

p -POINTS IN ITERATED FORCING EXTENSIONS

JUDY ROITMAN

ABSTRACT. Selective ultrafilters exist in direct iterated ccc extensions whose length has uncountable cofinality, as do p -points which are not selective. Nonselective p -points also exist e.g. in an iterated Laver or Mathias extension of length ω_2 over a model of CH.

0. Introduction. The question of whether p -points exist in $\beta N - N$ is an old one. In 1956 Walter Rudin showed that they exist under CH. About a dozen years later, Booth constructed a selective ultrafilter under MA, and soon thereafter Ketonen showed that every dominating family in ${}^\omega\omega$ has cardinality \mathfrak{C} iff every filter with a base of cardinality $< \mathfrak{C}$ extends to a p -point. The question became: could p -points or selective ultrafilters not exist? In [Ku], Kunen showed that no selective ultrafilters exist in a simultaneous extension by enough random reals; he also knew that selective ultrafilters exist in direct iterated ccc extensions whose length is regular and equals the new continuum. We extend this result by showing

THEOREM 1. *Selective ultrafilters exist in direct iterated ccc extensions whose length has uncountable cofinality.*

Also in [Ku], Kunen showed that under MA there exist p -points which are not selective. We show

THEOREM 2. *Nonselective p -points exist in direct iterated ccc extensions whose length has uncountable cofinality.*

These theorems are proved in §3. In §4 we indicate how to extend the method of proof to non-ccc extensions, proving

THEOREM 3. *Nonselective p -points exist in δ -cc extensions whose length has cofinality $\geq \delta$ and which have cofinally many Laver, Mathias, or Cohen reals.*

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1. Topological preliminaries. A p -point is a nonprincipal ultrafilter $x \in \beta N - N$ such that every countable intersection of neighborhoods of x contains a neighborhood of x . We will not use this definition, however, but rely instead

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on the following well-known equivalence:

$x \in \beta N - N$ is a *p*-point iff for every $P = \{P_i : i \in \omega\}$ a partition of ω , either (i) some $P_i \in x$, or (ii) $\exists A \in x (|A \cap P_i| < \omega$ for all i).

By replacing “ $< \omega$ ” in clause (ii) with “ ≤ 1 ” we define a selective ultrafilter.

It will be convenient to relativize these definitions as follows: if $M \subset V$ are models of set theory, then $x \in V$ is a *p*-point over M iff for every partition $P \in M$, either (i) or (ii) is satisfied. Selective over M is defined similarly.

For the proof of Theorem 2, we will need the following observation: if x is selective then $\exists A \in x (\sum_{k \in A} 1/k < \infty)$. This is an immediate consequence of 0.1 in [Ku].

2. Forcing preliminaries. Two elements of a Boolean algebra are said to be incompatible if their inf is 0. A Boolean algebra has the δ -chain condition (δ -cc) iff every collection of incompatible elements is of cardinality $< \delta$. The ω_1 -cc is usually called the ccc.

M is always assumed to be a transitive model of set theory. A function $f \in {}^\omega \omega$ is said to be Cohen over M iff f is generic over M with respect to the Cohen partial order (= finite functions from ω to ω with the order of reverse inclusion).

By an iterated Boolean algebra of length κ we mean a complete Boolean algebra $\mathbf{B} \in M$, intermediate algebras \mathbf{B}_α , $\alpha < \kappa$, and sets of terms \mathbf{C}_α , $\alpha < \kappa$, such that:

- (1) $M^{\mathbf{B}_\alpha} \subset M^{\mathbf{B}_\gamma}$ when $\alpha < \gamma$;
- (2) \mathbf{B} is the direct sum of each pair $\mathbf{B}_\alpha, \mathbf{C}_\alpha$;
- (3) \mathbf{C}_α is complete in $M^{\mathbf{B}_\alpha}$, for $\alpha < \kappa$;
- (4) each \mathbf{B}_α is complete in M , $\alpha < \kappa$;
- (5) \mathbf{B} is the completion of $\bigcup_{\alpha < \kappa} \mathbf{B}_\alpha$.

The iteration is direct if, for each limit $\gamma < \kappa$, \mathbf{B}_γ is the completion of $\bigcup_{\alpha < \gamma} \mathbf{B}_\alpha$. We write $M_\alpha = M^{\mathbf{B}_\alpha}$.

As usual we blur the line between sets and Boolean-valued objects, thus writing, e.g., $A \subset \omega$ instead of $[A \subset \omega] = 1$.

The salient facts we need include the well-known:

FACT 4. If $\mathbf{B} \in M$ is a δ -cc iteration of length κ and $\text{cf}(\kappa) \geq \delta$, then $A \subset \omega$ and $A \in M^{\mathbf{B}} \rightarrow A \in M_\alpha$ for some $\alpha < \kappa$.

As a trivial corollary, we have, under the same assumptions on \mathbf{B} , that every partition of ω in $M^{\mathbf{B}}$ sits in some M_α , for $\alpha < \kappa$, as does every $f \in {}^\omega \omega \cap M^{\mathbf{B}}$.

We also use

FACT 5 (KUNEN). If $\mathbf{B} \in M$ is a direct ccc iteration of length κ , $\text{cf}(\kappa) > \omega$, and each \mathbf{B}_α is not with value 1 the 2-element algebra, then for every $\alpha < \kappa$, $\exists f_\alpha \in M^{\mathbf{B}}$ (f_α is Cohen over M_α).

In the situation of Fact 5, we say that $M^{\mathbf{B}}$ has cofinally many Cohen reals and wlog we assume that $f_\alpha \in M_{\alpha+1}$.

Finally, as a notational convenience, a partition is understood to be a partition of ω into infinitely many pieces, and if P is a partition, P_j is the j th element of P . This convention has already been used in these introductory sections.

3. Proofs.

We fix M , and \mathbf{B} a complete direct ccc iterated Boolean algebra in M of length κ , $\text{cf}(\kappa) > \omega$. Since we will be inducting on $\text{cf}(\kappa)$, wlog assume κ is regular. We work in $M^{\mathbf{B}}$.

To prove Theorem 1 we will, by induction, construct an ascending chain of nonprincipal filters $x_\alpha \in M_\alpha$, $\alpha < \kappa$, such that

- (1) x_γ is an ultrafilter in M_γ extending $\bigcup_{\alpha < \gamma} x_\alpha$.
- (2) $x_{\alpha+1}$ is selective over M_α .

By Fact 4, $\bigcup_{\alpha < \kappa} x_\alpha$ will be our desired selective ultrafilter.

Clearly the only question is how to construct $x_{\alpha+1}$. For this we need more definitions.

DEFINITION 6. Let $P = \{P_i : i < \omega\}$ be a partition of ω , $f \in {}^\omega\omega$. Then we define

$$A^{f,P} = \{n : n \in P_k \rightarrow f(k) = n\} \quad \text{and} \quad A_{f,P} = \{n : n \in P_k \rightarrow f(k) \geq n\}.$$

(The $A_{f,P}$'s will be used to prove Theorem 2.)

The idea behind constructing $x_{\alpha+1}$ is to slice through a partition $P \in M_\alpha$ by putting $A^{f_\alpha,P}$ into $x_{\alpha+1}$, where $f_\alpha \in M_{\alpha+1}$ is the real, guaranteed by Lemma 5, Cohen over M_α . We have to exercise some discretion, however, since if we tried this for all partitions $P \in M_\alpha$ we would not have a nonprincipal filter.

Let $\mathcal{P} \in M_\alpha$ be the family of all partitions $P \in M_\alpha$ such that $\forall j (P_j \notin x_\alpha)$. We note that \mathcal{P} is maximal with respect to the following property:

(*) if S is a finite subset of \mathcal{P} , and $A \in x_\alpha$, then $A \cap \bigcap_{P \in S} \bigcup_{j > m} P_j$ is infinite, for all $m < \omega$.

We let $x_{\alpha+1}^* = x_\alpha \cup \{A^{f_\alpha,P} : P \in \mathcal{P}\}$. A standard genericity argument shows that $x_{\alpha+1}^*$ is a nonprincipal filter, since if p is a Cohen condition and A, S are as in the hypothesis of (*), p can be extended to make $A \cap \bigcap_{P \in S} A^{f_\alpha,P}$ arbitrarily large.

Let $x_{\alpha+1} \supset x_{\alpha+1}^*$ be a nonprincipal ultrafilter in $M_{\alpha+1}$. Condition (1) is trivially satisfied. By definition of \mathcal{P} , if $P \in M_\alpha - \mathcal{P}$, then some $P_j \in x_\alpha$. So (2) is satisfied. Thus $x_{\alpha+1}$ is selective relative to M_α , and Theorem 1 is proved.

Before proving Theorem 2 we make more ad hoc definitions.

DEFINITION 7. $A \subset \omega$ is said to be small if $\sum_{k \in A} 1/k < \infty$. A is large if $\omega - A$ is small. We define $\bar{x} = \{A \subset \omega : A \text{ is large}\}$.

From first year calculus we have

FACT 8. \bar{x} is a nonprincipal filter, as is the filter generated by $\bar{x} \cup \{A\}$ for any A which is not small, $A \subset \omega$.

By induction we construct an ascending chain of nonprincipal filters $x_\alpha \in M_\alpha$, $\alpha < \kappa$, such that:

(1) x_γ is an ultrafilter in M_γ extending $\bigcup_{\alpha < \gamma} x_\alpha$, and containing no small sets.

(2) $x_{\alpha+1}$ is a p -point over M_α .

(3) $A \in M_\alpha$ and A small $\rightarrow A \notin x_\alpha$.

Again by Fact 4, $\bigcup_{\alpha < \kappa} x_\alpha$ is our desired p -point, while by property (4) and the criterion mentioned in §1, it is not selective. It remains to actually perform the induction.

Again we only show the construction of $x_{\alpha+1}$, given $x_\beta, \beta < \alpha$, satisfying the inductive hypothesis. As indicated in the remark of Definition 6, $x_{\alpha+1}$ will be constructed by adding $A_{f_\alpha, P}$ to x_α for selected P , where f_α is as in the proof of Theorem 1.

First we let $\bar{x}_{\alpha+1} = x_\alpha \cup [\bar{x} \cap M_{\alpha+1}]$. By Fact 8, $\bar{x}_{\alpha+1}$ is nonprincipal. Now let $\mathcal{P} \in M_\alpha$ be the family of all partitions P such that $\forall j (P_j \notin \bar{x}_{\alpha+1})$. We note that \mathcal{P} is maximal with respect to $(**) = (*)$ with x_α replaced by $\bar{x}_{\alpha+1}$, and “infinite” replaced by “not small.”

We want to show that $x_{\alpha+1}^* = \bar{x}_{\alpha+1} \cup \{A_{f_\alpha, P} : P \in \mathcal{P}\}$ is nonprincipal. To do this, by Fact 8 it suffices to show that if $A \in x_\alpha$ and S is a finite subset of \mathcal{P} then $C = A \cap \bigcap_{P \in S} A_{f_\alpha, P}$ is not small. Another genericity argument does it: if p is a Cohen condition, by $(**)$ we can extend it to make $\sum_{k \in C} 1/k$ arbitrarily large.

Again let $x_{\alpha+1}$ be an ultrafilter in $M_{\alpha+1}$ extending $x_{\alpha+1}^*$. All properties are satisfied as before. Thus Theorem 2 is proved.

4. Extensions. In the previous section we actually proved the following:

THEOREM 9. *In an iterated γ -cc extension whose length has cofinality $\geq \delta$, and which has cofinally many Cohen reals,*

- (a) \exists a selective ultrafilter,
- (b) \exists a p -point which is not selective.

Furthermore, the properties of Cohen reals used are rather weak. They are: Property A. $\exists f \in {}^\omega \omega - M; \forall A \subset \omega, A \in M; \forall S$ a finite set of partitions, $S \in M$,

(†) if $A \cap \bigcap_{P \in S} \bigcup_{j > m} P_j$ is infinite, for all $m \in \omega$, then $A \cap \bigcap_{P \in S} A^{f, P}$ is infinite;

(††) if $A \cap \bigcap_{P \in S} \bigcup_{j > m} P_j$ is not small, for all $m < \omega$, then $A \cap \bigcap_{P \in S} A_{f, P}$ is not small.

The existence of a function satisfying A(††) is easily seen to be equivalent to the following statement:

(√) $\exists h \in {}^\omega \omega - M$ such that for every $g \in {}^\omega \omega \cap M, \{n: f(n) \geq g(n)\}$ is infinite.

(√) is verified by generic reals other than Cohen reals. As examples, we quote the definitions of Laver and Mathias reals, leaving it to the reader to prove that both satisfy the conclusion of (√).

The Mathias ordering is the set of all $\langle s, A \rangle$, where s is a finite increasing sequence on ω and A is an infinite increasing sequence on ω , with $\sup s <$

$\inf A$. We say $\langle s, A \rangle \leq \langle t, B \rangle$ iff $s \subseteq t$, $\text{range}(s - t) \subseteq \text{range } B$, and $\text{range } A \subseteq \text{range } B$. The object generic with respect to this ordering is an increasing sequence on ω ; the function listing that sequence in order is a Mathias real.

Similarly, Laver reals are defined by the following partial order: consider the set of all trees T on $\omega^{<\omega}$ (= finite sequences on ω) such that for some k , if $t \in T$ is a node on the n th level, $n > k$, then t has infinitely many successors on the $(n + 1)$ st level; in particular, each T is a subset of $\omega^{<\omega}$, and the order is reverse inclusion.

Neither Laver nor Mathias forcing have ccc. Thus Theorem 3, proved by the remark after the definition of Property A, really needs to relax ccc to some δ -cc. Furthermore, a corollary of Laver's use of Laver reals is that adding a Laver real does not add a Cohen real; and it can be shown that adding a Mathias real also does not add a Cohen real. Thus Theorem 3 is not a consequence of Theorem 9. (It is a peculiar fact that adding two Laver reals side by side does add a Cohen real; as does adding two Mathias reals side by side.)

The major question left is, of course, whether there is a model with no p -points. Other questions are: Do selective ultrafilters exist in the models of Theorem 3? (Galvin) Is there is model in which every p -point is selective and p -points exist? (Kunen has recently shown the consistency of " p -points exist but not selective ultrafilters.")

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66044