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THREE-LEVEL BDDC IN THREE DIMENSIONS*

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Abstract. Balancing domain decomposition by constraints (BDDC) methods are nonoverlapping iterative substructuring domain decomposition methods for the solution of large sparse linear algebraic systems arising from the discretization of elliptic boundary value problems. Their coarse problems are given in terms of a small number of continuity constraints for each subdomain, which are enforced across the interface. The coarse problem matrix is generated and factored by a direct solver at the beginning of the computation and it can ultimately become a bottleneck if the number of subdomains is very large. In this paper, two three-level BDDC methods are introduced for solving the coarse problem approximately for problems in three dimensions. This is an extension of previous work for the two-dimensional case. Edge constraints are considered in this work since vertex constraints alone, which work well in two dimensions, result in a noncompetitive algorithm in three dimensions. Some new technical tools are then needed in the analysis and this makes the three-dimensional case more complicated. Estimates of the condition numbers are provided for two three-level BDDC methods, and numerical experiments are also discussed.

Key words. BDDC, three-level, three dimensions, domain decomposition, coarse problem, condition number, Chebyshev iteration

AMS subject classifications. 65N30, 65N55

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1. Introduction.

Balancing domain decomposition by constraints (BDDC) methods, which were introduced and analyzed in [4, 11, 12], are similar to the balancing Neumann–Neumann algorithms. The coarse problem in a BDDC algorithm is given in terms of a set of primal constraints chosen for each subdomain, and the matrix of the coarse problem is generated and factored by using a direct solver at the beginning of the computation. We note that there are now computer systems with more than 100,000 powerful processors, which allow very large and detailed simulations. If there is a one to one or one to several relationship between processors and subdomains, then we can have a large number of subdomains. The coarse component of a two-level preconditioner can therefore become a bottleneck if the number of subdomains is very large. One way to remove this difficulty is to introduce one or more additional levels. In our recent paper [17], two three-level BDDC methods were introduced for two-dimensional problems with vertex constraints. We solve the coarse problem approximately by using the BDDC idea recursively and show that a good rate of convergence still can be maintained. However, in three dimensions, vertex constraints alone are not enough to obtain good polylogarithmic condition number bounds due to much weaker interpolation estimates, and constraints on the averages over edges or faces are needed. The new constraints lead to a considerably more complicated coarse problem and the need for new technical tools in the analysis. In this paper, we extend the two three-level BDDC methods in [17] to the three-dimensional

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case using primal edge average constraints. With the help of new technical tools, we are able to provide estimates of the condition numbers of the system with these two new preconditioners.

We note that, since this paper was submitted, several papers have appeared on inexact solvers for BDDC (see [5, 10, 16, 15]) and dual-primal finite element tearing and interconnecting (FETI-DP), which are iterative substructuring algorithms closely related to BDDC (see [6]). For the study of the convergence rates of the BDDC algorithms and their connection with the FETI-DP algorithms, see [11, 12, 9, 2].

The rest of the paper is organized as follows. We first review the two-level BDDC methods briefly in section 2. We introduce our first three-level BDDC method and the corresponding preconditioner $\widetilde{M}^{-1}$ in section 3. We give some auxiliary results in section 4. In section 5, we provide an estimate of the condition number for the system with the preconditioner $\widetilde{M}^{-1}$, which is of the form $C(1+\log\frac{\hat{H}}{H})^2(1+\log\frac{H}{h})^2$, where $\hat{H}$, H, and h are typical diameters of the subregions, subdomains, and elements, respectively. (We decompose the whole domain into subregions and each subregion is then partitioned into several subdomains; see section 3 for details.) In section 6, we introduce a second three-level BDDC method which uses Chebyshev iterations. We denote the corresponding preconditioner by $\widehat{M}^{-1}$. We show that the condition number bound of the system with the preconditioner $\widehat{M}^{-1}$ is of the form $CC(k)\left(1+\log\frac{H}{h}\right)^2$, where $C(k)$ is a function of k, the number of Chebyshev iterations, and also depends on the eigenvalues of the preconditioned coarse problem and on the two parameters chosen for the Chebyshev iteration. $C(k)$ goes to 1 as k goes to ∞; i.e., the condition number approaches that of the two-level case. Finally, some computational results are presented in section 7.

2. The two-level BDDC method.

The two-level BDDC methods have been studied extensively; see [4, 11, 12, 9]. In this section, we will briefly review this work and introduce notation which will be used in the rest of the paper.

We will consider a second order scalar elliptic problem in a three-dimensional region Ω as follows: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \rho \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where $\rho(x) > 0$ for all $x \in \Omega$. We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \ldots, N$, and set $H = \max_i H_i$. We then introduce a triangulation of all the subdomains. Let Γ be the interface between the subdomains and let the set of interface nodes Γ_h be defined by $\Gamma_h = (\cup_i \partial\Omega_{i,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$. The nodes of the different triangulations match across Γ.

Let $\mathbf{W}^{(i)}$ be the standard finite element space of continuous, piecewise trilinear functions on Ω_i; the algorithms and theory developed in this paper work for other lower order finite elements as well. We assume that these functions vanish on $\partial\Omega$. Each $\mathbf{W}^{(i)}$ can be decomposed into a subdomain interior part $\mathbf{W}_I^{(i)}$ and a subdomain interface part $\mathbf{W}_\Gamma^{(i)}$, i.e., $\mathbf{W}^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Gamma^{(i)}$, where the subdomain interface part $\mathbf{W}_\Gamma^{(i)}$ will be further decomposed into a primal subspace $\mathbf{W}_\Pi^{(i)}$ and a dual subspace $\mathbf{W}_\Delta^{(i)}$, i.e., $\mathbf{W}_\Gamma^{(i)} = \mathbf{W}_\Pi^{(i)} \bigoplus \mathbf{W}_\Delta^{(i)}$. (They are called primal and dual in earlier works on FETI algorithms, where the dual variables are controlled by Lagrange multipliers.) Here, we will consider only edge average constraints over all the edges of all subdomains as primal variables. We change the variables to make the edge average degrees of freedom explicit; see [7, section 4.2.1] and [9, section 2.3]. From now on, we assume that all the matrices are written in terms of the new variables.

We denote the associated product spaces by $\mathbf{W} := \prod_{i=1}^N \mathbf{W}^{(i)}$, $\mathbf{W}_\Gamma := \prod_{i=1}^N \mathbf{W}_\Gamma^{(i)}$, $\mathbf{W}_\Delta := \prod_{i=1}^N \mathbf{W}_\Delta^{(i)}$, $\mathbf{W}_\Pi := \prod_{i=1}^N \mathbf{W}_\Pi^{(i)}$, and $\mathbf{W}_I := \prod_{i=1}^N \mathbf{W}_I^{(i)}$. Correspondingly, we have $\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma$, and $\mathbf{W}_\Gamma = \mathbf{W}_\Pi \bigoplus \mathbf{W}_\Delta$.

The elements of $\mathbf{W}$ can be discontinuous across the interface. However, the finite element approximation of the elliptic problem is continuous across Γ. We denote the corresponding subspace of $\mathbf{W}$ by $\widehat{\mathbf{W}}$. Similarly, we denote the corresponding subspaces of $\mathbf{W}_\Gamma$, $\mathbf{W}_\Delta$, and $\mathbf{W}_\Pi$ by $\widehat{\mathbf{W}}_\Gamma$, $\widehat{\mathbf{W}}_\Delta$, and $\widehat{\mathbf{W}}_\Pi$, respectively.

In order to define the BDDC preconditioner, we further introduce an interface subspace $\widetilde{\mathbf{W}}_\Gamma \subset \mathbf{W}_\Gamma$, for which all the edge average primal constraints are enforced. The space $\widetilde{\mathbf{W}}_\Gamma$ can be decomposed into $\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta$. We also have $\widehat{\mathbf{W}}_\Gamma \subset \widetilde{\mathbf{W}}_\Gamma$.

The global problem has the following form: Find $(\mathbf{u}_I, \mathbf{u}_\Delta, \mathbf{u}_\Pi) \in (\mathbf{W}_I, \widehat{\mathbf{W}}_\Delta, \widehat{\mathbf{W}}_\Pi)$ such that

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & A_{\Pi I}^T \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Pi\Delta}^T \\ A_{\Pi I} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_\Pi \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{pmatrix}. \tag{2.2}$$

This problem is assembled from the subdomain problems

$$\begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^T} & A_{\Pi I}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Delta^{(i)} \\ \mathbf{u}_\Pi^{(i)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I^{(i)} \\ \mathbf{f}_\Delta^{(i)} \\ \mathbf{f}_\Pi^{(i)} \end{pmatrix}. \tag{2.3}$$

We also denote by $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$ the dual spaces, that is, the spaces of the right-hand sides corresponding to $\mathbf{W}_\Gamma$, $\widehat{\mathbf{W}}_\Gamma$, and $\widetilde{\mathbf{W}}_\Gamma$, respectively.

In order to describe the BDDC algorithms, we need to introduce several restriction, extension, and scaling operators between different spaces.

The restriction operators from the product spaces to the subdomain local spaces are

$$\widehat{\mathbf{W}}_\Gamma \xrightarrow{R_\Gamma^{(i)}} \mathbf{W}_\Gamma^{(i)}, \quad \widehat{\mathbf{W}}_\Gamma \xrightarrow{\widehat{R}_\Delta^{(i)}} \mathbf{W}_\Delta^{(i)}, \quad \widetilde{\mathbf{W}}_\Gamma \xrightarrow{\overline{R}_\Gamma^{(i)}} \mathbf{W}_\Gamma^{(i)}, \quad \widehat{\mathbf{W}}_\Pi \xrightarrow{R_\Pi^{(i)}} \mathbf{W}_\Pi^{(i)}, \quad \text{and} \quad \mathbf{W}_\Delta \xrightarrow{R_\Delta^{(i)}} \mathbf{W}_\Delta^{(i)}.$$

Additionally, there are three restriction operators:

$$\widehat{\mathbf{W}}_\Gamma \xrightarrow{\widehat{R}_\Pi} \widehat{\mathbf{W}}_\Pi, \quad \widetilde{\mathbf{W}}_\Gamma \xrightarrow{R_{\Gamma\Delta}} \mathbf{W}_\Delta, \quad \text{and} \quad \widetilde{\mathbf{W}}_\Gamma \xrightarrow{R_{\Gamma\Pi}} \widehat{\mathbf{W}}_\Pi.$$

We also introduce two extension operators:

$$\widehat{\mathbf{W}}_\Gamma \xrightarrow{\widetilde{R}_\Gamma} \widetilde{\mathbf{W}}_\Gamma \xrightarrow{\overline{R}_\Gamma} \mathbf{W}_\Gamma,$$

where $\widetilde{R}_\Gamma$ is the direct sum of the operators $\widehat{R}_\Delta^{(i)}$ and $\widehat{R}_\Pi$, and $\overline{R}_\Gamma$ is the direct sum of the operators $\overline{R}_\Gamma^{(i)}$.

Multiplying each element of the matrix $\widehat{R}_\Delta^{(i)}$, which corresponds to a node $x \in \partial\Omega_i$, with $\delta_i^\dagger(x)$ gives us $\widehat{R}_{D,\Delta}^{(i)}$. Here, we define the scale factor $\delta_i^\dagger(x)$ as follows: For $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x)}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h, \tag{2.4}$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$, and $\rho_j(x)$ is the coefficient of (2.1) at x in the subdomain Ω_j.

The scaled extension operator $\widetilde{R}_{D,\Gamma}$ is the direct sum of the operators $\widehat{R}_{D,\Delta}^{(i)}$ and $\widehat{R}_{\Pi}$. Equivalently, we can write $\widetilde{R}_{D,\Gamma} = D\widetilde{R}_{\Gamma}$, where D is a diagonal scaling matrix. The diagonal elements of D, corresponding to the primal variables, are 1, and all others are given by $\delta_i^{\dagger}(x)$.

We also use the same restriction, extension, and scaled extension operators for $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$.

We now reduce the global problem (2.2) to an interface problem. We first introduce the subdomain Schur complement $S_\Gamma^{(i)}$ by eliminating the subdomain interior variables $u_I^{(i)}$ in (2.3) as follows:

$$S_\Gamma^{(i)} = \begin{pmatrix} A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)^T} \\ A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{pmatrix} - \begin{pmatrix} A_{\Delta I}^{(i)} \\ A_{\Pi I}^{(i)} \end{pmatrix} A_{II}^{(i)^{-1}} \begin{pmatrix} A_{\Delta I}^{(i)^T} & A_{\Pi I}^{(i)^T} \end{pmatrix}$$

and let

$$S_\Gamma = \begin{pmatrix} S_\Gamma^{(1)} & & \\ & \ddots & \\ & & S_\Gamma^{(N)} \end{pmatrix}.$$

The partially assembled Schur complement $\widetilde{S}_\Gamma$ is obtained from S_Γ by assembling the primal variables on the subdomain interface, i.e.,

$$\widetilde{S}_\Gamma = \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma.$$

$\widetilde{S}_\Gamma$ can be further assembled with respect to the variables of the $\mathbf{W}_\Delta^{(i)}$ and the reduced interface problem of (2.2) can be written as follows: Find $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$ such that

$$\widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \mathbf{u}_\Gamma = \mathbf{g}_\Gamma,$$

where

$$\mathbf{g}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} \left\{ \begin{pmatrix} \mathbf{f}_\Delta^{(i)} \\ \mathbf{f}_\Pi^{(i)} \end{pmatrix} - \begin{pmatrix} A_{\Delta I}^{(i)} \\ A_{\Pi I}^{(i)} \end{pmatrix} A_{II}^{(i)^{-1}} \mathbf{f}_I^{(i)} \right\}.$$

The preconditioned two-level BDDC equation is of the form

$$M^{-1}\widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \mathbf{u}_\Gamma = M^{-1}\mathbf{g}_\Gamma,$$

where the preconditioner $M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}$ has the following form:

(2.5)
$$\widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \left(\sum_{i=1}^N \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} \right) R_{\Gamma\Delta} + \Phi S_\Pi^{-1} \Phi^T \right\} \widetilde{R}_{D,\Gamma}.$$

Here Φ is the matrix given by the coarse level basis functions of minimal energy defined by

$$\Phi = R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T \sum_{i=1}^N \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} R_\Pi^{(i)}.$$

The coarse level problem matrix S_Π is determined by

(2.6)
$$S_\Pi = \sum_{i=1}^N R_\Pi^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{pmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} \right\} R_\Pi^{(i)},$$

which is obtained by assembling subdomain matrices; for additional details, cf. [4, 11, 9].

We know that, under certain assumptions, and for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\mathbf{u}_\Gamma^T M \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \mathbf{u}_\Gamma \le C \left(1 + \log(H/h)\right)^2 \mathbf{u}_\Gamma^T M \mathbf{u}_\Gamma. \tag{2.7}$$

These estimates can be established directly by using methods very similar to those of certain studies of the FETI-DP algorithms. Denote by E_D and P_D, respectively, the average and jump operators (see [14, Formulas (6.4) and (6.38)]) on the space $\widetilde{\mathbf{W}}_\Gamma$. Central to obtaining the condition number estimate for the preconditioned two-level BDDC operator is a bound for the E_D operators (see [12, Theorem 25]). Since $E_D + P_D = I$ (see [14, Lemma 6.10]), we need only find a bound for the P_D operator. We obtain a bound for the P_D operator by using [14, Lemma 6.36] under [14, Assumption 4.3.1] for the triangulation and using [14, Assumption 6.27.2] for the coefficient $\rho(x)$ of (2.1).

3. A three-level BDDC method. For the three-level cases, as in [17], the coarse problem matrix S_Π defined in (2.6) will not be factored by a direct solver. Instead, a new level is introduced and the coarse problem is solved approximately. Call the new level the subregion level. To distinguish the spaces and operators for the subregion level from those for the subdomain level, we use the subscript c for the former.

We decompose Ω into N_c subregions Ω^j with diameters $\hat{H}^j$, $j = 1, \ldots, N_c$. Each subregion Ω^j is the union of N_j subdomains Ω_i^j with diameters H_i^j. Let $\hat{H} = \max_j \hat{H}^j$ and $H = \max_{i,j} H_i^j$, for $j = 1, \ldots, N_c$, and $i = 1, \ldots, N_j$. Then N, the total number of subdomains, can be written as $N = N_1 + \cdots + N_{N_c}$.

We introduce the subregional Schur complement as

$$S_\Pi^{(j)} = \sum_{i=1}^{N_j} R_\Pi^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{pmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} \right\} R_\Pi^{(i)} \tag{3.1}$$

and note that the coarse problem matrix S_Π can be assembled from the $S_\Pi^{(j)}$.

In the two-level case, S_Π is factored by a direct solver at the beginning of the computation; cf. (2.5). Here, we build $\widetilde{S}_\Pi^{-1}$ to approximate S_Π^{-1}. Replacing S_Π^{-1} in (2.5) with $\widetilde{S}_\Pi^{-1}$ gives us the three-level preconditioner $\widetilde{M}^{-1}$:

$$\widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \left(\sum_{i=1}^N \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} \right) R_{\Gamma\Delta} + \Phi \widetilde{S}_\Pi^{-1} \Phi^T \right\} \widetilde{R}_{D,\Gamma}.$$

To define $\widetilde{S}_\Pi^{-1}$ in detail, we need to introduce several spaces and operators.

Let Γ_c be the interface between the subregions; note that $\Gamma_c \subset \Gamma$. For each subregion Ω^i, we denote by $\mathbf{W}_c^{(i)}$ the space corresponding to the subdomain edge average variables in this subregion. Let $\mathbf{W}_c = \prod_{i=1}^{N_c} \mathbf{W}_c^{(i)}$ and let $\widehat{\mathbf{W}}_c$ be the subspace of $\mathbf{W}_c$ of elements that are continuous across Γ_c. $\mathbf{W}_c^{(i)}$ can be decomposed into a subregion interior part $\mathbf{W}_{c,I_c}^{(i)}$ and a subregion interface part $\mathbf{W}_{c,\Gamma_c}^{(i)}$, i.e., $\mathbf{W}_c^{(i)} = \mathbf{W}_{c,I_c}^{(i)} \bigoplus \mathbf{W}_{c,\Gamma_c}^{(i)}$. We further decompose the subregion interface part $\mathbf{W}_{c,\Gamma_c}^{(i)}$ into a primal subspace $\mathbf{W}_{c,\Pi_c}^{(i)}$ and a dual subspace $\mathbf{W}_{c,\Delta_c}^{(i)}$, i.e., $\mathbf{W}_{c,\Gamma_c}^{(i)} = \mathbf{W}_{c,\Pi_c}^{(i)} \bigoplus \mathbf{W}_{c,\Delta_c}^{(i)}$. Here, we will consider only the use of edge average constraints over subregion edges. Again, we should change the variables for all local coarse matrices corresponding to these edge average constraints. We will assume that all matrices are written in the new variables.

We denote the associated subregion interface product space by $\mathbf{W}_{c,\Gamma_c} := \prod_{i=1}^{N_c} \mathbf{W}_{c,\Gamma_c}^{(i)}$. We note that the elements in $\mathbf{W}_{c,\Gamma_c}$ can be discontinuous across the subregion interface Γ_c. Let $\widehat{\mathbf{W}}_{c,\Gamma_c}$ and $\widetilde{\mathbf{W}}_{c,\Gamma_c}$ be two subsets of $\mathbf{W}_{c,\Gamma_c}$. The elements are continuous across Γ_c in $\widehat{\mathbf{W}}_{c,\Gamma_c}$, whereas only the primal variables are continuous across Γ_c in $\widetilde{\mathbf{W}}_{c,\Gamma_c}$. We have $\widehat{\mathbf{W}}_{c,\Gamma_c} \subset \widetilde{\mathbf{W}}_{c,\Gamma_c} \subset \mathbf{W}_{c,\Gamma_c}$. We also need two extension operators $\widetilde{R}_{\Gamma_c}$ and $\overline{R}_{\Gamma_c}$,

$$\widehat{\mathbf{W}}_{c,\Gamma_c} \xrightarrow{\widetilde{R}_{\Gamma_c}} \widetilde{\mathbf{W}}_{c,\Gamma_c} \xrightarrow{\overline{R}_{\Gamma_c}} \mathbf{W}_{c,\Gamma_c},$$

which are similar to $\widetilde{R}_\Gamma$ and $\overline{R}_\Gamma$, respectively.

We denote by $\widehat{\mathbf{F}}_c$ and $\widehat{\mathbf{F}}_{\Gamma_c}$ the dual spaces of $\widehat{\mathbf{W}}_c$ and $\widehat{\mathbf{W}}_{c,\Gamma_c}$, respectively. We use the same operators for $\widehat{\mathbf{F}}_c$ and $\widehat{\mathbf{F}}_{\Gamma_c}$.

We are now ready to explain how $\widetilde{S}_\Pi^{-1}$ works on a vector in $\widehat{\mathbf{F}}_c$. Given a vector $\Psi \in \widehat{\mathbf{F}}_c$, let $\mathbf{y} = S_\Pi^{-1}\Psi$ and $\widetilde{\mathbf{y}} = \widetilde{S}_\Pi^{-1}\Psi$. We write Ψ, $\mathbf{y}$, and $\widetilde{\mathbf{y}}$ in terms of interior and interface parts, i.e., $\Psi = (\Psi_{I_c}^{(1)}, \dots, \Psi_{I_c}^{(N_c)}, \Psi_{\Gamma_c})^T$, $\mathbf{y} = (\mathbf{y}_{I_c}^{(1)}, \dots, \mathbf{y}_{I_c}^{(N_c)}, \mathbf{y}_{\Gamma_c})^T$, and $\widetilde{\mathbf{y}} = (\widetilde{\mathbf{y}}_{I_c}^{(1)}, \dots, \widetilde{\mathbf{y}}_{I_c}^{(N_c)}, \widetilde{\mathbf{y}}_{\Gamma_c})^T$.

To obtain $\mathbf{y}$, we can solve $S_\Pi \mathbf{y} = \Psi$ by block factorization. This vector satisfies

$$\begin{pmatrix} S_{\Pi_{I_cI_c}}^{(1)} & \mathbf{0} & \mathbf{0} & S_{\Pi_{\Gamma_cI_c}}^{(1)^T} R_{\Gamma_c}^{(1)} \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & S_{\Pi_{I_cI_c}}^{(N_c)} & S_{\Pi_{\Gamma_cI_c}}^{(N_c)^T} R_{\Gamma_c}^{(N_c)} \\ R_{\Gamma_c}^{(1)^T} S_{\Pi_{\Gamma_cI_c}}^{(1)} & \cdots & R_{\Gamma_c}^{(N_c)^T} S_{\Pi_{\Gamma_cI_c}}^{(N_c)} & \sum_{i=1}^{N_c} R_{\Gamma_c}^{(i)^T} S_{\Pi_{\Gamma_c\Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)} \end{pmatrix} \begin{pmatrix} \mathbf{y}_{I_c}^{(1)} \\ \vdots \\ \mathbf{y}_{I_c}^{(N_c)} \\ \mathbf{y}_{\Gamma_c} \end{pmatrix}$$
$$= \begin{pmatrix} \Psi_{I_c}^{(1)} \\ \vdots \\ \Psi_{I_c}^{(N_c)} \\ \Psi_{\Gamma_c} \end{pmatrix},$$

where $R_{\Gamma_c}^{(i)} : \widehat{\mathbf{W}}_{c,\Gamma_c} \to \mathbf{W}_{c,\Gamma_c}^{(i)}$ is a restriction operator.

We solve $\mathbf{y}_{I_c}^{(i)}$ in terms of $\mathbf{y}_{\Gamma_c}$ and have

$$\mathbf{y}_{I_c}^{(i)} = S_{\Pi_{I_cI_c}}^{(i)^{-1}} \left(\Psi_{I_c}^{(i)} - S_{\Pi_{I_c\Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)} \mathbf{y}_{\Gamma_c} \right). \tag{3.2}$$

We then obtain the subregion interface problem

$$\left(\sum_{i=1}^{N_c} R_{\Gamma_c}^{(i)^T}(S_{\Pi_{\Gamma_c\Gamma_c}}^{(i)} - S_{\Pi_{\Gamma_c I_c}}^{(i)} S_{\Pi_{I_c I_c}}^{(i)^{-1}} S_{\Pi_{\Gamma_c I_c}}^{(i)^T}) R_{\Gamma_c}^{(i)}\right) \mathbf{y}_{\Gamma_c} = \Psi_{\Gamma_c} - \sum_{i=1}^{N_c} R_{\Gamma_c}^{(i)^T} S_{\Pi_{\Gamma_c I_c}}^{(i)} S_{\Pi_{I_c I_c}}^{(i)^{-1}} \Psi_{I_c}^{(i)}. \tag{3.3}$$

Let

$$T^{(i)} = S_{\Pi_{\Gamma_c\Gamma_c}}^{(i)} - S_{\Pi_{\Gamma_c I_c}}^{(i)} S_{\Pi_{I_c I_c}}^{(i)^{-1}} S_{\Pi_{\Gamma_c I_c}}^{(i)^T}$$

be the subregion Schur complement in (3.3).

Denote their direct sum by T:

$$T = \begin{pmatrix} T^{(1)} & & \\ & \ddots & \\ & & T^{(N_c)} \end{pmatrix}.$$

As on the subdomain level case, we introduce a partially assembled Schur complement of S_Π, and denote it by $\widetilde{T}$. $\widetilde{T}$ can be written as

$$\widetilde{T} = \overline{R}_{\Gamma_c}^T T \overline{R}_{\Gamma_c}. \tag{3.4}$$

We define $\mathbf{h}_{\Gamma_c} \in \widehat{\mathbf{F}}_{\Gamma_c}$ by

$$\mathbf{h}_{\Gamma_c} = \Psi_{\Gamma_c} - \sum_{i=1}^{N_c} R_{\Gamma_c}^{(i)^T} S_{\Pi_{\Gamma_c I_c}}^{(i)} S_{\Pi_{I_c I_c}}^{(i)^{-1}} \Psi_{I_c}^{(i)}. \tag{3.5}$$

The reduced subregion interface problem (3.3) can be written as follows: Find $\mathbf{y}_{\Gamma_c} \in \widehat{\mathbf{W}}_{c,\Gamma_c}$ such that

$$\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c} \mathbf{y}_{\Gamma_c} = \mathbf{h}_{\Gamma_c}. \tag{3.6}$$

To obtain the approximation $\widetilde{\mathbf{y}} = \widetilde{S}_\Pi^{-1} \mathbf{\Psi}$, we do not solve (3.6) exactly. Instead, we compute $\widetilde{\mathbf{y}}_{\Gamma_c}$ as

$$\widetilde{\mathbf{y}}_{\Gamma_c} = \widetilde{R}_{D_c,\Gamma_c}^T \widetilde{T}^{-1} \widetilde{R}_{D_c,\Gamma_c} \mathbf{h}_{\Gamma_c}. \tag{3.7}$$

Here $\widetilde{R}_{D_c,\Gamma_c}$ is a scaled operator which is similar to $\widetilde{R}_{D,\Gamma}$; we can write $\widetilde{R}_{D_c,\Gamma_c} = D_c \widetilde{R}_{\Gamma_c}$, where D_c is a diagonal scaling matrix. The diagonal elements of D_c, corresponding to the primal variables, are 1, and all others are given by $\delta_{c,i}^\dagger(x)$. Here $\delta_{c,i}^\dagger(x)$ is similar to $\delta_i^\dagger(x)$, which is defined in (2.4), except that $\delta_{c,i}^\dagger(x)$ is defined for the subregion interface instead of the subdomain interface nodes. For an x on the subregion interface, $\delta_{c,i}^\dagger(x)$ is defined as follows: For $\gamma \in [\frac{1}{2}, \infty)$, $\delta_{c,i}^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j\in\mathcal{N}_x} \rho_j^\gamma(x)}$, where $\mathcal{N}_x$ is the set of indices j of the subregions such that $x \in \partial\Omega^j$ and $\rho_j(x)$ is the coefficient of (2.1) at $x \in \partial\Omega^j$. (In our theory, we assume the ρ_i are constant in the subregions.)

We will maintain the same relation between $\widetilde{\mathbf{y}}_{I_c}^{(i)}$ and $\widetilde{\mathbf{y}}_{\Gamma_c}$as for $\mathbf{y}_{I_c}^{(i)}$ and $\mathbf{y}_{\Gamma_c}$ in (3.2), i.e.,

$$\widetilde{\mathbf{y}}_{I_c}^{(i)} = S_{\Pi_{I_c I_c}}^{(i)^{-1}} \left(\Psi_{I_c}^{(i)} - S_{\Pi_{I_c \Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)} \widetilde{\mathbf{y}}_{\Gamma_c} \right). \tag{3.8}$$

4. Some auxiliary results. In this section, we will collect a number of results which are needed in our theory. In order to avoid a proliferation of constants, we will use the notation $A \approx B$. This means that there are two constants c and C, independent of any mesh parameter and the coefficients of (2.1), such that $cA \le B \le CA$, where $C < \infty$ and $c > 0$. For the definition of discrete harmonic functions, see [14, section 4.4].

LEMMA 4.1. *Let $\mathcal{D}$ be a cube with vertices $A_1 = (0,0,0)$, $B_1 = (H,0,0)$, $C_1 = (H,H,0)$, $D_1 = (0,H,0)$, $A_2 = (0,0,H)$, $B_2 = (H,0,H)$, $C_2 = (H,H,H)$, and $D_2 = (0,H,H)$ with a quasi-uniform triangulation of mesh size h. Then, there exists a discrete harmonic function v defined in $\mathcal{D}$ such that $\bar{v}_{A_1B_1} \approx 1+\log\frac{H}{h}$, where $\bar{v}_{A_1B_1}$ is the average of v over the edge A_1B_1, $|v|^2_{H^1(\mathcal{D})} \approx H\left(1+\log\frac{H}{h}\right)$, and v has a zero average over the other edges of the cube.*

Proof. This lemma follows from a result by Brenner and He [1, Lemma 4.2]. Let N be an integer and G_N the function defined on $(0,1)$ by

$$G_N(x) = \sum_{n=1}^{N}\left(\frac{1}{4n-3}\sin\left((4n-3)\pi x\right)\right).$$

$G_N(x)$ is even with respect to the midpoint of $(0,1)$, where it attains its maximum in absolute value. Moreover, we have

$$|G_N|^2_{H^{1/2}_{00}(0,1)} \approx 1+\log N \text{ and } \|G_N\|_{L^2(0,1)} \approx 1;$$

see [1, Lemma 3.7].

Let $[-H,0]$ and $[0,H]$ have a mesh inherited from the quasi-uniform meshes on D_1A_1 and A_1B_1, respectively, and let $g_h(x)$ be the nodal interpolation of $G_N(\frac{x+H}{2H})$. Then, we have

$$\|g_h\|_{L^\infty(-H,H)} \approx 1+\log\frac{H}{h}, \quad |g_h|^2_{H^{1/2}_{00}(-H,H)} \approx 1+\log\frac{H}{h}, \quad \text{and} \quad \|g_h\|_{L^2(-H,H)} \approx H; \tag{4.1}$$

see [1, Lemma 3.7] or [17, Lemma 1].

Let $\tau_h(x)$ be a function on $[0,H]$ defined as follows:

$$\tau_h(x) = \begin{cases} \frac{x}{h_1}, & 0 \le x \le h_1, \\ 1, & h_1 \le x \le H-h_2, \\ \frac{H-x}{h_2}, & H-h_2 \le x \le H, \end{cases}$$

where h_1 and h_2 are the lengths of the two end mesh intervals of $[0,H]$. Then the following estimates hold:

$$\|\tau_h\|^2_{L^2(0,H)} \approx H \quad \text{and} \quad |\tau_h|^2_{H^{1/2}_{00}(0,H)} \approx 1+\log\frac{H}{h}; \tag{4.2}$$

see [1, Lemma 3.6].

Define the discrete harmonic function v as 0 everywhere on the boundary of $\mathcal{D}$ except on the two open faces $A_1B_1C_1D_1$ and $A_1B_1B_2A_2$. On these two faces it is defined by

$$v(x_1,x_2,0) = g_h(x_2)\tau_h(x_1) \quad \text{for } (x_1,x_2) \in A_1B_1C_1D_1,$$

$$v(x_1,0,x_3) = g_h(-x_3)\tau_h(x_1) \quad \text{for } (x_1,x_3) \in A_1B_1B_2A_2.$$

It is clear that $\bar{v}_{A_1B_1} \approx 1 + \log \frac{H}{h}$ and that v has a zero average over the other edges. Since v is discrete harmonic in $\mathcal{D}$, we have

$$\begin{aligned} |v|^2_{H^1(\mathcal{D})} &\approx |v|^2_{H^{1/2}(\partial\mathcal{D})} \\ &\approx |g_h|^2_{H^{1/2}_{00}(-H,H)} \|\tau_h\|^2_{L^2(0,H)} + |\tau_h|^2_{H^{1/2}_{00}(0,H)} \|g_h\|^2_{L^2(-H,H)} \\ &\approx H\left(1+\log\frac{H}{h}\right), \end{aligned}$$

where we have used (4.1), (4.2), and [1, Corollary 3.5]. □

Remark. In Lemma 4.1, we have constructed the function v for a cube $\mathcal{D}$. By using similar ideas, we can construct functions v for other shape-regular polyhedra which will satisfy similar properties and bounds.

LEMMA 4.2. *Let Ω^i_j be the subdomains of a subregion Ω^i, $j = 1, \ldots, N_i$, and let $V^h_{i,j}$ be the standard continuous piecewise trilinear finite element function space for the subdomain Ω^i_j with a quasi-uniform fine mesh with mesh size of order h. Denote by $\mathcal{E}_k$, $k = 1, \ldots, K_j$, the edges of the subdomain Ω^i_j. Given the average values of u, $\bar{u}_{\mathcal{E}_k}$ over each edge, let $u \in V^h_{i,j}$ be the minimal energy extension in each subdomain Ω^i_j with these average values given on the edges of Ω^i_j, $j = 1, \ldots, N_i$. Then, we have*

$$\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i} |u|^2_{H^1(\Omega^i_j)}\right) \approx \sum_{j=1}^{N_i}\sum_{k_1,k_2=1}^{K_j} H|\bar{u}_{\mathcal{E}_{k_1}} - \bar{u}_{\mathcal{E}_{k_2}}|^2.$$

Proof. Without loss of generality, we assume that the subdomains are hexahedral. Denote the edges of the subdomain Ω^i_j by $\mathcal{E}_k$, $k = 1, \ldots, 12$, and denote the average values of u over these 12 edges by $\bar{u}_{\mathcal{E}_k}$, $k = 1, \ldots, 12$, respectively.

According to Lemma 4.1, we can construct 11 discrete harmonic functions ϕ_m, $m = 2, \ldots, 12$, on Ω^i_j such that

$$(\overline{\phi}_m)_{\mathcal{E}_k} = \begin{cases} (\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})\,(1+\log\frac{H}{h}), & m = k, \\ 0, & m \neq k, \end{cases}$$

and with

$$|\phi_m|^2_{H^1(\Omega^i_j)} \approx (\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})^2\, H\left(1+\log\frac{H}{h}\right), \quad m = 2, \ldots, 12. \tag{4.3}$$

Let $v_j = \frac{1}{1+\log\frac{H}{h}}(\sum_{m=2}^{12}\phi_m) + \bar{u}_{\mathcal{E}_1}$; we then have $(\bar{v}_j)_{\mathcal{E}_k} = \bar{u}_{\mathcal{E}_k}$, for $k = 1, \ldots, 12$, and

$$\begin{aligned} |v_j|^2_{H^1(\Omega^i_j)} &= |\frac{1}{1+\log\frac{H}{h}}\left(\sum_{m=2}^{12}\phi_m\right) + \bar{u}_{\mathcal{E}_1}|^2_{H^1(\Omega^i_j)} \\ &= \left(\frac{1}{1+\log\frac{H}{h}}\right)^2 \left|\sum_{m=2}^{12}\phi_m\right|^2_{H^1(\Omega^i_j)} \leq 11\left(\frac{1}{1+\log\frac{H}{h}}\right)^2 \sum_{m=2}^{12}|\phi_m|^2_{H^1(\Omega^i_j)} \\ &\leq \left(\frac{1}{c^{1/2}(1+\log\frac{H}{h})}\right)^2 H\left(1+\log\frac{H}{h}\right)\sum_{m=2}^{12}(\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})^2 \\ &\leq \frac{1}{c(1+\log\frac{H}{h})}\sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2. \end{aligned}$$

Here, we have used (4.3) for the penultimate inequality.

By the definition of u, we have

$$|u|^2_{H^1(\Omega^i_j)} \leq |v_j|^2_{H^1(\Omega^i_j)} \leq \frac{1}{c(1+\log\frac{H}{h})} \sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2.$$

Summing over all the subdomains in the subregion Ω^i, we have

$$c\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i} |u|^2_{H^1(\Omega^i_j)}\right) \leq \sum_{j=1}^{N_i}\sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2.$$

This proves one side of the equivalence.

We prove the other side as follows:

$$\begin{aligned}
&\sum_{j=1}^{N_i}\sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2 = \sum_{j=1}^{N_i}\sum_{k=1}^{12} H|\overline{(u-\bar{u}_{\mathcal{E}_1})}_{\mathcal{E}_k}|^2 \\
&\leq C\left(\sum_{j=1}^{N_i}\sum_{k=1}^{12} H\frac{1}{H}\|u-\bar{u}_{\mathcal{E}_1}\|^2_{L^2(\mathcal{E}_k)}\right) \leq C\left(\sum_{j=1}^{N_i}\left(1+\log\frac{H}{h}\right)|u|^2_{H^1(\Omega^i_j)}\right) \\
&\leq C\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i}|u|^2_{H^1(\Omega^i_j)}\right).
\end{aligned}$$

Here, we have used a standard finite element Sobolev inequality; see [14, Lemma 4.30] for the second inequality and [14, Lemma 4.16] for the penultimate inequality.

We complete the proof of the other side of the equivalence by using the triangle inequality. ∎

We now introduce a new mesh on each subregion; we follow [3, 13]. The purpose of introducing this mesh is to relate the quadratic form of Lemma 4.2 to one for a more conventional finite element space.

Given a subregion Ω^i and subdomains Ω^i_j, $j = 1, \ldots, N_i$, let $\mathcal{T}$ be a quasi-uniform subtriangulation of Ω^i such that its set of vertices includes the vertices and the midpoints of the edges of Ω^i_j. For the hexahedral case, we decomposed each hexahedron into eight hexahedra by connecting the midpoints of the edges. We then partition the vertices of the new mesh $\mathcal{T}$ into two sets. The midpoints of edges are called primary and the other vertices of the new mesh $\mathcal{T}$ are called secondary. We call two vertices in the triangulation $\mathcal{T}$ adjacent if there is an edge of $\mathcal{T}$ between them, as in the standard finite element context; see Figures 1 and 2.

Let $U_H(\Omega)$ be the continuous piecewise trilinear finite element function space with respect to the new triangulation $\mathcal{T}$. For a subregion Ω^i, $U_H(\Omega^i)$ and $U_H(\partial\Omega^i)$ are defined as restrictions:

$$U_H(\Omega^i) = \{u|_{\Omega^i} : \; u \in U_H(\Omega)\}, \qquad U_H(\partial\Omega^i) = \{u|_{\partial\Omega^i} : \; u \in U_H(\Omega)\}.$$

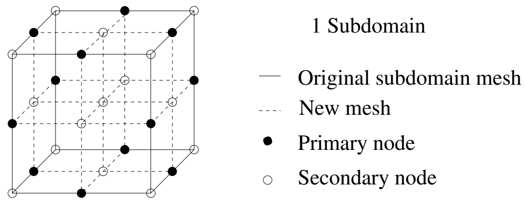

FIG. 1. *The new mesh and primary and secondary nodes in one subdomain of a subregion. (Note that all the lines of the original subdomain mesh are drawn in the same way.)*

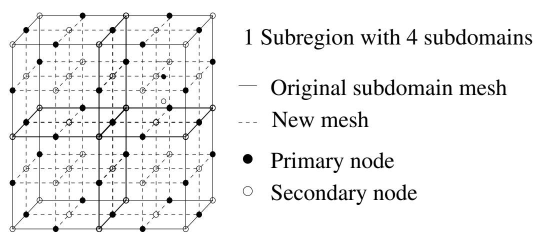

FIG. 2. *The new mesh and primary and secondary nodes in a subregion with four subdomains. (Note that all the lines of the original subdomain mesh are drawn in the same way.)*

We define a mapping $I_H^{\Omega^i}$ of any function ϕ, defined at the primary vertices in Ω^i, to $U_H(\Omega^i)$ by

(4.4)

$$I_H^{\Omega^i}\phi(x) = \begin{cases} \phi(x) \text{ if } x \text{ is a primary node;} \\ \text{the average of the values at all adjacent primary nodes} \\ \text{on the edges of } \Omega^i \text{ if } x \text{ is a vertex of } \Omega^i; \\ \text{the average of the values at two adjacent primary nodes} \\ \text{on the same edge of } \Omega^i \text{ if } x \text{ is an edge secondary node of } \Omega^i; \\ \text{the average of the values at all adjacent primary nodes on the} \\ \text{boundary of } \Omega^i \text{ if } x \text{ is a face secondary boundary node of } \Omega^i; \\ \text{the average of the values at all adjacent primary nodes} \\ \text{if } x \text{ is an interior secondary node of } \Omega^i \text{ with some adjacent} \\ \text{primary nodes;} \\ \text{the average of the values at all adjacent nodes} \\ \text{if } x \text{ is an interior secondary node of } \Omega^i \text{ without any adjacent} \\ \text{primary nodes;} \\ \text{the result of trilinear interpolation using the vertex values} \\ \text{if } x \text{ is not a vertex of } \mathcal{T}. \end{cases}$$

We recall that $\mathbf{W}_c^{(i)}$ is the discrete space of the values at the primary nodes given by the subdomain edge average values. $I_H^{\Omega^i}$ can be considered as a map from $\mathbf{W}_c^{(i)}$ to $U_H(\Omega^i)$ or as a map from $U_H(\Omega^i)$ to $U_H(\Omega^i)$.

Let $I_H^{\partial\Omega^i}$ be the mapping of a function ϕ defined at the primary vertices on the boundary of Ω^i to $U_H(\partial\Omega^i)$ and defined by $I_H^{\partial\Omega^i}\phi = (I_H^{\Omega^i}\phi_e)|_{\partial\Omega^i}$, where ϕ_e is any function in $\mathbf{W}_c^{(i)}$ such that $\phi_e|_{\partial\Omega^i} = \phi$. The map is well defined since the boundary values of $I_H^{\Omega^i}\phi_e$ depend only on the boundary values of ϕ_e.

Finally, let

$$\widetilde{U}_H(\Omega^i) = \{\psi = I_H^{\Omega^i}\phi,\ \phi \in U_H(\Omega^i)\}, \qquad \widetilde{U}_H(\partial\Omega^i) = \{\psi|_{\partial\Omega^i},\ \psi \in \widetilde{U}_H(\Omega^i)\}.$$

$I_H^{\partial\Omega^i}$ also can be considered as a map from $\mathbf{W}_{c,\Gamma_c}^{(i)}$ to $\widetilde{U}_H(\partial\Omega^i)$.

Remark. We carefully define the operators $I_H^{\Omega^i}$ and $I_H^{\partial\Omega^i}$ so that, if the edge averages of $w_i \in \mathbf{W}_{c,\Gamma_c}^{(i)}$ and $w_j \in \mathbf{W}_{c,\Gamma_c}^{(j)}$ over an edge $\mathcal{E}$ are the same, we have $(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}}$. Here we need to use a weighted average which has a larger weight at the two end points since we consider an edge as an open set and the two end primary points have only one neighboring secondary node on the edge. But this will not affect our analysis. We could also define a weighted edge average of w_i and w_j and obtain $(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}}$ for the usual average.

We list some useful lemmas from [3]. For proofs of Lemmas 4.3 and 4.4, see [3, Lemmas 6.1 and 6.2], respectively.

LEMMA 4.3. *There exists a constant $C > 0$, independent of H and $|\Omega^i|$, the volume of Ω^i, such that*

$$|I_H^{\Omega^i}\phi|_{H^1(\Omega^i)} \le C|\phi|_{H^1(\Omega^i)} \text{ and } \|I_H^{\Omega^i}\phi\|_{L^2(\Omega^i)} \le C\|\phi\|_{L^2(\Omega^i)} \quad \forall \phi \in U_H(\Omega^i).$$

LEMMA 4.4. *For $\hat{\phi} \in \widetilde{U}_H(\partial\Omega^i)$,*

$$\inf_{\phi\in\widetilde{U}_H(\Omega^i),\ \phi|_{\partial\Omega^i}=\hat{\phi}} \|\phi\|_{H^1(\Omega^i)} \approx \|\hat{\phi}\|_{H^{1/2}(\partial\Omega^i)},$$

$$\inf_{\phi\in\widetilde{U}_H(\Omega^i),\ \phi|_{\partial\Omega^i}=\hat{\phi}} |\phi|_{H^1(\Omega^i)} \approx |\hat{\phi}|_{H^{1/2}(\partial\Omega^i)}.$$

LEMMA 4.5. *For all $w_i \in \mathbf{W}_{c,\Gamma_c}^{(i)}$, we have*

$$\rho_i C_1 |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)} \approx \left(1 + \log\frac{H}{h}\right)(T^{(i)}w_i, w_i),$$

where $(T^{(i)}w_i, w_i) = w_i^T T^{(i)} w_i = |w_i|^2_{T^{(i)}}$ *and* $T^{(i)} = S^{(i)}_{\Pi_{\Gamma_c\Gamma_c}} - S^{(i)}_{\Pi_{\Gamma_c I_c}} S^{(i)^{-1}}_{\Pi_{I_c I_c}} S^{(i)^T}_{\Pi_{\Gamma_c I_c}}$.

Proof. By the definition of $T^{(i)}$, we have

$$\begin{aligned}
&\left(1+\log\frac{H}{h}\right)(T^{(i)}w_i,w_i) = \left(1+\log\frac{H}{h}\right)\inf_{v\in\mathbf{W}_c^{(i)},\ v|_{\partial\Omega^i}=w_i}|v|^2_{S_\Pi^{(i)}}\\
&= \inf_{v\in\mathbf{W}_c^{(i)},\ v|_{\partial\Omega^i}=w_i}\rho_i\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i}\inf_{u\in V_{i,j}^h,\ \bar{u}_{\mathcal{E}_l}=v_l,\ \mathcal{E}_l\subset\partial\Omega_j^i}|u|^2_{H^1(\Omega_j^i)}\right)\\
&\approx \inf_{v\in\mathbf{W}_c^{(i)},\ v|_{\partial\Omega^i}=w_i}\rho_i\sum_{j=1}^{N_i}\sum_{k_1,k_2=1}^{K_j}H|\bar{u}_{\mathcal{E}_{k_1}}-\bar{u}_{\mathcal{E}_{k_2}}|^2\\
&\approx \inf_{v\in\mathbf{W}_c^{(i)},\ v|_{\partial\Omega^i}=w_i}\rho_i\sum_{j=1}^{N_i}\sum_{k_1,k_2=1}^{K_j}H|v_{k_1}-v_{k_2}|^2\\
&\approx \inf_{v\in\mathbf{W}_c^{(i)},\ v|_{\partial\Omega^i}=w_i}\rho_i|I_H^{\Omega^i}v|^2_{H^1(\Omega^i)} \approx \inf_{\phi\in\widetilde{U}_H(\Omega^i),\ \phi|_{\partial\Omega^i}=I_H^{\partial\Omega^i}w_i}\rho_i|\phi|^2_{H^1(\Omega^i)}\\
&\approx \rho_i|I_H^{\partial\Omega^i}w_i|_{H^{1/2}(\partial\Omega^i)}.
\end{aligned}$$

We use Lemma 4.2 for the third bound, the definitions of $I_H^{\Omega^i}$ and $I_H^{\partial\Omega^i}$ for the fourth and fifth bounds, and Lemma 4.4 for the final bound. ∎

To be fully rigorous, we assume that there is a quasi-uniform coarse triangulation of each subregion. We can then obtain uniform upper and lower bounds for each subregion as is required in Lemma 4.5.

We define the interface average operator E_{D_c} on $\widetilde{\mathbf{W}}_{c,\Gamma_c}$ as $E_{D_c}=\widetilde{R}_{\Gamma_c}\widetilde{R}^T_{D_c,\Gamma_c}$, which computes the averages across the subregion interface Γ_c and then adopts these averages at the boundary points of the subregions.

The interface average operator E_{D_c} has the following property.

LEMMA 4.6.

$$|E_{D_c}\mathbf{w}_{\Gamma_c}|^2_{\widetilde{T}} \le C\left(1+\log\frac{\hat{H}}{H}\right)^2|\mathbf{w}_{\Gamma_c}|^2_{\widetilde{T}}$$

for any $\mathbf{w}_{\Gamma_c}\in\widetilde{\mathbf{W}}_{c,\Gamma_c}$, *where* C *is a positive constant independent of* $\hat{H}$, H, h, *and the coefficients of* (2.1). *Here* $\widetilde{T}$ *is defined in* (3.4).

Proof. Let $w_i=\overline{R}^{(i)}_{\Gamma_c}\mathbf{w}_{\Gamma_c}\in\mathbf{W}^{(i)}_{c,\Gamma_c}$, where $\overline{R}^{(i)}_{\Gamma_c}$ is the restriction operator from $\widetilde{\mathbf{W}}_{c,\Gamma_c}$ to $\mathbf{W}^{(i)}_{c,\Gamma_c}$. We rewrite the formula for $v:=w_{\Gamma_c}-E_{D_c}w_{\Gamma_c}$ for an arbitrary element $\mathbf{w}_{\Gamma_c}\in\widetilde{\mathbf{W}}_{c,\Gamma_c}$, and find that for $i=1,\dots,N_c$,

$$v_i(x):=(\mathbf{w}_{\Gamma_c}(x)-E_{D_c}\mathbf{w}_{\Gamma_c}(x))_i=\sum_{j\in\mathcal{N}_x}\delta^\dagger_{c,j}(w_i(x)-w_j(x)),\quad x\in\partial\Omega^i\cap\Gamma_c. \tag{4.5}$$

Here $\mathcal{N}_x$ is the set of indices of the subregions that have x on their boundaries.

We have

$$|E_{D_c}\mathbf{w}_{\Gamma_c}|^2_{\widetilde{T}}=\sum_{i=1}^{N_c}|w_i-v_i|^2_{T^{(i)}}\le 2\sum_{i=1}^{N_c}|w_i|^2_{T^{(i)}}+2\sum_{i=1}^{N_c}|v_i|^2_{T^{(i)}}\text{ and }|\mathbf{w}_{\Gamma_c}|^2_{\widetilde{T}}=\sum_{i=1}^{N_c}|w_i|^2_{T^{(i)}}.$$

We can therefore focus on the estimate of the contribution from a single subregion Ω^i and proceed as in the proof of [14, Lemma 6.36].

We will also use the simple inequality

$$\rho_i \delta_{c,j}^{\dagger^2} \le \min(\rho_i, \rho_j) \text{ for } \gamma \in [1/2, \infty). \tag{4.6}$$

By Lemma 4.5,

$$(T^{(i)} v_i, v_i) \le C \frac{1}{(1+\log \frac{H}{h})} \rho_i |I_H^{\partial\Omega^i}(v_i)|^2_{H^{1/2}(\partial\Omega^i)}. \tag{4.7}$$

Let $l_i = I_H^{\partial\Omega^i}(v_i)$. By using a partition of unity as in [14, Lemma 6.36], we have

$$l_i = \sum_{\mathcal{F} \subset \partial\Omega^i} I^H(\theta_{\mathcal{F}} l_i) + \sum_{\mathcal{E} \subset \partial\Omega^i} I^H(\theta_{\mathcal{E}} l_i) + \sum_{\mathcal{V} \in \partial\Omega^i} \theta_{\mathcal{V}} l_i(\mathcal{V}),$$

where I^H is the nodal piecewise linear interpolant on the coarse mesh $\mathcal{T}$. We note that the analysis of face and edge terms is almost identical to that in [14, Lemma 6.36]. But the vertex terms are different because of $I_H^{\partial\Omega^i}$. We need only consider the vertex term when two subregions share at least an edge. This make the analysis simpler than in the proof of [14, Lemma 6.36].

Face terms. First, consider

$$I^H(\theta_{\mathcal{F}} l_i) = I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^i}(\delta_{c,j}^{\dagger}(w_i - w_j))).$$

Similar to [14, Lemma 6.36], we obtain, by using (4.6),

$$\begin{aligned}
&\rho_i |I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^i}(\delta_{c,j}^{\dagger}(w_i - w_j)))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\quad = \rho_i \delta_{c,j}^{\dagger^2} |I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^i}(w_i - w_j))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\quad \le \min(\rho_i, \rho_j) |I^H(\theta_{\mathcal{F}}((I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}) - (I_H^{\partial\Omega^i} w_j - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}}) \\
&\qquad + ((\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}})))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\quad \le 3 \min(\rho_i, \rho_j) \Big(|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\qquad + |I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_j - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\qquad + |\theta_{\mathcal{F}}((\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}})|^2_{H^{1/2}(\partial\Omega^i)} \Big).
\end{aligned} \tag{4.8}$$

By the definition of $I_H^{\partial\Omega^i}$,

$$I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_j)) = I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^j} w_j)) \quad \text{and} \quad (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{F}}.$$

By [14, Lemma 4.26], the first and second terms in (4.8) can be estimated as follows:

$$\begin{aligned}
&\min(\rho_i, \rho_j)(|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)} \\
&+ |I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_j - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)}) \\
&\quad = \min(\rho_i, \rho_j)(|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)} \\
&+ |I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^j} w_j - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)}) \\
&\quad \le C \left(1 + \log \frac{\hat{H}}{H}\right)^2 \left(\rho_i |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)} + \rho_j |I_H^{\partial\Omega^j} w_j|^2_{H^{1/2}(\partial\Omega^j)} \right).
\end{aligned}$$

Let $\mathcal{E} \subset \partial\mathcal{F}$. Since the edge averages of w_i and w_j are the same, we have, by the definition of $I_H^{\partial\Omega^i}$ and $I_H^{\partial\Omega^j}$, that $(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}}$. As we have pointed out before, we use a weighted average which has a larger weight at the two end points.

We then have

$$\begin{aligned} &|(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{F}}|^2 \\ &\quad \le 2\left(|(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}|^2 + |(\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}} - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{F}}|^2\right). \end{aligned} \tag{4.9}$$

It is sufficient to consider the first term on the right-hand side. Using [14, Lemma 4.30], we find

$$\begin{aligned} &|(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}|^2 \\ &\quad = |\overline{(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}})_{\mathcal{E}}}|^2 \le C/\hat{H}_i \| I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} \|^2_{L^2(\mathcal{E})}, \end{aligned}$$

and, by using [14, Lemma 4.17] and the Poincaré inequality given as [14, Lemma A.17], we have

$$|(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}|^2 \le \frac{C}{\hat{H}_i}\left(1 + \log\frac{\hat{H}}{H}\right) |I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}|^2_{H^{1/2}(\mathcal{F})}.$$

Combining this with the bound for $\theta_{\mathcal{F}}$ in [14, Lemma 4.26], we have

$$\begin{aligned} &\min(\rho_i, \rho_j) |\theta_{\mathcal{F}}((\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}})|^2_{H^{1/2}(\partial\Omega^i)} \\ &\quad \le C\left(1 + \log\frac{\hat{H}}{H}\right)^2 \left(\rho_i |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)} + \rho_j |I_H^{\partial\Omega^j} w_j|^2_{H^{1/2}(\partial\Omega^j)}\right). \end{aligned}$$

Edge terms. We can develop the same estimate as in [14, Lemma 6.34]. For simplicity, we consider only an edge $\mathcal{E}$ common to four subregions Ω^i, Ω^j, Ω^k, and Ω^l. Then,

$$\begin{aligned} &\rho_i |I^H(\theta_{\mathcal{E}} l_i)|^2_{H^{1/2}(\partial\Omega^i)} \\ &\quad \le \rho_i \Big(|I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^i}(\delta^\dagger_{c,j}(w_i - w_j)))|^2_{H^{1/2}(\partial\Omega^i)} \\ &\quad + |I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^i}(\delta^\dagger_{c,k}(w_i - w_k)))|^2_{H^{1/2}(\partial\Omega^i)} \\ &\quad + |I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^i}(\delta^\dagger_{c,l}(w_i - w_l)))|^2_{H^{1/2}(\partial\Omega^i)} \Big). \end{aligned} \tag{4.10}$$

We recall that $\delta^\dagger_{c,j}$, $\delta^\dagger_{c,k}$, and $\delta^\dagger_{c,l}$ are constants.

By the definition of $I_H^{\partial\Omega^i}$, $I_H^{\partial\Omega^j}$, $I_H^{\partial\Omega^k}$, and $I_H^{\partial\Omega^l}$, we have

$$\theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_j) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^j} w_j), \quad \theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_k) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^k} w_k), \quad \theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_l) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^l} w_l),$$

and

$$(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^k} w_k})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^l} w_l})_{\mathcal{E}}.$$

We assume that Ω^i shares a face with Ω^j as well as Ω^l and shares an edge only with Ω^k.

First, we consider the second term in (4.10). By [14, Lemmas 4.19 and 4.17] and (4.6), we have

$$
\begin{aligned}
&\rho_i|I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^i}(\delta_{c,k}^{\dagger}(w_i - w_k)))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\quad \le C\rho_i {\delta_{c,k}^{\dagger}}^2 \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}}) - \theta_{\mathcal{E}}(I_H^{\partial\Omega^k} w_k - (\overline{I_H^{\partial\Omega^k} w_k})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})} \\
&\quad \le 2C \Big(\rho_i \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})} \\
&\qquad + \rho_k \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^k} w_k - (\overline{I_H^{\partial\Omega^k} w_k})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})}\Big) \\
&\quad \le 2C \Big(\rho_i \|I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}}\|^2_{L^2(\mathcal{E})} + \rho_k \|I_H^{\partial\Omega^k} w_k - (\overline{I_H^{\partial\Omega^k} w_k})_{\mathcal{E}}\|^2_{L^2(\mathcal{E})}\Big) \\
&\quad \le 2C \left(1 + \log \frac{\hat{H}}{H}\right) \Big(\rho_i |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\mathcal{F}^i)} + \rho_k |I_H^{\partial\Omega^k} w_k|^2_{H^{1/2}(\mathcal{F}^k)}\Big) \\
&\quad \le 2C \left(1 + \log \frac{\hat{H}}{H}\right) \Big(\rho_i |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)} + \rho_k |I_H^{\partial\Omega^k} w_k|^2_{H^{1/2}(\partial\Omega^k)}\Big),
\end{aligned}
$$

where $\mathcal{F}^i$ is a face of Ω^i, $\mathcal{F}^k$ is a face of Ω^k, and $\mathcal{F}^i$ and $\mathcal{F}^k$ share the edge $\mathcal{E}$.

The first and third terms can be estimated similarly.

Vertex terms. We can apply techniques similar to those of the proof in [14, Lemma 6.36]. We have

$$
\rho_i|\theta_{\mathcal{V}} l_i(\mathcal{V})|^2_{H^{1/2}(\partial\Omega^i)} = \rho_i|\theta_{\mathcal{V}}(I_H^{\partial\Omega^i} v_i)(\mathcal{V})|^2_{H^{1/2}(\partial\Omega^i)}. \tag{4.11}
$$

By (4.5) and the definition of $I_H^{\partial\Omega^i}$, we see that $(I_H^{\partial\Omega^i} v_i)(\mathcal{V})$ is nonzero only when two subregions share one or several edges with a common vertex $\mathcal{V}$.

In the definition of $I_H^{\Omega^i}$, we denote by $\mathcal{E}_{i,m}$, $m = 1, 2, 3, \ldots$, the edges in $\partial\Omega^i$ which share $\mathcal{V}$. Denote by $p_{i,m}$ the primary nodes on the edges $\mathcal{E}_{i,m}$ which are adjacent to $\mathcal{V}$.

By the definition of $I_H^{\Omega^i}$, (4.11), and $|\theta_{\mathcal{V}}|^2_{H^{1/2}(\partial\Omega^i)} \le CH_i$, we have

$$
\begin{aligned}
\rho_i|\theta_{\mathcal{V}}(I_H^{\partial\Omega^i} v_i)(\mathcal{V})|^2_{H^{1/2}(\partial\Omega^i)} &\le C\rho_i|\sum_m v_i(p_{i,m})|^2 |\theta_{\mathcal{V}}|^2_{H^{1/2}(\partial\Omega^i)} \\
&\le C\rho_i H_i \sum_m |v_i(p_{i,m})|^2. \qquad (4.12)
\end{aligned}
$$

Let us look at the first term in (4.12); the other terms can be estimated in the same way. We find that

$$
\begin{aligned}
&\rho_i H_i |v_i(p_{i,1})|^2 \\
&\quad = \rho_i H_i | \sum_{j,\ \mathcal{E}_{i,1} \subset \partial\Omega^j} \delta_{c,j}^{\dagger} (w_i(p_{i,1}) - w_j(p_{i,1}))|^2 \\
&\quad \le C \sum_{j,\ \mathcal{E}_{i,1} \subset \partial\Omega^j} \min(\rho_i, \rho_j) H_i |w_i(p_{i,1}) - w_j(p_{i,1})|^2 \\
&\quad = C \sum_{j,\ \mathcal{E}_{i,1} \subset \partial\Omega^j} \min(\rho_i, \rho_j) H_i |I_H^{\partial\Omega^i} w_i(p_{i,1}) - I_H^{\partial\Omega^j} w_j(p_{i,1})|^2 \\
&\quad \le C \sum_{j,\ \mathcal{E}_{i,1} \subset \partial\Omega^j} \min(\rho_i, \rho_j) H_i \left(|I_H^{\partial\Omega^i} w_i(p_{i,1}) - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}_{i,1}}|^2 \right. \\
&\qquad \left. + |I_H^{\partial\Omega^j} w_j(p_{i,1}) - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}_{i,1}}|^2 \right) \\
&\quad \le C \sum_{j,\ \mathcal{E}_{i,1} \subset \partial\Omega^j} \min(\rho_i, \rho_j) \left(H_i | \left(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}_{i,1}} \right) (p_{i,1})|^2 \right. \\
&\qquad \left. + H_i | \left(I_H^{\partial\Omega^j} w_j - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}i,1} \right) (p_{i,1})|^2 \right) \\
&\quad \le C \sum_{j,\ \mathcal{E}i,1 \subset \partial\Omega^j} \min(\rho_i, \rho_j) \left(\| I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}_{i,1}} \|_{L^2(\mathcal{E}_{i,1})}^2 \right. \\
&\qquad \left. + \| I_H^{\partial\Omega^j} w_j - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}_{i,1}} \|_{L^2(\mathcal{E}_{i,1})}^2 \right) \\
&\quad \le C \sum_{j,\ \mathcal{E}i,1 \subset \partial\Omega^j} \left(1 + \log \frac{\hat{H}}{H} \right) \left(\rho_i |I_H^{\partial\Omega^i} w_i|_{H^{1/2}(\partial\Omega^i)}^2 + \rho_j |I_H^{\partial\Omega^j} w_j|_{H^{1/2}(\partial\Omega^i)}^2 \right).
\end{aligned}
$$

For the third equality, we use that $p_{i,1}$ is a primary node. For the fourth inequality, we use that $(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}_{i,1}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}_{i,1}}$. We use [14, Lemmas B.5] for the sixth inequality and [14, Lemma 4.17] for the last inequality.

Combining all face, edge, and vertex estimates, we obtain

$$
(4.13) \quad \rho_i |I_H^{\partial\Omega^i}(v_i)|_{H^{1/2}(\partial\Omega^i)}^2 \le C \left(1 + \log \frac{\hat{H}}{H} \right)^2 \sum_{j:\ \partial\Omega^j \cap \partial\Omega^i \neq \emptyset} \rho_j |I_H^{\partial\Omega^i}(w_j)|_{H^{1/2}(\partial\Omega^j)}^2.
$$

Using (4.13), Lemma 4.5, and (4.7), we obtain

$$
\begin{aligned}
(T^{(i)} v_i, v_i) = |v_i|_{T^{(i)}}^2 &\le C \frac{1}{(1 + \log \frac{H}{h})} \rho_i |I_H^{\partial\Omega^i}(v_i)|_{H^{1/2}(\partial\Omega^i)}^2 \\
&\le C \frac{\left(1 + \log \frac{\hat{H}}{H}\right)^2}{(1 + \log \frac{H}{h})} \sum_{j:\ \partial\Omega^j \cap \partial\Omega^i \neq \emptyset} \rho_j |I_H^{\partial\Omega^i}(w_j)|_{H^{1/2}(\partial\Omega^j)}^2 \\
&\le \frac{C}{c} \left(1 + \log \frac{\hat{H}}{H} \right)^2 \sum_{j:\ \partial\Omega^j \cap \partial\Omega^i \neq \emptyset} (T^{(j)} w_j, w_j) \\
&= \frac{C}{c} \left(1 + \log \frac{\hat{H}}{H} \right)^2 \sum_{j:\ \partial\Omega^j \cap \partial\Omega^i \neq \emptyset} |w_j|_{T^{(j)}}^2. \qquad \square
\end{aligned}
$$

LEMMA 4.7. *Given any* $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$, *let* $\Psi = \Phi^T \widetilde{R}_{D,\Gamma}\mathbf{u}_\Gamma$. *We have,*

$$\Psi^T S_\Pi^{-1}\Psi \le \Psi^T \widetilde{S}_\Pi^{-1}\Psi \le C\left(1+\log\frac{\hat{H}}{H}\right)^2 \Psi^T S_\Pi^{-1}\Psi.$$

Proof. Using (3.2), (3.5), and (3.6), we have

$$\begin{aligned}
\Psi^T S_\Pi^{-1}\Psi &= \sum_{i=1}^{N_c} \Psi_{I_c}^{(i)^T}\mathbf{y}_{I_c}^{(i)} + \Psi_{\Gamma_c}^T\mathbf{y}_{\Gamma_c}\\
&= \sum_{i=1}^{N_c} \Psi_{I_c}^{(i)^T}\left(S_{\Pi_{I_cI_c}}^{(i)^{-1}}(\Psi_{I_c}^{(i)} - S_{\Pi_{I_c\Gamma_c}}^{(i)}R_{\Gamma_c}^{(i)}\mathbf{y}_{\Gamma_c})\right) + \left(\mathbf{h}_{\Gamma_c} + \sum_{i=1}^{N_c} R_{\Gamma_c}^{(i)^T} S_{\Pi_{\Gamma_cI_c}}^{(i)} S_{\Pi_{I_cI_c}}^{(i)^{-1}}\Psi_{I_c}^{(i)}\right)^T \mathbf{y}_{\Gamma_c}\\
&= \sum_{i=1}^{N_c} \Psi_{I_c}^{(i)^T} S_{\Pi_{I_cI_c}}^{(i)^{-1}}\Psi_{I_c}^{(i)} + \mathbf{h}_{\Gamma_c}^T\mathbf{y}_{\Gamma_c} = \sum_{i=1}^{N_c} \Psi_{I_c}^{(i)^T} S_{\Pi_{I_cI_c}}^{(i)^{-1}}\Psi_{I_c}^{(i)} + \mathbf{h}_{\Gamma_c}^T\left(\widetilde{R}_{\Gamma_c}^T\widetilde{T}\widetilde{R}_{\Gamma_c}\right)^{-1}\mathbf{h}_{\Gamma_c}.
\end{aligned}$$

Using (3.8), (3.5), and (3.7), we also have

$$\begin{aligned}
\Psi^T \widetilde{S}_\Pi^{-1}\Psi &= \sum_{i=1}^{N_c} \Psi_{I_c}^{(i)^T}\widetilde{\mathbf{y}}_{I_c^{(i)}} + \Psi_{\Gamma_c}^T\widetilde{\mathbf{y}}_{\Gamma_c}\\
&= \sum_{i=1}^{N_c} \Psi_{I_c}^{(i)^T}\left(S_{\Pi_{I_cI_c}}^{(i)^{-1}}(\Psi_{I_c}^{(i)} - S_{\Pi_{I_c\Gamma_c}}^{(i)}R_{\Gamma_c}^{(i)}\widetilde{\mathbf{y}}_{\Gamma_c})\right) + \left(\mathbf{h}_{\Gamma_c} + \sum_{i=1}^{N_c} R_{\Gamma_c}^{(i)^T} S_{\Pi_{\Gamma_cI_c}}^{(i)} S_{\Pi_{I_cI_c}}^{(i)^{-1}}\Psi_{I_c}^{(i)}\right)^T \widetilde{\mathbf{y}}_{\Gamma_c}\\
&= \sum_{i=1}^{N_c} \Psi_{I_c}^{(i)^T} S_{\Pi_{I_cI_c}}^{(i)^{-1}}\Psi_{I_c}^{(i)} + \mathbf{h}_{\Gamma_c}^T\widetilde{\mathbf{y}}_{\Gamma_c} = \sum_{i=1}^{N_c} \Psi_{I_c}^{(i)^T} S_{\Pi_{I_cI_c}}^{(i)^{-1}}\Psi_{I_c}^{(i)} + \mathbf{h}_{\Gamma_c}^T\left(\widetilde{R}_{D_c,\Gamma_c}^T\widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c}\right)\mathbf{h}_{\Gamma_c}.
\end{aligned}$$

We need only compare $\mathbf{h}_{\Gamma_c}^T(\widetilde{R}_{\Gamma_c}^T\widetilde{T}\widetilde{R}_{\Gamma_c})^{-1}\mathbf{h}_{\Gamma_c}$ and $\mathbf{h}_{\Gamma_c}^T(\widetilde{R}_{D_c,\Gamma_c}^T\widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c})\mathbf{h}_{\Gamma_c}$ for any $\mathbf{h}_{\Gamma_c} \in \widehat{\mathbf{F}}_{\Gamma_c}$. We follow the proofs of [8, Theorem 1].

Let

$$\mathbf{w}_{\Gamma_c} = \left(\widetilde{R}_{\Gamma_c}^T\widetilde{T}\widetilde{R}_{\Gamma_c}\right)^{-1}\mathbf{h}_{\Gamma_c} \in \widehat{\mathbf{W}}_{c,\Gamma_c} \text{ and } \mathbf{v}_{\Gamma_c} = \widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c}\mathbf{h}_{\Gamma_c} \in \widetilde{\mathbf{W}}_{c,\Gamma_c}. \tag{4.14}$$

Noting the fact that $\widetilde{R}_{\Gamma_c}^T\widetilde{R}_{D_c,\Gamma_c} = \widetilde{R}_{D_c,\Gamma_c}^T\widetilde{R}_{\Gamma_c} = I$ and using (4.14), we have

$$\begin{aligned}
\mathbf{h}_{\Gamma_c}^T\left(\widetilde{R}_{\Gamma_c}^T\widetilde{T}\widetilde{R}_{\Gamma_c}\right)^{-1}\mathbf{h}_{\Gamma_c} \;&=\; \mathbf{h}_{\Gamma_c}^T\mathbf{w}_{\Gamma_c} \;=\; \mathbf{h}_{\Gamma_c}^T\widetilde{R}_{D_c,\Gamma_c}^T\widetilde{R}_{\Gamma_c}\mathbf{w}_{\Gamma_c}\\
&= \mathbf{h}_{\Gamma_c}^T\widetilde{R}_{D_c,\Gamma_c}^T\widetilde{T}^{-1}\widetilde{T}\widetilde{R}_{\Gamma_c}\mathbf{w}_{\Gamma_c} \;=\; \left(\widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c}\mathbf{h}_{\Gamma_c}\right)^T\widetilde{T}\widetilde{R}_{\Gamma_c}\mathbf{w}_{\Gamma_c}\\
&= \mathbf{v}_{\Gamma_c}^T\widetilde{T}\widetilde{R}_{\Gamma_c}\mathbf{w}_{\Gamma_c} \;=\; \langle\mathbf{v}_{\Gamma_c}, \widetilde{R}_{\Gamma_c}\mathbf{w}_{\Gamma_c}\rangle_{\widetilde{T}}\\
&\le \langle\mathbf{v}_{\Gamma_c}, \mathbf{v}_{\Gamma_c}\rangle_{\widetilde{T}}^{1/2}\langle\widetilde{R}_{\Gamma_c}\mathbf{w}_{\Gamma_c}, \widetilde{R}_{\Gamma_c}\mathbf{w}_{\Gamma_c}\rangle_{\widetilde{T}}^{1/2}\\
&= \left(\mathbf{h}_{\Gamma_c}^T\left(\widetilde{R}_{D_c,\Gamma_c}^T\widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c}\right)\mathbf{h}_{\Gamma_c}\right)^{1/2}\left(\mathbf{h}_{\Gamma_c}^T\left(\widetilde{R}_{\Gamma_c}^T\widetilde{T}\widetilde{R}_{\Gamma_c}\right)^{-1}\mathbf{h}_{\Gamma_c}\right)^{1/2}.
\end{aligned}$$

We obtain

$$\mathbf{h}_{\Gamma_c}^T\left(\widetilde{R}_{\Gamma_c}^T\widetilde{T}\widetilde{R}_{\Gamma_c}\right)^{-1}\mathbf{h}_{\Gamma_c} \le \mathbf{h}_{\Gamma_c}^T\left(\widetilde{R}_{D_c,\Gamma_c}^T\widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c}\right)\mathbf{h}_{\Gamma_c}.$$

On the other hand,

$$\begin{aligned}
&\mathbf{h}_{\Gamma_c}^T \left(\widetilde{R}_{D_c,\Gamma_c}^T \widetilde{T}^{-1} \widetilde{R}_{D_c,\Gamma_c}\right) \mathbf{h}_{\Gamma_c} = \mathbf{w}_{\Gamma_c}^T \left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right) \left(\widetilde{R}_{D_c,\Gamma_c}^T \widetilde{T}^{-1} \widetilde{R}_{D_c,\Gamma_c}\right) \mathbf{h}_{\Gamma_c} \\
&= \left\langle \mathbf{w}_{\Gamma_c}, \widetilde{R}_{D_c,\Gamma_c}^T \left(\widetilde{T}^{-1} \widetilde{R}_{D_c,\Gamma_c} \mathbf{h}_{\Gamma_c}\right)\right\rangle_{\left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)} = \left\langle \mathbf{w}_{\Gamma_c}, \widetilde{R}_{D_c,\Gamma_c}^T \mathbf{v}_{\Gamma_c}\right\rangle_{\left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)} \\
&\le \langle \mathbf{w}_{\Gamma_c}, \mathbf{w}_{\Gamma_c}\rangle^{1/2}_{\left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)} \left\langle \widetilde{R}_{D_c,\Gamma_c}^T \mathbf{v}_{\Gamma_c}, \widetilde{R}_{D_c,\Gamma_c}^T \mathbf{v}_{\Gamma_c}\right\rangle^{1/2}_{\left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)} \\
&= \left(\mathbf{h}_{\Gamma_c}^T \left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)^{-1} \mathbf{h}_{\Gamma_c}\right)^{1/2} \left\langle \widetilde{R}_{\Gamma_c} \widetilde{R}_{D_c,\Gamma_c}^T \mathbf{v}_{\Gamma_c}, \widetilde{R}_{\Gamma_c} \widetilde{R}_{D_c,\Gamma_c}^T \mathbf{v}_{\Gamma_c}\right\rangle^{1/2}_{\widetilde{T}} \\
&= \left(\mathbf{h}_{\Gamma_c}^T \left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)^{-1} \mathbf{h}_{\Gamma_c}\right)^{1/2} |E_{D_c} \mathbf{v}_{\Gamma_c}|_{\widetilde{T}} \\
&\le C \left(1 + \log \frac{\hat{H}}{H}\right) \left(\mathbf{h}_{\Gamma_c}^T \left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)^{-1} \mathbf{h}_{\Gamma_c}\right)^{1/2} |\mathbf{v}_{\Gamma_c}|_{\widetilde{T}} \\
&= C \left(1 + \log \frac{\hat{H}}{H}\right) \left(\mathbf{h}_{\Gamma_c}^T \left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)^{-1} \mathbf{h}_{\Gamma_c}\right)^{1/2} \left(\mathbf{h}_{\Gamma_c}^T \left(\widetilde{R}_{D_c,\Gamma_c}^T \widetilde{T}^{-1} \widetilde{R}_{D_c,\Gamma_c}\right) \mathbf{h}_{\Gamma_c}\right)^{1/2},
\end{aligned}$$

where we use Lemma 4.6 for the penultimate inequality.

We finally obtain

$$\mathbf{h}_{\Gamma_c}^T \left(\widetilde{R}_{D_c,\Gamma_c}^T \widetilde{T}^{-1} \widetilde{R}_{D_c,\Gamma_c}\right) \mathbf{h}_{\Gamma_c} \le C \left(1 + \log \frac{\hat{H}}{H}\right)^2 \left(\mathbf{h}_{\Gamma_c}^T \left(\widetilde{R}_{\Gamma_c}^T \widetilde{T} \widetilde{R}_{\Gamma_c}\right)^{-1} \mathbf{h}_{\Gamma_c}\right). \qquad \square$$

5. Condition number estimate for the new preconditioner. In order to estimate the condition number for the system with the new preconditioner $\widetilde{M}^{-1}$, we compare it to the system with the preconditioner M^{-1}.

LEMMA 5.1. *Given any* $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T \widetilde{M}^{-1} \mathbf{u}_\Gamma \le C \left(1 + \log \frac{\hat{H}}{H}\right)^2 \mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma. \tag{5.1}$$

Proof. We have, for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\begin{aligned}
&\mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma \\
&= \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \sum_{i=1}^N \left(\mathbf{0}\ R_\Delta^{(i)^T}\right) \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} R_{\Gamma\Delta} \right\} \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma \\
&+ \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \Phi S_\Pi^{-1} \Phi^T \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{u}_\Gamma^T \widetilde{M}^{-1} \mathbf{u}_\Gamma \\
&= \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \sum_{i=1}^N \left(\mathbf{0}\ R_\Delta^{(i)^T}\right) \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} R_{\Gamma\Delta} \right\} \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma \\
&+ \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \Phi \widetilde{S}_\Pi^{-1} \Phi^T \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma.
\end{aligned}$$

We obtain our result by using Lemma 4.7. ∎

THEOREM 5.2. *The condition number for the system with the three-level preconditioner* $\widetilde{M}^{-1}$ *is bounded by* $C(1+\log\frac{\hat{H}}{H})^2(1+\log\frac{H}{h})^2$.

Proof. Combining the condition number bound, given in (2.7) for the two-level BDDC method, and Lemma 5.1, we find that the condition number for the three-level method is bounded by $C(1+\log\frac{\hat{H}}{H})^2(1+\log\frac{H}{h})^2$. ∎

6. Using Chebyshev iterations. Another approach to the three-level BDDC methods is to use an iterative method with a preconditioner to solve (3.6). Here, we consider a Chebyshev method with a fixed number of iterations and use $\widetilde{R}^T_{D_c,\Gamma_c}\widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c}$ as a preconditioner. Denoting the eigenvalues of $(\widetilde{R}^T_{D_c,\Gamma_c}\widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c})(\widetilde{R}^T_{\Gamma_c}\widetilde{T}\widetilde{R}_{\Gamma_c})$ by λ_j, we need two input parameters l and u, which are estimates for the minimum and maximum values of λ_j, for the Chebyshev iterations. From our analysis above, we know that $l=1$ and $\max_j\lambda_j \le C(1+\log\frac{\hat{H}}{H})^2(1+\log\frac{H}{h})^2$. We can use the conjugate gradient method to obtain an estimate for the largest eigenvalue at the beginning of the computation to choose a proper u.

Let $\alpha=\frac{2}{l+u}$, $\mu=\frac{u+l}{u-l}$, and $\sigma_j=1-\alpha\lambda_j$. As for the two-dimensional case in [17, section 6], we have the following theorem. No new ideas are required.

THEOREM 6.1. *The condition number using the three-level preconditioner* $\widehat{M}^{-1}$ *with* k *Chebyshev iterations is bounded by* $C\frac{C_2(k)}{C_1(k)}(1+\log\frac{H}{h})^2$, *where*

$$C_1(k)=\min_j\left(1-\frac{\cosh(k\cosh^{-1}(\mu\sigma_j))}{\cosh(k\cosh^{-1}(\mu))}\right),$$

$$C_2(k)=\max_j\left(1-\frac{\cosh(k\cosh^{-1}(\mu\sigma_j))}{\cosh(k\cosh^{-1}(\mu))}\right),$$

and $\frac{C_2(k)}{C_1(k)}\to 1$ *as* $k\to\infty$.

7. Numerical experiments. We have applied our two three-level BDDC algorithms to the model problem (2.1), where $\Omega=[0,1]^3$. We decompose the unit cube into $\widehat{N}\times\widehat{N}\times\widehat{N}$ subregions with the side-length $\hat{H}=1/\widehat{N}$ and each subregion into $N\times N\times N$ subdomains with the side-length $H=\hat{H}/N$. Equation (2.1) is discretized, in each subdomain, by conforming piecewise trilinear elements with an element diameter h. The preconditioned conjugate gradient iteration is stopped when the norm of the residual has been reduced by a factor of 10^{-6}.

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of experiments, we use the first preconditioner $\widetilde{M}^{-1}$. We take the coefficient $\rho\equiv 1$ in case 1. In case 2, ρ is constant in one direction with a checkerboard pattern in the cross sections, where we take $\rho=1$ or $\rho=100$. The coefficients in both cases satisfy [14, Assumption 6.27.2]; i.e., for all pairs of subdomains which have a vertex but not an edge in common, there exists an acceptable edge path (see [14, Definition 6.26]) between these two subdomains. Table 1 gives the iteration counts and condition number estimates with a change of the number of subregions. We find that the condition numbers are independent of the number of subregions. Table 2 gives results with a change of the number of subdomains and the size of the subdomain problems.

TABLE 1
Eigenvalue bounds and iteration counts with the preconditioner $\widetilde{M}^{-1}$ with a change of the number of subregions, $\frac{\hat{H}}{H} = 3$ and $\frac{H}{h} = 3$.

	Case 1		Case 2	
Num. of subregions	Iter.	Cond. #	Iter.	Cond. #
$3 \times 3 \times 3$	9	2.6603	9	2.2559
$4 \times 4 \times 4$	10	2.8701	10	2.5245
$5 \times 5 \times 5$	11	2.9668	11	2.8074
$6 \times 6 \times 6$	11	3.0190	11	2.8477

TABLE 2
Eigenvalue bounds and iteration counts with the preconditioner $\widetilde{M}^{-1}$ with a change of the number of subdomains and the size of subdomain problems with $3 \times 3 \times 3$ subregions.

	Case 1		Case 2			Case 1		Case 2	
$\frac{\hat{H}}{H}$	Iter.	Cond. #	Iter.	Cond. #	$\frac{H}{h}$	Iter.	Cond. #	Iter.	Cond. #
3	9	2.6603	9	2.2559	3	9	2.6603	9	2.2559
4	9	3.0446	10	2.5183	4	9	2.7261	10	2.3299
5	10	3.3570	11	2.7782	5	10	2.8381	10	2.4353
6	10	3.6402	11	3.0078	6	10	2.9601	11	2.5488

TABLE 3
Eigenvalue bounds and iteration counts with the preconditioner $\widehat{M}^{-1}$, $u = 2.3$, $3 \times 3 \times 3$ subregions, $\frac{\hat{H}}{H} = 6$, and $\frac{H}{h} = 3$.

k	Iter.	$C_1(k)$	$\lambda_{\min}$	$\lambda_{\max}$	Cond. #
1	13	0.6061	0.6167	2.3309	3.7797
2	9	0.9159	0.9255	1.8968	2.0496
3	8	0.9827	1.0000	1.8835	1.8836
4	8	0.9964	1.0016	1.8854	1.8825
5	8	0.9993	1.0009	1.8797	1.8780

TABLE 4
Eigenvalue bounds and iteration counts with the preconditioner $\widehat{M}^{-1}$, $u = 3$, $3\times3\times3$ subregions, $\frac{\hat{H}}{H} = 6$, and $\frac{H}{h} = 3$.

k	Iter.	$C_1(k)$	$\lambda_{\min}$	$\lambda_{\max}$	Cond. #
1	15	0.5000	0.5093	2.0150	3.9562
2	10	0.8571	0.8678	1.9744	2.2753
3	8	0.9615	0.9900	1.8821	1.9012
4	8	0.9897	1.0015	1.8955	1.8927
5	8	0.9972	1.0020	1.8903	1.8866

In the second set of experiments, we use the second preconditioner $\widehat{M}^{-1}$ and take the coefficient $\rho \equiv 1$. We use the preconditioned conjugate gradient (PCG) to estimate the largest eigenvalue of $(\widetilde{R}^T_{D_c,\Gamma_c} \widetilde{T}^{-1} \widetilde{R}_{D_c,\Gamma_c})(\widetilde{R}^T_{\Gamma_c} \widetilde{T} \widetilde{R}_{\Gamma_c})$, which is approximately 2.3249. For $18 \times 18 \times 18$ subdomains and $\frac{H}{h} = 3$, we have a condition number estimate of 1.8767 for the two-level preconditioned BDDC operator. We select different values of u, the upper bound eigenvalue estimate of the preconditioned system, and of k to see how the condition number changes. We take $u = 2.3$ and $u = 3$ in Tables 3 and 4, respectively. We also evaluate $C_1(k)$ for $k = 1, 2, 3, 4, 5$. From these two tables, we find that the smallest eigenvalue is bounded from below by $C_1(k)$ and the condition number estimate becomes closer to 1.8767, the value for the two-level case, as k increases. We also see that if we can get a more precise estimate for the largest eigenvalue of $(\widetilde{R}^T_{D_c,\Gamma_c}\widetilde{T}^{-1}\widetilde{R}_{D_c,\Gamma_c})(\widetilde{R}^T_{\Gamma_c}\widetilde{T}\widetilde{R}_{\Gamma_c})$, we need fewer Chebyshev iterations to get a condition number close to that of the two-level case. However, the iteration count is not very sensitive to the choice of u.

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