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A SINGULAR STOCHASTIC INTEGRAL EQUATION

DAVID NUALART AND MARTA SANZ

ABSTRACT. This note is devoted to the discussion of the stochastic differential equation $XdX + YdY = 0$, X and Y being continuous local martingales. A method to construct solutions of this equation is given.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t, t \geq 0\}$ a filtration on it satisfying the usual properties. That means, $\mathcal{F}_t$ is right continuous, and $\mathcal{F}_0$ contains the null sets of $\mathcal{F}$.

Let $X = \{X_t, t \geq 0\}$ be a continuous local martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$. The continuous local martingale $M_t = \int_0^t X_s\, dX_s$ has an associated increasing process given by $\langle M\rangle_t = \int_0^t X_s^2\, d\langle X\rangle_s$. Denoting by $\langle M\rangle$ and $\langle X\rangle$ the measures on $\mathbf{R}$ induced by the sample paths $\langle M\rangle_t$ and $\langle X\rangle_t$, respectively, we obviously have $\langle M\rangle \ll \langle X\rangle$, and $d\langle M\rangle/d\langle X\rangle = X^2$. Reciprocally, $\langle X\rangle \ll \langle M\rangle$, and $d\langle X\rangle/d\langle M\rangle = 1/X^2$. In fact, it is known that $\langle X\rangle$ does not charge the set $\{X = 0\}$.

By Itô's formula $X_t^2 - X_0^2 = 2M_t + \langle X\rangle_t$. So, applying the preceding result we have

$$X_t^2 - X_0^2 = 2M_t + \int_0^t \frac{1}{X_s^2}\, d\langle M\rangle_s.$$

Therefore, a continuous local martingale Y is a solution of the stochastic differential equation

$$XdX + YdY = 0, \tag{1}$$

if and only if

$$Y_t^2 - Y_0^2 = -2M_t + \int_0^t \frac{1}{Y_s^2}\, d\langle M\rangle_s. \tag{2}$$

Equation (1) arises in a natural way in the theory of two-parameter martingales with path independent variation adapted to the σ-fields generated by two independent couples of two-dimensional brownian motions (see [4]).

First we prove a lemma that will be used to construct Y_t^2.

LEMMA 1. *Let $b(t)$ be a continuous real function defined on $\mathbf{R}_+$ such that $b(0) \geq 0$ and μ a continuous measure on $\mathbf{R}_+$. Assume that $b(t)$ takes constant values on every interval $[a, c]$ such that $\mu([a, c]) = 0$. Then the integral equation*

$$r(t) = b(t) + \int_0^t \frac{1}{r(s)}\, d\mu_s \tag{3}$$

has a unique, nonnegative, continuous solution.

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PROOF. The case $d\mu_s = ds$ was proved by McKean in [3]. In the general case define $G(s) = \inf\{t\colon F(t) > s\}$, F being the distribution function of μ. G is right continuous and $F \circ G = \mathrm{Id}$.

$b \circ G$ is continuous. Indeed, fix a point $s > 0$, and suppose that $G(s) = c$, $G(s^-) = a$. Then $F(t) = s$ for all t in $[a, c]$ and, by hypothesis, b is constant on $[a, c]$, which proves the continuity of $b \circ G$ on s.

Using McKean's result, we know that equation

$$r'(t) = b(G(t)) + \int_0^t \frac{1}{r'(s)}\, ds$$

has a unique, nonnegative, continuous solution.

Then, if we define $r = r' \circ F$, r is a solution of (3). In fact,

$$r'(F(t)) = b(G(F(t))) + \int_0^{F(t)} \frac{1}{r'(s)}\, ds,$$

but

$$\int_0^{F(t)} \frac{1}{r'(s)}\, ds = \int_0^{F(t)} \frac{1}{r'(F(G(s)))}\, ds = \int_0^t \frac{1}{r'(F(s))}\, d\mu_s,$$

and $b \circ G \circ F = b$ as easily follows from the assumption made on b.

Now we can state the main result.

THEOREM 1. *Let X be a continuous local martingale. Assume that there exists a sequence ϕ_n of independent random variables, with $\phi_n \in \{-1, 1\}$ and $E(\phi_n) = 0$, such that they are independent of X and $\mathcal{F}_0$-measurable. Then there exists a continuous local martingale Y such that $\int_0^t (X_s\, dX_s + Y_s\, dY_s) = 0$ for all $t \geq 0$.*

PROOF. First we choose an integrable random variable Y_0 which will be the value of Y at the origin.

Let us consider the equation

$$r(t) = Y_0^2 - 2M(t) + \int_0^t \frac{1}{r(s)}\, d\langle M\rangle_s, \tag{4}$$

where $M_t = \int_0^t X_s\, dX_s$.

It is well known (see [2]) that there exists a null set $N \subset \Omega$ such that for all $\omega \notin N$, $\langle M\rangle([a, c]) = 0$ implies that M takes a constant value on the interval $[a, c]$.

So, for $\omega \notin N$ fixed, we can apply Lemma 1 to $b(t) = Y_0^2 - 2M(t)$ and $\mu = \langle M\rangle$, and state the existence of a unique, nonnegative, continuous solution of (4).

Now $r(t)$ is a local submartingale because it is the sum of a local martingale and the increasing process $\int_0^t (1/r(s))\, d\langle M\rangle_s$. We want to show that $\sqrt{r(t)}$ also is a local submartingale. To do this, apply Itô's formula to $f(r(t)) = \sqrt{r(t) + \lambda}$, where λ is any real positive number,

$$\begin{aligned}\sqrt{r(t) + \lambda} &= \sqrt{r(0) + \lambda} - \int_0^t (r(s) + \lambda)^{-1/2}\, dM(s) \\ &\quad + \int_0^t \frac{1}{2}(r(s) + \lambda)^{-1/2} \left(\frac{1}{r(s)} - \frac{1}{r(s) + \lambda}\right) d\langle M\rangle_s.\end{aligned}$$

Let $\{T_n, n \in N\}$ be an increasing sequence of stopping times such that $T_n \uparrow \infty$ and M^{T_n} is a martingale bounded by n. Set $R_n = \inf\{t, r(t) \geq n\}$ and $S_n = T_n \wedge$

R_n. Then $\sqrt{r(t \wedge S_n) + \lambda}$ is a positive submartingale, and letting λ tend to zero we obtain, by monotone convergence, the submartingale property of $\sqrt{r(t \wedge S_n)}$. Therefore $\sqrt{r(t)}$ is a local submartingale because $S_n \uparrow \infty$.

The increasing process given by the Doob decomposition of $\sqrt{r(t)}$ is $A_t = \sqrt{r(t)} - \sqrt{r(0)} + \int_0^t (r(s))^{-1/2}\, dM(s)$. In fact, for any $\lambda > 0$, $\sqrt{r(t \wedge S_n) + \lambda} - \sqrt{r(0) + \lambda} + \int_0^{t \wedge S_n} (r(s) + \lambda)^{-1/2}\, dM(s)$ is increasing.

We next show that $\int_0^t 1_{\{\sqrt{r(s)} > 0\}}\, dA_s = 0$. Indeed, we have

$$r(t) = (A_t + \sqrt{r(0)})^2 - 2(A_t + \sqrt{r(0)}) \int_0^t (r(s))^{-1/2}\, dM(s) + \left(\int_0^t (r(s))^{-1/2}\, dM(s) \right)^2 .$$

Computing the bounded variation part of each term we obtain

$$r(0) + \int_0^t \frac{1}{r(s)}\, d\langle M \rangle_s = (A_t + \sqrt{r(0)})^2 - 2 \int_0^t \left(\int_0^s (r(u))^{-1/2}\, dM(u) \right) dA_s + \int_0^t \frac{1}{r(s)}\, d\langle M \rangle_s .$$

Therefore,

$$0 = (A_t + \sqrt{r(0)})^2 - r(0) - 2 \int_0^t \left(\int_0^s (r(u))^{-1/2}\, dM(u) \right) dA_s = 2 \int_0^t \sqrt{r(s)}\, dA_s,$$

which implies the assertion.

Henceforth, for any $n \geq 1$, $\{\rho_n(t) = \sqrt{r(S_n + t) \wedge S_{n+1}}, \mathcal{F}_{S_n + t}, t \geq 0\}$ is a submartingale which satisfies $\int_0^t 1_{\{\rho_n(s) > 0\}}\, dA_s^n = 0$, where

$$A_t^n = A((S_n + t) \wedge S_{n+1})$$

is the increasing process associated to $\rho_n(t)$. Using Barlow's procedure (see [1]) it is possible to find a martingale $M_n(t)$ such that $|M_n(t)| = \rho_n(t)$ and $M_n(0) = M_{n-1}(S_n)$ (we take $S_0 = 0$ and $M_0 = Y_0$). Then $Y = \sum_n M_n 1_{[\![T_n, T_{n+1}]\!]}$ is a nonnegative local submartingale whose absolute value is $\sqrt{r(t)}$. Note that according to Barlow's method, the sign of Y is defined in terms of the sequence ϕ_n. So $Y_t^2 = r(t)$ satisfies equation (2) and this finishes the proof of the theorem. □

REMARKS. 1. Instead of assuming that the σ-field $\mathcal{F}_0$ is rich enough to contain the sequence ϕ_n, we can adjoin a new probability space and show the existence of the local martingale Y in the product space.

2. There is no uniqueness of the solution because $-Y$ also is a solution. Moreover the random variable Y_0 is arbitrary.

3. Let $T = \inf\{t, Y_t = 0\}$. If the initial value Y_0 is given, there is uniqueness of the solution in $[0, T]$. Furthermore, to construct the solution in this interval we do not need the sequence ϕ_n.

4. Suppose that $Y_0 = X_0$. In this case, the processes X^2 and Y^2 may have the same law. A sufficient condition for it to hold is the equality between the law of $\{(M_t, \langle M \rangle_t), t \geq 0\}$ and that of $\{(-M_t, \langle M \rangle_t), t \geq 0\}$. For example, this condition holds if M_t is a Wiener process.

5. As a consequence of Theorem 1 we obtain the existence of nonnull two-dimensonal martingales with increasing norm.

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DEPARTAMENT D'ESTADÍSTICA, FACULTAT DE MATEMÀTIQUES, UNIVERSITAT DE BARCELONA, BARCELONA, SPAIN

DEPARTAMENT D'ESTADÍSTICA, FACULTAT DE CIÈNCIES, UNIVERSITAT AUTÒNOMA DE BARCELONA, BELLATERRA (BARCELONA), SPAIN