by a proof of the theorems of alignment for a space of N dimensions assuming they have been proved for the cases $N = 2, 3, --- \underline{N-1}$. These definitions, assumptions and proofs are assumed in this paper.

3. <u>Method</u>

The method to be used for the proof of the proposition in this paper is the method of Mathematical Induction. According to Veblen and Young, one of the important advantages of this method of formal inference from explicitly stated assumptions is that it makes the theorem appear almost self-

(a) Veblen and Young, Projective Geometry, Boston 1910 volume I, page 29.

evident.

1st. The proof that the statement actually holds in the first instance, or that is in some particular instance. This part of the proof will be taken care of in this paper by the proof that the theorem holds in a space of four dimensions, page 7.

2nd. The proof that if the statement holds in any particular instance, it also holds in the next. This will be done by proving that if the theorem is true for an N-1 space, it is true for an N space.

4. Explanation of the special symbols used in this paper.

The starting point of a strictly logical treatment of any branch of mathematics must be a set of undefined elements and relations, and a set of unproved propositions involving them; and from these all other propositions are to be derived by a method of formal logic. The undefined elements are the OS, read "Zero space", which means a space of no dimensions or in ordinary language a point, and the 1S read "one space", which means a space of one dimension or more commonly a line. As long as a definitely numbered space is designated the number of the space will be give; but when an indefinitely numbered space is mentioned it will be called some letter space. For example; 1S, 5S, and 8S mean one space, five space and eight space respectively; while LS and NS may be used to mean any number space. To designate a particular number space, the number space will be called a particular letter, thus; 1S A, means the one space A. Another method used for the designation of a particular

NS is to call it NS AB, which means the NS that was determine either by the N-1 S A and the N-1 S B, or by the N+1 S A and the N+1 S B, that is by two spaces of index one lower or one higher. In order to make thinking as easy as possible the OS and N-1 S that determine the main NS's are always named OS B and N-1 S X.

A, means assumption; $\neq$, means is not equal to or identical with; C, means corollary; T, means theorem; E, means assumption of extension; CE, means colollary of extension. L, means lemma.

In order to make the principle as clear as possible the "on" terminology is used (a); that is a point is on, or lies on, or is a point of a line, is expressed by, the point is on the line, or in symbols the OS is on the 1S: similarly, to say that two lines intersect in or meet in a point, say the two lines are on a common point, symbolically two 1S's are on a common OS.

Hence we may use any of "(1+2+3+---N) such types of statements as are given below.

A OS is on a 1S	A 1S is on a OS
" " " " " 2S	" 2S " " " OS.
" 1S " " " 2S.	" 2S " " " 1S.
" OS " " " 3S	" 3S " " " OS.
" 1S " " " 3S.	" 3S " " " 1S.
" 2S " " " 3S.	" 3S " " " 2S.
etc. etc, and etc to	etc, etc, and etc to
an N-1Sis on an NS.	An NS is on an N-1S

(a) Cf Veblen & Young, Vol I pp. 14 thru 33

SECTION 2, 4-SPACE

PROOF OF THE THEOREM OF DUALITY IN A 4-SPACE

Definition 1: K OS's are said to be independent if no F of them ($F \leq K$) are all on the same (F-2)S.

Definition 2: An RS is on an HS and the HS is on the RS ($R<H$), if every OS of the RS is on the HS.

Definition 3: An RS is distinct from an HS, if the RS is not on the HS.

Definition 4: If OS A, OS B, OS C, OS D, and OS E are any five independent OS's and if 3S X is determined by any four of these OS's say OS A, OS C, OS D and OS E, the class of all OS's such that every OS is on a 1S with OS B and some OS of the 3S X, is called a 4S, (the 4S is said to be on and to be determined by OS B and 3S X).

Elements: The elements with which this argument is concerned are OS, 1S, 2S, 3S and 4S.

ASSUMPTIONS	THEOREMS
A1. If OS A and OS B are distinct, there is one and only one 1S on both OS A and OS B.	T5, C2: If 3S A and 3S B are distinct, there is one and only one 2S on both 3S A and 3S B.
A2. If OS A, OS B and OS C are not on the same 1S and if OS D and OS E ($D \neq E$) are two other OS's so located that OS A, OS C and OS D are on a 1S, and OS B, OS C and	T5, C4: If 3S A, 3S B and 3S C are not on the same 2S and if 3S D and 3S E ($D \neq E$) are two other 3S's so located that 3S A, 3S C and 3S D are on a 2S, and 3S B, 3S C and

OS E are on a 1S, there is a OS F such that OS D OS E and OS F are on a 1S and OS A, OS B and OS F are on a 1S.	3S E are on a 2S, there is a 3S F such that 3S D, 3S E and 3S F are on a 2S and 3S A, 3S B and 3S F are on a 2S.
E0. There are at least three OS's on every 1S.	CE, 1B: There are at least three 3S's on every 2S.
E1. There exists at least one 1S.	A: There exists at least one 2S.
E2. Not all OS's are on the same 1S.	CE, 2: Not all 3S's are on same 2S.
E3. Not all OS's are on the same 2S.	CE, 3: Not all 3S's are on the same 1S.
E4. Not all OS's are on the same 3S.	CE, 4: Not all 3S's are on the same OS.
E4'. All OS's are on the same 4S.	CE, 5: All 3S's are on the same 4S.

Lemma: There exists a 4S on any 5 independent OS's.

Proof: Let the 5 independent OS's be OS A, OS B, OS C, OS D and OS E. Then OS A, OS C, OS D and OS E are independent; for otherwise, there would exist a 2S containing all of them (def. 1), and this 2S with OS B would determine a 3S containing all 5 of the given OS's, contrary to the hypothesis that they are independent. Hence by the lemma to the theorems on a 3S there is a 3S, which we will call 3S X, on the OS A, OS C, OS D and OS E; and this 3S X with OS B determines a 4S which is on OS A, OS B, OS C, OS D and OS E (def. 4)

THEOREM 1: Any LS, determined by (L+1) independent OS's on the same NS (L<N=4), is on the NS.

Case 1: Any 1S RT is on the 4S, if OS R and OS T are independent OS's on the 4S.

Proof: 1st: If OS R and OS T are both on the 3S X the theorem is true (T1 on a 3S and def. 4).

2nd: If 1S RT is on OS B, since OSR and OS T are given on the 4S, there is a 1S RTB that will be on 3S X on some OS (def. 4). Then all OS's on 1S RT will be on a 1S with a OS of 3S X and OS B (A1). Hence they are all on the 4S (def. 2).

3rd: If OS R is on 3S X and OS T is not on 3S X and 1S RT is not on OS B, the 1S BT will be on 3S X on some OS T' (def. 4) and OS T' # OS R (A1). Select any OS M (E0) on 1S RT. 1S BM will be on 1S RT' on OS M' (A2, since OS R, OS T, OS T' are not on the same 1S, and OS M, OS B etc). Since OS M' is on 1S RT', it is on 3S X (T1 on 3S). Hence OS M is on 4S (def. 4). And since OS M was any OS of 1S RT, it is true of all OS's of 1S RT.

4th: If neither OS R nor OS R are on 3S X and 1S RT is not on OS B, 1S BR and 1S BT are on 3S X in some two OS's, OS R' and OS T' respectively (def. 4). Now 1S R'T' and 1S RT are on the same OS K (A2, since OS B, OS R, OS T are not on the same 1S, and OS R' and OS T' are so located etc.). But OS K is on 3S X (T1 on 3S), so any OS of 1S RT is on 4S (3rd) or 1S RT is on the 4S (def. 2).

Case 2: Any 2S on three independent OS's of a 4S is on the 4S.

Proof: Consider the 2S as determined by OS Q and 1S Z of the 3 given OS's. Every OS of the 2S is on a 1S with OS Q and some OS of the 1S Z (def. of 2S). Hence every OS of the 2S is on the 4S (T1 and def. 2).

Case 3: Any 3S on 4 independent OS's of a 4S is on the 4S.

Proof: Consider the 3S as determined by OS Q and 2S Z of the 4 given OS's. Every OS of the 3S is on the 1S with OS Q and some OS of 2S Z (def. of 3S). Hence every OS of the 3S is on the 4S (T1 and def. 2).

THEOREM 2: Any LS on an NS (N>L) not on the (N-1)S that determines that NS, has one and only one (L-1)S in common with the determining (N-1)S.

Case 1: Any 1S of a 4S, not on the 3S that determines that 4S is on one and only one OS of the determining 3S.

Proof: Consider any 1S K. If 1S K is on OS B the theorem is true (def. 4). But if 1S K is not on OS B, select any OS R and OS T on the 1S K (EO). Then 1S BR and 1S BT will be on 3S X in some OSR' and OS T' respectively (def. 4). Hence 1S R'T' will be on 1S RT in some OS M (A2). But 1S R'T' is on 3S X (T1 on 3S), so OS M is on 3S X as well as on 1S K (OS R and OS T were OS's on the 1S K). Hence any 1S of a 4S meets the 3S that determines that 4S on a OS.

There can be but one OS in common, for if there were one more, there would be at least two OS's on a 1S on a 3S, and thus (T1 on 3S) every OS of the 1S would be on the 3S X, which is contrary to the hypothesis.

Case 2: Any 2S on a 4S not on the 3S that determines that 4S is on one and only one 1S of the determining 3S.

Proof: Consider any 2S K. By the def. of a 2S, 2S K is determined by some OS R, not on 3S X, and 1S E. Select any 2 OS's on 1S E (E0), and by def. of 2S, each of these OS's with OS R form a distinct 1S (A1) on 2S K (T1 on 2S). Since 2S K is on the 4S these 1S's are on the 4S (def. 2 and T1). But each 1S is on 3S X in one and only one OS (Case 1) so these two OS's determine a 1S (def. of 1S). Hence the 2S K and the 3S X are on a 1S.

2S K and 3S X can have no more than a 1S in common because if they had as much as another OS in common the 2S K would be on the 3S X (T1 on 3S), which is contrary to the hypothesis.

Case 3: Any 3S on a 4S not on the 3S that determines that 4S is on one and only one 2S of the determining 3S.

Proof: Consider any 3S E on the 4S. If we think of 3S E as determined by OS Q, not on 3S X, and 2S R (def. of 2S), 2S R is on the 4S (def. 2). Select any 3 independent OS's on the 2S R (def. of 2S). Now each OS with OSQ will determine a distinct 1S (A1) in the 4S (T1 on 4S), which will meet the 3S X in a OS (case 1). These three independent OS's will determine a 2S (L on 2S) on both the 3S E and

3S X for all 3 OS's are on both. Hence 3S E is on 3S X in a 2S.

3S E and 3S X can have no more than a 2S in common or they will be identical (T3 on 3S), which is contrary to the hypothesis.

THEOREM 3: Any RS and any ES ($R \geq D$) having a common CS ($D \geq C$), but not a ($C+1$)S, are on a common <u>$R+D-C$</u> S and not both on the same NS ($N<R+D-C$).

Case 1: If any 1S and any other 1S are on a common OS but not on a common 1S, they are both on a 2S, and not on a 1S or OS. Proved (T3 on 2S).

Case 2: If any 1S and any 2S are on a common OS but not a 1S, they are both on a 3S, and not a smaller space. Proved (T3 on 3S).

Case 3: If any two 2S's are on a common 1S but not on a 2S they are both on a 3S, not a smaller space. Proved (T3 on 3S).

Case 4: If any two 2S's are on a common OS but not on a 1S, they are both on a 4S, and not a smaller space.

Proof: Consider any 2S A and 2S B on a common OS R. Select any OS E not OS R (E1 and E0) on 2S B. Now OS E and 2S A determine a 3S V (L on 3S). OS E and OS R determine a 1S RE (A1), which is on the 3S V and the 2S B (T1 on 3S and 2S). Now select any OS D on 2S B not on 1S RE (E2). Then OSD and 3S V determine a 4S (def. 4), while OS D and 1S RE determine a 2S (def. of 2S), which is on the 4S and the 2S B (T1 on 4S and 2S). This 2S is 2S B for it is determined by 3 OS's on 2S B (T 5 on 2S). Hence 2S B and 2S A are on the 4S (T1).

This argument shows that OS E and OS D are two independent OS's not on 2S A, which together with the 3 independent OS's of 2S A, make 5 independent OS's, each of which is either on 2S A or 2S B. If the 2S A and 2S B were not on a 4S, but were on a smaller space, there could not be 5 independent OS's.

Case 5: If any 1S and any 3S are on a common OS but not on a 1S, they are on a 4S.

Proof: Consider any 3S A and any 1S B on a common OS R. Select any OS E on 1S B not on OS R (E0). Now OS E and 3S A determine a 4S V (def. 1 of 4S). OS E and OS R determine a 1S ER (A1), which is on the 4S V and the 1S B (T1 on 4S and 1S).

3S A and 1S B can be on no smaller space by the same reasoning as that used in case 4.

Case 6: If any 4S and any 2S are on a common 1S but not on a common 2S, they are both on a 4S.

Proof: Consider any 3S A and any 2S B on a common 1S R. Select any OS E on 2S B not on 1S R (E2). Now OS E and 3S A determine a 4S V (def. 1). OS E and 1S R determine a 2S Q (def. of 2S), which is on 4S V and 2S B (T1 on 4S and 2S).

3S A and 2S B can be on no smaller space by the same reasoning as that used in case 4.

Case 7: Any two distinct 3S's on a common 2S, are on a 4S.

Proof: Consider any 3S A and any 3S B on a common 2S R. Select any OS E on 3S B not on 2S R (E3). Now OS E and 3S A determine a 4S V (def. 4), OS E and 2S R determine a 3S Q (def. of 3S), which is on 4S V and 3S B (T1 on 4S and 3S).

3S A and 3S B can be on no smaller space, by the same reasoning as that used in case 4.

THEOREM 4: (Converse of theorem 3). Any RS and any DS (R⩾D) on the same NS (N>R) must have at least an (R+D-N)S in common.

Case 1: Any 3S and any 1S, on a 4S are on a OS at least.

Proof: Consider any 3S A and any 1S Z. If 3S A is on 3S X, that determines the 4S, the theorem is true (T2, case,). It is also true if the 1S Z is determined by OS R and OS B, OS B being the OS with which 3S X determined the 4S, and if at the same time OS B is on 3S A. But if OS B is not on 3S A, the 1S RB meets 3S X in OS R' (def. 4S) and by (T2) in only one such OS. Now let OS C be any OS of the 3S A. The 1S BC meets 3S X in a OS C' (def. of 4S). By T2, 3S A has a common 2S B with 3S X. This 2S B has in common with 1S R'C' a OS D at least (T4 on 3S). Now 1S Z meets both the 1S C'D and the 1S CC', hence it meets the 1S CD (A2) and has at least one OS on the 3S A.

Now suppose that 1S Z is not on OS B; but is determined by OS R and OS T, also suppose that 3S A is not on OS B. By the case just considered, 1S BR and 1S BT meet 3S A in OS R' and OS T' respectively. The 1S Z which meets 1S BR' and 1S BT' must then meet 1S R'T' in a OS, which by (T1 on 3S) is on 3S A.

Suppose finally that OS B is on 3S A, still under the hypothesis that 1S Z is not on OS B. By T2, 3S A meets 3S X in a 2S B, and 2S L determined by OS B and 1S Z meets 3Z X in a 1S L'. By T4 on 3S, 1S L' and 2S B on 3S X have at least one OS P in common. Now the 1S Z and 1S BP are on the 2S L and hence have a common OS Q (T4 on 2S). By T1 on 3S, the OS Q is common to 3S A and 1S Z.

Case 2: Any 3S and any 2S, on a 4S are on a 1S at least.

Proof: Consider any 3S A and any 2S B on a 4S. If 3S A and 2S B do not have at least a 1S in common, they are on a common 5S (T3), contrary to the hypothesis.

Case 3: Any two 2S's on a 4S are on a OS at least.

Proof: If they are not on a OS at least, they are on a common 5S (T3), contrary to the hypothesis.

Case 4: Any two 3S's on a 4S are on a 2S at least.

Proof: If they are not on a 2S at least, they are on a common 5S (T3), contrary to the hypothesis.

THEOREM 5: (Or theorem 1 when L=N). Two NS's are identical if the (N-1)S are the OS, that determine one, are on the other.

Case 1: Two 2S's are identical etc. (T5 on 2S).

Case 2: Two 3S's are identical etc. (T5 on 3S).

Case 3: Two 4S's are identical if the 3S and the OS, that determine one of them,are on the other.

Proof: Consider the two four spaces 4S T and 4S S, and consider them as determined by OS T & 3S T' and OS S & 3S S' respectively. Now if OS T & 3S T' are on 4S S, any OS of 4S T is on a 1S joining OS T and some OS of 3S T' (def. of 4S). Hence, by T1, every OS of 4S T is on the 4S S. Let OS P be any OS of 4S S not on OS T (E1, E0). The 1S PT meets the 3S T' in a OS (T4), that is a (3+1-4)S. Every OS of the 4S S is on the 4S T (def. of 4S).

Corollary 1: There is one any only one 4S on five independent OS's.

Proof: This is true by def. of 4S, lemma (page 8) and T5.

Corollary 2: If two (N-1)S's, on an NS, are distinct, there is an (N-2)S on them, not a KS, (K>N-2).

Case 1: If two 1S's, on a 2S, are distinct, there is a OS on them, not a greater space (T5, C2 on 2S).

Case 2: If two 2S's, on a 3S, are distinct, there is a 1S on them, not a greater space (T5, C2 on 3S).

Case 3: If two 3S's, on a 4S, are distinct, there is a 2S on them, not a greater space.

Proof: By (T4) the two 3S's must be on a ($\underline{4-1}+\underline{4-1}-4$)S, that is a 2S. They can have no more than a 2S in common, for if they had even another OS, they would be identical (T5).

Corollary 3: Any three 3S's on a 4S but not on a common 2S are on a 1S, not a smaller space.

Proof: The given 3S A and 3S B have a 2S AB in common (T4 case 4). Likewise 3S B and 3S C have a 2S BC in common. 2S AB and 2S BC are both on 3S B, hence they meet on a 1S R (T2 on 2S). 1S R must be on each of the 3S's (T1).

They can have no more than a 1S in common for if they had another OS in common they would be on the same 2S, contrary to the hypothesis.

Corollary 4: If 3S A, 3S B and 3S C are not on the same 2S and if 3S D and 3S E (D#E) are so located that 3S A, 3S C and 3S D are on a 2S and 3S B, 3S C and 3S E are on a 2S, there is a 3S F such that 3S D, 3S E and 3S F are on a 2S and 3S A, 3S B and 3S F are on a 2S.

Proof: 3S D & 3S E are on a 2S L (T5, C2). The 3S A, 3S B and 3S C are all on the 1S P (T5, C3). Hence 2S AB, 2S BC and 2S AC (all being 2S's by T5, C2) all contain the 1S P. 3S D was given so located that it is on a 2S with 3S C and 3S B, and 3S E was given so located that it is on a 2S with 3S C and 3S A. Hence the 2S L, which is on both 3S D & 3S E, must be on the 1S P. Thus we have 2S L & 2S AB on a common 1S P, so they determine a (2+2-1)S (T3), which is the 3S F.

COROLLARIES OF EXTENSION

Corollary 1: There are at least three 1S's on every OS of a 4S.

Proof: Any three OS's of the 4S say OS A, OS B and OS C determine a 2S (def. of 2S). There are at least three OS's on any 1S AB (E0). All three will be on the 2S (T1 on 2S). Now each 1S thru these three OS's and OS C will determine a distinct 1S on the 2S (A1 & T1 on 2S). Hence there are three distinct 1S's on any OS C of the 4S (T1 on 4S).

Corollary 1a: There are at least three 2S's on every 1S of a 4S.

Proof: Consider the 3S determined by the 2S of the preceeding corollary and OS P (E3). Then each 1S would determine with OS P a distinct 2S (T5 on 2S) on 1S CP (T3). Hence there are three distinct 2S's on any 1S CP of the 4S.

Corollary 1b: There are at least three distinct 3S's on every 2S on a 4S.

Proof: Consider the 4S as determined by the 2S of the preceeding corollary and OS Q (E4). Then each 2S, with OS Q would determine a distinct 2S (T5 on 2S) on 2S CPQ (T3). Hence there are three distinct 3S's on any 2S CPQ.

Corollary 2: Not all 2S's are on the same 2S.

Proof: Take any 3S A and any 3S B on a 2S AB. Then take any 1S on each 3S on a OS of 2S AB. With any other OS (E3), they determine a 2S (def. of 2S), which is not on 2S AB or it will coincide with both 3S A and 3S B (T5), contrary to the hypothesis that 2S A and 3S B are distinct.

Corollary 3: Not all 3S's are on the same 1S.

Proof: C2, T3 and CE2 on 3S.

Corollary 4: Not all 3S's are on the same 0S.

Proof: C3, T3 and CE3 on 2S.

Corollary 5: All 3S's are on a 4S.

Proof: All 3S's are on a 4S, for otherwise the definition of a 3S and E4' would be violated.

THE THEOREM OF DUALITY IN A 4S.

THEOREM: Any proposition deducible from assumptions A and E concerning OS's, 1S's, 2S's, 3S's and 4S's of a 4S remain valid if the words OS and 3S and the words 1S and 2S are interchanged.

Proof: Any proposition that is deducible from the assumptions A & E on page 7 is obtained from the assumptions given on the left by a certain sequence of logival reasoning. Clearly the same sequence of logival reasoning may be applied to the corresponding propositions given on the right. They will of course, refer to the class of all 3S's on a 2S, when the original argument refers to the class of all OS's on a 1S. The steps of the original argument lead to a conclusion stated in terms of some or all of the first (1+2+3+4 and their duals) types of "on" statements given on page 6. Similarly the derived argument leads in the same way to a conclusion concerning these same "on" statements: But when the first states that a OS P is on a 1S L, the later will state that a 3S P is on a 2S L and whenever the original states that a 3S L is on a OS P, the later will state that a 1S L is on a 3S P.

Hence to every statement in the conclusion of the original argument, will correspond a statement in the conclusion of the derived argument in which the word 3S is used for the word OS and the word 2S is used for the word 1S, or visa versa.

SECTION 3, N-SPACE

PROOF OF THE THEOREM OF DUALITY IN AN N-SPACE

Definition 1: K OS's are said to be independent if no F of them ($F \leqq K$) are all on the same F-2S.

Definition 2: An RS is on an HS and the HS is on the RS ($R<H$), if every OS of the RS is on the HS.

Definition 3: An R S is distinct from an HS, if the RS is not on the HS.

Definition 4: If OS A, OS B, OS C ----OS N+1 are any N+1 independent OS's and if N-1S X is determined by N of these OS's say OS A, OS C, OS D--- OS N+1, the class of all OS's such that every OS is on a 1S with OS B and some OS of the N-1S X is called an NS, (the NS is said to be on and to be determined by OS B and N-1S X)

Elements: The elements with which this argument is concerned are OS, 1S, 2S, 3S ---NS.

ASSUMPTIONS

A1. If OS A and OS B are distinct, there is one and only one 1S on both OS A and OS B.

A2. If OS A, OS B and OS C are not on the same 1S and if OS D and OS E ($D \neq E$) are two other OS's so located that OS A,

THEOREMS

T5, C2. If N-1S A and N-1S B are distinct, there is one and only one N-2S on both N-1S A and N-1S B.

T5, C4. If N-1S A, N-1S B and N-1S C are not on the same N-2S and if N-1S D and N-1S E ($D \neq E$) are two other N-1S's so located that N-1S A,

OS C and OS D are on a 1S, and OS B, OS C and OS E are on a 1S, there is a OS F such that OS D OS E and OS F are on a 1S and OS A, OS B and OS F are on a 1S.	N-1S C and N-1S D are on an N-2S, and N-1S B, N-1S C and N-1S E are on an N-2S, there is an N-1S F such that N-1S D, N-1S E and N-1S F are on an N-2S and N-1S A, N-1S B and N-1S F are on an N-2S.
EO. There are at least 3 OS's on every 1S	CE, 1n. There are at least 3 N-1S's on every N-2S.

LEMMA: There exists an NS on any N+1 independent OS's.

Proof: Let the N+1 independent OS's be OS A, OS B, OS C ---OS N+1. Then OS A, OS C, OS D----OS N+1 are independent: for, otherwise, there would exist an N-2S containing them all (def. 1), and this N-2S with OS B would determine an N-1S containing all N+1 of the given OS's, contrary to the hypothesis that they are independent. Hence by the lemma of an N-1S there is an N-1S on the OS A, OS C, OS D---OS N+1; and this N-1S with OS B determines an NS which is on OS A, OS B, OS C--- OS N+1 (def. 4).

THEOREM L: Any LS determined by L+1 independent OS's on the same NS (L<N) is on the NS.

Case 1: Any 1S RT on the independent OS's, OS R and OS T of an NS, is on the NS.

Proof: 1st; If OS R and OS T are both on N-1S X, the theorem is true (def, 4 & T1 on an N-1S

2nd. If 1S RT is on OS B, since OS R and OS T are given on the NS, there is a 1S RTB that will be on the N-1S X at some OS (def. 4). Then all OS's on 1S RT will be on a 1S with some OS of N-1S X and OS B (A1), hence they are all on the NS (def. 4).

3rd: If OS R is on N-1S X and OS T is not on N-1S X and 1S RT is not on OS B, the 1S BT will be on N-1S X in some OS T' (T'#R, A1) by(def. 4) Select any OS M on 1S RT (EO), 1S BM will be on 1S RT' on OS M' (A2). But OS M' is on 1S RT', so it is on N-1S X (T1 on N-1S). Hence OS M is on NS (def. 4). Since OS M was any OS of 1S RT, it is true for all OS's of 1S RT.

4th, or in general: If neither OS R not OS T are on N-1S X and 1S RT is not on OS B, 1S BR and 1S BT are on N-1S B in some two OS's, OS R' and OS T' respectively (def. 4). Now 1S R'T' and 1S RT are on some OS K(A2) but OS K is on N-1S X (T1 on N-1S). Hence any OS of 1S RT is on the NS (3rd), or 1S RT is on the NS (def. 2).

Case L, or the general theorem:

Proof: (By mathematical induction). By case 1 the theorem is true when L=1. Assume it is true when L=K-1, prive it is true for L=K. All OS's of a KS on K+1 independent OS's of an NS are (def. 4 & T5 on KS) on a 1S joining one of these OS's to the OS's of the K-1S determined by the remaining K OS's. But under the hypothesis of the induction, this K-1S is on the NS and hence (case 1) all OS's of the KS are on the NS.

THEOREM 2: Any LS on any NS (N>L) not on the (N-1)S that determines that NS is on an L-1S and not a KS (K>L-1) of the determining N-1S.

Case 1: Any 1S on an NS not on the N-1S that determines that NS, is on one and only one OS of the determining N-1S.

Proof: Consider any 1S K. If 1S K is on OS B the theorem is true (def. 1). But if 1S K is not on OSBB, then select OS R and OS T on the 1S K (EO) and 1S BR and 1S BT will be on N-1S X in some OS R' and OS T' respectively (def. 1). Hence 1S R'T' will be on 1S RT in some OS M (A2). But 1S R'T' is on the N-1S X (T1 on N-1S).
Hency OS M is on N-1S X, and also on 1S K for OS R & OS T were OS's on the 1S K. So any 1S of an NS meets the N-1S that determines that NS in a OS.

1S K & N-1S X can be on only one OS, for if they were on as much as another OS, there would be 2 OS's of a 1S on an N-1S, and thus (T1 on N-1S) the 1S would be on the N-1S X, and that is contrary to the hypothesis.

Case 2: Any 2S on an NS, not on the N-1S that determines that NS, is on a 1S not a KS where K>1 of the N-1S.

Proof: Consider any 2S K, say it is determined by OS R not on N-1S X and 1S ER. Now there must be at least 3 OS's on 1S ER (EO) and by (def. of 2S) each of these

OS's with OS R form a distinct 1S (A1) on 2S K (T1 on 2S). since 2S K is on NS these 1S's are on the NS (def. 2 & T1). Hence each 1S is on the N-1S X in one and only one OS (case 1). Two of these OS's would determine a 1S (def. of 1S), hence they have a 1S in common.

2S K and N-1S X can be on only one 1S, which means that the 3rd OS must be on the same 1S. If they had as much as a 1S and a OS in common the 2S K would be on N-1S X (T1 on N-1S), and that is contrary to the hypothesis.

Case L, or in general: Now LS Z is (def. of an LS & def. 1) on L-1 independent OS'S of the NS. Hence by joining one of these OS's say OS R, not on N-1S X, with the remaining L OS's we would have L distinct 1S's (A1), which would therefore meet the N-1S X in L distinct OS's (case 1). These L OS's are independent, for if not they would contain all of the OS's of our LS, and that is impossible by (def. of an LS). So these L OS's would form an L-1S (def. of an L-1S), and this L-1S is on both the N-1S X and the LS (T1 & def. 2).

The LS can not be on a KS (K>L-1) of the N-1S X, for if they had as much as an L-1S and a OS in common, the LS would be on the N-1S X (T1 on N-1S), and that is contrary to the hypothesis.

THEOREM 3: Any RS and any DS ($R \geq D$) having a common CS ($D \geq C$); but not a common C+1S, are on a common R+D-CS and are not both on an NS ($N < R+D-C$).

Proof: If D=C, the proposition is true (T5 on a CS & def. 2). If D>C consider the several cases. When D=C+1 let OS P' be a OS on DS not on CS (EC+1 on DS). Then OS P' and the RS determine an R+1S and OS P' and the CS determine a C+1S, such that the C+1S is contained on the R+1S and (def. 2). If D=C+2, let OS P''be a OS of the DS not on the C+1S (EC 2 on DS). Then OSP'' and the R+1S determine a R+2S, while OS P'' and C+1S determine a C+2S, which is on the R+2S and the DS. This process can be continued until there results a C+yS determined by OS $P^{y'}$ and C+y-1S, containing all the OS's of the DS. And then it would be true that D=C+y (T5 on DS) or y=D-C. At this stage in the process we obtain an R+yS which contains both the RS and the DS, which by substitution would be an R+D-CS.

This argument shows that OS P', OS P'' etc---to OS $P^{y'}$, where y=D-C are D-C OS's, no 2 on a OS and no 3 on a 1S etc---to, no D-C-2 on a D-CS, which together with the R+1 OS's not on an R-1S of the RS, make D-C+R-1 OS's not on R+D-C-1S or that is D-C+R+1 independent OS's each of which is either on the RS or the DS. If the RS and the DS are on an NS, where

N<R+D-C they could not be on D-C+R+1 independent OS's for they would not exist.

THEOREM 4, or the converse of theorem 3: Any RS and any DS (R≥D) on the same NS (N>R) must have at least an R+D-NS in common.

Case 1: Any N-1S and any 1S on the same NS are on N-1+1-NS or a OS.

Proof: Consider any N-1S A and 1S Z. If N-1S A is the N-1S X that determines the NS, the theorem is true (T2 case 1). It is also true if the 1S Z is determined by OS R and OS B, OS B being the OS with which the N-1S X determined the NS, and if at the same time OS B is on the N-1S A. But if OS B is not on N-1S A, the 1S RB meets N-1S X in OS R' (def. NS), and by (T2) on only one such OS. Now let OS C be any OS of the N-1S A. The 1S BC meets the N-1S X in a OS C' (def. of NS). By T2, N-1S A has a common N-2S B with N-1S X. This N-2S B has in common with 1S R'C' a OS D' at least (T4 on N-1S). All OS's of the 1S D'C are then on N-1S A (T1 on N-1S). Now 1S Z meets both the 1S C'D' and the 1S CC', hence it meets the 1S CD' (A2) and has at least one OS on the N-1S A. Now suppose that 1S Z is not on OS B, but is determined by OS R and OS T; also suppose that N-1S A is not on OS B. By the case just considered, 1S BR and 1S BT meet N-1S A in OS R' and OS T' respectively. The 1S Z which meets 1S BR' and 1S BT' must then meet

1S R'T' in a OS, which by (T1 on N-1S) is on the N-1S A. Suppose finally that OS B is on N-1S A, still under the hypothesis that 1S Z is not on OS B. By (T2) N-1S A meets N-1S X in a N-2S B, and the 2S L, determined by OS B and 1S Z, meets N-1S X in a 1S L'. By (T4 on N-1S), 1S L' and N-2S B have at least one OS P in common. Now the 1S Z and 1S BP are on the 2S BZ, and hence have a common OS Q (T4 on 2S). By (def. 2 and T1 on an N-1S) the OS Q is common to N-1S A and 1S Z.

Case R, or the general case.

Proof: Assume that the RS and DS are on a R+D-N-1S. Then by (T3) they would be on a D+R-(R+D+N-1)S, or an N+1S, contrary to the hypothesis. Hence they must meet on an R+D-NS at least.

THEOREM 5, or proposition 1, when L=N. Two NS's are identical if the N-1S and the OS that determine one are on the other.

Proof: Call the two NS's, NS T and NS S and consider them as determined by OS T & N-1S T' and OS S & N-1S S' respectively. Now if OS T & N-1S T' are on NS S, any OS of NS T is on a 1S joining OS T and some OS of N-1ST' (def. of NS). Hence by (T1) every OS of NS T is on the NS S. Let OS P be any OS of NS S not on OS T (E1 & E0). The 1S PT meets the N-1S T' in a OS (T4) or in an N-1-(N-1)S. Every OS of the NS S is on the NS T (def. of NS).

Corollary 1: There is one and only one NS on N+1 independent OS's.

Proof: This is true by def. of NS, lemma on an NS (page 22) and T5.

Corollary 2: If two N-1S's on an NS are distinct, there is an N-2S on them, not a KS, K>N-2.

Proof: By (T4) the two N-1S's must be on an N-1+N-1=NS or an N-2S. They can have no more than an N-2S in common, for if they had even another OS, they would be identical (T5).

Corollary 3: Three N-1S's on an NS but not on an N-2S are on an N-3S, not a LS, L>N-3.

Proof: Let the three N-1S's be N-1S A, N-1S B and N-1S C. Now N-1S A & N-1S B are on an N-2S K and not on an N-1S (C1), N-1S C & N-2 S K are on at least an N-1+N-2-NS or an N-3S (T4). Hence there is one N-3S on them. If they are on even a OS more they could be on an N-2S (def. of N-2S), contrary to the hypothesis.

Corollary 4: If N-1S A, N-1S B and N-1S C are not on the same N-2S and if N-1S D and N-1S E (D≠E) are two other N-1S's so located that N-1S A, N-1S C and N-1S D are on an N-2S and N-1S B, N-1S C and N-1S E are on an N-2S, there is an N-1S F such that N-1S D, N-1S E and N-1S F are on an N-2S and N-1S A, N-1S B and N-1S F are on an N-2S.

Proof: N-1S D & N-1S E are on an N-2S L (T5 C2).

The N-1S A, N-1S B & N-1S C are on an N-3S P (T5 C3). Hence N-2S AB, N-2S BC & N-2S AC, all being N-2S's by (T5 C3), all contain the N-3S P. N-1S D was given so located that it is on an N-2S with N-1S B & N-1S C, and N-1S E was given so located that it is on an N-2S with N-1S C & N-1S A. Hence the N-2S L which is on both N-1S D & N-1S E must be on the N-2S P. Thus we have N-2S L & N-2S AB are on a common N-3S P, so they determine an N-2+N-2-(N-3)S (T3), or an N-1S F.

COROLLARIES OF EXTENSION

Corollary 1a: There are at least three 1S's on every OS on an NS.

Proof: Any three OS's of the N-1 OS's that determine the NS, say OS A OS B and OS C, determine a 2S (L on 2S). There are at least three OS's on any 1S AB (EO). All three will be on the 2S (T1 on 2S). Hence each 1S thru these three OS's and OS C will determine a distinct 1S on the OS C of the NS (T1 on NS).

Corollary 1b: There are at least three 2S's on every 1S of an NS.

Proof: Consider the 3S determined by the 2S of the preceeding corollary and OS P (E3). Then each 1S would determine with OS P a distinct 2S (T5 on 2S) on the 1S CP (T3).

Corollary 1c: There are at least three distinct 3S's on every 2S on an NS.

Proof: Consider the 4S determined by the 3S of (CE 1b) and a OS Q (E4). Then each 2S would with OS Q determine a 3S (T3) that is distinct (T5) and is on the 2S CPQ.

Corollary 1n: There are at least three NS's on every N-2S of an NS.

Proof: By def. an NS is determined by OS B & N-1S X. There are at least three N-2S's on every N-3S of an NS (CE n-3). Then each N-2S would determine with OS B an N-1S (T3). They will be distinct (T5) and each is on the N-3S common to the three N-2S's and the OS B, or an N-2S (def. of N-2S). Then there are three N-1S's on every N-2S of an NS.

THE THEOREM OF DUALITY IN AN NS

THEOREM: Any proposition deducible from assumptions A and E concerning OS's, 1S's, etc---to KS's of an NS remains valid if the words OS & <u>N-1</u>S, 1S & <u>N-2</u>S, 2S & <u>N-3</u>S etc---to KS & <u>N-K-1</u>S are interchanged.

Proof: Any proposition deducible from assumptions A and E is obtained from the assumptions given on the left of page 21 by a certain sequence of logical inferences. Clearly the same sequence of logical inferences may be applied to the corresponding propositions given on the right. They will of course, refer to the class of all <u>N-1</u>S's on an <u>N-2</u>S when the original argument refers to the class of all OS's on a 1S. The steps of the original argument lead to a conclusion necessarily stated in terms of some or all of the 2(1+2+3+4+--N) types of "on" statements enumerated on page 6 of section 1: The derived argument leads in the same way to a conclusion which whenever the original states that OS C is on a 2S L, says that an <u>N-1</u>S C is on an <u>N-3</u>S L or if the original argument states that a 2S L is on a OS C, says that <u>N-3</u>S L is on an <u>N-1</u>S C. Applying similar considerations to each of the 2(1+2+3+4+--N) types of "on" statements in succession, we see that to each statement in the conclusion arrived at by the original argument corresponds a statement arrived at by the derived argument in which the words OS & <u>N-1</u>S, 1S & <u>N-2</u>S, etc.--to KS & <u>N-K-1</u>S in the original statement have been interchanged.