ISOMORPHISM OF THE TOEPLITZ $C^*$-ALGEBRAS FOR THE HARDY AND BERGMAN SPACES ON CERTAIN REINHARDT DOMAINS

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Abstract. I. Raeburn has conjectured that the Toeplitz $C^*$-algebras $\mathcal{T}(D)$ and $\mathcal{T}(\partial D)$ defined on the Bergman space $H^2(D)$ and the Hardy space $H^2(\partial D)$ of an arbitrary strongly pseudoconvex domain $D$ in $\mathbb{C}^n$ are isomorphic. Applying the groupoid $C^*$-algebra approach of Curto, Muhly, and Renault to $C^*$-algebras of Toeplitz type, we prove that this conjecture holds for (not even necessarily pseudoconvex) Reinhardt domains in $\mathbb{C}^2$ satisfying a mild boundary condition.

Introduction

In [R] it was conjectured that the Toeplitz $C^*$-algebras $\mathcal{T}(D)$ and $\mathcal{T}(\partial D)$ defined on the Bergman space $H^2(D)$ and the Hardy space $H^2(\partial D)$ are isomorphic for arbitrary (strongly) pseudoconvex complex domains $D$. The conjecture has been known to hold for the unit ball in $\mathbb{C}^n$ by [C], for the weakly pseudoconvex domain $D = \{z \in \mathbb{C}^n| \sum_i |z_i|^{2p_i} < 1\}$ with $p_i$ in $\mathbb{N}$ by [CrR], and for weakly pseudoconvex domains of finite type with smooth boundary by [S]. But for general (strongly) pseudoconvex domains $D$, it remains unsolved. Applying the groupoid $C^*$-algebra approach of Curto, Muhly, and Renault [CuM, MRe], to $C^*$-algebras of Toeplitz type, and the results of [SShU, Sh], we can prove this conjecture for $D$, a (not even necessarily pseudoconvex) Reinhardt domain in $\mathbb{C}^2$ satisfying a mild boundary condition (essentially the condition used in [Sh]). We shall adapt the arguments used for the case of Bergman spaces in [Sh] to the case of Hardy spaces, and we shall use the notations and technical lemmas in [Sh] as well.

1. Toeplitz and groupoid $C^*$-algebras

Let $D$ be a bounded Reinhardt domain in $\mathbb{C}^2$, i.e., a bounded connected open region in $\mathbb{C}^2$ containing 0 that is invariant under componentwise multiplication by elements of the two-torus $\mathbb{T}^2$. The invariance under $\mathbb{T}^2$ implies that $D$ is completely determined by $|D| := \{|z|; z \in D\} \subseteq \mathbb{R}_+^2$, where $|z| = (|z_1|, |z_2|)$ and $\mathbb{R}_+ = \{x; x \in \mathbb{R}, x \geq 0\}$. Without loss of generality,
we may assume that $D$ is contained in the unit polydisk $\Delta^2$. We define the corresponding logarithmic domain
\[ C := \{ \ln(|z|) | z \in D, z_1 \neq z_2 \neq 0 \} \subseteq \mathbb{R}_+^2 \]
with $\ln$ acting componentwise. (It is known that a complete Reinhardt domain $D$ is pseudoconvex if and only if $C$ is convex [H].) In the following, we assume that $D$ satisfies the conditions (I)--(III) in [Sh, §4], and to avoid technical difficulties in defining the Hardy space, we shall also assume that the boundary $\partial C$ is a union of smooth curves such that the total (arc-length) measure of $\partial|D|$ (and hence the total three-dimensional surface measure of $\partial D$) is positive and finite. (In particular, we assume that the functions $\phi$ and $\psi$ defined on [Sh, pp. 276, 296] are piecewise smooth and have bounded first derivatives on $K_\epsilon$ for sufficiently small $\epsilon > 0$.) So for example, the complete pseudoconvex Reinhardt domains $D$ with piecewise smooth boundary (cf. [Sh, Remark 4.2]) and the complete Reinhardt domains $D$ with piecewise analytic boundary $\partial C$ such that the measure of $\partial|D|$ is finite are included.

Let $H^2(D)$ be the Bergman space over $D$, i.e., the Hilbert subspace of $L^2(D)$ consisting of holomorphic $L^2$-functions over $D$ (with respect to volumetric Lebesgue measure), and let $H^2(\partial D)$ be the Hardy space over $D$, i.e., the closure of the space of continuous functions on $\partial D$ that can be extended continuously to holomorphic functions on $D$, in the Hilbert space $L^2(\partial D)$ (with respect to the surface measure on $\partial D$). Let $P$ and $P'$ be the orthogonal projections from $L^2(D)$ onto $H^2(D)$ and from $L^2(\partial D)$ onto $H^2(\partial D)$, respectively. The Toeplitz $C^*$-algebra $\mathcal{T}(D)$ (resp. $\mathcal{T}(\partial D)$) is defined to be the $C^*$-algebra generated by the operators $T_\phi := PM_\phi$ restricted to $H^2(D)$ (resp. $T'_\phi := P'M'_\phi$ restricted to $H^2(\partial D)$) where $\phi$ is a continuous function on the closure $\overline{D}$ of $D$ and $M_\phi$ (resp. $M'_\phi$) is the multiplication operator by $\phi$ on $L^2(D)$ (resp. on $L^2(\partial D)$). In order to study $\mathcal{T}(D)$ and $\mathcal{T}(\partial D)$, we use the framework introduced by Curto and Muhly in [CuM] to relate them to groupoid $C^*$-algebras $C^*(\mathcal{G})$ and $C^*(\mathcal{G}')$, respectively, and then we show that $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic as topological groupoids (with Haar systems) and hence $\mathcal{T}(D)$ and $\mathcal{T}(\partial D)$ are isomorphic. First, let us sketch the result of [CuM].

With respect to the canonical orthonormal basis $\{e_\nu = z^\nu/\|z^\nu\|_2\}$ of $H^2(D)$ where $\nu \in \mathbb{Z}^2_+$ and $z^\nu = (z_1^\nu)(z_2^\nu)$, the Toeplitz operator $T_{zm}, m = 1, 2$, can be viewed as a multivariable weighted shift. More precisely, $T_{zm}(e_\nu) = w_m(\nu)e_{\nu + \epsilon_m}$ where the $\epsilon_m$'s are the standard basis of $\mathbb{R}^2$ and
\[ w_m(\nu) = \|z^\nu + \epsilon_m\|_2/\|z^\nu\|_2. \]

Let $\mathcal{A}$ be the $\mathbb{Z}^2$-invariant commutative $C^*$-subalgebra of $l^\infty(\mathbb{Z}^2)$ generated by $w_1$ and $w_2$. Then $\mathcal{A} \cong C_0(Y)$, where $Y$ is the maximal ideal space of $\mathcal{A}$ and $\mathbb{Z}^2$ is embedded in $Y$ in the canonical way. In particular, each $\nu$ in $\mathbb{Z}^2$ determines a character $\chi(\nu)$ in $Y$. Clearly the $\mathbb{Z}^2$-action on $Y$ induced by the $\mathbb{Z}^2$-action on $\mathcal{A} \subseteq l^\infty(\mathbb{Z}^2)$ coincides with the usual translation on $\mathbb{Z}^2$. The closure $X$ of $\mathbb{Z}^2$ in $Y$ consists of characters that are weak * limits of $\chi(\nu)$ with $\nu$ in $\mathbb{Z}^2$. Let $\mathcal{G}$ be the reduction [MRe] of the transformation group groupoid $Y \times \mathbb{Z}^2$ to $X$ endowed with the natural Haar system $\lambda^* = \delta_x \times \lambda$ where $\lambda$ is the counting measure on $\mathbb{Z}^2$. Then the (reduced) groupoid $C^*$-algebra $C^*(\mathcal{G})$ contains $\mathcal{T}(D)$ and equals $\mathcal{T}(D)$ under certain conditions.

Since $D$ is a Reinhardt domain, it is easy to show that $\{z^\nu\}$ also form a
complete orthogonal basis of $H^2(\partial D)$. In fact, it is clear that

$$
(z^\nu, z^\mu)' = \int_{\partial\{D\}} \int_0^{2\pi} \int_0^{2\pi} \exp(i\theta \cdot (\nu - \mu))|z|^\nu|z|^\mu |1\ d\theta_1 \ d\theta_2 \ d'd'z| \\
= \int_{\partial\{D\}} 0 \ d'd'|z| = 0
$$

if $\nu \neq \mu$ in $\mathbb{Z}^2_+$, where $\theta = (\theta_1, \theta_2), \ 1 = (1, 1), \ d'd'|z|$ is the arclength measure on the boundary $\partial\{D\}$ of $\{D\}$ in $\mathbb{R}^2_+$, and $\langle , \rangle'$ is the inner product on $H^2(\partial D)$. On the other hand, given $f$ in $C(\bar{D})$ such that $f|_D$ is holomorphic, by [H] we have a power series expansion $f(z) = \sum a_{-\nu}z^\nu$ absolutely converging on every compact subset of $\bar{D}$, the pseudoconvex hull of $D$ (whose logarithmic domain is the convex hull $\bar{C}$ of $C$). Define $f_{\varepsilon}(z) = f((1-\varepsilon)z)$ for $\varepsilon > 0$ and $z \in \bar{D}$. Then clearly $f_{\varepsilon}$ converges to $f$ uniformly on $\partial D$ and $\sum a_{-\nu}((1-\varepsilon)z)^\nu$ converges absolutely and uniformly to $f_{\varepsilon}(z)$ on the closure of $D$ that contains $\bar{D}$. So $f_{\varepsilon}|_D$ is in the closed subspace spanned by $z^\nu, \nu \in \mathbb{Z}^2_+$, in $L^2(\partial D)$. Thus $H^2(\partial D)$ equals the closed span of $z^\nu, \nu \in \mathbb{Z}^2_+$, in $L^2(\partial D)$.

Now we can apply the above procedure used for $H^2(D)$ to associate a groupoid $\mathcal{G}'$ to $H^2(\partial D)$, and we get the corresponding objects $w', \mathcal{A}', Y', X'$, and $\chi'(\nu)$. First we shall try to analyze the structure of $\mathcal{G}'$. In order to do this, we need to find all possible weak* limits of $\chi'(\nu)$ with $\nu$ a sequence in $\mathbb{Z}^2_+$.

### 2. Technical lemmas

In the following, for any vectors $y$ and $\nu$ in $\mathbb{R}^2$, we shall write $y\nu := y_1\nu_1 + y_2\nu_2$.

Let $u$ be a unit vector in $\mathbb{R}^2_+$, $u^\perp = (-u_2, u_1)$ and $\bar{F} = \bar{F}_u := (\mathbb{R}u^\perp - yu) \cap \partial\bar{C}$ be the face of $\bar{C}$ determined by $u$, where $\bar{C}$ is the convex hull of $C$ and $\gamma$ is the shortest distance between $\partial C$ and $\mathbb{R}u^\perp$. For a more detailed definition of notions used in the following we refer the reader to [Sh].

From now on, we shall use $\nu$ to denote a sequence of elements in $\mathbb{Z}^2_+$ satisfying the following properties, unless otherwise specified. We write $\nu \rightarrow (r', \rho', \omega') \in \mathbb{R}^3_+$ if (1) $\lim(\nu u^\perp / \nu u) = +\infty$, (2) $\lim(\nu u^\perp / \nu u) = 0$, (3) $((2\nu - 1)u^\perp, (2\nu - 1)u)$ (and hence $((2\nu - 1)u^\perp, (2\nu - 1)u)$ for any fixed $\mu$ in $\mathbb{Z}^2_+$) belongs to slope $r'$ [Sh], (4) $\rho(r', (2\nu - 1)u^\perp, (2\nu - 1)u) = \rho'$, and (5) if $r' = r_{\pm}$ then $\omega(r', (2\nu - 1)u^\perp, (2\nu - 1)u, A_{\pm}) = \omega'$ (cf. [Sh, §4]). Note that conditions (1) and (2) imply that $\nu u^\perp / \|\nu\|$ converges to $0$, $\nu / \|\nu\|$ converges to $u$, and $\|\nu\|$ diverges to $\infty$.

Clearly, we have

$$
\|z^{\nu-1}\|_2^2 = \int_{\partial\{C\}} \int_0^{2\pi} \int_0^{2\pi} e^{(2\nu-1)x} \ d\theta_1 \ d\theta_2 \ d'x = (2\pi)^2 \int_{\partial\{C\}} e^{(2\nu-1)x} \ d'x
$$

where $d'x$ is the pull-back measure on $\partial C$ of the measure $d'd'|z|$ on $\partial\{D\}$.
through the componentwise exponential map, and hence

\[
\frac{w'_m(\nu - 1)^2}{\int_{\partial C} \exp((2(\nu + \varepsilon_m) - 1)x) d'x} = \frac{\exp(-2\gamma \varepsilon_m u) \tilde{L}'(\partial C, (2(\nu + \varepsilon_m) - 1)u^+, (2(\nu + \varepsilon_m) - 1)u, -xu)}{\tilde{L}'(\partial C, (2\nu - 1)u^+, (2\nu - 1)u, -xu)}
\]

for any \( \nu \) in \( \mathbb{Z}_2 \), where

\[
\tilde{L}'(\mathcal{L}, b, c, h) := \int_{\mathcal{L}} \exp(b(xu^+) - ch(x) + cy) d'x
\]

for any measurable function \( h \) on \( \partial C \) and any measurable subset \( \mathcal{L} \) of \( \partial C \).

Some of the technical results of [Sh] need be modified when dealing with Hardy spaces. The following three lemmas either modify some results of [Sh] or relate the quantities for Hardy spaces (e.g., \( \tilde{L}' \)) to corresponding quantities (e.g., \( \tilde{L} \)) studied in [Sh], so we may apply some results of [Sh]. Lemma 1 is related to [Sh, Lemma A.3].

**Lemma 1.** Let \( b \) and \( c \) be two sequences of real numbers with \( c \) and \( |c/b| \) diverging to \( \infty \), and let \( f \) be a nonnegative measurable function on \( \mathbb{R} \) with the measure of \( f^{-1}([0, \varepsilon]) \) positive for all \( \varepsilon > 0 \). Given a measurable function \( h \) on \( \mathcal{L} \subseteq \partial C \) with \( h(x) \geq \varepsilon > 0 \) for all \( x \) in \( \mathcal{L} \), if there are \( \delta, M > 0 \) such that \( h(x) \geq \delta|xu^+| \) for all \( x \) in \( \mathcal{L} \) with \( \sigma xu^+ > M \) where \( \sigma \) is \( \lim \text{sign}(b) \) if the limit exists (i.e., \( b \) is eventually always positive or negative) and \( \sigma xu^+ = |xu^+| \) otherwise, then

\[
\lim \frac{\tilde{L}'(\mathcal{L}, b, c, h)}{\tilde{L}(\mathcal{L}, b, c, f)} = 0
\]

for any fixed \( \varepsilon > 0 \), where \( K_\varepsilon = f^{-1}([0, \varepsilon]) \) and \( \tilde{L}(S, b, c, f) = \int_S \exp(bt - cf(t)) dt \) as defined on [Sh, p. 269].

**Proof.** In this proof, we use \( |\mathcal{L}| \) to denote the measure of subsets \( \mathcal{L} \) of \( \partial C \). Without loss of generality, we may assume that \( M > 2\varepsilon/\delta \). Clearly there is a bounded subset \( K \) of positive measure \( |K| \) in \( K_{\varepsilon/3} \). We have \( bt - cf(t) = [(b/c)t - f(t)]c > -(\varepsilon/2)c \) for all \( t \in K \), for sufficiently large \( c \) since \( K \subseteq K_{\varepsilon/3} \) is bounded and \( \lim(b/c) = 0 \), by assumption. On the other hand, for \( x \) in \( \mathcal{L} \) with \( xu^+ \in (-M, M) \), we have

\[
bxu^+ - ch(x) < c(bc^{-1}xu^+ - \varepsilon) < -(2\varepsilon/3)c,
\]

for a similar reason. Furthermore, for \( x \) in \( \mathcal{L} \setminus [-M, M] \) with \( \sigma xu^+ > M \), we have

\[
bxu^+ - ch(x) < c(bc^{-1}xu^+ - \delta|xu^+|)
\]

\[
< -c\delta|xu^+|/2 < -c\delta M/2 < -ce < -2ce/3
\]

for \( c \) sufficiently large, and for \( x \) in \( \mathcal{L} \setminus [-M, M] \) with \( \sigma xu^+ < M \) (this can happen only when \( \sigma \) is well defined), we have \( \sigma xu^+ < -M \) and hence

\[
|bxu^+ - ch(x)| \leq |bxu^+| - ce < M|b| - ce \leq -ce < -2ce/3.
\]

Thus

\[
\tilde{L}(K_\varepsilon, b, c, f) > \exp(-ce/2)|K|
\]

while

\[
\tilde{L}'(\partial C, b, c, h) < \int_{\partial C} \exp(-2ce/3) d'x = \exp(-2ce/3)|\mathcal{L}|.
\]
Now it is easy to see that
\[
\lim \frac{L'(\xi, b, c, h)}{L(K, b, c, f)} = 0
\]
since \(|\xi| < |\partial C| = \text{meas}(\partial|D|) < \infty\) by assumption and \(|K| > 0\). Q.E.D.

Recall that on [Sh, p. 300] we have \(L(s, b, c, n) := \int_0^s \exp(bt - ct^n) \frac{l(t)}{l(t/c^n)} dt\) for \(s > 0\). Now we generalize this notion to include a weight function \(l\) by defining
\[
L(s, b, c, n, l) := \int_0^s \exp(bt - ct^n) l(t) dt.
\]
Lemma 2 relates \(L(s, b, c, n, l)\) to \(L(s, b, c, n)\) of [Sh]. Recall that two sequences \(b\) and \(c\) are similar, denoted by \(b \sim c\), if \(\lim(b/c) = 1\).

**Lemma 2.** Let \(l\) be a positive continuous function defined on \([0, s]\), \(0 < s < \infty\), and \(n \in \mathbb{N}\). Then

1. there are \(\alpha, \beta > 0\) such that \(\alpha \leq L(s, b, c, n, l)/L(s, b, c, n) \leq \beta\);
2. if \(\lim(b/c^{1/n}) = 0\), then we have
\[
L(s, b, c, n, l) \approx l(0)\lambda_n c^{-1/n}
\]
where \(\lambda_n = \int_0^\infty \exp(-t^n) dt\).

**Proof.** (1) is obvious, since \(l\) is positive and continuous on the compact set \([0, s]\) and we may take \(\alpha = \min(l[0, s])\) and \(\beta = \max(l[0, s])\).

(2) By a change of variable, we have
\[
L(s, b, c, n, l) = c^{-1/n} \int_0^{c^{1/n}} \exp((b/c^{1/n})t - t^n) l(t/c^{1/n}) dt.
\]
But since \(\exp(-t^n)\) decreases much faster than \(\exp((bc^{-1/n})t)\) increases as \(t\) goes to \(\infty\) (for \(c\) sufficiently large), it is easy to check, by the bounded convergence theorem, that
\[
\lim \int_0^{c^{1/n}} \exp((b/c^{1/n})t - t^n) l(t/c^{1/n}) dt = \int_0^\infty \exp(-t^n) l(0) dt
\]
(note that \(l\) is continuous at 0 and uniformly bounded on \([0, s]\)). Q.E.D.

Let \(\tilde{L}(S, b, c, f, l) := \int_S \exp(bt - cf(t)) l(t) dt\) for subsets \(S \subseteq [0, s]\). Lemma 3 is related to [Sh, Lemma A.4].

**Lemma 3.** Given sequences \(b\) and \(c\) with \(c\) and \(|c/b|\) diverging to \(\infty\) and a nonnegative continuous function \(f\) on \([0, s]\) with \(f^{-1}(0) = \{0\}\), we have

1. \(\tilde{L}([0, s], b, c, f, l) \approx \tilde{L}([0, s], b, c, f, l)\) for any \(s_1, s_2\) in \((0, s]\);
2. if \(f\) is of degree of contact \(n\) at 0 [Sh] with \(D^n f(0) = k\) and \(\lim(b/c^{1/n}) = 0\), then \(\tilde{L}([0, s], b, c, f, l) \approx \tilde{L}(s, b, \kappa c, n, l)\).

**Proof.** (1) Use Lemma A.3 in the appendix of [Sh] and the fact that the weight function \(l(t)\) is bounded away from both 0 and \(\infty\) for \(t\) in \([0, s]\).

(2) For each \(\varepsilon > 0\) there is \(\delta > 0\) such that
\[
(\kappa - \varepsilon)t^n < f(t) < (\kappa + \varepsilon)t^n
\]
for all $t$ in $[0, \delta]$, and hence by Lemma 2, we get
\[
I(0)\lambda_n((\kappa - \varepsilon)c)^{-1/n} \approx L(\delta, b, (\kappa - \varepsilon)c, n, l) \leq \tilde{L}([0, \delta], b, c, f, l)
\]
\[
\leq L(\delta, b, (\kappa + \varepsilon)c, n, l) \approx I(0)\lambda_n((\kappa + \varepsilon)c)^{-1/n}.
\]

Now use (1) and let $\varepsilon$ go to 0; it is easy to see that $\tilde{L}([0, s], b, c, f, l) \approx I(0)\lambda_n(\kappa c)^{-1/n} \approx L(s, b, \kappa c, n, l)$. Q.E.D.

Note that for a function $f$ of degree of contact $n$ we may assume that $\tilde{L}([0, s], b, c, f, l) \geq L(s, b, \theta c, n, l)$ for some $\theta > 0$ as in the proof of Lemma 3(2) by taking $s$ sufficiently small as long as it is allowed to shrink $s$.

3. The structure of $\Phi$

Applying Lemma 1 with $f = \phi$ on [Sh, pp. 276, 296] and $\mathcal{L} = \partial C \setminus K'_e$ where $K'_e = \{x \in \partial C \mid \text{dist}(x, R_\alpha) \leq \varepsilon + \gamma\}$, we have
\[
\exp(-2\gamma u_m) \lim w_m'(\nu - 1)^2
= \lim \frac{\tilde{L}'(K'_e, (2\nu + \varepsilon_m - 1)u^+ + (2\nu + \varepsilon_m - 1)u, -xu)}{\tilde{L}'(K'_e, (2\nu - 1)u^+, (2\nu - 1)u, -xu)}
\]
if the latter limit exists. Since points in $K'_e$ are parametrized by $\phi$ and $\psi$ (cf. [Sh, §4, conditions (I)–(III)]), we have
\[
\tilde{L}'(K'_e, (2\nu - 1)u^+, (2\nu - 1)u, -xu)
= L(S_\phi, (2\nu - 1)u^+, (2\nu - 1)u, \phi, l_\phi)
+ L(S_\psi, (2\nu - 1)u^+, (2\nu - 1)u, \psi, l_\psi)
\]
where $S_\phi := \{t | tu^+ - (\phi(t) + \gamma)u \in K'_e\}$ and $l_\phi(t) = \|\phi'(u(t))\|$ with $x(t) = tu^+ - (\phi(t) + \gamma)u$ (and similarly for $\psi$). (As in [Sh], we identify $\mathbb{R}u^+$ with $\mathbb{R}$ by identifying $y = tu^+$ with $t$.) Note that by the assumptions on $\phi$ and $\psi$ in condition (III), $S_\phi$ and $S_\psi$ are disjoint unions of intervals (or points) containing $F + \gamma u$ and $S_\psi \subseteq S_\phi$. However by Lemma 3, we may shrink $S_\phi$ and assume that $S_\phi = S_\psi$ without affecting the computation of $\lim(w_m(\nu - 1)^2)$.

Note that since $x(t)' = u^+ - \phi'(t)u$ is never 0 and $(e^{x(t)})' = (x_1(t) \exp(x_1(t)))$, by the general assumption on $\phi$, $l_\phi$ is uniformly bounded on $K_e$ and is bounded below away from 0 on any compact subset of $K_e$. Furthermore, $l_\phi$ has positive one-sided limits at each $\alpha_1$, say $\phi_1^\pm$. A similar statement holds for $\psi$.

When the slope $r'$ to which $((2\nu - 1)u)^\perp$, $(2\nu - 1)u$ belongs (see [Sh, p. 267] for definition) is $+\infty$ (resp. $-\infty$), we get
\[
\lim(w_m'(\nu - 1)^2) = \exp(2\varepsilon_m q_k) \quad (\text{resp. } \exp(2\varepsilon_m q_1))
\]
by Corollary A.2 in the appendix of [Sh] and a similar argument used in [Sh, §2] (and the remark after Lemma 3). So $\chi'(\nu - 1)$ determines a limit character corresponding to an endpoint of $F$ as in Case I on [Sh, pp. 284–285].

If $r' \in \mathbb{R}$ and $r' \neq 0$ when $N := \max\{n_i\} = \infty$, then by [Sh, Lemmas 2 and
3 and Corollary A.2], we have the similarity formula
\[
\exp(2\gamma u_m)w'_m(\nu - 1)^2 \approx \left\{ \sum_{i, \sigma} \left[ \tilde{I}'(\nu + \epsilon_m, f, i, \sigma) + \tilde{I}'(\nu + \epsilon_m, g, i, \sigma) \right] \right\} / \left\{ \sum_{i, \sigma} \left[ \tilde{I}'(\nu, f, i, \sigma) + \tilde{I}'(\nu, g, i, \sigma) \right] \right\}
\]
\[
\approx \left\{ \sum_{i, \sigma} \left[ \phi_{ia} \tilde{I}'(\nu + \epsilon_m, f, i, \sigma) + \psi_{ia} \tilde{I}'(\nu + \epsilon_m, g, i, \sigma) \right] \right\} / \left\{ \sum_{i, \sigma} \left[ \phi_{ia} \tilde{I}'(\nu, f, i, \sigma) + \psi_{ia} \tilde{I}'(\nu, g, i, \sigma) \right] \right\}
\]
where
\[
\tilde{I}'(\nu, f, i, \sigma) = \exp(a_i(2\nu - 1)u^\perp) \\
x \tilde{L}([0, \sigma(I_{ia} - a_i)], (2\nu - 1)u^\perp, (2\nu - 1)u, f_{ia}, l_{\phi_{ia}})
\]
with \( \sigma = \pm \), \( l_{\phi_{ia}}(t) = l_{\phi}(a_i + \sigma t) \), and \( \tilde{I}' \), \( f_{ia} \), \( g_{ia} \) are as defined in [Sh, §4] (recall that \( f_{ia}(t) = \phi(a_i + \sigma t) \)). In case of Bergman spaces, we have the same similarity formula except that the nontrivial positive coefficients \( \phi_{ia} \) and \( \psi_{ia} \) are replaced by 1 and the plus sign in front of the \( g \) terms is replaced by the minus sign (cf. [Sh, pp. 283, 298]). These differences will not matter and the procedure used in [Sh] to derive the limit characters \( \lim(\chi'(\nu - 1)) \) can be carried out here with only inessential modification. So we shall omit it.

If \( r' = 0 \) and \( N = \infty \), then by [Sh, Lemmas 2, 3 and Corollary A.2], we have
\[
\exp(2\gamma u_m)\lim(w'_m(\nu - 1)^2)
\]
\[
= \int \exp(t(\ln(\rho') + 2\epsilon_m u^\perp))l_{\phi}(t) dt / \int \exp(t\ln(\rho'))l_{\phi}(t) dt,
\]
an equation similar to the corresponding one gotten for \( H^2(D) \) on [Sh, p. 286] except the presence of weight \( L_{\phi} \), where the integral is over \( \phi^{-1}([0]) \) (or \( F_0 + \gamma u \)).

Thus we get a result similar to what was obtained in [Sh, §§2, 4]; namely, every sequence \( \nu \) in \( \mathbb{Z}_2 \) with \( \nu \to u \) \( (r', \rho', \omega') \) for some unit vector \( u \) in \( \mathbb{R}_+^2 \) determines a limit character \( \chi'(r', \rho', \omega') \) that depends on \( r', \rho', \omega' \) (and \( F_u \)) only, and so the groupoid \( \mathcal{G}' \) is parametrized by \( r', \rho', \omega' \), and \( F_u \) (with some redundancy as noted on [Sh, p. 291] for the case of Bergman spaces). Now by [Sh, Lemma 1.3], \( \nu \to u \) \( (r', \rho', \omega') \) if and only if \( \nu \to u \) \( (r', \rho', \omega') \) \( \exp(-1u^\perp) \). Thus the map sending \( (r', \rho', \omega') \) to
\[
(r, \rho, \omega) = (r', \rho' \exp(-1u^\perp), \omega' \exp(-1u^\perp))
\]
induces a one-to-one correspondence from \( \mathcal{G}' \) to \( \mathcal{G} \), which is a topological groupoid isomorphism, and hence \( C^*(\mathcal{G}) \cong C^*(\mathcal{G}') \). Then by the general framework of [CuM] about multivariable weighted shifts and the argument used
in [Sh, proof of Theorem 3.1], we have $\mathcal{I}(D) \cong \mathcal{I}(\partial D)$, which is the goal of this paper. Note that since the structure of $\mathcal{I}(D)$ is determined in [Sh], so is the structure of $\mathcal{I}(\partial D)$ by our result.

REFERENCES


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