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ON THE TYPE OF WIENER–HOPF C^*-ALGEBRAS

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ABSTRACT. In this paper, we show that if the positive cone P of a locally compact group G does not satisfy a regularity condition then the corresponding Wiener–Hopf C^*-algebra $\mathscr{W}(P)$ is not of type I while the converse does not hold, and that if $C^*(G)$ is not of type I then neither is $\mathscr{W}(P)$. Thus a conjecture and a question, both proposed by P. Muhly and J. Renault in their important systematic treatment of general Wiener–Hopf C^*-algebras using groupoid C^*-algebras, are settled.

INTRODUCTION

In [M-R], Muhly and Renault initiated a systematic study of the C^*-algebras $\mathscr{W}(P)$ of Wiener–Hopf operators on normal subsemigroups P of locally compact groups G, using a completely new approach; namely, the method of groupoid C^*-algebras [C, R1]. There they made a conjecture that if P does not satisfy a certain regularity condition (see below) then $\mathscr{W}(P)$ does not contain the algebra $\mathscr{K}$ of compact operators. In the first section of this paper, we prove that conjecture by showing that for such P, $\mathscr{W}(P)$ is a non-type-I C^*-algebra and contains no nontrivial compact operators. This result in particular implies a result of [D]. On the other hand, it is also asked in [M-R] whether the regularity condition of P implies that $\mathscr{W}(P)$ is of type I (i.e., the converse of the conjecture). In the second section of this paper, we give a negative answer to this question. In fact, we show that, whether P satisfies the regularity condition or not, $\mathscr{W}(P)$ is never of type I unless $C^*(G)$ is of type I.

1.

Let G be a second countable, locally compact group with identity e and left Haar measure λ fixed and P be a closed normal subsemigroup of G containing e such that P is the closure of its interior $\mathrm{Int}(P)$ (and hence of positive measure). We also assume that P generates G and $\{e\} = P \cap P^{-1}$. By definition,

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the Wiener–Hopf C^*-algebra $\mathscr{W}(P)$ (of the pair (G, P)) is the C^*-algebra generated by the Wiener–Hopf operators $\mathscr{W}(f)$ on $L^2(P, \lambda)$, $f \in C_c(G)$, where

$$\mathscr{W}(f)\xi(t) := \int_G f(s)\xi(ts)\chi_P(ts)\, d\lambda(s)$$

for $\xi \in L^2(P, \lambda)$. Let $\mathscr{A}$ be the (commutative) C^*-subalgebra of $C_b(G)$ generated by $\chi_P * f$ with $f \in L^1(G)$, where

$$\chi_P * f(t) = \int_G \chi_P(ts) f(s^{-1})\, d\lambda(s),$$

and let Y be the maximal ideal space of $\mathscr{A}$ (i.e., $\mathscr{A} = C_0(Y)$). It is easy to see that $\mathscr{A}$ is invariant under the G-action by (right) translation and hence we have a corresponding G-action on Y. Clearly G is imbedded in Y through evaluation and so we may define X to be the closure of P in Y. By [M-R, Lemma 3.3], X is compact. Now we define $\mathfrak{G}$ to be the transformation group groupoid $Y \times G$ reduced to X with the Haar system inherited from that of $Y \times G$ defined by $\{\delta_y \times \lambda\}_{y \in Y}$ [M-R]. Then one of the main results of [M-R] says that the (reduced) groupoid C^*-algebra $C^*(\mathfrak{G})$ is isomorphic to the Wiener–Hopf C^*-algebra $\mathscr{W}(P)$. More precisely, they proved that the induced representation $\operatorname{Ind}(\delta_e)$ of $C^*(\mathfrak{G})$ on $L^2(\mathfrak{G}, \lambda^e)$ is faithful and its image coincides with $\mathscr{W}(P)$ if we identify $L^2(\mathfrak{G}, \lambda^e)$ and $L^2(P, \lambda)$ suitably [M-R]. Recall that under this identification, $\operatorname{Ind}(\delta_e)$ becomes the representation π on $L^2(P, \lambda)$ defined by

$$\pi(f)\xi(t) = \int_G f(t, s)\xi(ts)\chi_P(ts)\, d\lambda(s)$$

for f in $C_c(\mathfrak{G})$ and ξ in $L^2(P, \lambda)$.

Recall that X is called a regular compactification of P if P is open in X and the embedding of P in X is a homeomorphism from P to the image. If X is not a regular compactification of P, it is conjectured in [M-R, 3.7.3] that $\mathscr{W}(P)$ does not contain the algebra $\mathscr{K}$ of compact operators. One of our main results is the proof of this conjecture.

For the convenience of discussion, we list three conditions on P (which will be shown to be equivalent in Lemma 2).

(1) X is not a regular compactification of P.

(2) There is a sequence $p_n \in P$ diverging to ∞ in G but converging to some $p \in P$ in X.

(3) For any $p \in \operatorname{Int}(P)$, there is a sequence $p_n \in \operatorname{Int}(P)$ diverging to ∞ in G but converging to p in X.

Lemma 1. *If $p_n \in G$ converges to a $p \in \operatorname{Int}(P)$ in Y, then $p_n \in \operatorname{Int}(P)$ for n sufficiently large.*

Proof. Now clearly $p_n p^{-1}$ converges to e in Y since Y is a G-space. Given any neighborhood V of e in G and any nonnegative $f \in C_c(V)$ with $f(e) > 0$, it is easy to see that $\operatorname{supp}(\chi_P * f) \subseteq PV$ and $\chi_P * f(e) > 0$ (since P is the closure of $\operatorname{Int}(P)$). Now since $\chi *_P f(p_n p^{-1})$ converges to $\chi *_P f(e)$, we have $p_n p^{-1} \in PV$ for n large. Choosing V sufficiently small, we have $Vp \subseteq \operatorname{Int}(P)$ since $p \in \operatorname{Int}(P)$, and hence $p_n \in PVp \subseteq P\operatorname{Int}(P) \subseteq \operatorname{Int}(P)$. Q.E.D.

Lemma 2. *Conditions* (1), (2), *and* (3) *are all equivalent.*

Proof. Condition (1) implies that either (i) there is a sequence $\tilde{q}_n$ in $X \backslash P$ converging to some $q \in P$ in X (or equivalently in Y), or (ii) the embedding of P in X is not a homeomorphism from P to its image in X. First we assume (i), and then by considering a sequence K_n of compact subsets of G such that $G = \bigcup_n K_n$ and $K_n \subseteq \operatorname{Int}(K_{n+1})$, we can easily get a sequence $q_n \in P \backslash K_n$ (and close to $\tilde{q}_n$) converging to q in X such that q_n diverges in G to ∞. Next we assume (ii), and then there is a sequence $p_n \in P$ converging to some $p \in P$ in X but not in G. Since, by elementary facts about compact Hausdorff spaces, the embedding restricted to any compact subset of P is a homeomorphism, it is easy to see that p_n has to diverge to ∞ in G (note that P is closed in G). So, in both cases, condition (2) holds and hence (1) implies (2).

Now assuming condition (2), we get, say, a sequence $q_n \in P$ converging to some $q \in P$ in X such that q_n diverges in G to ∞. Since G acts on Y by homeomorphisms, we have $q_n q^{-1} \in G$ converges to e in Y. Now for any $p \in \operatorname{Int}(P)$, $q_n q^{-1} p \in G$ converges to p in Y and diverges to ∞ in G. So all we need to show is that $q_n q^{-1} p$ is in $\operatorname{Int}(P)$ for n large, but this comes from Lemma 1. Thus (2) implies (3).

It is obvious that condition (3) implies (1) since $\operatorname{Int}(P)$ is not empty. Q.E.D.

Theorem 1. *With the notations as above, if X is not a regular compactification of P, then $\mathscr{W}(P)$ is not of type* I *and contains no nontrivial compact operators.*

Proof. First let us point out that the induced representation $\operatorname{Ind}(\delta_e)$ is in fact irreducible. This can be seen through the results of [R2]. In fact, it is easy to check that $\operatorname{Ind}(\delta_e)$ can be disintegrated into the integral of the representation $(m, \mathscr{H})$ of the dynamical system $(\mathfrak{G}, \mathfrak{G}, \mathbf{C})$ (c.f. [R2, Theorem 4.1(i)]), where m is the transverse measure of $\mathfrak{G}$ determined by the measure $\mu = \lambda|_P$ on $\mathfrak{G}^{(0)} = X$ (c.f. [R2, Proposition 2.2]) and $\mathscr{H}$ is the trivial one-dimensional Hilbert bundle over X. Clearly $(m, \mathscr{H})$ is irreducible and hence, by [R2, Theorem 4.1(ii)], $\operatorname{Ind}(\delta_e)$ is irreducible. Thus by [Ar, Corollary 2 of Theorem 1.4.2], the image of $\operatorname{Ind}(\delta_e)$ either contains $\mathscr{K}$ or intersects with $\mathscr{K}$ on $\{0\}$. We shall prove the latter to be true.

For any $(p_0, g_0) \in \mathfrak{G} \subseteq X \times G$ such that $p_0 \in \operatorname{Int}(P)$ and $p_0 g_0 \in \operatorname{Int}(P)$ (note that such points are dense in $\mathfrak{G}$), we can find a sequence $p_n \in \operatorname{Int}(P)$ converging to $p_0 \in \operatorname{Int}(P)$ in X, by Lemma 2. We claim that there is a neighborhood V of e in G such that $p_n V \subseteq \operatorname{Int}(P) \subseteq X$ for all $n \geq 0$, since otherwise, there are a subsequence $p_{n(k)}$ of p_n and a sequence $g_{n(k)} \in G$ converging to e in G such that $p_{n(k)} g_{n(k)} \notin \operatorname{Int}(P)$ for all k, but clearly $p_{n(k)} g_{n(k)}$ converges to $p_0 e = p_0 \in \operatorname{Int}(P)$ in X which contradicts Lemma 1. Let U be a relative compact neighborhood of e in G with the closure $\overline{U} \subseteq V$ (so $0 < \lambda(U) < \infty$). Note that since p_n diverges to ∞ in G, by taking a subsequence of p_n if necessary, we may assume that $p_n U$'s are all disjoint.

Now for any $f \in C_c(\mathfrak{G})$, we have $f(p_n p_0^{-1} t, t^{-1} s)$ converging uniformly to $f(t, t^{-1} s)$ on $(t, s) \in (p_0 U) \times (p_0 U)$, since the map sending (p, t, s) to $(p p_0^{-1} t, t^{-1} s)$ is a continuous map from the compact $\{p_n | n \geq 0\} \times (p_0 \overline{U}) \times (p_0 \overline{U}) \subseteq X \times G \times G$ to $\mathfrak{G}$.

Let $\pi_n(f)$ be the operator $\pi(f)$ compressed to $L^2(p_n U, \lambda)$ for all $n \geq 0$. Then

$$\pi_n(f)\xi(t) = \int f_n(t, t^{-1}s)\xi(s)\, d\lambda(s)$$

for $\xi \in L^2(p_n U, \lambda)$ and $t \in p_n U$ (it is understood that ξ is extended by 0 outside $p_n U$). Let τ_n be the unitary operator from $L^2(p_n U, \lambda)$ to $L^2(p_0 U, \lambda)$ defined by

$$(\tau_n \xi)(t) = \xi(p_n p_0^{-1} t)$$

for $t \in p_0 U$. Then the operator $\tau_n \pi_n(f) \tau_n^{-1}$ on $L^2(p_0 U, \lambda)$ satisfies

$$(\tau_n \pi_n(f) \tau_n^{-1} \xi)(t) = \int f(p_n p_0^{-1} t, t^{-1} s)\xi(s)\, d\lambda(s)$$

for $\xi \in L^2(p_0 U, \lambda)$ and $t \in p_0 U$. By the above uniform convergence, it is easy to see that $\|\tau_n \pi_n(f) \tau_n^{-1} - \pi_0(f)\|$ converges to 0 and hence $\|\pi_n(f)\|$ converges to $\|\pi_0(f)\|$ as n goes to infinity. Since the $p_n U$'s are all disjoint and hence the $L^2(p_n U, \lambda)$'s are orthogonal to one another, we have $\|\pi(f) - T\| \geq \|\pi_0(f)\|$ for any compact operator T on $L^2(P, \lambda)$.

Now let $W = p_0 U$ be as above and $\xi = \lambda(U)^{-1/2} \chi_W$ be a unit vector in $L^2(p_0 U, \lambda) \subseteq L^2(P, \lambda)$. If the rank one projection $T := \langle \cdot, \xi \rangle \xi$ is in $\mathscr{W}(P)$, then there is a sequence $\pi(f_m)$ with $f_m \in C_c(\mathfrak{G})$ such that $\lim \|\pi(f_m) - T\| = 0$ and hence $\lim \|\pi_0(f_m)\| = 1$. But then by the above result $\|\pi(f_m) - T\| \geq \|\pi_0(f_m)\| \geq 1/2$ for all large m, a contradiction. So we have proved that $T \notin \mathscr{W}(P)$, and hence $\mathscr{W}(P) \cap \mathscr{K} = \{0\}$ and $\mathscr{W}(P)$ is not of type I since $\pi = \mathrm{Ind}(\delta_e)$ is irreducible. Q.E.D.

A consequence of Theorem 1 is the following result of [D].

Corollary. *For $P = \{z \in \mathbf{Z}^n | \iota_\alpha(z) \geq 0\} \subseteq G = \mathbf{Z}^n$ with $n \geq 2$, the Wiener–Hopf C^*-algebra $\mathscr{W}(P)$ is non-type-*I *and contains no nontrivial compact operators, where $\iota_\alpha(z) := \sum_{i=1}^n z_i \alpha_i$ for any fixed* **Q***-linearly independent real numbers α_i and for all $z \in \mathbf{Z}^n$.*

Proof. Since G is discrete, $\mathscr{A}$ is the (commutative) C^*-subalgebra of $l^\infty(G)$ generated by the characteristic functions χ_{P+z} with $z \in \mathbf{Z}^n$. Since α_i's are **Q**-linearly independent and $n \geq 2$, we have the closure of $\iota_\alpha(P)$ in **R** equal to $[0, \infty)$, and hence we can find a sequence $\{y(k)\}_{k \in \mathbf{N}} \subseteq P$ diverging to infinity such that $\lim \iota_\alpha(y(k)) = 0$. Now clearly $y \in P + z$ if and only if

$\iota_\alpha(y) \geq \iota_\alpha(z)$, so $0 \in P+z$ if and only if $y(k) \in P+z$ for k sufficiently large (since $\iota_\alpha(y(k)) \geq 0$ for all k). Thus $\lim \chi_{P+z}(y(k)) = \chi_{P+z}(0)$ for all z in $\mathbf{Z}^n$, and hence $y(k)$ converges to 0 in Y as characters of $\mathscr{A}$. Now by Lemma 2, X is not a regular compactification of P and so by Theorem 1, we get the statement. Q.E.D.

2.

In [M-R, 3.7.3], it is asked whether X being a regular compactification of P implies that $\mathscr{W}(P)$ is of type I. We shall give a negative answer to this question. The point is that unless (the reduced) $C^*(G)$ itself is of type I, the algebra $\mathscr{W}(P)$ cannot be of type I.

Theorem 2. *If $C^*(G)$ is not of type* I, *then $\mathscr{W}(P)$ is not of type* I.

Proof. We claim that for any compact $K \subseteq G$, there is a $g \in G$ such that $Pg \subseteq \bigcap_{k\in K} Pk$.

First we check this for finite K. Since P is assumed to be normal and generate G, we have $G = P^{-1}P$. So, for any k, h in G, $Pk \cap Ph$ is not empty and hence contains Pg for some $g \in G$ (in fact, for all $g \in Pk \cap Ph$). Thus, by iteration, any intersection of finitely many Pk's, $k \in G$, is not empty and contains some Pg.

Now if we can show that for any $h \in G$ there is a (compact) neighborhood U with the property of the claim (i.e., $\bigcap_{u\in U} Pu$ contains some Pk), then we get the claim, since then we can cover K by finitely many such U's and then, by the claim for the finite case, the intersection of the corresponding finitely many Pk's will contain some Pg as desired. In fact, for any $h \in G$, we can find $k \in G$ such that $kh^{-1} \in \operatorname{Int}(P)$ and so $kU^{-1} \subseteq \operatorname{Int}(P)$ for a neighborhood U of h. Thus $k \in Pu$ for all $u \in U$ and hence $Pk \subseteq \bigcap_{u\in U} Pu$. So the claim is proved.

Now let $e = g_1, g_2, g_3, \ldots$ be a sequence dense in G and with $C_n := \bigcap_{i=1}^n Pg_i \subseteq P$ nonempty for all n. Note that for any $g \in G$, since Pg contains the open set $\operatorname{Int}(P)g$, we have Pg containing some g_i and hence Pg_i, and so $C_n \subseteq Pg$ for all $n \geq i$. Thus, using the claim, we get that for any compact $K \subseteq G$, there is an i such that $C_n \subseteq \bigcap_{k\in K} Pk$ for all $n \geq i$.

Let $p_n \in C_n$; then by the above result, it is easy to check that

$$\lim \chi_P * f(p_n) = \int_G \tilde{f}(s)\, d\lambda(s)$$

and similarly for any $g \in G$

$$\lim \chi_P * f(p_n g) = \int_G (\tau_g \tilde{f})(s)\, d\lambda(s) = \int_G \tilde{f}(s)\, d\lambda(s)$$

for all $f \in C_c(G)$, where $\tilde{f}(s) = f(s^{-1})$ and $\tau_g f(s) := f(g^{-1}s)$. So p_n converges to a G-invariant limit character q in X.

Now by the general theory of groupoid C^*-algebra developed in [R1], the invariant closed subspace $\{q\}$ gives rise to a closed subgroupoid $\mathfrak{G}_q = \{q\} \times G$

of $\mathfrak{G}$ whose groupoid C^*-algebra $C^*(\mathfrak{G}_q) \cong C^*(G)$ is a quotient of $C^*(\mathfrak{G})$. So $C^*(\mathfrak{G})$ cannot be of type I since $C^*(G)$ is not. Q.E.D.

Now we use Theorem 2 to show that regularity of (X, P) need not imply $\mathscr{W}(P)$ being of type I.

Consider the discrete Heisenberg group H $(= \mathbf{Z}^3$ as sets) whose group C^*-algebra has been extensively studied [An-Pas]. Recall that the group operation is defined by $(i, j, k)\cdot(a, b, c) = (a+i, b+ci+j, c+k)$ for a, b, c, i, j, k in $\mathbf{Z}$. Let

$$P = \{(a, b, c) | a, c > 0, a, b, c \in \mathbf{Z}\} \cup \{(0, 0, 0)\}.$$

Then one can check that P satisfies all the requirements (i.e., P is a normal subsemigroup generating H with $P \cap P^{-1} = \{e\}$). Since H is discrete, $\mathscr{A}$ is the commutative C^*-algebra generated by $\chi_{P\cdot z}$, $z \in \mathbf{Z}^3$, in $l^\infty(\mathbf{Z}^3)$. Clearly, $\chi_{P\cdot(0,1,0)}(0, 0, 0) = 0$ while $\chi_{P\cdot(0,1,0)}(a, b, c) = 1$ for all $(a, b, c) \in P\backslash\{(0, 0, 0)\}$. Thus $(0, 0, 0)$ is an isolated point in X and hence by Lemma 2, X is a regular compactification of P. But by Theorem 2, $\mathscr{W}(P)$ is not of type I since $C^*(H)$ is well known to be non-type-I. So this example gives a negative answer to the question of Muhly and Renault, and one may then ask, instead, whether the regularity of (X, P) and the condition that $C^*(G)$ is of type I imply that $\mathscr{W}(P)$ is also of type I. But the answer to this substitute question is still negative. In fact, the algebras $\mathscr{W}(P_{\alpha,\beta})$ (which have been studied closely in [Par]) provide such counterexamples when one of α and β is irrational, where $P_{\alpha,\beta}$ is the cone $\{(m, n) \in \mathbf{Z}^2 | -\alpha m + n \geq 0$ and $-\beta m + n \leq 0\}$ (with $0 < \alpha < \beta$) in $G = \mathbf{Z}^2$ whose $C^*(G) = C(\mathbf{T}^2)$ is of type I. It can be checked that $\mathscr{W}(P_{\alpha,\beta})$ is of type I if and only if both α and β are rational, but the corresponding compactification $X_{\alpha,\beta}$ of $P_{\alpha,\beta}$ is always regular for any real α and β (with $0 < \alpha < \beta$).

References

[An-Pas] J. Anderson and W. Paschke, *The rotation algebra*, Houston J. Math (to appear).

[Ar] W. B. Arveson, *An invitation to C^*-algebras*, Graduate Texts in Math., vol. 34, Springer-Verlag, Berlin, 1976.

[C] A. Connes, *Sur la théorie non commutative de l'intégration*, Lecture Notes in Math. vol. 725, Springer-Verlag, Berlin, 1979, pp. 19–143.

[D] R. G. Douglas, *On the C^*-algebra of a one-parameter semi-group of isometries*, Acta Math. **128** (1972), 143–151.

[M-R] P. S. Muhly and J. N. Renault, *C^*-algebras of multivariable Wiener–Hopf operators*, Trans. Amer. Math. Soc. **274** (1982), 1–44.

[Par] E. Park, *Index theory and Toeplitz algebras on certain cones in* $\mathbf{Z}^2$, preprint.

[R1] J. N. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, New York, 1980.

[R2] ——, *Représentation des produits croisés d'algèbres de groupoïdes*, J. Operator Theory **18** (1987), 67–97.

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